The Heath-Jarrow-Morton Framework

The Heath-Jarrow-Morton framework refers to a class of models that are derived by directly modeling the dynamics of instantaneous forward-rates. The central insight of this framework is to recognize that there is an explicit relationship between the drift and volatility parameters of the forward-rate dynamics in a no-arbitrage world.

The familiar short-rate models can be derived in the HJM framework but in general, however, HJM models are non-Markovian. As a result, it is not possible to use the PDE-based computational approach for pricing derivatives. Instead, discrete-time HJM models and Monte-Carlo methods are often used in practice.

1 Forward Rates

We use $f(t; s, s + \delta)$ to denote the forward rate at time $t$ for lending between times $s$ and $s + \delta$, where $t < s$ and $\delta > 0$. A simple no-arbitrage argument then implies that

$$\frac{Z^s_t}{Z^{s+\delta}_t} = \exp (\delta f(t; s, s + \delta))$$

so that in particular,

$$f(t; s, s + \delta) = -\log(Z^{s+\delta}_t) - \log(Z^s_t) / \delta$$.

**Definition 1** The forward rate, $f(t, s)$, is then defined to be the instantaneous interest rate agreed upon at time $t$ for lending or borrowing at time $s$. It satisfies

$$f(t, s) = \lim_{\delta \to 0} f(t; s, s + \delta)$$

$$= -\frac{\partial}{\partial s} \log(Z^s_t)$$ (1)

By integrating equation (1) we obtain

$$\int_t^s f(t, u) \, du = -\int_t^s \frac{\partial}{\partial u} \log(Z^u_t) \, du$$

$$= -\log(Z^s_t)$$

which implies

$$Z^s_t = \exp \left( -\int_t^s f(t, u) \, du \right).$$ (2)

We finally assume\footnote{This assumption makes intuitive sense and may be justified mathematically.} that the short rate, $r_t$, satisfies $r_t = f(t, t)$. 

2 The HJM No-Arbitrage Drift Restriction

Motivated in part by (2), the HJM framework considers forward rates to be the fundamental building block of term-structure models. We therefore model the evolution of forward rates explicitly and assume that each time $t$ forward rate, $f(t, s)$ for $t < s < T$, has $Q$-dynamics given by

$$f(t, s) = f(0, s) + \int_0^t \alpha(u, s) \, du + \int_0^t \sigma(u, s)^T \, dW_u$$

or, in differential notation,

$$df(t, s) = \alpha(t, s) \, dt + \sigma(t, s)^T \, dW_t$$

where $W_t$ is a $d$-dimensional standard $Q$-Brownian motion. Here $\alpha(u, s)$ and $\sigma(u, s)$ are $\mathcal{F}_u$-adapted processes for all $s > 0$.

Remark 1 It is worth emphasizing that since each forward rate, $f(t, s)$ for $s \in [t, T]$, is a stochastic process (with dynamics of the form given in (4)), we actually have infinitely many processes in the HJM framework. All of these processes, however, are driven by the same $d$-dimensional standard $Q$-Brownian motion, $W_t$.

The central insight of Heath-Jarrow-Morton was to recognize that the assumption of no-arbitrage implied a particular relationship between the drift, $\alpha(t, \cdot)$, and the volatility, $\sigma(t, \cdot)$. This relationship is given in the following theorem.

**Theorem 1** If there are no-arbitrage opportunities and if the $Q$-dynamics of the forward rates are given by (4), then

$$\alpha(t, s) = \sigma(t, s) \int_t^s \sigma(t, u) \, du.$$  

**Proof:** The proof proceeds by finding the $Q$-dynamics of $Z_t^s$ as implied by (2) and (4). We know that under the no-arbitrage assumption, it must be the case that the expected instantaneous return on $Z_t^s$ is equal to $r_t$ and this observation is then sufficient to derive (5).

