Martingale Pricing Applied to Options, Forwards and Futures

We now apply martingale pricing theory to the problems of pricing European and American options, as well as forwards and futures contracts. We first consider the problem of pricing European and American options. We will interpret the classic binomial model in the context of martingale pricing theory and use it to further develop our intuition. Then after using the binomial model to derive the famous Black-Scholes formula, we will consider American options. After a brief discussion on hedging, we then end with forwards and futures prices, and in the process we will determine under what situations the two coincide.

Options Theory: Put-Call Parity and No-Arbitrage Arguments

Definition 1 *A* European call option *on a security with strike K and expiration T is worth* $\max(0,\ S_T - K)$ *at expiration where S_T is the time T value of the security. We will use C_t to denote the date t value of a call option.*

Definition 2 *A* European put option *on a security with strike K and expiration T is worth* $\max(0,\ K - S_T)$ *at expiration. We will use P_t to denote the date t value of a put option.*

Option pricing theory refers to the methodology and theory that is used to compute C_t and P_t, as well as the prices of other classes of options. The general theory of arbitrage pricing is of fundamental importance in this regard. We begin with the famous put-call parity equation that relates European call option prices to the corresponding European put option prices.

Theorem 1 (Put-Call Parity) *The call and put options prices, C_t and P_t respectively, satisfy*

$$S_t \ = \ C_t \ - \ P_t \ + \ d(t,T)K \tag{1}$$

where $d(t,T)$ is the discount factor for lending between dates t and the expiration date of the options, T. K is the strike of both the call option and the put option, and the security is assumed to pay no intermediate dividends.

Proof: Consider two portfolios: at date t the first portfolio is long a call option, short a put option and invests $\$d(t,T)K$ at the risk-free rate until T. At date t the second portfolio is long a single share of stock. It is easy to see that at time T, the value of both portfolios is exactly equal to S_T, the terminal stock price. This means that both portfolios must have the same value at date t for otherwise there would be an arbitrage opportunity. In particular, this implies (1). ■

As with put-call parity, simple applications of the no-arbitrage assumption often allow us to derive results that must hold independently of the underlying probability model. Example 1 provides some other instances of these applications.

Example 1 (Luenberger Exercise 12.4)

Consider a family of call options on a non-dividend paying stock, each option being identical except for its strike price. The value of a call option with strike price K is denoted by $C(K)$. Prove the following three general relations using arbitrage arguments.

(a) $K_2 > K_1$ implies $C(K_1) \geq C(K_2)$.

(b) $K_2 > K_1$ implies $K_2 - K_1 \geq C(K_1) - C(K_2)$.

(b) $K_3 > K_2 > K_1$ implies

$$C(K_2) \leq \left(\frac{K_3 - K_2}{K_3 - K_1}\right) C(K_1) \ + \ \left(\frac{K_2 - K_1}{K_3 - K_1}\right) C(K_3).$$

Solution

(a) is trivial and **(b)** is also very simple if you consider the maximum possible difference in value at maturity of two options with strikes K_1 and K_2.

(c) Perhaps the easiest way to show this (and solve other problems like it) is to construct a portfolio whose properties imply the desired inequality. For example, if a portfolio's value at maturity is non-negative in every possible state of nature, then the value of the portfolio today must be also be non-negative. Otherwise, there would be an arbitrage opportunity. What would it be? We will use this observation to solve this question.

Construct a portfolio as follows: short one option with strike K_2, go long $(K_3 - K_2)/(K_3 - K_1)$ options with strike K_1, and go long $(K_2 - K_1)/(K_3 - K_1)$ options with strike K_3. Now check that the value of this portfolio is non-negative in every possible state of nature, i.e., for every possible value of the terminal stock price. As a result, the value of the portfolio must be non-negative today and that gives us the desired inequality. ▮

The Binomial Lattice Model for Security Prices

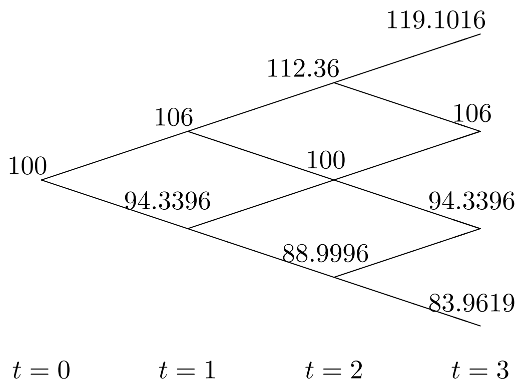

Figure 1

The binomial model is a discrete-time, discrete space model that describes the price evolution of a single risky stock[1] that does not pay dividends. If the stock price at the beginning of a period is X then it will either increase to uX or decrease to dX at the beginning of the next period. In Figure 1 above we have set $S_0 = 100$, $u = 1.06$ and $d = 1/u$.

