**Foreign Exchange and “Quantos”**

These notes consider foreign exchange markets and the pricing of derivative securities in these markets. As is the case with equity derivatives, the implied volatility surface corresponding to vanilla European FX option prices is neither flat nor constant. It is therefore widely accepted that the Black-Scholes GBM model is a poor model for FX markets. As is the case with equity derivatives, however, vanilla FX option prices are quoted and their Greeks are calculated using the Black-Scholes framework. It is therefore necessary to understand how Black-Scholes applies to the FX markets when working with derivatives in these markets.

We will also consider assets that are denominated in a foreign currency but whose value we wish to determine in units of a domestic currency. There are two different means of converting the asset’s foreign currency value into domestic currency: (i) using the prevailing exchange rate and (ii) using a fixed predetermined exchange rate. The latter method leads to the concept of a *quanto* security. We will study both methods in some detail. Because such securities are exotic, they are generally not priced in practice using the Black-Scholes framework. Nonetheless, we will consider them in this framework for two reasons: (i) given our knowledge of Black-Scholes, this is the easiest way of introducing the concepts and (ii) it will afford us additional opportunities to work with martingale pricing using different numeraire and EMMs.

A particular feature of FX markets are the triangular relationships that exist between currency triples. For example, the USD/JPY exchange rate can be expressed in terms of the USD/EUR and EURO/JPY exchange rates. These implications ought to be borne in mind when working with FX derivatives. They and other peculiarities of FX modeling will be studied further in the assignments.

## 1 Foreign Exchange Modeling

Unless otherwise stated we will call one currency the *domestic* currency and the other the *foreign* currency. We will then let $X_t$ denote the exchange rate at time $t$, representing the time $t$ cost in units of the domestic currency of 1 unit of the foreign currency. For example, taking the usual market convention, if the USD/EUR exchange rate is 1.2, then USD is the domestic currency, EUR is the foreign currency and 1 EUR costs $1.20 USD.

Suppose instead we expressed the exchange rate as the cost in Euro of 1 USD. Then USD would be the foreign currency and EUR would be the domestic currency. Note that the designations “domestic” and “foreign” have nothing to do with where you are living or where the transaction takes place. They are a function only of how the exchange rate is quoted.

The terms base and quote are also often used in practice. The exchange rate then represents how much of the quote currency is needed to purchase one unit of the base currency. So we will continue to use “domestic” and “foreign” in these lecture notes. Later on we will use “base currency” to refer to the accounting currency. For example, a US based company would take the USD as its base currency whereas a German company would have the EUR as its base currency.

We will let $r_d$ and $r_f$ denote the domestic and foreign risk-free rates, respectively. Note that the foreign currency plays the role of the “stock” and $r_f$ is the “dividend yield” of the stock. Note also that holding the

---

1In these notes when we write “Curr1/Curr2” we will always take Curr1 to be the domestic currency and Curr2 to be the foreign currency. Unfortunately, in practice this is not necessarily the case and the domestic and foreign designations will depend on the currency pair convention.

2But beware: I have seen “base” used in place of the domestic currency as well!

3We will assume that $r_d$ and $r_f$ are constant in these notes. When pricing long-dated FX derivatives, these rates are often taken to be stochastic. The same applies for long-dated equity derivatives.
Foreign cash account is only risk-free from the perspective of someone who uses the foreign currency as their unit of account, i.e. accounting currency. A domestic investor who invests in the foreign cash account, or any other asset denominated in the foreign currency, is exposed to exchange rate risk.

Currency Forwards

Let $F_t^{(T)}$ denote the time $t$ price of a forward contract for delivery of the foreign currency at time $T$. Then since the initial value of the forward contract is 0, martingale pricing implies

$$0 = E_t^Q \left[ \frac{F_t^{(T)} - X_T}{B_T} \right]$$

which then implies

$$F_t^{(T)} = E_t^Q \left[ \frac{X_T}{B_T} \right].$$

Assuming that interest rates are constant\(^4\) we obtain

$$F_t^{(T)} = E_t^Q [X_T] = e^{r_d(T-t)} E_t^Q \left[ e^{-r_d(T-t)} X_T \right] = e^{(r_d-r_f)(T-t)} X_t \quad (1)$$

a relationship that is sometimes referred to as covered interest parity.

