Market Models

One of the principal disadvantages of short rate models, and HJM models more generally, is that they focus on unobservable instantaneous interest rates. The so-called market models that were developed[1] in the late 90's overcome this problem by directly modeling observable market rates such as LIBOR[2] and swap rates. These models are straightforward to calibrate and have quickly gained widespread acceptance from practitioners. The first market models were actually developed in the HJM framework where the dynamics of instantaneous forward rates are used via Itô's Lemma to determine the dynamics of zero-coupon bonds. The dynamics of zero coupon bond prices were then used, again via Itô's Lemma, to determine the dynamics of LIBOR. Market models are therefore not inconsistent with HJM models. In these lecture notes, however, we will prefer to specify the market models directly rather than derive them in the HJM framework. In the process, we will derive Black's formulae for caplets and swaptions thereby demonstrating the consistency of these formulae with martingale pricing theory.

Throughout these notes, we will ignore the possibility of default or counter-party risk and treat LIBOR interest rates as the fundamental rates in the market. Zero-coupon bond prices are then computed using LIBOR rather than the default-free rates implied by the prices of government securities. This does result in a minor inconsistency in that we price derivative securities assuming no possibility of default yet the interest rates themselves that play the role of "underlying security", i.e. LIBOR and swap rates, implicitly incorporate the possibility of default. This inconsistency actually occurs in practice when banks trade caps, swaps and other instruments with each other, and ignore the possibility of default when quoting prices. Instead, the associated credit risks are kept to a minimum through the use of *netting* agreements and by counter-parties limiting the total size of trades they conduct with one another. This approach can also be justified when counter-parties have a similar credit rating and similar exposures to one another. Finally, we should mention that it is indeed possible[3], and sometimes necessary, to explicitly model credit risk even when we are pricing 'standard' securities such as caps and swaps. It goes without saying of course, that default risk needs to be modeled explicitly when pricing credit derivatives and related securities.

1 LIBOR, Swap Rates and Black's Formulae for Caps and Swaptions

We now describe two particularly important market interest rates, namely LIBOR and swap rates. We first define LIBOR and forward LIBOR, and then describe Black's formula for caplets. After defining LIBOR we then proceed to discuss swap rates and forward swap rates as well as describing Black's formula for swaptions. In practice, the "underlying security" for caps and swaptions are LIBOR and LIBOR-based swap rates. Therefore by modeling the dynamics of these rates directly we succeed in obtaining more realistic models than those developed in the short-rate or HJM framework.

[1] See Miltersen, Sandmann and Sondermann (1997), Brace, Gatarek and Musiela (1997), Jamshidian (1997) and Musiela and Rutkowski (1997).

[2] These models apply equally well to Euribor rates.

[3] See chapter 11 of Cairns for a model where swaps are priced taking the possibility of default explicitly into account.

LIBOR

The forward rate at time t based on simple interest for lending in the interval $[T_1, T_2]$ is given by[4]

$$F(t, T_1, T_2) \;=\; \frac{1}{T_2 - T_1}\left(\frac{Z_t^{T_1} - Z_t^{T_2}}{Z_t^{T_2}}\right) \tag{1}$$

where, as before, Z_t^T is the time t price of a zero-coupon bond maturing at time T. Note also that if we measure time in years, then (1) is consistent with $F(t, T_1, T_2)$ being quoted as an annual rate.

LIBOR rates are quoted as **simply-compounded** interest rates, and are quoted on an annual basis. The accrual period or *tenor*, $T_2 - T_1$, is usually fixed at $\delta = 1/4$ or $\delta = 1/2$ corresponding to 3 months and 6 months, respectively. With a fixed value of δ in mind we can define the δ-year forward rate at time t with maturity T as

$$L(t, T) \;:=\; F(t, T, T + \delta) \;=\; \frac{1}{\delta}\left(\frac{Z_t^T - Z_t^{T+\delta}}{Z_t^{T+\delta}}\right). \tag{2}$$

Note that the δ-year spot LIBOR rate at time t is then given by $L(t, t)$.

