A Cournot-Stackelberg Model of Supply Contracts with Financial Hedging

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Abstract

We study the performance of a stylized supply chain where N retailers and a single producer compete in a Cournot-Stackelberg game. At time $t = 0$ the retailers order a single product from the producer and upon delivery at time $\tau > 0$, they sell it in the retail market at a stochastic clearance price. We assume the retailers' profits depend on the realized path of some tradeable stochastic process such as a foreign exchange rate, commodity price or more generally, some tradeable economic index. Because production and delivery do not take place until time τ, the producer offers an $\mathcal{F}_\tau$-measurable *menu* of wholesale prices to the retailers and in response the retailers' choose ordering quantities that are also $\mathcal{F}_\tau$-measurable. We also assume, however, that the retailers are budget-constrained and are therefore limited in the number of units they may purchase from the producer. The supply chain might therefore be more profitable if the retailers were able to reallocate their budgets across different states of nature. In order to affect such a reallocation, we assume the retailers are also able to trade dynamically in the financial market. After solving for the Cournot-Stackelberg equilibrium when the retailers have identical budgets we study: (i) the impact of hedging on the supply chain and (ii) the interaction between hedging and retail competition on the various players including the firms themselves, the end consumers and society as a whole. We show, for example, that given a fixed aggregate budget, the expected profits of the producer increases with N while the aggregate expected profits of the retailers decrease with N irrespective of whether or not hedging is possible. Perhaps more surprisingly, we show that when the retailers can hedge there exists a level of competition, $\bar{N}$, that is often finite and optimal from the perspective of the consumers, the firms and society as a whole. In contrast, when the retailers cannot hedge, these welfare measures are uniformly increasing in N. When the retailers have non-identical budgets we solve for their Cournot game and show how the general Cournot-Stackelberg game can be solved in certain instances.

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1 Introduction

We study the performance of a stylized supply chain where multiple retailers and a single producer compete in a Cournot-Stackelberg game. At time $t = 0$ the retailers order a single product from the producer and upon delivery at time $\tau > 0$, they sell it in the retail market at a stochastic clearance price that depends on the realized path or terminal value of some observable and tradeable financial process. Because delivery does not take place until time τ, the producer offers a *menu* of wholesale prices to the retailer, one for each realization of the process up to some time, τ. The retailers' ordering quantities can therefore also be contingent upon the realization of the process up to time τ. Production of the good is then completed on or after time τ and profits in the supply chain are then realized.

We also assume, however, that the retailers are budget-constrained and are therefore limited in the number of units they may purchase from the producer. As a result, the supply chain might be more profitable if the retailers were able to reallocate their financial resources, i.e. their budgets, across different states. By allowing the retailers to trade dynamically in the financial markets we enable such a reallocation of resources. The producer has no need to trade in the financial markets as he is not budget constrained and, like the retailers, is assumed to be risk neutral.

After solving for the Cournot-Stackelberg equilibrium when the retailers have identical budgets we study: (i) the impact of hedging on the supply chain and (ii) the interaction between hedging and retail competition on the various players including the firms themselves, the end consumers and society as a whole. We show, for example, that given a fixed aggregate budget, the expected profits of the producer increases with N while the aggregate expected profits of the retailers decrease with N irrespective of whether or not hedging is possible. Perhaps more surprisingly, we show that when the retailers can hedge there exists a level of competition, $\bar{N}$, that is often finite and optimal from the perspective of the consumers, the firms and society as a whole. In contrast, when the retailers cannot hedge, these welfare measures are uniformly increasing in N. When the retailers have non-identical budgets we solve for their Cournot game and show how the general Cournot-Stackelberg game can be solved in certain instances. These instances include (i) the case where the budget constraints are not binding in equilibrium and (ii) the case where all of the retailers order a strictly positive quantity in each state of the world. In each case we find that a linear price contract is optimal, a result that also holds for the equibudget case. Our model can easily handle variations where, for example, the retailers are located in a different currency area to the producer or where the retailers must pay the producer before their budgets are available.

In contributing to the work on financial hedging in supply chains, our main contributions are: (i) solving for the Cournot-Stackelberg equilibrium when the retailers' have identical budgets (ii) our analysis of the interaction between hedging and retail competition on the various players including the firms themselves, the end consumers and society as a whole and (iii) the solution of the Cournot game in the case of non-identical budgets and the corresponding solution of the Cournot-Stackelberg game in special instances. This latter contribution is technical in nature since solving for just the Cournot game when the retailers have different budgets is challenging. Indeed there appears to be relatively few instances of solutions to Cournot games in the literature where the players are not identical.

A distinguishing feature of our model with respect to most of the literature in supply chain management is the budget constraint that we impose on the retailers' procurement decisions. Some recent exceptions include Buzacott and Zhang (2004), Caldentey and Chen (2012), Caldentey and Haugh (2009), Dada and Hu (2008), Hu and Sobel (2005), Gupta and Chen (2016), Kouvelis and Zhao (2012), Kouvelis and Zhao (2016), Wang and Yao (2016a), Wang and Yao (2016b) and Xu and Birge (2004); see also Part Three in Kouvelis et al. (2012). These papers generally consider various mechanisms such as asset-backed financing or bank borrowing to mitigate the impact of the budget constraint.

The work by Caldentey and Haugh (2009), Kouvelis and Zhao (2012) and Caldentey and Chen (2012) are the most closely related to this paper. They all consider a two-echelon supply chain system in which there is a single budget constrained retailer and investigate different types of procurement contracts between the agents using a Stackelberg equilibrium concept. In Kouvelis and Zhao (2012) the supplier offers different type of contracts designed to provide financial services to the retailer. They analyze a set of alternative financing schemes including *supplier early payment discount*, *open account financing*, *joint supplier financing with bank*, and *bank financing* schemes. In a similar setting, Caldentey and Chen (2012) discuss two alternative forms of financing for the retailer: (a) *internal financing* in which the supplier offers a procurement contract that allows the retailer to pay *in arrears* a fraction of the procurement cost after demand is realized and (b) *external financing* in which a third party financial institution offers a commercial loan to the retailer. They conclude that in an optimally designed contract it is in the supplier's best interest to offer financing to the retailer and that the retailer will always prefer internal rather than external financing.

In Caldentey and Haugh (2009) the supplier offers a modified wholesale price contract to a single budget constrained retailer and the contract is executed at a future time τ. The terms of the contract are such that the actual wholesale price charged at time τ depends on information publicly available at this time. Delaying the execution of the contract is important because in this model the retailer's demand depends in part on a financial index that the retailer and supplier can observe through time. As a result, the retailer can dynamically trade in the financial market to adjust his budget to make it contingent upon the evolution of the index. Their model shows how financial markets can be used as (i) a source of public information upon which procurement contracts can be written and (ii) as a means for financial hedging to mitigate the effects of the budget constraint. In this paper, we therefore extend the model in Caldentey and Haugh (2009) by considering a market with *multiple retailers* in Cournot competition as well as a Stackelberg leader. One of the distinguishing features of this work is that it allows us to study the impact of hedging upon competition in the retailers' market. Our extended model can also easily handle variations where, for example, the retailers are located in a different currency area to the producer or where the retailers must pay the producer before their budgets are available.

A second related stream of research considers Cournot-Stackelberg equilibria. There is an extensive economics literature on this topic that focuses on issues of existence and uniqueness of the Nash equilibrium. See Okoguchi and Szidarovsky (1999) for a comprehensive review. In the context of supply chain management, there has been some recent research that investigates the design of efficient contracts between the supplier and the retailers. For example, Bernstein and Federgruen (2003) derive a perfect coordination mechanism between the supplier and the retailers. This mechanism takes the form of a nonlinear wholesale pricing scheme. Zhao et al. (2005) investigate inventory sharing mechanisms among competing dealers in a distribution network setting. Li (2002) studies a Cournot-Stackelberg model with asymmetric information in which the retailers are endowed with some private information about market demand. In contrast, the model we present in this paper uses the *public* information provided by the financial markets to improve the supply chain coordination.

There also exists a related stream of research that investigates the use of financial markets and instruments to hedge operational risk exposure. See Boyabatli and Toktay (2004) and the survey paper by Zhao and Huchzermeier (2015) for detailed reviews. For example, Caldentey and Haugh (2006) consider the general problem of dynamically hedging the profits of a risk-averse corporation when these profits are partially correlated with returns in the financial markets. Chod et al. (2009) examine the joint impact of operational flexibility and financial hedging on a firm's performance and their complementarity/substitutability with the firm's overall risk management strategy. Ding et al. (2007) and Dong et al. (2014) examine the interaction of operational and financial decisions from an integrated risk management standpoint. Boyabatli and Toktay (2011) analyze the effect of capital market imperfections on a firm's operational and financial decisions in a capacity investment setting. Babich and Sobel (2004) propose an infinite-horizon discounted Markov decision process in which an IPO event is treated as a stopping time. They characterize an optimal capacity-expansion and financing policy so as to maximize the expected present value of the firm's IPO. Babich et al. (2012) consider how trade credit financing affects the relationships among firms in the supply chain, supplier selection, and supply chain performance.

Finally, there is an extensive stream of research in the corporate finance literature that relates to financial risk management and that is closely related to this paper. Of course the Modigliani-Miller theorem (Modigliani and Miller, 1958) states that firms, in the absence of market frictions, do not need to hedge since individual shareholders can do so themselves. In practice, however, there are many frictions that necessitate firm hedging and it is well known (see, for example, Boyle and Boyle, 2001) that many firms do so. These frictions include taxes and the costs of financial distress (Smith and Stulz, 1985), managerial motives (Stulz, 1984) as well as the costs associated with external financing (Stulz, 1990, Lessard, 1991). The work of Froot et al. (1993) was particularly influential and, building upon the earlier work of Lessard (1991), argues that the most important driver of firm hedging are the costs associated with external financing. In a two-period model they explicitly derive the optimal hedging strategy together with optimal financing and investing decisions for a single firm with costly external financing. This is very much in the spirit of our paper where the retailers are budget constrained and financial hedging allows them to mitigate the effect of these constraints. In contrast to Froot et al. (1993), however, we do not allow for the possibility of external financing and we do this to avoid further complicating our model. We do note that allowing external financing in our framework should be relatively straightforward at least in the case of homogeneous retailers with identical budgets. But because it's costly, allowing such financing would still leave the retailers effectively budget constrained albeit with higher effective budgets. As a result we don't believe that our results would change qualitatively if external financing were permitted.

Adam et al. (2007) also assume a two-period model with firms that are identical ex-ante. They focus on determining what percentage of the firms will hedge in a Cournot equilibrium framework. In contrast to our work, it is therefore not the case that every firm will have an incentive to hedge in equilibrium. For tractability reasons, they also assume external financing is not possible. In addition to Adam et al. (2007), other more recent papers also consider firm hedging in a game-theoretic framework. For example, Pelster (2015) considers hedging in a duopoly framework with mean-variance preferences while Loss (2012) also considers a duopoly and concludes that the firms' hedging demands decrease with the correlation between their internal funds and investment opportunities. Liu and Parlour (2009) considers a Cournot hedging framework where players can hedge the cash-flows from an indivisible project but not the probability of winning the project in an auction setting. In contrast to our work, none of these papers consider a Cournot-Stackelberg framework and they generally assume homogenous players where only very simple forms of hedging, e.g. via forward contracts, are allowed.

The remainder of this paper is organized as follows. In Section 2 we describe our model, focussing in particular on the supply chain, the financial markets and the contractual agreement between the producer and the retailers. In Section 3 we describe and analyze two benchmark models: (i) a decentralized supply chain where financial hedging is not possible and (ii) a centralized system where all decisions are made by a central planner with the objective of maximizing the overall supply chain's expected profits. These benchmarks will be used as comparison points in Section 4 where we solve for the full Cournot-Stackelberg equilibrium when the retailers have identical budgets. In Section 4 we also study the interaction between hedging and competition among the retailers, and also consider various measures of supply chain welfare in equilibrium. We analyze the case of non-identical retailer budgets in Section 5 and obtain explicit expressions for the retailers' purchasing decisions in the Cournot equilibrium as a function of the producer's price menu. In Section 5 we also obtain the producer's optimal price menu, i.e. the Stackelberg equilibrium, in some interesting special cases. Motivated by these results, we also propose a class of linear wholesale price contracts and show by way of example that it is straightforward for the producer to optimize numerically over this class. We conclude in Section 6. Most of the proofs are contained in the Appendix A while various extensions to the model can be found in Appendix B. These extensions include variations where the retailers are located in a different currency area to the producer and where the retailers must pay the producer before their budgets are available.

2 Model Description

We now describe the model in further detail. We begin with the supply chain description and then discuss the role of the financial markets. At the end of the section we define the contract which specifies the agreement between the producer and the retailers. Throughout this section we will assume[1] for ease of exposition that both the producer and the retailers are located in the same currency area and that interest rates are identically zero.

2.1 The Supply Chain

We model an isolated segment of a competitive supply chain with one producer that produces a single product and N competing retailers that face a stochastic clearance price[2] for this product. This clearance price, and the resulting cash-flow to the retailers, is realized at a fixed future time. The retailers and producer, however, negotiate the terms of a procurement contract at time $t = 0$. This contract specifies three quantities:

[1]In Appendix B we will relax these assumptions and still maintain the tractability of our model using change of measure arguments.

[2]Similar models are discussed in detail in Section 2 of Cachon (2003). See also Lariviere and Porteus (2001).

(i) A production time $\tau > 0$.

(ii) A rule that specifies the size of the order, q_i, chosen by the i^{th} retailer where $i = 1, \ldots, N$. In general, q_i will depend upon market information available at time τ.

(iii) The payment, $\mathcal{W}(q_i)$, that the i^{th} retailer pays to the producer for fulfilling the order. Again, $\mathcal{W}(q_i)$ will generally depend upon market information available at time τ.

We will restrict ourselves to transfer payments that are linear on the ordering quantity. That is, we consider the so-called *wholesale price* contract where $\mathcal{W}(q) = w\, q$ and where w is the per-unit wholesale price charged by the producer. We assume that the producer offers the same contract to each retailer. We also assume that during the negotiation of the contract the producer acts as a Stackelberg leader. That is, for a fixed procurement time τ, the producer moves first and at $t = 0$ proposes a wholesale price menu, w_τ, to which the retailers then respond by selecting their ordering levels, q_i, for $i = 1, \ldots, N$. Note that the N retailers also compete among themselves in a Cournot-style game to determine their optimal ordering quantities and trading strategies.

We assume that the producer has unlimited production capacity and that production takes place at time τ with a per-unit production cost of c. We will assume that c is constant but many of our results, however, go through when c is stochastic. The producer's payoff as a function of the wholesale price, w_τ, and the ordering quantities, q_i, is given by

$$\Pi_{\mathrm{P}|\tau} := (w_\tau - c) \sum_{i=1}^{N} q_i. \tag{1}$$

We assume that each retailer is restricted by a budget constraint that limits his ordering decisions. In particular, we assume that each retailer has an initial budget, B_i, that may be used to purchase product units from the producer. Without loss of generality, we order the retailers so that $B_1 \geq B_2 \geq \ldots \geq B_N$ and let $B_{\mathrm{C}} := \sum_i B_i$ be the cumulative budget available in the retailers' market. We assume each of the retailers can trade in the financial markets during the time interval $[0, \tau]$, thereby transferring cash resources from states where they are not needed to states where they are. As in Adam et al. (2007) we assume there is no external financing available. While of course this is not always a realistic assumption it is often the case that external financing is very expensive. Indeed many researchers, including the influential work of Froot et al. (1993), argue that the main motivation for firms' hedging activities is the high cost of external financing. Forbidding external financing aids the tractability of our problem and we believe is not a serious limitation since the possibility of *costly* external financing would still mean that firms were essentially budget constrained, albeit with higher effective budgets than would be the case without the possibility of external financing.

For a given set of order quantities, the i^{th} retailer collects a random revenue at time τ. We compute this revenue using a linear *clearance price* model. That is, the market price at which the retailer sells these units is a random variable, $P(Q) := A_\tau - (q_i + Q_{i-})$, where A_τ is a non-negative random variable, $Q_{i-} := \sum_{j \neq i} q_j$ and $Q := \sum_j q_j$. The random variable A_τ models the market size that we assume is unknown. The realization of A_τ, however, will depend on the realization of the financial markets between times 0 and τ. The payoff of the i^{th} retailer, as a function of w_τ, and the order quantities, then takes the form

$$\Pi_{\mathrm{R}_i|\tau} \;:=\; \left(A_\tau - \,(q_i + Q_{i-})\right)\, q_i - w_\tau\, q_i. \tag{2}$$

A stochastic clearance price is easily justified since in practice unsold units are generally liquidated using secondary markets at discount prices. Therefore, we can view our clearance price as the average selling price across all units and markets. As stated earlier, w_τ and the q_i's will in general depend upon market information available at time τ. Since $\mathcal{W}(q)$, $\Pi_{\mathrm{P}|\tau}$ and the $\Pi_{\mathrm{R}_i|\tau}$'s are functions of w_τ and the q_i's, these quantities will also depend upon market information available at time τ.

The linear clearance price in (2) is commonly assumed in the operations and economics literature for reasons of tractability and estimation. It also helps ensure that the game will have a unique Nash equilibrium. (For further details see Chapter 4 of Vives, 2001.) We also note that the assumption of instantaneous production at time τ as well as market clearing (and realization) of profits at that time can easily be generalized with no loss in tractability. For example, we could have assumed there exists a "final" time $T > \tau$ with production taking place in the interval $[\tau, T]$. The market size could then be represented by a random variable A which is not observed until time T and which depends on both financial *and* non-financial noise. In that case, it can easily be seen that all of our analysis will still go through and that A_τ will then equal $\mathbb{E}_\tau[A]$, the risk-neutral expectation of A conditional on the time τ information in the financial market.

A key aspect of our model is the dependence between the payoffs of the supply chain and returns in the financial market. Other than assuming the existence of A_τ, we do not need to make any assumptions regarding the nature of this dependence. We will, however, make the following assumption.

Assumption 1 *$A_\tau \geq c$ with probability 1.*

This condition ensures that for every state there is a total production level, $Q \geq 0$, for which the retailers' expected market price exceeds the producer's production cost. In particular, this assumption implies that it is possible to profitably operate the supply chain in every state. Note also that we do not assume $P(Q)$ will be strictly positive in all states. This for example, may not be the case when hedging is not allowed. It may be more accurate then to interpret $P(Q)$ as a net profit.

2.2 The Financial Market

The financial market is modeled as follows. Let X_t denote[3] the time t value of a tradeable security and let $\{\mathcal{F}_t\}_{0 \leq t \leq \tau}$ be the filtration generated by X_t on a probability space, $(\Omega, \mathcal{F}, \mathbb{Q})$ with $\mathcal{F} = \mathcal{F}_\tau$. There is also a risk-less *cash account* available from which cash may be borrowed or in which cash may be deposited. Since we have assumed zero interest rates, the time τ gain (or loss), $G_\tau(\theta)$, that results from following a self-financing[4] $\mathcal{F}_t$-adapted trading strategy, θ_t, can be represented as a stochastic integral with respect to X. In a continuous-time setting, for example, we have

$$G_\tau(\theta) := \int_0^\tau \theta_s \, dX_s. \tag{3}$$

[3] All of our analysis goes through if we assume X_t is a multi-dimensional price process. For ease of exposition we will assume X_t is one-dimensional.

[4] A trading strategy, θ_s, is self-financing if cash is neither deposited with nor withdrawn from the portfolio during the trading interval, $[0, \tau]$. In particular, trading gains or losses are due to changes in the values of the traded securities. Note that θ_s represents the number of units of the tradeable security held at time s. The self-financing property then implicitly defines the position at time s in the cash account. Because we have assumed interest rates are identically zero, there is no term in (3) corresponding to gains or losses from the cash account holdings. See Duffie (2004) for a technical definition of the self-financing property.

We assume that $\mathbb{Q}$ is an equivalent martingale measure (EMM) so that discounted security prices are $\mathbb{Q}$-martingales. Since we are assuming that interest rates are identically zero, however, it is therefore the case that X_t is a $\mathbb{Q}$-martingale. Subject to integrability constraints on the set of feasible trading strategies, we also see that $G_t(\theta)$ is a $\mathbb{Q}$-martingale for every $\mathcal{F}_t$-adapted self-financing trading strategy, θ_t. In what follows, $\mathbb{E}[\cdot]$ denotes expectation with respect to $\mathbb{Q}$.

Our analysis will be simplified considerably by making a *complete financial markets* assumption. In particular, let G_τ be any suitably integrable contingent claim that is $\mathcal{F}_\tau$-measurable. Then a complete financial markets assumption amounts to assuming the existence of an $\mathcal{F}_t$-adapted self-financing trading strategy, θ_t, such that $G_\tau(\theta) = G_\tau$. That is, G_τ is *attainable*. This assumption is very common in the financial literature; see for example Smith and McCardle (1998). Moreover, many incomplete financial models can be made complete by simply expanding the set of tradeable securities. When this is not practical, we can simply assume the existence of a market-maker with a known pricing function or pricing kernel who is willing to sell G_τ in the market-place. In this sense, we could then claim that G_τ is indeed attainable.

Regardless of how we choose to justify it, assuming complete financial markets means that we will never need to solve for an optimal dynamic trading strategy, θ. Instead, we will only need to solve for an optimal contingent claim, G_τ, safe in the knowledge that any such claim is attainable. For this reason we will drop the dependence of G_τ on θ in the remainder of the paper. The only restriction that we will impose on any such trading gain, G_τ, is that it satisfies $\mathbb{E}[G_\tau] = G_0$ where G_0 is the initial amount of capital that is devoted to trading in the financial market. Without any loss of generality we will assume $G_0 = 0$. This assumption will be further clarified in Section 2.3.

2.3 The Flexible Procurement Contract with Financial Hedging

The final component of our model is the contractual agreement between the producer and the retailers. We consider a variation of the traditional wholesale price contract in which the terms of the contract are specified contingent upon the public history, $\mathcal{F}_\tau$, that is available at time τ. Specifically, at time $t = 0$ the producer offers an $\mathcal{F}_\tau$-measurable wholesale price, w_τ, to the retailers. In response to this offer, the i^{th} retailer decides on an $\mathcal{F}_\tau$-measurable ordering quantity[5], $q_i = q_i(w_\tau)$, for $i = 1, \ldots, N$. Note that the contract itself is negotiated at time $t = 0$ whereas the actual order quantities are only realized at time $\tau \geq 0$. Note that an alternative and equivalent interpretation is that w_τ is announced at time $t = 0$ and that the retailers don't respond until time τ.