We start by defining

$$Y_t := -\int_t^s f(t, u) \, du.$$

Standard calculus and (4) then implies

$$dY_t = f(t, t) \, dt - \int_t^s df(t, u) \, du = r_t \, dt - \int_t^s \left[ \alpha(t, u) \, dt + \sigma(t, u)^T \, dW_t \right] \, du.$$  

If we now define

$$\alpha^*(t, s) := \int_t^s \alpha(t, u) \, du$$

$$\sigma^*(t, s)^T := \int_t^s \sigma(t, u)^T \, du$$

and assume\(^\text{2}\) it is ok to switch the order of the stochastic and Riemann integration, then we obtain from (6)

$$dY_t = r_t \, dt - \alpha^*(t, s) \, dt - \sigma^*(t, s)^T \, dW_t.$$  

\(^\text{2}\)The result justifying this interchange of the order of integration is known as the Stochastic Fubini Theorem.
Now we know from (2) that $Z^s_t = \exp(Y_t)$. Itô’s Lemma then implies
\[
dZ^s_t = \exp(Y_t) \, dY_t + \frac{1}{2} \exp(Y_t) \, (dY_t)^2
\]
\[
= Z^s_t \left[ r_t - \alpha^*(t, s) + \frac{1}{2} \sigma^*(t, s)^T \sigma^*(t, s) \right] \, dt - Z^s_t \sigma^*(t, s)^T \, dW_t.
\]
But the $Q$-dynamics for $Z^s_t$ must have a drift coefficient equal to $r_t Z^s_t$. We therefore have
\[
\alpha^*(t, s) = \frac{1}{2} \sigma^*(t, s)^T \sigma^*(t, s).
\]
(8)
We then obtain (5) by differentiating across (8) with respect to $s$.

Theorem 1 implies that the arbitrage-free $Q$-dynamics of $f(t, s)$ for $t < s < T$ are given by
\[
df(t, s) = \left( \sigma(t, s)^T \int_t^s \sigma(t, u) \, du \right) \, dt + \sigma(t, s)^T \, dW_t
\]
(9)
\[
= \sum_{i=1}^d \left( \sigma_i(t, s) \int_t^s \sigma_i(t, u) \, du \right) \, dt + \sum_{i=1}^d \sigma_i(t, s)^T \, dW_{it}.
\]
(10)

Remark 2 Note that the HJM framework refers to a class of models. A particular HJM model is only specified once $f(0, s)$ and $\sigma(t, s)$ for $0 < t < s$ have been specified.

Remark 3 In applications it is sometimes assumed that $\sigma(t, s)$ is deterministic. In such circumstances we recover the familiar Gaussian class of models.

Example 1 (Ho-Lee)
Let us assume $d = 1$ and that $\sigma(t, u) = \sigma$, a constant, for all $t < u < T$. Then (10) implies
\[
f(t, s) = f(0, s) + \sigma^2 \int_0^t (s - u) \, du + \sigma W_t
\]
\[
= f(0, s) + \sigma^2 \left( ts - t^2/2 \right) + \sigma W_t.
\]
Since $r_t = f(t, t)$ we therefore have
\[
r_t = f(0, t) + \frac{\sigma^2}{2} t^2 + \sigma W_t
\]
and recognize this as the Ho-Lee model with a deterministic time-varying drift.

Example 2 (Hull-White)
Again we assume $d = 1$ but now take $\sigma(t, u) = \sigma \exp(-\alpha(u - t))$ where $\sigma$ and $\alpha$ are positive constants. Equation (10) and straightforward integration imply
\[
r_t = f(0, t) + \frac{\sigma^2}{2\alpha^2} \left( e^{-\alpha t} - 1 \right)^2 + \sigma \int_0^t e^{-\alpha(t-u)} \, dW_u
\]
which is consistent with the Hull-White model or the Vasicek model with time-varying drift parameters.
Remark 4 It should be stated that the above two examples where we obtained explicit expressions for the short-rate are not typical. More generally, \( r_t \) will not be a Markov process.

