A Cournot-Stackelberg Model of Supply Contracts with Financial Hedging

René Caldentey
Stern School of Business, New York University, New York, NY 10012, rcaldent@stern.nyu.edu.

Martin B. Haugh
Department of IE and OR, Columbia University, New York, NY 10027, mh2078@columbia.edu.

Abstract

We study the performance of a stylized supply chain where multiple retailers and a single producer compete in a Cournot-Stackelberg game. At time $t = 0$ the retailers order a single product from the producer and upon delivery at time $T > 0$, they sell it in the retail market at a stochastic clearance price. We assume the retailers' profits depend in part on the realized path of some tradeable stochastic process such as a foreign exchange rate, interest rate or more generally, some tradeable economic index. Because production and delivery do not take place until time T, the producer offers a *menu* of wholesale prices to the retailer, one for each realization of the process up to some time, τ, where $0 \leq \tau \leq T$. The retailers' ordering quantities therefore depend on the realization of the process until time τ. We also assume, however, that the retailers are budget-constrained and are therefore limited in the number of units they may purchase from the producer. The supply chain might therefore be more profitable if the retailers were able to reallocate their budgets across different states of nature. In order to affect a (partial) reallocation, we assume that the retailers are also able to trade dynamically in the financial market. After solving for the Nash equilibrium we address such questions as: (i) whether or not the players would be better off if the retailers merged and (ii) whether or not the players are better off when the retailers have access to the financial markets. Our model can easily handle variations where, for example, the retailers are located in a different currency area to the producer or where the retailers must pay the producer before their budgets are available. Finally, we consider the case where the producer can choose the optimal timing, τ, of the contract and we formulate this as an optimal stopping problem.

Subject Classifications: Finance: portfolio, management. Optimal control: applications. Production: applications.

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1 Introduction

We study the performance of a stylized supply chain where multiple retailers and a single producer compete in a Cournot-Stackelberg game. At time $t = 0$ the retailers order a single product from the producer and upon delivery at time $T > 0$, they sell it in the retail market at a stochastic clearance price that depends in part on the realized path or terminal value of some observable and tradeable financial process. Because production and delivery do not take place until time T, the producer offers a *menu* of wholesale prices to the retailer, one for each realization of the process up to time some time, τ, where $0 \leq \tau \leq T$. The retailers' ordering quantities are therefore contingent upon the realization of the process up to time τ.

We also assume, however, that the retailers are budget-constrained and are therefore limited in the number of units they may purchase from the producer. As a result, the supply chain might be more profitable if the retailers were able to reallocate their financial resources, i.e. their budgets, across different states. By allowing the retailers to trade dynamically in the financial markets we enable such a (partial) reallocation of resources. The producer has no need to trade in the financial markets as he is not budget constrained and, like the retailers, is assumed to be risk neutral. After solving for the Cournot-Stackelberg equilibrium we address such questions as whether or not the players would be better off if the retailers merged and whether or not the players are better off when the retailers have access to the financial markets.

We now attempt to position our paper within the vast literature on supply chain management. We refer the reader to the books by de Kok and Graves (2003) and Simchi-Levi et al. (2004) for a general overview of supply chain management issues and to the survey article by Cachon (2003) for a review of supply chain management contracts.

A distinguishing feature of our model with respect to most of the literature in supply chain management is the budget constraint that we impose on the retailers' procurement decisions. Some recent exceptions include Buzacott and Zhang (2004), Caldentey and Haugh (2009), Dada and Hu (2008), Kouvelis and Zhao (2008), Xu and Birge (2004) and Caldentey and Chen (2009).

Xu and Birge (2004) analyze a single-period newsvendor model which is used to illustrate how a firm's inventory decisions are affected by the existence of a budget constraint and the firm's capital structure. In a multi-period setting, Hu and Sobel (2005) examine the interdependence of a firm's capital structure and its short-term operating decisions concerning inventory, dividends, and liquidity. In a similar setting, Dada and Hu (2008) consider a budget-constrained newsvendor that can borrow from a bank that acts strategically when choosing the interest rate applied to the loan. They characterize the Stackelberg equilibrium and investigate conditions under which channel coordination, *i.e.*, where the ordering quantities of the budget-constrained and non budget-constrained newsvendors coincide, can be achieved.

Buzacott and Zhang (2004) incorporate asset-based financing in a deterministic multi-period production/inventory control system by modeling the available cash in each period as a function of the firm's assets and liabilities. In their model a retailer finances its operations by borrowing from a commercial bank. The terms of the loans are contingent upon the retailer's balance sheet and income statement and in particular, upon the inventories and accounts receivable. The authors conclude that asset-based financing allows retailers to enhance their cash return over what it would otherwise be if they were only able to use their own capital.

The work by Caldentey and Haugh (2009), Kouvelis and Zhao (2008) and Caldentey and Chen (2009) are the most closely related to this paper. They all consider a two-echelon supply chain system in which there is a single budget constrained retailer and investigate different types of procurement contracts between the agents using a Stackelberg equilibrium concept. In Kouvelis and Zhao (2008) the supplier offers different type of contracts designed to provide financial services to the retailer. They analyze a set of alternative financing schemes including *supplier early payment discount*, *open account financing*, *joint supplier financing with bank*, and *bank financing* schemes. In a similar setting, Caldentey and Chen (2009) discuss two alternative forms of financing for the retailer: (a) *internal financing* in which the supplier offers a procurement contract that allows the retailer to pay *in arrears* a fraction of the procurement cost after demand is realized and (b) *external financing* in which a third party financial institution offers a commercial loan to the retailer. They conclude that in an optimally designed contract it is in the supplier's best interest to offer financing to the retailer and that the retailer will always prefer internal rather than external financing.

In Caldentey and Haugh (2009) the supplier offers a modified wholesale price contract which is executed at a future time τ. The terms of the contract are such that the actual wholesale price charged at time τ depends on information publicly available at this time. Delaying the execution of the contract is important because in this model the retailer's demand depends in part on a financial index that the retailer and supplier can observe through time. As a result, the retailer can dynamically trade in the financial market to adjust his budget to make it contingent upon the evolution of the index. Their model shows how financial markets can be used as (i) a source of public information upon which procurement contracts can be written and (ii) as a means for financial hedging to mitigate the effects of the budget constraint. In this paper, we therefore extend the model in Caldentey and Haugh (2009) by considering a market with *multiple retailers* in Cournot competition as well as a Stackelberg leader. Our extended model can also easily handle variations where, for example, the retailers are located in a different currency area to the producer or where the retailers must pay the producer before their budgets are available. In addition we consider the case where the producer can choose the optimal timing, τ, of the contract and we formulate this as an optimal stopping problem.

A second related stream of research considers Cournot-Stackelberg equilibria. There is an extensive economics literature on this topic that focuses on issues of existence and uniqueness of the Nash equilibrium. See Okoguchi and Szidarovsky (1999) for a comprehensive review. In the context of supply chain management, there has been some recent research that investigates the design of efficient contracts between the supplier and the retailers. For example, Bernstein and Federgruen (2003) derive a perfect coordination mechanism between the supplier and the retailers. This mechanism takes the form of a nonlinear wholesale pricing scheme. Zhao et al. (2005) investigate inventory sharing mechanisms among competing dealers in a distribution network setting. Li (2002) studies a Cournot-Stackelberg model with asymmetric information in which the retailers are endowed with some private information about market demand. In contrast, the model we present in this paper uses the *public* information provided by the financial markets to improve the supply chain coordination.

Finally, we mention that there exists a related stream of research that investigates the use of financial markets and instruments to hedge operational risk exposure. See Boyabatli and Toktay (2004) for a detailed review. For example, Caldentey and Haugh (2006) consider the general problem of dynamically hedging the profits of a risk-averse corporation when these profits are partially correlated

with returns in the financial markets. Chod et al. (2009) examine the joint impact of operational flexibility and financial hedging on a firm's performance and their complementarity/substitutability with the firm's overall risk management strategy. Ding et al. (2007) and Dong et al. (2006) examine the interaction of operational and financial decisions from an integrated risk management standpoint. Boyabatli and Toktay (2010) analyze the effect of capital market imperfections on a firm's operational and financial decisions in a capacity investment setting. Babich and Sobel (2004) propose an infinite-horizon discounted Markov decision process in which an IPO event is treated as a stopping time. They characterize an optimal capacity-expansion and financing policy so as to maximize the expected present value of the firm's IPO. Babich et al. (2008) consider how trade credit financing affects the relationships among firms in the supply chain, supplier selection, and supply chain performance.

The remainder of this paper is organized as follows. Section 2 describes our model, focussing in particular on the supply chain, the financial markets and the contractual agreement between the producer and the retailers. We analyze this model in Section 3 in the special case where all of the retailers have identical budgets. We then consider the more general case in Section 4 where we focus on characterizing the Cournot equilibrium of the retailers. In Section 5 we discuss the value of the financial markets and we conclude in Section 6. Most of our proofs as well as our discussion of the optimal timing of the contract are contained in the Appendices.

2 Model Description

We now describe the model in further detail. We begin with the supply chain description and then discuss the role of the financial markets. At the end of the section we define the contract which specifies the agreement between the producer and the retailers. Throughout this section we will assume for ease of exposition that both the producer and the retailers are located in the same currency area and that interest rates are identically zero. In Section 3 we will relax these assumptions and still maintain the tractability of our model using change of measure arguments.

2.1 The Supply Chain

We model an isolated segment of a competitive supply chain with one producer that produces a single product and N competing retailers that face a stochastic clearance price[1] for this product. This clearance price, and the resulting cash-flow to the retailers, is realized at a fixed future time $T > 0$. The retailers and producer, however, negotiate the terms of a procurement contract at time $t = 0$. This contract specifies three quantities:

(i) A production time, τ, with $0 \leq \tau \leq T$. While τ will be fixed for most of our analysis, we will also consider the problem of selecting an optimal τ in Appendix C.

(ii) A rule that specifies the size of the order, q_i, for the i^{th} retailer where $i = 1, \ldots, N$. In general, q_i may depend upon market information available at time τ.

[1]Similar models are discussed in detail in Section 2 of Cachon (2003). See also Lariviere and Porteus (2001).

(*iii*) The payment, $\mathcal{W}(q_i)$, that the i^{th} retailer pays to the producer for fulfilling the order. Again, $\mathcal{W}(q_i)$ will generally depend upon market information available at time τ. The timing of this payment is not important when we assume that interest rates are identically zero. In Section 3.4, however, we will assume interest rates are stochastic when we consider the case where the retailers must pay the producer *before* their budgets are available. It will then be necessary to specify exactly when the retailers pay the producer.

We will restrict ourselves to transfer payments that are linear on the ordering quantity. That is, we consider the so-called *wholesale price* contract where $\mathcal{W}(q) = w\,q$ and where w is the per-unit wholesale price charged by the producer. We assume that the producer offers the same contract to each retailer and while this simplifies the analysis considerably, it is also realistic. For example, it is often illegal for a producer to price-discriminate among its customers. We also assume that during the negotiation of the contract the producer acts as a Stackelberg leader. That is, for a fixed procurement time τ, the producer moves first and at $t = 0$ proposes a wholesale price menu, w_τ, to which the retailers then react by selecting their ordering levels, q_i, for $i = 1, \ldots, N$. Note that the N retailers also compete among themselves in a Cournot-style game to determine their optimal ordering quantities and trading strategies.

We assume that the producer has unlimited production capacity and that if production takes place at time τ then the per-unit production cost is c_τ. We will generally assume that c_τ is constant but[2] many of our results, however, go through when c_τ is stochastic. The producer's payoff as a function of the procurement time, τ, the wholesale price, w_τ, and the ordering quantities, q_i, is given[3] by

$$\Pi_{\mathrm{P}} := \sum_{i=1}^{N} (w_\tau - c_\tau)\, q_i. \tag{1}$$

We assume that each retailer is restricted by a budget constraint that limits his ordering decisions. In particular, we assume that each retailer has an initial budget, B_i, that may be used to purchase product units from the producer. Without loss of generality, we order the retailers so that $B_1 \geq B_2 \geq \ldots \geq B_N$. We assume each of the retailers can trade in the financial markets during the time interval $[0, \tau]$, thereby transferring cash resources from states where they are not needed to states where they are.

For a given set of order quantities, the i^{th} retailer collects a random revenue at time T. We compute this revenue using a linear *clearance price* model. That is, the market price at which the retailer sells these units is a random variable, $P(Q) := A - (q_i + Q_{i-})$, where A is a non-negative random variable, $Q_{i-} := \sum_{j \neq i} q_j$ and $Q := \sum_j q_j$. The random variable A models the market size that we assume is unknown. The realization of A, however, will depend in part on the realization of the financial markets between times 0 and T. The payoff of the i^{th} retailer, as a function of τ, w_τ, and the order quantities, then takes the form

$$\Pi_{\mathrm{R}_i} \;:=\; (A - (q_i + Q_{i-}))\; q_i - w_\tau\, q_i. \tag{2}$$

[2] When we consider the optimal timing of τ in Appendix C we will assume that c_τ is deterministic and increasing in τ so that production postponement comes at a cost.

[3] In Section 3.3 we will assume that the producer and retailers are located in different currency areas. We will then need to adjust (2) appropriately.

A stochastic clearance price is easily justified since in practice unsold units are generally liquidated using secondary markets at discount prices. Therefore, we can view our clearance price as the average selling price across all units and markets. As stated earlier, w_τ and the q_i's will in general depend upon market information available at time τ. Since $\mathcal{W}(q)$, Π_{P} and the Π_{R_i}'s are functions of w_τ and the q_i's, these quantities will also depend upon market information available at time τ.

The linear clearance price in (2) is commonly assumed in the economics literature for reasons of tractability. It also helps ensure that the game will have a unique Nash equilibrium. (For further details see Chapter 4 of Vives, 2001.)

2.2 The Financial Market

The financial market is modeled as follows. Let X_t denote[4] the time t value of a tradeable security and let $\{\mathcal{F}_t\}_{0\le t\le T}$ be the filtration generated by X_t on a probability space, $(\Omega, \mathcal{F}, \mathbb{Q})$. We do not assume that $\mathcal{F}_T = \mathcal{F}$ since we want the non-financial random variable, A, to be $\mathcal{F}$-measurable but not $\mathcal{F}_T$-measurable. There is also a risk-less *cash account* available from which cash may be borrowed or in which cash may be deposited. Since we have assumed[5] zero interest rates, the time τ gain (or loss), $G_\tau(\theta)$, that results from following a self-financing[6] $\mathcal{F}_t$-adapted trading strategy, θ_t, can be represented as a stochastic integral with respect to X. In a continuous-time setting, for example, we have

$$G_\tau(\theta) := \int_0^\tau \theta_s \, dX_s. \tag{3}$$

We assume that Q is an equivalent martingale measure (EMM) so that discounted security prices are Q-martingales. Since we are currently assuming that interest rates are identically zero, however, it is therefore the case that X_t is a Q-martingale. Subject to integrability constraints on the set of feasible trading strategies, we also see that $G_t(\theta)$ is a Q-martingale for every $\mathcal{F}_t$-adapted self-financing trading strategy, θ_t.

Our analysis will be simplified considerably by making a *complete financial markets* assumption. In particular, let G_τ be any suitably integrable contingent claim that is $\mathcal{F}_\tau$-measurable. Then a complete financial markets assumption amounts to assuming the existence of an $\mathcal{F}_t$-adapted self-financing trading strategy, θ_t, such that $G_\tau(\theta) = G_\tau$. That is, G_τ is *attainable*. This assumption is very common in the financial literature. Moreover, many incomplete financial models can be made complete by simply expanding the set of tradeable securities. When this is not practical, we can simply assume the existence of a market-maker with a known pricing function or pricing kernel[7] who is willing to sell G_τ in the market-place. In this sense, we could then claim that G_τ is indeed attainable.

[4] All of our analysis goes through if we assume X_t is a multi-dimensional price process. For ease of exposition we will assume X_t is one-dimensional.

[5] As mentioned earlier, we will relax this assumption in Section 3.4.

[6] A trading strategy, θ_t, is self-financing if cash is neither deposited with nor withdrawn from the portfolio during the trading interval, $[0, T]$. In particular, trading gains or losses are due to changes in the values of the traded securities. Note that θ_t represents the number of units of the tradeable security held at time s. The self-financing property then implicitly defines the position at time s in the cash account. Because we have assumed interest rates are identically zero, there is no term in (3) corresponding to gains or losses from the cash account holdings. See Duffie (2004) for a technical definition of the self-financing property.

[7] See Duffie (2004). More generally, Duffie may be consulted for further technical assumptions (that we have omitted to specify) regarding the filtration, $\{\mathcal{F}_t\}_{0\le t\le T}$, feasible trading-strategies, etc.

Regardless of how we choose to justify it, assuming complete financial markets means that we will never need to solve for an optimal dynamic trading strategy, θ. Instead, we will only need to solve for an optimal contingent claim, G_τ, safe in the knowledge that any such claim is attainable. For this reason we will drop the dependence of G_τ on θ in the remainder of the paper. The only restriction that we will impose on any such trading gain, G_τ, is that the corresponding trading gain process, $G_s := \mathbb{E}_s^{\mathbb{Q}}[G_\tau]$ be a Q-martingale[8] for $s < \tau$. In particular we will assume that any feasible trading gain, G_τ, satisfies $\mathbb{E}_0^{\mathbb{Q}}[G_\tau] = G_0$ where G_0 is the initial amount of capital that is devoted to trading in the financial market. Without any loss of generality we will typically assume $G_0 = 0$. This assumption will be further clarified in Section 2.3.

A key aspect of our model is the dependence between the payoffs of the supply chain and returns in the financial market. Other than assuming the existence of $\mathbb{E}_\tau^{\mathbb{Q}}[A]$, the expected value of A conditional on the information available in the financial markets at time τ, we do not need to make any assumptions regarding the nature of this dependence. We will make the following assumption regarding $\mathbb{E}_\tau^{\mathbb{Q}}[A]$.