The retailers' order quantities at time τ are constrained by their available budgets at this time. Besides the initial budget, B_i, the i^{th} retailer has access to the financial markets where he can hedge his budget constraint by purchasing at date $t = 0$ a contingent claim, $G_\tau^{(i)}$, that is realized at date τ and that satisfies $\mathbb{E}[G_\tau^{(i)}] = 0$. Given an $\mathcal{F}_\tau$-measurable wholesale price, w_τ, the retailer purchases an $\mathcal{F}_\tau$-measurable contingent claim, $G_\tau^{(i)}$, and selects an $\mathcal{F}_\tau$-measurable ordering quantity, $q_i = q_i(w_\tau)$, in order to maximize the economic value of his profits. Because of his access to the financial markets, the retailer can therefore mitigate his budget constraint so that it becomes

$$w_\tau \, q_i \leq B_i + G_\tau^{(i)} \qquad \text{for all } \omega \in \Omega \;\text{ and }\; i = 1, \ldots, N.$$

[5]There is a slight abuse of notation here and throughout the paper when we write $q_i = q_i(w_\tau)$. This expression should not be interpreted as implying that q_i is a function of w_τ. We only require that q_i be $\mathcal{F}_\tau$-measurable and so a more appropriate interpretation is to say that $q_i = q_i(w_\tau)$ is the retailer's response to w_τ.

Before proceeding to analyze this contract a number of further clarifying remarks are in order.

1. The model assumes a common knowledge framework in which all parameters of the model are known to all agents. Because of the Stackelberg nature of the game, this assumption implies that the producer knows the retailers' budgets and the distribution of the market demand. It also implies that the retailers know each others budgets and that all players know the retailers can hedge in the financial markets.

2. In this model the producer does not trade in the financial markets because, being risk-neutral and not restricted by a budget constraint, he has no incentive to do so.

3. A potentially valid criticism of this model is that, in practice, a retailer is often a small entity and may not have the ability to trade in the financial markets. There are a number of responses to this. First, we use the word 'retailer' in a loose sense so that it might in fact represent a large entity. For example, an airline purchasing aircraft is a 'retailer' that certainly does have access to the financial markets. Second, it is becoming ever cheaper and easier for even the smallest 'player' to trade in the financial markets and many of them do so routinely to hedge interest rate risk, foreign exchange rate risk etc.

4. We claimed earlier that, without loss of generality, we could assume $G_0^{(i)} = 0$. This is clear for the following reason. If $G_0^{(i)} = 0$ then then the i^{th} retailer has a terminal budget of $B_\tau^{(i)} := B_i + G_\tau^{(i)}$ with which he can purchase product units at time τ and where $\mathbb{E}[G_\tau^{(i)}] = 0$. If he allocated $a > 0$ to the trading strategy, however, then he would have a terminal budget of $B_\tau^{(i)} = B_i - a + G_\tau^{(i)}$ at time τ but now with $\mathbb{E}[G_\tau^{(i)}] = a$. That the retailer is indifferent between the two approaches follows from the fact any terminal budget, $B_\tau^{(i)}$, that is feasible under one modeling approach is also feasible under the other and vice-versa.

5. Another potentially valid criticism of this framework is that the class of contracts is too complex. In particular, by only insisting that w_τ is $\mathcal{F}_\tau$-measurable we are permitting wholesale price contracts that might be too complicated to implement in practice. If this is the case then we can easily simplify the set of feasible contracts. By using appropriate conditioning arguments, for example, it would be straightforward to impose the tighter restriction that w_τ be $\sigma(X_\tau)$-measurable instead where $\sigma(X_\tau)$ is the σ-algebra generated by X_τ. In section 5.2, for example, we will consider wholesale price contracts that are linear in c and A_τ.

We complete this section with a summary of the notation and conventions that will be used throughout the remainder of the paper. The subscripts R, P, and C are used to index quantities related to the retailers, producer and central planner, respectively. The subscript τ is used to denote the value of a quantity conditional on time τ information. For example, $\Pi_{\mathrm{P}|\tau}$ is the producer's payoff conditional on time τ information. The expected value, $\mathbb{E}[\Pi_{\mathrm{P}|\tau}]$, is simply denoted by Π_{P} and similar expressions hold for the retailers and central planner. Any other notation will be introduced as necessary.

3 Benchmarks

In this section we consider two special cases of the model that we will use as benchmarks to better understand the effects of (i) access to financial markets and (ii) decentralization and competition on the overall performance of the firms as well as on the efficiency of the entire supply chain.

3.1 Decentralized Supply Chain with No Hedging

First, we consider the special case in which the retailers are not able to hedge their budget constraints. In this case, for a given wholesale price menu w_τ set by the producer, we can determine retailer i's order quantity by solving the best-response optimization problem:

$$\Pi_{\mathrm{R}_i}(w_\tau) = \max_{q_i \geq 0} \; \mathbb{E}\left[(A_\tau - (q_i + Q_{i-}) - w_\tau)\; q_i\right] \tag{4}$$

$$\text{subject to} \qquad w_\tau\, q_i \leq B_i, \qquad \text{for all } \omega \in \Omega. \tag{5}$$

Each of the N retailers must solve this problem and our goal is to characterize the resulting Cournot equilibrium. To this end, note that the budget constraint in (5) must be imposed pathwise since the retailers are not able to hedge. As a result, problem (4)-(5) decouples and we can determine the retailers' optimal ordering strategy separately for each outcome $\omega \in \Omega$. Indeed, it is not hard to see that

$$q_i = \min\left\{ \frac{(A_\tau - Q_{-i} - w_\tau)^+}{2}\;,\; \frac{B_i}{w_\tau} \right\}$$

solves (4)-(5). In what follows, and without loss of optimality, we will assume that $w_\tau \leq A_\tau$ for otherwise $q_i = 0$ for all $i = 1, \ldots, N$ and the supply chain would effectively shut down.

Let $Q = \sum_{i=1}^{N} q_i$ be the cumulative order quantity (or total output) in the retail market. Then, one can show that in equilibrium, the optimality condition above is equivalent to

$$q_i = \min\left\{ A_\tau - Q - w_\tau\;,\; \frac{B_i}{w_\tau} \right\}. \tag{6}$$

A direct consequence of (6) is that in equilibrium the order quantity of a retailer is weakly increasing in his budget. It follows that the set of retailers can be partioned into two groups: "low budget" and "high budget" retailers for whom the budget constraint is and is not binding, respectively. Proposition 1 below builds on this property to solve the Cournot game among the retailers. The following definitions will be useful in the statement of this and other results.

First, we define the sequence $\{\mathcal{B}_k \colon k = 0, \ldots, N\}$ by

$$\mathcal{B}_0 = \infty \qquad \text{and} \qquad \mathcal{B}_k := k\, B_k + \sum_{i=k}^{N} B_i, \quad k = 1, \ldots, N. \tag{7}$$

Recall that, without loss of generality, the retailers have been ordered so that $B_1 \geq B_2 \geq \ldots \geq B_N$. It follows that the sequence $\{\mathcal{B}_k\}$ is also non-increasing in k. Another important property is that $\mathcal{B}_k$ does not depend on $B_1, \ldots, B_{k-1}$. Our second definition is the (random) mapping $m : \mathbb{R}_+ \to \{1, 2 \ldots, N+1\}$ given by

$$m(w) := \max\Big\{ k \in \{1, 2 \ldots, N+1\} \colon w\,(A_\tau - w) \leq \mathcal{B}_{k-1} \Big\}. \tag{8}$$

The practical meaning of $m(w)$ is explained in the following result. We recall that $B_{\mathrm{C}} = \sum_i B_i$ is the combined budget of all retailers.

Proposition 1 (Cournot Equilibrium with No Hedging) *Let w_τ be the wholesale price menu set by the producer. Then there is a unique equilibrium in the retailers' market given by*

$$q_i(w_\tau, B_i) = \begin{cases} \frac{1}{m(w_\tau)}\left[A_\tau - w_\tau - \sum_{i=m(w_\tau)}^{N} \frac{B_i}{w_\tau}\right] & \text{for } i = 1, \dots, m(w_\tau) - 1 \\ \frac{B_i}{w_\tau} & \text{for } i = m(w_\tau), \dots, N. \end{cases}$$

In equilibrium, only the first $m(w_\tau) - 1$ retailers are not constrained by their budgets. In particular, $m(w_\tau) = N + 1$, i.e., none of the retailers are budget constrained, if $B_N \geq w_\tau\,(A_\tau - w_\tau)/(N + 1)$. On the other hand, $m(w_\tau) = 1$, i.e., all retailers are budget constrained, if $B_1 + B_{\mathrm{C}} < w_\tau\,(A_\tau - w_\tau)$.

Using this result, we can study the impact of the budget constraints – and in particular, the distribution of the budgets among the retailers – on the end-consumers' market. Towards this end, we focus on the aggregate output $Q(w_\tau, B_1, \dots, B_N)$ offered by the retailers as a function of w_τ and the vector of budgets $(B_1, \dots, B_N)$, that is

$$Q(w_\tau, B_1, \dots, B_N) := \sum_{i=1}^{N} q_i(w_\tau, B_i) = \frac{1}{m(w_\tau)}\left[(m(w_\tau) - 1)\,(A_\tau - w_\tau) + \sum_{i=m(w_\tau)}^{N} \frac{B_i}{w_\tau}\right]. \tag{9}$$

On the one hand, $Q(w_\tau, B_1, \dots, B_N)$ represents the demand function that the producer faces for a given vector of budget $(B_1, \dots, B_N)$. In addition, we can view $Q(w_\tau, B_1, \dots, B_N)$ as a measure of the social welfare in the market. Indeed, given the linear demand function $P_\tau = A_\tau - Q$ and linear production costs $C(Q) = c\,Q$, the social welfare, *i.e.*, the sum of the firms' and end consumers' surplus, of an output Q is equal to

$$\mathcal{S}(Q) = (A_\tau - c)\,Q - \frac{Q^2}{2}. \tag{10}$$

It follows that $\mathcal{S}(Q)$ is maximized at $Q_\tau^* = A_\tau - c$ or equivalently when $P_\tau^* = c$. Since in equilibrium both the producer and the retailers make non-negative profits, we have $P_\tau \geq w_\tau \geq c$ and so $Q(w_\tau, B_1, \dots, B_N) \leq Q_\tau^*$. We conclude that both the producer and a social welfare maximizer have the same preferences over the vector of retailers' budgets $(B_1, \dots, B_N)$ for a fixed w_τ. That is, both prefer vectors that maximize the market output $Q(w_\tau, B_1, \dots, B_N)$. Below, we show that for a given cumulative budget B_{C}, the total output is maximized when B_{C} is evenly distributed among the N retailers. To formalize this observation the following intermediate result will be useful.

Proposition 2 *The producer's demand function $Q(w_\tau, B_1, \dots, B_N)$ satisfies*

$$Q(w_\tau, B_1, \dots, B_N) = \min_{k \in \{0,1,\dots,N\}} \left\{ \frac{1}{k+1}\left[k\,(A_\tau - w_\tau) + \sum_{i=k+1}^{N} \frac{B_i}{w_\tau}\right]\right\}.$$

A trivial implication of this proposition is that $Q(w_\tau, B_1, \dots, B_N)$ is decreasing in w_τ. On the other hand, a possibly less obvious consequence is that

$$Q(w_\tau, B_1, \dots, B_N) \geq Q(w_\tau, B_1 + \epsilon, \dots, B_N - \epsilon), \qquad \forall\, 0 \leq \epsilon \leq B_N.$$

That is, transferring budget from the lowest budget retailer to the highest budget retailer (weakly) decreases the total output. This suggests that total output in the market is maximized if the total budget is evenly distributed among the retailers. We formalize this observation in the following result.

Proposition 3 *Let $B_{\mathrm{C}} = \sum_i B_i$ be the cumulative budget available for procurement in the retailers' market and set $\bar{B} = B_{\mathrm{C}}/N$. Then, for a fixed wholesale price menu w_τ*

$$Q(w_\tau, \bar{B}, \ldots, \bar{B}) \geq Q(w_\tau, B_1, \ldots, B_N) \geq Q(w_\tau, B_{\mathrm{C}}, 0, \ldots, 0).$$

It follows that the social welfare of the system is maximized when B_{C} is uniformly distributed among the retailers and it is minimized when there is a single retailer that controls the entire budget B_{C}.

We conclude this subsection by turning to the producer's problem of determining the wholesale price to charge. An optimal wholesale price menu solves the producer's optimization problem $\Pi_{\mathrm{P}} = \max_{w_\tau} \mathbb{E}\Big[(w_\tau - c)\, Q(w_\tau, B_1, \ldots, B_N)\Big]$. Similar to the retailers' problem, we can solve this optimization problem pathwise, that is, solving

$$\Pi_{\mathrm{P}|\tau} = \max_{w_\tau}\, (w_\tau - c)\, Q(w_\tau, B_1, \ldots, B_N). \tag{11}$$

In general, there is no simple closed-form solution for the optimal wholesale price w_τ in (11). This is, however, a one-dimensional optimization problem that can be easily solved numerically. Furthermore, for some special instances, we can use the result in Proposition 2 to efficiently solve the producer's optimization problem. Indeed, using the representation of $Q(w_\tau, B_1, \ldots, B_N)$ in Proposition 2 we have that

$$\Pi_{\mathrm{P}|\tau} = \max_{w_\tau \in [c, A_\tau]}\ \min_{k \in \{0,1,\ldots,N\}} \left\{ \frac{(w_\tau - c)}{k+1} \left[k\,(A_\tau - w_\tau) + \sum_{i=k+1}^{N} \frac{B_i}{w_\tau} \right] \right\}. \tag{12}$$

The argument inside the braces in (12) is concave in w_τ for all $k \in \{0, 1, \ldots, N\}$. The minimum over $k \in \{0, 1, \ldots, N\}$ is therefore also concave and so finding the optimal w_τ in (12) is an easy numerical task.

3.2 Centralized System

We now consider the special case in which the supply chain is controlled by a single firm – a central planner – that decides both production and retail sales. As is customary in the supply chain management literature, we view this vertically integrated system as a benchmark to assess the inefficiencies of a decentralized system, in particular those arising from the double marginalization phenomenon induced by a two-tier system, i.e., retailers acting as middlemen, and the level of competition (or lack thereof) in the retailers' market. In order to have a fair comparison between our decentralized system and a vertically integrated one, we will assume that the central planner is also budget constrained and endowed with a budget $B_{\mathrm{C}} = \sum_i B_i$.

Let us first consider the case in which the central planner has access to the financial market to hedge the budget constraint. In this setting, the central planner is interested in solving:

$$\Pi_{\mathrm{C}} = \max_{Q \geq 0,\, G_\tau}\ \mathbb{E}\left[(A_\tau - Q - c)\; Q\right] \tag{13}$$

$$\text{subject to} \qquad c\,Q \leq B_{\mathrm{C}} + G_\tau, \qquad \text{for all } \omega \in \Omega \tag{14}$$

$$\mathbb{E}\left[G_\tau\right] = 0. \tag{15}$$

A variation of problem (13)-(15) was studied in Caldentey and Haugh (2009). The next result summarizes the solution.

Proposition 4 (Central Planner's Optimal Strategy with Hedging) *The central planner's optimal production strategy, $Q_{\rm C}$, is equal to*

$$Q_{\rm C} = \left(\frac{A_\tau - \delta_{\rm C}}{2} \right)^+ \tag{16}$$

where $\delta_{\rm C}$ is the minimum $\delta \geq c$ that solves

$$\mathbb{E}\left[c \left(\frac{A_\tau - \delta}{2} \right)^+ \right] \leq B_{\rm C}.$$

Also, the central planner's optimal expected payoff satisfies $\Pi_C = \mathbb{E}[\Pi_{C|\tau}]$, where $\Pi_{C|\tau}$ is the central planner's payoff conditional on the information available at time τ and is given by

$$\Pi_{C|\tau} = \frac{(A_\tau + \delta_{\rm C} - 2c)\,(A_\tau - \delta_{\rm C})^+}{4}. \tag{17}$$

Note that in the absence of a budget constraint (*e.g.*, if $B_{\rm C} = \infty$) the central planner would produce an optimal quantity $Q^*_\tau = (A_\tau - c)/2$ and collect an optimal payoff $\Pi^*_{{\rm C}|\tau} = (A_\tau - c)^2/4$. This first-best solution is in general not feasible because of the limited budget. However, the previous proposition reveals that it can be achieved as long as the budget constraint is satisfied *in expectation*, that is, as long as $\mathbb{E}[c\,(A_\tau - c)/2] \leq B_{\rm C}$. In this case, financial hedging allows the central planner to fully circumvent the budget constraint.

Let us suppose now that the central planner has no access to the financial market. Mathematically, this is equivalent to making $G_\tau = 0$ in the optimization problem (13)-(15) above. Similar to our derivation in Section 3.1, we can solve this modified optimization pathwise. The following result summarizes this solution. (The proof is straightforward and is omitted.)

Proposition 5 (Central Planner's Optimal Strategy without Hedging) *The central planner's optimal production strategy $Q_{{\rm C}|\tau}$ and payoff $\Pi_{C|\tau}$, conditional on the value A_τ at time τ, are equal to*

$$Q_{{\rm C}|\tau} = \min\left\{ \frac{A_\tau - c}{2}, \frac{B_{\rm C}}{c} \right\} \quad \textit{and} \quad \Pi_{C|\tau} = (A_\tau - c - Q_{{\rm C}|\tau})\, Q_{{\rm C}|\tau}.$$

4 Identical Budgets

In this section we consider the case where all retailers have identical budgets. While not a realistic assumption in practice, we can solve for the producer's optimal price menu and therefore provide a full characterization of the Cournot-Stackelberg equilibrium in this case. Moreover, we can: (i) address questions regarding the impact of hedging on supply chain performance by comparing to the no-hedging results in Section 3.1 (ii) identify the potential benefits for the retailers from merging or remaining in competition and (iii) also compare the equilibrium solution to the solution of the central planner's problem from Section 3.2 and therefore determine the efficiency of the supply chain when financial hedging is possible. Finally, according to Proposition 3, the equibudget case is also important in its own right as it corresponds to a socially efficient distribution of the total procurement budget among the retailers (at least when hedging is not possible).

Consider then the case where each of the retailers has the same budget so that $B_i = B$ for all $i = 1, \ldots, N$. For a given price menu, w_τ, the i^{th} retailer's problem is

$$\begin{aligned} \Pi_{\mathrm{R}}(w_\tau) = & \max_{q_i \geq 0,\, G_\tau} \; \mathbb{E}\left[(A_\tau - (q_i + Q_{i-}) - w_\tau)\; q_i\right] && (18) \\ \text{subject to} \quad & w_\tau\, q_i \leq B + G_\tau, \qquad \text{for all } \omega \in \Omega && (19) \\ & \mathbb{E}\left[G_\tau\right] = 0. && (20) \end{aligned}$$

While the equibudget problem is a special case of the game we will study in Section 5, it is instructive to see an alternative solution. In the equibudget case, each of the N retailers has the following solution:

Proposition 6 (Optimal Strategy for the N Retailers in the Equibudget Case)
Let w_τ be an $\mathcal{F}_\tau$-measurable wholesale price offered by the producer and let $\mathcal{Q}_\tau$, $\mathcal{X}$ and $\mathcal{X}^c$ be defined as follows. $\mathcal{Q}_\tau := \frac{(A_\tau - w_\tau)^+}{(N+1)}$, $\quad \mathcal{X} := \{\omega \in \Omega \,:\, B \geq \mathcal{Q}_\tau\, w_\tau\}$ and $\mathcal{X}^c := \Omega - \mathcal{X}$. The following two cases arise in the computation of the optimal ordering quantities and the financial claims:

> **Case 1**: *Suppose that $\mathbb{E}\left[\mathcal{Q}_\tau\, w_\tau\right] \leq B$. Then $q_i(w_\tau) = \mathcal{Q}_\tau$ and there are infinitely many choices of the optimal claim, $G_\tau = G_\tau^{(i)}$, for $i = 1, \ldots, N$. One natural choice is to take*
>
> $$G_\tau = \left[\mathcal{Q}_\tau\, w_\tau - B\right] \cdot \begin{cases} \delta & \text{if } \omega \in \mathcal{X} \\ 1 & \text{if } \omega \in \mathcal{X}^c \end{cases} \qquad \text{where} \qquad \delta := \frac{\int_{\mathcal{X}^c} \left[\mathcal{Q}_\tau\, w_\tau - B\right] \mathrm{d}\mathbb{Q}}{\int_{\mathcal{X}} \left[B - \mathcal{Q}_\tau\, w_\tau\right] \mathrm{d}\mathbb{Q}}.$$
>
> *In this case (possibly due to the ability to trade in the financial market), the budget constraint is not binding for any of the N retailers.*
>
> **Case 2**: *Suppose $\mathbb{E}\left[\mathcal{Q}_\tau w_\tau\right] > B$. Then*
>
> $$q_i(w_\tau) \;=\; q(w_\tau) \;=\; \frac{(A_\tau - w_\tau\,(1+\lambda))^+}{(N+1)} \qquad \text{and} \qquad G_\tau := q(w_\tau) w_\tau - B \tag{21}$$
>
> *is optimal for each i where $\lambda \geq 0$ solves $\mathbb{E}\left[q(w_\tau) w_\tau\right] \;=\; B$.*

The manufacturer's problem is straightforward to solve. Given the best response of the N retailers, his problem may be formulated as

$$\begin{aligned} \Pi_{\mathrm{P}} \;=\; & \max_{w_\tau,\, \lambda \geq 0} \; N \; \mathbb{E}\left[(w_\tau - c)\, \frac{(A_\tau - w_\tau\,(1+\lambda))^+}{(N+1)}\right] && (22) \\ \text{subject to} \quad & \mathbb{E}\left[w_\tau\, \frac{(A_\tau - w_\tau\,(1+\lambda))^+}{(N+1)}\right] \;\leq\; B. && (23) \end{aligned}$$

Note that the factor N outside the expectation in (22) is due to the fact that there are N retailers and that the producer earns the same profit from each of them. Note also that there should be N constraints in this problem, one corresponding to each of the N retailers. However, by Proposition 6, these N constraints are identical since each retailer solves the same problem. The producer's problem then only requires the one constraint given by (23). We can easily re-write this problem as

$$\begin{aligned} \Pi_{\mathrm{P}} \;=\; & \max_{w_\tau,\, \lambda \geq 0} \; \frac{2N}{N+1} \; \mathbb{E}\left[(w_\tau - c)\, \frac{(A_\tau - w_\tau\,(1+\lambda))^+}{2}\right] && (24) \\ \text{subject to} \quad & \mathbb{E}\left[w_\tau\, \frac{(A_\tau - w_\tau\,(1+\lambda))^+}{2}\right] \;\leq\; \frac{(N+1)}{2} B && (25) \end{aligned}$$

and now it is clearly identical[6] to the producer's problem where the budget constraint has been replaced by $(N+1)B/2$ and there is just one retailer. In particular, the solution of the producer's problem and of the Cournot-Stackelberg game follows immediately from Proposition 7 in Caldentey and Haugh (2009). We have the following result.