Exercise 1 How would you specify \( \sigma(t, s) \) so as to recover the CIR model?

Example 3 (A 2-Factor Gaussian Model)

A popular\(^4\) 2-factor Gaussian model is given by

\[
df(t, s) = \sum_{i=1}^{2} \left( \sigma_i(t, s) \int_t^s \sigma_i(t, u) \, du \right) \, dt + \sum_{i=1}^{2} \sigma_i(t, s)^T \, dW_t^{(i)}
\]

where

\[
\begin{align*}
\sigma_1(t, s) &= \sigma_{11} e^{-\alpha_1(s-t)} \\
\sigma_2(t, s) &= \sigma_{21} e^{-\alpha_1(s-t)} + \sigma_{22} e^{-\alpha_2(s-t)}.
\end{align*}
\]

This model is motivated in part by a principal components analysis\(^5\) (PCA) of changes in forward rates. This analysis suggests that the primary sources of randomness in the forward-rate curve are random changes in the slope of the curve followed by random twists in the curve. We can model this in our 2-factor Gaussian model as follows:

(i) Interpret \( W_t^{(1)} \) as driving changes in the slope of the curve.

(ii) Interpret \( W_t^{(2)} \) as driving twists in the curve. For this interpretation to make sense, we need to impose further constraints. For example, we might impose \( 0 < \alpha_2 < \alpha_1, \sigma_{22} > 0 \) and \( \sigma_{21} < -\sigma_{22} \).

Exercise 2 Convince yourself that the parameter constraints we impose in Example 3 do indeed allow us to interpret \( W_t^{(2)} \) as driving twists in the forward-rate curve.

Example 4 (The Drift Restriction for Foreign Bond Markets)

We assume that there are two bond markets, one domestic and one foreign. For simplicity, we take \( d = 1 \) and assume the \( Q \)-dynamics\(^6\) of the domestic and foreign forward rates are respectively given by

\[
\begin{align*}
df_d(t, s) &= \alpha_d(t, s) \, dt + \sigma_d(t, s) \, dW_t \\
df_f(t, s) &= \alpha_f(t, s) \, dt + \sigma_f(t, s) \, dW_t
\end{align*}
\]

We also assume that the exchange rate, \( X_t \), has \( Q \)-dynamics\(^7\) given by

\[
dX_t = (r^d_t - r^f_t) X_t \, dt + \sigma^x X_t \, dW_t
\]

where \( r^d_t \) and \( r^f_t \) are the domestic and foreign short-rates, respectively. \( X_t \) is the value of one unit of foreign currency in terms of domestic currency. We established earlier that under a no-arbitrage assumption, the relationship between \( \alpha_d(t, s) \) and \( \sigma_d(t, s) \) is described by (5). We now wish to establish a similar relationship between \( \alpha_f(t, s) \) and \( \sigma_f(t, s) \).

Let \( U_{s,t}^x \) denote the time \( t \) value of the foreign zero-coupon bond maturing at time \( s \) in units of the domestic currency. We then have

\[
\frac{U_{s,t}^x}{B_t} = X_t e^{-\int_t^s f_f(t, u) \, du - \int_0^t r^d_u \, du} := V_t
\]

\(^4\)But see Brigo and Mercurio, Section 5.2, or James and Webber, Section 8.2, for circumstances where \( r_t \) will be Markovian.

\(^5\)See Cairns Example 6.6

\(^6\)A question in Assignment 5 explains how PCA may be used to choose the volatility functions, \( \sigma_i(t, s) \).