Assumption 1 *For all* $\tau \in [0, T]$, $\bar{A}_\tau := \mathbb{E}_\tau^{\mathbb{Q}}[A] \geq c_\tau$.

This condition ensures that for every time and state there is a total production level, $Q \geq 0$, for which the retailers' expected market price exceeds the producer's production cost. In particular, this assumption implies that it is possible to profitably operate the supply chain.

2.3 The Flexible Procurement Contract with Financial Hedging

The final component of our model is the contractual agreement between the producer and the retailers. We consider a variation of the traditional wholesale price contract in which the terms of the contract are specified contingent upon the public history, $\mathcal{F}_\tau$, that is available at time τ. Specifically, at time $t = 0$ the producer offers an $\mathcal{F}_\tau$-measurable wholesale price, w_τ, to the retailers. In response to this offer, the i^{th} retailer decides on an $\mathcal{F}_\tau$-measurable ordering quantity[9], $q_i = q_i(w_\tau)$, for $i = 1, \ldots, N$. Note that the contract itself is negotiated at time $t = 0$ whereas the actual order quantities are only realized at time $\tau \geq 0$.

The retailers' order quantities at time τ are constrained by their available budgets at this time. Besides the initial budget, B_i, the i^{th} retailer has access to the financial markets where he can hedge his budget constraint by purchasing at date $t = 0$ a contingent claim, $G_\tau^{(i)}$, that is realized at date τ and that satisfies $\mathbb{E}_0^{\mathbb{Q}}[G_\tau^{(i)}] = 0$. Given an $\mathcal{F}_\tau$-measurable wholesale price, w_τ, the retailer purchases an $\mathcal{F}_\tau$-measurable contingent claim, $G_\tau^{(i)}$, and selects an $\mathcal{F}_\tau$-measurable ordering quantity, $q_i = q_i(w_\tau)$, in order to maximize the economic value of his profits. Because of his access to the financial markets, the retailer can therefore mitigate his budget constraint so that it becomes

$$w_\tau \, q_i \leq B_i + G_\tau^{(i)} \qquad \text{for all } \omega \in \Omega \;\text{ and }\; i = 1, \ldots, N.$$

[8] Whenever we write $\mathbb{E}_s^{\mathbb{Q}}[\cdot]$ it should be understood as $\mathbb{E}^{\mathbb{Q}}[\cdot|\mathcal{F}_s]$.

[9] There is a slight abuse of notation here and throughout the paper when we write $q_i = q_i(w_\tau)$. This expression should not be interpreted as implying that q_i is a function of w_τ. We only require that q_i be $\mathcal{F}_\tau$-measurable and so a more appropriate interpretation is to say that $q_i = q_i(w_\tau)$ is the retailer's response to w_τ.

Since the no-trading strategy with $G_\tau^{(i)} \equiv 0$ is always an option, it is clear that for a given wholesale price, w_τ, the retailers are always better-off having access to the financial market. Whether or not the retailers will remain better off in equilibrium will be discussed in Section 3.

Before proceeding to analyze this contract a number of further clarifying remarks[10] are in order.

1. The model assumes a common knowledge framework in which all parameters of the models are known to all agents. Because of the Stackelberg nature of the game, this assumption implies that the producer knows the retailers' budgets and the distribution of the market demand. We also make the implicit assumption that the only information available regarding the random variable, A, is what we can learn from the evolution of X_t in the time interval $[0, \tau]$. If this were not the case, then the trading strategy in the financial market could depend on some non-financial information and so it would not be necessary to restrict the trading gains to be $\mathcal{F}_\tau$-measurable. More generally, if Y_t represented some non-financial noise that was observable at time t, then the trading strategy, θ_t, would only need to be adapted with respect to the filtration generated by X *and* Y. In this case the complete financial market assumption is of no benefit and it would be necessary for the retailers to solve the much harder problem of finding the optimal θ in order to find the optimal $G_\tau^{(i)}$'s.

2. In this model the producer does not trade in the financial markets because, being risk-neutral and not restricted by a budget constraint, he has no incentive to do so.

3. A potentially valid criticism of this model is that, in practice, a retailer is often a small entity and may not have the ability to trade in the financial markets. There are a number of responses to this. First, we use the word 'retailer' in a loose sense so that it might in fact represent a large entity. For example, an airline purchasing aircraft is a 'retailer' that certainly does have access to the financial markets. Second, it is becoming ever cheaper and easier for even the smallest 'player' to trade in the financial markets. Finally, even if the retailer does not have access to the financial market, then the producer, assuming he is a big 'player', can offer to trade with the retailer or act as his financial broker.

4. We claimed earlier that, without loss of generality, we could assume $G_0^{(i)} = 0$. This is clear for the following reason. If $G_0^{(i)} = 0$ then then the i^{th} retailer has a terminal budget of $B_\tau^{(i)} := B_i + G_\tau^{(i)}$ with which he can purchase product units at time τ and where $\mathbb{E}_0^{\mathbb{Q}}[G_\tau^{(i)}] = 0$. If he allocated $a > 0$ to the trading strategy, however, then he would have a terminal budget of $B_\tau^{(i)} = B_i - a + G_\tau^{(i)}$ at time τ but now with $\mathbb{E}_0^{\mathbb{Q}}[G_\tau^{(i)}] = a$. That the retailer is indifferent between the two approaches follows from the fact any terminal budget, $B_\tau^{(i)}$, that is feasible under one modeling approach is also feasible under the other and vice-versa.

5. Another potentially valid criticism of this framework is that the class of contracts is too complex. In particular, by only insisting that w_τ is $\mathcal{F}_\tau$-measurable we are permitting wholesale price contracts that might be too complicated to implement in practice. If this is the case then we can easily simplify the set of feasible contracts. By using appropriate conditioning arguments, for example, it would be straightforward to impose the tighter restriction that w_τ be $\sigma(X_\tau)$-measurable instead where $\sigma(X_\tau)$ is the σ-algebra generated by X_τ.

[10] These clarifications were also made in Caldentey and Haugh (2009) who study the single-retailer case.

We complete this section with a summary of the notation and conventions that will be used throughout the remainder of the paper. The subscripts R, P, and C are used to index quantities related to the retailers, producer and central planner, respectively. The subscript τ is used to denote the value of a quantity conditional on time τ information. For example, $\Pi_{\mathrm{P}|\tau}$ is the producer's expected payoff conditional on time τ information. The expected value, $\mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{P}|\tau}]$, is simply denoted by Π_{P} and similar expressions hold for the retailers and central planner. Any other notation will be introduced as necessary.

3 The Equibudget Case

We begin with the special case where all of the retailers have identical budgets. While not a realistic assumption in practice, we can solve for the producer's optimal price menu in this case and therefore solve for the overall Cournot-Stackelberg equilibrium. Moreover, we can completely address questions regarding whether or not the retailers should merge or remain in competition. We can also compare the equilibrium solution to the solution of the centralized planner in this case and therefore determine the efficiency of the supply chain. Some of the single-retailer results of Caldentey and Haugh (2009) will prove useful in this multi-retailer equibudget case.

Consider then the case where each of the retailers has the same budget so that $B_i = B$ for all $i = 1, \ldots, N$. For a given price menu, w_τ, the i^{th} retailer's problem is

$$\Pi_{\mathrm{R}}(w_\tau) = \max_{q_i \geq 0,\, G_\tau} \mathbb{E}_0^{\mathbb{Q}}\left[\left(\bar{A}_\tau - (q_i + Q_{i-}) - w_\tau\right) q_i\right] \tag{4}$$

$$\text{subject to} \qquad w_\tau\, q_i \leq B + G_\tau, \qquad \text{for all } \omega \in \Omega \tag{5}$$

$$\mathbb{E}_0^{\mathbb{Q}}\left[G_\tau\right] = 0. \tag{6}$$

While the equibudget problem is a special case of the game we will solve in Section 4, it is instructive to see an alternative solution. In the equibudget case, each of the N retailers has the following solution:

Proposition 1 (Optimal Strategy for the N Retailers in the Equibudget Case)

Let w_τ be an $\mathcal{F}_\tau$-measurable wholesale price offered by the producer and let $\mathcal{Q}_\tau$, $\mathcal{X}$ and $\mathcal{X}^c$ be defined as follows. $\mathcal{Q}_\tau := \frac{\left(\bar{A}_\tau - w_\tau\right)^+}{(N+1)}$, $\mathcal{X} := \{\omega \in \Omega \,:\, B \geq \mathcal{Q}_\tau\, w_\tau\}$ *and* $\mathcal{X}^c := \Omega - \mathcal{X}$. *The following two cases arise in the computation of the optimal ordering quantities and the financial claims:*

> **Case 1**: *Suppose that $\mathbb{E}_0^{\mathbb{Q}}\left[\mathcal{Q}_\tau\, w_\tau\right] \leq B$. Then $q_i(w_\tau) = \mathcal{Q}_\tau$ and there are infinitely many choices of the optimal claim, $G_\tau = G_\tau^{(i)}$, for $i = 1, \ldots, N$. One natural choice is to take*
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> $$G_\tau = \left[\mathcal{Q}_\tau\, w_\tau - B\right] \cdot \begin{cases} \delta & \text{if } \omega \in \mathcal{X} \\ 1 & \text{if } \omega \in \mathcal{X}^c \end{cases} \qquad \text{where} \qquad \delta := \frac{\int_{\mathcal{X}^c}\left[\mathcal{Q}_\tau\, w_\tau - B\right]\, \mathrm{d}\mathbb{Q}}{\int_{\mathcal{X}}\left[B - \mathcal{Q}_\tau\, w_\tau\right]\, \mathrm{d}\mathbb{Q}}.$$
>
> *In this case (possibly due to the ability to trade in the financial market), the budget constraint is not binding for any of the N retailers.*
>
> **Case 2**: *Suppose $\mathbb{E}_0^{\mathbb{Q}}\left[\mathcal{Q}_\tau w_\tau\right] > B$. Then*
>
> $$q_i(w_\tau) \;=\; q(w_\tau) \;=\; \frac{\left(\bar{A}_\tau - w_\tau\,(1+\lambda)\right)^+}{(N+1)} \qquad \text{and} \qquad G_\tau := q(w_\tau) w_\tau - B \tag{7}$$

is optimal for each i where $\lambda \geq 0$ solves $\mathbb{E}_0^{\mathbb{Q}}[q(w_\tau)w_\tau] = B$.

PROOF: See Appendix A.

The manufacturer's problem is straightforward to solve. Given the best response of the N retailers, his problem may be formulated as

$$\Pi_{\text{P}} = \max_{w_\tau,\, \lambda \geq 0} \; N \; \mathbb{E}_0^{\mathbb{Q}}\left[(w_\tau - c_\tau)\,\frac{\left(\bar{A}_\tau - w_\tau\,(1+\lambda)\right)^+}{(N+1)}\right] \tag{8}$$

$$\text{subject to} \qquad \mathbb{E}_0^{\mathbb{Q}}\left[w_\tau\,\frac{\left(\bar{A}_\tau - w_\tau\,(1+\lambda)\right)^+}{(N+1)}\right] \leq B. \tag{9}$$

Note that the factor N outside the expectation in (8) is due to the fact that there are N retailers and that the producer earns the same profit from each of them. Note also that there should be N constraints in this problem, one corresponding to each of the N retailers. However, by Proposition 1, these N constraints are identical since each retailer solves the same problem. The producer's problem then only requires the one constraint given in (9). We can easily re-write this problem as

$$\Pi_{\text{P}} = \max_{w_\tau,\, \lambda \geq 0} \; \frac{2N}{N+1} \; \mathbb{E}_0^{\mathbb{Q}}\left[(w_\tau - c_\tau)\,\frac{\left(\bar{A}_\tau - w_\tau\,(1+\lambda)\right)^+}{2}\right] \tag{10}$$

$$\text{subject to} \qquad \mathbb{E}_0^{\mathbb{Q}}\left[w_\tau\,\frac{\left(\bar{A}_\tau - w_\tau\,(1+\lambda)\right)^+}{2}\right] \leq \frac{(N+1)}{2}B \tag{11}$$

and now it is clearly identical[11] to the producer's problem where the budget constraint has been replaced by $(N+1)B/2$ and there is just one retailer. In particular, the solution of the producer's problem and of the Cournot-Stackelberg game follows immediately from Proposition 7 in Caldentey and Haugh (2009). We have the following result.

Proposition 2 (Producer's Optimal Strategy and the Cournot-Stackelberg Solution)

Let ϕ_P be the minimum $\phi \geq 1$ that solves $\mathbb{E}_0^{\mathbb{Q}}\left[\left(\frac{\bar{A}_\tau^2 - (\phi\, c_\tau)^2}{8}\right)^+\right] \leq \frac{(N+1)}{2}B$ and let $\delta_P := \phi_P\, c_\tau$. Then the optimal wholesale price and ordering level for each retailer satisfy

$$w_\tau = \frac{\bar{A}_\tau + \delta_P}{2} \qquad \textit{and} \qquad q_\tau = \frac{\left(\bar{A}_\tau - \delta_P\right)^+}{2(N+1)}. \tag{12}$$

The players' expected payoffs conditional on time τ information satisfy

$$\Pi_{P|\tau} = \frac{2N}{(N+1)}\,\frac{(\bar{A}_\tau + \delta_P - 2c_\tau)\,(\bar{A}_\tau - \delta_P)^+}{8} \qquad \textit{and} \qquad \Pi_{R|\tau} = \frac{((\bar{A}_\tau - \delta_P)^+)^2}{4(N+1)^2}. \tag{13}$$

PROOF: The statements regarding the producer follow immediately from Proposition 7 in Caldentey and Haugh (2009) with the budget replaced by $(N+1)B/2$ and the objective function multiplied by $2N/(N+1)$. The statements regarding the retailers are due to the fact that the optimal value of λ in (10) is 0. This value of λ and the optimal value of w_τ can then be substituted into the expression for the optimal ordering quantity in either[12] Case 1 or Case 2 of Proposition 1. The expressions for q_τ and $\Pi_{\text{R}|\tau}$ then follow immediately. □

[11] The factor $2N/(N+1)$ in the objective function has no bearing on the optimal λ and w_τ.

[12] Both cases lead to the same value of q_τ as the producer chooses the price menu in such a way that the budget is at the cutoff point between being binding and non-binding with $\lambda = 0$.

3.1 Should the Retailers Merge in the Equibudget Case?

In the equibudget case we can answer the question as to whether or not the producer and retailers would be better off if the retailers were to merge into a single entity with a combined budget of $N \times B$. In this subsection[13] we will use the superscripts C and M to denote quantities associated with the *competitive* retailers and *merged* retailers, respectively. The constraint in (11) implies that from the perspective of the producer's optimization problem, the merged entity's budget would increase by only a factor of $2N/(N+1)$. Similarly it is clear from (10) that the producer's objective function would be reduced by this same factor of $2N/(N+1)$. As before, the subscripts P and R refer to the producer and retailer, respectively. We will use the subscript AR to denote a quantity that is summed across *all* retailers. This will only apply in the competitive retailer case so, for example, $\Pi^{\mathrm{C}}_{\mathrm{AR}|\tau}$ refers to the total profits of the N retailers when they remain in competition. Our first result is that the producer always prefers the retailers to remain in competition when they have identical budgets.

Proposition 3 (Producer Prefers Competitive Retailers) *The expected profits of the producer when there are N retailers, each with a budget of B, is greater than or equal to his expected profits when there is just one retailer with a budget of $N \times B$.*

PROOF: See Appendix A.

It is worth emphasizing that the producer is only better off in expectation when there are multiple competing retailers. On a path-by-path basis, the producer will not necessarily be better off. In particular, there will be some outcomes where the ordering quantity is zero under the competing retailers model and strictly positive under the merged retailer model. The producer will earn zero profits on such paths under the competing retailer model, but will earn strictly positive profits under the merged retailer model.

Proposition 4 (Retailers Are Always Better Off Merging) *The profits of the merged retailer are greater than the total profits of the N competing retailers on a path-by-path basis.*

PROOF: The profits of the merged retailer is given by $\Pi^{\mathrm{M}}_{\mathrm{R}|\tau} = \frac{((\bar{A}_\tau - \delta_{\mathrm{M}})^+)^2}{16}$ where δ_{M} is the value of δ_{H} in Proposition 7 of Caldentey and Haugh (2009) but with B replaced by $N \times B$. The total profits of the retailers in the Cournot version of the game, however, is given by $\Pi^{\mathrm{C}}_{\mathrm{AR}|\tau} = \frac{N((\bar{A}_\tau - \delta_{\mathrm{P}})^+)^2}{4(N+1)^2}$ where δ_{P} is given by Proposition 2. It is clear that $\delta_{\mathrm{P}} \geq \delta_{\mathrm{M}}$ and so the result follows immediately. □

3.2 Efficiency of the Supply Chain in the Equibudget Case

In this section we briefly discuss the efficiency of the supply chain in the equibudget case. To do this we need to solve the central planner's problem when he has a budget of NB. We can do this by appealing again to the results of Caldentey and Haugh (2009). We focus on production levels,

[13] In Section 3.2 we will use C to refer to the central planner.

double marginalization and the competition penalty. Towards this end, we define the following performance measures, all of which are conditional on $\mathcal{F}_\tau$:

$$\mathcal{Q}_\tau := \frac{Nq_\tau}{q_{\mathrm{C}|\tau}} = \frac{N(\bar{A}_\tau - \delta_{\mathrm{P}})^+}{(N+1)(\bar{A}_\tau - \delta_{\mathrm{C}})^+}, \qquad \mathcal{W}_\tau := \frac{w_\tau}{c_\tau} = \frac{\bar{A}_\tau + \delta_{\mathrm{P}}}{2c_\tau}, \quad \text{and}$$

$$\begin{aligned}\mathcal{P}_\tau &:= 1 - \frac{\mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{P}|\tau}] + N\mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{R}|\tau}]}{\mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{C}|\tau}]} \\ &= 1 - \frac{N}{(N+1)^2}\,\frac{\left[(N+2)\bar{A}_\tau + N\delta_{\mathrm{P}} - 2(N+1)c_\tau\right]\,(\bar{A}_\tau - \delta_{\mathrm{P}})^+}{(\bar{A}_\tau + \delta_{\mathrm{C}} - 2c_\tau)(\bar{A}_\tau - \delta_{\mathrm{C}})^+}\end{aligned}$$

where $\mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{C}|\tau}]$ is the central planner's expected profits, δ_{C} is the smallest value[14] of $\delta \geq c_\tau$ such that $\mathbb{E}_0^{\mathbb{Q}}[c_\tau \left(\frac{\bar{A}_\tau - \delta}{2}\right)^+] \leq NB$, and $q_{\mathrm{C}|\tau}$ is the optimal ordering quantity of the central planner.