Proposition 7 (Producer's Optimal Strategy and the Cournot-Stackelberg Solution)
Let ϕ_P be the minimum $\phi \geq 1$ that solves $\mathbb{E}\left[\left(\frac{A_\tau^2-(\phi\, c)^2}{8}\right)^+\right] \leq \frac{(N+1)}{2}B$ and let $\delta_P := \phi_P\, c$. Then the optimal wholesale price and ordering level for each retailer satisfy

$$w_\tau = \frac{A_\tau + \delta_P}{2} \qquad \textit{and} \qquad q_\tau = \frac{(A_\tau - \delta_P)^+}{2(N+1)}. \tag{26}$$

The players' expected payoffs conditional on time τ information satisfy

$$\Pi_{P|\tau} \;=\; \frac{2N}{(N+1)}\,\frac{(A_\tau + \delta_P - 2c)\,(A_\tau - \delta_P)^+}{8} \qquad \textit{and} \qquad \Pi_{R|\tau} = \frac{((A_\tau - \delta_P)^+)^2}{4(N+1)^2}. \tag{27}$$

As mentioned above, the solution of the Cournot-Stackelberg equilibrium in our model with multiple symmetric retailers is similar to the one in Caldentey and Haugh (2009) with one retailer. As a result, a number of properties of the equilibrium are the same. Here we summarize a few important ones for completeness.

1. In equilibrium, the retailers can fully hedge away their budget constraints if and only if their initial budget satisfies $B \geq \mathbb{E}[(A_\tau^2 - c_\tau^2)]/(4(N+1))$. On the other hand, if the retailers have no access to the financial markets, then their budget constraint is not binding if and only if $B \geq (A_\tau^2 - c_\tau^2)/(4(N+1))$ w.p.1. It follows that if the random variable A_τ has unbounded support, then it is not possible to completely circumvent the budget constraint in the absence of financial hedging.

2. The producer is always better off if the N retailers have access to the financial markets. The situation is more complicated for the retailers. In particular, the retailers may or may not prefer having access to the financial markets in equilibrium. The relationship between c and $\delta_{\rm P}$ (as defined in Proposition 7) is key: if $\delta_{\rm P} = c$ the retailers also prefer having access to the financial markets. If $c < \delta_{\rm P}$, however, then their preferences can go either way.

3. Access to financial hedging can increase or decrease the total output $Q_\tau = N\, q_\tau$ in the market.

4. In terms of the efficiency of the supply chain, we can compare the equilibrium in Proposition 7 to the central planner's solution in Section 3.2 assuming the central planner has a Budget $B_{\rm C} = NB$. We focus on production levels ($\mathcal{Q}_\tau$), double marginalization ($\mathcal{W}_\tau$) and the competition penalty ($\mathcal{P}_\tau$). These performance measures are defined conditional on $\mathcal{F}_\tau$ as follows:

$$\mathcal{Q}_\tau := \frac{N q_\tau}{q_{{\rm C}|\tau}} = \frac{N(A_\tau - \delta_{\rm P})^+}{(N+1)(A_\tau - \delta_{\rm C})^+}, \qquad \mathcal{W}_\tau := \frac{w_\tau}{c} = \frac{A_\tau + \delta_{\rm P}}{2c}, \quad \text{and}$$

$$\mathcal{P}_\tau \;:=\; 1 \;-\; \frac{\Pi_{{\rm P}|\tau} + N\,\Pi_{{\rm R}|\tau}}{\Pi_{{\rm C}|\tau}} = 1 \;-\; \frac{N}{(N+1)^2}\;\frac{[(N+2)A_\tau + N\delta_{\rm P} - 2(N+1)c]\;(A_\tau - \delta_{\rm P})^+}{(A_\tau + \delta_{\rm C} - 2c)(A_\tau - \delta_{\rm C})^+},$$

[6] The factor $2N/(N+1)$ in the objective function has no bearing on the optimal λ and w_τ.

where $\Pi_{\mathrm{C}|\tau}$ is the central planner's profits conditional on time τ information, δ_{C} is the smallest value (see Section 3.2 for details) of $\delta \geq c$ such that $\mathbb{E}[c\left(\frac{A_\tau-\delta}{2}\right)^+] \leq NB$, and $q_{\mathrm{C}|\tau}$ is the optimal ordering quantity of the central planner.

It is interesting to note that, conditional on $\mathcal{F}_\tau$, the centralized supply chain is not necessarily more efficient than the decentralized operation. For instance, we know that in some cases $\delta_{\mathrm{P}} < \delta_{\mathrm{C}}$ and so for all those outcomes, ω, with $\delta_{\mathrm{P}} < A_\tau < \delta_{\mathrm{C}}$, $q_{\mathrm{C}|\tau} = 0$ and $q_\tau > 0$ and the competition penalty is minus infinity. We mention that this only occurs because of the retailers' ability to trade in the financial markets. If $\delta_{\mathrm{P}} \geq \delta_{\mathrm{C}}$, however, then it is easy to see that the centralized solution is always more efficient than the decentralized supply chain so that $\mathcal{Q}_\tau \leq 1$ and $\mathcal{P}_\tau \geq 0$. We also note that if the budget is large enough so that both the decentralized retailers and central planner can hedge away the budget constraint then $\delta_{\mathrm{P}} = \delta_{\mathrm{C}} = c$ and

$$\mathcal{Q}_\tau = \frac{N}{(N+1)} \quad \text{and} \quad \mathcal{P}_\tau = \frac{1}{(N+1)^2}.$$

Hence, in this case, as N increases the decentralized solution approaches the centralized solution.

4.1 Competition in the Retailers' Market

We now investigate how the equilibrium is affected by the degree of competition in the retailers' market, specifically by the number of retailers N. To this end, we find convenient to make explicit the dependence of the different components of the model on N. So, for example, we will write $B(N)$ for the budget of a retailer or $w_\tau(N)$ for the wholesale price, and so on. In order to isolate the impact of N on the equilibrium outcome, we assume that the cumulative budget B_{C} remains constant as we vary N, that is, we assume that each retailer has a budget $B(N) := B_{\mathrm{C}}/N$. Thus, our sensitivity analysis and results in this section are exclusively driven by the degree of competition in the retailers' market and not by an increase in the cumulative budget available as the number of retailers grows large.

From Proposition 7, we obtain that the value of $\phi_{\mathrm{P}}(N)$ is the minimum $\phi \geq 1$ that satisfies the budget constraint

$$\mathbb{E}\left[\left(A_\tau^2 - (\phi\, c)^2\right)^+\right] \leq 4\, B_{\mathrm{C}}\left(1 + \frac{1}{N}\right).$$

We use this inequality to determine the maximum value of N for which this budget constraint is non binding, i.e., for which $\phi_{\mathrm{P}}(N) = 1$. By Assumption 1, $A_\tau \geq c$ and so the value of this threshold is given by

$$\bar{N} := \left\lfloor \frac{4\, B_{\mathrm{C}}}{(\mathbb{E}[A_\tau^2 - c^2] - 4\, B_{\mathrm{C}})^+} \right\rfloor. \tag{28}$$

In this definition, we allow for $\bar{N} = \infty$ if $\mathbb{E}[A_\tau^2 - c^2] \leq 4\, B_{\mathrm{C}}$. (The positive part in the denominator of (28) is used to ensure that $\bar{N} \geq 0$.)

For $N \leq \bar{N}$ the budget constraint is not binding so the equilibrium outcome is independent of N in this region. On the other hand, for all $N > \bar{N}$ the budget constraint is binding and $\phi_{\mathrm{P}}(N)$ is strictly increasing in N in this case. A direct implication of this and of equation (26) is that the wholesale price $w_\tau(N)$ is also constant in $N \leq \bar{N}$ and increasing in $N > \bar{N}$. Figure 1 shows an example that compares the expected equilibrium wholesale price, $\mathbb{E}[w_\tau(N)]$, and the expected

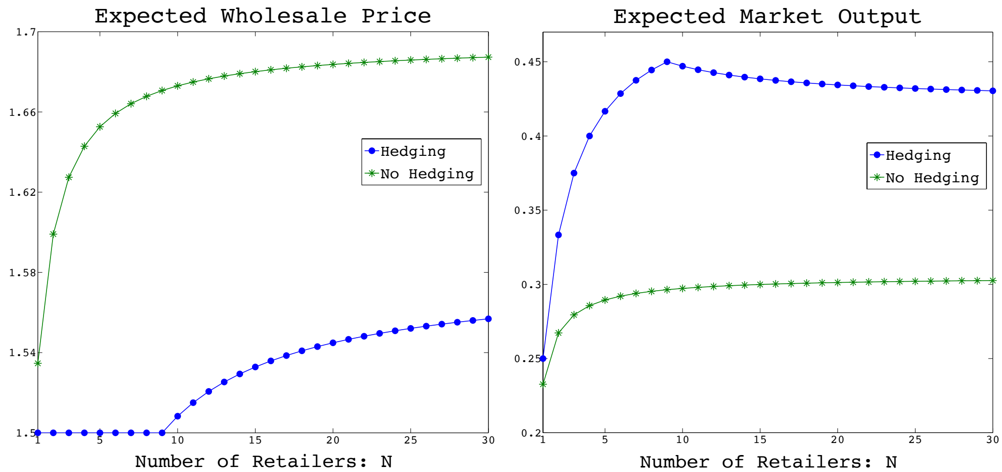

Figure 1: Expected wholesale price $\mathbb{E}[w_\tau]$ as a function of the number of retailers N for the cases where the retailers can ($\circ$) and cannot ($\star$) hedge their budget constraints. The demand A_τ is uniformly distributed in $[1, 3]$, the per unit production cost is $c = 1$ and the total cumulative budget is $B_{\mathrm{C}} = 0.75$.

total market output, $\mathbb{E}[Q_\tau(N)]$, with and without hedging as a function of N. It is worth noting that in this example $\bar{N} = 9$ so that if there are nine retailers or less who can hedge then their budget constraints are not binding and the equilibrium wholesale price is independent of N while the market output is monotonically increasing in N. If the number of retailers is greater than nine, however, then their budget constraints will be binding and the wholesale price and the total market output become monotonically increasing and decreasing, respectively, in N. On the other hand, if the retailers cannot hedge then their budget constraints are always binding independent of N and both expected wholesale price and total output are monotonically increasing in N. The following proposition formalizes these observations for the case in which the retailers hedge their budget constraints.

Proposition 8 *Suppose the retailers have access to the financial markets and therefore hedge their budget constraints. Then in equilibrium:*

a) *The expected wholesale price $\mathbb{E}[w_\tau(N)]$ does not depend on N for $N \leq \bar{N}$ and is increasing in N for $N > \bar{N}$.*

b) *Suppose that A_τ admits a smooth density $f \in \mathcal{C}^2[0, \infty)$ such that $\lim_{x\to\infty} x^2 f(x) = 0$. The expected market output $\mathbb{E}[Q_\tau(N)] = \mathbb{E}[N\, q_\tau(N)]$ is increasing in N for $N \leq \bar{N}$ and decreasing in N for $N > \bar{N}$.*

We now consider the impact of N on the firms' expected payoff. Figure 2 displays the producer's expected profits $\Pi_{\mathrm{P}}(N) := \mathbb{E}[\Pi_{\mathrm{P}|\tau}]$ (left panel) and the retailers' aggregate expected profits $N\,\Pi_{\mathrm{R}}(N) := N\mathbb{E}[\Pi_{\mathrm{R}|\tau}]$ (right panel) as a function of N for the cases with and without hedging. We see in this example that the producer's expected profits increases with N while the retailers' aggregate expected profits decreases with N. This is somewhat intuitive since the oligopsony power of the retailers decreases (thereby making the producer better off) as their number increases. The following result formalizes this intuition.

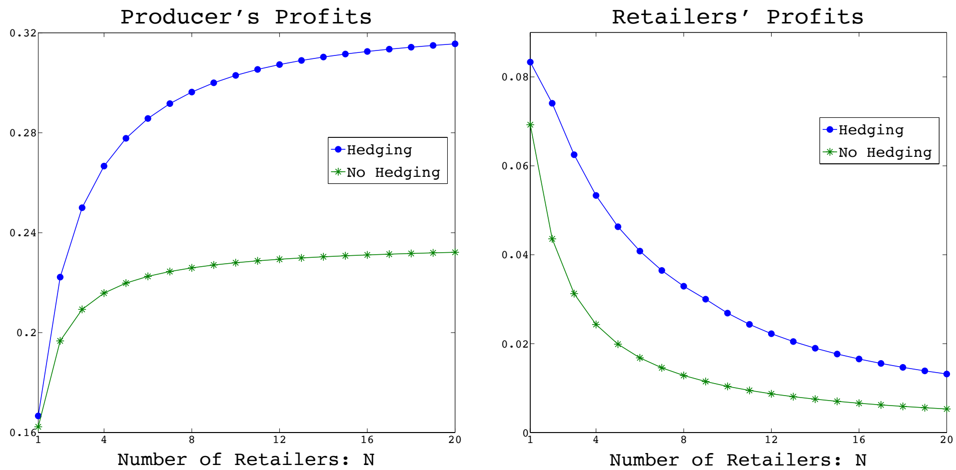

Figure 2: Producer's expected payoff $\Pi_{\mathsf{P}}(N)$ (left panel) and retailers' aggregate expected payoff $N\,\Pi_{\mathsf{R}}(N)$ (right panel) as a function of N for the cases with and without hedging. The parameters were as in Figure 1.

Proposition 9 *Consider the equibudget case with a fixed aggregate budget B_{C}. Then, the expected profits of the producer increases with N while the aggregate expected profits of the retailers decrease with N. This result is independent of whether or not the retailers have access to the financial markets to hedge their budget constraints.*

An immediate corollary of this result is that consolidations in the retail market always benefit the retailers and hurt the producer. It is also worth emphasizing that when the retailers hedge their budgets constraints the results in Proposition 9 hold only in expectation. On a path-by-path basis the producer will not necessarily be better off as the number of retailers increases. In particular, there will be some outcomes where the ordering quantity is zero under the multiple competing retailers and strictly positive when there is a single (merged) retailer for example. The producer will earn zero profits on such paths under the competing retailer model, but will earn strictly positive profits under the merged retailer model.

The result in Proposition 9 raises the more general question of whether the supply chain as a whole is better or worse off as the number of retailers increases. To investigate this question, in Figure 3 we compare the cumulative profits of the decentralized supply chain (SC) with those of a central planner (CP) for the cases in which the retailers (or the central planner) do and do not hedge the budget constraints. The two panels in the figure differ only in the value of the cumulative budget, B_{C}: in the left panel $B_{\text{C}} = 0.75$ and $\bar{N} = 9$; in the right panel $B_{\text{C}} = 1.5$ and $\bar{N} = \infty$.

First, we note that the cumulative profits of the supply chain are higher when the retailers hedge their budgets constraints than when they do not. Interestingly, in the left panel there is a range of values of N for which the profits of a decentralized supply chain with hedging exceed the profits of a central planner without hedging. It is also worth noting that the profits of the supply chain with hedging increases in N for $N \leq \bar{N}$ and decreases for $N > \bar{N}$. This suggests that $\bar{N}$ defines an "optimal" level of competition in the retailer's market. Furthermore, on the right panel, $\bar{N} = \infty$ and the profits of the decentralized system with hedging converge to the profits of a central planner that also hedges the budget constraint as $N \to \infty$. The following proposition establishes this result.

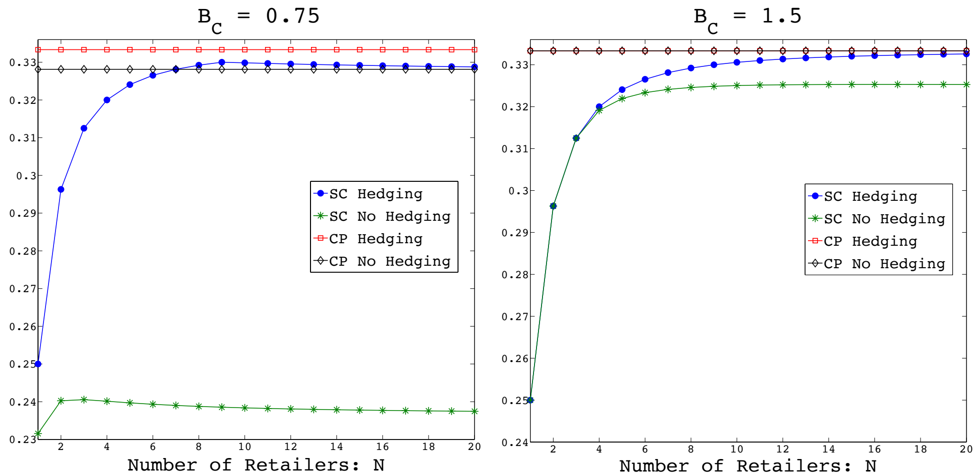

Figure 3: Expected profits for the decentralized supply chain (SC) and a central planner (CP) as a function of N for the cases with and without hedging. The demand and production cost parameters were as in Figure 1.

Proposition 10 *Suppose $\bar{N} = \infty$. Then the Cournot-Stackelberg equilibrium of the game with equi-budget retailers is given by*

$$w_\tau = \frac{A_\tau + c}{2} \qquad \textit{and} \qquad q_\tau = \frac{A_\tau - c}{2\,(N+1)}, \quad i = 1, \ldots, N$$

and the firms' expected payoffs are given by

$$\Pi_{\mathrm{P}}(N) = \frac{N}{4(N+1)}\,\mathbb{E}\big[(A_\tau - c)^2\big] \quad \textit{and} \quad \Pi_{\mathrm{R}}(N) = \frac{1}{4(N+1)^2}\,\mathbb{E}\big[(A_\tau - c)^2\big].$$

It follows that

$$\lim_{N\to\infty} \Pi_{\mathrm{P}}(N) = \frac{1}{4}\,\mathbb{E}\big[(A_\tau - c)^2\big] = \Pi_{\mathrm{C}} \qquad \textit{and} \qquad \lim_{N\to\infty} N \times \Pi_{\mathrm{R}}(N) = 0.$$

According to Proposition 10, if the cumulative budget B_{C} if sufficiently large so that $\bar{N} = \infty$, then the decentralized supply chain's profits, $\Pi_{\mathrm{P}}(N) + N \times \Pi_{\mathrm{R}}(N)$, achieve the central planner profits, Π_{C}, as competition in the retailer's market increases, i.e., as $N \to \infty$. Interestingly, this supply chain efficiency is reached despite the double marginalization that persists in equilibrium as $N \to \infty$ since $w_\tau = (A_\tau + c)/2 > c$ for all outcomes ω for which $A_\tau > c$.

We conclude this section by comparing the market equilibrium with and without hedging from the perspective of the end consumers and society as a whole. To this end, we define the consumers' expected surplus, $\mathcal{C}(N)$, and the total social welfare, $\mathcal{S}(N)$, as a function of the number of retailers N:

$$\mathcal{C}(N) := \frac{\mathbb{E}[Q_\tau(N)^2]}{2} \quad \text{and} \quad \mathcal{S}(N) := \mathcal{C}(N) + \Pi_{\mathrm{P}}(N) + N\,\Pi_{\mathrm{R}}(N) = \mathbb{E}[(A_\tau - c)\,Q_\tau(N)] - \frac{\mathbb{E}[Q_\tau(N)^2]}{2}.$$

Figure 4 depicts these measures for the same numerical example of Figure 1. First, we note that the end consumers and society as a whole are better off when retailers are able to hedge their budget constraints. It is also interesting to note that when the retailers are able to hedge, the consumers'

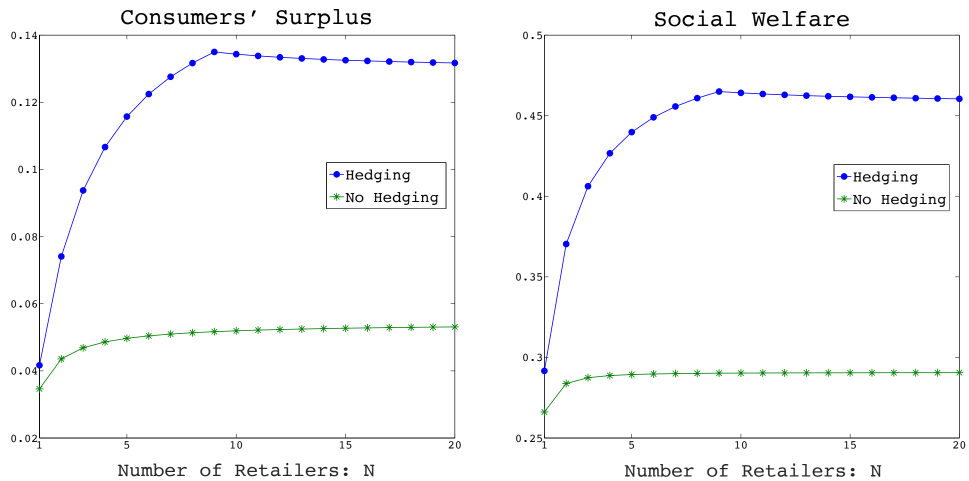

Figure 4: Consumers' expected surplus (left panel) and social welfare (right panel) as a function of N for the cases with and without hedging. The parameters were as in Figure 1.

surplus and society welfare are maximized when $N = \bar{N}$. This is interesting as one might have expected that consumers' surplus would have been maximized when $N = \infty$, *i.e.*, when retailers' competition is most intense. However, when hedging is available there is a socially optimal number of retailers ($N = 9$ in this example) and excessive competition in the retailers' market can end up hurting consumers, the supply chain and society as a whole. This is in direct contrast with the outcome when hedging is not possible for in this case the welfare increases as N gets large. That is, in the absence of hedging more competition in the retailers' market is better for both consumers and society as a whole.

In general, we have not been able to prove that $\bar{N}$ is the *socially* optimal number of retailers, although all our numerical experiments suggest that this is indeed the case. The following proposition provides some partial support for this claim.

Proposition 11 *In the equibudget case with fixed aggregate budget B_{C}, consumers' surplus $\mathcal{C}(N)$, the firm's cumulative profits, $\Pi_{\mathrm{P}}(N) + N\,\Pi_{\mathrm{R}}(N)$, and the social welfare, $\mathcal{S}(N)$, are all strictly increasing in N for $N \leq \bar{N}$ when the retailers hedge in the financial markets.*

Hence, independently of the measure of welfare that we might adopt, i.e. consumers', firms' or society's, $\bar{N}$ is a lower bound on the minimum number of retailers that should be operating in the system from a welfare standpoint.

5 Non-Identical Budgets

In this section we study the more general case in which the retailers have different budgets and can hedge in the financial markets. We therefore seek to extend the no-hedging results from Section 3.1 here. Our main focus will be on the solving the Cournot game played by the retailers as this is considerably more challenging than in the equibudget case. After solving for the Cournot game we will discuss some special cases where we can also solve for the equilibrium wholesale price contract

w_τ. Because of its difficulty, we consider the main contribution of this section to be the solution of the retailers' Cournot game.