\(^7\)The EMM \( Q \) corresponds to taking the domestic cash account as numeraire.
and note that \( V_t \) must be a \( Q \)-martingale. If we apply Itô’s Lemma and again interchange the order of stochastic and Riemann integration, then we see that

\[
d\left\{-\int_t^s f(t,u) \, du - \int_0^t r^d_u \, du\right\} = f(t,t) \, dt - \int_t^s [\alpha_f(t,u) \, dt + \sigma_f(t,u) \, dW_t] \, du - r^d_t \, dt
\]

\[
= (r^d_t - r^d_s) \, dt - \left[\int_t^s \alpha_f(t,u) \, du\right] \, dt - \left[\int_t^s \sigma_f(t,u) \, du\right] \, dW_t.
\]

We now apply Itô’s Lemma to \( V_t \) and use the previous expression to obtain

\[
dV_t = V_t \left[\frac{1}{2} \left(\int_t^s \sigma_f(t,u) \, du\right)^2 - \int_t^s \alpha_f(t,u) \, du - \sigma^2_t \left(\int_t^s \sigma_f(t,u) \, du\right)\right] \, dt + \text{(other terms)} \, dW_t. \tag{15}
\]

Since \( V_t \) is a martingale, the drift terms in (15) must sum to zero, i.e.,

\[
\frac{1}{2} \left(\int_t^s \sigma_f(t,u) \, du\right)^2 - \int_t^s \alpha_f(t,u) \, du - \sigma^2_t \left(\int_t^s \sigma_f(t,u) \, du\right) = 0. \tag{16}
\]

Differentiating across (16) with respect to \( s \) then implies

\[
\alpha_f(t,s) = \sigma_f(t,s) \left[\int_t^s \sigma_f(t,u) \, du - \sigma^2_t\right]. \tag{17}
\]

Remark 5 It is clear that we can easily derive an analogous restriction in Example 4 when we take \( d > 1 \). In practice, we would indeed assume \( d > 1 \). This class of models is useful, for example, if we wish to price a foreign bond option with time \( t \) value, \( C_t \), given by

\[
C_t = E^Q_t \left[e^{-\int_t^T r_u \, du} (U_T - K)^+\right]
\]

where \( T \) is the option’s expiration and \( S > T \). These models also often arise in the context of pricing hybrid securities.

Example 5 (Log-Normal Forward Rates?)

In specifying a HJM model we must specify the volatility function \( \sigma(t,s) \). An obvious choice would be to select

\[
\sigma(t,s) = \sigma f(t,s) \tag{18}
\]

where \( \sigma \) is a positive constant, leading presumably to log-normal forward rates. In their original paper, however, Heath, Jarrow and Morton showed that choosing \( \sigma(t,s) \) according to (18) would cause the forward rates to explode. In particular, the no-arbitrage drift restriction of (5) implies

\[
\alpha(t,s) = \sigma^2 f(t,s) \int_t^s f(t,u) \, du \\
\approx \sigma^2 f(t,s)^2 (t-s) \tag{19}
\]

for \( t \) close to \( s \). It is the \( f(t,s)^2 \) term in (19) that causes the explosion problem. Shreve\(^8\) provides intuition for this observation by considering the ordinary differential equation, \( f'(t) = \sigma^2 f^2(t) \), whose solution is given by

\[
f(t) = \frac{f(0)}{1 - \sigma^2 f(0)t}. \tag{20}
\]

It is clear that this solution explodes at \( t = 1/\sigma^2 f(0) \).

\(^8\)See Section 10.4.1 Stochastic Calculus for Finance II: Continuous Time Models.
Remark 6 It is worth pointing out that the explosive behavior of Example 5 when we define $\sigma(t,s)$ according to (18) does not occur in the discrete-time world of Section 5.

Remark 7 The desire to generate log-normal interest rates and therefore provide a theoretical basis for Black’s caplet formula helped spur the development of market LIBOR models. These models are not inconsistent with HJM models and indeed some of the earlier market LIBOR models, e.g. the BGM model, were developed in the HJM framework. They focus, however, on simple forward rates rather than continuously compounded forward rates.