The binomial model assumes that there is also a cash account available that earns risk-free interest at a gross rate of R per period. We assume R is constant[2] and that the two securities (stock and cash account) may be

[1] Binomial models may also be used to model commodity and foreign exchange prices, as well as dividend-paying stock prices and interest rate dynamics.

[2] This assumption may easily be relaxed. For example, in the *Martingale Pricing Theory* lecture notes, we saw the market in Example 9 was a complete market with stochastic interest rates. We could therefore allow R to vary from node to node in the binomial model here and still maintain market completeness.

purchased or sold[3] short. We let $B_k = R^k$ denote the value at time k of \$1 that was invested in the cash account at date 0.

In order to avoid arbitrage opportunities, it must be the case (why?) that

$$\boxed{d \; < \; R \; < \; u.} \tag{2}$$

We will always assume that (2) is satisfied and so we can therefore conclude that a risk-neutral or equivalent martingale measure (EMM), Q, exists. Moreover, the binomial model is a *complete* model[4] and so Q is unique. We will usually use the cash account, B_k, as the numeraire security. The price of any security can therefore be computed by evaluating the discounted expected payoff of the security using the risk-neutral probability measure, Q. Thus the time t price of a security[5] that is worth X_T at time T (and does not provide any cash flows in between) is given by

$$X_t \;=\; B_t \, \mathrm{E}_t^Q\left[\frac{X_T}{B_T}\right] \;=\; \frac{1}{R^{T-t}} \, \mathrm{E}_t^Q[X_T]. \tag{3}$$

We restate one of our martingale pricing results in the specific case of the binomial model:

Theorem 2 *The deflated price process, X_t/B_t, is a Q-martingale.*

As we stated in the *Martingale Pricing Theory* lecture notes, this is an extremely important result that holds in just about every model where arbitrage opportunities are precluded.

Exercise 1 *Show that the Q-probability of an up-move is $q \;=\; \frac{R-d}{u-d}$. This then implies that the Q-probability of a down-move is $1-q \;=\; \frac{u-R}{u-d}$.*

The binomial model is one of the workhorses of financial engineering. In addition to being a complete model (which means we can identify a unique Q), it is also *recombining*. For example, an up-move followed by a down-move leads to the same node as a down-move followed by an up-move. This recombining feature implies that the number of nodes in the tree grows linearly with the number of time periods rather than exponentially. This leads to a considerable gain in computational efficiency when it comes to pricing *path-independent* securities.

Example 2 (Pricing a Call Option)

Compute the price of a European call option on the security of Figure 1 with expiration at $T = 3$, and strike $K = 95$. Assume also that $R = 1.02$.

Solution: First, we find $q = \frac{R-d}{u-d} = \frac{1.02-1.06^{-1}}{1.6-1.06^{-1}} = 0.657$ which is the Q-probability of an up-move. If C_0 denotes the date 0 price of the option then (3) implies that it is given by

$$C_0 \;=\; \frac{1}{R^3} \, \mathrm{E}_0^Q[C_T] \;=\; \frac{1}{R^3} \, \mathrm{E}_0^Q[\max(0, \; S_3 - 95)]. \tag{4}$$

At this point, there are two possible ways in which we can proceed:

(i) Compute the Q-probabilities of the terminal nodes and then use (4) to determine C_0. This method does not bother to compute the intermediate prices, C_t.

(ii) Alternatively, we can work backwards in the tree one period at a time to find C_t at each node and at each time t. This procedure is sometimes referred to as *backwards recursion* and is no more than a simple application of dynamic programming.

[3] Short selling the cash account is equivalent to borrowing.

[4] The binomial model is complete since each embedded one-period model is complete. This is true because the matrix of payoffs corresponding to each embedded one-period model has rank 2 which is equal to the number of possible outcomes.

[5] X_t can also be the value process of a self-financing trading strategy.

Stock Price				European Option Price			
			119.10				24.10
		112.36	106.00			19.22	11.00
	106.00	100.00	94.34		14.76	7.08	0.00
100.00	94.34	89.00	83.96	11.04	4.56	0.00	0.00
t=0	t=1	t=2	t=3	t=0	t=1	t=2	t=3

For example in the *European Option Payoff* table above, we see that $14.76 = \frac{1}{R}\left(q(19.22) + (1-q)(7.08)\right)$, i.e., the value of the option at any node is the discounted expected value of the option one time period ahead. This is just restating the Q-martingale property of discounted security price processes. We find that the call option price at $t = 0$ is given by $\$11.04$. ∎

Example 3 (A Counter-Intuitive Result)

Consider the same option-pricing problem of Example 2 except that we now take $R = 1.04$. We then obtain a European call option price of 15.64 as may be seen from the lattices given below. Note that this price is *greater* than the option price, 11.04, that we obtained in Example 2.

