IEOR E4706 Foundations of Financial Engineering
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Assignment 4 (Mandatory)

1. **(A Surprising Result?)** Consider an N-period binomial model for a non-dividend paying stock where the true probability of an up-move in each period is given by $p = .5$. The initial value of the stock is $S_0 = \$100$. Let C_0 denote the time $t = 0$ price of a European call option on the stock with strike K that expires after N periods. Now suppose that some extremely favorable news about the stock has just been announced so that while S_0 still remains at \$100, the probability of an up-move in each period has increased dramatically so that now $p = .999$. What will happen to C_0? Justify your answer. Is this what you would expect in practice? Again, justify your answer.

2. **(Handling Discrete Dividends)** A common method of adapting the binomial model to handle discrete dividends is as follows. Let S_0 denote the time $t = 0$ price of the stock and $D :=$ present values of dividends that are paid between now and option maturity, T. Let $X_0 := S_0 - D_0$ so that X_0 represents the stochastic component of the stock price. We then build a regular recombing binomial lattice to model the dynamics of X_t. At each node on the lattice for X_t we can calculate the stock price as $S_t = X_t + PV(\text{Dividends yet to be paid in } [t,T])$. Options can therefore be priced using this lattice for X_t. (Note that the use of the closed interval $[t,T]$ rather than $(t,T]$ implicitly implies the option-holder gets the dividend if the option is exercised at time t and a dividend is paid at that date.)

   Use this method to price a 6-month American call option on a stock with initial value $S_0 = 100$, strike $K = 100$ and a continuously compounded risk-free rate of $r = 5\%$. You can assume $\sigma = 30\%$ for X_t and that the stock pays a dividend of \$5 in 4 months time. A 6-period lattice will suffice. Is the option price the same as the price of the corresponding European option? While this method of handling discrete dividends is often used in practice, it does have some flaws. Can you identify any of them?

3. **(Using State Prices)** Use the forward equations to compute the state prices for $t = 0, \ldots, 6$ in the short-rate lattice of Example 1 in the *"Term Structure Lattice Models"* lecture notes. Then use the state prices to price the zero-coupon bond, European option, forward contract and caplet of Examples 1, 2, 4 and 6 respectively.

4. **(Sensitivity to Calibration: Very Important!!)** Reproduce the short-rate and payer-swaption lattices of Example 8 in the *"Term Structure Lattice Models"* lecture notes. Now experiment with different values of b (but re-calibrating each time so that the spot rate curve remains unchanged) to see how sensitive the swaption price is to the particular value of b that you choose. (Among other things, this should highlight the importance of calibrating models correctly and understanding whether or not your model is appropriate for the problem at hand. For example, we know that option prices are sensitive to volatility and since b is a volatility parameter in the BDT model, it is clearly important to calibrate it accurately. In

fact we may wish to choose and calibrate a separate b_i for each time period. If we were pricing a swaption where the underlying swap expired after 10 years instead of just 10 months we would also want to consider whether or not a model that cannot, for example, incorporate mean-reversion should be used for such a task.)

5. **(Pricing a Bermudan Swaption)** Price the payer-swaption of Example 8 in the *"Term Structure Lattice Models"* lecture notes but now assume that it may be exercised at any time, $t \in \mathcal{T} = \{2, 3, \ldots, 9\}$, and that the fixed rate in the underlying swap contract is now set at 11.65%. If exercised at time t then the first cash flow occurs at $t+1$ based on the short rate prevailing at time t. (Such an instrument is called a *Bermudan* swaption.)

6. **(Forward Start Swaps)** Consider a payer forward start swap where the swap begins at some fixed time T_n in the future and expires at time $T_M > T_n$. We assume the accrual period is of length δ, measured in years. Since payments are made in arrears, the first payment occurs at $T_{n+1} = T_n + \delta$ and the final payment at T_{M+1}. First convince yourself that at time $t < T_n$ the value, SW_t, of this forward start swap is

$$SW_t \;=\; \mathrm{E}_t^Q \left[\delta \sum_{j=n}^{M} \frac{B_t}{B_{T_{j+1}}} (L(T_j, T_j) - R) \right]$$

where R is the fixed rate (annualized) specified in the contract and $L(T_j, T_j)$ is the spot LIBOR rate applying to the interval $[T_j, T_{j+1}] = [T_j, T_j + \delta]$.

Note that LIBOR rates are *annualized* rates based on *simple* compounding. This means that 1 dollar invested at time T_j until time $T_j + \delta$ at the LIBOR rate $L(T_j, T_j)$ will be worth $1 + \delta L(T_j, T_j)$ dollars at time $T_j + \delta$.

(a) Assuming a notional principal of \$1, show that

$$SW_{T_n} \;=\; 1 - Z_{T_n}^{T_{M+1}} \;-\; R\delta \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j}$$

where $Z_{t_i}^{t_j}$ is the time t_i price of a zero-coupon bond that matures at time $t_j \geq t_i$.

(b) Now use martingale pricing to show that for $t < T_n$

$$SW_t \;=\; Z_t^{T_n} - Z_t^{T_{M+1}} \;-\; R\delta \sum_{j=n+1}^{M+1} Z_t^{T_j}.$$

(c) The *forward swap rate* is the value $R = R(t, T_n, T_M)$ for which $SW_t = 0$. Find the forward rate, again in terms of zero-coupon bond prices.