IEOR E4602: Quantitative Risk Management

Basic Concepts and Techniques of Risk Management

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References: Chapter 2 of 2^{nd} ed. of MFE by McNeil, Frey and Embrechts.

Outline

Risk Factors and Loss Distributions

Notation (to be used throughout the course):

- Δ a fixed period of time such as 1 day or 1 week.
- Let V_t be the value of a portfolio at time $t\Delta$.
- So portfolio *loss* between $t\Delta$ and $(t+1)\Delta$ is given by

$$L_{t+1} \; := -\left(V_{t+1} - V_t\right)$$

  - note that a loss is a **positive** quantity
  - it (of course) depends on change in values of the securities.

More generally, may wish to define a set of d risk factors

$$\mathbf{Z_t} := (Z_{t,1}, \ldots, Z_{t,d})$$

so that

$$V_t \;=\; f(t, \mathbf{Z_t}).$$

for some function $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$.

Risk Factors and Loss Distributions

e.g. In a stock portfolio might take the stock prices or some function of the stock prices as our risk factors.

e.g. In an options portfolio $\mathbf{Z_t}$ might contain stock factors together with implied volatility and interest rate factors.

Let $\mathbf{X_t} := \mathbf{Z_t} - \mathbf{Z_{t-1}}$ denote the change in values of the risk factors between times t and $t+1$.

Then have

$$L_{t+1}(\mathbf{X_{t+1}}) \; = \; -\left(f(t+1, \mathbf{Z_t} + \mathbf{X_{t+1}}) - f(t, \mathbf{Z_t})\right)$$

Given the value of $\mathbf{Z_t}$, the distribution of L_{t+1} depends **only on the distribution of X_{t+1}.**

Estimating the (conditional) distribution of X_{t+1} is then a very important goal in much of risk management.

Linear Approximations to the Loss Function

Assuming $f(\cdot,\cdot)$ is differentiable, can use a first order Taylor expansion to approximate L_{t+1}:

$$\hat{L}_{t+1}(\mathbf{X_{t+1}}) \; := \; -\left(f_t(t, \mathbf{Z_t})\Delta + \sum_{i=1}^{d} f_{z_i}(t, \mathbf{Z_t}) \; X_{t+1,i} \right) \tag{1}$$

where f-subscripts denote partial derivatives.

First order approximation commonly used when X_{t+1} is likely to be small

- often the case when Δ is small, e.g. $\Delta = 1/365 \equiv 1$ day, and market not too volatile.

Second and higher order approximations also based on Taylor's Theorem can also be used.

Important to note, however, that if X_{t+1} is likely to be very large then **Taylor approximations can fail.**

Conditional and Unconditional Loss Distributions

Important to distinguish between the conditional and unconditional loss distributions.

Consider the series $\mathbf{X_t}$ of risk factor changes and assume that they form a stationary time series with stationary distribution $F_{\mathbf{X}}$.

Let $\mathcal{F}_t$ denote all information available in the system at time t including in particular $\{\mathbf{X_s} : s \leq t\}$.

Definition: The **unconditional** loss distribution is the distribution of L_{t+1} given the time t composition of the portfolio and assuming the CDF of $\mathbf{X_{t+1}}$ is given by $F_{\mathbf{X}}$.

Definition: The **conditional** loss distribution is the distribution of L_{t+1} given the time t composition of the portfolio and conditional on the information in $\mathcal{F}_t$.

Conditional and Unconditional Loss Distributions

If the $\mathbf{X_t}$'s are *IID* then the conditional and unconditional distributions coincide.

For long time horizons, e.g. $\Delta = 6$ months, we might be more inclined to use the unconditional loss distribution.

However, for short horizons, e.g. 1 day or 10 days, then the conditional loss distribution is clearly the appropriate distribution

- true in particular in times of high market volatility when the unconditional distribution would bear little resemblance to the true conditional distribution.

Example: A Stock Portfolio

Consider a portfolio of d stocks with $S_{t,i}$ denoting time t price of the i^{th} stock and λ_i denoting number of units of i^{th} stock.