Stock Price				European Option Price			
			119.10				24.10
		112.36	106.00			21.01	11.00
	106.00	100.00	94.34		18.19	8.76	0.00
100.00	94.34	89.00	83.96	15.64	6.98	0.00	0.00
t=0	t=1	t=2	t=3	t=0	t=1	t=2	t=3

This observation seems counterintuitive: after all, we are dealing only with positive cash flows, the values of which have not changed, i.e. the option payoffs upon expiration at $t = 3$ have not changed. On the other hand, the interest rate that is used to discount cash flows has increased in this example and so we might have expected the value of the option to have *decreased*. What has happened? (Certainly this situation would never have occurred in a deterministic world!)

First, from a purely mechanical viewpoint we can see that the risk-neutral probabilities have changed. In particular, the risk-neutral probability of an up-move, $q = (R - d)/(u - d)$, has *increased* since R has increased. This means that we are more likely to end up in the higher-payoff states. This increased likelihood of higher payoffs more than offsets the cost of having a larger discount factor and so we ultimately obtain an increase in the option value.[6]

This, however, is only one aspect of the explanation. It is perhaps more interesting to look for an intuitive explanation as to why q should increase when R increases. You should think about this! ∎

[6] Recall that in a recent assignment we introduced the concept of a stochastic discount factor. Increasing R from 2% to 4% does not mean we have increased the stochastic discount factor. In fact, since the value of the option has increased, it must be the case that the stochastic discount factor decreased at least in some states!

Calibrating the Binomial Model and Convergence to Black-Scholes

In continuous-time models, it is often assumed that security price processes are *geometric Brownian motions*. In that case we write $S_t \sim \mathsf{GBM}(\mu, \sigma)$ if

$$S_{t+s} \;=\; S_t \; e^{(\mu - \sigma^2/2)s \;+\; \sigma(B_{t+s} - B_t)} \tag{5}$$

where B_t is a standard Brownian motion. Note that this model has the nice property that the gross return, $R_{t,t+s}$, in any period, $[t, t+s]$, is independent of returns in earlier periods. In particular, it is independent of S_t. This follows by noting

$$R_{t,t+s} \;=\; \frac{S_{t+s}}{S_t} \;=\; e^{(\mu - \sigma^2/2)s \;+\; \sigma(B_{t+s} - B_t)}$$

and recalling the independent increments property of Brownian motion. It is appealing[7] that $R_{t,t+s}$ is independent of S_t since it models real world markets where investors care only about returns and not the absolute price level of securities. The binomial model has similar properties since the gross return in any period of the binomial model is either u or d, and this is independent of what has happened in earlier periods.

Calibrating the Binomial Model

We often wish to *calibrate* the binomial model so that its dynamics match that of the geometric Brownian motion in (5). To do this we need to choose u, d and p, the real-world probability of an up-move, appropriately. There are many possible ways of doing this, but one of the more common choices[8] is to set

$$p_n \;=\; \frac{e^{\mu T/n} - d_n}{u_n - d_n} \tag{6}$$

$$u_n \;=\; \exp(\sigma\sqrt{T/n}) \tag{7}$$

$$d_n \;=\; 1/u_n \;=\; \exp(-\sigma\sqrt{T/n}) \tag{8}$$

where T is the expiration date and n is the number of periods. (This calibration becomes more accurate as n increases.) Note then, for example, that $\mathrm{E}[S_{i+1}|S_i] = p_n u_n S_i + (1-p_n) d_n S_i = S_i \exp(\mu T/n)$, as desired.

We will choose the gross risk-free rate per period, R_n, so that it corresponds to a continuously-compounded rate, r, in continuous time. We therefore have

$$R_n \;=\; e^{rT/n}. \tag{9}$$

Remark 1 *Recall*[9] *that the true probability of an up-move, p, has no bearing upon the risk-neutral probability, q, and therefore it does not* directly *affect how securities are priced. From our calibration of the binomial model, we therefore see that μ, which enters the calibration only through p, does not impact security prices. On the other hand, u and d depend on σ which therefore does impact security prices. This is a recurring*[10] *theme in derivatives pricing as you will see in* Financial Engineering: Continuous-Time Asset Pricing *when you study continuous-time models.*

Remark 2 *We just stated that p does not* directly *affect how securities are priced. This means that if p should suddenly change but S_0, R, u and d remain unchanged, then q, and therefore derivative prices, would also remain unchanged. This seems very counter-intuitive but an explanation is easily given. In practice, a change in p would generally cause one or more of S_0, R, u and d to also change. This would in turn cause q, and therefore derivative prices, to change. We could therefore say that p has an* indirect *effect on derivative security prices.*

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[7] More sophisticated models will sometimes allow for *return predictability* where $R_{t,t+s}$ is not independent of S_t. Even then, it is still appropriate to model returns rather than absolute security values.

[8] We write p_n, u_n and d_n to emphasize that their values depend explicitly on the number of periods, n, for a fixed expiration, T.

[9] We saw this in the *Martingale Pricing Theory* lecture notes, though we do insist that $q_i > 0 \Leftrightarrow p_i > 0$.