It is interesting to note that, conditional on $\mathcal{F}_\tau$, the centralized supply chain is not necessarily more efficient than the decentralized operation. For instance, we know that in some cases $\delta_{\mathrm{P}} < \delta_{\mathrm{C}}$ and so for all those outcomes, ω, with $\delta_{\mathrm{P}} < \bar{A}_\tau < \delta_{\mathrm{C}}$, $q_{\mathrm{C}|\tau} = 0$ and $q_\tau > 0$ and the competition penalty is minus infinity. We mention that this only occurs because of the retailers' ability to trade in the financial markets. If $\delta_{\mathrm{P}} \geq \delta_{\mathrm{C}}$, however, then it is easy to see that the centralized solution is always more efficient than the decentralized supply chain so that $\mathcal{Q}_\tau \leq 1$ and $\mathcal{P} \geq 0$. We also note that if the budget is large enough so that both the decentralized and centralized operations can hedge away the budget constraint then $\delta_{\mathrm{P}} = \delta_{\mathrm{C}} = c_\tau$ and

$$\mathcal{Q}_\tau = \frac{N}{(N+1)} \quad \text{and} \quad \mathcal{P}_\tau = \frac{1}{(N+1)^2}.$$

3.3 Retailers Based in a Foreign Currency Area

We now assume that the retailers and producer are located in different currency areas and use change-of-numeraire arguments to show that our analysis still[15] goes through. Without any loss of generality, we will assume that the retailers and producer are located in the "foreign" and "domestic" currency areas, respectively. The exchange rate, Z_t say, denotes the time t domestic value of one unit of the foreign currency. When the producer proposes a contract, w_τ, we assume that he does so in units of the foreign currency. Therefore the i^{th} retailer pays $q_i w_\tau$ units of foreign currency[16] to the producer. The retailers' problem is therefore unchanged from the problem we considered at the beginning of Section 3 if we take $\mathbb{Q}$ to be an EMM of a *foreign* investor who takes the foreign cash account as his numeraire security. As explained in Appendix B, this same $\mathbb{Q}$ can also be used by the producer as a *domestic* EMM where he takes the *domestic* value of the *foreign* cash account as the numeraire security.

We could take our financial process, X_t, to be equivalent to Z_t so that the retailers hedge their foreign exchange risk in order to mitigate the effects of their budget constraints. This would only make sense if $\bar{A}_\tau$ and the exchange rate, Z_t, were dependent. More generally, we could allow X_t

[14] See Caldentey and Haugh (2009) for details.

[15] See Ding et al. (2007) for a comprehensive review of the literature discussing exchange rate uncertainty in a production/inventory context.

[16] Which is the domestic currency from the retailers' perspective.

to be multi-dimensional so that it includes Z_t as well as other tradeable financial processes that influence $\bar{A}_\tau$.

The producer must convert the retailers' payments into units of the domestic currency and he therefore earns a per-unit profit of either (i) $w_\tau Z_\tau - c_\tau$ if production costs are in units of the domestic currency or (ii) $Z_\tau(w_\tau - c_\tau)$ if production costs are in units of the foreign currency. Case (i) would apply if production takes place domestically whereas case (ii) would apply if production takes place in the foreign currency area. We will assume[17] that interest rates in both the domestic and foreign currency areas are identically zero.

Analogously to (10) and (11) we find in the equibudget case that the producer's problem in case (i) is given by

$$\Pi_{\text{P}} \;=\; \max_{w_\tau,\,\lambda\geq 0} \;\; Z_0 \frac{2N}{N+1} \;\; \mathbb{E}_0^{\mathbb{Q}} \left[\frac{(w_\tau Z_\tau - c_\tau)}{Z_\tau} \, \frac{\left(\bar{A}_\tau - w_\tau\,(1+\lambda)\right)^+}{2} \right] \tag{14}$$

$$\text{subject to} \qquad \mathbb{E}_0^{\mathbb{Q}} \left[w_\tau \, \frac{\left(\bar{A}_\tau - w_\tau\,(1+\lambda)\right)^+}{2} \right] \;\leq\; \frac{(N+1)}{2} B. \tag{15}$$

Note that Z_τ appears in the denominator inside the expectation in (14) because, as explained above, the domestic value of the foreign cash account is the appropriate numeraire corresponding to the EMM, $\mathbb{Q}$. Since we have assumed interest rates are identically zero, the foreign value of the foreign cash-account is identically one and so its domestic value is Z_t at time t. For the same reason, Z_0 appears outside the expectation in (14). Solving the producer's problem in (14) and (15) is equivalent to solving the problem he faced earlier in this section but now with a *stochastic* cost, $\hat{c}_\tau := c_\tau / Z_\tau$. However, it can easily be seen that the proof of Proposition 2, or more to the point, Proposition 7 in Caldentey and Haugh (2009), goes through unchanged when c_τ is stochastic. We therefore obtain the same result as Proposition 2 with c_τ replaced by $\hat{c}_\tau$ and $\mathbb{Q}$ interpreted as a foreign EMM with the domestic value of the foreign cash account as the numeraire security.

Remark: If instead case (ii) prevailed so that the producer's per-unit profit was $Z_\tau(w_\tau - c_\tau)$ then the Z_τ term in both the numerator and denominator of (14) would cancel, leaving the producer with an identical problem to that of Section 3 albeit with different EMMs. So while the analysis for case (ii) is identical to that of Section 3, the probability measures under which the solutions are calculated are different.

3.4 Stochastic Interest Rates and Paying the Producer in Advance

We now consider the problem where the retailers' budgets are only available at time T but that the producer must be paid at time $\tau < T$. We will assume that interest rates are stochastic and no longer identically zero so that the retailers' *effective* time τ budgets are also stochastic. In particular, we will assume that the $\mathbb{Q}$-dynamics of the short rate are given by the Vasicek[18] model so that

$$dr_t \;=\; \alpha(\mu - r_t)\, dt + \sigma dW_t \tag{16}$$

[17] We assume zero interest rates only so that we can focus on the issues related to foreign exchange.

[18] See, Duffie (2004) for a description of the Vasicek model and other related results that we use in this subsection. Note that it is not necessary to restrict ourselves to the Vasicek model. We have done so in order to simplify the exposition but our analysis holds for more general models such as the multi-factor Gaussian and CIR processes that are commonly employed in practice.

where α, μ and σ are all positive constants and W_t is a $\mathbb{Q}$-Brownian motion. The short-rate, r_t, is the instantaneous continuously compounded risk-free interest rate that is earned at time t by the 'cash account', *i.e.*, cash placed in a deposit account. In particular, if \$1 is placed in the cash account at time t then it will be worth $\exp\left(\int_t^T r_s\, ds\right)$ at time $T > t$. It may be shown that the time τ value of a zero-coupon-bond with face value \$1 that matures at time $T > \tau$ satisfies

$$Z_\tau^T \;:=\; e^{a(T-\tau)+b(T-\tau)r_\tau} \tag{17}$$

where $a(\cdot)$ and $b(\cdot)$ are known deterministic functions of the time-to-maturity, $T-\tau$. In particular, Z_τ^T is the appropriate discount factor for discounting a known deterministic cash flow from time T to time $\tau < T$.

Returning to our competitive supply chain, we assume as before that the N retailers' profits are realized at time $T \geq \tau$. Since the producer now demands payment from the retailers at time τ when production takes place this implies that the retailers will be forced to borrow against the capital B that is not available until time T. As a result, the i^{th} retailer's effective budget at time τ is given by

$$B_i(r_\tau) \;:=\; B_i Z_\tau^T \;=\; B_i e^{a(T-\tau)+b(T-\tau)r_\tau}.$$

As before, we assume that the stochastic clearance price, $A-Q$, depends on the financial market through the co-dependence of the random variable A, and the financial process, X_t. To simplify the exposition, we could assume that $X_t \equiv r_t$ but this is not necessary. If X_t is a financial process other than r_t, we simply need to redefine our definition of $\{\mathcal{F}_t\}_{0\leq t\leq T}$ so that it represents the filtration generated by X_t *and* r_t. Before formulating the optimization problems of the retailers and the producer we must adapt our definition of feasible $\mathcal{F}_\tau$-measurable financial gains, G_τ. Until this point we have insisted that any such G_τ must satisfy $\mathbb{E}_0^{\mathbb{Q}}[G_\tau] = 0$, assuming as before that zero initial capital is devoted to the financial hedging strategy. This was correct when interest rates were identically zero but now we must replace that condition with the new condition[19]

$$\mathbb{E}_0^{\mathbb{Q}}\left[D_\tau G_\tau\right] \;=\; 0 \tag{18}$$

where $D_\tau := \exp\left(-\int_0^\tau r_s\, ds\right)$ is the stochastic discount factor. The i^{th} retailer's problem for a given $\mathcal{F}_\tau$-measurable wholesale price, w_τ, is therefore given[20] by

$$\Pi_{\mathrm{R}}(w_\tau) \;=\; \max_{q_i\geq 0,\, G_\tau} \mathbb{E}_0^{\mathbb{Q}}\left[D_T\left(A_T - \left(q_i + Q_{i-}\right)\right) q_i - D_\tau w_\tau q_i\right] \tag{19}$$

$$\text{subject to} \qquad w_\tau\, q_i \leq B_i(r_\tau) + G_\tau, \qquad \text{for all } \omega \in \Omega \tag{20}$$

$$\mathbb{E}_0^{\mathbb{Q}}\left[D_\tau G_\tau\right] \;=\; 0 \tag{21}$$

$$\text{and } \mathcal{F}_\tau - \text{measurability of } q_i. \tag{22}$$

Note that both D_T and D_τ appear in the objective function (19) and reflect the times at which the retailer makes and receives payments. We also explicitly imposed the constraint that q_i be $\mathcal{F}_\tau$-measurable. This was necessary[21] because of the appearance of D_T in the objective function.

[19] See the first paragraph of Appendix B for why this is the case.

[20] We write A_T for A to emphasize the timing of the cash-flow.

[21] To be precise, terms of the form $D_T(A_T - q_i)$ should also have appeared in the problem formulations of earlier sections in this paper. In those sections, however, $D_t \equiv 1$ for all t and so the conditioning argument we use above allows us to replace A_T with $\bar{A}_\tau$ in those sections.

We can easily impose the $\mathcal{F}_\tau$-measurability of q_i by conditioning with respect to $\mathcal{F}_\tau$ inside the expectation appearing in (19). We then obtain

$$\mathbb{E}_0^{\mathbb{Q}}\left[D_\tau\left(\bar{A}_\tau^{\mathrm{D}} - (q_i + Q_{i-})\right) - w_\tau)\ q_i\right] \tag{23}$$

as our new objective function where $\bar{A}_\tau^{\mathrm{D}} := \mathbb{E}_\tau^{\mathbb{Q}}[D_T A_T]/D_\tau$. With this new objective function it is no longer necessary to explicitly impose the $\mathcal{F}_\tau$-measurability of q_i.

It is still straightforward to solve for the retailers' Cournot equilibrium. One could either solve the problem directly as before or alternatively, we could use the change-of-numeraire method of Section 3.3 that is described in Appendix B. In particular, we could switch to the so-called *forward measure* where the EMM, $\mathbb{Q}_\tau$, now corresponds to taking the zero-coupon bond maturing at time τ as the numeraire. In that case the i^{th} retailer's objective function in (23) can be written[22] as

$$Z_0^\tau\ \mathbb{E}_0^{\mathbb{Q}_\tau}\left[\left(\bar{A}_\tau^{\mathrm{D}} - (q_i + Q_{i-})\right) - w_\tau)\ q_i\right]. \tag{24}$$

We can therefore solve for the retailers' Cournot equilibrium using our earlier analysis but with $\bar{A}_\tau$ and $\mathbb{Q}$ replaced by $\bar{A}_\tau^{\mathrm{D}}$ and $\mathbb{Q}_\tau$, respectively. Note that the constant factor, Z_0^τ, in (24) is the same for each retailer and therefore makes no difference to the analysis. Following the first approach we obtain the following solution to the retailer's problem. We omit the proof as it is very similar to the proof of Proposition 1.

Proposition 5 (Retailers' Optimal Strategy)

Let w_τ be an $\mathcal{F}_\tau$-measurable wholesale price offered by the producer and define $\mathcal{Q}_\tau := \frac{\left(\bar{A}_\tau^D - w_\tau\right)^+}{(N+1)\,Z_\tau^T}$. This is the optimal ordering quantity for each retailer in the absence of any budget constraints. The following two cases arise:

Case 1: *Suppose $\mathbb{E}_0^{\mathbb{Q}}\left[D_\tau \mathcal{Q}_\tau w_\tau\right] \le \mathbb{E}_0^{\mathbb{Q}}\left[D_\tau B(r_\tau)\right] = Z_0^T B$. Then $q_i(w_\tau) := q_\tau := \mathcal{Q}_\tau$ for all i and (possibly due to the ability to trade in the financial market) the budget constraints are not binding.*

Case 2: *Suppose $\mathbb{E}_0^{\mathbb{Q}}\left[D_\tau \mathcal{Q}_\tau w_\tau\right] > Z_0^T B$. Then*

$$q_i(w_\tau) := q_\tau := \frac{\left(\bar{A}_\tau^D - w_\tau\,(1+\lambda)\right)^+}{(N+1)Z_\tau^T} \quad \text{for all} \quad i = 1, \ldots, N \tag{25}$$

where $\lambda \ge 0$ solves

$$\mathbb{E}_0^{\mathbb{Q}}\left[D_\tau w_\tau q_\tau\right] \;=\; \mathbb{E}_0^{\mathbb{Q}}\left[B(r_\tau) D_\tau\right] \;=\; Z_0^T B. \tag{26}$$

Given the retailers' best response, the producer's problem may now be formulated[23] as

$$\Pi_{\mathrm{P}} = \max_{w_\tau,\, \lambda \ge 0}\ \mathbb{E}_0^{\mathbb{Q}}\left[D_\tau (w_\tau - c_\tau)\,\frac{\left(\bar{A}_\tau^{\mathrm{D}} - w_\tau\,(1+\lambda)\right)^+}{(N+1)\,Z_\tau^T}\right] \tag{27}$$

$$\text{subject to} \qquad \mathbb{E}_0^{\mathbb{Q}}\left[D_\tau w_\tau\,\frac{\left(\bar{A}_\tau^{\mathrm{D}} - w_\tau\,(1+\lambda)\right)^+}{(N+1)\,Z_\tau^T}\right] \le Z_0^T B. \tag{28}$$

[22] The condition (21) can also be written in terms of $\mathbb{Q}_\tau$ as $Z_0^\tau\ \mathbb{E}_0^{\mathbb{Q}_\tau}[G_\tau] = 0$, i.e., $\mathbb{E}_0^{\mathbb{Q}_\tau}[G_\tau] = 0$.

[23] We assume here and in the foreign retailer setting that the production costs, c_τ, are paid at time τ.

The Cournot-Stackelberg equilibrium and solution of the producer's problem in the equibudget case is given by the following proposition. We again omit the proof of this proposition as it it very similar to the proof of Proposition 2.

Proposition 6 (The Equilibrium Solution)
Let ϕ_P *be the minimum* $\phi \geq 1$ *that satisfies* $\mathbb{E}_0^{\mathbb{Q}}\left[\frac{D_\tau}{8Z_\tau^T}\left((\bar{A}_\tau^D)^2 - (\phi c_\tau)^2\right)^+\right] \leq \frac{(N+1)}{2} Z_0^T B$ *and let* $\delta_P := \phi_P c_\tau$. *Then the optimal wholesale price and ordering level satisfy*

$$w_\tau = \frac{\delta_P + \bar{A}_\tau^D}{2} \quad \textit{and} \quad q_\tau = \frac{\left(\bar{A}_\tau^D - \delta_P\right)^+}{2(N+1)Z_\tau^T}.$$

4 The Cournot Game in the Non-Equibudget Case

We now consider the more general and interesting problem where the retailers are no longer assumed to have identical budgets. We will focus on the case where interest rates are identically zero but note that the change of measure argument of Section 3.4 can easily be applied to handle stochastic interest rates. Taking Q_{i-} and the producer's price menu, w_τ, as fixed, the i^{th} retailer's problem is formulated as

$$\Pi_{\mathrm{R}_i}(w_\tau) = \max_{q_i \geq 0,\, G_\tau} \mathbb{E}_0^{\mathbb{Q}}\left[\left(\bar{A}_\tau - (q_i + Q_{i-}) - w_\tau\right) q_i\right] \tag{29}$$

$$\text{subject to} \quad w_\tau\, q_i \leq B_i + G_\tau, \qquad \text{for all } \omega \in \Omega \tag{30}$$

$$\mathbb{E}_0^{\mathbb{Q}}\left[G_\tau\right] = 0. \tag{31}$$

Each of the N retailers must solve this problem and our goal is to characterize the resulting Cournot equilibrium. We also assume that the retailers have been ordered so that $B_1 \geq B_2 \geq \ldots \geq B_N$. The following proposition, whose statement requires some additional notation, computes the retailers' equilibrium order quantities as a function of w_τ. First, we define the random variable $\alpha_\tau := \bar{A}_\tau / w_\tau$. Since $\bar{A}_\tau$ is the expected maximum clearing price (corresponding to $Q = 0$) and w_τ is the procurement cost, we may interpret $\alpha_\tau - 1$ as the expected maximum per unit margin of the retail market. It follows that in equilibrium the producer chooses w_τ so that $\alpha_\tau \geq 1$ and we will assume that this condition is satisfied. We also define the auxiliary function, $H(\cdot)$, which plays an important role in solving the retailers' problem:

$$H(B) := \inf\{x \geq 1 \quad \text{such that} \quad \mathbb{E}_0^{\mathbb{Q}}[w_\tau^2\,(\alpha_\tau - x)^+] \leq B\}.$$

Note that $H(B)$ is a non-increasing function in $B > 0$.