**Exercise 1** Prove (1) directly using a no-arbitrage argument.

**Exercise 2** Compute the time $t$ value of a forward contract that was initiated at time $t = 0$ with strike $K$ for delivery at time $T > t$.

The forward currency markets\(^5\) are very liquid and play an important role in currency trading. A forward FX rate, $F_t^{(T)}$, is usually quoted as a premium or discount to the spot rate, $X_t$, via the forward points.

FX Swaps

An FX swap is a simultaneous purchase and sale of identical amounts of one currency for another with two different value dates. The value dates are the dates upon which delivery of the currencies take place. In an FX swap, the first date is usually\(^6\) the spot date and the second date is some forward date, $T$. An FX swap is then a regular spot FX trade combined with a forward trade, both of which are executed simultaneously for the same quantity. FX swaps are often called forex swaps. They should not be confused with currency swaps which are considerably less liquid. A currency swap is typically a long-dated instrument where interest payments and principal in one currency are exchanged for interest payments and principal in another currency. FX swaps are regularly used by institutions to fund their FX balances.

Forwards and FX swaps are typically quoted in terms of forward points which are the difference between the forward price and the spot price. Using (1) we see

$$F_t^{(T)} - X_t = \left( e^{(r_d-r_f)(T-t)} - 1 \right) X_t$$

\(^4\)We could also allow interest rates that are (i) deterministic or (ii) stochastic but $Q$-independent of $X_T$ in our derivation of (1). But a no-arbitrage argument using zero-coupon bonds could still be used to derive (1) in general. See Exercise 1.

\(^5\)Exchange-traded currency futures also exist.

\(^6\)But other value dates are also possible. It is possible, for example, to trade forward-forward swaps where both value dates are beyond the spot date, or tom-next swaps.
When interest rates are identical the forward points are zero. As the interest rate differential gets larger, the absolute value of the forward points increases.

**Currency Options**

A European call option on the exchange rate, $X$, with maturity $T$ and strike $K$ pays $\max(0, X_T - K)$ at time $T$. If we assume that $X_t$ has GBM dynamics then the same Black-Scholes analysis that we applied to options on stocks will also apply here. In particular the $Q$-dynamics of the exchange rate are given by

$$\frac{dX_t}{X_t} = (r_d - r_f)dt + \sigma_x dW_t^{(x)}$$

and the price of the currency option satisfies

$$C(X, t) = e^{-r_f(T-t)}X_t\Phi(d_1) - e^{-r_d(T-t)}K\Phi(d_2)$$

where $d_1 = \frac{\log \left( \frac{X_t}{K} \right) + (r_d - r_f + \sigma_x^2/2)(T-t)}{\sigma_x \sqrt{T-t}}$ and $d_2 = d_1 - \sigma_x \sqrt{T-t}$.

where $\sigma_x$ is the exchange rate volatility. All the usual Black-Scholes Greeks apply. It is worth bearing in mind, however, that foreign exchange markets typically assume a sticky-by-delta implied volatility surface. This means that as the exchange rate moves, the volatility of an option with a given strike is also assumed to move in such a way that the volatility skew, as a function of delta, does not move.

The notional of the option is the number of foreign currency units that the option holder has the right to buy or sell at maturity. So for example, if the notional of a call option is $N$, then the time $t$ value of the option is $N \times C(X, t)$.

**Delta and the Option Premium**

When trading a currency option, the price of the option may be paid in units of the domestic currency or it may be paid in units of the foreign currency. This situation never arises when trading stock options. For example, if you purchase an option on IBM then you will pay for that option in dollars, i.e. the domestic currency, and not in IBM stock which plays the role of the foreign currency. Note that this is a matter of practice as there is nothing in theory to stop me paying for the IBM option in IBM stock. In currency markets, however, and depending on the currency pair, it might be quite natural to pay for the currency option in the foreign currency.