Remark 1 *LIBOR or the* London Inter-Bank Offered Rate, *is determined on a daily basis when the* British Bankers' Association *(BBA) polls a pre-defined list of banks with strong credit ratings for their interest rates. The highest and lowest responses are dropped and then the average of the remainder is taken to be the LIBOR rate. Because there is some credit risk associated with these banks, LIBOR will be higher than the corresponding rates on government treasuries. However, because the banks that are polled have strong credit ratings the spread between LIBOR and treasury rates is generally not very large and is often less than* 100 *basis points. Moreover, the pre-defined list of banks is regularly updated so that banks whose credit ratings have deteriorated are replaced on the list with banks with superior credit ratings. This has the practical impact of ensuring that forward LIBOR rates will still only have a very modest degree of credit risk associated with them.*

Black's Formula for Caplets

Consider now a caplet with payoff $\delta(L(T, T) - K)^+$ at time $T + \delta$. The time t price, C_t, is given by

$$\begin{aligned} C_t &= B_t \mathrm{E}_t^Q\left[\frac{\delta(L(T, T) - K)^+}{B_{T+\delta}}\right] \\ &= \delta Z_t^{T+\delta} \mathrm{E}_t^{P_{T+\delta}}\left[(L(T, T) - K)^+\right]. \end{aligned}$$

where (B_t, Q) is an arbitrary numeraire-EMM pair and $(Z_t^{T+\delta}, P_{T+\delta})$ is the forward measure-numeraire pair. The market convention is to quote caplet prices using Black's formula which equates C_t to a Black-Scholes like formula so that

$$C_t \;=\; \delta Z_t^{T+\delta}\left[L(t, T)\Phi\left(\frac{\log(L(t, T)/K) + \sigma^2(T - t)/2}{\sigma\sqrt{T - t}}\right) \;-\; K\Phi\left(\frac{\log(L(t, T)/K) - \sigma^2(T - t)/2}{\sigma\sqrt{T - t}}\right)\right] \tag{3}$$

where $\Phi(\cdot)$ is the CDF of a standard normal random variable. Note that (3) is what you would get for C_t if you assumed that

$$dL(t, T) \;=\; \sigma L(t, T)\; dW^{T+\delta}(t)$$

where $W^{T+\delta}(t)$ is a $P_{T+\delta}$-Brownian motion and σ is an 'implied' volatility that is used to quote prices.

Black's formula for caps is to equate the cap price with the sum of caplet prices given by (3) but where a common σ is assumed. Similar formulae exist for floorlets and floors.

[4] This follows from a simple arbitrage argument. Prove it!

Swap Rates

Consider a payer *forward start* swap where the swap begins at some fixed time T_n in the future and expires at time $T_M \geq T_n$. We assume the accrual period is of length δ. Since payments are made in arrears, the first payment occurs at $T_{n+1} = T_n + \delta$ and the final payment at $T_{M+1} = T_M + \delta$. Then martingale pricing implies that the time $t < T_n$ value, SW_t, of this forward start swap is

$$SW_t \;=\; \mathrm{E}_t^Q \left[\delta \sum_{j=n}^{M} \frac{B_t}{B_{T_{j+1}}} (L(T_j, T_j) - R) \right]$$

where R is the fixed (annualized) rate specified in the contract. A standard argument using the properties of floating-rate bond prices implies that

$$SW_{T_n} \;=\; 1 - Z_{T_n}^{T_{M+1}} \;-\; R\delta \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j}. \tag{4}$$

Exercise 1 *Prove (4).*

Equation (4) in turn easily implies (why?) that for $t < T_n$ we have

$$SW_t \;=\; Z_t^{T_n} - Z_t^{T_{M+1}} \;-\; R\delta \sum_{j=n+1}^{M+1} Z_t^{T_j}.$$

Definition 1 *The* forward swap rate *at time t is the value $R = R(t, T_n, T_M)$ for which $SW_t = 0$. In particular, we obtain*

$$R = R(t, T_n, T_M) = \frac{Z_t^{T_n} - Z_t^{T_{M+1}}}{\delta \sum_{j=n+1}^{M+1} Z_t^{T_j}}. \tag{5}$$

The spot *swap rate is then obtained by taking $t = T_n$ in (5).*

Now consider the time t price[5] of a *payer-swaption* that expires at time $T_n > t$ and with payments of the underlying swap taking place at times $T_{n+1}, \ldots, T_{M+1}$. Assuming a fixed rate of $\widehat{R}$ (annualized) and a notional principle of $1, the value of the option at expiration is given by the positive part of (4). It satisfies

$$C_{T_n} \;=\; \left(1 - Z_{T_n}^{T_{M+1}} \;-\; \widehat{R}\delta \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j} \right)^+ . \tag{6}$$

Using (5) at $t = T_n$ we can substitute for $1 - Z_{T_n}^{T_{M+1}}$ in (6) and find that

$$\begin{aligned} C_{T_n} \;&=\; \left(\delta \left[R(T_n, T_n, T_M) - \widehat{R} \right] \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j} \right)^+ \\ &=\; \left(\delta \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j} \right) \left[R(T_n, T_n, T_M) - \widehat{R} \right]^+ . \end{aligned} \tag{7}$$

[5] Note that in (6) we have implicitly assumed that the strike is $k = 0$.

