Financial Engineering I, IEOR E4706, Fall 2003

Columbia University
Instructor: Martin Haugh
Final Examination, December 16th 2003
Total Marks: 100
Time: 3 hours

Notes that are on both sides of a single sheet of A4 paper may be used during the exam.

Question 1 (40 marks)

(a) Consider two 5-year bonds: one has an 8% coupon and sells for 106 while the other has a 5% coupon and sells for 98. Find the price of a 5-year zero-coupon bond with face value 100. (5 marks)

(b) The current price of a forward contract for delivery of copper in 6 months is 96 cents per pound and the current price of copper in the spot market is 93 cents per pound. Assuming that there are no storage costs and that 6-month T-bills are currently selling for \$960, do you think there is an arbitrage opportunity? If so, explain briefly how you would take advantage of it. (5 marks)

(c) The current price of gold is \$430 per ounce, and storage costs are \$2 per ounce per year, payable quarterly in advance. Assuming a constant interest rate of 8% compounded quarterly, what is the forward price of gold for delivery in 6 months? (5 marks)

(d) Suppose we are in a one period world where $t = 0$ at the beginning of the period, and $t = 1$ at the end. There are three possible states at $t = 1$ and there are two securities available for trading at $t = 0$. The payoffs of these securities at $t = 1$ are $(1, 3, 2)$ and $(3, 6, 4)$, respectively, so for example, the payoff of the second security at $t = 1$ is 6 if the second state occurs. In this world, what, if any, elementary securities are available? (5 marks)

(e) A 7-year 8% coupon bond with semi-annual coupon payments and face value \$100 is currently trading at \$98. The next coupon payment is due in exactly six months. Is the yield-to-maturity of this bond less than 8%, equal to 8% or greater than 8%? Explain your answer. (5 marks)

(f) A Treasury bond with several years to maturity, a face value of $\$10,000$ and a coupon of 10% that is paid semi-annually, costs $\$9,100$ today. The latest coupon has just been paid. If interest rates for 1 year out are flat at 12%, what is the forward price for delivery of this bond in 1 year? (5 marks)

(g) A forward contract for delivery of a security in exactly 1 year is being *sold* by the *ABC Banking Corporation*. (The underlying security may be sold short and does not provide any intermediate cash-flows). Markets are complete and interest rates are deterministic. A futures contract for delivery of the same security also expires in exactly 1 year and arbitrage pricing theory suggests that the current futures price should equal the forward price of the security. However, the forward price quoted by the *ABC Banking Corporation* is *less* than the current futures price. Can you give a (good) possible explanation for this observation? (5 marks)

(h) A particular stock with price S_0 pays yearly dividends and the next such dividend will be paid in exactly 6 months. The price today at $t = 0$ of an American call option on the stock with strike K and expiration $T = 5$ months is C_0. The discount factor, $d(0, T)$, is also known. Do you have enough information to compute the price of a European put option on this stock with the same strike, K, and expiration, T? Explain your answer. (5 marks)

Question 2 (30 marks)
(a) Consider the binomial lattice in Figure 2.1 that describes the evolution of a non-dividend paying stock in a 3 period world. You may assume that there is a risk-free asset which pays a total return of $R = 1.03$ per period.