When computing your delta it is important to know what currency was used to pay for the currency option. Returning to the stock analogy, suppose you paid for an IBM call option in IBM stock that you borrowed in the stock-borrow market. Then I would inherit a long delta position from the option and a short delta position from the option payment. My overall net delta position will still be long (why?), but less long that it would have been if I had paid for it in dollars. The same is true if you pay for a currency option in units of foreign currency. When an option premium is paid in units of the foreign currency and the delta is adjusted to reflect this, we sometimes refer to it as the premium-adjusted delta.

As is probably clear by now, currencies can be quite confusing! And it takes time working in the FX markets before most people can get completely comfortable with the various market conventions. The one advantage that currencies have over stocks from a modeling perspective is that currencies do not pay discrete dividends and so the (often) ad-hoc methods that have been developed for handling discrete dividends are not needed when modeling currencies.

**Other Deltas**

There are other definitions of “delta” that are commonly used in FX-space. These alternative definitions arise because (i) the value of currency derivatives can be expressed naturally in domestic or foreign currency units and (ii) forwards play such an important role in FX markets. In addition to the regular Black-Scholes delta (which is the usual delta sensitivity) and the premium-adjusted delta we also have:
• The forward delta. This is the sensitivity of the option price with respect to changes in the value of the underlying forward contact with the same strike and maturity as the option.

• The premium-adjusted forward delta. This is the same as the forward delta but you need to adjust for the fact that the option premium was paid in units of the foreign currency.

Note that all of the delta definitions we have discussed thus far have implicitly assumed that we want to hedge the domestic value of the option. However, we might also want to hedge the foreign value of the option.

Example 1 (Hedging Domestic Value v Foreign Value)

Suppose an FX trader in Australia buys a JPY/AUD call option from an FX trader in Japan. If the Australian trader hedges the AUD value of the option and the JPY trader hedges the JPY value of the option, then they will not put on equal and opposite hedges. For example, suppose the value of the option is 100 AUD and the current AUD/JPY exchange rate is 65 JPY per AUD. If the Australian trader has hedged the AUD value of the option and the exchange rate moves from 65 to 67, say, his position will still be worth 100 AUD. However, the JPY value of his position will have changed from $100 \times 65 = 6,500$ JPY to $67 \times 100 = 6,700$ JPY. So hedging the AUD value of his position clearly does not hedge the JPY value of his position.

As a result, we have another set of deltas that correspond to hedging the foreign value of the option. There are also “sticky-by-delta” versions of all these deltas. Clearly then, one must be very careful in specifying and interpreting a delta. The default delta that is quoted in the market place will, as always, depend on the currency pair and market conventions. Trading systems will generally be capable of reporting these different deltas.

What Does At-the-Money (ATM) Mean?

The usual definition of “at-the-money” in equity-space is that the strike of the option under consideration is equal to the current spot price. But there are other alternative definitions that commonly occur in FX-space. They are:

1. At-the-money spot which is our usual definition.

2. At-the-money forward.

3. At-the-money value neutral. This is the strike, $K'$, such that the call value with strike $K'$ equals the put value with the same strike, $K$.

4. At-the-money delta-neutral. This is the strike, $K$, where the delta of the call option with strike $K$ is equal to minus the delta of the put option with the same strike $K$. But as there many different definitions of delta, we have many possibilities here.

Building an FX Volatility Surface

The construction of a volatility surface in FX space is unfortunately much more difficult than in equity-space. The simple reason for this is that the liquid options in the FX markets are quoted for a given delta and not for a given strike. In particular, FX markets typically quote volatility prices for

1. ATM options. However, the particular definition of ATM will depend on the currency-pair and market conventions.

2. 25-delta and 10-delta strangles. A strangle is a long call option together with a long put option.