We begin by solving the Cournot game played by the retailers. Taking Q_{i-} and the producer's price menu, w_τ, as fixed, the i^{th} retailer's problem is formulated as

$$\Pi_{\mathrm{R}_i}(w_\tau) = \max_{q_i \geq 0,\, G_\tau} \; \mathbb{E}\left[\left(A_\tau - (q_i + Q_{i-}) - w_\tau\right) q_i\right] \tag{29}$$

$$\text{subject to} \qquad w_\tau\, q_i \leq B_i + G_\tau, \qquad \text{for all } \omega \in \Omega \tag{30}$$

$$\mathbb{E}\left[G_\tau\right] = 0. \tag{31}$$

A first key step in solving (29)-(31) is to note that we can replace the set of pathwise budget constraints in (30) by a single average budget constraint, namely, $\mathbb{E}[w_\tau\, q_i] \leq B_i$. To see this, consider the relaxed retailer's problem:

$$\widetilde{\Pi}_{\mathrm{R}_i}(w_\tau) = \max_{q_i \geq 0} \; \mathbb{E}\left[\left(A_\tau - (q_i + Q_{i-}) - w_\tau\right) q_i\right] \tag{32}$$

$$\text{subject to} \qquad \mathbb{E}\left[w_\tau\, q_i\right] \leq B_i, \qquad \text{for all } \omega \in \Omega. \tag{33}$$

It should be clear that the feasible region of (32)-(33) contains the feasible region of (29)-(31) and so $\widetilde{\Pi}_{\mathrm{R}_i}(w_\tau) \geq \Pi_{\mathrm{R}_i}(w_\tau)$. On the other hand, for any feasible solution q_i of (32)-(33), we can set a trading strategy such that $G_\tau = w_\tau\, q_i - \mathbb{E}[w_\tau\, q_i]$. But the pair (q_i, G_τ) is feasible for (29)-(31) and generates the same expected payoff. It follows that $\widetilde{\Pi}_{\mathrm{R}_i}(w_\tau) = \Pi_{\mathrm{R}_i}(w_\tau)$ and we can safely focus on solving the simpler optimization problem (32)-(33) to determine the Cournot equilibrium in the retailer's market.

Taking Q_{i-} and the producer's price menu, w_τ, as fixed, it is straightforward to obtain

$$q_i \;=\; \frac{\left(A_\tau - w_\tau\,(1+\lambda_i) - Q_{i-}\right)^+}{2} \tag{34}$$

where $\lambda_i \geq 0$ is the deterministic Lagrange multiplier corresponding to the i^{th} retailer's budget constraint in (33). In particular, $\lambda_i \geq 0$ is the smallest real such that $\mathbb{E}\left[w_\tau\, q_i\right] \leq B_i$. Given the ordering of the budgets, B_i, it follows that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ when they are chosen optimally. Equation (34) and the ordering of the Lagrange multipliers then implies that for each outcome $\omega \in \Omega$, there is a function $n_\tau(\omega) \in \{0, 1, \ldots, N\}$ such that $q_j(\omega) = 0$ for all $j > n_\tau$. In other words, $n_\tau(\omega)$ is the number of active retailers in state ω.

Continuing to drop the dependence of random variables on ω (e.g., writing n_τ for $n_\tau(\omega)$), we therefore obtain the following system of equations

$$q_i \;=\; A_\tau - w_\tau\,(1+\lambda_i) - Q, \qquad \text{for} \quad i = 1, \ldots, n_\tau \tag{35}$$

where $Q = \sum_{i=1}^{n_\tau} q_i$. For each $\omega \in \Omega$, this is a system with n_τ linear equations in n_τ unknowns which we can easily solve. Summing the q_i's we obtain

$$Q \;=\; \frac{1}{n_\tau + 1}\left[n_\tau A_\tau - w_\tau \sum_{i=1}^{n_\tau}(1+\lambda_i)\right]. \tag{36}$$

Substituting this value of Q in (35), and using the fact that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$, we see the optimal ordering quantities, q_i for $i = 1, \ldots, N$, satisfy

$$q_i = \frac{\left[A_\tau - w_\tau \left((n_\tau + 1)(1 + \lambda_i) - \sum_{j=1}^{n_\tau} (1 + \lambda_j) \right)\right]^+}{(n_\tau + 1)}, \quad i = 1, 2 \ldots, N. \tag{37}$$

To complete the characterization of the Cournot equilibrium in the retailers' market, we must compute the values of the Lagrange multipliers $\{\lambda_i,\ i = 1, \ldots, N\}$ as well as the random variable n_τ. For reasons that will soon become apparent, it will be convenient to replace the Lagrange multipliers by an equivalent set of unknowns $\{\alpha_i,\ i = 1, \cdots N\}$ that we define below.

Suppose $q_i(\omega) = 0$ in some outcome, ω. Then (34) implies $A_\tau - w_\tau (1 + \lambda_i) - Q \ \leq\ 0$ which, after substituting for Q using (36), implies that

$$(1 + \lambda_i)(1 + n_\tau) \ \geq\ \alpha_\tau \ + \sum_{j=1}^{n_\tau} (1 + \lambda_j), \tag{38}$$

where $\alpha_\tau := A_\tau / w_\tau$. Since A_τ is the expected maximum clearing price (corresponding to $Q = 0$) and w_τ is the procurement cost, we may interpret $\alpha_\tau - 1$ as the expected maximum per unit margin of the retail market. It follows that in equilibrium the producer chooses w_τ so that $\alpha_\tau \geq 1$. We also note that equation (38) implies that n_τ depends on ω only through the value of α_τ, that is, $n_\tau = n_\tau(\alpha_\tau)$.

Let α_i denote that value of α_τ where the i^{th} retailer moves from ordering zero to ordering a positive quantity. Abusing notation slightly, we see[7] that $n(\alpha_i) = i - 1$ and so (38) implies

$$\alpha_i \ = \ i(1 + \lambda_i) \ - \ \sum_{j=1}^{i-1} (1 + \lambda_j) \quad \text{ for } \ i = 1, \ldots, N. \tag{39}$$

Using (39) recursively, one can show that

$$1 + \lambda_i = \frac{\alpha_i}{i} + \sum_{j=1}^{i-1} \frac{\alpha_j}{j\,(j+1)}. \tag{40}$$

Substituting this expression in (37), it follows that for all $i = 1, \ldots, N$

$$\begin{aligned} q_i \ &= \ w_\tau \left[\frac{\alpha_\tau}{1 + n_\tau} - (1 + \lambda_i) + \sum_{j=1}^{n_\tau} \frac{1 + \lambda_j}{1 + n_\tau} \right]^+ = w_\tau \left[\frac{\alpha_\tau}{1 + n_\tau} - \frac{\alpha_i}{i} + \sum_{j=1}^{n_\tau} \frac{1 + \lambda_j}{1 + n_\tau} - \sum_{j=1}^{i-1} \frac{\alpha_j}{j\,(j+1)} \right]^+ \\ &= \ w_\tau \left[\frac{\alpha_\tau}{1 + n_\tau} - \frac{\alpha_i}{i+1} + \sum_{j=1}^{n_\tau} \frac{1 + \lambda_j}{1 + n_\tau} - \sum_{j=1}^{i} \frac{\alpha_j}{j\,(j+1)} \right]^+ = w_\tau \left[\frac{\alpha_\tau}{n_\tau + 1} - \frac{\alpha_i}{i+1} + \sum_{j=i+1}^{n_\tau} \frac{\alpha_j}{j\,(j+1)} \right]^+ \end{aligned}$$

where the last equality follows from the identity:

$$\sum_{j=1}^{n_\tau} \frac{1 + \lambda_j}{1 + n_\tau} = \frac{1}{1 + n_\tau} \sum_{j=1}^{n_\tau} \left(\frac{\alpha_j}{j} + \sum_{k=1}^{j-1} \frac{\alpha_k}{k(k+1)} \right) = \sum_{j=1}^{n_\tau} \frac{\alpha_j}{j(j+1)},$$

[7] We are assuming that the N budgets are distinct so that $B_{k-1} > B_k$. This then implies $q_i(\alpha_k) > 0$ for all $i \leq k - 1$. The case where some budgets coincide is straightforward to handle.

which in turns follows from (40).

It should be clear from the discussion above that

$$n_\tau = \max\{i \in \{0, 1, \ldots, N\} \quad \text{such that} \quad \alpha_i \le \alpha_\mathcal{T}\} \tag{41}$$

and we therefore only need to derive the values of the α_i's. We have relegated this derivation to Appendix A and we summarize the main results in Proposition 12 below. We first need the following definition.

Definition 5.1 *Let w_τ be an $\mathcal{F}_\tau$-measurable wholesale price contract. For any $\mathcal{B} \ge 0$, we define*

$$H(\mathcal{B}) := \inf\{x \ge 1 \quad \textit{such that} \quad \mathbb{E}[w_\tau^2\,(\alpha_\mathcal{T} - x)^+] \le \mathcal{B}\}.$$

Note that $H(\mathcal{B})$ is a non-increasing function in $\mathcal{B} > 0$.

Proposition 12 (Cournot Equilibrium in the Retailers' Market)
For a given $\mathcal{F}_\tau$-measurable wholesale price menu, w_τ, the optimal ordering quantities, q_i, satisfy

$$q_i = w_\tau \left[\frac{\alpha_\mathcal{T}}{n_\tau + 1} - \frac{\alpha_i}{i+1} + \sum_{j=i+1}^{n_\tau} \frac{\alpha_j}{j\,(j+1)} \right]^+ \quad \textit{for all} \quad i = 1, 2, \ldots, N$$

with the budget constraint $\mathbb{E}[w_\tau\, q_i] \le B_i$ binding if $\alpha_i > 1$ where

$$\begin{aligned} \alpha_i \quad &:= \quad H(\mathcal{B}_i) \qquad \textit{for all} \;\; i = 1, 2, \ldots, N, & (42)\\ n_\tau \quad &:= \quad \max\{i \in \{0, 1, \ldots, N\} \quad \textit{such that} \quad \alpha_i \le \alpha_\mathcal{T}\}, & (43) \end{aligned}$$

where the $\{\mathcal{B}_i\}$ are defined in equation (7).

<u>Proof:</u> See Appendix A.

The ordering $B_1 \ge B_2 \ge \cdots \ge B_N$ implies that $q_i > 0$ if and only if $i \le n_\tau$. The parameter α_i is therefore the cutoff[8] point such that the i^{th} retailer orders a positive quantity only if $\alpha_\mathcal{T} \ge \alpha_i$. It is interesting to note that equation (42) implies that α_i does not depend on the $i-1$ highest budgets, B_j, for $j = 1, \ldots, i-1$. In fact α_i only depends on B_i, the sum of the $N - i$ smallest budgets and the *number* of retailers, $i - 1$, that have a budget larger than B_i. As a result, q_i only depends on B_i, $(B_{i+1} + \cdots + B_N)$ and i. In other words, the procurement decisions of small retailers are unaffected by the size (but not the number) of larger retailers for a given wholesale price w_τ. In equilibrium, however, we expect the wholesale price w_τ to depend on the entire vector of budgets. Proposition 12 also implies that

$$q_i - q_{i+1} = w_\tau \left(\frac{(\alpha_\mathcal{T} - \alpha_i)^+ - (\alpha_\mathcal{T} - \alpha_{i+1})^+}{i+1} \right), \quad i = 1, 2, \ldots, N$$

and this confirms our intuition that larger retailers order more than smaller ones so that q_i is non-increasing in i. This follows from the fact that $H(B)$ is non-increasing in B which implies that the α_i's are non-decreasing in i. Having characterized the Cournot equilibrium of the N retailers, we can now determine the producer's expected profits, $\Pi_P = \mathbb{E}[(w_\tau - c)\, Q(w_\tau)]$, for a fixed price menu, w_τ. We have the following proposition.

[8] This construction has similarities to the equilibrium constructions found in Golany and Rothblum (2008) and Ledvina and Sircar (2012) who also obtain cutoff points below which firms produce and above which firms are costed out.

Proposition 13 (Producer's Expected Profits)

An $\mathcal{F}_\tau$-measurable wholesale price menu w_τ is a Stackelberg equilibrium in the producer's market if it maximizes the producer's expected payoff, $\Pi_{\rm P}$, given by

$$\Pi_{\rm P} \;=\; \left(\frac{m-1}{m}\right) \mathbb{E}[(w_\tau - c)\,(A_\tau - w_\tau)^+] \;+\; \sum_{j=m}^{N} \left(\frac{B_j}{m} \;-\; \frac{c}{j(j+1)}\, \mathbb{E}[(A_\tau - \alpha_j\, w_\tau)^+]\right) \tag{44}$$

where

$$m \;=\; m(\{w_\tau\}) \;:=\; \max\{i \geq 1 \quad \textit{such that} \quad \alpha_{i-1} = 1\} \tag{45}$$

and $\alpha_i = \;H(\mathcal{B}_i)$ (as defined in Proposition 12) with $\alpha_0 := 1$.

Proof: See Appendix A.

The producer's problem is then to maximize $\Pi_{\rm P}$ in (44) over price menus w_τ. A first important observation regarding this problem is that it cannot be solved path-wise since the α_i's are deterministic and depend implicitly in a non-trivial way on w_τ through the function $H(\cdot)$. Note also that m is the index of the first retailer whose budget constraint is binding[9] with the understanding that if $m = N+1$ then all N retailers are non-binding. We can characterize those values of $m \in \{1, \ldots, N+1\}$ that are possible. In particular, if the producer sets $w_\tau = A_\tau$ then all of the retailers are non-binding and so $m = N+1$. We can also find the smallest possible value of m, $m_{\min}$ say, by setting $w_\tau = c$, solving for the resulting α_i's using (42) and then taking $m_{\min}$ according to (45). Assuming the B_i's are distinct, the achievable values of m are given by the set $M_{\rm feas} := \{m_{\min}, \ldots, N+1\}$. This can be seen by taking $w_\tau = \gamma c + (1-\gamma)A_\tau$ with $\gamma = 0$ initially and then increasing it to 1. In the process each of the values in $M_{\rm feas}$ will be obtained.

We could use this observation to solve numerically for the producer's optimal menu, w_τ^*, by solving a series of sub-problems. In particular we could solve for the optimal price menu subject to the constraint that $m = m^*$ for each possible value of $m^* \in M_{\rm feas}$. Each of these $N - m^* + 2$ sub-problems could be solved numerically after discretizing the probability space. The overall optimal price menu, w_τ^*, is then simply the optimal price menu in the sub-problem whose objective function is maximal. In an effort to be more concrete in identifying the structure of an optimal wholesale price menu, we consider a special case in the next subsection. Specifically, we consider those instances of the problem in which, in equilibrium, all retailers produce a positive amount in each state of the world. Besides providing some concrete theoretical insight regarding the form of an optimal wholesale price, this case is also of practical relevance as we expect it matches the operating condition of most supply chains.

5.1 Special Case: $n_\tau = N$

We now consider the special case in which the Cournot-Stackelberg equilibrium is such that every retailer produces a positive quantity pathwise, i.e. for every realization of the demand forecast A_τ. Mathematically, this condition is equivalent to requiring that in equilibrium $n_\tau = N$ a.s. in (41), or equivalently, that $\alpha_N \leq \alpha_\tau$ a.s. given our ranking of the retailers' budgets.

[9] We say a player is binding if his budget constraint is binding in the Cournot equilibrium. Otherwise a player is non-binding.

A direct consequence of this condition is that $A_\tau \geq \alpha_j\, w_\tau$ a.s. for all $j = 1, \ldots, N$. Hence, from Proposition 13, it follows that the producer's optimization problem is given by

$$\Pi_{\mathrm{P}} = \max_{w_\tau} \left(\frac{m-1}{m}\right) \mathbb{E}[(w_\tau - c)\,(A_\tau - w_\tau)^+] \; + \; \sum_{j=m}^{N} \left(\frac{B_j}{m} \; - \; \frac{c}{j(j+1)}\, \mathbb{E}[A_\tau - \alpha_j\, w_\tau]\right)$$
$$\text{subject to} \qquad m \;=\; \max\{i \geq 1 \quad \text{such that} \quad \alpha_{i-1} = 1\}.$$

We can further simply this optimization problem by noticing that in equilibrium an optimal w_τ satisfies $w_\tau \leq A_\tau$ a.s. Indeed, for any $w_\tau \geq A_\tau$, the cumulative order quantity will be $Q_\tau = 0$ so restricting the wholesale price to satisfy $w_\tau \leq A_\tau$ is without loss of generality.

We solve the producer's problem in two steps. First, we fix the value of m and compute an optimal wholesale price $w_\tau(m)$ under this restriction. Then, we determine the optimal solution by maximizing over the value of m. We have relegated to Appendix A most of the technical details of this two-step program. Here we summarize the main results beginning with the next proposition.

Proposition 14 *Let $\Pi_{\mathrm{P}}(m)$ be the producer's optimal expected profit under the additional constraint that only the first $m-1$ retailers are not budged constrained in equilibrium. Then, under the assumption $n_\tau = N$ a.s., $\Pi_{\mathrm{P}}(m)$ solves the following optimization problem:*

$$\Pi_{\mathrm{P}}(m) = \max_{\mu,\sigma} \left\{ \left(\frac{m-1}{m}\right) [\mu\, \mathbb{E}_0[A_\tau] + \sigma\, \sigma_{A_\tau} + c\mu - \sigma^2 - \mu^2] + \sum_{j=m}^{N} \frac{c\,\mu\big(\mu\, \mathbb{E}_0[A_\tau] + \sigma\, \sigma_{A_\tau} - \mathcal{B}_j\big)}{j(j+1)(\sigma^2+\mu^2)} \right\} + \delta(m)$$
$$\textit{subject to} \qquad \mathcal{B}_m + \mu^2 + \sigma^2 < \mu\, \mathbb{E}_0[A_\tau] + \sigma\, \sigma_{A_\tau} \leq \mathcal{B}_{m-1} + \mu^2 + \sigma^2, \quad \mu \geq 0, \quad \textit{and} \quad \sigma \geq 0,$$

where $\delta(m) := \sum_{j=m}^{N} \frac{B_j}{m} - \frac{N}{N+1}\, c\, \mathbb{E}[A_\tau]$ and $\sigma_{A_\tau} := \sqrt{\mathbb{V}\mathrm{ar}(A_\tau)}$. Furthermore, if we let $\mu(m)$ and $\sigma(m)$ denote the optimal solution of this optimization problem then the retailer's optimal wholesale price $w_\tau(m)$ is given by

$$w_\tau(m) = \mu(m) + \frac{\sigma(m)}{\sigma_{A_\tau}}\, (A_\tau - \mathbb{E}[A_\tau]).$$

<u>Proof</u>: See Appendix A.

According to Proposition 14, we can reduce the complexity of computing an optimal wholesale price menu w_τ to a sequence of optimizations in two scalars μ and σ. From the proof of Proposition 14, μ corresponds to the mean of w_τ while σ corresponds to its standard deviation. Equipped with this result, we can complete our derivation of an optimal solution by identifying

$$m^* := \arg\max\{\Pi_{\mathrm{P}}(m) : m = 1, \ldots, N+1\}$$

as well as $\mu^* := \mu(m^*)$ and $\sigma^* := \sigma(m^*)$. From this, it follows that a candidate solution for the optimal wholesale price is given by

$$w_\tau^* = \mu^* + \frac{\sigma^*}{\sigma_{A_\tau}}\, (A_\tau - \mathbb{E}[A_\tau]). \tag{46}$$

Note that we say that w_τ^* is only a candidate solution because we still need to very that our assumption $n_\tau = N$ a.s. is satisfied under the proposed w_τ^*. The following result takes care of this final step in our derivation.

Proposition 15 *If $m^* < N+1$ then a sufficient condition for the optimality of w^*_τ in (46) is that*

$$\left(\frac{\mu^*\,\mathbb{E}[A_\tau] + \sigma^*\,\sigma_{A_\tau} - (N+1)\,B_N}{(\mu^*)^2 + (\sigma^*)^2}\right)\left(\mu^* + \frac{\sigma^*}{\sigma_{A_\tau}}\,(A_\tau - \mathbb{E}[A_\tau])\right) \le A_\tau \quad \text{a.s.}$$

On the other hand, if $m^* = N+1$ then a sufficient condition for the optimality of w^*_τ in (46) is that

$$\mu^* + \frac{\sigma^*}{\sigma_{A_\tau}}\,(A_\tau - \mathbb{E}[A_\tau]) \le A_\tau \quad \text{a.s.}$$

PROOF: See Appendix A.

Besides providing a mathematically tractable reformulation of the producer's problem, Propositions 14 and 15 reveal that at optimality the optimal wholesale price is a linear function of the demand forecast A_τ. We will use this observation to support the numerical analysis that we conduct in the next subsection where we propose a linear wholesale price approximation for the producer's problem in the general case in which $n_\tau < N$.

5.2 Constant and Linear Wholesale Price Contracts

Solving for the wholesale price contract that optimizes the producer's expected profits in Proposition 13 for an arbitrary vector of budgets $(B_1, \ldots, B_N)$ is a complex optimization problem in general. But in the equi-budget case and the case where $n_\tau = N$ a.s. we have seen that a wholesale price contract that is linear in A_τ is optimal. It is easy to see that this result also holds in the non-equibudget case where none of the budgets are binding in equilibrium. Motivated by this observation, we restrict ourselves in this section to the family of price contracts that are linear[10] in A_τ and c. Through a numerical example we demonstrate that it is straightforward to optimize numerically for the optimal contract within this family for an arbitrary vector of budgets. We also note that since c is deterministic this family includes the important case of a constant wholesale price contract. Our example will also consider the benefit (to the producer) of offering an optimal linear contract versus an optimal constant wholesale price contract. We first discuss the constant wholesale price contract as a special case as some additional explicit results are available in this case.

A Constant Wholesale Price

The problem of optimizing the expected payoff in (44) is considerably more tractable if the producer offers a constant wholesale price $\bar{w}$ instead of a random menu w_τ. From a practical standpoint, this is an important special case since a constant wholesale price is also a much simpler contract to implement. Note, however, that even with a constant wholesale price the retailers are still free to choose $\mathcal{F}_\tau$-measurable responses, q_τ, and to engage in financial hedging. In this case, Proposition 13 can be specialized as follows.

[10] Since c is a constant these contracts include the optimal contracts for the equibudget case (26) and the case (46) where $n_\tau = N$ a.s.

Corollary 1 *Under a constant wholesale price, $\bar{w}$, the producer's expected payoff is given by*

$$\Pi_{\mathrm{P}} = \frac{\bar{w} - c}{m} \left((m-1)\ \mathbb{E}[(A_\tau - \bar{w})^+]\ + \sum_{j=m}^{N} \frac{B_j}{\bar{w}} \right). \tag{47}$$

Let $\bar{w}^*(B_1, \ldots, B_N)$ be the constant wholesale price that maximizes Π_P as a function of the budgets $B_1 \geq B_2 \geq \cdots \geq B_N$. Then it follows that $\bar{w}^*(B_1, \ldots, B_N) \geq \bar{w}^*(\infty, \ldots, \infty)$ for all such budgets.