[10] While a technical explanation is beyond the scope of this course, it is related to the fact that in continuous time *diffusion models*, the set of equivalent martingale measures depends explicitly on σ and does not depend at all on μ.

Convergence of the Binomial Model to Black-Scholes

The Black-Scholes formula for the price of a call option on a non-dividend paying security with initial price S_0, strike K, time to expiration T, continuously compounded interest rate r, and volatility parameter σ, is given by

$$C(S_0, T) \;=\; S_0 N(d_1) \;-\; K e^{-rT} N(d_2) \tag{10}$$

where

$$\begin{aligned} d_1 &= \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

and $N(\cdot)$ is the CDF of a standard Normal random variable.

For a fixed expiration, T, we consider the sequence of binomial models, M_n, that are parameterized by (6), (7), (8) and (9). The following steps outline how (10) may be obtained by letting $n \to \infty$ in this sequence.

Step 1: First observe that the call option price, C, in the model, M_n, is given by

$$\begin{aligned} C_n &= \frac{1}{R_n^n} \sum_{j=\eta_n}^{n} \binom{n}{j} q_n^j (1-q_n)^{n-j} (u_n^j d_n^{n-j} S_0 - K) \\ &= S_0 D\left(n, \eta_n; \frac{q_n u_n}{R_n}\right) \;-\; \frac{K}{R_n^n} D(n, \eta_n; q_n) \end{aligned} \tag{11}$$

where $q_n = (R_n - d_n)/(u_n - d_n)$ is the risk-neutral probability of an up-move in M_n,

$$\begin{aligned} D(n, \eta_n; q_n) &:= \sum_{j=\eta_n}^{n} \binom{n}{j} q_n^j (1-q_n)^{n-j} \quad \text{and} \\ \eta_n &:= \text{int}\left(\frac{\ln(K/S_0 d_n^n)}{\ln(u_n/d_n)}\right) + 1 \end{aligned}$$

is the minimum number of up-moves required for the call option to expire in the money.

Step 2: Note the similarity between the Black-Scholes formula, (10), and (11). Clearly all that is now required is to show that $D\left(n, \eta_n; \frac{q_n u_n}{R_n}\right)$ and $D(n, \eta_n; q_n)$ converge to the appropriate normal CDF probabilities.

Step 3: For example, $D(n, \eta_n; q_n) = 1 - \mathbf{P}(X_n < \eta_n)$ where X_n is a sum of n independent Bernouilli random variables with parameter, q_n. We can therefore write

$$\mathbf{P}(X_n < \eta_n) \;=\; \mathbf{P}\left(\frac{X_n - nq_n}{\sqrt{nq_n(1-q_n)}} < \frac{\eta_n - nq_n}{\sqrt{nq_n(1-q_n)}}\right) \tag{12}$$

and apply the Central Limit Theorem[11] to obtain the appropriate normal CDF term in (10). (It will also be necessary when substituting for η_n and q_n in the right-hand-side of (12) to take $\exp(a\sqrt{T}) \approx 1 + a\sqrt{T} + a^2T/2$. Higher order terms can be neglected as they will vanish as $n \to \infty$.)

Step 4: We can deal in a similar manner with $D\left(n, \eta_n; \frac{q_n u_n}{R_n}\right)$ and obtain $C_n \to C$ as defined by (10) when $n \to \infty$.

The price of a put option, $P(S_0, T)$, in the Black-Scholes model can be obtained either by a similar argument or

[11] We cannot apply the standard Central Limit Theorem (CLT) as the distribution of the random variables, i.e. the Bernouilli random variables in this case, depend on n. Instead we have to apply a version of the CLT known as Lindeberg's CLT.

by invoking put-call parity. In particular, we have

$$P(S_0, T) \;=\; Ke^{-rT}N(-d_2) \;-\; S_0N(-d_1). \tag{13}$$

It may also be seen that the binomial stock price process in M_n converges to a (μ, σ) geometric Brownian motion, and in the limit satisfies $S_t \;=\; S_0\ e^{(\mu-\sigma^2/2)t+\sigma B_t}$, where B_t is a standard Brownian motion.

American Options

An American option on a security gives the holder of the option the right to *exercise* it at any date on or before the expiration date, T. An exercise *strategy*, τ, is a rule specifying when the option should be exercised. Because τ can depend on the evolution of information in the model, it is in general a random variable. However, it should not be possible to base the decision to exercise or not on information that is not yet available. We therefore insist that τ is a *stopping time*.

Let V_t denote the date s price of an American option that expires at date $T > t$. The owner of the option may exercise it any date $s \in \{t, \ldots, T\}$. If exercised at time s, the owner of the option then receives Y_s. We assume that markets are complete and that there are no arbitrage opportunities so there exists a unique equivalent martingale measure, Q, relative to some numeraire security, B_t. We want to determine an expression for V_t.