3 The Drift Restriction under The Forward Measure

Suppose we wish to work under the forward measure, $P_{T_1}$, that corresponds to taking $Z_{T_1}$ as numeraire. Then Section 11 of the Introduction to Stochastic Calculus and Mathematical Finance lecture notes implies

$$ \frac{dP_{T_1}}{dQ} = \frac{1}{B_{T_1}Z_{T_1}^0} = \exp \left( -\int_0^{T_1} [\alpha(v, u) du + \int_0^u \sigma(v, u)^T dW_v] \right) $$

$$ = \exp \left( -\int_0^{T_1} \left[ \int_v^T \alpha(v, u) du + \int_0^u \sigma(v, u)^T dW_v \right] dv \right) \tag{21} $$

$$ = \exp \left( -\int_0^{T_1} \left[ \int_v^{T_1} \alpha(v, u) du - \int_0^T \sigma(v, u)^T dW_v \right] dv \right) \tag{22} $$

$$ = \exp \left( -\frac{1}{2} \int_0^{T_1} \sigma^*(v, T_1) T \sigma^*(v, T_1) dv - \int_0^{T_1} \sigma^*(v, T_1) T dW_v \right) \tag{23} $$

Note that in going from (21) to (22) we needed to change the limits of integration as we switched the order of stochastic and Riemann integration. Girsanov’s Theorem now implies that

$$ W_{P_{T_1}} := W_t + \int_0^t \sigma^*(v, T_1) dv $$

is a standard $d$-dimensional $P_{T_1}$ Brownian motion. Equation (9) now implies

$$ df(t,s) = (\sigma(t,s)^T \sigma^*(t,s)) dt + \sigma(t,s)^T dW_t $$

$$ = [\sigma(t,s)^T \sigma^*(t,s) - \sigma(t,s)^T \sigma^*(t,T_1)] dt + \sigma(t,s)^T dW_{P_{T_1}} $$

$$ = \sigma(t,s)^T \sigma^*(t,s) - \sigma^*(t,T_1) \right] dt + \sigma(t,s)^T dW_{P_{T_1}} $$

$$ = -\sigma(t,s)^T \left[ \int_s^{T_1} \alpha(t, u) du \right] dt + \sigma(t,s)^T dW_{P_{T_1}}. $$

Remark 8 In contrast with LIBOR market models, the HJM framework is not usually applied under the forward measure.

4 Model Calibration in the HJM Framework

We need to specify $f(0,t)$ and $\sigma(t,s)$ in order to identify a particular HJM model. For the purposes of pricing, it is more convenient to calibrate using the EMM, $Q$, rather than using the true data generating measure, $P$. This is certainly true when we wish to use the model to price securities since all pricing is done under $Q$ rather than under $P$. There are times, however, when it may be desirable to calibrate under $P$. For example, if we wish to use the model for risk-management purposes, e.g. computing a VaR or CVaR, then we typically want to work...
with \( P \) and therefore need to calibrate the model under \( P \). The discussion below assumes that we are calibrating under \( Q \).

**Specifying \( f(0, t) \):** Recalling equation (2), we see that

\[
f(0, t) = -\frac{\partial \log Z^t_0}{\partial t}.
\]

(24)

If we therefore choose \( f(0, t) \) to satisfy (24) (interpreting the \( Z^t_0 \)'s in (24) as the time 0 market prices of zero-coupon bonds), then our HJM model will be automatically calibrated to the market term-structure. This contrasts with the short-rate models with deterministic time-varying parameters, where some work is required to match the term-structure.

**Remark 9** In practice, zero-coupon bond prices for only a finite number of maturities are observed in the market. This means estimating \( f(0, t) \) for all \( 0 < t < T \) requires care, especially since differentiation (as required in (24)) is a numerically unstable procedure. In practice, market rates such as swap rates or interest rate futures are also used along with zero-coupon bond prices to help construct the yield curve.