PROOF: See Appendix A.

While m is a function of $\bar{w}$ it is nonetheless straightforward to check that Π_{P} in (47) is a continuous function of $\bar{w}$. Note also that if some budget is transferred from one non-binding player to another non-binding player and both players remain non-binding after the transfer then Π_{P} is unchanged. Similarly if some budget is transferred from one binding player to another binding player and both players remain binding after the transfer then Π_{P} is again unchanged. Both of these statements follow from (47) and because it is easy to confirm that in each case the value of m is unchanged.

If some budget is transferred from a binding player to a non-binding player, however, then the ordering of the B_i's and the definition of the α_i's imply that both players remain binding and non-binding, respectively, after the transfer. Therefore m remains unchanged and Π_{P} decreases according to (47).

Conversely, we can increase Π_{P} by transferring budget from a non-binding player to a binding player in such a way that both players remain non-binding and binding, respectively, after the transfer. It is also possible to increase Π_{P} if budget is transferred from one non-binding player to another non-binding player so that the first player becomes binding after the transfer. Note that the statements above are consistent with the idea that the producer would like to see the budgets spread evenly among the various retailers.

The final part of Corollary 1 asserts that it is in the producer's best interest to increase the wholesale price when selling to budget-constraint retailers. By doing so the producer is inducing the retailers to reallocate their limited budgets into those states in which demand is high and for which the retailers have the incentives to invest more of their budgets in procuring units from the producer. As a result, the producer is able to extract a larger fraction of the retailers initial budgets.

Optimizing Over the Family of Linear Wholesale Price Contracts

We return now to linear price contracts of the form $w_\tau = w_c c + w_A A_\tau$ for constants w_c and w_A. The constant wholesale price contract is of course a special case corresponding to $w_A \equiv 0$ since c is assumed constant. Motivated by our earlier results, it seems reasonable to conjecture that this class should contain contracts that are optimal or close to optimal for the producer. We now demonstrate through a numerical example that it is straightforward for the producer to optimize over (w_c, w_A) and therefore compute the Cournot-Stackelberg equilibrium solution for this class of contracts. This example also highlights some of the qualitative features of the equilibrium and in particular, how the expected profits of the producer and retailers can be very sensitive to the volatility of the financial index.

Example 1 We consider a simple[11] additive model of the form $A_\tau = F(X_\tau)$ where X_t follows a geometric Brownian motion with dynamics $\mathrm{d}X_t = \sigma X_t \mathrm{d}W_t$ and where W_t is a $\mathbb{Q}$-Brownian motion. Note that since interest rates are zero, X_t must be a $\mathbb{Q}$-martingale and therefore does not have a drift. To model the dependence between the market clearance price and the process, X_t, we assume a linear model so that $F(x) = A_0 + A_1 x$ where $A_0 > c$ and A_1 are positive constants. We therefore have

$$A_\tau = A_0 + A_1 X_\tau \tag{48}$$

with $\bar{A} := \mathbb{E}[A_\tau] = A_0 + A_1 X_0$. Note that with $A_0 > c$ we are assured that Assumption 1 is satisfied.

We consider a problem with $N = 5$ retailers with budgets given by $B = [1400\ \ 1000\ \ 500\ \ 300\ \ 100]$. Other problem parameters are $X_0 = \sigma = 1$, $\tau = .5$, $A_1 = 100$, $A_0 = 10$ and $c = 7$. In Figure 5 below we have plotted the producer's expected profits Π_P as a function of w_c and w_A on the grid $[0,\, 5] \times [0,\, 1.2]$. Note that Π_P was computed numerically using the results of Proposition 13. We first

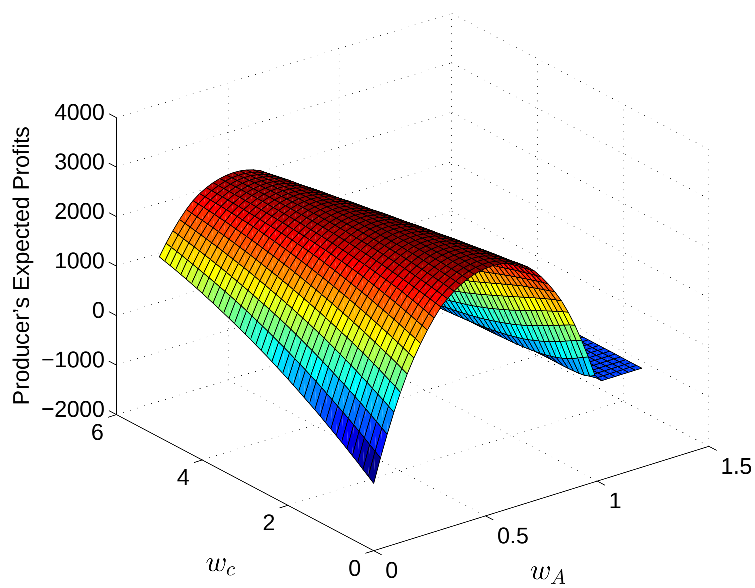

Figure 5: Producer's expected profits as a function of w_a and w_c with $w_\tau = w_c c + w_A A_\tau$. The other parameters are $N = 5$ retailers, $X_0 = \sigma = 1$, $\tau = .5$, $A_1 = 100$, $A_0 = 10$, $c = 7$ and $B = [1400\ \ 1000\ \ 500\ \ 300\ \ 100]$.

observe that Π_P is very sensitive to w_A while it is less sensitive to w_c. This of course is due in part to our selected parameter values. Moreover, while not shown in the figure, it is indeed the case that Π_P decreases to zero for sufficiently large values of w_c. The behavior of Π_P as a function of w_A (with w_c fixed) is perhaps more interesting: we see that Π_P initially increases in w_A but eventually decreases to zero at which point the producer earns zero profits and the supply chain has been shut down. We also see that the producer incurs losses for values of (w_c, w_A) close to $(0, 0)$ which of course is not surprising since the producer is undercharging at these values.

For this example, we found an optimal equilibrium value of $\Pi_\mathsf{P} = 3,016$ with corresponding optimal parameters $w_c^* = .975$ and $w_A^* = .48$. At this equilibrium solution we found that the budget constraints

[11] As discussed at the end of Section 2.1, this model is consistent with a more general model where the true market size is a random variable A that is not observed until time $T > \tau$ and that can depend on both X_T and non-financial noise. For example, we could take $A = F(X_T) + \varepsilon$ with $\mathbb{E}[\varepsilon] = 0$ and ε capturing the non-financial component of the market price uncertainty. In this case A_τ is as given in (48) and the analysis would not change.

were binding for all but the largest retailer. It is also interesting to note that the optimal value of w_A was very close to $.5$ which we know to be the optimal coefficient when the budget constraints aren't binding. This lends further support to our consideration of linear contracts for the general problem with non-symmetric and possibly binding budgets. Incidentally, we also solved the symmetric problem with $N = 5$ retailers where each $B_i = 660$. In this case the total retailer budget in both problems are identical and equal to $3,300$. In this symmetric case we found optimal producer profits of $\Pi_{\mathsf{P}} = 3,131$ corresponding to $w_c^* = 4.15$ and $w_A^* = .487$. In this case the (identical) budget constraints for all five retailers were binding. We were also not surprised to see the optimal value of Π_{P} increases in the symmetric case since this intuitively corresponds to a situation of increased competition among the retailers. Note that we proved such a result in the no-hedging case of Section 3.1.

It is also interesting to compare the optimal producer profits, $\Pi_{\mathsf{P}}^{\mathsf{L}}$, assuming *linear* wholesale contracts are available, with his optimal profits, $\Pi_{\mathsf{P}}^{\mathsf{C}}$, when only *constant* wholesale price contracts are available. We perform this comparison as a function of σ, the volatility parameter driving the dynamics of the financial market in Figure 6 below.

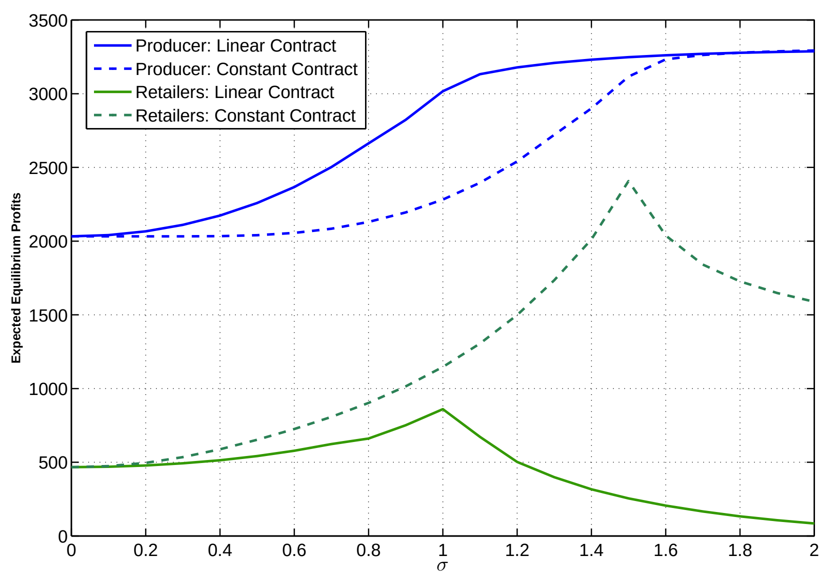

Figure 6: Expected equilibrium profits as a function of σ for the producer and retailers: optimal linear wholesale price contract with $w_\tau = w_c c + w_A A_\tau$ versus the optimal constant contract (with $w_A \equiv 0$). The other parameters are $N = 5$ retailers, $X_0 = 1$, $\tau = .5$, $A_1 = 100$, $A_0 = 10$, $c = 7$ and $B = [1400 \;\; 1000 \;\; 500 \;\; 300 \;\; 100]$.

Our first observation is that $\Pi_{\mathsf{P}}^{\mathsf{L}} = \Pi_{\mathsf{P}}^{\mathsf{C}}$ when $\sigma = 0$. This is as expected, however, since if $\sigma = 0$ then A_τ is a constant and so the ability to offer price menus that depend on A_τ therefore has no value. It is also interesting to note that $\Pi_{\mathsf{P}}^{\mathsf{L}}$ also equals $\Pi_{\mathsf{P}}^{\mathsf{C}}$ for large values of σ. This may at first appear surprising but we must first recall that even when the producer offers a constant wholesale price, the retailers' ordering quantities can and will depend on $A_\tau = A_0 + A_1 X_0 e^{-\sigma^2\tau/2 + \sigma W_\tau}$. Since $\text{Var}\,(A_\tau) = (A_1 X_0)^2 \left(e^{\sigma^2 \tau} - 1\right)$ increases in σ it is clear that larger values of σ imply that larger values of A become increasingly probable. The producer can take advantage of this by charging a very large constant wholesale price and forcing the retailers to purchase at that large price only in states where A_τ is sufficiently large. For large values of σ the producer therefore becomes indifferent between offering a simple constant wholesale price menu and offering a linear pricing menu. This can be seen clearly in Figure 7 below where we again plot the producer's expected profits as a function of w_c and w_A but now on the grid $[300,\, 600] \times [0,\, 1]$. The parameters are identical to those of Figure 5 with the exception that we now take $\sigma = 2$, corresponding to the largest value of σ in Figure 6. We see there is a large range of (w_c, w_A) values which optimize the producer's profits in Figure 7. Most importantly, and consistent with our observations above, this range contains a strip of values for which $w_A = 0$ which of course corresponds to constant wholesale price contracts.

A similar line of reasoning also explains why $\Pi_{\mathsf{P}}^{\mathsf{L}}$ and $\Pi_{\mathsf{P}}^{\mathsf{C}}$ are both increasing in σ in Figure 6. This is despite the fact that $\bar{A} = \mathbb{E}[A_\tau] = A_0 + A_1\, X_0$ does *not* depend on σ. We note that for intermediate values of σ the linear price contract outperforms (from the producer's perspective) the constant wholesale price contract by up to approximately 50%, demonstrating the potential value of such contracts. We note in passing that the volatility σ of financial indices might typically range from 20% to 50% depending on market conditions and the indices in question. We also note that the linear contract performs very close to optimal for intermediate and large values of σ because in this range his expected profits are close to 3,300 which is the total budget of the retailers. (For very small values of σ the constant – and therefore also the linear – contract is very close to optimal as there is nothing to be gained by contracting on the financial market in that case.)

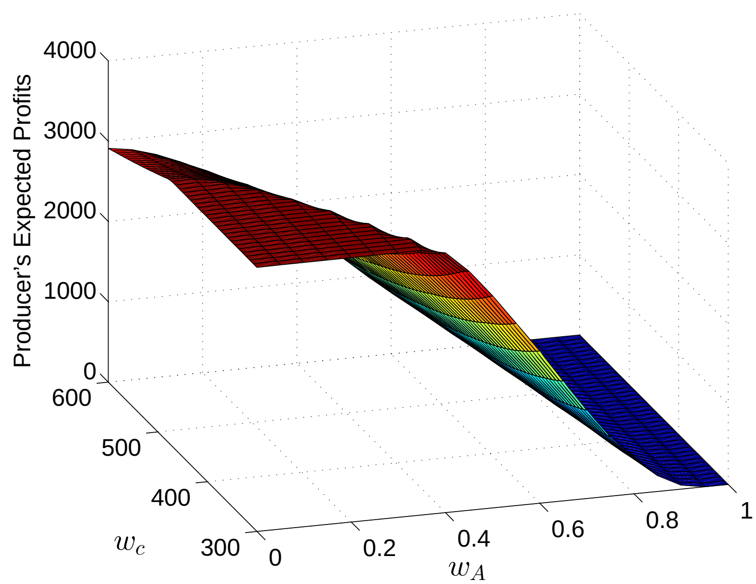

Figure 7: Producer's expected profits as a function of w_a and w_c with $w_\tau = w_c c + w_A A_\tau$. The parameters are identical to those of Figure 5 with the exception that $\sigma = 2$ and we have changed the values of w_a and w_c over which the producer's expected profits are plotted.

In Figure 6 we have also plotted the total equilibrium profits of the retailers as a function of σ. The first main observation is that the retailers always do better with the constant contract than with the linear contract. This is because with a constant contract, the producer cannot force the retailers to only order in certain highly profitable states (for the producer) and shut the supply chain down in other less profitable states. We also note a kink in both of the retailers' curves after which their expected equilibrium profits start to decrease with σ. Some intuition for this (in the symmetric retailer case) is provided by the expression in (27) for the retailers' expected payoffs conditional on time τ information. Since $\mathsf{Var}\,(A_\tau)$ is increasing in σ we can argue (because of the positive part) that the expectation of A_τ component in (27) will increase in σ. In contrast, the value of ϕ_{P} decreases in σ. The expectation of the retailers' equilibrium conditional profits in (27) therefore consists of two competing terms. The first term dominates initially which is why we see the retailers' curves in Figure 6 increasing for smaller values of σ. Beyond a certain point, however, the second term starts to dominate and so the retailers' expected profits start to decrease in σ at this point.

We also mention that the results of Figure 6 were computed on a relatively coarse grid beginning at $\sigma = 0$ and using a step-size of 0.2. It would have been easy of course to use a much finer grid but for values of σ greater than 1, we encountered some numerical instabilities in computing the retailers' total equilibrium profits for both types of contracts. The main reason for this is the need to compute the α_i's numerically and the relatively large range of (w_c, w_A) values which are (more or less) optimal[12] for the producer. While the producer is *almost* indifferent to which contract he chooses within this optimal range, the retailers are not, so that the reported equilibrium profits of the retailers (for large values of σ) are very sensitive to the particular solution chosen by the producer, i.e. found by the numerical optimizer. Had we therefore chosen a finer grid for σ, we would have obtained curves for the expected total retailer profits that were less smooth (due to the aforementioned numerical instabilities) than those of Figure 6 for values of σ greater than 1.

On a closely related note, given that the producer appears to be indifferent to the choice of constant or linear wholesale contracts for values of σ close to 2, one might expect the total expected retailers' profits for the two contracts to also coincide for values of σ close to 2. We do not see this, however, because while the producer is almost indifferent between the two contract types, he still marginally prefers the linear contract (although this is not apparent from Figure 6) and this results (for the reasons given earlier) in the retailers being considerably worse off than if only a constant contract was available.

6 Conclusions and Further Research

We have studied the performance of a stylized supply chain where multiple retailers and a single producer compete in a Cournot-Stackelberg game. At time $t = 0$ the retailers order a single product from the producer and upon delivery at time $\tau > 0$, they sell it in the retail market at a stochastic clearance price that depends on the realized path or terminal value of some observable and tradeable financial process. Because delivery does not take place until time τ, the producer offers a *menu* of wholesale prices to the retailer, one for each realization of the process up to time some time, τ. The retailers' ordering quantities are therefore contingent upon the realization of the process up to time τ. Production of the good is then completed on or after time τ at which point profits in the supply chain are also realized. We also assumed, however, that the retailers were budget-constrained and were therefore limited in the number of units they could purchase from the producer. Because the supply chain is potentially more profitable if the retailers can allocate their budgets across different states we allow them to trade dynamically in the financial market.

In contributing to the work on financial hedging in supply chains, our main contributions are: (i) solving for the Cournot-Stackelberg equilibrium when the retailers' have identical budgets (ii) our analysis of the interaction between hedging and retail competition on the various players including the firms themselves, the end consumers and society as a whole and (iii) the solution of the Cournot game in the case of non-identical budgets and the corresponding solution of the Cournot-Stackelberg game in special instances. This latter contribution is technical in nature since solving for just the Cournot game when the retailers have different budgets is challenging. Indeed there appears to be

[12]We saw precisely this phenomenon in Figure 7 for the case $\sigma = 2$.

relatively few instances of solutions to Cournot games in the literature where the players are not identical. However, we did identify important instances for which an analytical solution is available, namely, the case in which the retailers all order a strictly positive quantity almost surely, and the case in which retailers are symmetric. In both cases, the optimal wholesale price menu is linear in the maximum expected retail price (A_τ). Motivated by this pattern, we investigate numerically the performance of an optimal linear (in c_τ and A_τ) wholesale price. Our results suggest that this simple family of contract performs well especially when market volatility is high. (When market volatility is low there is much less to be gained from contracting on the financial index and so constant contract would be close to optimal in that case).

There are several possible directions for future research. First, it would be interesting to allow for the possibility of external financing as we suspect this problem should be tractable in the equi-budget case. It would also be of interest to identify and calibrate settings where supply chain payoffs are strongly dependent on markets. We would then like to estimate just how much value is provided by the financial markets in its role as (i) a source of public information upon which contracts may be written and (ii) as a means of mitigating the retailers' budget constraints. A further direction is to consider alternative contracts such as 2-part tariffs for coordinating the supply chain. Of course we would still like to have these contracts be contingent upon the outcome of the the financial markets. Other directions include amending the model to the one where the retailers compete via strategic complements rather than via strategic substitutes which is the case in this work. Finally, we would like to characterize the producer's optimal price menu in the general non-equibudget case and to see that if the optimal contract turns out to be linear in A_τ, as suspected.

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A Appendix: Proofs

PROOF OF PROPOSITION 1: Let $\mathcal{S}_b \subseteq \{1, \ldots, N\}$ be the subset of retailers for which the budget constraint is binding in equilibrium. It follows that

$$q_i = \frac{B_i}{w_\tau} \qquad \forall i \in \mathcal{S}_b.$$

Similarly, let $S_{nb} = \{1, \ldots, N\} \setminus \mathcal{S}_b$ be the set of retailers for which the budget constraint is not binding. It follows from the first-order optimality condition for problem (4)-(5) that the order quantity for retailer $i \in \mathcal{S}_{nb}$ is also uniquely given by

$$q_i = (A_\tau - Q_{-i} - w_\tau)/2 \qquad \forall i \in \mathcal{S}_{nb}.$$

If we let $Q = q_1 + q_2 + \cdots + q_N$ then $Q_{-i} = Q - q_i$ and we get

$$q_i = A_\tau - Q - w_\tau \qquad \forall i \in \mathcal{S}_{nb}.$$

Hence, retailers whose budget constraint is not binding order the same amount in equilibrium. It follows that there exists an integer $m \in \{0, 1, \ldots, N\}$ such that $\mathcal{S}_{nb} = \{1, \ldots, m-1\}$ and $\mathcal{S}_b = \{m, \ldots, N\}$.