Therefore we see that the swaption is like a call option on the swap rate. The time t value of the swaption, C_t, is then given by the Q-expectation of the right-hand-side of (7), suitably deflated by the numeraire.

Black's Formula for Swaptions

Market convention, however, is to quote swaption prices via Black's formula which equates C_t to a Black-Scholes-like formula so that

$$C_t = \left(\delta \sum_{j=n+1}^{M+1} Z_t^{T_j}\right) \left[R(t,T_n,T_M)\Phi\left(\frac{\log(R(t,T_n,T_M)/\widehat{R}) + \sigma^2(T_n - t)/2}{\sigma\sqrt{T_n - t}}\right) - \widehat{R}\Phi\left(\frac{\log(R(t,T_n,T_M)/\widehat{R}) - \sigma^2(T_n - t)/2}{\sigma\sqrt{T_n - t}}\right)\right] \qquad (8)$$

where again σ is an 'implied' volatility that is used to quote prices. Note that the expression in (8) is what we would obtain for the expectation of

$$\left(\delta \sum_{j=n+1}^{M+1} Z_t^{T_j}\right) \left[R(T_n,T_n,T_M) - \widehat{R}\right]^+$$

if $dR(t,T_n,T_M) = \sigma R(t,T_n,T_M)\ dW_t$.

It should be stated that Black's formulae for caps and swaptions did not originally correspond to prices that arise from the application of martingale pricing theory to some particular model. As originally conceived, they merely provided a framework for quoting market prices. The market models of these lecture notes will provide a belated justification for these formulae. We shall see that the justifications are mutually inconsistent, however, in that it is impossible for both formulae to hold simultaneously within the one model.

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2 The Term Structure of Volatility

The term structure of volatility[6] is a graph of volatility plotted against time to maturity, τ. There are of course many definitions of volatility and care is needed in specifying which definition is intended. Some commonly used definitions of the term structure of volatility at time t include:

1. The volatility of spot rates $Y_t^{t+\tau}$ as a function of τ. Depending on the model under consideration, this volatility may be available in closed form and the model calibrated to historical or implied rates.

2. The volatility, $\sigma(t, t+\tau)$, of instantaneous forward rates, $f(t, t+\tau)$.

3. The implied volatility, σ, given by Black's formula for caplets. This will vary with time to maturity and can be computed at any time from caplet prices in the market. The implied volatility also varies with the strike of the caplet, i.e. there is a volatility skew for each maturity. The term structure of caplet volatilities if therefore strike dependent.

4. The implied volatility, σ, given by Black's formula for caps. Again this will vary with time to maturity and strike. It can can be computed at any time from market prices for caps.

When calibrating term structure models it is common to calibrate using both market prices or rates, and the term structure of volatility. As a result we often want to work with models that allow for a rich variety of term structures of volatility as well of course, as a rich variety of term structures of interest rates.

[6]'Quants' in the fixed-income industry commonly refer to the 'term-structure of volatility' when discussing fixed-income derivatives and models. In this section we briefly give some possible definitions of the 'term-structure of volatility' but we will not need these definitions elsewhere in the course.

3 Numeraires and Zero-Coupon Bond Prices

While the cash account with $B_t := \exp\left(-\int_0^t r_s \, ds\right)$ has been the default numeraire to date, we will not work with the cash account as our numeraire in the context of market models. The reason is clear: in market models we take LIBOR rates (or swap rates) with a fixed tenor, δ, in mind, as our fundamental interest rates. It would therefore be very inconvenient (as well as defeating the purpose) if we had to determine the instantaneous short rate at each point in time. As a result we will generally work with other numeraire-EMM pairs as described below.