Theorem 3 **(a)** *Let τ denote a generic stopping time and define $Z_t := \max_{\tau\in\{t,\ldots,T\}} E_t^Q\left[\frac{Y_\tau B_t}{B_\tau}\right]$. Then Z_t/B_t is the smallest supermartingale satisfying $Z_t \geq Y_t$ for all t. Moreover $\tau^*(t) := \min\{s \geq t \;:\; Z_s = Y_s\}$ is an optimal stopping time.*

(b) *The American option price, V_t, satisfies $V_t = Z_t$ for all $t \in \{0, \ldots, T\}$ and $\tau^*(0)$ is an optimal exercise strategy.*

Proof: **(a)** First note that $Z_T/B_T = Y_T/B_T$. This therefore implies

$$\frac{Z_{T-1}}{B_{T-1}} \;=\; \max\left\{\frac{Y_{T-1}}{B_{T-1}},\ E_{T-1}^Q\left[\frac{Z_T}{B_T}\right]\right\}.$$

More generally, we can easily see that

$$\begin{aligned}\frac{Z_t}{B_t} \;&=\; \max\left\{\frac{Y_t}{B_t},\ E_t^Q\left[\max_{\tau\geq t+1} E_{t+1}^Q\left[\frac{Y_\tau}{B_\tau}\right]\right]\right\} \\ &=\; \max\left\{\frac{Y_t}{B_t},\ E_t^Q\left[\frac{Z_{t+1}}{B_{t+1}}\right]\right\}\end{aligned} \tag{14}$$

so it is clear[12] that Z_t/B_t is a supermartingale. Moreover, it is clear that $Z_t \geq Y_t$. Now suppose that U_t/B_t is any other supermartingale satisfying $U_t \geq Y_t$. Since $Z_T = Y_T$ it is clear that $Z_T \leq U_T$. Moreover, by hypothesis U_{t-1} clearly satisfies

$$\frac{U_{t-1}}{B_{t-1}} \geq \max\left\{\frac{Y_{t-1}}{B_{t-1}},\ E_{t-1}^Q\left[\frac{U_t}{B_t}\right]\right\}.$$

Iterating backwards from $t = T$ it is clear that $U_t \geq Z_t$. Finally, it is clear[13] that $\tau^*(t) := \min\{s \geq t \;:\; Z_s = Y_s\}$ is an optimal stopping time.

(b) Note that since markets are complete, we know that Y_τ is attainable for every stopping time, τ. We need to consider two situations: (i) $V_t < Z_t$ and (ii) $V_t > Z_t$. If (i) prevails, you should purchase the option, adopt an optimal exercise policy and adopt a trading strategy that is equivalent to selling the security with payoff $Y_{\tau^*(t)}$

[12]Note that a simple dynamic programming argument could also be used to derive (14).

[13]This does take a little work to prove rigorously. See, for example, Pliska.

at time $\tau^*(t)$. It is easy to show that this leads to arbitrage profits. If (ii) prevails, you should sell the option and invest the proceeds appropriately to construct an arbitrage. The details are left as an exercise but note that you have no control over the exercise strategy that the purchaser of the option might adopt. ■

We can use the result of Theorem 3 and in particular, equation (14), to price American options. For example, in the binomial model we can use dynamic programming to compute the optimal strategy. As usual, we will use the cash account (with value B_k at date k), as the numeraire security.

Example 4 (Pricing an American Put Option)

Compute the price of an American put option on the security of Figure 1 with expiration at $T = 3$, and strike $K = 95$. Assume again that $R = 1.02$.

Solution: We know from Example 2 that $q = 0.657$, and we know the value of the American option at date $T = 3$. We then work backwards in the tree until we have found C_0^a, the date $t = 0$ value of the American option. For example at date $t = 2$ when the stock price is \$89 we know that the option value is given by

$$C_2^a(\$89) \;=\; \max\left(K - 89,\; \frac{1}{R}\left[q(.66) + (1-q)(11.04)\right]\right) \;=\; K - 89 \;=\; 6.$$

That is, the value of the option at that node is \$6 and it is optimal to exercise the option there. Continuing in this recursive manner, we find that $C_0^a = \$.77$.

Stock Price

			119.10
		112.36	106.00
	106.00	100.00	94.34
100.00	94.34	89.00	83.96
t=0	t=1	t=2	t=3

American Option Price

			0.00
		0.00	0.00
	0.07	0.22	0.66
0.77	2.16	6.00	11.04
t=0	t=1	t=2	t=3

■

In Example 4 we found that it was sometimes optimal to exercise the option *prior* to expiration. This raises the interesting question of whether or not there are American options for which it is *never* optimal to exercise them prior to maturity. We have the following result.