Proposition 7 *For a given wholesale price menu,* w_τ, *the optimal ordering quantities,* q_i, *satisfy*

$$q_i = w_\tau \left[\frac{\alpha_\tau}{n_\tau + 1} - \frac{\alpha_i}{i+1} + \sum_{j=i+1}^{n_\tau} \frac{\alpha_j}{j\,(j+1)}\right]^+ \qquad \textit{for all} \quad i = 1, 2, \ldots, N$$

$$\mathbb{E}_0^{\mathbb{Q}}[w_\tau\, q_i] = B_i \quad \textit{if} \quad \alpha_i > 1,$$

where

$$\alpha_i \;:=\; H((i+1)B_i + B_{i+1} + \cdots + B_N) \qquad \textit{for all} \;\; i = 1, 2, \ldots, N, \tag{32}$$

$$n_\tau \;:=\; \max\{i \in \{0, 1, \ldots, N\} \quad \textit{such that} \quad \alpha_i \leq \alpha_\tau\}. \tag{33}$$

PROOF: See Appendix A.

It is clear from the proof of Proposition 7 that the random variable n_τ is the number of retailers that order a positive quantity given the wholesale price, w_τ. Furthermore, the ordering $B_1 \geq B_2 \geq \cdots \geq B_N$ implies that $q_i > 0$ if and only if $i \leq n_\tau$. The parameter α_i is therefore the cutoff point such that the i^{th} retailer orders a positive quantity only if $\alpha_\tau \geq \alpha_i$. It follows from equation (32) that α_i does not depend on the $i-1$ highest budgets, B_j, for $j = 1, \ldots, i-1$. In fact α_i only depends on B_i, the sum of the $N-i$ smallest budgets and the *number* of retailers, $i-1$, that have a budget larger than B_i. As a result, q_i only depends on B_i, $(B_{i+1} + \cdots + B_N)$ and i. In other words, the procurement decisions of small retailers are unaffected by the size of larger retailers for a given wholesale price w_τ. In equilibrium, however, we expect the wholesale price w_τ to depend on the entire vector of budgets. Proposition 7 also implies that

$$q_i - q_{i+1} = w_\tau \left(\frac{(\alpha_\tau - \alpha_i)^+ - (\alpha_\tau - \alpha_{i+1})^+}{i+1} \right), \quad i = 1, 2, \ldots, N$$

and this confirms our intuition that larger retailers order more than smaller ones so that q_i is non-increasing in i. This follows from the fact that $H(B)$ is non-increasing in B which implies that the α_i's are non-increasing in i. Having characterized the Cournot equilibrium of the N retailers, we can now determine the producer's expected profits, $\Pi_P = \mathbb{E}_0^{\mathbb{Q}}[(w_\tau - c_\tau)\, Q(w_\tau)]$, for a fixed price menu, w_τ. We have the following proposition.

Proposition 8 *The producer's expected payoff satisfies*

$$\Pi_P \;=\; \frac{m-1}{m}\, \mathbb{E}_0^{\mathbb{Q}}[(w_\tau - c_\tau)\,(\bar{A}_\tau - w_\tau)] \;+\; \sum_{j=m}^{N} \left(\frac{B_j}{m} \;-\; \frac{c_\tau}{j(j+1)}\, \mathbb{E}_0^{\mathbb{Q}}[(\bar{A}_\tau - \alpha_j\, w_\tau)^+] \right) \tag{34}$$

where

$$m \;=\; m(\{w_\tau\}) \;:=\; \max\{i \geq 1 \quad \textit{such that} \quad \alpha_{i-1} = 1\} \tag{35}$$

with $\alpha_0 := 1$.

PROOF: See Appendix A.

Note that m is the index of the first retailer whose budget constraint is binding[24] with the understanding that if $m = N+1$ then all N retailers are non-binding. We can characterize those values of $m \in \{1, \ldots, N+1\}$ that are possible. In particular, if the producer sets $w_\tau = \bar{A}_\tau$ then all of the retailers are non-binding and so $m = N+1$. We can also find the smallest possible value of m, $m_{\min}$ say, by setting $w_\tau = c_\tau$, solving for the resulting α_i's using (32) and then taking $m_{\min}$ according to (35). Assuming the B_i's are distinct, the achievable values of m are given by the set $M_{\text{feas}} := \{m_{\min}, \ldots, N+1\}$. This can be seen by taking $w_\tau = \gamma c_\tau + (1-\gamma)\bar{A}_\tau$ with $\gamma = 0$ initially and then increasing it to 1. In the process each of the values in M_{feas} will be obtained.

We could use this observation to solve numerically for the producer's optimal menu, w_τ^*, by solving a series of sub-problems. In particular we could solve for the optimal price menu subject to the constraint that $m = m^*$ for each possible value of $m^* \in M_{\text{feas}}$. Each of these $N - m^* + 2$ sub-problems could be solved numerically after discretizing the probability space. The overall optimal price menu, w_τ^*, is then simply the optimal price menu in the sub-problem whose objective function is maximal.

[24] We say a player is binding if his budget constraint is binding in the Cournot equilibrium. Otherwise a player is non-binding.

4.1 A Constant Wholesale Price

The problem of numerically optimizing the expected payoff in (34) is considerably more tractable if the producer offers a constant wholesale price $\bar{w}$ instead of a random menu w_τ. From a practical standpoint, this is an important special case since a constant wholesale price is also a much simpler contract to implement. In this case, Proposition 8 can be specialized as follows.

Corollary 1 *Under a constant wholesale price, $\bar{w}$, the producer's expected payoff is given by*[25]

$$\Pi_P = \frac{\bar{w} - c_\tau}{m} \left((m-1)\ \mathbb{E}_0^{\mathbb{Q}}[(\bar{A}_\tau - \bar{w})^+] \ + \sum_{j=m}^{N} \frac{B_j}{\bar{w}} \right). \tag{36}$$

Let $\bar{w}^*(B_1, \ldots, B_N)$ be the constant wholesale price that maximizes the value of Π_P. Then, it follows that $\bar{w}^*(B_1, \ldots, B_N) \geq \bar{w}^*(\infty, \ldots, \infty)$ for all $B_1 \geq B_2 \geq \cdots \geq B_N$.

PROOF: See Appendix A.

The last part of Corollary 1 asserts that it is in the producer's best interest to increase the wholesale price when selling to budget-constraint retailers. By doing so the producer is inducing the retailers to reallocate their limited budgets into those states in which demand is high and for which the retailers have the incentives to invest more of their budgets in procuring units from the producer. As a result, the producer is able to extract a larger fraction of the retailers initial budgets.

While m is a function of $\bar{w}$ it is nonetheless straightforward to check that Π_P in (36) is a continuous function of $\bar{w}$. Note also that if some budget is transferred from one non-binding player to another non-binding player and both players remain non-binding after the transfer then Π_P is unchanged. Similarly if some budget is transferred from one binding player to another binding player and both players remain binding after the transfer then Π_P is again unchanged. Both of these statements follow from (36) and because it is easy to confirm that in each case the value of m is unchanged.

If some budget is transferred from a binding player to a non-binding player, however, then the ordering of the B_i's and the definition of the α_i's imply that both players remain binding and non-binding, respectively, after the transfer. Therefore m remains unchanged and Π_P decreases according to (36).

Conversely, we can increase Π_P by transferring budget from a non-binding player to a binding player in such a way that both players remain non-binding and binding, respectively, after the transfer. It is also possible to increase Π_P if budget is transferred from one non-binding player to another non-binding player so that the first player becomes binding after the transfer.

Note that the statements above are consistent with the idea that the producer would like to see the budgets spread evenly among the various retailers. See Proposition 9 below for a similar result.

4.2 Should the Retailers Merge or Remain in Competition?

A question of particular interest is whether or not the retailers should merge or remain in competition. We now give a partial[26] answer to that question from the producer's perspective. The

[25] The positive part in (36) is needed because in this case with $w_\tau = \bar{w}$ it is not necessarily true that $\alpha_\tau = \bar{A}_\tau / w_\tau \geq 1$.

[26] We were able to give a more complete answer in Section 3 when we specialized to the case of equal budgets.

following proposition, which we prove in Appendix A, describes conditions under which the producer always[27] prefers the retailers to remain in competition.

Proposition 9

(a) For any w_τ, the producer prefers the N retailers to remain in competition rather than merging and combining their budgets when the marginal production cost, c_τ, is zero. In particular, this is true in the Cournot-Stackelberg equilibrium where the producer optimizes over w_τ.

(b) If w_τ is restricted to a constant, then the producer prefers the N retailers to remain in competition rather than merging and combining their budgets. In particular, this is true in the Cournot-Stackelberg equilibrium where the producer optimizes over the constant, w_τ.

PROOF: See Appendix A for the proof of (a). The proof of (b) follows from the discussion immediately following Corollary 1.

5 The Value of Financial Markets

In this section we discuss the value that the financial markets add to the competitive supply chain. There are two means by which the financial markets add value: (i) as a mechanism for mitigating the retailers' budget constraints via dynamic trading and (ii) as a source of public information upon which the ordering quantities and prices are contingent. We begin with (i) and towards this end we need to discuss the so-called[28] *F-contract*. The F-contract is in fact identical to our earlier contract but we now assume that the retailers can no longer trade in the financial markets.

5.1 The F-Contract

Drawing on the results of Caldentey and Haugh (2009) in the single-retailer case, we can compare the performance of the supply chain across the two contracts in the equibudget case as well as determining the players' preferences over each contract. We begin with a brief discussion of the retailers' problem in the general case where budgets are not identical across the N retailers.

The General Case: Non-Identical Budgets

For a fixed price menu, w_τ, it is straightforward to solve for the retailers' Cournot equilibrium. In particular, the i^{th} player solves

$$\Pi^{\mathrm{F}}_{\mathrm{R}}(w_\tau) = \max_{q_i \geq 0} \mathbb{E}^{\mathbb{Q}}_0 \left[\left(\bar{A}_\tau - (q_i + Q_{i-}) - w_\tau \right) q_i \right] \tag{37}$$

$$\text{subject to} \qquad w_\tau \, q_i \leq B_i \qquad \text{for all } \omega \in \Omega. \tag{38}$$

[27] While we expect this result to be true in general, we have been unable to prove it. We also prove this result in Section 3 when the retailers have identical budgets.

[28] The term "F-contract" was introduced by Caldentey and Haugh (2009) and so we will use the same term here.

This problem decouples and is solved separately for each outcome, ω. The first order conditions imply

$$q_i \;=\; \min\left(\frac{B_i}{w_\tau},\, \frac{(\bar{A}_\tau - Q_{i-} - w_\tau)^+}{2}\right).$$

We see that there is a function $m(\omega) \in \{0, 1, \ldots, N\}$ so that the budget constraints are not binding in state ω for the first m retailers only. The solution then takes the form

$$q_i^{(m)} \;=\; \begin{cases} q^{(m)} \;:=\; \dfrac{\left(\bar{A}_\tau \;-\; \sum_{j=m+1}^{N} \frac{B_j}{w_\tau} \;-\; w_\tau\right)^+}{m+1}, & i \;=\; 1, \ldots, m \\ B_i / w_\tau, & i \;=\; m+1, \ldots, N. \end{cases} \tag{39}$$

Note that m was a constant in Section 4 whereas here m is random. In order to determine the value of $m = m(\omega)$, we must determine that value of m whereby the m^{th} retailer can afford to order $q^{(m)}$ units but where the $(m+1)^{th}$ retailer cannot afford $q^{(m)}$ units. Mathematically, this translates to determining the value of m such that $B_{m+1} < q^{(m)} w_\tau \leq B_m$ with the understanding that $B_{N+1} := 0$. If no such $m \geq 1$ exists then we take $m = 0$ and the budget constraints bind for all N retailers. It is also necessary to check that there is not more than one value of m for which the above conditions hold. While this may seem intuitively clear, it is not immediately obvious and so we state it as a Lemma which we prove in Appendix A.

Lemma 1 *There is at most one value of $m \in \{1, \ldots, N\}$ satisfying $B_{m+1} < q^{(m)} w_\tau \leq B_m$.*

The retailers' problem is then solved separately for each $\omega \in \Omega$ by determining the number of non-binding retailers, $m(\omega)$. The producer's problem also decouples and he simply chooses $w_\tau(\omega)$ to optimize his expected profits given the retailers reaction function. We could characterize the values of w_τ for which exactly i retailers are non-binding for $i = 0, \ldots, N$ and then determine an expression for the producer's expected profits. Since our focus in this paper is not on the F-contract, however, we will move instead to the equibudget case where it is possible to make statements concerning the players preferences over the two contracts.

The Equibudget Case

When the N retailers all have the same budget, B, then (39) is easily seen to reduce to

$$q_i \;=\; q_\tau \;:=\; \min\left(\frac{\left(\bar{A}_\tau \;-\; w_\tau\right)^+}{N+1},\, \frac{B}{w_\tau}\right) \quad \text{for all } i \;=\; 1, \ldots, N. \tag{40}$$

The producer's optimal objective function then becomes

$$\Pi_{\mathrm{P}}^{\mathrm{F}} \;=\; N\, \mathbb{E}_0^{\mathbb{Q}}\left[\max_{w_\tau \geq c_\tau} \left\{(w_\tau - c_\tau)\, \min\left(\frac{\left(\bar{A}_\tau \;-\; w_\tau\right)^+}{N+1},\, \frac{B}{w_\tau}\right)\right\}\right] \tag{41}$$

$$\;=\; \mathbb{E}_0^{\mathbb{Q}}\left[\max_{w_\tau \geq c_\tau} \left\{(w_\tau - c_\tau)\, \min\left(\frac{N\left(\bar{A}_\tau \;-\; w_\tau\right)^+}{N+1},\, \frac{NB}{w_\tau}\right)\right\}\right]$$

$$\;\geq\; \mathbb{E}_0^{\mathbb{Q}}\left[\max_{w_\tau \geq c_\tau} \left\{(w_\tau - c_\tau)\, \min\left(\frac{\left(\bar{A}_\tau \;-\; w_\tau\right)^+}{2},\, \frac{NB}{w_\tau}\right)\right\}\right]. \tag{42}$$

But (42) is the producer's problem when the N retailers merge and have a combined budget of NB. We have therefore shown that the producer also prefers the retailers to remain in competition when the flexible contract is under consideration. Explicit solutions for the maximization problems in (41) and (42) are easily computed and are given in Caldentey and Haugh (2009). We also obtain the following result.

Proposition 10 *In the equibudget case the producer is always better off if the N retailers have access to the financial markets.*

PROOF: When the retailers have access to the financial markets the producer's objective function is given by (10). But this is equivalent to $2N/(N+1)$ times the objective function of the producer when there is just a single retailer with a budget of $(N+1)B/2$. Similarly, the producer's objective function in (41) is equivalent to $2N/(N+1)$ times the producer's objective function in the flexible setting with just a single retailer having a budget of $(N+1)B/2$. But then the result follows immediately from Proposition 8 in Caldentey and Haugh (2009) who show in the single retailer setting that the producer always prefers the retailer to have access to the financial markets. □

Caldentey and Haugh (2009) show that the situation is more complicated for the retailers. In particular, the retailers may or may not prefer having access to the financial markets in equilibrium. The relationship between c_τ and δ_{P} (as defined in Proposition 2) is key: if $c_\tau = \delta_{\mathrm{P}}$ the retailers also prefer having access to the financial markets. If $c_\tau < \delta_{\mathrm{P}}$, however, then their preferences can go either way.

5.2 The Value of Information in the Financial Markets

The financial markets also add value to the supply chain by allowing the retailers to mitigate their budget constraints via dynamic trading. The next proposition emphasizes the value of information in a competitive supply chain. Under the assumption of zero marginal production costs, it states that for an $\mathcal{F}_{\tau_1}$-measurable price menu, w_τ, the producer is always better off when the retailers' orders are allowed to be contingent upon time τ_2 information where $\tau_2 > \tau_1$. Later in Appendix C we will discuss the optimal timing, τ, of the contract. Clearly the optimal τ achieves the optimal tradeoff between the value of additional information and the cost associated with delaying production.

Proposition 11 *Consider two times $\tau_1 < \tau_2$ and let w_τ be an $\mathcal{F}_{\tau_1}$-measurable price menu. Consider the following two scenarios: (1) the producer offers price menu w_τ and the retailers choose their Cournot-optimal $\mathcal{F}_{\tau_1}$-measurable ordering quantities which is then produced at time τ_1 and (2) the producer again offers price menu w_τ but the retailers now choose their Cournot-optimal $\mathcal{F}_{\tau_2}$-measurable ordering quantities which is then produced at time τ_2. If $c_{\tau_1} = c_{\tau_2} = 0$ then the producer always prefers scenario (2).*

PROOF: See Appendix A.

The conclusion of Proposition 11 might appear to be obvious as it is clearly true that the *retailers* would prefer scenario (2). After all, scenario (2) gives them (at no extra cost) additional information

upon which to base their ordering decisions and additional time to run their financial hedging strategy. However, it is not immediately clear that the producer should also benefit from this delay. Proposition 11 states that the producer does indeed benefit from this delay, at least when marginal production costs are zero.