3. And 25-delta and 10-delta risk-reversals. A risk reversal is a long call option together with a short put option.

In each of the strangles and risk reversals, the strike of the put option will be lower than the strike of the call option. These strikes then need to be determined numerically using combinations of the quoted volatilities and the designated deltas. Once the various strikes and their implied volatilities have been calculated, curve fitting
techniques are then used to build the entire volatility surface. So building an FX implied volatility surface is considerably more complicated than building an equity implied volatility surface. This appears to be a legacy of how FX options markets developed and in particular, of how risk reversals and strangles became the liquid securities in the market.

Until recently there have been very few (if any) detailed descriptions of the various conventions in FX markets and how FX volatility surfaces are built. More recently, however, some papers\(^7\) have been published on this topic. They can be consulted for further (tedious but necessary) details.

\section{The Triangle Relationships}

The fact that all currency pair combinations can be traded imposes restrictions on the dynamics of the various exchange rates. To see this we will introduce some new notation. In particular, let

\[ X_{t}^{a/b} := \text{# of units of currency A required to purchase 1 unit of currency B}. \]

Then it is easy to see that

\[ X_{t}^{a/b} = X_{t}^{a/c} \times X_{t}^{c/b} \] \hspace{1cm} (4)
\[ X_{t}^{a/b} = \frac{1}{X_{t}^{b/a}} \] \hspace{1cm} (5)

The advantage of this notation is that it follows the usual rules of division, i.e. \( a/c \times c/b = a/b \). The identity in (4) clearly defines the price process for any currency pair once the price processes for the other two pairs are specified.

\begin{example}[Geometric Brownian Motions]
Suppose \( X_{t}^{a/c} \) and \( X_{t}^{c/b} \) both follow GBM’s under some probability measure, \( \mathcal{P} \), so that

\[ dX_{t}^{a/c} = \mu^{a/c} X_{t}^{a/c} dt + \sigma^{a/c} X_{t}^{a/c} dW_{t}^{a/c} \]
\[ dX_{t}^{c/b} = \mu^{c/b} X_{t}^{c/b} dt + \sigma^{c/b} X_{t}^{c/b} dW_{t}^{c/b} \]

where \( W_{t}^{a/c} \) and \( W_{t}^{c/b} \) are \( \mathcal{P} \)-Brownian motions with \( dW_{t}^{a/c} \times dW_{t}^{c/b} = \rho dt \). Then Itô’s Lemma applied to (4) implies

\[ dX_{t}^{a/b} = \left( \mu^{a/c} + \mu^{c/b} + \sigma^{a/c} \sigma^{c/b} \rho \right) X_{t}^{a/b} dt + \sigma^{a/b} X_{t}^{a/b} dW_{t}^{a/b} \]

where \( \sigma^{a/b} := \sqrt{(\sigma^{a/c})^2 + (\sigma^{c/b})^2 + 2\rho \sigma^{a/c} \sigma^{c/b}} \) \hspace{1cm} (6)

and where \( W_{t}^{a/b} \) is also a \( \mathcal{P} \)-Brownian motion. The volatility, \( \sigma^{a/b} \), is therefore completely determined by the volatilities and correlation of the other two exchange rates.

In practice, we of course know that FX rates do not follow GBM’s. In particular, the implied volatilities for each of the three currency pairs will be a function of strike and time-to-maturity and so (6) is of little practical value. However, we emphasize again that if we have a good model for the dynamics of \( X_{t}^{a/c} \) and \( X_{t}^{c/b} \) then (4) uniquely determines the dynamics of \( X_{t}^{a/b} \).

\end{example}

\section{Converting to Base Currency Deltas}

Suppose now that you have bought a call option on \( X_{T}^{a/b} \) and that the option payoff is denominated in the base domestic currency, A. Let \( C(X_{T}^{a/b}) \) denote the time \( t \) value (in currency A) of this option as a function of the

\footnote{See, for example, “FX Volatility Smile Construction” by D. Reiswich and U. Wystup (2009). See also the forthcoming book “Foreign Exchange Option Pricing: A Practitioner’s Guide” by I.J. Clark and published by Wiley.}

\footnote{See (13) and the paragraph following it for a justification of this statement.}
current exchange rate, \( X_{t}^{a/b} \). Then this option could be delta-hedged in the usual Black-Scholes manner. Note, however, that if we delta-hedge the option we are hedging the currency A value of the option. In certain circumstances, however, and as mentioned earlier, we may wish to hedge or monitor the currency C value of this option. For example, a trading desk of a US bank may be responsible for all EUR/JPY trading. The trading book and its P&L will then be denominated in EUR, say. The desk, however, may also want to monitor the USD value of its positions and P&L if USD is the base currency of the bank. After all, there is not much point in being the greatest EUR/JPY trader in the world and earning billions of JPY if the value of JPY against the USD has collapsed and is now worthless.