Now, summing over i the q_i, we can solve for Q as a function of m to get

$$Q = \frac{1}{m}\left[(m-1)\left(A_\tau - w_\tau\right) + \sum_{i=m}^{N} \frac{B_i}{w_\tau}\right].$$

Plugging back this value of Q on the expression for q_i above we get the equilibrium order for retailer $i \in \mathcal{S}_{nb}$. Finally, by the definition of m as the threshold retailer whose budget is binding, we must have $B_m < w_\tau\left(A_\tau - Q - w_\tau\right) \leq B_{m-1}$. Plugging the value of Q above and after some straightforward manipulations, we get that this condition is equivalent to

$$\mathcal{B}_m < w_\tau\left(A_\tau - w_\tau\right) \leq \mathcal{B}_{m-1},$$

where the definition of $\mathcal{B}_k$ is given in (7). Finally, the uniqueness of m follows now from the fact that the sequence $\{\mathcal{B}_k\}$ is non increasing in k. $\square$

Proof of Proposition 2: For fixed values of w_τ and $B_1, \ldots, B_N$, let us define the sequence

$$Q_k := \left\{ \frac{1}{k+1} \left[k\,(A_\tau - w_\tau) + \sum_{i=k+1}^{N} \frac{B_i}{w_\tau} \right] \right\}.$$

From the definition of $Q(w_\tau, B_1, \ldots, B_N)$ in (9), we need to show that $Q_{m(w_\tau)-1} \leq Q_k$ for all $k = 0, \ldots, N$. We prove this by showing the following stronger result: $Q_{k-1} \geq Q_k$ for $k = 1, \ldots, m(w_\tau) - 1$ and $Q_{k-1} \leq Q_k$ for $k = m(w_\tau), \ldots, N$. To see this, note that the condition $Q_{k-1} \geq Q_k$ is equivalent to

$$(k+1) \left[(k-1)\,(A_\tau - w_\tau) + \sum_{i=k}^{N} \frac{B_i}{w_\tau} \right] \geq k \left[k\,(A_\tau - w_\tau) + \sum_{i=k+1}^{N} \frac{B_i}{w_\tau} \right].$$

After some straightforward manipulations, this inequality is equivalent to (under the trivial assumption that $w_\tau \geq 0$)

$$k\,B_k + \sum_{i=k}^{N} B_i \geq w_\tau\,(A_\tau - w_\tau).$$

Finally, by the definition of $m(w)$ in (8), this inequality is equivalent to $m(w_\tau) \geq k + 1$. $\square$

Proof of Proposition 3: Let $\bar{Q} = Q(w_\tau, \bar{B}, \ldots, \bar{B})$ and $Q = Q(w_\tau, B_1, \ldots, B_N)$ be the market total output when the retailers' budgets are $\bar{B} = (\bar{B}, \ldots, \bar{B})$ and $B = (B_1, \ldots, B_N)$, respectively. We want to show that $\bar{Q} \geq Q$. To this end, we take advantage of the fact that the retailers are symmetric under $\bar{B}$ to identify two cases:

1. Budget constraints are binding under $\bar{B}$: In this case, it follows trivially that

$$\bar{Q} = \frac{B_C}{w_\tau} = \sum_{i=1}^{N} \frac{B_i}{w_\tau} \geq \sum_{i=1}^{N} q_i = Q.$$

2. Budget constraints are not binding under $\bar{B}$: In this case, it is not hard to see that

$$\bar{Q} = \frac{N}{N+1}(A_\tau - w_\tau).$$

   On the other hand, by from Proposition 1 we know that there exists an $m = m(w_\tau) \in \{0, 1, \ldots, N\}$ such that

$$Q = \frac{1}{m+1} \left[m\,(A_\tau - w_\tau) + \sum_{i=m+1}^{N} \frac{B_i}{w_\tau} \right].$$

   Now, if $m = N$ then $\bar{Q}$ and the result follows. Let us then assume that $m \leq N_1$. From the definition of m in (8) it follows that

$$w_\tau\,(A_\tau - w_\tau) > \mathcal{B}_k, \qquad \text{for } k = m+1, \ldots, N. \tag{A-1}$$

Next, we use this condition recursively to show that

$$\begin{aligned} Q &= \frac{1}{m+1}\left[m\left(A_\tau - w_\tau\right) + \sum_{i=m+1}^{N} \frac{B_i}{w_\tau}\right] \\ &\leq \frac{1}{m+2}\left[(m+1)\left(A_\tau - w_\tau\right) + \sum_{i=m+2}^{N} \frac{B_i}{w_\tau}\right] \\ &\vdots \\ &\leq \frac{1}{N+1}\left[N\left(A_\tau - w_\tau\right)\right] = \bar{Q}. \end{aligned}$$

We now use condition (A-1) to prove the first inequality and the others follows the same steps. We can write the first inequality as

$$[(m+1)^2 - m(m+1)](A_\tau - w_\tau) \geq (m+2)\sum_{i=m+1}^{N} \frac{B_i}{w_\tau} - (m+1)\sum_{i=m+2}^{N} \frac{B_i}{w_\tau},$$

which after some manipulations is equivalent to

$$w_\tau\left(A_\tau - w_\tau\right) \geq (m+1)B_{m+1} + \sum_{i=m+1}^{N} B_i.$$

Note that the term on the right is equal to $\mathcal{B}_{m+1}$ and so the inequality follows from condition (A-1). $\square$

Proof of Proposition 6: It is straightforward to see that $\mathcal{Q}_\tau$ is each of the N retailer's optimal ordering level given the wholesale price menu, w_τ, in the absence of a budget constraint. This follows from a standard Cournot-style analysis in which all of the retailers, owing to their identical budgets, order the same quantity. In order to implement this solution, each retailer would need a budget $\mathcal{Q}_\tau\, w_\tau$ for all $\omega \in \Omega$. Therefore, if each retailer can generate a financial gain, G_τ, such that $\mathcal{Q}_\tau\, w_\tau \leq B + G_\tau$ for all $\omega \in \Omega$ then he would be able to achieve his unconstrained optimal solution.

By definition, $\mathcal{X}$ contains all those states for which $B \geq \mathcal{Q}_\tau\, w_\tau$. That is, the original budget B is large enough to cover the optimal purchasing cost for all $\omega \in \mathcal{X}$. However, for $\omega \in \mathcal{X}^c$, the initial budget is not sufficient. The financial gain, G_τ, then allows the retailer to transfer resources from $\mathcal{X}$ to $\mathcal{X}^c$.

Suppose the condition in Case 1 holds so that $\mathbb{E}\left[\mathcal{Q}_\tau\, w_\tau\right] \leq B$. Note that according to the definition of G_τ in this case, we see that $B + G_\tau = \mathcal{Q}_\tau\, w_\tau$ for all $\omega \in \mathcal{X}^c$. For $\omega \in \mathcal{X}$, however, $B + G_\tau = (1-\delta)\, B + \delta\, \mathcal{Q}_\tau\, w_\tau \geq \mathcal{Q}_\tau\, w_\tau$. The inequality follows since $\delta \leq 1$. G_τ therefore allows the retailer to implement the unconstrained optimal solution. The only point that remains to check is that G_τ satisfies $\mathbb{E}[G_\tau] = 0$. This follows directly from the definition of δ.

Suppose now that the condition specified in Case 2 holds. We solve the i^{th} retailer's optimization problem in (18) by relaxing the gain constraint with a Lagrange multiplier, λ_i. We also relax the budget constraint for each realization of X up to time τ. The corresponding multiplier for each such realization is denoted by $\beta_\tau^{(i)}\, d\mathbb{Q}$ where $\beta_\tau^{(i)}$ plays the role of a Radon-Nikodym derivative of a positive measure that is absolutely continuous with respect to $\mathbb{Q}$. The first-order optimality conditions for the relaxed version of the retailer's problem are then given by

$$q_i = \frac{(A_\tau - w_\tau\,(1+\beta_\tau^{(i)}) - Q_{i-})^+}{2}$$

$$\beta_\tau^{(i)} = \lambda_i, \qquad \beta_\tau^{(i)}\left(w_\tau\, q_i - B + G_\tau\right) = 0, \qquad \beta_\tau^{(i)} \geq 0, \qquad \text{and } \mathbb{E}[G_\tau] = 0.$$

We look for a symmetric equilibrium of the above system of equations where $\lambda_i = \lambda$ and $q_i = q$ for all $i = 1, \ldots, N$.

It is straightforward to show that the solution given in Case 2 of the proposition satisfies these optimality conditions; only the non-negativity of $\beta_\tau^{(i)}$ needs to be checked separately. To prove this, note that $\beta_\tau^{(i)} = \lambda_i = \lambda$, therefore it suffices to show that $\lambda \geq 0$. This follows from three observations

(a) Since $0 \leq w_\tau$ the function $\mathbb{E}\left[w_\tau\,\left(\frac{A_\tau - w_\tau\,(1+\lambda)}{(N+1)}\right)^+\right]$ is decreasing in λ_i.

(b) In Case 2, by hypothesis, we have

$$\mathbb{E}\left[w_\tau\,\frac{(A_\tau - w_\tau)^+}{(N+1)}\right] = \mathbb{E}\left[\mathcal{Q}_\tau\, w_\tau\right] > B$$

(c) Finally, we know that λ solves

$$\mathbb{E}\left[w_\tau\,\frac{(A_\tau - w_\tau\,(1+\lambda))^+}{(N+1)}\right] = B.$$

(a) and (b) therefore imply that we must have $\lambda \geq 0$. $\square$

<u>Proof of Proposition 7:</u> The statements regarding the producer follow immediately from Proposition 7 in Caldentey and Haugh (2009) with the budget replaced by $(N+1)B/2$ and the objective function multiplied by $2N/(N+1)$. The statements regarding the retailers are due to the fact that the optimal value of λ in (24) is 0. This value of λ and the optimal value of w_τ can then be substituted into the expression for the optimal ordering quantity in either[13] Case 1 or Case 2 of Proposition 6. The expressions for q_τ and $\Pi_{\mathrm{R}|\tau}$ then follow immediately. $\square$

<u>Proof of Proposition 8:</u> Let first us prove part (a). Suppose that $N \leq \bar{N}$. Then, we have $\delta_{\mathrm{P}}(N) = c$ in Proposition 7 and so

$$w_\tau(N) = \frac{A_\tau + c}{2} \qquad \text{and} \qquad Q_\tau(N) = \frac{N\,(A_\tau - c)^+}{2(N+1)}.$$

It follows trivially that $w_\tau(N)$ is constant and $Q_\tau(N)$ is increasing in N, pathwise and therefore on expectation. Turning to case (b), when $N > \bar{N}$ then $\delta_{\mathrm{P}}(N) > c$ solves the equation

$$\mathbb{E}[(A_\tau^2 - \delta_{\mathrm{P}}(N)^2)^+] = 4\,B_{\mathrm{C}}\,(N+1)/N. \tag{A-2}$$

[13] Both cases lead to the same value of q_τ as the producer chooses the price menu in such a way that the budget is at the cutoff point between being binding and non-binding with $\lambda = 0$.

Since the right-hand side is decreasing in N, it follows that $\delta_{\mathrm{P}}(N)$ is increasing in N. As a result, the wholesale price

$$w_\tau(N) = \frac{A_\tau + \delta_{\mathrm{P}}(N)}{2}$$

is also increasing in N pathwise and therefore on expectation.

To complete the proof, we need to show that the total expected output $\mathbb{E}[Q_\tau(N)]$ is decreasing in N when $N > \bar{N}$. To show this, we first rewrite (A-2) as follows $\mathbb{E}[Q_\tau(N)\,(A_\tau + \delta_{\mathrm{P}}(N))] = 2\,B_{\mathrm{C}}$ or equivalently

$$\delta_{\mathrm{P}}(N) + \frac{\mathbb{E}[A_\tau\, Q_\tau(N)]}{\mathbb{E}[Q_\tau(N)]} = \frac{2\,B_{\mathrm{C}}}{\mathbb{E}[Q_\tau(N)]}.$$

From this identity, and the fact that $\delta_{\mathrm{P}}(N)$ is increasing in N, we will show that $\mathbb{E}[Q_\tau(N)]$ is decreasing in N by showing that the term $\mathbb{E}[A_\tau\, Q_\tau(N)]/\mathbb{E}[Q_\tau(N)]$ is increasing in N. For this, we use the the following lemma.

Lemma 1 *Let A be a nonnegative random variable that admits a smooth density $f \in \mathcal{C}^2[0,\infty)$ such that $\lim_{x\to\infty} x^2\, f(x) = 0$. Then, the function*

$$\bar{\mathcal{A}}(x) := \frac{\mathbb{E}[A\,(A-x)^+]}{\mathbb{E}[(A-x)^+]}$$

is increasing in x.

Before proving the lemma, note that $\mathbb{E}[A_\tau\, Q_\tau(N)]/\mathbb{E}[Q_\tau(N)] = \bar{\mathcal{A}}(\delta_{\mathrm{P}}(N))$. Hence, since $\delta_{\mathrm{P}}(N)$ in increasing in N, proving the lemma will complete the proof.

Proof of the Lemma: First, let us rewrite $\bar{\mathcal{A}}(x)$ as follows:

$$\bar{\mathcal{A}}(x) = x + \frac{\mathbb{E}[(A-x)\,(A-x)^+]}{\mathbb{E}[(A-x)^+]} = x + \frac{\int_x^\infty (y-x)^2\, f(y)\,\mathrm{d}y}{\int_x^\infty (y-x)\, f(y)\,\mathrm{d}y} = x + \frac{\int_0^\infty y^2\, f(y+x)\,\mathrm{d}y}{\int_0^\infty y\, f(y+x)\,\mathrm{d}y}.$$

Taking derivative with respect to x, and using the fact that $f \in \mathcal{C}^2[0,\infty)$ so that we can interchange derivation and integration, we get

$$\frac{\mathrm{d}\bar{\mathcal{A}}(x)}{\mathrm{d}x} = 1 + \frac{\left(\int_0^\infty y^2\, f'(y+x)\,\mathrm{d}y\right)\,\left(\int_0^\infty y\, f(y+x)\,\mathrm{d}y\right) - \left(\int_0^\infty y^2\, f(y+x)\,\mathrm{d}y\right)\,\left(\int_0^\infty y\, f'(y+x)\,\mathrm{d}y\right)}{\left(\int_0^\infty y\, f(y+x)\,\mathrm{d}y\right)^2}.$$

Now using integration by parts and the assumption on f that $\lim_{x\to\infty} x^2\, f(x) = 0$ we have

$$\int_0^\infty y^2\, f'(y+x)\,\mathrm{d}y = -2\int_0^\infty y\, f(y+x)\,\mathrm{d}y \qquad \text{and} \qquad \int_0^\infty y\, f'(y+x)\,\mathrm{d}y = -\int_0^\infty f(y+x)\,\mathrm{d}y.$$

Plugging these identities above we get

$$\begin{aligned}
\frac{\mathrm{d}\bar{\mathcal{A}}(x)}{\mathrm{d}x} &= 1 + \frac{\left(\int_0^\infty y^2\, f(y+x)\,\mathrm{d}y\right)\,\left(\int_0^\infty f(y+x)\,\mathrm{d}y\right) - 2\left(\int_0^\infty y\, f(y+x)\,\mathrm{d}y\right)^2}{\left(\int_0^\infty y\, f(y+x)\,\mathrm{d}y\right)^2} \\
&= \frac{\left(\int_0^\infty y^2\, f(y+x)\,\mathrm{d}y\right)\,\left(\int_0^\infty f(y+x)\,\mathrm{d}y\right) - \left(\int_0^\infty y\, f(y+x)\,\mathrm{d}y\right)^2}{\left(\int_0^\infty y\, f(y+x)\,\mathrm{d}y\right)^2} \\
&= \left(\frac{\int_0^\infty f(y+x)\,\mathrm{d}y}{\int_0^\infty y\, f(y+x)\,\mathrm{d}y}\right)^2 \left[\frac{\int_0^\infty y^2\, f(y+x)\,\mathrm{d}y}{\int_0^\infty f(y+x)\,\mathrm{d}y} - \left(\frac{\int_0^\infty y\, f(y+x)\,\mathrm{d}y}{\int_0^\infty f(y+x)\,\mathrm{d}y}\right)^2\right] \geq 0.
\end{aligned}$$

The inequality follows by noticing that the term inside the square brackets is the equal to the variance of a nonnegative random variable Y with density

$$f_Y(y) = \frac{f(y+x)}{\int_0^\infty f(z+x)\,\mathrm{d}z}. \quad \square$$

<u>PROOF OF PROPOSITION 9:</u> Let us consider first the case in which the retailers hedge their budget constraints. From the solution to the producer's problem in Proposition 7 we have that

$$\Pi_{\mathrm{P}}(N) = \mathbb{E}\left[(w_\mathcal{T}(N) - c)\, Q_\mathcal{T}(N)\right], \text{ where } w_\mathcal{T}(N) = \frac{A_\tau + \phi(N)\, c}{2}, \quad Q_\mathcal{T}(N) = \frac{N\,(A_\tau - \phi(N)\, c)^+}{2(N+1)}$$

and $\phi(N)$ is the minimum $\phi \geq 1$ such that $\mathbb{E}[(A_\tau^2 - (\phi\, c)^2)^+] \leq 4\, B_{\mathrm{C}}\,(N+1)/N$.

In this solution, the fact that N is an integer is immaterial and so we will prove the monotonicty of $\Pi_{\mathrm{P}}(N)$ assuming N is a continuous variable.

First, suppose that $N < \bar{N}$, then the budget constraint is not binding and $\phi(N) = 1$. It follows that

$$\Pi_{\mathrm{P}}(N) = \frac{N}{N+1}\, \mathbb{E}\left[\frac{(A_\tau - c)^2}{4}\right].$$

Hence, $\Pi_{\mathrm{P}}(N)$ is increasing in N when the budget constraint is strictly not binding.

Suppose now that $N > \bar{N}$ and the budget constraint is strictly binding N. Then,

$$\Pi_{\mathrm{P}}(N) = B_{\mathrm{C}} - c\,\mathbb{E}[Q_\mathcal{T}(N)].$$

Hence, in this case it suffices to show that $\mathbb{E}[Q_\mathcal{T}(N)]$ is nonincreasing in N. To prove this, we first note that there exists a threshold $\bar{\delta}_N$ such that

$$Q_\mathcal{T}(N+1) - Q_\mathcal{T}(N) > 0 \quad \text{if and only if} \quad A_\tau > \bar{\delta}_N.$$

It is a matter of simple calculation to derive the value of $\bar{\delta}_N = \Big[(N{+}1)^2\,\phi(N{+}1) - N(N{+}2)\,\phi(N)\Big]\, c$.

Next, let us suppose by contradiction that $\mathbb{E}[Q_\mathcal{T}(N+1)] > \mathbb{E}[Q_\mathcal{T}(N)]$ then multiplying both side of the inequality by $(\bar{\delta}_N + \phi(N+1)\, c)/2$ we get

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\bar{\delta}_N + \phi(N+1)\, c}{2}\right) Q_\mathcal{T}(N+1)\right] \;&>\; \mathbb{E}\left[\left(\frac{\bar{\delta}_N + \phi(N+1)\, c}{2}\right) Q_\mathcal{T}(N)\right] \\
&\geq\; \mathbb{E}\left[\left(\frac{\bar{\delta}_N + \phi(N)\, c}{2}\right) Q_\mathcal{T}(N)\right],
\end{aligned}$$

where the weak inequality uses the fact that $\phi(N+1) \geq \phi(N)$. Now, let us add and subtract $A_\tau/2$ on both side of the inequalities to get

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{A_\tau + \phi(N+1)\, c}{2}\right) Q_\mathcal{T}(N+1) + \left(\frac{\bar{\delta}_N - A_\tau}{2}\right) Q_\mathcal{T}(N+1)\right] \\
&> \mathbb{E}\left[\left(\frac{A_\tau + \phi(N)\, c}{2}\right) Q_\mathcal{T}(N) + \left(\frac{\bar{\delta}_N - A_\tau}{2}\right) Q_\mathcal{T}(N)\right].
\end{aligned}$$

But since $N > \bar{N}$, the budget constraint is binding and

$$\mathbb{E}\left[\left(\frac{A_\tau + \phi(N+1)\,c}{2}\right)\, Q_\tau(N+1)\right] = \mathbb{E}\left[\left(\frac{A_\tau + \phi(N)\,c}{2}\right)\, Q_\tau(N)\right] = B_{\mathrm{C}}.$$

As a result, the previous inequality becomes

$$\mathbb{E}\left[\left(\frac{\bar{\delta}_N - A_\tau}{2}\right)\, \Big[Q_\tau(N+1) - Q_\tau(N)\Big]\right] > 0.$$

But, by the definition of $\bar{\delta}_N$ above, it follows that

$$\left(\frac{\bar{\delta}_N - A_\tau}{2}\right)\, \Big[Q_\tau(N+1) - Q_\tau(N)\Big] \leq 0 \quad \text{w.p.1.}$$

From this contradiction we conclude that $\mathbb{E}[Q_\tau(N+1)] \leq \mathbb{E}[Q_\tau(N)]$ and so the producer's expected payoff is non-decreasing in N.

Let us now turn to the cumulative payoff of the retailers. Again, from the solution in Proposition 7, it follows that

$$N \times \Pi_{\mathrm{R}}(N) = \frac{\mathbb{E}[Q_\tau^2(N)}{N]}.$$

Using a similar line of arguments as the one used above, we will show that $N \times \Pi_{\mathrm{R}}(N)$ is nonincreasing in N. First, if the retailers' budget constraint is not binding, *i.e.* $N \leq \bar{N}$, then

$$\frac{\mathbb{E}[Q_\tau^2(N)]}{N} = \frac{N}{4(N+1)^2}\, \mathbb{E}[(A_\tau - c)^2],$$

which is nonincreasing in N for $N \geq 1$.

Suppose now that the retailers' budget is binding, that is, $N \geq \bar{N}$. Consider the incremental payoff of the retailers as the number of retailers' increases from N to $N+1$.

$$\frac{Q_\tau^2(N+1)}{N+1} - \frac{Q_\tau^2(N)}{N} = \left(\frac{Q_\tau(N+1)}{\sqrt{N+1}} + \frac{Q_\tau(N)}{\sqrt{N}}\right) \left(\frac{Q_\tau(N+1)}{\sqrt{N+1}} - \frac{Q_\tau(N)}{\sqrt{N}}\right).$$

Hence, whether the retailers' payoff increases or decreases with N depends on the sign of the last multiplicand on the right. We will show that this term is always nonpositive. Indeed, from the definition of $Q_\tau(N)$ it follows that

$$\frac{Q_\tau(N+1)}{\sqrt{N+1}} - \frac{Q_\tau(N)}{\sqrt{N}} > 0 \quad \text{if and only if} \quad \frac{\sqrt{N+1}}{N+2}\,(A_\tau - \delta(N+1))^+ - \frac{\sqrt{N}}{N+1}\,(A_\tau - \delta(N))^+ > 0.$$

Since $\delta(N+1) \geq \delta(N)$, a necessary condition for this last inequality to hold is $A_\tau > \delta(N+1)$. This together with the fact that the function $\sqrt{N}/(N+1)$ is decreasing in N implies that

$$\frac{Q_\tau(N+1)}{\sqrt{N+1}} - \frac{Q_\tau(N)}{\sqrt{N}} > 0 \quad \text{if and only if} \quad \delta(N+1) < A_\tau < \frac{\frac{\sqrt{N}}{N+1}\delta(N) - \frac{\sqrt{N+1}}{N+2}\delta(N+1)}{\frac{\sqrt{N}}{N+1} - \frac{\sqrt{N+1}}{N+2}}.$$

Finally, using again the fact that $\delta(N+1) \geq \delta(N)$, this last condition is equivalent to

$$\delta(N+1) < A_\tau < \frac{\frac{\sqrt{N}}{N+1}\delta(N) - \frac{\sqrt{N+1}}{N+2}\delta(N+1)}{\frac{\sqrt{N}}{N+1} - \frac{\sqrt{N+1}}{N+2}} \leq \delta(N).$$

We conclude that
$$\frac{Q_\tau^2(N+1)}{N+1} - \frac{Q_\tau^2(N)}{N} \leq 0$$
pathwise and therefore in expectation.

Let us consider now the case in which the retailers do not hedge their budget constraint. In this case, the equilibrium of the game can be computed pathwise, for each value of A_τ.Furthermore, using Proposition 1 in in Caldentey and Haugh (2009) we have that

$$\Pi_{\mathrm{P}|\tau}(N) = \frac{N}{N+1}\left[\frac{(A_\tau+\delta(N)-2\,c)\,(A_\tau-\delta(N))}{4}\right] \quad \text{and} \quad N\times\Pi_{\mathrm{R}|\tau}(N) = \frac{N}{(N+1)^2}\left[\frac{(A-\delta(N))^2}{4}\right],$$

where
$$\delta(N) = \max\left\{c, \sqrt{A_\tau^2 - 8\,\frac{(N+1)}{N}\,B_{\mathrm{C}}}\right\}.$$

Some tedious, but straightforward calculations, show that $\Pi_{\mathrm{P}|\tau}(N)$ is non-decreasing in N while $N\times\Pi_{\mathrm{R}|\tau}(N)$ is non-increasing in N. $\square$

<u>Proof of Proposition 10:</u> If $\bar{N}=\infty$, then $\phi_{\mathrm{P}}(N)=1$ for all $N\geq 1$ and the result follows from Propostion 7. $\square$

<u>Proof of Proposition 11:</u> From the definition of $\bar{N}$, it follows that $\phi(N)=1$ for all $N\leq\bar{N}$. Hence,
$$Q_\tau(N) = \frac{N\,(A_\tau-c)^+}{2(N+1)}$$
and so

$$\mathcal{C}(N) = \frac{N^2\,\mathbb{E}[(A_\tau-c)]}{8\,(N+1)^2} \qquad \text{and} \qquad \Pi_{\mathrm{P}}(N) + N\,\Pi_{\mathrm{R}}(N) = \frac{N\,(N+2)}{4\,(N+1)^2}\mathbb{E}[(A_\tau-c)^2].$$

We conclude that both $\mathcal{C}(N)$ and $\Pi_{\mathrm{S}}(N)$ are increasing in N for $N\leq\bar{N}$. The prof is complete by noticing that social welfare $\mathcal{S}(N)=\mathcal{C}(N)+\Pi_{\mathrm{S}}(N)$. $\square$

<u>Proof of Proposition 12:</u>

The proof of the proposition makes use of the following preliminary result.