6 Conclusions and Further Research

We have studied the performance of a stylized supply chain where multiple retailers and a single producer compete in a Cournot-Stackelberg game. At time $t = 0$ the retailers order a single product from the producer and upon delivery at time $T > 0$, they sell it in the retail market at a stochastic clearance price that depends in part on the realized path or terminal value of some tradeable financial process. Because production and delivery do not take place until time T, the producer offers a *menu* of wholesale prices to the retailer, one for each realization of the process up to time some time, τ, where $0 \leq \tau \leq T$. The retailers' ordering quantities can therefore depend on the realization of the process until time τ. We also assumed, however, that the retailers were budget-constrained and were therefore limited in the number of units they could purchase from the producer. Because the supply chain is potentially more profitable if the retailers can allocate their budgets across different states we allow them to trade dynamically in the financial market. After solving for the Nash equilibrium we addressed such questions as: (i) whether or not the players would be better off if the retailers merged and (ii) whether or not the players are better off when the retailers have access to the financial markets. We also considered variations of the model where, for example, the retailers were located in a different currency area to the producer. Finally in Appendix C we consider the situation where the producer could choose the optimal timing, τ, of the contract and we formulated this as an optimal stopping problem.

There are several possible directions for future research. First, it would be interesting to model and solve the game where each retailer's budget constitutes private information that is known only to him. This problem formulation would therefore require us to solve for a Bayesian Nash equilibrium. It would also be of interest to identify and calibrate settings where supply chain payoffs are strongly dependent on markets. We would then like to estimate just how much value is provided by the financial markets in its role as (i) a source of public information upon which contracts may be written and (ii) as a means of mitigating the retailers' budget constraints. A further direction is to consider alternative contracts such as 2-part tariffs for coordinating the supply chain. Of course we would still like to have these contracts be contingent upon the outcome of the the financial markets. Finally, we would like to characterize the producer's optimal price menu in the general non-equibudget case and where the marginal cost, c_τ, is not zero.

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References

Babich, V., G. Aydm, Y. Brunet, J. Keppo, R. Saigal. 2008. Risk, financing and the optimal number of suppliers. Tech. rep., Industrial and Operations Engineering, University of Michigan, Ann Arbor, MI.

Babich, V., M. Sobel. 2004. Pre-ipo operational and financial decisions. *Management Sci.* **50** 935–948.

Bernstein, F., A. Federgruen. 2003. Pricing and replenishment strategies in a distribution system with competing retailers. *Oper. Res.* **51**(3) 409–426.

Boyabatli, O., L.B. Toktay. 2004. Operational hedging: A review with discussion. Working paper, Insead, Fontainebleau, France.

Boyabatli, O., L.B. Toktay. 2010. Stochastic capacity investment and flexible versus dedicated technology choice in imperfect capital markets. Working paper, Lee Kong Chian School of Business, Singapore Management University, Singapore., Available at SSRN: http://ssrn.com/abstract=1003551.

Buzacott, J.A., R. Q. Zhang. 2004. Inventory management with asset-based financing. *Management Sci.* **50**(9) 1274–1292.

Cachon, G. 2003. Supply chain coordination with contracts. A.G. de Kok, S.A. Graves, eds., *Supply Chain Management: Design, Coordination and Operation*, *Handbooks in Operations Research and Management Science.*, vol. 11, chap. 6. Elsevier, Amsterdam, The Netherlands.

Caldentey, R., X. Chen. 2009. The role of financial services in procurement contracts. Tech. rep., New York University, New York.

Caldentey, R., M. Haugh. 2006. Optimal control and hedging of operations in the presence of financial markets. *Math. Oper. Res.* **31**(2) 285–304.

Caldentey, R., M. Haugh. 2009. Supply contracts with financial hedging. *Oper. Res.* **57**(1) 47–65.

Chod, J., N. Rudi, J. Van Mieghem. 2009. Operational flexibility and financial hedging: Complements or substitutes? *Management Science* **56**(6) 1030–1045.

Dada, M., Q.J. Hu. 2008. Financing the newsvendor inventory. *Operations Research Letters* **36**(5).

de Kok, A.G., S.A. Graves, eds. 2003. *Supply Chain Management: Design, Coordination and Operation*, *Handbooks in Operations Research and Management Science*, vol. 11. Elsevier, Amsterdam, The Netherlands.

Ding, Q., L. Dong, P. Kouvelis. 2007. On the integration of production and financial hedging decisions in global markets. *Oper. Res.* **55**(3) 470–489.

Dong, L., P. Kouvelis, P. Su. 2006. Operational hedging strategies and competitive exposure to exchange rates. Working paper, Olin Business School, St. Louis.

Duffie, D. 2004. *Dynamic Asset Pricing Theory*. Princeton University Press, New Jersey.

Hu, Q.M., M.J. Sobel. 2005. Capital structure and inventory management. Working paper, Case Western Reserve University.

Kouvelis, P., W. Zhao. 2008. Financing the newsvendor: Supplier vs. bank, optimal rates, and alternative schemes. Working paper, Olin Business School, Washington University, St. Louis.

Lariviere, M.A., E.L. Porteus. 2001. Selling to the newsvendor: An analysis of price-only contracts. *Manufacturing Service Oper. Management* **3**(4) 293–305.

Li, L. 2002. Information sharing in a supply chain with horizontal competition. *Management Sci.* **48**(9) 1196–1212.

Okoguchi, K., F. Szidarovsky. 1999. *The Theory of Oligopoly with Multi-Product Firms*. Springer, New York.

Øksendal, B. 1998. *Stochastic Differential Equations*. Springer Verlag, New York.

Simchi-Levi, D., S. D. Wu, Z. Shen, eds. 2004. *Handbook of Quantitative Supply Chain Analysis: Modeling in the eBusiness Era*. Springer Science, New York, NY.

Vives, X. 2001. *Oligopoly Pricing: Old Ideas and New Tools*. Prentice Hall, Boston, Massachusetts.

Xu, X.D., J.R. Birge. 2004. Joint production and financing decisions: modeling and analysis. Working paper, Booth School of Business, University of Chicago, available at SSRN: http://ssrn.com/abstract=652562.

Zhao, H., V. Deshpande, J. K. Ryan. 2005. Inventory sharing and rationing in decentralized dealer networks. *Management Sci.* **51**(4) 531–547.

A Proofs

PROOF OF PROPOSITION 1: It is straightforward to see that $\mathcal{Q}_\tau$ is each of the N retailer's optimal ordering level given the wholesale price menu, w_τ, in the absence of a budget constraint. This follows from a standard Cournot-style analysis in which all of the retailers, owing to their identical budgets, order the same quantity. In order to implement this solution, each retailer would need a budget $\mathcal{Q}_\tau\, w_\tau$ for all $\omega \in \Omega$. Therefore, if each retailer can generate a financial gain, G_τ, such that $\mathcal{Q}_\tau\, w_\tau \leq B + G_\tau$ for all $\omega \in \Omega$ then he would be able to achieve his unconstrained optimal solution.

By definition, $\mathcal{X}$ contains all those states for which $B \geq \mathcal{Q}_\tau\, w_\tau$. That is, the original budget B is large enough to cover the optimal purchasing cost for all $\omega \in \mathcal{X}$. However, for $\omega \in \mathcal{X}^c$, the initial budget is not sufficient. The financial gain, G_τ, then allows the retailer to transfer resources from $\mathcal{X}$ to $\mathcal{X}^c$.

Suppose the condition in Case 1 holds so that $\mathbb{E}_0^{\mathbb{Q}}\left[\mathcal{Q}_\tau\, w_\tau\right] \leq B$. Note that according to the definition of G_τ in this case, we see that $B + G_\tau = \mathcal{Q}_\tau\, w_\tau$ for all $\omega \in \mathcal{X}^c$. For $\omega \in \mathcal{X}$, however, $B + G_\tau = (1-\delta)\, B + \delta\, \mathcal{Q}_\tau\, w_\tau \geq \mathcal{Q}_\tau\, w_\tau$. The inequality follows since $\delta \leq 1$. G_τ therefore allows the retailer to implement the unconstrained optimal solution. The only point that remains to check is that G_τ satisfies $\mathbb{E}_0^{\mathbb{Q}}[G_\tau] = 0$. This follows directly from the definition of δ.

Suppose now that the condition specified in Case 2 holds. We solve the i^{th} retailer's optimization problem in (29) by relaxing the gain constraint (31) with a Lagrange multiplier, λ_i. We also relax the budget constraint in (30) for each realization of X up to time τ. The corresponding multiplier for each such realization is denoted by $\beta_\tau^{(i)}\, \mathrm{d}\mathbb{Q}$ where $\beta_\tau^{(i)}$ plays the role of a Radon-Nikodym derivative of a positive measure that is absolutely continuous with respect to $\mathbb{Q}$. The first-order optimality conditions for the relaxed version of the retailer's problem are then given by

$$q_i = \frac{(\bar{A}_\tau - w_\tau\,(1+\beta_\tau^{(i)}) - Q_{i-})^+}{2}$$

$$\beta_\tau^{(i)} = \lambda_i, \qquad \beta_\tau^{(i)}\left(w_\tau\, q_i - B + G_\tau\right) = 0, \qquad \beta_\tau^{(i)} \geq 0, \qquad \text{and } \mathbb{E}_0^{\mathbb{Q}}[G_\tau] = 0.$$

We look for a symmetric equilibrium of the above system of equations where $\lambda_i = \lambda$ and $q_i = q$ for all $i = 1, \ldots, N$.

It is straightforward to show that the solution given in Case 2 of the proposition satisfies these optimality conditions; only the non-negativity of $\beta_\tau^{(i)}$ needs to be checked separately. To prove this, note that $\beta_\tau^{(i)} = \lambda_i = \lambda$, therefore it suffices to show that $\lambda \geq 0$. This follows from three observations

(a) Since $0 \leq w_\tau$ the function $\mathbb{E}_0^{\mathbb{Q}}\left[w_\tau\left(\frac{\bar{A}_\tau - w_\tau\,(1+\lambda)}{(N+1)}\right)^+\right]$ is decreasing in λ_i.

(b) In Case 2, by hypothesis, we have

$$\mathbb{E}_0^{\mathbb{Q}}\left[w_\tau\,\frac{\left(\bar{A}_\tau - w_\tau\right)^+}{(N+1)}\right] = \mathbb{E}_0^{\mathbb{Q}}\left[\mathcal{Q}_\tau\, w_\tau\right] > B$$

(c) Finally, we know that λ solves

$$\mathbb{E}_0^{\mathbb{Q}}\left[w_\tau\,\frac{\left(\bar{A}_\tau - w_\tau\,(1+\lambda)\right)^+}{(N+1)}\right] = B.$$

(a) and (b) therefore imply that we must have $\lambda \geq 0$. $\square$

PROOF OF PROPOSITION 3: Let $f(B)$ be the producer's optimal expected profits as a function of the retailers' common budget. Then

$$\begin{aligned} f(B) \;&:=\; \max_{w_\tau,\,\lambda\geq 0} \quad \mathbb{E}_0^{\mathbb{Q}}\left[(w_\tau - c_\tau)\,\frac{\left(\bar{A}_\tau - w_\tau\,(1+\lambda)\right)^+}{2}\right] \\ \text{subject to} \quad & \mathbb{E}_0^{\mathbb{Q}}\left[w_\tau\,\frac{\left(\bar{A}_\tau - w_\tau\,(1+\lambda)\right)^+}{2}\right] \;\leq B. \end{aligned}$$

From (10) and (11) we see that the expected payoff, Π_{P}, that the producer achieves by serving N competing retailers with individual budget B is therefore equal to

$$\Pi_{\text{P}} \;=\; \frac{2N}{N+1}\,f\left(\frac{(N+1)}{2}\,B\right).$$

When there is just one retailer with a budget of NB, the producer's expected profit, $\widetilde{\Pi}_{\text{P}}$ say, satisfies $\widetilde{\Pi}_{\text{P}} = f(NB)$. In order to prove the proposition we must therefore show show that for all $B \geq 0$ and N

$$f(NB) \;\leq\; \frac{2N}{N+1}\,f\left(\frac{(N+1)}{2}\,B\right).$$

Since $f(0) = 0$, a sufficient condition for this inequality to hold is that $f(B)$ is a concave function in $[0,\infty)$. We can use the results in Proposition 3 to rewrite $f(B)$ as

$$f(B) \;=\; \mathbb{E}_0^{\mathbb{Q}}\left[\frac{(\bar{A}_\tau + (1\wedge\phi)\,c_\tau - 2c_\tau)\,(\bar{A}_\tau - (1\wedge\phi)\,c_\tau)^+}{8}\right] \tag{A-1}$$

where ϕ is the positive root of the equation $\mathbb{E}_0^{\mathbb{Q}}\left[(\bar{A}_\tau^2 - (\phi\,c_\tau)^2)^+\right] = 8\,B$. We will prove the concavity of $f(B)$ by first proving it for a discrete approximation to $\bar{A}_\tau$ and then using a convergence argument to prove it for $\bar{A}_\tau$. Specifically, if we define $A_\theta := \theta\lfloor \bar{A}_\tau/\theta \rfloor$ for an arbitrary $\theta > 0$, then A_θ takes values in $\{0,\theta,2\theta,\dots\}$ and

$$\mathbb{Q}(A_\theta = k\,\theta) \;=\; \mathbb{Q}(k\,\theta \leq \bar{A}_\tau < (k+1)\,\theta) \qquad \text{for} \quad k = 0,1,2\dots$$

We define the auxiliary function

$$f_\theta(B) \;:=\; \mathbb{E}_0^{\mathbb{Q}}\left[\frac{(A_\theta + (1\wedge\phi_\theta)\,c_\tau - 2c_\tau)\,(A_\theta - (1\wedge\phi_\theta)\,c_\tau)^+}{8}\right] \tag{A-2}$$

where ϕ_θ is the positive root of the equation $\mathbb{E}_0^{\mathbb{Q}}\left[(A_\theta^2 - (\phi_\theta\,c_\tau)^2)^+\right] = 8\,B$. Since $\lim_{\theta\downarrow 0} A_\theta = \bar{A}_\tau$, we can apply the Dominated convergence Theorem[29] to see that $\lim_{\theta\downarrow 0} f_\theta(B) = f(B)$. And since the limit of concave functions is itself concave, it therefore suffices to prove the concavity of $f_\theta(B)$. To show this, let us define $\bar{B}$ such that $\phi_\theta \geq 1$ for all $B \leq \bar{B}$. It follows that $f_\theta(B) = f_\theta(\bar{B})$ for all $B \geq \bar{B}$. Hence, since $f_\theta(B)$ is continuous and nondecreasing, we only need to prove that $f_\theta(B)$ is concave in the domain $[0,\bar{B}]$.

[29] The expressions inside the expectations in (A-1) and (A-2) are dominated by $\mathbb{E}_0^{\mathbb{Q}}\left[\frac{A_\theta^2}{8}\right]$ so that it is sufficient for convergence that $\bar{A}_\tau$ has a second moment.

For $B \leq \bar{B}$ the budget constraint in the definition of $f_\theta(B)$ is tight and so we can rewrite $f_\theta(B)$ as

$$f_\theta(B) \;=\; B - \frac{c_\tau}{4}\, H_\theta(B), \quad B \leq \bar{B},$$

where

$$H_\theta(B) \;:=\; \mathbb{E}_0^{\mathbb{Q}}\left[(A_\theta - \delta)^+\right] \qquad \text{and } \delta \geq 1 \text{ solves} \qquad \mathbb{E}_0^{\mathbb{Q}}\left[\left(A_\theta^2 - \delta^2\right)^+\right] = 8\,B.$$

For a fixed $B \in [0, \bar{B}]$, let us define $\delta_\theta(B) \geq 0$ as the positive root in the budget constraint above and define a sequence of budgets $0 = B_0 \leq B_1 \leq \cdots \leq B_m = \bar{B}$ such that $k_i := \lceil \delta_\theta(B_i)/\theta \rceil$ for all $B \in [B_i, B_{i+1})$. It then follows that for all $B \in [B_i, B_{i+1})$,

$$\delta_\theta(B) \;=\; \sqrt{\frac{\sum_{k \geq k_i} (k\,\theta)^2\, \mathbb{Q}(A_\theta = k\,\theta) \;-\; 8\,B}{\mathbb{Q}(A_\theta \geq k_i\,\theta)}} \quad \text{and} \quad H_\theta(B) = \sum_{k \geq k_i} k\,\theta - \delta_\theta(B)\,\mathbb{Q}(A_\theta \geq k_i\,\theta).$$

We therefore see that $H_\theta(B)$ is convex in each interval, $[B_i, B_{i+1})$, since $\delta_\theta(B)$ is concave in these intervals. To complete the proof, it suffices to show that $H_\theta(B)$ is continuously differentiable in $[0, \bar{B}]$ which is equivalent to showing that $H_\theta(B)$ is differentiable at each B_i, $i = 1, \ldots, m$. Since $\delta_\theta(B)$ is continuous in B by construction, the continuously differentiability of $H_\theta(B)$ follows by observing that the derivative of $\delta_\theta(B)$ with respect to B is proportional to $1/\delta_\theta(B)$. □

The following lemma is used in proving Proposition 7.

