IEOR E4706 Foundations of Financial Engineering
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Due: 5pm Tuesday 4^{th} October 2016

Assignment 3 (Mandatory)

The Examples discussed in the questions below refer to the examples in the *"Martingale Pricing Theory in Discrete-Time and Discrete-Space Models"* lecture notes.

1. In our discrete-time, discrete-state framework, prove that the market is complete if and only if every embedded one-period model is complete.

2. **(No-Arbitrage Bounds)** Suppose we find ourselves at node $I_1^{6,7,8,9}$ at $t = 1$ in Example 10. The one-period model beginning at this node is incomplete and so not every security is attainable. Is a call option on the second security with strike $k = 2$ attainable? If so, what is the price of this call option? If not, can you bound the price of the option, i.e. find bounds on the option price that must be satisfied if there is to be no arbitrage? (Recall that the securities are numbered from 0 to 2.)

3. Referring to Example 9, compute the date $t = 0$ value of a European style option that expires at $t = 2$ with a payoff function given by $\max\left(0,\ S_2^{(1)} - S_2^{(2)} - 1\right)$.

4. Find the replicating portfolio for the contingent claim $X = \max\left(0,\ S_2^{(1)} - S_2^{(2)} - 1\right)$, i.e. the option of the previous question.

5. **(Pricing an Equity Swap)** Consider an equity swap based on the first security of Example 9. (Recall that the securities are numbered 0, 1 and 2 so that the first security refers to the security whose value at node $I_1^{1,2,3}$, for example, is 1.4346.) Party A is long the swap and receives $\$100 \times S_t^{(1)}$ from party B at dates $t = 1$ and $t = 2$. In return, party B is short the swap and receives $\$M$ at dates $t = 1$ and $t = 2$ from party A. Compute the value of the swap at $t = 0$ and determine what value of M will make the swap worth 0 at $t = 0$. (Note that you may answer this question using either the state prices or the unique EMM, Q, that corresponds to your choice of numeraire security.)

6. Compute the date $t = 0$ price of a forward contract for delivery at date $T = 2$ of the second security in Example 9.

7. Compute the date $t = 0$ price of a futures contract that expires at $T = 2$ on the second security in Example 9. Is the futures price different to the forward price? Are you surprised?

8. **(Futures Prices and Incomplete Markets)** Referring to Example 10, suppose we changed the payoffs of the 1^{st} and 2^{nd} securities in states ω_4 and ω_5 so that the model was arbitrage-free. (By leaving the payoff of the 0^{th} security unchanged at 1.1025 in those states, we maintain its interpretation as the cash account.) Now recall that this model is incomplete, implying in particular that there does not exist a unique EMM, Q.

   (a) Show that equation $F_0 = \mathrm{E}_0^Q[S_n]$ where F_0 is the time 0 futures price, still holds, irrespective of which Q we use. In particular, conclude that a unique arbitrage-free futures price process is computable. (Note that $F_0 = \mathrm{E}_0^Q[S_n]$ only holds when we take the cash account as numeraire so when we say "irrespective of which Q we use" it is implicit that we always have the cash account as the numeraire.)

(b) Give an example of a two-period financial market where a unique arbitrage-free futures price process on a particular underlying security does *not* exist.