**Specifying \( \sigma(t, s) \):** There are a number of ways to specify the volatility functions, \( \sigma(t, s) \). One method is to use historical data which, of course, is generated under the true probability measure, \( P \). Though we wish to calibrate the model under \( Q \), this does not create any difficulty as the volatility functions are the same under both \( P \) and \( Q \) (or any EMM for that matter). This is clear from Girsanov's Theorem. One of the questions in Assignment 5 demonstrates how this may be done using principal components analysis when certain simplifying assumptions are made about the volatility functions.

One disadvantage that arises from using historical data is that model prices for commonly traded securities such as caps and swaptions will generally not match the model’s prices for those securities. As a result, it is often preferable to ignore historical data and instead choose \( \sigma(t, s) \) so that the model prices caps or swaptions consistently with the market. This may be achieved by specifying a functional form for \( \sigma(t, s) \), e.g., as in Example 3, and then choosing the parameters to match security prices in the market.

It is also possible to combine the two approaches using historical data and market prices to calibrate. Such an approach is also described in Assignment 5.

## 5 HJM in Discrete-Time with a View to Monte-Carlo Simulation

Since HJM models are generally non-Markovian, the PDE-based techniques associated with the Feynman-Kac formula are not available as a tool for pricing securities. As a result, Monte Carlo simulation is the computational tool of choice. In order to simulate \( f(t, s) \) for \( s > t \), it is necessary to discretize both time, \( t \), and maturity, \( s \). There are many ways to do this, with perhaps the most obvious method being to simulate the continuous-time model (4) using an Euler scheme on \( 0 = t_0 < t_1 < \cdots < t_m = T \). Since we need to discretize both time and maturity, however, it would be very computationally expensive to simulate the Euler scheme for small values of \( h \) where \( h := t_{i+1} - t_i \). Instead we might choose to simulate the Euler scheme for larger values of \( h \). However, the quality of the approximation then deteriorates and so it might be preferable to directly develop discrete-time (but still continuous-state) arbitrage-free HJM models instead. This approach is commonly used in practice and we now describe\(^9\) it.

We will assume that the same partition, \( 0 = t_0 < t_1 < \cdots < t_m = T \), is used to discretize both time, \( t \), and maturity, \( s \), for \( 0 \leq t \leq s \leq T \). Let \( \hat{f}(t_i, t_j) \) denote\(^10\) the forward rate at time \( t_i \) for borrowing or lending

\(^9\)Our description closely follows that of Section 3.6.2 in *Monte Carlo Methods in Financial Engineering* by Glasserman.

\(^10\)We will use “hat” notation to denote discretized variables.
between \( t_j \) and \( t_{j+1} \). Then the time \( t_i \) zero-coupon bond price for maturity, \( t_j \), is given by

\[
\hat{Z}_i^j = \exp \left( -\sum_{k=i}^{j-1} \hat{f}(t_i, t_k)h_k \right)
\]  

(25)

where \( h_k := t_{k+1} - t_k \).

**Exercise 3** Let \( f(0, u) \), for \( 0 \leq u \leq T \), denote the observed term-structure in the market place. Explain why we would choose

\[
\hat{f}(0, t_k) = \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} f(0, u) \, du = \frac{1}{h_k} \int_{t_k}^{h_k+1} f(0, u) \, du
\]

if we want to calibrate our discrete-time model to the market term-structure.

For ease of exposition, we will assume\(^{11}\) that \( d = 1 \). As in the continuous-time HJM framework, we then assume that the \( Q \)-dynamics of forward rates are given by

\[
\hat{f}(t_i, t_j) = \hat{f}(t_{i-1}, t_j) + \hat{\mu}(t_{i-1}, t_j)h_{i-1} + \hat{\sigma}(t_{i-1}, t_j)\sqrt{h_{i-1}} \, Z_i, \quad j = i, \ldots, m
\]

(26)

where \( Z_1, \ldots, Z_m \) are IID \( N(0, 1) \) random variables. Note also that as before, all forward rates are driven by the same source of randomness, i.e., \( \{Z_i\}_{i=1}^{m} \). In general, \( \hat{\mu}(t_{i-1}, t_j) \) and \( \hat{\sigma}(t_{i-1}, t_j) \) are adapted processes that may depend on the entire forward rate-curve at time \( t_{i-1} \). The volatility term, \( \hat{\sigma}(t_{i-1}, t_j) \), is usually chosen by matching the prices of commonly traded derivatives in the model to those in the market place.