Lemma 2 *The optimal ordering quantities, q_i for $i = 1, \ldots, N$, satisfy*

$$q_i \;=\; \begin{cases} \frac{\left[\bar{A}_\tau - w_\tau\left((n_\tau+1)(1+\lambda_i) - \sum_{j=1}^{n_\tau}(1+\lambda_j)\right)\right]}{(n_\tau+1)}, & \text{for } i \leq n_\tau \\ 0, & \text{otherwise,} \end{cases} \tag{A-3}$$

where $n_\tau = i - 1$ for all outcomes, $\omega \in \Omega$, satisfying $\alpha_{i-1} \leq \alpha_\tau(\omega) < \alpha_i$. Indeed α_i is the value of α_τ where the i^{th} retailer moves from ordering zero to ordering a positive quantity. This cutoff point satisfies

$$\alpha_i \;=\; i(1+\lambda_i) \;-\; \sum_{j=1}^{i-1}(1+\lambda_j) \quad \text{for } i = 1, \ldots, N \tag{A-4}$$

where λ_j is the Lagrange multiplier for the j^{th} retailer's optimization problem. Moreover, if V_τ is an $\mathcal{F}_\tau$-measurable random variable then

$$(i+1)\,\mathbb{E}_0^{\mathbb{Q}}[V_\tau\, q_i] \;+\; \sum_{j=i+1}^{N} \mathbb{E}_0^{\mathbb{Q}}[V_\tau\, q_j] \;=\; \int_{\alpha_i < \alpha_\tau} V_\tau\,(\bar{A}_\tau - \alpha_i\, w_\tau)\,\mathrm{d}\mathbb{Q}. \tag{A-5}$$

Proof of Lemma 2: Taking Q_{i-} and the producer's price menu, w_τ, as fixed, it is straightforward to obtain

$$q_i \;=\; \frac{\left(\bar{A}_\tau - w_\tau\,(1+\lambda_i) - Q_{i-}\right)^+}{2} \tag{A-6}$$

where $\lambda_i \geq 0$ is the deterministic Lagrange multiplier corresponding to the i^{th} retailer's budget constraint. In particular, $\lambda_i \geq 0$ is the smallest real such that $\mathbb{E}_0^{\mathbb{Q}}[w_\tau\, q_i] \leq B_i$. Given the ordering of the budgets, B_i, it follows that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ when they are chosen optimally. Equation

(A-6) and the ordering of the Lagrange multipliers then implies that for each outcome ω, there is a function $n_\tau \in \{0, 1, \ldots, N\}$ such that $q_j(\omega) = 0$ for all $j > n_\tau$. Continuing to write q_i for $q_i(\omega)$, we therefore obtain the following system of equations

$$q_i \;=\; \bar{A}_\tau - w_\tau\,(1+\lambda_i) - Q, \qquad \text{for} \quad i = 1, \ldots, n_\tau \tag{A-7}$$

where we recall that $Q = \sum q_i$. For a fixed ω we have n_τ linear equations in n_τ unknowns which we can easily solve to obtain (A-3). Summing the q_i's we also obtain

$$Q \;=\; \frac{1}{n_\tau + 1}\left[n_\tau \bar{A}_\tau - w_\tau \sum_{i=1}^{n_\tau}(1+\lambda_i)\right]. \tag{A-8}$$

Now suppose $q_i(\omega) = 0$ in some outcome, ω. Then (A-6) implies $\bar{A}_\tau - w_\tau\,(1+\lambda_i) - Q \;\leq\; 0$ which, after substituting for Q using (A-8), implies that

$$1 + \lambda_i \;\geq\; \frac{\alpha_\tau \;+\; \sum_{j=1}^{n_\tau}(1+\lambda_j)}{n_\tau + 1}. \tag{A-9}$$

Let α_i be the cutoff point where the i^{th} retailer moves from ordering zero to ordering a positive quantity. Abusing notation slightly, we see[30] that $n(\alpha_i) = i - 1$ and so (A-9) implies (A-4). Since the λ_i's are non-decreasing in i it is easy to see that the α_i's are also non-decreasing in i, as we would expect. We also see that $n(\alpha_\tau) = k - 1$ for all α_τ satisfying $\alpha_{k-1} \leq \alpha_\tau < \alpha_k$. Setting $\alpha_{N+1} := \infty$, we can combine these results and (A-3) to write

$$q_i \;=\; \sum_{k=i}^{N} \frac{1}{k+1}\left[\bar{A}_\tau - w_\tau\left((k+1)\,(1+\lambda_i) - \sum_{j=1}^{k}(1+\lambda_j)\right)\right] \mathbb{1}\Big(\alpha_\tau \in [\alpha_k, \alpha_{k+1})\Big). \tag{A-10}$$

Letting $\Omega_k := \{\omega :\; \alpha_k \leq \alpha_\tau < \alpha_{k+1}\}$, we see that (A-10) implies

$$\begin{aligned}
\sum_{j=i+1}^{N} \mathbb{E}_0^{\mathbb{Q}}[V_\tau\, q_j] \;&=\; \sum_{j=i+1}^{N}\sum_{k=j}^{N}\int_{\Omega_k} \frac{V_\tau}{k+1}\left[\bar{A}_\tau - w_\tau\left((k+1)\,(1+\lambda_j) - \sum_{s=1}^{k}(1+\lambda_s)\right)\right] \mathrm{d}\mathbb{Q} \\
&=\; \sum_{k=i+1}^{N}\sum_{j=i+1}^{k}\int_{\Omega_k} \frac{V_\tau}{k+1}\left[\bar{A}_\tau - w_\tau\left((k+1)\,(1+\lambda_j) - \sum_{s=1}^{k}(1+\lambda_s)\right)\right] \mathrm{d}\mathbb{Q} \\
&=\; \sum_{k=i+1}^{N}\int_{\Omega_k} \frac{V_\tau}{k+1}\left[(k-i)\,\bar{A}_\tau - w_\tau\left((k+1)\sum_{j=i+1}^{k}(1+\lambda_j) - (k-i)\sum_{s=1}^{k}(1+\lambda_s)\right)\right] \mathrm{d}\mathbb{Q}.
\end{aligned}$$

Combining this last identity and the fact that

$$\mathbb{E}_0^{\mathbb{Q}}[V_\tau\, q_i] \;=\; \sum_{k=i}^{N}\int_{\Omega_k} \frac{V_\tau}{k+1}\left[\bar{A}_\tau - w_\tau\left((k+1)\,(1+\lambda_i) - \sum_{j=1}^{k}(1+\lambda_j)\right)\right] \mathrm{d}\mathbb{Q}$$

we obtain

$$\mathbb{E}_0^{\mathbb{Q}}[V_\tau\, q_i] + \frac{1}{i+1}\sum_{j=i+1}^{N}\mathbb{E}_0^{\mathbb{Q}}[V_\tau\, q_j] \;=\; \sum_{k=i}^{N}\int_{\Omega_k}\frac{V_\tau}{k+1}\,Z_{ik}\,\mathrm{d}\mathbb{Q} \tag{A-11}$$

[30] We are assuming that the N budgets are distinct so that $B_{k-1} > B_k$. This then implies $q_i(\alpha_k) > 0$ for all $i \leq k-1$. The case where some budgets coincide is straightforward to handle.

where

$$Z_{ik} \;:=\; \left[\bar{A}_\tau - w_\tau\left((k+1)\,(1+\lambda_i) - \sum_{j=1}^{k}(1+\lambda_j)\right) + \frac{(k-i)\,\bar{A}_\tau - w_\tau\left((k+1)\,\sum_{j=i+1}^{k}(1+\lambda_j) - (k-i)\,\sum_{s=1}^{k}(1+\lambda_s)\right)}{i+1}\right].$$

and where we have used the convention $\sum_{j=i+1}^{i}(1+\lambda_j) = 0$. After some straightforward manipulations, one can show that

$$Z_{ik} \;=\; \frac{k+1}{i+1}\,(\bar{A}_\tau - \alpha_i\, w_\tau)$$

and so by the definition of Ω_k we can substitute for Z_{ik} in (A-11) and obtain (A-5). □

Proof of Proposition 7: Using (A-4) recursively, one can show that

$$1+\lambda_i = \frac{\alpha_i}{i} + \sum_{j=1}^{i-1}\frac{\alpha_j}{j\,(j+1)}.$$

Substituting this expression in (A-3) and using the fact that $\alpha_i > \alpha_\tau$ for $i > n_\tau$, we obtain the expression for q_i in Proposition 7. In addition, it follows from the proof of Lemma 2 that $\alpha_i - 1 = \lambda_i$ is the Lagrange multiplier for the i^{th} retailer's budget constraint. Hence, if $\alpha_i > 1$ the budget constraint is binding and $\mathbb{E}_0^{\mathbb{Q}}[w_\tau q_i] = B_i$.

To complete the proof, we need to show that α_i and n_τ are given by (32) and (33). The expression for n_τ follows from the proof of Lemma 2.

With V_τ set to w_τ, Lemma 2 implies

$$\mathbb{E}_0^{\mathbb{Q}}[w_\tau\, q_i] + \frac{1}{i+1}\sum_{j=i+1}^{N}\mathbb{E}_0^{\mathbb{Q}}[w_\tau\, q_j] \;=\; \int_{\alpha_i<\alpha_\tau}\frac{w_\tau}{i+1}\,(\bar{A}_\tau - \alpha_i\, w_\tau)\mathrm{d}\mathbb{Q} \tag{A-12}$$

for $i = 1, \ldots, N$. But the budget constraints for the N retailers also imply

$$\mathbb{E}_0^{\mathbb{Q}}[w_\tau\, q_i] + \frac{1}{i+1}\sum_{j=i+1}^{N}\mathbb{E}_0^{\mathbb{Q}}[w_\tau\, q_j] \;\le\; B_i + \frac{1}{i+1}\sum_{j=i+1}^{N} B_j \tag{A-13}$$

which, when combined with (A-12), leads to

$$\int_{\alpha_i<\alpha_\tau} w_\tau\,(\bar{A}_\tau - \alpha_i\, w_\tau)\mathrm{d}\mathbb{Q} \;\le\; (i+1)\,B_i + \sum_{j=i+1}^{N} B_j \tag{A-14}$$

for $i = 1, \ldots, N$. We can use (A-14) sequentially to determine the α_i's. Beginning at $i = N$, we see that the N^{th} retailer's budget constraint is equivalent to

$$\int_{\alpha_N<\alpha_\tau} w_\tau\,(\bar{A}_\tau - \alpha_N\, w_\tau)\mathrm{d}\mathbb{Q} \;\le\; (N+1)\,B_N. \tag{A-15}$$

The optimality condition on λ_i implies that it is the smallest non-negative real that satisfies the i^{th} budget constraint. Since the optimal λ_i's are non-decreasing in i, we see from (A-13) that α_i is therefore the smallest real greater than or equal to 1 satisfying the i^{th} budget constraint. Therefore, beginning with $i = N$ we can check if $\alpha_N = 1$ satisfies (A-15) and if it does, then we know the N^{th} budget constraint is not binding. If $\alpha_N = 1$ does not satisfy (A-15) then we set α_N equal to that value (greater than one) that makes (A-15) an equality. In particular, we obtain that the optimal value of α_N is $H((N+1)B_N)$, as desired.

Note that if $\alpha_N = 1$ then none of the budget constraints are binding. In particular, this implies $\alpha_i = 1$ and $\lambda_i = 0$ for all $i = 1, \ldots N$. Moreover, (32) must be satisfied for all i since it is true for $i = N$ and since the B_i's are decreasing. Suppose now that the budget constraint is binding for retailers $i+1, \ldots, N$ and consider the i^{th} retailer. Then the i^{th} retailer's budget constraint is equivalent[31] to (A-14) and we can again use precisely the same argument as before to argue that (32) holds. □

Proof of Proposition 8: To compute the value of Π_P we will compute the expected revenue $\mathbb{E}_0^{\mathbb{Q}}[w_\tau\, Q]$ and expected cost, $c_\tau \mathbb{E}_0^{\mathbb{Q}}[Q]$, separately. The first step is to determine those retailers that will be using their entire budgets in the Cournot equilibrium. We know from Proposition 7 and the definition of m in (35) that only the budget constraints of the first $m-1$ retailers will not be binding and so $\mathbb{E}_0^{\mathbb{Q}}[w_\tau\, q_i] = B_i$ for $i = m, m+1, \ldots, N$. It also follows that $\lambda_1 = \lambda_2 = \cdots = \lambda_{m-1} = 1$ and so (A-7) implies that

$$q_i = \left(\bar{A}_\tau - Q - w_\tau\right)^+, \qquad i = 1, \ldots, m-1. \tag{A-16}$$

Using this identity we obtain

$$\begin{aligned}
\mathbb{E}_0^{\mathbb{Q}}[w_\tau\, Q] &= \sum_{j=1}^{m-1} \mathbb{E}_0^{\mathbb{Q}}[w_\tau\, q_j] + \sum_{j=m}^{N} \mathbb{E}_0^{\mathbb{Q}}[w_\tau\, q_j] \\
&= (m-1)\, \mathbb{E}_0^{\mathbb{Q}}[w_\tau\, (\bar{A}_\tau - Q - w_\tau)^+] + \sum_{j=m}^{N} B_j \\
&= (m-1)\, \mathbb{E}_0^{\mathbb{Q}}[w_\tau\, (\bar{A}_\tau - w_\tau)^+ - w_\tau\, Q] + \sum_{j=m}^{N} B_j
\end{aligned} \tag{A-17}$$

where we have used the observation that $(\bar{A}_\tau - Q - w_\tau)^+ = (\bar{A}_\tau - w_\tau)^+ - Q$. This observation follows because (i) if $\bar{A}_\tau \le w_\tau$ then by (A-6) $Q = 0$ and (ii) if $Q > 0$ then $\bar{A}_\tau \ge w_\tau$ and we can argue using (A-7), say, that $(\bar{A}_\tau - Q - w_\tau)^+ = \bar{A}_\tau - Q - w_\tau$. We can now re-arrange (A-17) to obtain

$$\mathbb{E}_0^{\mathbb{Q}}[w_\tau\, Q] = \frac{1}{m} \left(\sum_{j=m}^{N} B_j + (m-1)\, \mathbb{E}_0^{\mathbb{Q}}[w_\tau\, (\bar{A}_\tau - w_\tau)^+] \right). \tag{A-18}$$

In order to calculate the expected cost, we can use Lemma 2 with $V_\tau \equiv 1$ to see that for any i we have

$$(i+1)\, \mathbb{E}_0^{\mathbb{Q}}[q_i] \;+\; \sum_{j=i+1}^{N} \mathbb{E}_0^{\mathbb{Q}}[q_j] \;=\; \mathbb{E}_0^{\mathbb{Q}}[(\bar{A}_\tau - \alpha_i\, w_\tau)^+]. \tag{A-19}$$

[31] Equivalence follows because the second terms on either side of the inequality sign in (A-13) are equal by assumption.

(A-19) then defines a system of N linear equations in the N unknowns, $\mathbb{E}_0^{\mathbb{Q}}[q_i], \quad i = 1, \ldots, N$. If we let $M = [M_{ij}]$ be the $N \times N$ upper-triangular matrix defined as

$$M_{ij} := \begin{cases} 0 & \text{if } i > j \\ i+1 & \text{if } i = j \\ 1 & \text{if } i < j \end{cases}$$

then it is easy to check that $[M_{ij}^{-1}] = 1_{\{j=i\}}/(j+1) - 1_{\{j>i\}}/(j(j+i))$. The system (A-19) then implies

$$\begin{aligned} \sum_{i=1}^{N} \mathbb{E}_0^{\mathbb{Q}}[q_i] &= \sum_{i=1}^{N}\sum_{j=1}^{N} [M_{ij}^{-1}]\, \mathbb{E}_0^{\mathbb{Q}}[(\bar{A}_\tau - \alpha_j\, w_\tau)^+] \\ &= \sum_{j=1}^{N} \frac{1}{j(j+1)}\, \mathbb{E}_0^{\mathbb{Q}}[(\bar{A}_\tau - \alpha_j\, w_\tau)^+] \end{aligned} \tag{A-20}$$

and so we obtain

$$\mathbb{E}_0^{\mathbb{Q}}[c_\tau Q] = \frac{(m-1)c_\tau}{m}\, \mathbb{E}_0^{\mathbb{Q}}[(\bar{A}_\tau - w_\tau)^+] + \sum_{j=m}^{N} \frac{c_\tau}{j(j+1)}\, \mathbb{E}_0^{\mathbb{Q}}[(\bar{A}_\tau - \alpha_j\, w_\tau)^+]. \tag{A-21}$$

where we have used the fact that $\alpha_1 = \ldots = \alpha_{m-1} = 1$. We can now combine (A-18) and (A-21) to obtain (34) as desired. □

Proof of Corollary 1:

Using the definition of α_j for $j \geq m$ and assuming a constant $\bar{w}$, we see that the expectation $\mathbb{E}_0^{\mathbb{Q}}[(\bar{A}_\tau - \alpha_j\, w_\tau)^+]$ in equation (34) can be replaced by $((j+1)B_j + B_{j+1} + \cdots + B_N)/\bar{w}$. The rest of the derivation of (36) follows directly after some simple calculations.

