Monte Carlo Simulation, IEOR E4703, Spring 2003

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Solutions to Final Examination

Question 1 (20 marks)
(a) $f(\cdot)$ must integrate to 1 so $k = 2$.

(b) The CDF is given by

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^4}{2} + \frac{x^8}{2} & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$$

It is difficult to use the inverse transform method because there is not a closed form expression available for $F^{-1}(x)$ and we would therefore have to invert $F(\cdot)$ numerically.

(c) We can write $f(x) = .5f_1(x) + .5f_2(x)$ where $f_1(x) = 4x^3$ and $f_2(x) = 8x^7$ for $0 \leq x \leq 1$. It is easy to simulate a value of X then as follows:

generate U_1, U_2 IID $U(0,1)$
if $U_1 \leq .5$
    set $X = U_2^{1/4}$
else
    set $X = U_2^{1/8}$

(d) We can take $g(\cdot)$ to be the uniform density on $(0,1)$ so that $g(x) = 1$ for $0 \leq x \leq 1$. We must have $f(x)/g(x) \leq a$ for all x, so this means that we can take $a = 6$. The acceptance-rejection algorithm is given by:

generate Y, U IID $U(0,1)$
while $U \geq (2Y^3 + 4Y^7)/6$
    generate Y, U IID $U(0,1)$
set $X = Y$

Question 2 (24 marks)
(a) Σ is given by

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & -0.7 \\ -0.7 & 1 \end{pmatrix}$$

(b) We can take

$$\mathbf{C} = \begin{pmatrix} 1 & -0.7 \\ 0 & \sqrt{.51} \end{pmatrix}$$

(c) Generate U_1, U_2 IID $U(0,1)$. Set $X_1 = \Phi^{-1}(U_1)$ and $X_2 = \Phi^{-1}(U_2)$. Then set $(Z_1\ Z_2)^T = \mathbf{C}^T\ (X_1\ X_2)^T$.

(d) Here we will assume that (Z_1, Z_2) are generated according to the procedure in (c). We have the following algorithm:

for $i = 1$ **to** n

  generate (Z_1, Z_2)

  set $S_i = S_i(0)e^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}Z_i}$ for $i = 1, 2$

  set $h_i = \exp(-rT)\max(S_1 - S_2,\ 0)$

  set $\tilde{S}_i = S_i(0)e^{(r-\sigma_i^2/2)T-\sigma_i\sqrt{T}Z_i}$ for $i = 1, 2$

  set $\tilde{h}_i = \exp(-rT)\max(\tilde{S}_1 - \tilde{S}_2,\ 0)$

  set $\overline{h}_i = (h_i + \tilde{h}_i)/2$

end for

set $\widehat{\theta}_n = \sum \overline{h}_i/n$

set $\widehat{\sigma}_n^2 = \sum(\overline{h}_i - \widehat{\theta}_n)^2/(n-1)$

set Approx $100(1-\alpha)\%$ CI equal to $\widehat{\theta}_n \pm z_{1-\alpha/2}\ \widehat{\sigma}_n/\sqrt{n}$

(e) This is tricky. If you want to apply the monotonicity result then you need to view the payoff as a function of the *independent* random variables X_1, X_2 of part (c). Then it can be seen that the payoff is monotonic in X_1 and X_2 so that a variance reduction is guaranteed.

(f) You could, for example, condition on the value of X_1. Then use the method of conditional Monte Carlo to estimate θ. This would amount to computing the expectation of the Black-Scholes put option price where the strike *and* initial stock price are functions of X_1.

