IEOR E4703: Monte Carlo Simulation

Columbia University
Instructor: Martin Haugh
Midterm Examination October 27th 2004
Total Marks: 100
Time: 2 hours

Notes that are on either side of a single letter size sheet of paper may be used during the exam.

Question 1 (30 marks)
Suppose the random variable, X, has the triangular probability density function

$$f(x) = \begin{cases} 0 & x < 0 \\ kx & 0 \leq x \leq 2 \\ k(4-x) & 2 \leq x \leq 4 \\ 0 & x > 4 \end{cases}$$

(a) What is the value of k? (6 marks)

(b) Describe in detail the inverse transform method to generate a sample of X, given a uniform $U(0,1)$ random number generator. (12 marks)

(c) Describe in detail an acceptance - rejection algorithm for generating a sample of X, given a uniform random number generator. How many uniform random variables on average will be required to generate one sample of X? (12 marks)

Question 2 (40 marks)
Let S_t^a and S_t^b denote the time t prices of stocks A and B, respectively. We assume that there is also a riskless cash account available so that \$1 invested at $t = 0$ is worth \$$e^{rt}$ at time t. For option pricing purposes, we will *assume* that the stock prices follow correlated geometric Brownian motions with drift rate, r. In particular, we assume

$$S_{t+s}^a = S_t^a e^{(\mu_a - \sigma_a^2/2)s + \sigma_a (B_{t+s}^a - B_t^a)} \quad (1)$$

$$S_{t+s}^b = S_t^b e^{(\mu_b - \sigma_b^2/2)s + \sigma_b (B_{t+s}^b - B_t^b)} \quad (2)$$

where B_t^a and B_t^b are standard Brownian motions with correlation coefficient, ρ.

(a) Explain in detail how you would generate (S_{t+s}^a, S_{t+s}^b) *conditional* on knowing (S_t^a, S_t^b) using the Cholesky decomposition method. (15 marks)

Suppose now that your portfolio at $t = 0$ consists of N_a units of stock A and N_b units of stock B. You plan to hold this portfolio until time T. At each date $t \in \{0 < t_1 <$

$t_2 < \ldots < t_n = T\}$ you will check the value of your portfolio. In particular, you want to know the probability that the value of your portfolio on any of these dates is ever less than some fixed quantity, L. That is, you want to estimate

$$P_L \ := \ P\left(\min_{1 \le i \le n} \ \left(N_a S^a_{t_i} + N_b S^b_{t_i}\right) \le L\right).$$

(b) Describe in detail an algorithm for estimating P_L using Monte Carlo simulation. You should also describe how to estimate confidence intervals for your estimate. (It is best to present your algorithm using pseudo-code. You do not need to repeat the details of your answer from part (a). Indeed, you may assume that you now know how to generate values of (S^a_{t+s}, S^b_{t+s}) *conditional* on knowing (S^a_t, S^b_t).) (15 marks)

(c) Suppose that you also wish to compute the price of an option that expires at date T with payoff given by $\max\left(0,\ S^a_T/S^b_T - K\right)$. Explain how you would estimate the time 0 value of the option, C_0 say, using a Monte Carlo simulation algorithm that only generates *one* random variate per sample outcome. (For this part you should assume that the stock prices follow the same processes as in (1) and (2) except that both μ_a and μ_b are now replaced with r.) (10 marks)

Question 3 (30 marks)
Consider estimating the integral

$$\theta = \int_0^\infty e^{-x^2} \ dx$$

(a) Describe the Monte Carlo integration method for estimating θ. (15 marks)

(b) Describe another Monte Carlo simulation method to estimate θ that does not require a change of variable in the integration. Hint: You might consider writing

$$\theta = \int_0^\infty e^{-x^2} \frac{h(x)}{h(x)} \ dx$$

for an appropriate density function, $h(x)$. You do not necessarily have to specify a particular $h(.)$, but you should state what properties $h(.)$ should possess and how you would use it to estimate θ. (15 marks.)