Let us now prove the second part. For notational convenience, we will write $\bar{w}^* := \bar{w}^*(B_1, \ldots, B_N)$ and $\bar{w}^\infty := \bar{w}^*(\infty, \ldots, \infty)$. Note that the producer payoff can be rewritten as follows

$$\Pi_{\text{P}}(\bar{w}) = \frac{m-1}{m}\, \Pi_{\text{P}}^{\infty}(\bar{w}) + \left(1 - \frac{c_\tau}{\bar{w}}\right) \sum_{j=m}^{N} \frac{B_j}{m}, \tag{A-22}$$

where $\Pi_{\text{P}}^{\infty}(\bar{w}) := (\bar{w} - c_\tau)\, \mathbb{E}[(\bar{A}_\tau - \bar{w})^+]$ is the producer's payoff when each retailer has an infinite budget (modulo the constant $(N-1)/N$). It follows that $\bar{w}^\infty$ is the unique maximizer of $\Pi_{\text{P}}^{\infty}(\bar{w})$. The uniqueness of $\bar{w}^\infty$ follows from the fact that $\Pi_{\text{P}}^{\infty}(\bar{w})$ is a unimodal function of $\bar{w}$. Hence, both $\Pi_{\text{P}}^{\infty}(\bar{w})$ and $(1 - c_\tau/\bar{w})$ are increasing functions of $\bar{w}$ in $[c_\tau, \bar{w}^\infty)$. This observation together with the facts that m is a piecewise constant function of $\bar{w}$ and $\Pi_{\text{P}}(\bar{w})$ is a continuous function of $\bar{w}$ imply that $\bar{w}^* \geq \bar{w}^\infty$. (The continuity of $\Pi_{\text{P}}^{\infty}(\bar{w})$ is not immediate since m is a discontinuous function of $\bar{w}$. However, checking this property is straightforward and is left to the reader.) □

Proof of Proposition 9:

(a) We will prove a more general result where we compare the N-retailer case to the $(N-1)$-retailer case that is created by merging the largest two retailers. We will use the notation in Proposition 8 to describe quantities associated with the N retailer case and use the 'hat' notation to describe

quantities associated with the $(N-1)$-retailer case. We must therefore show that $\Pi_{\text{P}} - \widehat{\Pi}_{\text{P}} \geq 0$. Proposition 8 then implies

$$\Pi_{\text{P}} - \widehat{\Pi}_{\text{P}} \;=\; \frac{(m-\hat{m})}{m\,\hat{m}}\,\mathbb{E}_0^{\mathbb{Q}}[w_\tau\,(\bar{A}_\tau - w_\tau)^+] \;+\; \sum_{j=m}^{N}\frac{B_j}{m} \;-\; \sum_{j=\hat{m}}^{N-1}\frac{\hat{B}_j}{\hat{m}}. \tag{A-23}$$

Note that $\hat{B}_1 = B_1 + B_2$ and that $\hat{B}_i = B_{i+1}$ for $i = 2, \ldots, N-1$. We also define C_i and $\hat{C}_i$ as

$$C_i \;:=\; (i+1)B_i + \;\cdots\; + B_N \quad \text{for } i = 1, \ldots, N$$

$$\hat{C}_i \;:=\; \begin{cases} 2(B_1+B_2) + \;\cdots\; + B_N, & \text{for } i = 1 \\ (i+1)B_{i+1} + \;\cdots\; + B_N, & \text{for } i = 2, \ldots, N-1. \end{cases}$$

Note that C_i and $\hat{C}_i$ are the arguments of the function, H, that defines the corresponding α_i's and $\hat{\alpha}_i$'s in (32). By the ordering assumption on the budgets we see that $C_1 \leq \hat{C}_1$ which implies $\alpha_1 \geq \hat{\alpha}_1$. Similarly for $i = 2, \ldots, N-1$ we have $C_i \geq \hat{C}_i$ which implies $\alpha_i \leq \hat{\alpha}_i$. In fact it is also clear that $C_{i+1} \geq \hat{C}_i$ for $i \geq 2$ so that $\alpha_{i+1} \leq \hat{\alpha}_i$. We now consider the various possible values of m and $\hat{m}$. The following cases follow from our previous observations:

1. $m = 1$: in this case all of the budget constraints in the original system are binding. The only possible values of $\hat{m}$ are 1 and 2. In particular $\hat{m} - m \in \{0, 1\}$.
2. $m = 2$: in this case only the first budget constraint in the original system is non-binding, $\hat{m}$ must also equal 2 and so $\hat{m} - m = 0$.
3. $3 \leq m \leq N+1$: in this case at least three budget constraints in the original system are non-binding, $\hat{m}$ can take on any value in $\{2, \ldots, m-1\}$ and $\hat{m} - m \in \{2-m, \ldots, -1\}$.

We now prove the result:

Case (i): Suppose $m = 1$ and $\hat{m} = 2$. Then (A-23) reduces to

$$\begin{aligned} \Pi_{\text{P}} - \widehat{\Pi}_{\text{P}} \;&=\; -\frac{1}{2}\,\mathbb{E}_0^{\mathbb{Q}}[w_\tau\,(\bar{A}_\tau - w_\tau)^+] \;+\; \sum_{j=1}^{N} B_j \;-\; \sum_{j=2}^{N-1}\frac{\hat{B}_j}{2} \\ &=\; \frac{1}{2}\left[2(B_1+B_2) \;+\; \sum_{j=3}^{N} B_j \;-\; \mathbb{E}_0^{\mathbb{Q}}[w_\tau\,(\bar{A}_\tau - w_\tau)^+]\right] \;\geq 0 \end{aligned} \tag{A-24}$$

since $\hat{m} = 2$, implying the first constraint is the new system is non-binding.

Case (ii): Suppose $m = \hat{m}$. Then (A-23) clearly implies $\Pi_{\text{P}} - \widehat{\Pi}_{\text{P}} \geq 0$. Together with Case (i), we have now covered the first two possibilities above.

Case (iii): Suppose $m \geq 3$ so that $\hat{m} - m < 0$. Then (A-23) implies

$$\begin{aligned}
\Pi_{\text{P}} - \widehat{\Pi}_{\text{P}} &= \frac{(m-\hat{m})}{m\,\hat{m}}\,\mathbb{E}_0^{\mathbb{Q}}[w_\tau\,(\bar{A}_\tau - w_\tau)^+] \;+\; \left(\sum_{j=m}^{N} B_j\right)\left(\frac{1}{m} - \frac{1}{\hat{m}}\right) \;-\; \frac{1}{\hat{m}}\sum_{j=\hat{m}+1}^{m-1} B_j \\
&= \frac{(m-\hat{m})}{m\,\hat{m}}\left[\mathbb{E}_0^{\mathbb{Q}}[w_\tau\,(\bar{A}_\tau - w_\tau)^+] \;-\; \sum_{j=m}^{N} B_j\right] \;-\; \frac{1}{\hat{m}}\sum_{j=\hat{m}+1}^{m-1} B_j \\
&\geq \frac{(m-\hat{m})}{m\,\hat{m}}\left[(\hat{m}+1)B_{\hat{m}+1} + \cdots + B_N \;-\; \sum_{j=m}^{N} B_j\right] \;-\; \frac{1}{\hat{m}}\sum_{j=\hat{m}+1}^{m-1} B_j \qquad \text{(A-25)}
\end{aligned}$$

where (A-25) follows since $\hat{\alpha}_{\hat{m}} > 1$ and so $\mathbb{E}_0^{\mathbb{Q}}[w_\tau\,(\bar{A}_\tau - w_\tau)^+] \geq \hat{C}_m$. Note that the right-hand-side of (A-25) equals $B_m/m > 0$ if $\hat{m} + 1 = m$. Otherwise the right-hand-side of (A-25) equals

$$\begin{aligned}
&\frac{(m-\hat{m})}{m\hat{m}}\left[(\hat{m}+1)B_{\hat{m}+1} + \cdots + B_{m-1}\right] \;-\; \frac{1}{\hat{m}}\sum_{j=\hat{m}+1}^{m-1} B_j \\
=\;& \frac{1}{\hat{m}}\left[(\hat{m}+1)B_{\hat{m}+1} + \cdots + B_{m-1}\right] \;-\; \frac{1}{m}\left[(\hat{m}+1)B_{\hat{m}+1} + \cdots + B_{m-1}\right] \;-\; \frac{1}{\hat{m}}\sum_{j=\hat{m}+1}^{m-1} B_j \\
=\;& B_{\hat{m}+1} \;-\; \frac{1}{m}\left[(\hat{m}+1)B_{\hat{m}+1} + \cdots + B_{m-1}\right] \\
\geq\;& B_{\hat{m}+1} \;-\; \frac{m-1}{m}B_{\hat{m}+1} \;=\; \frac{B_{\hat{m}+1}}{m} > 0.
\end{aligned}$$

and so the result follows. □

Proof of Lemma 1: The proof is by contradiction so suppose[32] $w_\tau > 0$, $m < n$, and that both n and m satisfy

$$B_{n+1} \;<\; q^{(n)} w_\tau \;\leq\; B_n \qquad \text{(A-26)}$$

$$B_{m+1} \;<\; q^{(m)} w_\tau \;\leq\; B_m. \qquad \text{(A-27)}$$

If we use (39) to substitute for $q^{(n)}$ and $q^{(m)}$ in (A-26) and (A-27), and then rearrange terms we obtain

$$(n+1)\frac{B_{n+1}}{w_\tau} \;+\; \sum_{j=n+1}^{N}\frac{B_j}{w_\tau} \;<\; \bar{A}_\tau - w_\tau \;\leq\; (n+1)\frac{B_n}{w_\tau} \;+\; \sum_{j=n+1}^{N}\frac{B_j}{w_\tau} \qquad \text{(A-28)}$$

$$(m+1)\frac{B_{m+1}}{w_\tau} \;+\; \sum_{j=m+1}^{N}\frac{B_j}{w_\tau} \;<\; \bar{A}_\tau - w_\tau \;\leq\; (m+1)\frac{B_m}{w_\tau} \;+\; \sum_{j=m+1}^{N}\frac{B_j}{w_\tau} \qquad \text{(A-29)}$$

Since $m < n$, however, we see that the ordering of the B_i's implies the expression on the right of the second inequality in (A-28) is less than or equal to the expression on the left of the first inequality in (A-29). A contradiction follows immediately since we then obtain $\bar{A}_\tau - w_\tau < \bar{A}_\tau - w_\tau$. □

[32] When $w_\tau = 0$, none of the budget constraints are binding and we obtain the standard Cournot equilibrium with $n = N$ in (39).

PROOF OF PROPOSITION 11: Let m_j, $\alpha_i^{(j)}$ for $i = 1, \ldots, N$ and $j = 1, 2$ denote the usual Cournot optimal quantities for scenario j and retailer i. Since $\mathbb{E}^{\mathbb{Q}}_{\tau_1}[\bar{A}_{\tau_2}]/x = \bar{A}_{\tau_1}/x$ for any $x > 0$ Jensen's Inequality implies

$$\mathbb{E}^{\mathbb{Q}}_0[w_\tau\,(\bar{A}_{\tau_2}/x - w_\tau)^+] \;\geq\; \mathbb{E}^{\mathbb{Q}}_0[w_\tau\,(\bar{A}_{\tau_1}/x - w_\tau)^+]. \tag{A-30}$$

After multiplying across (A-30) by x it then follows from the definition of the α_i's in (32) that

$$\alpha_i^{(2)} \;\geq\; \alpha_i^{(1)} \quad \text{for} \quad i = 1, \ldots, N.$$

This in turn implies that $m_2 \leq m_1$. Let $\Pi_{\text{P}}^{(1)}$ and $\Pi_{\text{P}}^{(2)}$ denote the producer's expected revenue in scenarios (1) and (2) respectively. Then (A-18) implies

$$\begin{aligned}
\Pi_{\text{P}}^{(2)} - \Pi_{\text{P}}^{(1)} &= \frac{(m_1 - m_2)}{m_1 m_2} \sum_{j=m_1}^{N} B_j \;+\; \frac{1}{m_2} \sum_{j=m_2}^{m_1-1} B_j \;+\; \frac{(m_2 - 1)}{m_2} \mathbb{E}^{\mathbb{Q}}_0[w_\tau\,(\bar{A}_{\tau_2} - w_\tau)^+] \\
&\quad - \frac{(m_1 - 1)}{m_1} \mathbb{E}^{\mathbb{Q}}_0[w_\tau\,(\bar{A}_{\tau_1} - w_\tau)^+] \\
&\geq \frac{(m_1 - m_2)}{m_1 m_2} \sum_{j=m_1}^{N} B_j \;+\; \frac{1}{m_2} \sum_{j=m_2}^{m_1-1} B_j \;+\; \frac{(m_2 - m_1)}{m_1 m_2} \mathbb{E}^{\mathbb{Q}}_0[w_\tau\,(\bar{A}_{\tau_1} - w_\tau)^+] \qquad (\text{(A-30) with } x = 1) \\
&= \frac{(m_1 - m_2)}{m_1 m_2} \left[\sum_{j=m_1}^{N} B_j \;-\; \mathbb{E}^{\mathbb{Q}}_0[w_\tau\,(\bar{A}_{\tau_1} - w_\tau)^+] \right] \;+\; \frac{1}{m_2} \sum_{j=m_2}^{m_1-1} B_j \\
&\geq \frac{(m_1 - m_2)}{m_1 m_2} \left[\sum_{j=m_1}^{N} B_j \;-\; (m_1 + 1) B_{m_1} - \cdots - B_N \right] \;+\; \frac{1}{m_2} \sum_{j=m_2}^{m_1-1} B_j \qquad (\text{by def. of } m_1) \\
&= \frac{(m_2 - m_1)}{m_2} B_{m_1} \;+\; \frac{1}{m_2} \sum_{j=m_2}^{m_1-1} B_j.
\end{aligned} \tag{A-31}$$

Note that the right-hand-side of (A-31) equals zero if $m_1 = m_2$. Otherwise $m_2 < m_1$ and the right-hand-side of (A-31) is greater than or equal to $(m_1 - m_2)(B_{m_2} - B_{m_1-1})/m_2$ which in turn is non-negative. The result therefore follows. □

B Martingale Pricing with Foreign Assets

Martingale pricing theory states that the time 0 value, G_0, of a security that is worth G_τ at time τ and does not pay any intermediate cash-flows, satisfies $G_0/N_0 = \mathbb{E}^{\mathbb{Q}}_0[G_\tau/N_\tau]$ where N_t is the time t price of the *numeraire* security and $\mathbb{Q}$ is an *equivalent martingale measure* (EMM) associated with that numeraire. It is common to take the cash account as the numeraire security and this is the approach we have followed in most of this paper. With the exception of Section 3.4, however, the value of the cash account at time t was always \$1 since we assumed interest rates were identically zero. We therefore had $G_0 = \mathbb{E}^{\mathbb{Q}}_0[G_\tau]$ and since we insisted $G_0 = 0$ we obtained $\mathbb{E}^{\mathbb{Q}}_0[G_\tau] = 0$. When interest rates are non-zero we still have $N_0 = 1$ but now $N_\tau = \exp\left(\int_0^\tau r_s\, ds\right)$ and so, for example, we have (18). In the main text we take $D_t = N_t^{-1}$. See Duffie (2004) for a development of martingale pricing theory.

Martingale Pricing with Foreign Assets

Suppose now that there is a *domestic* currency and a *foreign* currency with Z_t denoting the exchange rate between the two currencies at time t. In particular, $Z_t Y_t$ is the time t domestic currency value of a foreign asset that has a time t foreign currency value of Y_t. Let $B_t^{(f)}$ denote the time t value of the foreign cash account and let $\mathbb{Q}$ denote the EMM of a foreign investor taking the foreign cash account as numeraire. This implies

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{Y_T}{B_T^{(f)}}\right] = \frac{Y_t}{B_t^{(f)}} \tag{B-32}$$

for all $t \leq T$ and where we assume again that the asset with foreign currency value Y_t at time t does not[33] pay any intermediate cash flows. But equation (B-32) can be re-written as

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{Z_T Y_T}{Z_T B_T^{(f)}}\right] = \frac{Z_t Y_t}{Z_t B_t^{(f)}}. \tag{B-33}$$

Note that $Z_t Y_t$ is the domestic currency value of the foreign asset and $Z_t B_t^{(f)}$ is the domestic currency value of the foreign cash account. Note also that if V_t is the time t price of a domestic asset then V_t/Z_t is the foreign currency value of the asset at time t. We therefore obtain by martingale pricing that

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{V_T/Z_T}{B_T^{(f)}}\right] = \frac{V_t/Z_t}{B_t^{(f)}}$$

or equivalently,

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{V_T}{Z_T B_T^{(f)}}\right] = \frac{V_t}{Z_t B_t^{(f)}} \tag{B-34}$$

We can therefore conclude from (B-33) and (B-34) that $\mathbb{Q}$ is also the EMM of a domestic investor, but now with the *domestic* value of the foreign cash account as the corresponding. We use these observations in Section 3.3.

C Optimal Production Postponement

We now extend the contract so that τ, the time at which the physical transaction takes place, is an endogenous decision variable that is determined as part of the solution to the Nash equilibrium. Our discussion will focus on the single-retailer case, which as we have seen in Section 3, includes the multi-retailer case when the retailers have identical budgets. Later we will assume that the budget, B, is sufficiently large so that the budget constraint is non-binding (possibly due to the ability to trade in the financial markets). Our interest in the optimal timing of the contract is motivated by our desire to understand the tradeoff between delaying production when the production cost, c_τ, is increasing, and allowing the ordering quantities and price menu to be contingent upon a larger information set. Allowing the producer to choose the optimal timing of the contract therefore allows him to take optimal advantage of the information made available by the financial markets.

[33] If it did pay intermediate cash-flows between t and T then they would have to be included inside the expectation in (B-32).

We consider two alternatives formulations. In the first alternative, τ is restricted to be a deterministic time in $[0,T]$ that is selected at time $t=0$. Motivated by the terminology of dynamic programming, we refer to this alternative as the optimal *open-loop* production postponement model. In the second alternative, we permit τ to be an $\mathcal{F}_t$-stopping time that is bounded above by T. We call this alternative the optimal *closed-loop* production postponement model. In both cases, the procurement contract offered by the producer takes the form of a pair, (τ, w_τ), where the wholesale price menu, w_τ, is required to be $\mathcal{F}_\tau$-measurable. We note that the producer always prefers the closed-loop model though from a practical standpoint the open-loop model may be easier to implement in practice.

Independently of whether τ is a deterministic time or a stopping time, the optimal ordering level for the retailer, given a contract (τ, w_τ), is an $\mathcal{F}_\tau$-measurable menu, q_τ, that satisfies[34] the conditions in Proposition 1 with $N=1$.

The producer's problem of selecting the optimal time τ is given by

$$\Pi_{\text{P}} = \max_{\tau,\phi\geq 1} \mathbb{E}_0^{\mathbb{Q}}\left[\frac{(\bar{A}_\tau+\phi\,c_\tau-2c_\tau)\,(\bar{A}_\tau-\phi\,c_\tau)^+}{8}\right] \tag{C-35}$$

$$\text{subject to} \quad \mathbb{E}_0^{\mathbb{Q}}\left[\left(\frac{\bar{A}_\tau^2-\phi^2\,c_\tau^2}{8}\right)^+\right] \leq B. \tag{C-36}$$

Note that the expression for Π_{P} in (C-35) is the expression given for $\Pi_{\text{P}|\tau}$ in Proposition 2 while (C-36) is the corresponding constraint from the same Proposition. Of course τ should be restricted to either a deterministic time or a stopping time depending on which model (open-loop or closed-loop) is under consideration. For a given τ, the objective in (C-35) is decreasing in ϕ so that the producer's problem reduces to

$$\Pi_{\text{P}} = \max_{\tau} \mathbb{E}_0^{\mathbb{Q}}\left[\frac{(\bar{A}_\tau+\phi\,c_\tau-2c_\tau)\,(\bar{A}_\tau-\phi\,c_\tau)^+}{8}\right] \tag{C-37}$$

$$\text{subject to} \quad \phi = \inf\left\{\psi\geq 1 \;:\; \mathbb{E}_0^{\mathbb{Q}}\left[\left(\frac{\bar{A}_\tau^2-\psi^2\,c_\tau^2}{8}\right)^+\right] \leq B\right\}. \tag{C-38}$$

To solve this optimization problem we would first need to specify the functional forms of $\bar{A}_\tau$ and c_τ and depending on these specifications, the solution may or may not be easy to find. For the remainder of this section, however, we will show how this problem may be solved when additional assumptions are made. In particular, we make the following three assumptions:

1. X_t is a diffusion process with dynamics satisfying

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}W_t, \tag{C-39}$$

where W_t a $\mathbb{Q}$-Brownian motion. Note that we have not included a drift term in the dynamics of X_t since it must be the case that X_t is a $\mathbb{Q}$-martingale. This is not a significant assumption and we could easily consider alternative processes for X_t.

