IEOR E4706 Foundations of Financial Engineering
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Due: Tuesday 15^{th} November 2016

Assignment 7 (Mandatory)

1. **(The Deflated Gains Process is a Q-Martingale)**
   Consider a stock that has a continuous dividend yield of q with risk-neutral dynamics $dS_t = (r-q)S_t\,dt + \sigma S_t\,dW_t^Q$ where W_t^Q is a $\mathcal{Q}$-Brownian motion corresponding to taking the cash account as the numeraire. Show that, as expected,

$$S_0 \;=\; \mathrm{E}_0^Q\left[\int_0^T e^{-rt} q S_t\,dt + e^{-rT} S_T\right]. \tag{1}$$

   (You can assume that exchanging the order of integrations in (1) is justified.)

2. **(Deriving Black-Scholes When the Underlying Pays a Dividend Yield)**
   Derive the Black-Scholes PDE when the underlying stock has a constant dividend yield of q.

   Hint: The stock is assumed to pay a dividend of $qS_t\,dt$ at each time instant t. The instantaneous change in value from holding y_t units of the stock at time t is therefore given by $y_t(dS_t + qS_t\,dt)$.

3. **(Futures Contracts and Black-Scholes)**

   (a) Use martingale pricing to derive the time t price, $F_t^{(T)}$, of a futures contract for delivery of a stock at time T. You can assume that the Black-Scholes model holds and that the stock pays a constant dividend yield of q at each time instant. (You can use your knowledge of futures prices from discrete-time models to justify your answer.)

   (b) Compute the fair price of an option on a futures contract in the Black-Scholes model. You should assume that the futures contract expires at time T and that the option expires at time $\tau < T$. (This is straightforward using the original Black-Scholes formula and your answer from part (a). In particular, no messy calculations are required!)

   (c) Confirm directly that the option price you derived in part (b) satisfies the Black-Scholes PDE of Question 2.

4. **(Delta-Hedging)**
   Write a computer program that simulates m sample paths of the delta-hedging of a long position in a European option in the Black-Scholes model. Your code should take as inputs the initial stock price S_0, option expiration T, implied volatility σ_{imp}, risk-free rate r, dividend yield q and strike K as well as whether the option is a call or put. You should assume that at time $t = 0$ you have a short position in the option as well as an initial cash position equal to the value for which you sold the option. The total P&L at time T is then equal to the

total value of your portfolio (including the short option position) at that time. (Note that the total initial value is 0.)

Your code should also take as inputs: (i) the number of re-balancing periods N and (ii) the drift and volatility, μ and σ respectively, of the geometric Brownian motion used to simulate a path of the underlying stock price. Note that σ_{imp} and σ need not be the same.

At the very least your code should output the average option payoff and the average terminal total P&L (across the m paths) from holding the option and executing the delta-hedging strategy. See Section 4 of the *Black-Scholes Model* lecture notes for further details.

Once you have tested your code answer the following questions:

(a) When $\sigma_{\text{imp}} = \sigma$ how does the total P&L behave as a function of N? What happens *on average* if $\sigma_{\text{imp}} < \sigma$? If $\sigma_{\text{imp}} > \sigma$?

(b) For a fixed N, how does the total P&L behave as $\sigma_{\text{imp}} = \sigma$ increases?

(c) How does the drift, μ, affect the total P&L when $\sigma_{\text{imp}} = \sigma$?

5. **(OPTIONAL! Computing the Implied Volatility Surface)** The current index price is \$100 and the term structure of interest rates is constant at 3%. European call and put option prices of various strikes and maturities are presented below.

Call Prices

Strike \ T	.25	.5	1	1.5
60	40.2844	42.4249	50.8521	59.1664
70	30.5281	33.5355	42.6656	51.2181
80	21.0415	24.9642	34.4358	42.9436
90	12.2459	16.9652	26.4453	34.7890
100	5.2025	10.1717	19.4706	27.8938
110	1.3448	5.4318	14.4225	23.3305
120	0.2052	2.7647	11.2103	20.7206
130	0.0216	1.4204	9.1497	19.1828
140	0.0019	0.7542	7.7410	18.1858

Put Prices

Strike \ T	.25	.5	1	1.5
60	0.0858	2.1546	10.6907	19.3603
70	0.2548	3.1164	12.2087	20.9720
80	0.6934	4.3962	13.6833	22.2575
90	1.8232	6.2483	15.3972	23.6629
100	4.7050	9.3060	18.1270	26.3276
110	10.7725	14.4171	22.7834	31.3243
120	19.5582	21.6012	29.2757	38.2744
130	29.2999	30.1080	36.9195	46.2965

140	39.2055	39.2929	45.2152	54.8595

(a) Use put-call parity to determine a piece-wise constant dividend yield implied by the option prices. Does your dividend yield depend on the strikes you choose?

<u>*Hint*</u>: Put-call parity with a piecewise constant dividend yield implies

$$\begin{aligned} C_0(K,T_i) - P_0(K,T_i) &= e^{-\int_0^{T_i} q(t)\,\mathrm{d}t} S_0 - e^{-r\,T_i} K \\ &= e^{-\sum_{j=1}^{i} q_j (T_j - T_{j-1})} S_0 - e^{-rT_i} K \end{aligned} \tag{2}$$

for i = 1, ... , 4, and T = [0.25, 0.5, 1, 1.5].

(b) Write a piece of code to determine the Black-Scholes implied volatility for each option and plot the volatility surface.

(In practice, only the bid and ask prices of options are available in the market place and some pre-processing will be necessary to build the volatility surface and calibrate the implied dividends. For example, some options will have very wide bid-offers and are therefore less informative. Moreover, because these options are less liquid it is also the case that these bid-offers may not have been updated as recently as the more liquid options. It is often preferable then to ignore them when building the volatility surface.)