Monte Carlo Simulation, IEOR E4703, Spring 2003

Columbia University
Instructor: Martin Haugh
Final Examination, May 12, 2003
Total Marks: 100
Time: 3 hours

Question 1 (20 marks)
Consider the problem of generating a random variable X that has density function

$$f(x) = \begin{cases} 0 & x < 0 \\ k(x^3 + 2x^7) & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

(a) Find the value of k. (2 marks)

(b) Find the CDF of X and explain whether or not it is difficult to generate samples of X using the inverse transform method. (6 marks)

(c) Describe a composition method for generating X. (6 marks)

(d) Describe an acceptance-rejection algorithm for generating X. (Be sure to specify the sampling density, $g(\cdot)$, and the constant, a, that you use.) (6 marks)

Question 2 (24 marks)
Consider an exchange option where one has the option to exchange one asset for another. If $S_1(t)$ and $S_2(t)$ are the prices of the two assets at time t, then the discounted payoff of the option, X, is given by

$$X = e^{-rT} \max\left(S_1(T) - S_2(T),\ 0\right).$$

The price of the option is then given by $\theta := \mathrm{E}[X]$ where we assume that $S_i(t) \sim \mathrm{GBM}(r, \sigma_i)$ for $i = 1, 2$ and where we also assume that returns have correlation coefficient equal to ρ. That is,

$$S_i(t) = S_i(0) e^{(r - \sigma_i^2/2)t + \sigma_i \sqrt{t} Z_i}$$

where (Z_1, Z_2) are $\mathrm{N}(0,1)$ with correlation coefficient, ρ. Take $S_1(0) = 100$, $S_2(0) = 80$, $T = 1$, $\sigma_1 = 0.3$, $\sigma_2 = 0.2$ and $\rho = -0.7$.

(a) Write down the covariance matrix, Σ, for (Z_1, Z_2). (2 marks)

(b) Manually determine a 2×2 matrix C such that $C^T C = \Sigma$. (3 marks)

(c) Give the procedure to generate a sample of (Z_1, Z_2). You may assume that you can compute $\Phi^{-1}(\cdot)$ where $\Phi(\cdot)$ is the CDF of a standard normal random variable. You may also assume that a uniform random number generator is available to you. (3 marks)

(d) Give a Monte Carlo simulation algorithm that makes use of antithetic variates for estimating θ. You should also show how to construct approximate confidence intervals. (10 marks)

(e) Is your antithetic variate procedure guaranteed to produce a variance reduction? Why or why not? (3 marks)

(f) Explain clearly how you could estimate θ using a simulation algorithm that only required you to generate one random variable per sample. (3 marks)

Question 3 (18 marks)
Suppose that V is an exponentially distributed random variable with mean 1, and given that $V = v$, W is exponentially distributed with mean v. Describe an algorithm for estimating $\mathrm{P}(VW \leq 3)$ that only needs to generate *one* random variable per sample.
(You may need to use the following fact: the PDF of an exponentially distributed random variable with mean $1/\mu$ is given by $f(x) = \mu e^{-\mu x},\ x \geq 0$.)

Question 4 (20 marks)
(a) Minerva wished to estimate $\theta := \mathrm{E}[h(X)]$ where she believed that X had an exponential density with mean $\mu = 1$. To do this, she generated and *stored* n samples of X from the exponential distribution with mean 1, and used these samples to estimate θ. After doing this, however, she realized that X really has a Gamma$(2, 1)$ density so that its true PDF is given by $f(x) = x\exp(-x)$. As a result, a new estimate of θ needed to be computed. Unfortunately, Minerva did not know how to generate Gamma$(2, 1)$ random variables. But all was not lost and Minerva still managed to construct an unbiased estimate of θ.

(i) Explain how Minerva managed to do this **and** give an expression for her estimate, $\hat{\theta}_n$. (10 marks)

(ii) Suppose now that the true distribution of X had been other than the Gamma$(2, 1)$ distribution. Under what circumstances would Minerva still have been able to estimate θ? (5 marks)

(b) Suppose we wish to use importance sampling to estimate $\theta := \mathrm{E}[h(X)]$ where X is a random variable with PDF, $f(\cdot)$. When h is a non-negative function we saw that we would like to choose the importance sampling density, g, so that $g(x) = ch(x)f(x)$ where c is a constant such that g integrates to 1. In practice, of course, it is not possible to choose g this way since computing c is equivalent to computing θ. This observation, however, did suggest that we should choose g so that it has a similar shape to hf.

Suppose now that h is *not* restricted to be non-negative. Can you think of an importance sampling method for estimating θ that still makes uses of the above observation? (Hint:

You may wish to estimate more than one expectation.) (5 marks)

Question 5 (18 marks)
(a) (7 marks) Describe in detail a Monte Carlo simulation algorithm for estimating

$$\theta = \int_0^\infty e^{-x^2} \, dx.$$

(b) Could you estimate θ by using a stratified simulation algorithm? If so, what would your stratification variable be and show why you can in fact stratify using this variable? (4 marks)

(c) Describe in detail a stratified sampling algorithm for estimating θ. You may assume that m equi-probable strata are to be used and that the number of samples in each stratum, n_i, is constant for $i = 1, \ldots, m$. That is, $n_1 + \ldots + n_m = n$ where n is the total number of samples. You do **not** need to describe how to construct approximate confidence intervals. (Marks will be lost if your algorithm is not clearly explained. And the easiest way to explain your algorithm is to write it using pseudo-code!) (7 marks)