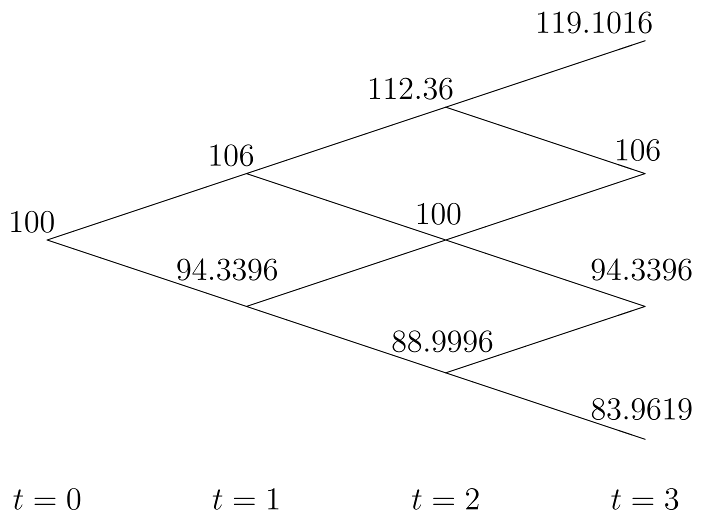


Figure 2.1

Please be clear when answering the following questions. It might be helpful to explain exactly what you are doing when you need to perform computations.

Making the usual assumptions, compute the price of an American put option on the stock with strike = 94 and expiration date $t = 3$. (7 marks)

(b) Use backward recursion to compute the price of a security that pays 1 at $t = 3$

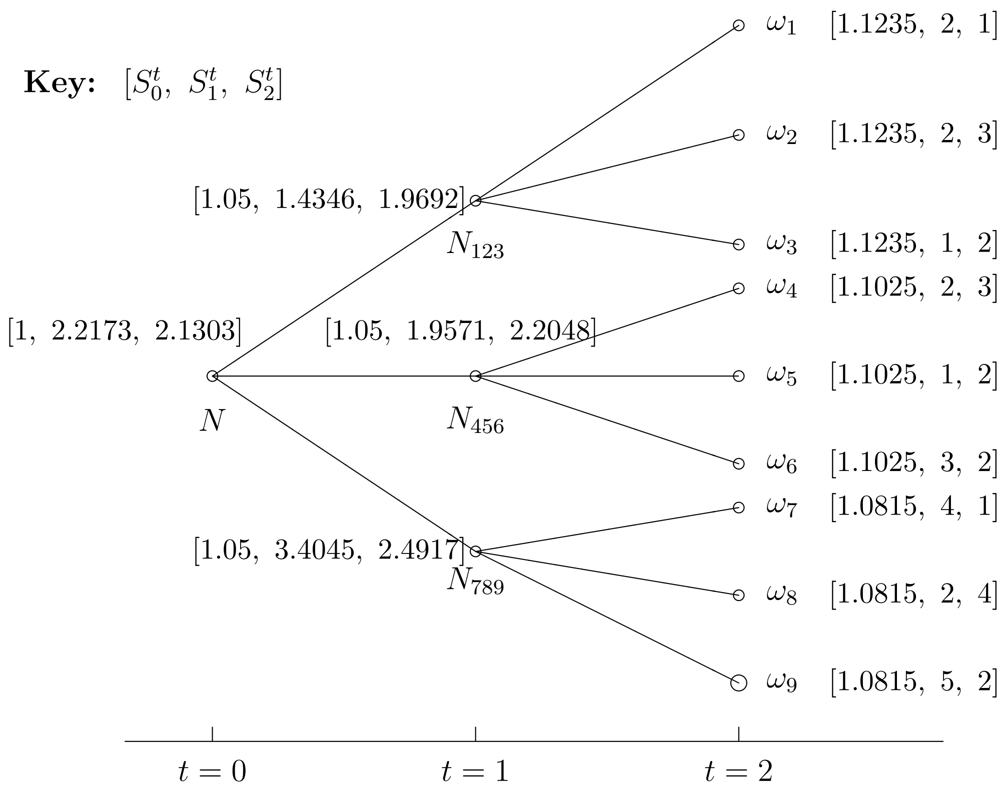
if the terminal stock price is 94.3396. Do the same for a security that pays 1 at $t = 3$ if the terminal stock price is 83.9619. (7 marks)

(c) Without doing backward recursion, and without enumerating paths, compute the price of a European put option with expiration date $t = 3$ and strike = 103. Show your work. (6 marks)