Take log stock prices as risk factors so

$$X_{t+1,i} \;=\; \ln S_{t+1,i} - \ln S_{t,i}$$

and

$$L_{t+1} \;=\; -\sum_{i=1}^{d} \lambda_i S_{t,i} \left(e^{X_{t+1,i}} - 1 \right).$$

Linear approximation satisfies

$$\hat{L}_{t+1} \;=\; -\sum_{i=1}^{d} \lambda_i S_{t,i} X_{t+1,i} \;=\; -V_t \sum_{i=1}^{d} \omega_{t,i} X_{t+1,i}$$

where $\omega_{t,i} \;:=\; \lambda_i S_{t,i} / V_t$ is the i^{th} portfolio weight.

If $\mathsf{E}[X_{t+1,i}] = \mu$ and $\mathsf{Covar}(X_{t+1,i}) = \Sigma$, then $\mathsf{E}_t\left[\hat{L}_{t+1}\right] = \; -V_t\, \omega' \mu$ and $\mathsf{Var}_t\left(\hat{L}_{t+1}\right) = \; V_t^2\; \omega' \Sigma \omega$.

Example: An Options Portfolio

Recall the Black-Scholes formula for time t price of a European call option with strike K and maturity T on a non-dividend paying stock satisfies

$$C(S_t, t, \sigma) \;=\; S_t \Phi(d_1) \;-\; e^{-r(T-t)} K \Phi(d_2)$$

$$\text{where} \quad d_1 \;=\; \frac{\log\left(\frac{S_t}{K}\right) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 \;=\; d_1 - \sigma\sqrt{T-t}$$

and where:

- $\Phi(\cdot)$ is the standard normal distribution CDF
- $S_t =$ time t price of underlying security
- $r =$ continuously compounded risk-free interest rate.

In practice use an implied volatility, $\sigma(K, T, t)$, that depends on strike, maturity and current time, t.

Example: An Options Portfolio

Consider a portfolio of European options all on the same underlying security.

If the portfolio contains d different options with a position of λ_i in the i^{th} option, then

$$L_{t+1} = -\lambda_0(S_{t+1} - S_t) - \sum_{i=1}^{d} \lambda_i \left(C(S_{t+1}, t+1, \sigma(K_i, T_i, t+1) - C(S_t, t, \sigma(K_i, T_i, t)) \right)$$

where λ_0 is the position in the underlying security.

Note that by put-call parity we can assume that all options are call options.

Can also use linear approximation technique to approximate L_{t+1}
- would result in a delta-vega-theta approximation.

For derivatives portfolios, the linear approximation based on 1^{st} order Greeks is often inadequate
- 2^{nd} order approximations involving gamma, volga and vanna might then be used – but see earlier warning regarding use of Taylor approximations.

Risk Factors in the Options Portfolio

Can again take $\log$ stock prices as risk factors but not clear how to handle the implied volatilities.

There are several possibilities:

1. Assume the $\sigma(K, T, t)$'s simply do not change
   - not very satisfactory but commonly assumed when historical simulation is used to approximate the loss distribution and historical data on the changes in implied volatilities are not available.