So let \( V_{t}^{c} \left( X_{t}^{c/a}, X_{t}^{c/b} \right) \) denote the time \( t \) currency C value of the option. Then, using (4) and (5) we obtain

\[
V_{t}^{c} \left( X_{t}^{c/a}, X_{t}^{c/b} \right) = X_{t}^{c/a} \times C \left( t, X_{t}^{a/b} \right) = X_{t}^{c/a} \times C \left( t, X_{t}^{a/c} X_{t}^{c/b} \right)
\]

(7)

\[
= \frac{C \left( t, X_{t}^{a/c} X_{t}^{c/b} \right)}{X_{t}^{a/c}}
\]

\[
= X_{t}^{c/a} \times C \left( t, \frac{X_{t}^{c/b}}{X_{t}^{c/a}} \right).
\]

(8)

According to (8), or (7) if we prefer, the currency C value of the option now has two delta exposures: (i) a delta to the \( X_{t}^{c/a} \) exchange rate and (ii) a delta to the \( X_{t}^{c/b} \) exchange rate. We obtain

\[
\frac{\partial V_{t}^{c}}{\partial X_{t}^{c/a}} = C - \frac{X_{t}^{c/b}}{X_{t}^{c/a}} \frac{\partial C}{\partial X}
\]

(9)

\[
\frac{\partial V_{t}^{c}}{\partial X_{t}^{c/b}} = \frac{\partial C}{\partial X}
\]

(10)

where \( \frac{\partial C}{\partial X} \) is the usual Black-Scholes delta. These quantities can then be used to hedge the currency C value of the option. Or we might just hedge the \( X_{t}^{c/a} \) exposure and leave the \( X_{t}^{c/b} \) exposure unhedged. This way of viewing the value of the option is particularly useful if the trading book contains positions in many currency pairs and we ultimately only care about the value of the portfolio or the P&L in one particular currency. It is also useful when some of the currency pairs, e.g. \( X_{t}^{a/b} \), are not liquid and we can only hedge them by trading in \( X_{t}^{c/a} \) and \( X_{t}^{c/b} \).

Remark 1 If we are assuming sticky-by-delta implied volatility surfaces then we might want to include additional terms in (9) and (10) corresponding to \( \frac{\partial^2 C}{\partial X^2} \). For example, if \( X_{t}^{c/a} \) changes (and we hold constant \( X_{t}^{c/b} \)) then \( X_{t}^{a/b} \) will change accordingly with a corresponding change in \( \sigma_{a/b} \) due to the sticky-by-delta assumption. This would require an additional term on the right-hand-side of (9). Similarly, a change in \( X_{t}^{c/b} \) whilst holding \( X_{t}^{c/a} \) constant would require an additional term on the right-hand-side of (10). Whether or not such terms would be included in practice probably depends on the currencies in question, the preferences of the trader and the flexibility of the systems.

Exercise 3 Create an Excel spreadsheet to price and hedge currency options. You can assume the options are on the JPY/EUR exchange rate. Simulate values of the exchange rate through time and execute the usual delta-hedging strategy. Compare the final USD value of the hedge portfolio (which is denominated in JPY) with the final USD value of the portfolio where the USD-value of the option was dynamically hedged. You can assume the exchange rates all follow GBM’s with constant interest rates, volatilities and correlation. These parameters should be inputs in the spreadsheet.