Theorem 4 *If interest rates are non-negative, it is never optimal to exercise an American call option on a non-dividend paying security prior to expiration.*

Proof: Using the Q-martingale property of the deflated security price process and the non-negativity of interest rates, we have

$$\begin{aligned}
\mathrm{E}_t^Q\left[(S_{t+s}/B_{t+s} \;-\; K/B_{t+s})^+\right] \;&\geq\; \mathrm{E}_t^Q\left[S_{t+s}/B_{t+s} \;-\; K/B_{t+s}\right] \\
&=\; S_t/B_t \;-\; K\mathrm{E}_t^Q[1/B_{t+s}] \\
&\geq\; S_t/B_t \;-\; K/B_t.
\end{aligned}$$

Since it is also the case that $(S_{t+s}/B_{t+s} \;-\; K/B_{t+s})^+ \geq 0$, we therefore also have

$$\mathrm{E}_t^Q\left[(S_{t+s}/B_{t+s} \;-\; K/B_{t+s})^+\right] \;\geq\; (S_t/B_t \;-\; K/B_t)^+ .$$

That is, $(S_t/B_t \;-\; K/B_t)^+$ is a *sub-martingale*. Now the *Optional Sampling Theorem* for sub-martingales states that if Y_t is a sub-martingale and $\tau \leq T$ is a stopping time, then $\mathrm{E}^Q[Y_\tau] \leq \mathrm{E}^Q[Y_T]$. If we apply this result to $(S_t/B_t \;-\; K/B_t)^+$ and recall that the price, C_0^a, of the American option is given by

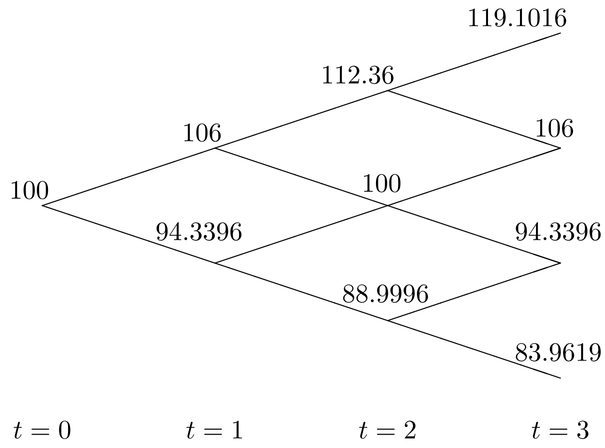
$$C_0^a \;=\; \max_{\tau}\; \mathrm{E}_0^Q\left[(S_\tau/B_\tau \;-\; K/B_\tau)^+\right]$$

we see that $C_0^a = \mathrm{E}_0^Q\left[(S_T/B_T \;-\; K/B_T)^+\right]$ and it is never optimal to exercise early. ■

Remark 3 *Note that if the security paid dividends in $[0,T]$, then S_t/B_t would not be a Q-martingale and the above proof would not go through. The result generalizes to other types of American options where the underlying security, S_t, again does not pay dividends in $[0,T]$, and where the payoff function is a* convex *function of S. The proof is similar and relies on the application of Jensen's Inequality.*

Example 5

(a) Consider the binomial lattice below that describes the evolution of a non-dividend paying stock in a 3 period world. You may assume that there is a risk-free asset which pays a total return of $R = 1.005$ per period.



Making the usual assumptions, compute the price of an American call option on the stock with strike $= 110$ and expiration date $t = 3$.

(b) Without doing any backward recursions, compute the price of a European put option on the stock with strike $= 110$ and expiration date $t = 3$. Explain your reasoning.

(c) Again without doing any backward recursions, compute the price of a security that pays out $\$100$ in every state at time $t = 3$ except in the state where $S_3 = 119.1016$, when it only pays out $\$50$. Explain your reasoning.

(d) Compute the price of an $Asian$ call option that expires at time $t = 2$ with strike $k = \$100$. The payoff of this option at time $t = 2$ is given by $\max\left(0, \frac{S_1+S_2}{2} - k\right)$.

<u>*Solution*</u>

(a) Work backwards in the lattice to see that the call price, C, is given by

$$C = \frac{1}{R^3}\, q^3\; 9.1016 = 1.3222$$

(b) American call price equals European call price so we can use put-call parity to compute put price, P. We find that $P = C - S + 1/R^3\ K = 1.3222 - 100 + 1/1.005^3\ 110 = 9.6886$

(c) This security is the same as being long a zero-coupon bond with face value \$100 and short $50/9.1016$ of the call options of part (a). Therefore it's price is $100/R^3 - 50/9.1016 \times 1.3222 = 91.2513$.

(d) There are 4 possible paths: uu, ud, du and dd, each with risk-neutral probabilities q^2, $q(1-q)$, $(1-q)q$ and $(1-q)^2$, respectively. The paths have payoffs 9.18, 3, 0 and 0 respectively. Hence the arbitrage free price is

$$\frac{1}{R^2}\left(q^2\ 9.18 + q(1-q)\ 3\right) = 3.2771.$$

∎

Hedging and the Greeks

In the binomial model we know that we can compute the price of any[14] derivative security by constructing a replicating portfolio.[15] Besides being important for pricing securities, replicating portfolios are also important for *hedging*. For example, suppose you have written a put option that expires at date T, and you have received P_0 for this at date 0. If you do not wish to take on the risk associated with this position you could either purchase the same put option or, alternatively, you could choose to *synthetically* purchase the put option. Synthetically purchasing the put option refers to adopting the self-financing trading strategy that replicates a long position in the put option. This strategy will cost P_0 initially and will have a final payoff that will exactly cancel your short position in the put that you sold. Hence you will have eliminated all risk associated with your original position.