The first step in the discrete-time HJM framework is to derive the drift restriction that is analogous to the continuous-time restriction, (5), of Theorem 1. As usual, we do this by insisting that discounted security prices are \( Q \)-martingales. By applying this in particular to zero-coupon bond prices, we see that we must have

\[
E_t^Q \left[ \hat{Z}_i^j \exp \left( -\sum_{k=0}^{i-1} \hat{f}(t_k, t_k)h_k \right) \right] = \hat{Z}_i^{j-1} \exp \left( -\sum_{k=0}^{i-2} \hat{f}(t_k, t_k)h_k \right).
\]  

(27)

By first canceling common terms on both sides of (27), then substituting (25) into (27) and then canceling again on both sides, we obtain

\[
E_t^Q \left[ \exp \left( -\sum_{k=i}^{j-1} \hat{f}(t_i, t_k)h_k \right) \right] = \exp \left( -\sum_{k=i}^{j-1} \hat{f}(t_{i-1}, t_k)h_k \right).
\]  

(28)

For each \( \hat{f}(t_i, t_k) \) on the left-hand-side of (28), we make a substitution using (26) and find

\[
E_t^Q \left[ \exp \left( -\sum_{k=i}^{j-1} \left( \hat{f}(t_{i-1}, t_k) + \hat{\mu}(t_{i-1}, t_k)h_{i-1} + \hat{\sigma}(t_{i-1}, t_k)\sqrt{h_{i-1}} \, Z_i \right)h_k \right) \right] = \exp \left( -\sum_{k=i}^{j-1} \hat{f}(t_{i-1}, t_k)h_k \right).
\]  

(29)

Canceling terms in (29) and noting that \( \hat{\mu}(t_{i-1}, t_k) \) is known at time \( t_{i-1} \), we obtain

\[
E_t^Q \left[ \exp \left( -\sum_{k=i}^{j-1} \hat{\sigma}(t_{i-1}, t_k)\sqrt{h_{i-1}} \, Z_i \right)h_k \right] = \exp \left( \sum_{k=i}^{j-1} \hat{\mu}(t_{i-1}, t_k)h_{i-1}h_k \right)
\]

which then implies

\[
\frac{1}{2} \left( \sum_{k=i}^{j-1} \hat{\sigma}(t_{i-1}, t_k)h_k \right)^2 = \sum_{k=i}^{j-1} \hat{\mu}(t_{i-1}, t_k)h_k.
\]  

(30)

Since (30) must hold for all \( j \), it must also be the case that

\[
\hat{\mu}(t_{i-1}, t_j)h_j = \frac{1}{2} \left( \sum_{k=i}^{j} \hat{\sigma}(t_{i-1}, t_k)h_k \right)^2 - \frac{1}{2} \left( \sum_{k=i}^{j-1} \hat{\sigma}(t_{i-1}, t_k)h_k \right)^2.
\]  

(31)

\(^{11}\)The derivation for \( d > 1 \) proceeds in exactly the same manner.
Remark 10 Equation (31) is the drift restriction for the discrete-time HJM model. Moreover, it can easily be seen\(^{12}\) that (31) is consistent with its continuous-time counterpart, (5).