---

9Although it could just as easily be JPY.
3 Options on Foreign Assets Struck in Domestic Currency

Let \( S_t \) denote the time \( t \) price\(^{10} \) of the foreign asset in units of the foreign currency. Then the payoff of the option is given by

\[
\text{Payoff} = \max(0, X_T S_T - K).
\]  

(11)

Let us write the domestic risk-neutral dynamics for \( S_t \), with the usual domestic cash account as numeraire, as

\[
dS_t = \mu_s S_t \, dt + \sigma S_t \, dW_t
\]  

(12)

with \( dW_t \times dW_t = \rho \, dt \). We assume the exchange rate dynamics continue to follow (2). In general we could let the correlation coefficient, \( \rho \), be stochastic but we will assume here that it is a constant, \( \rho \), for all \( t \) Note that the underlying price, \( X_t S_t \) is the time \( t \) value of the foreign stock measured in units of the domestic currency. We will also assume that \( S_t \) pays a continuous dividend yield of \( q \).

Remark 2 From the perspective of a foreign investor with foreign cash account as numeraire, the foreign asset’s risk-neutral drift in (12) would be \( \mu_s = r_f - q \). This is not true for the domestic investor, however, since \( S_t \) is not the (domestic) price of a traded asset nor\(^{11} \) is the foreign cash account a domestic numeraire. In fact it is \( S_t X_t \) that is the (domestic) price of a traded asset. This traded asset pays a dividend yield of \( q \) and so it is this security that has a drift of \( r_d - q \) under the domestic risk-neutral dynamics, i.e. the \( Q \)-dynamics taking the domestic cash account as numeraire.

A simple application of Itô’s Lemma implies that \( Z_t := X_t S_t \) has dynamics given by

\[
\frac{dZ_t}{Z_t} = (r_d - r_f + \mu_s + \rho \sigma_x \sigma) \, dt + \sigma_x \, dW_t^{(x)} + \sigma \, dW_t
\]  

(13)

where \( \sigma_z := \sqrt{\sigma_x^2 + \sigma^2 + 2 \rho \sigma_x \sigma} \). The quadratic variation of the second term in parentheses on the right-hand-side of (13) equals \( dt \). We can therefore apply Levy’s Theorem and write \( Z_t \) as a \( Q \)-GBM with drift \( r_d - q \) and volatility, \( \sigma_z \). Note that we didn’t care about the drift in (13) since the \( Q \)-drift of \( Z_t \) must be \( r_d - q \). The option price is then given by the Black-Scholes formula with risk-free rate \( r_d \), dividend yield \( q \) and volatility \( \sigma_z \).

Exercise 4 Argue that the value of \( \mu_s \) satisfies

\[
\mu_s = r_f - q - \rho \sigma_x \sigma.
\]  

(14)

Remark 3 In practice, a more sophisticated model would generally be used to price this option, depending as it does on the joint dynamics of \( X_t \) and \( S_t \). This is not true of our first example where we priced a vanilla currency option. In this case the correct volatility to be used in the Black-Scholes formula could be read off the volatility surface. It is also not true of the option in the exercise below where the volatility surface can again be used to obtain the correct option price.

Exercise 5 How would you price an option on a foreign asset that has been struck in foreign currency. Hint: This is easy!

\(^{10}\)Note that in the U.S a security whose time \( t \) value is given by \( X_t S_t \) is called an American Depositary Receipt or ADR. If, for example, \( S_t \) represents the time \( t \) value of a share of Volkswagen denominated in Euro, then a Volkswagen ADR allows US investors to easily invest in Volkswagen without having to trade in the foreign exchange markets. Similarly a European Depositary Receipt or EDR allows European investors to invest directly in non-Euro-denominated securities. Note, however, that the holder of an ADR or EDR is still exposed to foreign exchange risk. This is in contrast to the holder of a quanto security which we will discuss later.

\(^{11}\)In particular, an EMM-numeraire pair for a foreign investor is not an EMM-numeraire for a domestic investor. But see Exercise 6.
Exercise 6 Let $Q^f$ be an EMM for a foreign investor corresponding to some foreign numeraire, $X^f$ say. Note that $X^f_t$ is then denominated in units of the foreign currency. Show that $Q^f$ can also be used as an EMM for a domestic investor. What is the corresponding domestic numeraire? Is the converse also true, i.e., can any domestic EMM also be used as an EMM for a foreign investor?

