IEOR E4703: Monte Carlo Simulation, Fall 2004
Columbia University
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Outline Solutions to Midterm Examination

Question 1
(a) The density must integrate to 1 so $k = 1/4$.

(b) THE CDF is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2/8 & 0 \le x \le 2 \\ x - x^2/8 - 1 & 2 \le x \le 4 \\ 1 & x > 4 \end{cases}$$

We can generate a sample of X by first generating $U \sim U(0,1)$ and then setting

$$X = \begin{cases} \sqrt{8U} & 0 \le U \le 1/2 \\ 4 - \sqrt{8(1-U)} & 1/2 < U \le 1 \end{cases}$$

(c) We can select $g(x) = 1/4$ on $[0,4]$. Then $f(x)/g(x) \le 2 =: a$ for all $x \in [0,4]$. Now generate a sample of X as follows:

Generate $Y \sim U(0,4)$
Generate $U \sim U(0,1)$
While $U > f(Y)/ag(Y) = 2f(Y)$
    Generate Y
    Generate U
set $X = Y$

Question 2
(a) The variance-covariance matrix of $\mathbf{X} := (B^a_{t+s} - B^a_t,\ B^b_{t+s} - B^b_t)$ is

$$\mathbf{\Sigma} = \begin{pmatrix} s & s\rho \\ s\rho & s \end{pmatrix}$$

and we have $\mathbf{X} \sim \mathrm{MN}(\mathbf{0}, \mathbf{\Sigma})$. So to generate a value of $\mathbf{X}$ we

1. compute the Cholesky Decomposition, $\mathbf{C}$, of $\mathbf{\Sigma}$
2. generate $\mathbf{Z} \sim \mathrm{MN}(\mathbf{0}, \mathbf{I}_2)$
3. set $\mathbf{X} = \mathbf{C}^T\mathbf{Z}$.

Now we can generate (S^a_{t+s}, S^b_{t+s}) conditional on knowing (S^a_t, S^b_t) by setting

$$S^a_{t+s} = S^a_t e^{(r-\sigma_a^2/2)s + \sigma_a \mathbf{X}(1)}$$

$$S^b_{t+s} = S^b_t e^{(r-\sigma_b^2/2)s + \sigma_b \mathbf{X}(2)}.$$

(b)

set $sum = 0$
for $i = 1$ **to** m

> **set** $Ind = 0, \;\; j = 1$
> **while** $Ind = 0$ and $j \le n$
>
> > **generate** $(S^a_{t_j}\,,\; S^b_{t_j})$
> > **if** $N_a S^a_{t_j} + N_b S^b_{t_b} \le L$ **set** $Ind = 1$
> > **set** $j = j + 1$
>
> **set** $sum = sum + Ind$

set $\widehat{P}_L = sum/m$
set $(1-\alpha)\%$ CI equal to $\left[\widehat{P}_L \pm z_{1-\alpha/2}\frac{\widehat{\sigma}}{\sqrt{m}}\right]$

Note that in our algorithm, $\sum Ind^2 = \sum Ind$ so we can easily compute $\widehat{\sigma}$ given $\sum Ind$ and hence confidence intervals for $\widehat{P}_L$.

(c) Note first that

$$\begin{aligned} \frac{S^a_T}{S^b_T} &= \frac{S^a_0}{S^b_0}\,\exp\left[\left(\sigma_a^2/2 - \sigma_b^2/2\right)T + \sigma_a B^a_T - \sigma_b B^b_T\right] \\ &= \frac{S^a_0}{S^b_0}\,\exp\left[\left(\sigma_a^2/2 - \sigma_b^2/2\right)T + Z\right] \end{aligned}$$

where $Z \sim N(0, \sigma^2)$ for $\sigma^2 := (\sigma_a^2 + \sigma_b^2 - 2\rho\sigma_a\sigma_b)T$. It is clear that we can therefore generate a sample payoff using only one random variate by generating a sample of Z.

Question 3
(a) Use the substitution $y = x/(1+x)$ and solve to get $x = y/(1-y)$ and $dy/dx = (1+x)^{-2} = (1-y)^2$. This then implies

$$\theta = \int_0^1 \frac{e^{-y^2/(1-y)^2}}{(1-y)^2}\,dy$$

and this can now be estimated using the standard Monte Carlo integration method (which you should have described in one or two additional lines).

(b) There are several ways to do this but the easiest is to recognize that

$$\theta = E^h\left[\frac{e^{-X^2}}{h(X)}\right]$$

where $E^h[\cdot]$ denotes that the expectation is taken with respect to a random variable with density, $h(\cdot)$. In order for this to work it is necessary that $h(x) > 0$ for $x > 0$. (Acceptance-rejection methods would also work for this question.)