First, however, we will fix the maturities or *tenor dates* to which our market models will apply. At time t we could in principal have LIBOR rates, $L(t, T)$, available for all $T > t$. This is unnecessary, however, as the prices of most important securities, e.g. caps, floors, swaps, swaptions, Bermudan swaptions, etc., are determined by the rates (LIBOR or swap) applying to only a finite set of maturities. We therefore fix in advance a set of tenor dates[7]

$$0 := T_0 < T_1 < T_2 < \ \ldots \ < T_M < T_{M+1} \quad \text{with}$$

$$\delta_i := T_{i+1} - T_i, \qquad i = 0, 1, \ldots, M.$$

While the δ_i's are usually nominally equal, e.g. $1/4$ or $1/2$, day-count conventions will results in slightly different values for each δ_i. We let Z_t^n denote the time t price of a zero-coupon bond maturing at time $T_n > t$ for $n = 1, \ldots, M$. Similarly, we use $L_n(t)$ to denote the time t forward rate applying to the period $[T_n, T_{n+1}]$ for $n = 0, 1, \ldots, M$. In particular, (2) then states

$$L_n(t) \;=\; \frac{Z_t^n - Z_t^{n+1}}{\delta_n Z_t^{n+1}}, \quad \text{for } 0 \le t \le T_n, \quad n = 0, 1, \ldots, M. \tag{9}$$

With some work we can invert (9) to obtain an expression for bond prices in terms of LIBOR rates. We find

$$Z_{T_i}^n \;=\; \prod_{j=i}^{n-1} \frac{1}{1 + \delta_j L_j(T_i)} \quad \text{for } n = i+1, \ldots, M+1. \tag{10}$$

Equation (10) only determines the bonds prices at the fixed maturity dates. However, for an arbitrary date t we can easily check that

$$Z_t^n \;=\; Z_t^{\phi(t)} \prod_{j=\phi(t)}^{n-1} \frac{1}{1 + \delta_j L_j(t)} \quad \text{for } 0 \le t \le T_n. \tag{11}$$

where we define $\phi(t)$ to be next tenor date after time t. That is,

$$\phi(t) \;:=\; \min_{i=1,\ldots,M+1} \{i \;:\; t < T_i\}.$$

Remark 2 *The presence of $Z_t^{\phi(t)}$ in (11) suggests that it may not be sufficient to model only the dynamics of the forward LIBOR rates, $L_n(t)$, when we specify a market model since they are not sufficient to determine $Z_t^{\phi(t)}$ at an arbitrary time t. However, as we shall see below, this will not prove to be a problem as the $\phi(t)$ factor vanishes upon deflating by the numeraire.*

Exercise 2 *Prove equations (10) and (11).*

[7] The notation and setup in this section and the next will borrow heavily from Section 3.7 in *Monte Carlo Methods in Financial Engineering* by Glasserman.

Numeraire-EMM Pairs

The following numeraire-EMM pairs are commonly used in market models:

1. The spot measure, Q, assumes that B_t^* is the numeraire where B_t^* is defined as follows.
   - start with $\$1$ at $t = 0$ and then purchase $1/Z_0^1$ of the zero-coupon bonds maturing at time T_1
   - at time T_1 reinvest the funds in the zero-coupon bond maturing at time T_2
   - by continuing in this way, we see that at time t the spot numeraire will be worth

   $$B_t^* = Z_t^{\phi(t)} \prod_{j=0}^{\phi(t)-1} [1 + \delta_j L_j(T_j)]. \tag{12}$$

   Note the similarity between this numeraire and our usual cash account.

2. The forward measure, P^T, takes the zero-coupon bond maturing at time T as numeraire. We have seen this numeraire-EMM pair already.

3. The swap measure, P^X, is useful for pricing swaptions analytically. It takes the numeraire to be $X_t = \delta \sum_{k=1}^{M} Z_t^k$, which is indeed a positive security price process.

Deflating Zero-Coupon Bond Prices by the Spot Numeraire

Equations (11) and (12) show that deflated[8] zero-coupon bond prices, $D_n(t)$, satisfy

$$D_n(t) = \left(\prod_{j=0}^{\phi(t)-1} \frac{1}{1 + \delta_j L_j(T_j)} \right) \prod_{j=\phi(t)}^{n-1} \frac{1}{1 + \delta_j L_j(t)} \quad \text{for } 0 \leq t \leq T_n. \tag{13}$$

In particular, we see that the factor, $Z_t^{\phi(t)}$, has vanished.