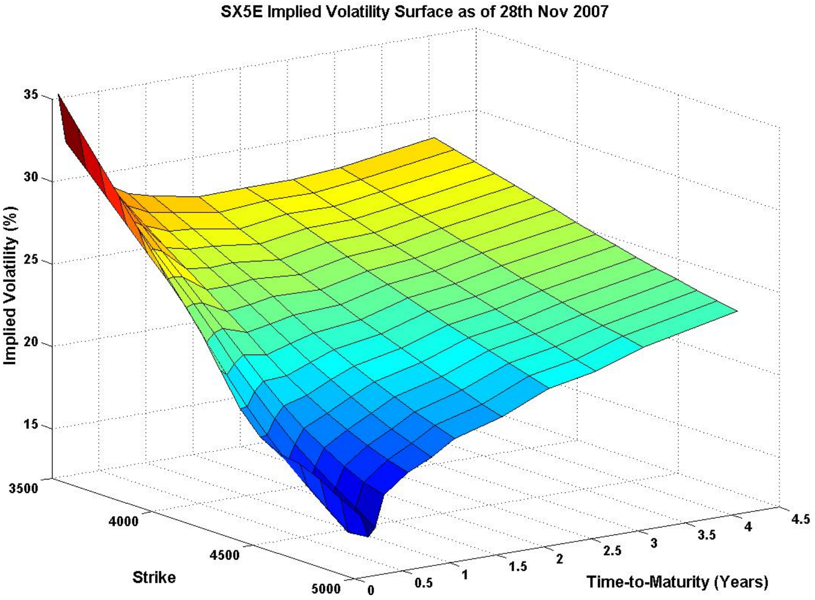
2. Let each $\sigma(K, T, t)$ be a separate factor. Not good for two reasons:
   (a) It introduces a large number of factors.
   (b) Implied volatilities are **not** free to move independently since no-arbitrage assumption imposes strong restrictions on how volatility surface may move.

Therefore important to choose factors in such a way that no-arbitrage restrictions are easily imposed when we estimate the loss distribution.

Risk Factors in the Options Portfolio

3. In light of previous point, it may be a good idea to parameterize the volatility surface with just a few parameters
   - and assume that only those parameters can move from one period to the next
   - parameterization should be so that no-arbitrage restrictions are easy to enforce.

4. Use dimension reduction techniques such as principal components analysis (PCA) to identify just two or three factors that explain most of the movements in the volatility surface.


SX5E Implied Volatility Surface as of 28th Nov 2007
Implied Volatility (%)
35
30
25
20
15
3500
4000
4500
5000
Strike
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
Time-to-Maturity (Years)

Example: A Bond Portfolio

Consider a portfolio containing quantities of d different default-free zero-coupon bonds.

Thei^{th} bond has price $P_{t,i}$, maturity T_i and face value 1.

s_{t,T_i} is the continuously compounded **spot interest rate** for maturity T_i so that

$$P_{t,i} = \exp(-s_{t,T_i}(T_i - t)).$$

There are λ_i units of i^{th} bond in the portfolio so total portfolio value given by

$$V_t = \sum_{i=1}^{d} \lambda_i \exp(-s_{t,T_i}(T_i - t)).$$

Example: A Bond Portfolio

Assume now only parallel changes in the spot rate curve are possible

- while unrealistic, a common assumption in practice
- this is the assumption behind the use of duration and convexity.

Then if spot curve moves by δ the portfolio loss satisfies

$$
\begin{aligned}
L_{t+1} &= -\sum_{i=1}^{d} \lambda_i \left(e^{-(s_{t+\Delta, T_i} + \delta)(T_i - t - \Delta)} - e^{-s_{t, T_i}(T_i - t)} \right) \\
&\simeq -\sum_{i=1}^{d} \lambda_i \left(s_{t, T_i}(T_i - t) - (s_{t+\Delta, T_i} + \delta)(T_i - t - \Delta) \right).
\end{aligned}
$$

Therefore have a **single** risk factor, δ.

Approaches to Risk Measurement

1. Notional Amount Approach.
2. Factor Sensitivity Measures.
3. Scenario Approach.
4. Measures based on loss distribution, e.g. Value-at-Risk (VaR) or Conditional Value-at-Risk (CVaR).

An Example of Factor Sensitivity Measures: the Greeks

Scenario analysis for derivatives portfolios is often combined with the Greeks to understand the riskiness of a portfolio

- and sometimes to perform a P&L attribution.

Suppose then we have a portfolio of options and futures

- all written on the same underlying security.

Portfolio value is the sum of values of individual security positions

- and the same is true for the portfolio Greeks, e.g. the portfolio delta, portfolio gamma and portfolio vega. Why?

Consider now a single option in the portfolio with price $C(S, \sigma, \ldots)$.

Will use a delta-gamma-vega approximation to estimate risk of the position

- but approximation also applies (why?) to the entire portfolio.