Question 3 (18 marks)
Use conditional Monte Carlo. We have $\theta = \mathrm{E}[Y]$ where $Y := I_{\{VW \leq 3\}}$. This implies $\theta = \mathrm{E}[\mathrm{E}[Y|V]]$. Now it is easily seen that $\mathrm{E}[Y|V = v] = P(W \leq 3/V|V = v) = 1 - \exp(-3/v^2)$. Finally, this implies that $\theta = \mathrm{E}[1 - \exp(-3/V^2)]$ where $V \sim \mathrm{Exp}(1)$. So to estimate θ, generate n samples of V and set

$$\widehat{\theta}_n = 1 - \frac{\sum \exp(-3/V_i^2)}{n}.$$

Question 4 (20 marks)
(a)
(i) Minerva used the concept of importance sampling to do this. In particular, let $f(\cdot)$ and $g(\cdot)$ denote the Gamma$(2, 1)$ and Exp(1) PDFs respectively. Then since $f(x)/g(x) = x$, Minerva could use

$$\widehat{\theta}_n = \frac{\sum h(X_i)X_i}{n}.$$

as an unbiased estimate of θ. (Some people said they could use the fact that a Gamma$(2, 1)$ is the sum of two independent exponentials to generate an unbiased estimate of θ. While this is true, it is not as efficient as importance sampling since you can get at most $n/2$ independent samples this way. As a result, this method, if described correctly, obtained 7 out of 10 marks.)

(ii) Minerva could still have estimated θ as long as the true PDF, $f(x)$, was such that $f(x) = 0$ for $x < 0$. That is, the support of $f(\cdot)$ had to coincide with (or in fact be a subset of) the support of the Exp(1) density which is $[0, \infty)$. (If you used the alternative approach of part (i), then you probably said that you could do it for Gamma$(r, 1)$ for $r \leq n$ and r a positive integer. This is true, but much too restrictive. Furthermore, as r increases, the number of independent samples you can obtain quickly goes to 1! Little, if any, credit was given for this answer.)

(b) Let $h^+(x) := \max(h(x), 0)$ and $h^-(x) := \max(-h(x), 0)$. Then $h(x) = h^+(x) - h^-(x)$. This implies

$$\theta = \mathrm{E}[h^+(X)] - \mathrm{E}[h^e(X)].$$

Now h^+ and h^- are both non-negative so we can estimate the two expectations separately using importance sampling. We could use the observation in the statement of the question to then choose the sampling densities, g^+ and g^-, that we would use to estimate each expectation.

Question 5 (18 marks)
(a) Let $u = 1/(1+x)$. Then $x^2 = (1-u)^2/u^2$ and $du/dx = -1/(1+x)^2 = -u^2$. Therefore we may write

$$\theta = \int_1^0 -\frac{e^{-(1-u)^2/u^2}}{u^2}\, du \;=\; \int_0^1 \frac{e^{-(1-u)^2/u^2}}{u^2}\, du \;=\; \mathrm{E}\left[\exp(-(1-U)^2/U^2) \;/\; U^2\right]$$

where $U \sim U(0,1)$. We can therefore estimate θ using Monte Carlo simulation as follows:

for $i = 1$ **to** n
    generate $U \sim U(0,1)$
    set $Y_i = \exp(-(1-U)^2/U^2) \;/\; U^2$
end for
set $\widehat{\theta}_n = \sum Y_i/n$

(b) Yes, we can stratify using U as the stratification variable. We can do this since it is easy to compute $P(U \in \Delta)$ for any interval, Δ, and because it is easy to generate $U^{(j)} := (U \mid U \in \Delta_j)$. For example if $\Delta_j = [(j-1)/m,\ j/m)$ then we can generate $U^{(j)}$ by setting $U^{(j)} = (j-1)/m + U/m$ where $U \sim U(0,1)$.

(c)

Stratification Simulation Algorithm for Estimating θ

set $\widehat{\theta}_{n,st} = 0$;
for $j = 1$ to m
    set $sum_j = 0$;
    for $i = 1$ to n_j
        generate $U \sim U(0,1)$
        set $U = (j-1)/m + U/m$
        set $sum_j = sum_j + \exp(-(1-U)^2/U^2) \;/\; U^2$
    end for
    set $\theta_j = sum_j/n_j$
    set $\widehat{\theta}_{n,st} = \widehat{\theta}_{n,st} + \theta_j/m$
end for