In discrete-time, discrete-space models it is easy to construct the replicating portfolio when the security in question is replicable. In continuous-time complete models we can also construct replicating portfolios. We briefly describe how to do this in the context[16] of the Black-Scholes model when the underlying security follows a geometric Brownian motion. First, we define the *delta* of an option at time t to be

$$\Delta_t = \frac{\partial C_t}{\partial S_t}$$

where C_t is the price of the option at time t, and S_t is the price of the underlying security at time t. Δ_t measures the sensitivity of the option price to changes in the price of the underlying security. The option may then be replicated by adopting the following self-financing trading strategy:

1. Commit the initial value, C_0, to the replicating portfolio.

2. At each time t, adjust your portfolio so that you have a position of Δ_t units in the underlying security and the remainder of your cash invested in the risk-free asset. (Sometimes this will require borrowing rather than investing in the risk-free asset.)

This strategy will replicate the terminal payoff of the option. In practice, however, the presence of transactions costs and other *market frictions* implies that it is not possible to adjust the portfolio (i.e. re-hedge) at each instant, t. Instead the portfolio is hedged periodically and because of this, hedging is sometimes conducted by also matching the second derivative, $\Gamma := \partial^2 C_t/\partial S_t^2$, of the replicating portfolio with that of the option to be hedged. The second derivative is referred to as the *gamma* of the option. Using Δ and Γ to hedge is analogous to using duration and convexity to immunize bond portfolios.

Exercise 2 *When Γ is also used to hedge, why is it necessary to include a derivative security, other than the derivative being hedged, in the replicating portfolio.*

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[14] Since the binomial model is a *complete* model.

[15] Recall that a replicating portfolio for a derivative security is a self-financing trading strategy whose terminal value is equal to the terminal value of the derivative security.

[16] Similar comments apply to other more general complete market models.

Other price sensitivities that are often mentioned include $\Theta := \partial C_t/\partial t$, Vega $:= \partial C_t/\partial \sigma$ and $\rho := \partial C_t/\partial r$. Interestingly, vega and ρ are defined despite the fact that σ and r are assumed to be constant in the original Black-Scholes framework. Collectively, delta, gamma, theta, vega[17] and rho are known as the *Greeks*.

While derivative securities in general cannot be replicated in incomplete models, similar strategies that match the Greeks can be used to reduce the risk associated with holding a particular position. This is true for both discrete and continuous models.

Forwards and Futures

We consider a general discrete-time, discrete-space financial market. For ease of exposition we will assume that the market is complete[18] and that none of the securities pay dividends. We will also assume that the numeraire security is the cash account with time k value, B_k.

Computing Futures Prices

Let F_k be the date k price of a futures contract written on a particular underlying security. We assume that the contract expires after n periods and we let S_k denote the time k price of the security. Then we know that $F_n = S_n$, i.e., at expiration the futures price and the security price must coincide. We can compute the futures price at $t = n - 1$ by recalling that anytime we enter a futures contract, the initial value of the contract is 0. Therefore the futures price, F_{n-1}, at date $t = n - 1$ must satisfy

$$0 \quad = \quad \mathrm{E}_{n-1}^{Q}\left[\frac{F_n - F_{n-1}}{B_n}\right].$$

Since B_n and F_{n-1} are both known at date $t = n-1$, we therefore have $F_{n-1} = \mathrm{E}_{n-1}^{Q}[F_n]$. By the same argument, we also have more generally that $F_k = \mathrm{E}_k^Q[F_{k+1}]$ for $0 \leq k < n$. We can then use the law of iterated expectations to see that $F_0 \;=\; \mathrm{E}_0^Q[F_n]$, implying in particular that the futures price process is a martingale. Since $F_n = S_n$ it is also worth observing that

$$\boxed{F_0 \quad = \quad \mathrm{E}_0^Q\left[S_n\right].} \tag{15}$$

Exercise 3 *What property of the cash account did we use in deriving (15)? (Note as a result that (15) only holds when Q is the EMM corresponding to taking the cash account as numeraire.)*

Exercise 4 *Does (15) change if the underlying security pays dividends? You can assume that S_i is then the ex-dividend price of the security at time i.*

Computing Forward Prices

Now let us consider the date 0 price, G_0, of a forward contract for delivery of the security at the same date, $t = n$. We recall that G_0 is chosen in such a way that the contract is initially worth zero. In particular, our general Martingale Pricing Theorem implies

$$0 \quad = \quad \mathrm{E}_0^Q\left[\frac{S_n - G_0}{B_n}\right].$$

Rearranging terms and using the fact that G_0 is known at date $t = 0$ we obtain

$$G_0 \quad = \quad \frac{\mathrm{E}_0^Q\left[S_n/B_n\right]}{\mathrm{E}_0^Q\left[1/B_n\right]}. \tag{16}$$

[17] Vega is the only one that is *not* a Greek letter.