Note approximation only holds for **"small"** moves in underlying risk factors

- a very important observation that is lost on many people!

Delta-Gamma-Vega Approximations to Option Prices

A simple application of Taylor's Theorem yields

$$\begin{aligned} C(S+\Delta S, \sigma+\Delta\sigma) &\approx C(S,\sigma) + \Delta S \frac{\partial C}{\partial S} + \frac{1}{2}(\Delta S)^2 \frac{\partial^2 C}{\partial S^2} + \Delta\sigma \frac{\partial C}{\partial \sigma} \\ &= C(S,\sigma) \ + \ \Delta S \ \delta \ + \ \frac{1}{2}(\Delta S)^2 \ \Gamma \ + \ \Delta\sigma \ \text{vega}. \end{aligned}$$

Therefore obtain

$$\begin{aligned} \text{P\&L} &\approx \delta \Delta S \ + \ \frac{\Gamma}{2}(\Delta S)^2 \ + \ \text{vega } \Delta\sigma \\ &= \text{delta P\&L} \ + \ \text{gamma P\&L} \ + \ \text{vega P\&L} \ . \end{aligned}$$

When $\Delta\sigma = 0$, obtain the well-known delta-gamma approximation
- often used, for example, in historical Value-at-Risk (VaR) calculations.

Can also write

$$\begin{aligned} \text{P\&L} &\approx \delta S \left(\frac{\Delta S}{S}\right) + \frac{\Gamma S^2}{2}\left(\frac{\Delta S}{S}\right)^2 + \text{vega } \Delta\sigma \\ &= \text{ESP} \times \text{Return} \ + \ \$ \text{ Gamma} \times \text{Return}^2 \ + \ \text{vega } \Delta\sigma \qquad (2) \end{aligned}$$

where ESP denotes the equivalent stock position or "dollar" delta.

Scenario Analysis and Stress Testing

Underlying	SX5E Index

Underlying and Volatility Stress Table

Sum of PnL	Volatility Stress								
Underlying Stress	-10	-5	-2	-1	0	1	2	5	10
-20	7,234	(3,488)	(10,770)	(13,305)	(15,885)	(18,508)	(21,168)	(29,351)	(43,504)
-10	29,406	13,520	3,663	341	(2,995)	(6,345)	(9,706)	(19,845)	(36,862)
-5	35,537	17,725	6,928	3,319	(293)	(3,908)	(7,525)	(18,380)	(36,458)
-2	37,552	18,874	7,647	3,905	165	(3,574)	(7,310)	(18,505)	(37,095)
-1	37,948	19,032	7,684	3,905	129	(3,645)	(7,416)	(18,709)	(37,450)
0	38,207	19,079	7,623	3,810	0	(3,806)	(7,609)	(18,992)	(37,873)
1	38,330	19,017	7,463	3,619	(220)	(4,056)	(7,888)	(19,354)	(38,365)
2	38,318	18,846	7,207	3,335	(531)	(4,394)	(8,251)	(19,794)	(38,925)
5	37,495	17,698	5,868	1,934	(1,995)	(5,920)	(9,839)	(21,566)	(41,000)
10	33,611	13,757	1,824	(2,153)	(6,128)	(10,102)	(14,073)	(25,969)	(45,722)
20	17,546	(923)	(12,382)	(16,244)	(20,124)	(24,018)	(27,925)	(39,709)	(59,483)

– Stress testing an options portfolio written on the Eurostoxx 50.

Scenario Analysis and Stress Testing

In general we want to stress the risk factors in our portfolio.

Therefore very important to understand the **dynamics** of the risk factors.