Lemma 2 *Let V_τ be an $\mathcal{F}_\tau$-measurable random variable then*

$$(i+1)\,\mathbb{E}[V_\tau\,q_i] \;+\; \sum_{j=i+1}^{N}\mathbb{E}[V_\tau\,q_j] \;=\; \int_{\alpha_i<\alpha_\tau} V_\tau\,(A_\tau-\alpha_i\,w_\tau)\,\mathrm{d}\mathbb{Q}. \tag{A-3}$$

Proof of the Lemma: From equation (39) and the fact that the λ_i's are non-decreasing in i, it is easy to see that the α_i's are also non-decreasing in i (as we would expect). We also see that $n(\alpha_\tau)=k-1$ for all α_τ satisfying $\alpha_{k-1}\leq\alpha_\tau<\alpha_k$. Setting $\alpha_{N+1}:=\infty$, we can combine these results and (37) to write

$$q_i \;=\; \sum_{k=i}^{N}\frac{1}{k+1}\left[A_\tau - w_\tau\left((k+1)\,(1+\lambda_i) - \sum_{j=1}^{k}(1+\lambda_j)\right)\right]\mathbb{1}\Big(\alpha_\tau\in[\alpha_k,\alpha_{k+1})\Big). \tag{A-4}$$

Letting $\Omega_k := \{\omega : \ \alpha_k \le \alpha_\tau < \alpha_{k+1}\}$, we see that (A-4) implies

$$\begin{aligned}
\sum_{j=i+1}^{N} \mathbb{E}[V_\tau\, q_j] &= \sum_{j=i+1}^{N} \sum_{k=j}^{N} \int_{\Omega_k} \frac{V_\tau}{k+1} \left[A_\tau - w_\tau \left((k+1)\,(1+\lambda_j) - \sum_{s=1}^{k} (1+\lambda_s) \right) \right] \mathrm{d}\mathbb{Q} \\
&= \sum_{k=i+1}^{N} \sum_{j=i+1}^{k} \int_{\Omega_k} \frac{V_\tau}{k+1} \left[A_\tau - w_\tau \left((k+1)\,(1+\lambda_j) - \sum_{s=1}^{k} (1+\lambda_s) \right) \right] \mathrm{d}\mathbb{Q} \\
&= \sum_{k=i+1}^{N} \int_{\Omega_k} \frac{V_\tau}{k+1} \left[(k-i)\, A_\tau - w_\tau \left((k+1) \sum_{j=i+1}^{k} (1+\lambda_j) - (k-i) \sum_{s=1}^{k} (1+\lambda_s) \right) \right] \mathrm{d}\mathbb{Q}.
\end{aligned}$$

Combining this last identity and the fact that

$$\mathbb{E}[V_\tau\, q_i] \ = \sum_{k=i}^{N} \int_{\Omega_k} \frac{V_\tau}{k+1} \left[A_\tau - w_\tau \left((k+1)\,(1+\lambda_i) - \sum_{j=1}^{k} (1+\lambda_j) \right) \right] \mathrm{d}\mathbb{Q}$$

we obtain

$$\mathbb{E}[V_\tau\, q_i] + \frac{1}{i+1} \sum_{j=i+1}^{N} \mathbb{E}[V_\tau\, q_j] \ = \ \sum_{k=i}^{N} \int_{\Omega_k} \frac{V_\tau}{k+1}\, Z_{ik}\, \mathrm{d}\mathbb{Q} \tag{A-5}$$

where

$$\begin{aligned}
Z_{ik} \ :=\ & \Bigg[A_\tau - w_\tau \left((k+1)\,(1+\lambda_i) - \sum_{j=1}^{k} (1+\lambda_j) \right) \\
& \quad + \frac{(k-i)\, A_\tau - w_\tau \left((k+1) \sum_{j=i+1}^{k} (1+\lambda_j) - (k-i) \sum_{s=1}^{k} (1+\lambda_s) \right)}{i+1} \Bigg].
\end{aligned}$$

and where we have used the convention $\sum_{j=i+1}^{i} (1+\lambda_j) = 0$. After some straightforward manipulations, one can show that

$$Z_{ik} \ = \ \frac{k+1}{i+1}\, (A_\tau - \alpha_i\, w_\tau)$$

and so by the definition of Ω_k we can substitute for Z_{ik} in (A-5) and obtain (A-3). □

Let us now return to the proof of Propositon 12. Let us recall that the Lagrange multipliers for the retailers's budget constraints $\{\lambda_i\}$ and the cutoff values $\{\alpha_i\}$ satisfy the relationship

$$1 + \lambda_i = \frac{\alpha_i}{i} + \sum_{j=1}^{i-1} \frac{\alpha_j}{j\,(j+1)}.$$

Recall also that we have ordered the retailers budget in decreasing order $B_1 \ge B_2 \ge \cdots \ge B_N$. As a result, if retailer's i budget constraint is not binding then retailer's j budget constraint is also not binding, for all $j = 1, \ldots, i$. It follows that if $\mathbb{E}[w_\tau q_i] < B_i$ then $\lambda_j = 0$ for all $j = 1, \ldots, i$ and from the the previous equation we conclude that $\alpha_j = 1$ for all $j = 1, \ldots, i$. Hence, if $\alpha_i > 1$ the budget constraint is binding and $\mathbb{E}[w_\tau q_i] = B_i$ as required.

To complete the proof, we need to show that α_i is given by (42). With V_τ set to w_τ, Lemma 2 above implies

$$\mathbb{E}[w_\tau\, q_i] + \frac{1}{i+1} \sum_{j=i+1}^{N} \mathbb{E}[w_\tau\, q_j] \ = \ \int_{\alpha_i < \alpha_\tau} \frac{w_\tau}{i+1}\, (A_\tau - \alpha_i\, w_\tau) \mathrm{d}\mathbb{Q} \tag{A-6}$$

for $i = 1, \ldots, N$. But the budget constraints for the N retailers also imply

$$\mathbb{E}[w_\tau \, q_i] + \frac{1}{i+1} \sum_{j=i+1}^{N} \mathbb{E}[w_\tau \, q_j] \;\leq\; B_i + \frac{1}{i+1} \sum_{j=i+1}^{N} B_j \tag{A-7}$$

which, when combined with (A-6), leads to

$$\int_{\alpha_i < \alpha_\tau} w_\tau \, (A_\tau - \alpha_i \, w_\tau) \mathrm{d}\mathbb{Q} \;\leq\; (i+1) \, B_i + \sum_{j=i+1}^{N} B_j \tag{A-8}$$

for $i = 1, \ldots, N$. We can use (A-8) sequentially to determine the α_i's. Beginning at $i = N$, we see that the N^{th} retailer's budget constraint is equivalent to

$$\int_{\alpha_N < \alpha_\tau} w_\tau \, (A_\tau - \alpha_N \, w_\tau) \mathrm{d}\mathbb{Q} \;\leq\; (N+1) \, B_N. \tag{A-9}$$

The optimality condition on λ_i implies that it is the smallest non-negative real that satisfies the i^{th} budget constraint. Since the optimal λ_i's are non-decreasing in i, we see from (A-7) that α_i is therefore the smallest real greater than or equal to 1 satisfying the i^{th} budget constraint. Therefore, beginning with $i = N$ we can check if $\alpha_N = 1$ satisfies (A-9) and if it does, then we know the N^{th} budget constraint is not binding. If $\alpha_N = 1$ does not satisfy (A-9) then we set α_N equal to that value (greater than one) that makes (A-9) an equality. In particular, we obtain that the optimal value of α_N is $H((N+1)B_N)$, as desired.

Note that if $\alpha_N = 1$ then none of the budget constraints are binding. In particular, this implies $\alpha_i = 1$ and $\lambda_i = 0$ for all $i = 1, \ldots N$. Moreover, (42) must be satisfied for all i since it is true for $i = N$ and since the B_i's are decreasing. Suppose now that the budget constraint is binding for retailers $i+1, \ldots, N$ and consider the i^{th} retailer. Then the i^{th} retailer's budget constraint is equivalent[14] to (A-8) and we can again use precisely the same argument as before to argue that (42) holds. $\square$

<u>Proof of Proposition 13:</u> To compute the value of Π_{P} we will compute the expected revenue $\mathbb{E}[w_\tau \, Q]$ and expected cost, $c\mathbb{E}[Q]$, separately. The first step is to determine those retailers that will be using their entire budgets in the Cournot equilibrium. We know from Proposition 12 and the definition of m in (45) that only the budget constraints of the first $m-1$ retailers will not be binding and so $\mathbb{E}[w_\tau \, q_i] = B_i$ for $i = m, m+1, \ldots, N$. It also follows that $\lambda_1 = \lambda_2 = \cdots = \lambda_{m-1} = 1$ and so (35) implies that

$$q_i = (A_\tau - Q - w_\tau)^+, \qquad i = 1, \ldots, m-1. \tag{A-10}$$

Using this identity we obtain

$$\begin{aligned}
\mathbb{E}[w_\tau \, Q] &= \sum_{j=1}^{m-1} \mathbb{E}[w_\tau \, q_j] + \sum_{j=m}^{N} \mathbb{E}[w_\tau \, q_j] \\
&= (m-1) \, \mathbb{E}[w_\tau \, (A_\tau - Q - w_\tau)^+] + \sum_{j=m}^{N} B_j \\
&= (m-1) \, \mathbb{E}[w_\tau \, (A_\tau - w_\tau)^+ - w_\tau \, Q] + \sum_{j=m}^{N} B_j
\end{aligned} \tag{A-11}$$

[14] Equivalence follows because the second terms on either side of the inequality sign in (A-7) are equal by assumption.

where we have used the observation that $(A_\tau - Q - w_\tau)^+ = (A_\tau - w_\tau)^+ - Q$. This observation follows because (i) if $A_\tau \leq w_\tau$ then by (34) $Q = 0$ and (ii) if $Q > 0$ then $A_\tau \geq w_\tau$ and we can argue using (35), say, that $(A_\tau - Q - w_\tau)^+ = A_\tau - Q - w_\tau$. We can now re-arrange (A-11) to obtain

$$\mathbb{E}[w_\tau\, Q] = \frac{1}{m}\left(\sum_{j=m}^{N} B_j + (m-1)\,\mathbb{E}[w_\tau\,(A_\tau - w_\tau)^+]\right). \tag{A-12}$$

In order to calculate the expected cost, we can use Lemma 2 with $V_\tau \equiv 1$ to see that for any i we have

$$(i+1)\,\mathbb{E}[q_i] \;+\; \sum_{j=i+1}^{N} \mathbb{E}[q_j] \;=\; \mathbb{E}[(A_\tau - \alpha_i\, w_\tau)^+]. \tag{A-13}$$

(A-13) then defines a system of N linear equations in the N unknowns, $\mathbb{E}[q_i], \quad i = 1, \ldots, N$. If we let $M = [M_{ij}]$ be the $N \times N$ upper-triangular matrix defined as

$$M_{ij} := \begin{cases} 0 & \text{if } i > j \\ i+1 & \text{if } i = j \\ 1 & \text{if } i < j \end{cases}$$

then it is easy to check that $[M_{ij}^{-1}] = 1_{\{j=i\}}/(j+1) - 1_{\{j>i\}}/(j(j+i))$. The system (A-13) then implies

$$\begin{aligned} \sum_{i=1}^{N} \mathbb{E}[q_i] \;&=\; \sum_{i=1}^{N}\sum_{j=1}^{N} [M_{ij}^{-1}]\,\mathbb{E}[(A_\tau - \alpha_j\, w_\tau)^+] \\ &=\; \sum_{j=1}^{N} \frac{1}{j(j+1)}\,\mathbb{E}[(A_\tau - \alpha_j\, w_\tau)^+] \end{aligned} \tag{A-14}$$

and so we obtain

$$\mathbb{E}[cQ] \;=\; \frac{(m-1)c}{m}\,\mathbb{E}[(A_\tau - w_\tau)^+] \;+\; \sum_{j=m}^{N} \frac{c}{j(j+1)}\,\mathbb{E}[(A_\tau - \alpha_j\, w_\tau)^+]. \tag{A-15}$$

where we have used the fact that $\alpha_1 = \ldots = \alpha_{m-1} = 1$. We can now combine (A-12) and (A-15) to obtain (44) as desired. $\square$

<u>PROOF OF PROPOSITION 14</u>: Suppose we fix the value of m. Using Definition 5.1 and our assumption that $n_\tau = N$ (a.s.), it follows that fixing the value of m is equivalent to imposing the condition $\alpha_{m-1} = 1 < \alpha_m$ or

$$\mathcal{B}_m < \mathbb{E}[w_\tau\,(A_\tau - w_\tau)] \leq \mathcal{B}_{m-1}.$$

In addition, we can calculate the value of α_j for all $j = m, \ldots, N$ by solving $\alpha_j = H(\mathcal{B}_j)$ and using our assumption that $A_\tau \geq \alpha_j\, w_\tau$ (a.s.). It follows that

$$\alpha_j = \frac{\mathbb{E}[A_\tau\, w_\tau] - \mathcal{B}_j}{\mathbb{E}[w_\tau^2]}, \qquad j = m, \ldots, N.$$

Plugging these values of α_j in (44), the producer's problem for a fix m is given by

$$\max_{w_\tau} \left(\frac{m-1}{m}\right) \mathbb{E}[A_\tau\, w_\tau + c\, w_\tau - w_\tau^2 - c\, A_\tau] \;+\; \sum_{j=m}^{N} \left(\frac{B_j}{m} \;-\; \frac{c}{j(j+1)}\, \mathbb{E}\left[A_\tau - \left(\frac{\mathbb{E}[A_\tau\, w_\tau] - \mathcal{B}_j}{\mathbb{E}[w_\tau^2]}\right) w_\tau\right]\right)$$
$$\text{subject to} \quad \mathcal{B}_m < \mathbb{E}[w_\tau\,(A_\tau - w_\tau)] \leq \mathcal{B}_{m-1}.$$

To solve this optimization problem, let us first fix the value of $\mathbb{E}[w] = \mu \geq c$ and $\mathbb{E}[w^2] = \sigma^2 + \mu^2$, for some scalars μ and σ to be determined later. Then, the producer objective function can be written as

$$\left(\frac{m-1}{m}\right) [c\,\mu - \sigma^2 - \mu^2] - \frac{c\,\mu}{\sigma^2+\mu^2} \sum_{j=m}^{N} \frac{\mathcal{B}_j}{j(j+1)} + \left(\frac{m-1}{m} + \frac{\mu}{\sigma^2+\mu^2} \sum_{j=m}^{N} \frac{c}{j(j+1)}\right) \mathbb{E}[A_\tau\, w_\tau] + \delta(m)$$

where $\delta(m) := \sum_{j=m}^{N} \frac{B_j}{m} - \frac{N}{N+1}\, c\, \mathbb{E}[A_\tau]$. Since the term inside the parenthesis is non-negative, it follows that the producer's problem reduces to

$$\max_{w_\tau} \mathbb{E}[A_\tau\, w_\tau] \quad \text{s.t.} \quad \mathbb{E}[w] = \mu, \quad \mathbb{E}[w^2] = \sigma^2 + \mu^2 \quad \text{and} \quad \mathcal{B}_m < \mathbb{E}[w_\tau\,(A_\tau - w_\tau)] \leq \mathcal{B}_{m-1}. \tag{A-16}$$

Lemma 3 *Problem (A-16) is feasible if and only if* $\mathcal{B}_m < \mu\,(\mathbb{E}[A_\tau] - \mu) + \sigma\,(\sigma_{A_\tau} - \sigma)$, *where* $\sigma_{A_\tau} = \sqrt{\mathbb{V}\mathrm{ar}_0(A_\tau)}$. *If this feasibility condition is satisfied, then at optimality*

$$\mathbb{E}[A_\tau\, w_\tau] = \min\left\{\mu\, \mathbb{E}_0[A_\tau] + \sigma\, \sigma_{A_\tau}\;,\; \mathcal{B}_{m-1} + \mu^2 + \sigma^2\right\}.$$

Furthermore, if $\mu\, \mathbb{E}_0[A_\tau] + \sigma\, \sigma_{A_\tau} \leq \mathcal{B}_{m-1} + \mu^2 + \sigma^2$ *(i.e., if an interior solution exists) then an optimal solution is given by*

$$w_\tau = \mu + \frac{\sigma}{\sigma_{A_\tau}}\,(A_\tau - \mathbb{E}[A_\tau]).$$

Proof of Lemma 3: Consider the following relaxed version of the optimization problem A-16:

$$\max_{w_\tau} \mathbb{E}[A_\tau\, w_\tau] \quad \text{s.t.} \quad \mathbb{E}[w] = \mu, \quad \text{and} \quad \mathbb{E}[w^2] = \sigma^2 + \mu^2.$$

We can solve this problem using Lagrangian relaxation and solving for w_τ pathwise. After some straightforward calculation we get that an optimal solution is given by

$$w_\tau = \mu + \frac{\sigma}{\sigma_{A_\tau}}\,(A_\tau - \mathbb{E}[A_\tau]).$$

Given this solution, let us check it feasibility on the original problem (A-16), that is, we need to check the constraint $\mathcal{B}_m < \mathbb{E}[w_\tau\,(A_\tau - w_\tau)] \leq \mathcal{B}_{m-1}$ or equivalently $\mathcal{B}_m + \mu^2 + \sigma^2 < \mathbb{E}[A_\tau\, w_\tau] \leq \mathcal{B}_{m-1} + \mu^2 + \sigma^2$. Since, at optimality $\mathbb{E}[A_\tau\, w_\tau] = \mu\, \mathbb{E}[A_\tau] + \sigma\, \sigma_{A_\tau}$, we conclude that problem (A-16) is infeasible if $\mu\, \mathbb{E}[A_\tau] + \sigma\, \sigma_{A_\tau} < \mathcal{B}_m + \mu^2 + \sigma^2$. Otherwise, the optimal value is equal to

$$\mathbb{E}[A_\tau\, w_\tau] = \min\left\{\mu\, \mathbb{E}_0[A_\tau] + \sigma\, \sigma_{A_\tau}\;,\; \mathcal{B}_{m-1} + \mu^2 + \sigma^2\right\}. \quad \square$$

Using the result in the previous lemma, we can replace the value of $\mathbb{E}[A_\tau\, w_\tau]$ in producer's objective function and solve terms of μ and σ. The following Lemma shows that at optimality an optimal μ and σ always lead to an interior solution in the sense of Lemma 3.

Lemma 4 *At optimality, the values of μ and σ are such that*

$$\mu\,\mathbb{E}_0[A_\tau] + \sigma\,\sigma_{A_\tau} \le \mathcal{B}_{m-1} + \mu^2 + \sigma^2.$$

PROOF OF LEMMA 4: Suppose at optimality the solution is not an interior one, that is, $\mu\,\mathbb{E}_0[A_\tau] + \sigma\,\sigma_{A_\tau} > \mathcal{B}_{m-1} + \mu^2 + \sigma^2$ and so

$$\mathbb{E}[A_\tau\, w_\tau] = \mathcal{B}_{m-1} + \mu^2 + \sigma^2.$$

Plugging this value in the producer's optimization above we get

$$\Pi_{\mathrm{P}}(m) = \left(\frac{m-1}{m}\right)[c\,\mu + \mathcal{B}_{m-1}] + \frac{c\,\mu}{\sigma^2+\mu^2}\sum_{j=m}^{N}\frac{\mathcal{B}_{m-1}-\mathcal{B}_j}{j(j+1)} + \sum_{j=m}^{N}\frac{c\,\mu}{j(j+1)} + \delta(m).$$

Since $\mathcal{B}_{m-1} \ge \mathcal{B}_j$ for $j = m,\ldots,N$, it follows that $\Pi_{\mathrm{P}}(m)$ is decreasing in σ and so at optimality we would like to set $\sigma = 0$. But for $\sigma = 0$, we would have $w_\tau = \mu$ (a.s.) and, as a result, $\mu\,\mathbb{E}_0[A_\tau] + \sigma\,\sigma_{A_\tau} \le \mathcal{B}_{m-1} + \mu^2 + \sigma^2$. This last inequality follows from the fact that we are assuming that the $(m-1)^{\text{th}}$ retailer is not binding and therefore $\mathbb{E}[w_\tau\,(A_\tau - w_\tau)] \le \mathcal{B}_{m-1}$. We conclude that at optimality, we must have $\mu\,\mathbb{E}_0[A_\tau] + \sigma\,\sigma_{A_\tau} \le \mathcal{B}_{m-1} + \mu^2 + \sigma^2$. $\square$

Finally, we can combine the results in the previous two lemmas to complete the proof of Proposition 14. $\square$

PROOF OF PROPOSITION 15: The condition $n_\tau = N$ (a.s.) is equivalent to $\alpha_N\, w^*_\tau \le A_\tau$ (a.s.). The value of α_N depends on the value of m^*. If $m^* = N+1$ then we at optimality all retailers are not budget constrained and $\alpha_N = 1$. On the other hand, if $m^* < N+1$ then retailer N is budget constrained and the value of α_N is given by

$$\alpha_N = \frac{\mathbb{E}[A_\tau\, w^*_\tau] - \mathcal{B}_N}{\mathbb{E}[(w^*_\tau)^2]} = \left(\frac{\mu^*\,\mathbb{E}[A_\tau] + \sigma^*\,\sigma_{A_\tau} - (N+1)\,B_N}{(\mu^*)^2 + (\sigma^*)^2}\right),$$

where in the second equality we have used the identity $\mathcal{B}_N = (N+1)\,B_N$ from equation (7). This complete the proof. $\square$

PROOF OF COROLLARY 1: Using the definition of α_j for $j \ge m$ and assuming a constant $\bar{w}$, we see that the expectation $\mathbb{E}[(A_\tau - \alpha_j\, w_\tau)^+]$ in equation (44) can be replaced by $((j+1)B_j + B_{j+1} + \cdots + B_N)/\bar{w}$. The rest of the derivation of (47) follows directly after some simple calculations.