[34] It is easy to check that the proof of Proposition 1 remains unchanged if τ is allowed to be a stopping time.

2. We adopt a specific functional form to model the dependence between the market clearance price and the financial market. In particular, we assume that there behaves a well-behaved[35] function, $F(x)$, and a random variable, ε, such that one of the following two models holds.

$$\text{Additive Model:} \qquad A = F(X_T) + \varepsilon, \qquad \text{with } \mathbb{E}^{\mathbb{Q}}[\varepsilon] = 0, \text{ or} \tag{C-40}$$

$$\text{Multiplicative Model:} \qquad A = \varepsilon\, F(X_T), \qquad \text{with } \varepsilon \geq 0 \text{ and } \mathbb{E}^{\mathbb{Q}}[\varepsilon] = 1. \tag{C-41}$$

The random perturbation ε captures the non-financial component of the market price uncertainty and is assumed to be independent of X_t. Note that if $F(x) = \bar{A}$, we recover a model for which demand is independent of the financial market.

3. We assume that the initial budget, B, is sufficiently large so that the retailer is able to hedge away the budget constraint for every stopping time, τ. That is, $\phi = 1$ for every $\tau \in \mathcal{T}$. This is a significant assumption[36] and effectively reduces the problem to one of finding the optimal (random) timing of the contract when there is no budget constraint.

C.1 Optimal Open-Loop Production Postponement

We now restrict τ to be a deterministic time in $[0, T]$. Based on the third assumption above, the producer's optimization problem in (C-37) reduces to

$$\max_{\tau\in[0,T]} \mathbb{E}_0^{\mathbb{Q}}\left[(\bar{A}_\tau - c_\tau)^2\right] = \max_{\tau\in[0,T]} \operatorname{Var}(\bar{A}_\tau) + (\bar{A} - c_\tau)^2. \tag{C-42}$$

We note that in this optimization problem there is a trade-off between demand learning as represented by the variance term, $\operatorname{Var}(\bar{A}_\tau)$, and production costs as represented by $(\bar{A} - c_\tau)^2$. The first term is increasing in τ while the second term is decreasing in τ so that, in general, the optimization problem in (C-42) does not admit a trivial solution and depends on the particular form of the functions $\operatorname{Var}(\mathbb{E}^{\mathbb{Q}}[A \,|\, X_\tau])$ and c_τ.

The Itô Representation Theorem[37] implies the existence of an $\mathcal{F}_t$-adapted process, $\{\theta_t : t \in [0, T]\}$, such that

$$A = \bar{A} + \int_0^T \theta_t \,\mathrm{d}X_t + \varepsilon \qquad \text{or} \qquad A = \varepsilon\left(\bar{A} + \int_0^T \theta_t \,\mathrm{d}X_t\right)$$

for the additive or multiplicative model, respectively. In both cases the Q-martingale property of X_t implies

$$\bar{A}_\tau = \bar{A} + \int_0^\tau \theta_t \,\mathrm{d}X_t. \tag{C-43}$$

In order to compute the variance of $\bar{A}_\tau$ we use the Q-martingale property of the stochastic integral and invoke Itô's isometry to obtain

$$\operatorname{Var}(\bar{A}_\tau) = \mathbb{E}_0^{\mathbb{Q}}\left[\left(\int_0^\tau \theta_t \,\mathrm{d}X_t\right)^2\right] = \mathbb{E}_0^{\mathbb{Q}}\left[\int_0^\tau \theta_t^2 \,\mathrm{d}[X]_t\right],$$

[35] It is necessary, for example, that $F(\cdot)$ satisfy certain integrability conditions so that the stochastic integral in (C-43) be a Q-martingale. In order to apply Itô's Lemma it is also necessary to assume that $F(\cdot)$ is twice differentiable. Because this section is intended to be brief, we omit the various technical conditions that are required to make our arguments completely rigorous.

[36] If we only wanted to solve for the open-loop policy it would not be necessary to make this assumption. In that case we could solve for the optimal τ and ϕ in (C-37) and (C-38) numerically.

[37] See Øksendal (1998) for a formal statement. Øksendal (1998) may also be consulted for a statement of Itô's isometry.

where the process $[X]_t$ is the quadratic variation of X_t with dynamics $\mathrm{d}[X]_t = \sigma^2(X_t)\,\mathrm{d}t$. It follows that

$$\mathrm{Var}(\bar{A}_\tau) \;=\; \int_0^\tau \mathbb{E}_0^{\mathbb{Q}}[(\theta_t\,\sigma(X_t))^2]\,\mathrm{d}t.$$

The open-loop optimal problem therefore reduces to solving

$$\max_{\tau\in[0,T]} \left\{ \int_0^\tau \mathbb{E}_0^{\mathbb{Q}}[(\theta_t\,\sigma(X_t))^2]\,\mathrm{d}t + (\bar{A} - c_\tau)^2 \right\}. \tag{C-44}$$

If there is an interior solution to this problem (*i.e.*, $\tau^* \in (0,T)$), then it must satisfy the first-order optimality condition

$$\mathbb{E}_0^{\mathbb{Q}}[(\theta_\tau\,\sigma(X_\tau))^2] - 2\,(\bar{A} - c_\tau)\,\dot{c}_\tau = 0, \qquad\qquad \text{where } \dot{c}_\tau := \frac{\mathrm{d}c_\tau}{\mathrm{d}\tau}.$$

Example 1 Consider the case in which the security price, X_t, follows a geometric Brownian motion with dynamics

$$\mathrm{d}X_t = \sigma\,X_t\,\mathrm{d}W_t$$

where $\sigma \neq 0$ and W_t is a Q-Brownian motion. The quadratic variation process then satisfies $\mathrm{d}[X]_t = \sigma^2\,X_t^2\,\mathrm{d}t$. To model the dependence between the market clearance price and the process, X_t, we assume a linear model for $F(\cdot)$ so that $F(X) = A_0 + A_1\,X$ where A_0 and A_1 are positive constants. Therefore, depending on whether we consider the additive or multiplicative model, we have

$$A = A_0 + A_1\,X_T + \varepsilon \qquad \text{or} \qquad A = \varepsilon\,(A_0 + A_1\,X_T),$$

where ε is a zero-mean or unit-mean random perturbation, respectively, that is independent of the process, X_t. It follows that $\bar{A}_\tau = A_0 + A_1\,X_\tau$ and $\bar{A} = \mathbb{E}_0^{\mathbb{Q}}[A] = A_0 + A_1\,X_0$. In addition, it is clear that θ_t is identically equal to A_1 for all $t \in [0,T]$. We assume that the per unit production cost increases with time and is given by

$$c_\tau = c_0 + \alpha\,\tau^\kappa, \qquad\qquad \text{for all } \tau \in [0,T],$$

where α and κ are positive constants.

To impose the additional constraint that $\bar{A}_\tau \geq c_\tau$ for all τ (Assumption 1), we restrict our choice of the parameters A_0, T, c_0, κ, and α so that $A_0 \geq c_0 + \alpha\,T^\kappa$. Since $\mathbb{E}_0^{\mathbb{Q}}[X_t^2] = X_0^2\,\exp(\sigma^2\,t)$ the optimization problem in (C-44) reduces to

$$\max_{\tau\in[0,T]} \left\{ (A_1\,X_0)^2\,(\exp(\sigma^2\,\tau) - 1) + (\bar{A} - c_0 - \alpha\,\tau^\kappa)^2 \right] \right\}.$$

In general, a closed form solution is not available unless $\kappa = 0$. This is a trivial case in which c_τ is constant and the optimal strategy is to postpone production until time T so that $\tau^* = T$. Figure 1 shows the value of the objective function as a function of τ for four different values of κ. The cost functions are such that it becomes cheaper to produce as κ increases. Note that for $\kappa \in \{4, 8\}$, it is convenient to postpone production. For the more expensive production cost functions that occur when $\kappa \in \{0.25, 1\}$, production postponement is not profitable and it is optimal to produce immediately. □

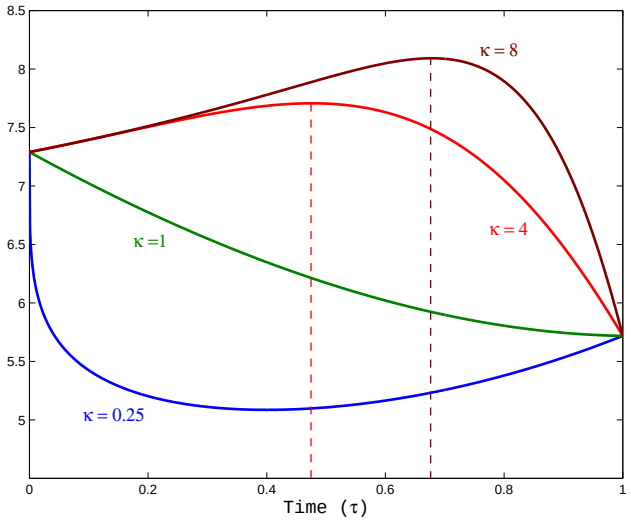

Figure 1: Optimal open-loop production postponement for four different production cost functions parameterized by κ. The other parameters are $X_0 = \sigma = T = 1$, $A_1 = 8$, $A_0 = 16$, $c_0 = 2.4$ and $\alpha = 5.6$.

C.2 Optimal Closed-Loop Production Postponement

Instead of selecting a fixed transaction time, τ, at $t = 0$, the producer now optimizes over the set of stopping times bounded above by T. In this case, the optimization problem in (C-37) reduces to solving for

$$\max_{\tau \in \mathcal{T}} \mathbb{E}_0^{\mathbb{Q}} \left[(\bar{A}_\tau - c_\tau)^2 \right], \tag{C-45}$$

where $\mathcal{T}$ is the set of $\mathcal{F}_t$-adapted stopping times bounded above by T. Again, the third assumption above has resulted in this simplified form of the objective function. According to the modeling of A in (C-40) or (C-41), it follows that $v(\tau, X_\tau) := \bar{A}_\tau = \mathbb{E}_\tau^{\mathbb{Q}}[F(X_T)]$ is a $\mathbb{Q}$-martingale that satisfies

$$\frac{v(t,x)}{\partial t} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 v(t,x)}{\partial x^2} = 0, \qquad v(T,x) = F(x).$$

We define U to be the set $\{(t,x) : \mathcal{G}g(t,x) > 0\}$ where $g(t,x) := (v(t,x) - c_t)^2$ is the payoff function and $\mathcal{G}$ is the generator

$$\mathcal{G} := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}.$$

We then obtain

$$U = \left\{ (t,x) : (\sigma(x)\, v_x(t,x))^2 > 2(v(t,x) - c_t)\, \dot{c}_t \right\},$$

where v_x is the first partial derivative of v with respect to x. In general, the set U is a proper subset of the optimal continuation region for the stopping problem in (C-45). Computing the optimal stopping time analytically is a difficult task and is usually done numerically. However, if U turns out to equal the entire state space then it is clear that it is always optimal to continue so that $\tau = T$.

Example 3: (Continued)

Consider the setting of Example 1 but where now τ is a stopping time instead of a deterministic time. For the linear function $F(X) = A_0 + A_1 X$, the auxiliary function v satisfies $v(t,x) = A_0 + A_1 x$, and the region U is given by

$$U = \left\{(t,x) \,:\, (\sigma\, x\, A_1)^2 > 2(A_0 + A_1\, x - c_t)\, \dot{c}_t\right\}.$$

Straightforward calculations allow us to rewrite U as

$$U = \left\{(t,x) \,:\, x > \frac{\dot{c}_t + \sqrt{\dot{c}_t^2 + 2\sigma^2\,(A_0 - c_t)\,\dot{c}_t}}{\sigma^2\, A_1}\right\}.$$

Let us define the auxiliary function

$$\rho(t) := \frac{\dot{c}_t + \sqrt{\dot{c}_t^2 + 2\sigma^2\,(A_0 - c_t)\,\dot{c}_t}}{\sigma^2\, A_1}.$$

Since U is a subset of the optimal continuation region, we know that it is never optimal to stop if $X_t > \rho(t)$. Of course, it is possible that $X_t < \rho(t)$ and yet still be optimal to continue.

We solved for the optimal continuation region numerically by using a binomial model to approximate the dynamics of X_t. In so doing, we can assess the quality of the (suboptimal) strategy that uses $\rho(t)$ to define the continuation region. Figure 2 shows the optimal continuation region and the threshold $\rho(t)$ for four different cost functions. These cost function are given by $c_\tau = c_0 + \alpha\,\tau^\kappa$ with $\kappa = 0.25, 1, 4,$ and 8. When $X(\tau)$ is above the optimal threshold it is optimal to continue. The vertical dashed line corresponds to the optimal open-loop deterministic time computed in Figure 1. For $\kappa = 0.25$ or $\kappa = 1$ this optimal deterministic time equals 0 since X_0 lies below the optimal threshold. For $\kappa = 4$ it equals 0.476, and for $\kappa = 8$ it equals 0.678.

Interestingly, for high values of κ the auxiliary threshold $\rho(t)$ is a good approximation for the optimal solution. However, as κ decreases the quality of the approximation deteriorates rapidly. Except for the case where $\kappa = 0.25$, the optimal threshold increases with time. This reflects the fact that the producer becomes more likely to stop and exercise the procurement contract as the end of the horizon approaches.

We conclude this example by computing the optimal expected payoff for the producer under both the optimal open-loop policy and the optimal closed-loop policy.

κ	Open-Loop Payoff	Closed-Loop Payoff	% Increase
0.25	7.29	7.29	0.0%
1	7.29	7.305	0.2%
4	7.71	7.99	3.7%
8	8.09	8.33	3.8%

Producer's expected payoff for four different production cost functions parameterized by κ.
The other parameters are $X_0 = \sigma = T = 1$, $A_1 = 8$, $A_0 = 16$, $c_0 = 2.4$ and $\alpha = 5.6$.

Naturally, the optimal stopping time (closed-loop) policy produces a higher expected payoff than the optimal deterministic time (open-loop) policy. The improvement, however, is only a few percentage points which might suggest that a simpler contract based on a deterministic time captures most of the benefits of allowing τ to be a decision variable. In practice, of course, it would be necessary to model the operations and financial markets more accurately and to calibrate the resulting model correctly before such conclusions could be drawn. □

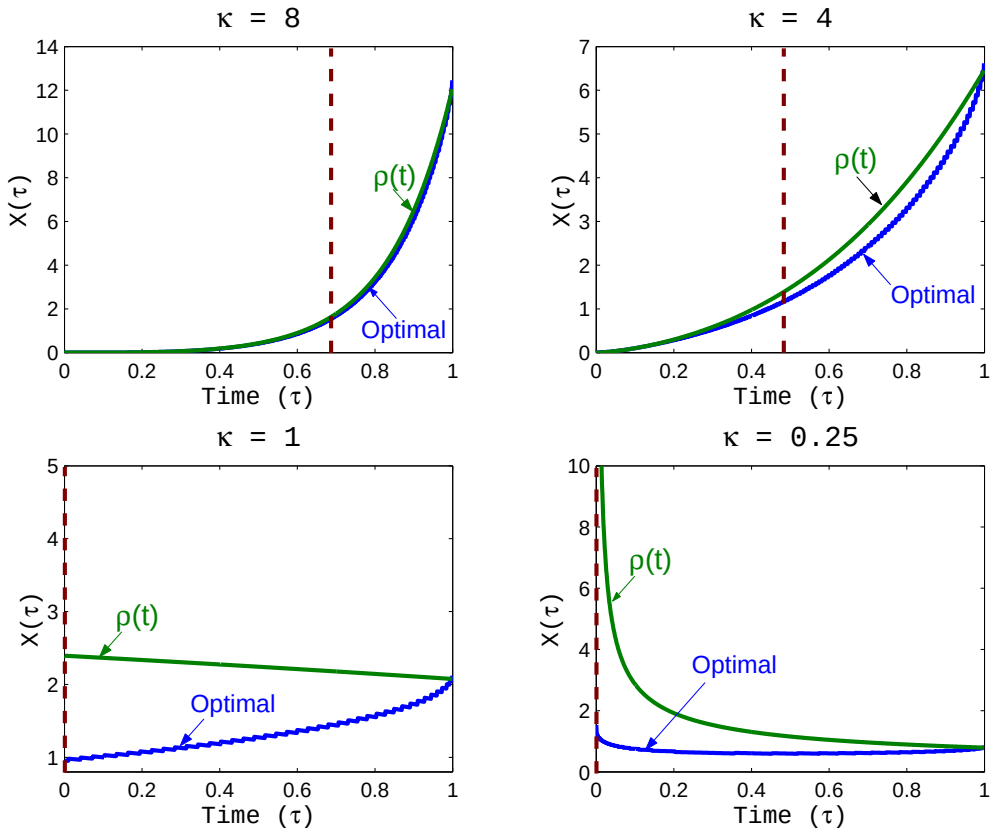

Figure 2: Optimal continuation region for four different manufacturing cost functions parameterized by κ. The other parameters are $X_0 = \sigma = T = 1$, $A_1 = 8$, $A_0 = 16$, $c_0 = 2.4$ and $\alpha = 5.6$.