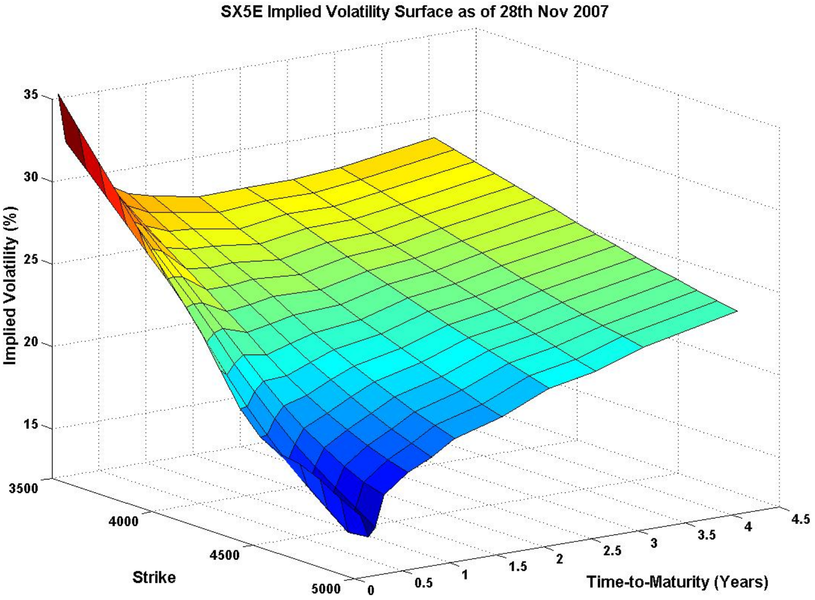
e.g. The implied volatility surface almost never experiences parallel shifts. Why?

- In fact changes in volatility surface tend to follow a **square root of time** rule.

When stressing a portfolio, it is also important to understand what risk factors the portfolio is exposed to.

e.g. A portfolio may be neutral with respect to the two most “important" risk factors but have very significant exposure to a third risk factor

- Important then to conduct stresses of that third risk factor
- Especially if the trader or portfolio manager knows what stresses are applied!


SX5E Implied Volatility Surface as of 28th Nov 2007
Implied Volatility (%)
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Time-to-Maturity (Years)

Value-at-Risk

Value-at-Risk (VaR) the most widely (mis-)used risk measure in the financial industry.

Despite the many weaknesses of VaR, financial institutions are required to use it under the Basel II capital-adequacy framework.

And many institutions routinely report VaR numbers to shareholders, investors or regulatory authorities.

VaR is calculated from the loss distribution

- could be conditional or unconditional
- could be a true loss distribution or some approximation to it.

Will assume that horizon Δ has been fixed so that L represents portfolio loss over time interval Δ.

Will use $F_L(\cdot)$ to denote the CDF of L.

Value-at-Risk

Definition: Let $F : \mathbb{R} \to [0,1]$ be an arbitrary CDF. Then for $\alpha \in (0,1)$ the α-quantile of F is defined by

$$q_\alpha(F) \;:=\; \inf\{x \in \mathbb{R} \;:\; F(x) \geq \alpha\}.$$

- If F is continuous and strictly increasing, then $q_\alpha(F) = F^{-1}(\alpha)$.
- For a random variable L with CDF $F_L(\cdot)$, will often write $q_\alpha(L)$ instead of $q_\alpha(F_L)$.
- Since any CDF is by definition right-continuous, immediately obtain the following result:

Lemma: A point $x_0 \in \mathbb{R}$ is the α-quantile of F_L if and only if

$$\begin{array}{lccl} (i) \;\; F_L(x_0) & \geq & \alpha & \text{and} \\ (ii) \;\; F_L(x) & < & \alpha & \text{for all } x < x_0. \end{array}$$

Definition: Let $\alpha \in (0,1)$ be some fixed confidence level. Then the VaR of the portfolio loss at the confidence interval, α, is given by $\text{VaR}_\alpha := q_\alpha(L)$, the α-quantile of the loss distribution.

VaR for the Normal Distributions

Because the normal CDF is both continuous and strictly increasing, it is straightforward to calculate VaR_α.

So suppose $L \sim \mathsf{N}(\mu, \sigma^2)$. Then

$$\mathsf{VaR}_\alpha \;=\; \mu + \sigma \Phi^{-1}(\alpha) \tag{3}$$

where $\Phi(\cdot)$ is the standard normal CDF.

