IEOR E4703: Monte Carlo Simulation Fall 2004
Columbia University
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Outline Solutions to Final Examination

Question 1 (20 marks)
(a) The only way the PDF can integrate to 1 is if $c = 6/5$.

(b)

$$F(x) = \begin{cases} 0, & x < 0 \\ (2x^3 + 3x^4)/5, & 0 \le x \le 1 \\ 1, & \text{for } x > 1 \end{cases} .$$

There is no closed form expression for $F^{-1}(\cdot)$ and so numerical methods would be required.

(c) Let $Y \sim U(0,1)$ with PDF $g(x) = 1$ for $x \in [0,1]$ and 0 otherwise. Need a such that $f(x)/g(x) \le a$ for all $x \in [0,1]$. Clearly $a = 18/5$ will suffice. The acceptance-rejection algorithm then continues to generate independent pairs of $U(0,1)$ random variables, (U_1, U_2), until $f(U_1)/ag(U_1) \ge U_2$. Then set $X = U_1$.

(d) We can write $f(x) = p_1 f_1(x) + p_2 f_2(x)$ where $p_1 = 2/5$, $p_2 = 3/5$, $f_1(x) = 3x^2$ for $x \in [0,1]$, and $f_2(x) = 4x^3$ for $x \in [0,1]$. The details are straightforward from here.

Question 2 (10 marks)
This is a 'composition' type algorithm. We have

$$\begin{aligned} F_X(x) = P(X \le x) &= \frac{1}{2}\left[P(X \le x | U_1 \le .5) + P(X \le x | U_1 > .5)\right] \\ &= \frac{1}{2}\left[P(U_2 \le \exp(x)) + P(U_2 \le 1 - \exp(-x))\right]. \end{aligned}$$

There are two cases to consider: (i) $x \ge 0$ and (ii) $x < 0$. We find

$$F(x) = \begin{cases} \exp(x)/2, & x < 0 \\ 1 - \exp(-x)/2, & x \ge 0 \end{cases} .$$

Question 3 (20 marks)
We know

$$C_0 = \mathrm{E}\left[e^{-rT}\left(\max(0, S_T - K_1) I_{\{S_{T/2} \le L\}} + \max(0, S_T - K_2) I_{\{S_{T/2} > L\}}\right)\right]$$

where $S_t \sim GBM(r, \sigma^2)$. Using conditional Monte Carlo (as we saw in the lecture notes), we can write $C_0 = \mathrm{E}[Y]$ where

$$Y := e^{-rT/2}\left(c(S_{T/2}, T/2, K_1, r, \sigma) I_{\{S_{T/2} \le L\}} + c(S_{T/2}, T/2, K_2, r, \sigma) I_{\{S_{T/2} \ge L\}}\right). \quad (1)$$

We could now use stratified sampling by stratifying along $Z \sim N(0,1)$ where

$$S_{T/2}(Z) = S_0 \exp\left((r - \sigma^2/2)T/2 + \sigma\sqrt{T/2}\ Z\right).$$

Note that since $Z \sim N(0,1)$ the two conditions for stratified sampling are satisfied: (i) $P(Z \in [a,b])$ is easily computed and (ii) We can generate $\left(S_{T/2} | Z \in [a,b]\right)$ either using an acceptance-rejection type algorithm or using the algorithm described in the lecture notes: first generate $U \sim U(\Phi(a), \Phi(b))$, set $Z = \Phi^{-1}(U)$ and then set $S_{T/2} = S_{T/2}(Z)$.

Note that we could also have chosen to stratify first and to then use conditional Monte Carlo. The algorithm would be unchanged.

Question 4 (15 marks)
(i) This is standard and is shown in the lecture notes.

(ii) Note that $0 \leq I_{\{W \geq a\}} e^{\mu^2/2 - \mu W} \leq e^{\mu^2/2 - \mu a}$. Since $\widehat{\theta}_\mu$ is the average of n such samples and is unbiased, we can conclude that $\theta \leq e^{\mu^2/2 - \mu a}$.

(iii) Minimizing $e^{\mu^2/2 - \mu a}$ with respect to μ we find $\mu = a$. This value is consistent with the maximum principle.

Question 5 (25 marks)
(a) This is standard and is shown in the lecture notes.

(b) This is also standard and should now be automatic for everyone!

(c) This is also standard. We know $E[S_T]$ and would expect S_T to be correlated with the option payoff. However, we would not expect a significant variance reduction due to the non-monotonicity of the option-payoff with respect to S_T. This is due to the presence of the barrier, B.

(d) The truncated $N(T)$ may be used a truncation variable since: (i) we can compute $P(N(T) = k)$ for any k and (ii) we can easily simulate samples of $\exp(-rT)(S_T - K)^+ 1_{\{S_T \leq B\}}$ *conditional* on $N(T)$. (You should also explain how to do this.)

(e) After stratifying with $N(T)$ we could use conditional Monte Carlo (condition on the jump sizes) and the Black-Scholes formula to obtain a more accurate estimate. (In fact, we could compute the option price as a weighted sum of Black-Scholes prices without needing to simulate at all. To do this, stratify with $N(T)$ and then recognize that conditional on $N(T) = k$, we can explicitly compute the option-price using the Black-Scholes formula by observing that conditional distribution of S_T is lognormal.)

Question 6 (10 marks)
(a) This is standard and is shown in the lecture notes.

(b) There are two methods that we accepted for controlling the discretization error: (i) the procedure as described in the lecture notes where the number of discretization points, m, was doubled until $\widehat{\theta}^m$ was sufficiently close to $\widehat{\theta}^{2m}$ and (ii) the procedure outlined in Assignment 7 where we doubled m and quadrupled n (the sample size) successively until we obtained a sufficiently accurate estimate of θ.

(c) A good transformation would be one that results in a constant volatility term, C say, in the SDE for Y_t so that

$$dY_t \;=\; \alpha(t, Y_t)\ dt + C\ dB_t.$$