Let us now prove the second part. For notational convenience, we will write $\bar{w}^* := \bar{w}^*(B_1,\ldots,B_N)$ and $\bar{w}^\infty := \bar{w}^*(\infty,\ldots,\infty)$. Note that the producer payoff can be rewritten as follows

$$\Pi_{\mathrm{P}}(\bar{w}) = \frac{m-1}{m}\,\Pi^\infty_{\mathrm{P}}(\bar{w}) + \left(1 - \frac{c}{\bar{w}}\right)\sum_{j=m}^{N}\frac{B_j}{m}, \tag{A-17}$$

where $\Pi^\infty_{\mathrm{P}}(\bar{w}) := (\bar{w} - c)\,\mathbb{E}[(A_\tau - \bar{w})^+]$ is the producer payoff when each retailers has an infinity budget (modulo the constant $(N-1)/N$). It follows that $\bar{w}^\infty$ is the unique maximizer of $\Pi^\infty_{\mathrm{P}}(\bar{w})$. The uniqueness of $\bar{w}^\infty$ follows from the fact that $\Pi^\infty_{\mathrm{P}}(\bar{w})$ is unimodal. Hence, both $\Pi^\infty_{\mathrm{P}}(\bar{w})$ and

$(1-c/\bar{w})$ are increasing functions of $\bar{w}$ in $[c, \bar{w}^{\infty})$. This observation together with the facts that m is a piece-wise constant function of $\bar{w}$ and $\Pi_{\mathrm{P}}(\bar{w})$ is a continuous function of $\bar{w}$ imply that $\bar{w}^* \geq \bar{w}^{\infty}$. The continuity of $\Pi_{\mathrm{P}}^{\infty}(\bar{w})$ is not immediate since m is a discontinuous function of $\bar{w}$. However, checking this property is straightforward and is left to reader. $\square$

B Appendix: Extensions

We now briefly discuss some extensions to our basic model. We consider three extensions relating to foreign exchange risk, interest rate risk and the preferred timing of the contract.

B.1 Retailers Based in a Foreign Currency Area

We now assume that the retailers and producer are located in different currency areas and use change-of-numeraire arguments to show that our analysis still[15] goes through. Without any loss of generality, we will assume that the retailers and producer are located in the "foreign" and "domestic" currency areas, respectively. The exchange rate, Z_t say, denotes the time t domestic value of one unit of the foreign currency. When the producer proposes a contract, w_τ, we assume that he does so in units of the foreign currency. Moreover, in order to simplify the exposition we will only consider here the case of identical budgets. Therefore the i^{th} retailer pays $q_i w_\tau$ units of foreign currency[16] to the producer. The retailers' problem is therefore unchanged from the problem we considered in Section 4 if we take $\mathbb{Q}$ to be an EMM of a *foreign* investor who takes the foreign cash account as his numeraire security. As explained in Appendix B.4, this same $\mathbb{Q}$ can also be used by the producer as a *domestic* EMM where he takes the *domestic* value of the *foreign* cash account as the numeraire security.

We could take our financial process, X_t, to be equivalent to Z_t so that the retailers hedge their foreign exchange risk in order to mitigate the effects of their budget constraints. This would only make sense if A_τ and the exchange rate, Z_t, were dependent. More generally, we could allow X_t to be multi-dimensional so that it includes Z_t as well as other tradeable financial processes that influence A_τ.

The producer must convert the retailers' payments into units of the domestic currency and he therefore earns a per-unit profit of either (i) $w_\tau Z_\tau - c$ if production costs are in units of the domestic currency or (ii) $Z_\tau(w_\tau - c)$ if production costs are in units of the foreign currency. Case (i) would apply if production takes place domestically whereas case (ii) would apply if production takes place in the foreign currency area. For ease of exposition we will assume that interest rates in both the domestic and foreign currency areas are identically zero.

Analogously to (24) and (25) we find in the equibudget case that the producer's problem in case

[15] See Ding et al. (2007) for a comprehensive review of the literature discussing exchange rate uncertainty in a production/inventory context.

[16] Which is the domestic currency from the retailers' perspective.

(i) is given by

$$\Pi_{\mathrm{P}} \;=\; \max_{w_\tau,\,\lambda\geq 0} \;\; Z_0 \frac{2N}{N+1} \;\; \mathbb{E}\left[\frac{(w_\tau Z_\tau - c)}{Z_\tau}\,\frac{(A_\tau - w_\tau\,(1+\lambda))^+}{2}\right] \tag{B-18}$$

$$\text{subject to} \qquad \mathbb{E}\left[w_\tau\,\frac{(A_\tau - w_\tau\,(1+\lambda))^+}{2}\right] \;\leq\; \frac{(N+1)}{2}B. \tag{B-19}$$

Note that Z_τ appears in the denominator inside the expectation in (B-18) because, as explained above, the domestic value of the foreign cash account is the appropriate numeraire corresponding to the EMM, $\mathbb{Q}$. Since we have assumed interest rates are identically zero, the foreign value of the foreign cash-account is identically one and so its domestic value is Z_t at time t. For the same reason, Z_0 appears outside the expectation in (B-18). Solving the producer's problem in (B-18) and (B-19) is equivalent to solving the problem he faced earlier in this section but now with a *stochastic* cost, $c_\tau := c/Z_\tau$. However, it can easily be seen that the proof of Proposition 7, or more to the point, Proposition 7 in Caldentey and Haugh (2009), goes through unchanged when c is stochastic. We therefore obtain the same result as Proposition 7 with c replaced by c_τ and $\mathbb{Q}$ interpreted as a foreign EMM with the domestic value of the foreign cash account as the numeraire security.

Remark: If instead case (ii) prevailed so that the producer's per-unit profit was $Z_\tau(w_\tau - c)$ then the Z_τ term in both the numerator and denominator of (B-18) would cancel, leaving the producer with an identical problem to that of Section 4 albeit with different EMMs. So while the analysis for case (ii) is identical to that of Section 4, the probability measures under which the solutions are calculated are different.

B.2 Stochastic Interest Rates and Paying the Producer in Advance

At the end of Section 2.1 we noted that there was no loss in generality in assuming that production did not take place until after time τ and that profits were realized at time T with the maximum size of the market being a random variable A that depended on both X_T and some non-financial noise. We will take that interpretation here so that $A_\tau = \mathbb{E}_\tau[A]$.

We now consider the problem where the retailers' budgets are only available at time T but that the producer must be paid at time $\tau < T$. Once again we only consider the case of identical budgets. We will assume that interest rates are stochastic and no longer identically zero so that the retailers' *effective* time τ budgets are also stochastic. In particular, we will assume that the $\mathbb{Q}$-dynamics of the short rate are given by the Vasicek[17] model so that

$$dr_t \;=\; \alpha(\mu - r_t)\; dt + \sigma dW_t \tag{B-20}$$

where α, μ and σ are all positive constants and W_t is a $\mathbb{Q}$-Brownian motion. The short-rate, r_t, is the instantaneous continuously compounded risk-free interest rate that is earned at time t by the 'cash account', *i.e.*, cash placed in a deposit account. In particular, if \$1 is placed in the cash

[17]See, Duffie (2004) for a description of the Vasicek model and other related results that we use in this subsection. Note that it is not necessary to restrict ourselves to the Vasicek model. We have done so in order to simplify the exposition but our analysis holds for more general models such as the multi-factor Gaussian and CIR processes that are commonly employed in practice.

account at time t then it will be worth $\exp\left(\int_t^T r_s\, ds\right)$ at time $T > t$. It may be shown that the time τ value of a zero-coupon-bond with face value \$1 that matures at time $T > \tau$ satisfies

$$Z_\tau^T \;:=\; e^{a(T-\tau)+b(T-\tau)r_\tau} \tag{B-21}$$

where $a(\cdot)$ and $b(\cdot)$ are known deterministic functions of the time-to-maturity, $T-\tau$. In particular, Z_τ^T is the appropriate discount factor for discounting a known deterministic cash flow from time T to time $\tau < T$.

Returning to our competitive supply chain, we assume that the retailers' profits are realized at time $T \geq \tau$. Since the producer now demands payment from the retailers at time τ when production amounts are determined, this implies that the retailers will be forced to borrow against the capital B that is not available until time T. As a result, the i^{th} retailer's effective budget at time τ is given by

$$B_i(r_\tau) \;:=\; B_i Z_\tau^T \;=\; B_i e^{a(T-\tau)+b(T-\tau)r_\tau}.$$

We assume that the stochastic clearance price, $A-Q$, depends on the financial market through the co-dependence of the random variable A, and the financial process, X_t. To simplify the exposition, we could assume that $X_t \equiv r_t$ but this is not necessary. If X_t is a financial process other than r_t, we simply need to redefine our definition of $\{\mathcal{F}_t\}_{0\leq t\leq T}$ so that it represents the filtration generated by X_t *and* r_t. Before formulating the optimization problems of the retailers and the producer we must adapt our definition of feasible $\mathcal{F}_\tau$-measurable financial gains, G_τ. Until this point we have insisted that any such G_τ must satisfy $\mathbb{E}[G_\tau] = 0$, assuming as before that zero initial capital is devoted to the financial hedging strategy. This was correct when interest rates were identically zero but now we must replace that condition with the new condition[18]

$$\mathbb{E}\left[D_\tau G_\tau\right] \;=\; 0 \tag{B-22}$$

where $D_\tau := \exp\left(-\int_0^\tau r_s\, ds\right)$ is the stochastic discount factor. The i^{th} retailer's problem for a given $\mathcal{F}_\tau$-measurable wholesale price, w_τ, is therefore given[19] by

$$\Pi_{\mathrm{R}}(w_\tau) \;=\; \max_{q_i\geq 0,\, G_\tau} \;\mathbb{E}\left[D_T\left(A_T - \;(q_i+Q_{i-})\right) q_i - D_\tau w_\tau q_i\right] \tag{B-23}$$

$$\text{subject to} \qquad w_\tau\, q_i \leq B_i(r_\tau) + G_\tau, \qquad \text{for all } \omega\in\Omega \tag{B-24}$$

$$\mathbb{E}\left[D_\tau G_\tau\right] \;=\; 0 \tag{B-25}$$

$$\text{and } \mathcal{F}_\tau - \text{measurability of } q_i. \tag{B-26}$$

Note that both D_T and D_τ appear in the objective function (B-23) and reflect the times at which the retailer makes and receives payments. We also explicitly imposed the constraint that q_i be $\mathcal{F}_\tau$-measurable. This was necessary because of the appearance of D_T in the objective function. We can easily impose the $\mathcal{F}_\tau$-measurability of q_i by conditioning with respect to $\mathcal{F}_\tau$ inside the expectation appearing in (B-23). We then obtain

$$\mathbb{E}\left[D_\tau\left(A_\tau^{\mathrm{D}} - (q_i+Q_{i-}) - w_\tau\right) q_i\right] \tag{B-27}$$

as our new objective function where $A_\tau^{\mathrm{D}} := \mathbb{E}_\tau[D_T A]/D_\tau$. With this new objective function it is no longer necessary to explicitly impose the $\mathcal{F}_\tau$-measurability of q_i.

[18]See the first paragraph of Appendix B.4 for why this is the case.

[19]We write A_T for A to emphasize the timing of the cash-flow.

It is still straightforward to solve for the retailers' Cournot equilibrium. One could either solve the problem directly as before or alternatively, we could use the change-of-numeraire method of Section B.1 that is described in Appendix B.4. In particular, we could switch to the so-called *forward measure* where the EMM, $\mathbb{Q}_\tau$, now corresponds to taking the zero-coupon bond maturing at time τ as the numeraire. In that case the i^{th} retailer's objective function in (B-27) can be written[20] as

$$Z_0^\tau \, \mathbb{E}^{\mathbb{Q}_\tau}\left[\left(A_\tau^{\mathrm{D}} - (q_i + Q_{i-}) - w_\tau\right) q_i\right]. \tag{B-28}$$

We can therefore solve for the retailers' Cournot equilibrium using our earlier analysis but with A_τ and $\mathbb{Q}$ replaced by A_τ^{D} and $\mathbb{Q}_\tau$, respectively. Note that the constant factor, Z_0^τ, in (B-28) is the same for each retailer and therefore makes no difference to the analysis. Following the first approach we obtain the following solution to the retailer's problem. We omit the proof as it is very similar to the proof of Proposition 6.

Proposition 16 (Retailers' Optimal Strategy)
Let w_τ be an $\mathcal{F}_\tau$-measurable wholesale price offered by the producer and define $\mathcal{Q}_\tau := \frac{(A_\tau^D - w_\tau)^+}{(N+1)\, Z_\tau^T}$. This is the optimal ordering quantity for each retailer in the absence of any budget constraints. The following two cases arise:

> **Case 1***: Suppose $\mathbb{E}\left[D_\tau \mathcal{Q}_\tau w_\tau\right] \leq \mathbb{E}\left[D_\tau B(r_\tau)\right] = Z_0^T B$. Then $q_i(w_\tau) := q_\tau := \mathcal{Q}_\tau$ for all i and (possibly due to the ability to trade in the financial market) the budget constraints are not binding.*
>
> **Case 2***: Suppose $\mathbb{E}\left[D_\tau \mathcal{Q}_\tau w_\tau\right] > Z_0^T B$. Then*
>
> $$q_i(w_\tau) := q_\tau := \frac{\left(A_\tau^D - w_\tau\,(1+\lambda)\right)^+}{(N+1) Z_\tau^T} \quad \textit{for all} \;\; i = 1, \ldots, N \tag{B-29}$$
>
> *where $\lambda \geq 0$ solves*
>
> $$\mathbb{E}\left[D_\tau w_\tau q_\tau\right] \;=\; \mathbb{E}\left[B(r_\tau) D_\tau\right] \;=\; Z_0^T B. \tag{B-30}$$

Given the retailers' best response, the producer's problem may now be formulated[21] as

$$\Pi_{\mathrm{P}} = \max_{w_\tau,\, \lambda \geq 0} \; \mathbb{E}\left[D_\tau (w_\tau - c)\, \frac{\left(A_\tau^{\mathrm{D}} - w_\tau\,(1+\lambda)\right)^+}{(N+1)\, Z_\tau^T}\right] \tag{B-31}$$

$$\text{subject to} \qquad \mathbb{E}\left[D_\tau w_\tau\, \frac{\left(A_\tau^{\mathrm{D}} - w_\tau\,(1+\lambda)\right)^+}{(N+1)\, Z_\tau^T}\right] \leq Z_0^T B. \tag{B-32}$$

The Cournot-Stackelberg equilibrium and solution of the producer's problem in the equibudget case is given by the following proposition. We again omit the proof of this proposition as it it very similar to the proof of Proposition 7.

Proposition 17 (The Equilibrium Solution)
Let ϕ_P be the minimum $\phi \geq 1$ that satisfies $\mathbb{E}\left[\frac{D_\tau}{8 Z_\tau^T}\; \left((A_\tau^D)^2 - (\phi c)^2\right)^+\right] \;\leq\; \frac{(N+1)}{2} Z_0^T B$ and let $\delta_P := \phi_P c$. Then the optimal wholesale price and ordering level satisfy

$$w_\tau = \frac{\delta_P + A_\tau^D}{2} \quad \textit{and} \quad q_\tau = \frac{(A_\tau^D - \delta_P)^+}{2(N+1) Z_\tau^T}.$$

[20] The condition (B-25) can also be written in terms of $\mathbb{Q}_\tau$ as $Z_0^\tau\, \mathbb{E}_0^{\mathbb{Q}_\tau}\left[G_\tau\right] = 0$, i.e., $\mathbb{E}_0^{\mathbb{Q}_\tau}\left[G_\tau\right] = 0$.

[21] We assume here and in the foreign retailer setting that the production costs, c, are paid at time τ.

B.3 On the Optimal Timing of the Contract

As we have seen, the financial markets add value to the supply chain by allowing the retailers to mitigate their budget constraints via dynamic trading. The next proposition emphasizes the value of information in a competitive supply chain. Under the assumption of zero marginal production costs, i.e. $c = 0$, it states that for an $\mathcal{F}_{\tau_1}$-measurable price menu, w_τ, the producer is always better off when the retailers' orders are allowed to be contingent upon time τ_2 information where $\tau_2 > \tau_1$.

Proposition 18 *Consider two times $\tau_1 < \tau_2$ and let w_τ be an $\mathcal{F}_{\tau_1}$-measurable price menu. Consider the following two scenarios: (1) the producer offers price menu w_τ and the retailers choose their Cournot-optimal $\mathcal{F}_{\tau_1}$-measurable ordering quantities which is then produced at time τ_1 and (2) the producer again offers price menu w_τ but the retailers now choose their Cournot-optimal $\mathcal{F}_{\tau_2}$-measurable ordering quantities which is then produced at time τ_2. If $c_{\tau_1} = c_{\tau_2} = 0$ then the producer always prefers scenario (2).*

<u>Proof of Proposition 18:</u> Let m_j, $\alpha_i^{(j)}$ for $i = 1, \ldots, N$ and $j = 1, 2$ denote the usual Cournot optimal quantities for scenario j and retailer i. Since $\mathbb{E}_{\tau_1}[A_{\tau_2}]/x = A_{\tau_1}/x$ for any $x > 0$ Jensen's Inequality implies

$$\mathbb{E}[w_\tau\,(A_{\tau_2}/x - w_\tau)^+] \;\geq\; \mathbb{E}[w_\tau\,(A_{\tau_1}/x - w_\tau)^+]. \tag{B-33}$$

After multiplying across (B-33) by x it then follows from the definition of the α_i's in (42) that

$$\alpha_i^{(2)} \;\geq\; \alpha_i^{(1)} \quad \text{for } \; i = 1, \ldots, N.$$

This in turn implies that $m_2 \leq m_1$. Let $\Pi_{\mathrm{P}}^{(1)}$ and $\Pi_{\mathrm{P}}^{(2)}$ denote the producer's expected revenue in scenarios (1) and (2) respectively. Then (A-12) implies

$$\begin{aligned}
\Pi_{\mathrm{P}}^{(2)} - \Pi_{\mathrm{P}}^{(1)} \;&=\; \frac{(m_1 - m_2)}{m_1 m_2} \sum_{j=m_1}^{N} B_j \;+\; \frac{1}{m_2} \sum_{j=m_2}^{m_1-1} B_j \;+\; \frac{(m_2-1)}{m_2}\mathbb{E}[w_\tau\,(A_{\tau_2} - w_\tau)^+] \\
&\quad -\; \frac{(m_1-1)}{m_1}\mathbb{E}[w_\tau\,(A_{\tau_1} - w_\tau)^+] \\
&\geq\; \frac{(m_1 - m_2)}{m_1 m_2} \sum_{j=m_1}^{N} B_j \;+\; \frac{1}{m_2} \sum_{j=m_2}^{m_1-1} B_j \;+\; \frac{(m_2-m_1)}{m_1 m_2}\mathbb{E}[w_\tau\,(A_{\tau_1} - w_\tau)^+] \qquad (\text{(B-33) with } x = 1) \\
&=\; \frac{(m_1 - m_2)}{m_1 m_2} \left[\sum_{j=m_1}^{N} B_j \;-\; \mathbb{E}[w_\tau\,(A_{\tau_1} - w_\tau)^+] \right] \;+\; \frac{1}{m_2} \sum_{j=m_2}^{m_1-1} B_j \\
&\geq\; \frac{(m_1 - m_2)}{m_1 m_2} \left[\sum_{j=m_1}^{N} B_j \;-\; (m_1+1)B_{m_1} - \cdots - B_N \right] \;+\; \frac{1}{m_2} \sum_{j=m_2}^{m_1-1} B_j \qquad (\text{by def. of } m_1) \\
&=\; \frac{(m_2 - m_1)}{m_2} B_{m_1} \;+\; \frac{1}{m_2} \sum_{j=m_2}^{m_1-1} B_j.
\end{aligned} \tag{B-34}$$

Note that the right-hand-side of (B-34) equals zero if $m_1 = m_2$. Otherwise $m_2 < m_1$ and the right-hand-side of (B-34) is greater than or equal to $(m_1 - m_2)(B_{m_2} - B_{m_1 - 1})/m_2$ which in turn is non-negative. The result therefore follows. □

B.4 Martingale Pricing with Foreign Assets

Martingale pricing theory states that the time 0 value, G_0, of a security that is worth G_τ at time τ and does not pay any intermediate cash-flows, satisfies $G_0/N_0 = \mathbb{E}^{\mathbb{Q}}[G_\tau/N_\tau]$ where N_t is the time t price of the *numeraire* security and $\mathbb{Q}$ is an *equivalent martingale measure* (EMM) associated with that numeraire. It is common to take the cash account as the numeraire security and this is the approach we have followed in most of this paper. With the exception of the material in Section B.2, however, the value of the cash account at time t was always \$1 since we assumed interest rates were identically zero. We therefore had $G_0 = \mathbb{E}^{\mathbb{Q}}[G_\tau]$ and since we insisted $G_0 = 0$ we obtained $\mathbb{E}^{\mathbb{Q}}[G_\tau] = 0$. When interest rates are non-zero we still have $N_0 = 1$ but now $N_\tau = \exp\left(\int_0^\tau r_s \, ds\right)$ and so, for example, we have (B-22). In the main text we take $D_t = N_t^{-1}$. See Duffie (2004) for a development of martingale pricing theory.

Suppose now that there is a *domestic* currency and a *foreign* currency with Z_t denoting the exchange rate between the two currencies at time t. In particular, Z_tY_t is the time t domestic currency value of a foreign asset that has a time t foreign currency value of Y_t. Let $B_t^{(f)}$ denote the time t value of the foreign cash account and let $\mathbb{Q}$ denote the EMM of a foreign investor taking the foreign cash account as numeraire. This implies

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{Y_T}{B_T^{(f)}}\right] = \frac{Y_t}{B_t^{(f)}} \tag{B-35}$$

for all $t \leq T$ and where we assume again that the asset with foreign currency value Y_t at time t does not[22] pay any intermediate cash flows. But equation (B-35) can be re-written as

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{Z_TY_T}{Z_TB_T^{(f)}}\right] = \frac{Z_tY_t}{Z_tB_t^{(f)}}. \tag{B-36}$$

Note that Z_tY_t is the domestic currency value of the foreign asset and $Z_tB_t^{(f)}$ is the domestic currency value of the foreign cash account. Note also that if V_t is the time t price of a domestic asset then V_t/Z_t is the foreign currency value of the asset at time t. We therefore obtain by martingale pricing that

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{V_T/Z_T}{B_T^{(f)}}\right] = \frac{V_t/Z_t}{B_t^{(f)}}$$

or equivalently,

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{V_T}{Z_TB_T^{(f)}}\right] = \frac{V_t}{Z_tB_t^{(f)}} \tag{B-37}$$

We can therefore conclude from (B-36) and (B-37) that $\mathbb{Q}$ is also the EMM of a domestic investor, but now with the *domestic* value of the foreign cash account as the corresponding.

[22]If it did pay intermediate cash-flows between t and T then they would have to be included inside the expectation in (B-35).