This follows from previous lemma if we can show $F_L(\mathsf{VaR}_\alpha) = \alpha$

- but this follows immediately from (3).

VaR for the t Distributions

The t CDF also continuous and strictly increasing so again straightforward to calculate VaR_α.

So let $L \sim \text{t}(\nu, \mu, \sigma^2)$, i.e. $(L-\mu)/\sigma$ has a standard t distribution with $\nu > 2$ degrees-of-freedom (dof). Then

$$\text{VaR}_\alpha \;=\; \mu + \sigma\, t_\nu^{-1}(\alpha)$$

where t_ν is the CDF for the t distribution with ν dof.

Note that now $\text{E}[L] = \mu$ and $\text{Var}(L) = \nu\sigma^2/(\nu-2)$.

Weaknesses of VaR

1. VaR attempts to describe the entire loss distribution with just a single number!
   - so significant information is lost
   - this criticism applies to all **scalar** risk measures
   - one way around it is to report VaR_α for several values of α.
2. Significant model risk attached to VaR
   - e.g. if loss distribution is heavy-tailed but a light-tailed, e.g. normal, distribution is assumed, then VaR_α will be severely underestimated as $\alpha \to 1$.
3. A fundamental problem with VaR is that it can be very difficult to estimate the loss distribution
   - true of all risk measures based on the loss distribution.
4. VaR is not a sub-additive risk measure so that it doesn't lend itself to aggregation.

(Non-) Sub-Additivity of VaR

e.g. Let $L = L_1 + L_2$ be the total loss associated with two portfolios, each with respective losses, L_1 and L_2.

Then

$$q_\alpha(F_L) \; > \; q_\alpha(F_{L_1}) \; + \; q_\alpha(F_{L_2}) \quad \text{is possible!}$$

- An undesirable property as we would expect some **diversification benefits** when we combine two portfolios together.
- Such a benefit would be reflected by the combined portfolio having a **smaller** risk measure than the sum of the two individual risk measures.

Will discuss sub-additivity property when we study coherent risk measures later in course.

Advantages of VaR

VaR is generally "easier" to estimate:

- True of quantile estimation in general since quantiles are not very sensitive to outliers.
  - not true of other risk measures such as Expected Shortfall / CVaR
- Even then, it becomes progressively more difficult to estimate VaR_α as $\alpha \rightarrow 1$
  - may be able to use Extreme Value Theory (EVT) in these circumstances.

But VaR easier to estimate only if we have correctly specified the appropriate probability model

- often an unjustifiable assumption!

Value of Δ that is used in practice generally depends on the application:

- For credit, operational and insurance risk Δ often on the order of 1 year.
- For financial risks typical values of Δ are on the order of days.

Expected Shortfall (ES)

Definition: For a portfolio loss, L, satisfying $\mathsf{E}[|L|] < \infty$ the expected shortfall at confidence level $\alpha \in (0,1)$ is given by

$$\mathsf{ES}_\alpha \;:=\; \frac{1}{1-\alpha} \int_\alpha^1 q_u(F_L)\; du. \tag{4}$$

Relationship between ES_α and VaR_α is therefore given by

$$\mathsf{ES}_\alpha \;:=\; \frac{1}{1-\alpha} \int_\alpha^1 \mathsf{VaR}_u(L)\; du$$

- so clear that $\mathsf{ES}_\alpha(L) \geq \mathsf{VaR}_\alpha(L)$.

Expected Shortfall (ES)

A more well known representation of $\text{ES}_\alpha(L)$ holds when F_L is continuous:

Lemma: If F_L is a continuous CDF then

$$\text{ES}_\alpha := \frac{\text{E}\left[L;\ L \geq q_\alpha(L)\right]}{1-\alpha}$$

$$= \text{E}\left[L \mid L \geq \text{VaR}_\alpha\right]. \qquad (5)$$

Proof: See Lemma 2.13 in *McNeil, Frey and Embrechts* (MFE). □

Expected Shortfall also known as Conditional Value-at-Risk (CVaR)

- when there are **atoms** in the distribution CVaR is defined slightly differently
- but we will continue to take (4) as our definition.