[18] We do not actually require market completeness to derive (15) and (16). Indeed, it is only necessary that certain securities are available, i.e. replicable. Can you identify what these securities are?

Remark 4 *Note that (16) holds regardless of whether or not the underlying security pays dividends or coupons (storage costs may be viewed as negative dividends). Dividends (or other intermediate cash-flows) influence G_0 through the evaluation of* $\mathrm{E}_0^Q[S_n/B_n]$.

Remark 5 *If the underlying security does not pay dividends then we obtain* $G_0 = S_0/\mathrm{E}_0^Q[1/B_n]$. *This is consistent with the expression, $S/d(0,n)$, that was given in the* Forwards, Futures and Swaps *lecture notes. This is clear since $\mathrm{E}_0^Q[1/B_n]$ is the time 0 value of \$1 at time n and this, by definition, is equal to $d(0,n)$.*

Exercise 5 *If the underlying security does pay dividends (or have storage costs) then show that the expression in (16) is consistent with the expression given in the* Forwards, Futures and Swaps *lecture notes.*

We are also in a position now to identify when forwards and futures price coincide. In particular, if we compare (16) with (15) then we immediately obtain the following result.

Theorem 5 *If B_n and S_n are Q-independent, then $G_0 = F_0$. In particular, if interest rates are deterministic, we have $G_0 = F_0$.*

Corollary 1 *In the binomial model with a constant (or deterministic) gross interest rate, R, we must have $G_0 = F_0$.*

Exercise 6 *In practice, interest rates are stochastic and often tend to be positively correlated with movements in the stock market. In such circumstances, convince yourself by considering (15) and (16) that the futures price, F_0, will generally be greater than the forward price, G_0.*

Example 6 (Futures and American Options)

Consider the commodity price lattice below where $u = 1.03$ and $d = .98$. If we assume $R = 1.01$ per period and there are no storage costs, then we easily find that the risk-neutral probabilities of an up-move and down-move are given by $q = .6$ and $1 - q = .4$, respectively.

Commodity Price

						119.41
					115.93	113.61
				112.55	110.30	108.09
			109.27	107.09	104.95	102.85
		106.09	103.97	101.89	99.85	97.85
	103.00	100.94	98.92	96.94	95.00	93.10
100.00	98.00	96.04	94.12	92.24	90.39	88.58
t=0	t=1	t=2	t=3	t=4	t=5	t=6

We will use S_k to denote the commodity price at time k and assume that a futures contract on the commodity exists and that it expires after six periods. We wish to compute the futures price lattice and we can do this in a number of different (though essentially equivalent) ways. The method we choose is to use our earlier observation that any time time k, $0 \leq k \leq 6$, the futures price, F_k satisfies $F_k = \mathrm{E}_k^Q[S_6]$ where Q is the EMM corresponding to the cash account as numeraire. But $\mathrm{E}_k^Q[S_6] = R^{6-k} S_k$ (why?) and so we have

$$F_k = R^{6-k} S_k.$$

For example, $F_0 = R^6 S_0 = (1.01^6)100 = 106.15$ and the futures prices at all times may be computed[19] this way. We obtain the futures price lattice below.

[19] See the *Equity_Options.xls* Excel workbook for other related examples.

Futures Price

						119.41
					117.09	113.61
				114.81	111.40	108.09
			112.58	109.24	105.99	102.85
		110.40	107.12	103.94	100.85	97.85
	108.25	105.04	101.92	98.89	95.95	93.10
106.15	103.00	99.94	96.97	94.09	91.30	88.58
t=0	t=1	t=2	t=3	t=4	t=5	t=6

Suppose now that we wish to find the price of an American put option with the futures price as the underlying security. If we assume that the option expires at the same time as the futures contract, i.e. after 6 periods, and the strike is $\$105$, then we obtain the price lattice below for the option. In particular, we obtain an initial price of $\$2$.

Price of American Put Option on Futures Price

						0.00
					0.00	0.00
				0.00	0.00	0.00
			0.13	0.34	0.85	2.15
		0.50	1.05	2.15	4.15	7.15
	1.12	2.09	3.70	6.11	9.05	11.90
2.00	3.37	5.38	8.03	10.91	13.70	16.42
t=0	t=1	t=2	t=3	t=4	t=5	t=6

■

Challenge Question

One of the most important results in financial economics is the Modigliani-Miller-Williams theorem. One version of this theorem states that in a *perfect* world, there is no need for companies to hedge their financial risks as individual shareholders can do this themselves. In practice, however, many companies hedge their financial risks and consider this to be a very important activity. Presumably then, the world has some imperfections. What might some of these imperfections be?