4 Quanto-Securities

We now consider securities whose payoff is based on $S_T$ converted into units of domestic currency at a fixed exchange rate, $\bar{X}$. Such a security is called a quanto. We will consider quanto forwards and quanto options and price them within the Black-Scholes framework. Quantos are traded very frequently in practice, particularly in the structured products market. Consider, for example, an option on a basket of three stocks: IBM, Toyota and Siemens. The three stock prices are denominated in USD, JPY and EUR, respectively. But what currency should the option payoff be denominated in? Suppose we choose US dollars but we don’t want any exchange rate exposure. Then a call option with strike $K$ might have a payoff given by

$$\max(0, 1/3 \text{IBM}_T + 1/3 \text{ST}_T + 1/3 \text{TOY}_T - K)$$

and we say that Siemens and Toyota have been quanto’ed into US dollars at fixed exchange rates of $1/(3 \times \text{ST}_0)$ and $1/(3 \times \text{TOY}_0)$, respectively.

Another market where quantos are frequently traded is the commodity market. Most commodities are priced in US dollars but non-US investors often wish to trade commodities denominated in units of their domestic currency. For example, a European investor who wishes to buy an option on crude oil but without the USD/EUR exchange risk could buy a quanto option where the oil is quanto’ed into Euro. For example, a call option on oil with strike $K$ and maturity $T$ would normally have a time $T$ payoff of $\max(0, O_T - K)$ which is denominated in USD since the oil price, $O_T$, is denominated in USD. If the option is quanto’ed into Euro, however, then the payoff is again $\max(0, O_T - K)$, but now it is denominated in Euro despite the fact that $O_T$ is denominated in USD.

Quanto Forwards

Consider a quanto-forward contract where $\bar{X} S_T$ is exchanged at maturity, $T$, for $F^q_t$ units of the domestic currency. If we enter into this contract at time $t$, then we know that $F^q_t := F^q$ is chosen so that the initial value of the contract is 0. Using the usual domestic cash account as numeraire, we immediately obtain that

$$0 = E^Q_t \left[ e^{-r_d(T-t)} (\bar{X} S_T - F^q_t) \right]$$

so that $F^q_t = \bar{X} E^Q_t \left[ S_T \right]$. We therefore need the $Q$-dynamics of $S_t$ in order to compute $F^q_t$. But from (12) and (14), we know the $Q$-dynamics of $S_t$ are

$$dS_t = (r_f - q - \rho \sigma_x \sigma) S_t dt + \sigma S_t dW_t$$

and so we obtain (why?)

$$F^q_t = \bar{X} S_t e^{(r_f - q - \rho \sigma_x \sigma)(T-t)}.$$  \hspace{1cm} (16)

Exercise 7 Show that the fair time-$t$ value of receiving $\bar{X} S_T$ at time $T$ is given by

$$V_t = \bar{X} S_t e^{(r_f - r_d - q - \rho \sigma_x \sigma)(T-t)}.$$  \hspace{1cm} (17)

Hint: This requires a one-line argument given (15) or (16).
Note that \( V_t \) is the value in units of the domestic currency of a non-dividend\(^{12} \) paying traded asset. Note also that
\[
F^q_t = \bar{e}^{r_f(T-t)} V_t
\]
which is the usual form of a forward price. That is, \( F^q_t \) is the time-\( t \) forward price for time-\( T \) delivery of a domestic non-dividend paying security that is worth \( V_T \) at time \( T \). The standard replicating portfolio argument using \( V_t \) and the domestic cash account can be used as an alternative proof of (18).

The question also arises as to how to dynamically replicate the time-\( T \) payoff \( V_T = \bar{X} S_T \) using the domestic cash account, the foreign cash account and the foreign asset. In fact at each time, \( t \), the replicating strategy consists of

1. Investing \( V_t \) units of the domestic currency in the foreign asset.
2. Investing \(-V_t \) units of the domestic currency in the foreign-cash account, i.e. borrowing in the foreign currency to fund the purchase of the foreign asset in step 1.
3. Investing \( V_t \) units of the domestic currency in the domestic cash account.