Example: Expected Shortfall for a Normal Distribution

Can use (5) to compute expected shortfall of an $\mathsf{N}(\mu, \sigma^2)$ random variable.

We find

$$\mathsf{ES}_\alpha \;=\; \mu \;+\; \sigma \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha} \tag{6}$$

where $\phi(\cdot)$ is the PDF of the standard normal distribution.

Example: Expected Shortfall for a t Distribution

Let $L \sim \mathrm{t}(\nu, \mu, \sigma^2)$ so that $\tilde{L} := (L - \mu)/\sigma$ has a standard t distribution with $\nu > 2$ dof.

Then easy to see that $\mathrm{ES}_\alpha(L) = \mu + \sigma \mathrm{ES}_\alpha(\tilde{L})$.

Straightforward using direct integration to check that

$$\mathrm{ES}_\alpha(\tilde{L}) \;=\; \frac{g_\nu\left(t_\nu^{-1}(\alpha)\right)}{1-\alpha}\;\left(\frac{\nu + (t_\nu^{-1}(\alpha))^2}{\nu - 1}\right) \tag{7}$$

where $t_\nu(\cdot)$ and $g_\nu(\cdot)$ are the CDF and PDF, respectively, of the standard t distribution with ν dof.

Remark: The t distribution is a much better model of stock (and other asset) returns than the normal model. In empirical studies, values of ν around 5 or 6 are often found to fit best.

The Shortfall-to-Quantile Ratio

Can compare VaR_α and ES_α by considering their ratio as $\alpha \to 1$.

Not too difficult to see that in the case of the **normal distribution**

$$\frac{\text{ES}_\alpha}{\text{VaR}_\alpha} \;\to\; 1 \;\text{ as } \alpha \to 1.$$

However, in the case of the t distribution with $\nu > 1$ dof we have

$$\frac{\text{ES}_\alpha}{\text{VaR}_\alpha} \;\to\; \frac{\nu}{\nu - 1} \;>\; 1 \;\text{ as } \alpha \to 1.$$

Standard Techniques for Risk Measurement

1. Historical simulation.
2. Monte-Carlo simulation.
3. Variance-covariance approach.

Historical Simulation

Instead of using a probabilistic model to estimate distribution of $L_{t+1}(\mathbf{X_{t+1}})$, we could estimate the distribution using a historical simulation.

In particular, if we know the values of $\mathbf{X_{t-i+1}}$ for $i = 1, \ldots, n$, then can use this data to create a set of historical losses:

$$\{\tilde{L}_i \; := \; L_{t+1}(\mathbf{X_{t-i+1}}) \; : \; i = 1, \ldots, n\}$$

- so $\tilde{L}_i$ is the portfolio loss that would occur if the risk factor returns on date $t - i + 1$ were to recur.

To calculate value of a given risk measure we simply assume the distribution of $L_{t+1}(\mathbf{X_{t+1}})$ is discrete and takes on each of the values $\tilde{L}_i$ w.p. $1/n$ for $i = 1, \ldots, n$, i.e., we use the empirical distribution of the $\mathbf{X_t}$'s.

e.g. Suppose we wish to estimate VaR$_\alpha$. Then can do so by computing the α-quantile of the $\tilde{L}_i$'s.

Historical Simulation

Suppose the $\tilde{L}_i$'s are ordered by

$$\tilde{L}_{n,n} \leq \cdots \leq \tilde{L}_{1,n}.$$

Then an estimator of $\mathrm{VaR}_\alpha(L_{t+1})$ is $\tilde{L}_{[n(1-\alpha)],n}$ where $[n(1-\alpha)]$ is the largest integer not exceeding $n(1-\alpha)$.

Can estimate ES_α using

$$\widetilde{\mathrm{ES}}_\alpha = \frac{\tilde{L}_{[n(1-\alpha)],n} + \cdots + \tilde{L}_{1,n}}{[n(1-\alpha)]}.$$

Historical simulation approach generally difficult to apply for derivative portfolios. Why?

