IEOR E4706 Foundations of Financial Engineering
Martin Haugh   Due: 5pm Friday 28^{th} October 2016

Assignment 6 (Mandatory)

1. **(Polya's Urn)**
   Consider an urn which contains red balls and green balls. Initially there is just one green ball and one red ball in the urn. At each time step a ball is chosen randomly from the urn:

   (a) If the ball is red, then it's returned to the urn with an *additional* red ball.

   (b) If the ball is green, then it's returned to the urn with an *additional* green ball.

   Let X_n denote the number of red balls in the urn after n draws. Then

$$\begin{aligned} \mathrm{P}(X_{n+1} = k+1 \,|\, X_n = k) &= \frac{k}{n+2} \\ \mathrm{P}(X_{n+1} = k \,|\, X_n = k) &= \frac{n+2-k}{n+2}. \end{aligned}$$

   Show that $M_n := X_n/(n+2)$ is a martingale.

2. **(Brownian Motion and Geometric Brownian Motion)**
   Let W_t be a standard Brownian motion.

   (a) What is $\mathrm{E}_0[W_{t+s}W_s]$?

   (b) The geometric Brownian motion (GBM) model for a security price assumes its time t price is given by

$$S_t = S_0\, e^{(\mu - \sigma^2/2)t + \sigma W_t}.$$

   Compute $\mathrm{E}_t\left[S_T\right]$ where as usual $\mathrm{E}_t[\cdot]$ denotes the expectation conditional on all time t information.

3. **(From Chapter 2 in Kerry Back's "*A Course in Derivative Securities*")**

   (a) Consider a discrete partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the time interval $[0, T]$ with $t_i - t_{i-1} = T/n$ for each i. Consider the function

$$X(t) = e^t.$$

   Write a function (in `Matlab`, `R` or some other language of your choice) that takes T and N as inputs, and then computes and prints $\sum_{i=1}^{N}[\Delta X(t_i)]^2$, where

$$\Delta X(t_i) = X(t_i) - X(t_{i-1}) = e^{t_i} - e^{t_{i-1}}.$$

   Hint: The sum can be computed as follows

```
sum = 0
For i = 1 To N
    DeltaX = Exp(iT/N) - Exp((i-1)T/N)
    sum = sum + DeltaX * DeltaX
Next i
```

(b) Repeat part (a) for the function $X(t) = t^3$. In both cases what happens to $\sum_{i=1}^{N}[\Delta X(t_i)]^2$ as $N \to \infty$, for a given T?

(c) Repeat part (b) to compute $\sum_{i=1}^{N}[\Delta W(t_i)]^2$ where W is a simulated Brownian motion. For a given T, what happens to the sum as $N \to \infty$?

(d) Repeat part (c), computing instead $\sum_{i=1}^{N} |\Delta W(t_i)|$ where $|\cdot|$ denotes the absolute value. What happens to this sum as $N \to \infty$, for a given T?

4. Use Itô's Lemma to prove that $\int_0^t W_s^2 \, dW_s \;=\; \frac{W_t^3}{3} - \int_0^t W_s \, ds$.

5. **(Oksendal Exercise 5.5)**

(a) Solve the Ornstein-Uhlenbeck equation $dX_t \;=\; \mu X_t \, dt \;+\; \sigma \, dW_t$ where μ and σ are real constants. (Hint: Use the integrating factor $e^{-\mu t}$ and consider $d(e^{-\mu t} X_t)$.)

(b) Find: (i) $\mathrm{E}[X_t]$ (ii) $\mathrm{Var}(X_t)$ (iii) $\mathrm{Cov}(X_t, X_{t+s})$.

6. **(Hedging Strategies)**
Let $G : \Omega \to R$ be a random variable. Ignoring technical restrictions, we can write G as

$$G \;=\; \mathrm{E}[G] \;+\; \int_0^T \theta_s \, dW_s.$$

By computing $\mathrm{E}_t[G]$ and then using Itô's Lemma, explicitly calculate the process, θ_t, for each of the following random variables, G.

Hint: Note that $M_t := \mathrm{E}_t[G]$ is martingale for any integrable random variable, G. Now what does this tell you about the dt term when you apply Itô's Lemma to M_t?

(a) $G = 1_A$ where $A = \{\exp(W_T) > K\}$.

(b) $G = W_T^2$.

(c) $G = \exp(a^2 T + aW_T)$ for $a > 0$.