Note that the first two components of the hedging strategy ensure that the strategy has no FX exposure. This must be the case as the asset we seek to replicate, \( V_T = \bar{X} S_T \), has no FX exposure. Note also that the time-\( t \)-value of the replicating portfolio is \( V_t \), as expected.

**Exercise 8** Use Itô’s Lemma and (17) to show that the replicating strategy is indeed as we have described.

Hint: You might start by assuming that \( a_t \) units of the domestic currency are invested in the foreign asset, \( b_t \) units of domestic currency are invested in the foreign cash account and \( c_t \) units of domestic currency are invested in the domestic cash account. The time-\( t \) domestic currency value of this portfolio is \( W_t = a_t + b_t + c_t \). The replicating strategy must be self-financing\(^{\text{13}} \) and so the dynamics of the strategy must satisfy
\[
dW_t = a_t \ d(\text{Return on foreign asset})_t + b_t \ d(\text{Return on foreign cash account})_t + c_t \ d(\text{Return on domestic cash account})_t.
\]

Note, for example, that \( d(\text{Return on foreign asset})_t = (X_t e^{qT S_t})/(X_t e^{qT S_t}) \). Why? Use this and other corresponding terms to compute \( dW_t \) and then compare with the dynamics of \( V_t \) which are obtained using (17).

(See Back’s “A Course in Derivative Securities” Section 6.6 if you are unsure.)

**Quanto Options**

We are now in a position to price quanto options. In fact, given our earlier analysis pricing a quanto option is straightforward. Let us assume in particular that we seek to price a call option with payoff in units of the domestic currency given by
\[
\text{Quanto Option Payoff} = \max(0, \bar{X} S_T - K).
\]

Martingale pricing states that the time-\( t \) price, \( P_t \), say, of this security is given by
\[
P_t = \mathbb{E}_t^Q \left[ e^{-r_f(T-t)} \max(0, \bar{X} S_T - K) \right]
\]
where the \( Q \)-dynamics of \( S_t \) satisfy (15). Note that we can rewrite (15) as
\[
dS_t = (r_d - q_f)S_t \ dt + \sigma S_t \ dW_t
\]
where \( q_f := q + r_d - r_f + \rho \sigma \sigma \). But then we see that the quanto option price is just \( \bar{X} \) times the Black-Scholes price of a call option on a stock with initial value \( S_t \), time-to-maturity \( T - t \), volatility \( \sigma \), strike \( K/\bar{X} \) and dividend yield \( q_f \).

**Remark 4** An alternative method of calculating the quanto option price is to view it as an option on the non-dividend security with time \( t \)-value \( V_t \). We can again use the Black-Scholes formula but this time using a zero-dividend yield and an initial value of \( V_0 = \bar{X} S_0 e^{(r_f - r_d - q_f - \rho \sigma \sigma)T} \).

\(^{12}\)Note that even though \( V_t \) is non-dividend paying, the underlying security \( S_t \), may well pay dividends.

\(^{13}\)All replicating strategies are, by definition, assumed to be self-financing.
It is also straightforward to replicate the payoff of the quanto option using the foreign asset, the foreign cash account and the domestic account. This is done by adopting the usual Black-Scholes delta-hedging strategy using the domestic cash account and the portfolio with value $V_t$ as the underlying risky asset. At each time $t$, we hold delta units of the portfolio and borrow the difference between the cost of the delta units and the value of the option. The portfolio is constructed using steps 1 to 3 as described earlier.

**Remark 5** Note that the key to valuing quanto’ed securities is obtaining the domestic risk-neutral dynamics of these securities. Everything else follows from the usual arguments.

**Remark 6** Banks and other market makers are very reluctant to quanto assets from “high-risk” economies. After all, who would want to sell an option that pays in US dollars $\max(0, Z_T - K)$ where $Z_T$ is the return on the Zimbabwe stock market!?