(d) A particular **chooser option** gives the owner the right to obtain (at no cost) at time $t = 1$ either a European call option *or* a European put option. The call and put options in question both have the same strike k and expiration $t = 2$. How much is the chooser option worth at date $t = 0$ when $k = 100$? (7 marks)

(e) Can you find an expression for the time $t = 0$ value of the chooser option for general values of k that can be written as the sum of the original call option and a put option with different strike and expiration. (This is tricky so don't waste time on it. Put-call parity helps!) (3 marks)

Question 3 (30 marks)



Consider the 2-period, 3-security financial model above. There are no arbitrage opportunities, the market is complete and the 0^{th} security may be interpreted as a cash account. It may be checked that the state prices and the corresponding risk-neutral probabilities (with the 0^{th} asset as numeraire) in the embedded one-period models are given by:

$$
\begin{aligned}
\text{At } N_{123} \ &: \ \pi_1^2(\omega_1) = .2 \ \ \pi_1^2(\omega_2) = .3 \ \ \pi_1^2(\omega_3) = .4346 \\
& \quad q_1^2(\omega_1) = .214 \ \ q_1^2(\omega_2) = .321 \ \ q_1^2(\omega_3) = .465 \\
\text{At } N_{456} \ &: \ \pi_1^2(\omega_4) = .3 \ \ \pi_1^2(\omega_5) = .3 \ \ \pi_1^2(\omega_6) = .3524 \\
& \quad q_1^2(\omega_4) = .315 \ \ q_1^2(\omega_5) = .315 \ \ q_1^2(\omega_6) = .37 \\
\text{At } N_{789} \ &: \ \pi_1^2(\omega_7) = .25 \ \ \pi_1^2(\omega_8) = .4 \ \ \pi_1^2(\omega_9) = .3209 \\
& \quad q_1^2(\omega_7) = .2575 \ \ q_1^2(\omega_8) = .412 \ \ q_1^2(\omega_9) = .3305 \\
\text{At } N \ &: \ \pi_0^1(N_{123}) = .3 \ \ \pi_0^1(N_{456}) = .3 \ \ \pi_0^1(N_{789}) = .3524 \\
& \quad q_0^1(N_{123}) = .315 \ \ q_0^1(N_{456}) = .315 \ \ q_0^1(N_{789}) = .37
\end{aligned}
$$

Note that $\pi_s^t(A)$ is the state price at time s for a security that pays \$1 at time t if the event A has occurred.

Please be clear when answering the following questions. It might be helpful to explain exactly what you are doing when you need to perform computations.

(a) Compute $\pi_0^2(\omega_1)$. (5 marks)

(b) Compute the one-period risk-neutral probabilities at node N_{123} when the second asset is the numeraire security. (5 marks)

(c) Compute the forward price at date $t = 0$ for delivery of the first security at $t = 2$. (5 marks)

(d) Explain clearly how you would find the replicating portfolio for a security with date $t = 2$ payoff $X \in R^9$. (No math is required but your answer should make it clear that you understand how to do this.) (5 marks)

(e) Suppose now that you are at node N_{456} at date $t = 1$ and that short-selling in the 0^{th} security is prohibited. Given this, can you explain how to find the best arbitrage-free upper bound on the date $t = 1$ price of the security that has date t=2 payoff $X = [x_1 \ x_2 \ x_3]$ in states ω_7, ω_8 and ω_9, respectively? (5 marks)

(f) You have a fixed initial wealth, $W_0 \geq 0$ at t=0 and you wish to choose a trading strategy to maximize $P(W_2 \geq \alpha)$ subject to $P(W_2 \geq 0) = 1$. (W_2 is your wealth at date $t = 2$ and $\alpha > 0$ is a fixed constant.) Explain how you would do this and what the structure of the optimal solution would look like. (This is tricky so don't waste time on it. It might help if you consider ranking the states in a particular order.)(5 marks)