But if applicable, then easy to apply.

Historical simulation estimates the unconditional loss distribution

- so not good for financial applications!

Monte-Carlo Simulation

Monte-Carlo approach similar to historical simulation approach.

But now use some **parametric** distribution for the change in risk factors to generate sample portfolio losses.

The (conditional or unconditional) distribution of the risk factors is estimated and m portfolio loss samples are generated.

Free to make m as large as possible

- Subject to constraints on computational time.
- Variance reduction methods often employed to obtain improved estimates of required risk measures.

While Monte-Carlo is an excellent tool, it is only as good as the model used to generate the data: if the estimated distribution of $\mathbf{X_{t+1}}$ is poor, then Monte-Carlo of little value.

The Variance-Covariance Approach

In the variance-covariance approach assume that $\mathbf{X_{t+1}}$ has a multivariate normal distribution so that

$$\mathbf{X_{t+1}} \;\sim\; \mathsf{MVN}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right).$$

Also assume the linear approximation

$$\hat{L}_{t+1}(\mathbf{X_{t+1}}) \;:=\; -\left(f_t(t, \mathbf{Z_t})\Delta + \sum_{i=1}^{d} f_{z_i}(t, \mathbf{Z_t})\; X_{t+1,i}\right)$$

is sufficiently accurate. Then can write

$$\hat{L}_{t+1}(\mathbf{X_{t+1}}) = -(c_t + \mathbf{b_t}^\top X_{t+1})$$

for a constant scalar, c_t, and constant vector, $\mathbf{b_t}$.

Therefore obtain

$$\hat{L}_{t+1}(\mathbf{X_{t+1}}) \;\sim\; \mathsf{N}\left(-c_t - \mathbf{b_t}^\top \boldsymbol{\mu},\; \mathbf{b_t}^\top \boldsymbol{\Sigma} \mathbf{b_t}\right).$$

and can calculate any risk measures of interest.

The Variance-Covariance Approach

This technique can be either conditional or unconditional

- depends on how $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are estimated.

The approach provides straightforward analytically tractable method of determining the loss distribution.

But it has several weaknesses: risk factor distributions are often fat- or heavy-tailed but the normal distribution is light-tailed

- this is easy to overcome as there are other multivariate distributions that are also **closed under linear operations**.

e.g. If $\mathbf{X_{t+1}}$ has a multivariate t distribution so that

$$\mathbf{X_{t+1}} \sim \; t\left(\nu, \boldsymbol{\mu}, \boldsymbol{\Sigma}\right)$$

then

$$\hat{L}_{t+1}(\mathbf{X_{t+1}}) \;\sim\; t\left(\nu, -c_t - \mathbf{b_t}^\top \boldsymbol{\mu},\; \mathbf{b_t}^\top \boldsymbol{\Sigma} \mathbf{b_t}\right).$$

A more serious problem is that the linear approximation will often not work well

- particularly true for portfolios of derivative securities.

Evaluating Risk Measurement Techniques

Important for any risk manger to constantly evaluate the reported risk measures.

e.g. If daily 95% VaR is reported then should see daily losses exceeding the reported VaR approximately 95% of the time.

So suppose reported VaR numbers are correct and define

$$Y_i \;:=\; \begin{cases} 1, & L_i \;\geq\; \text{VaR}_i \\ 0, & \text{otherwise} \end{cases}$$

where VaR_i and L_i are the reported VaR and realized loss for period i.

If Y_i's are *IID*, then

$$\sum_{i=1}^{n} Y_i \sim \text{Binomial}(n, .05)$$

Can use standard statistical tests to see if this is indeed the case.

Similar tests can be constructed for ES and other risk measures.

Other Considerations

- Risk-Neutral and Data-Generating (Empirical) Probability Measures.
- Data Risk.
- Multi-Period Risk Measures and Scaling.
- Model Risk.
- Data Aggregation.
- Liquidity Risk.
- P&L Attribution.