In the multi-factor case, i.e., when \(d > 1\), we assume a model of the form

\[
\hat{f}(t_i, t_j) = \hat{f}(t_{i-1}, t_j) + \hat{\mu}(t_{i-1}, t_j)h_{i-1} + \frac{1}{2} \sum_{l=1}^{d} \hat{\sigma}_l(t_{i-1}, t_j)\sqrt{h_{i-1}} Z_{i,l}, \quad j = i, \ldots, m
\]

where \(Z_i := (Z_{i,1}, \ldots, Z_{i,d})\) for \(i = 1, \ldots, m\), are independent \(N(0, I)\) random vectors. By repeating steps (27) to (30) we easily find that in a no-arbitrage world it must be the case that

\[
\hat{\mu}(t_{i-1}, t_j)h_j = \sum_{l=1}^{d} \left[ \frac{1}{2} \left( \sum_{k=i}^{j} \hat{\sigma}_l(t_{i-1}, t_k)h_k \right)^2 - \frac{1}{2} \left( \sum_{k=i}^{j-1} \hat{\sigma}_l(t_{i-1}, t_k)h_k \right)^2 \right]
\]

Equation (32) is the general drift restriction for the multi-factor discrete-time HJM model.

Pricing Derivatives

Derivative securities in discrete-time HJM models can now be priced using simulation. In particular, the time \(t = 0\) price, \(C_0\), of a derivative with payoff \(C_T\) at time \(T = t_m\) is given by

\[
C_0 = \mathbb{E}^Q \left[ \exp \left( - \sum_{k=0}^{m-1} \hat{f}(t_k, t_k)h_k \right) C_T \right].
\]

Example 6 (Pricing a Caplet)

Suppose we wish to price a caplet maturing a time \(T\), with strike \(K\) and the spot rate \(Y_T^{T + \tau}\) as the underlying. Then the payoff, typically occurring at time \(T + \tau\), is given by \(\max(Y_T^{T + \tau} - K, 0)\). In the continuous-time HJM framework we have

\[
Y_T^{T + \tau} = \frac{\int_T^{T + \tau} f(T, u) \, du}{\tau},
\]

In our discrete-time framework, if \(T = t_i\) and \(T + \tau = t_{i+1}\), then the analogue of (33) is a time \(t_{i+1}\) payoff given by \(\max(Y_{t_i}^{t_{i+1}} - K, 0)\) where now

\[
Y_{t_i}^{t_{i+1}} = \frac{\exp \left( \hat{f}(t_i, t_i)[t_{i+1} - t_i] \right) - 1}{t_{i+1} - t_i}.
\]

More generally, if the caplet applies to an interval \([t_i, t_{i+n}]\) then the relevant\(^{13}\) spot rate is given by

\[
Y_{t_i}^{t_{i+n}} = \frac{\exp \left( \sum_{k=0}^{n-1} \hat{f}(t_i, t_{i+k})[t_{i+k+1} - t_{i+k}] \right) - 1}{t_{i+n} - t_i}.
\]

In order to price caplets, all we now need to do is to simulate (26) and average (suitably discounted) the cash flows on each simulated path.

\[\hat{f}(t, t) = f(t, t) + \sum_{j=1}^{p} \hat{\sigma}_j(t, t)\sqrt{t - t_j} Z_{j}, \quad j = 1, \ldots, m\]

Remark 11 A particular advantage of the HJM framework over short rate models occurs in the context of simulation. For example, if we are pricing a derivative that matures at time \(T\) with a payoff \(f(Z_T^{T + \tau})\) then the payoff is easily evaluated at time \(T\). This is clear since time \(T\) zero-coupon bond prices, determined as they are by the forward rate curve at time \(T\), are immediately available at time \(T\) in any HJM model. This contrasts with short-rate models where an additional simulation may be required at time \(T\) to determine \(Z_T^{T + \tau}\). Recalling that the payoffs to caplets and swaps may be expressed in terms of zero-coupon bonds we see that this can be a significant advantage.

\(^{12}\)See Section 3.6.2 of Glasserman.

\(^{13}\)See Glasserman Section 3.6.3 for further details including pseudo-code for simulating efficiently the evolution of the forward curve according to (26) and (32).