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4 The LIBOR Market Model

Dynamics under the Spot Measure

We assume that the dynamics of the LIBOR rates satisfy

$$dL_n(t) = \mu_n(t) L_n(t)\, dt + L_n(t) \sigma_n(t)^T\, dW(t), \quad 0 \leq t \leq T_n, \quad n = 1, \ldots, M \tag{14}$$

where $W(t)$ is a d-dimensional Brownian motion, and $\mu_n(t)$ and $\sigma_n(t)$ are adapted processes that may depend on the current vector of interest rates $L(t) := (L_1(t), \ldots, L_M(t))$. The assumption of no arbitrage and the positivity of deflated bond prices implies the existence of an R^d-valued process $\nu_n(t)$ such that

$$dD_n(t) = D_n(t) \nu_n^T(t)\, dW(t). \tag{15}$$

We could apply Itô's Lemma directly to our expression for $D_n(t)$ in (13) but this would be awkward. Instead we will apply Itô's Lemma to $Y_n(t) := \log D_n(t)$. We see from (15) that

$$dY_n(t) = -\frac{1}{2} ||\nu_n(t)||^2\, dt + \nu_n^T(t)\, dW(t) \tag{16}$$

[8]We will take the spot numeraire to be the default numeraire.

We can also find an alternative expression for $dY_n(t)$ using (13). In particular, noting that the first factor in (13) is constant between maturities, we obtain via Itô's Lemma

$$\begin{aligned} dY_n(t) &= -\sum_{j=\phi(t)}^{n-1} d\log\left(1+\delta_j L_j(t)\right) \\ &= -\sum_{j=\phi(t)}^{n-1} \left(\frac{\delta_j \mu_j(t) L_j(t)}{1+\delta_j L_j(t)} - \frac{\delta_j^2 L_j(t)^2 \sigma_j^T(t)\sigma_j(t)}{2\left(1+\delta_j L_j(t)\right)^2} \right) dt - \left(\sum_{j=\phi(t)}^{n-1} \frac{\delta_j L_j(t) \sigma_j^T(t)}{1+\delta_j L_j(t)} \right) dW(t). \quad (17) \end{aligned}$$

Comparing the volatility terms in (16) and (17) then gives us

$$\nu_n(t) = -\sum_{j=\phi(t)}^{n-1} \frac{\delta_j L_j(t)\sigma_j(t)}{1+\delta_j L_j(t)}. \quad (18)$$

We would now like to find an expression for the μ_j's. Towards this end, we could compare the drift terms in (16) and (17), and this is easy to do when $n = 2$ and $\phi(t) = 1$. After some straightforward algebra, we easily find[9]

$$\mu_1(t) = -\sigma_1^T(t)\nu_2(t), \qquad 0 \le t \le T_1.$$

More generally, we obtain

$$\mu_n(t) = -\sigma_n^T(t)\ \nu_{n+1}(t) = \sum_{j=\phi(t)}^{n} \frac{\delta_j L_j(t)\sigma_n^T(t)\sigma_j(t)}{1+\delta_j L_j(t)}. \quad (19)$$

We could have obtained (19) by again comparing the drift terms in (16) and (17) but this appears to be very cumbersome. Exercise 3 instead provides a more elegant approach.

Exercise 3 *Use induction to establish*[10] *that the drifts, $\mu_n(t)$, must satisfy (19) under the no-arbitrage assumption. In particular, first assume $\mu_1, \ldots, \mu_{n-1}$ have been chosen in a manner that is consistent with the Q-martingale assumption on $D_1, \ldots, D_n$. Then show that D_{n+1} is a martingale if and only if $L_n D_{n+1}$ is a martingale. Finally, apply Itô's Lemma to $L_n D_{n+1}$ and use the martingale property to obtain (19).*

We therefore obtain that the arbitrage free Q-dynamics of the forward LIBOR rates are given by

$$dL_n(t) = \left(\sum_{j=\phi(t)}^{n} \frac{\delta_j L_j(t)\sigma_n^T(t)\sigma_j(t)}{1+\delta_j L_j(t)} \right) L_n(t)\ dt + L_n(t)\sigma_n(t)^T\ dW(t), \quad 0 \le t \le T_n, \quad n = 1, \ldots, M. \quad (20)$$

Dynamics under the Forward Measure

Consider now the case where we use the forward measure, P_{M+1}, and the associated numeraire, Z_t^{M+1}. We use $\widehat{D}_n(t)$ to denote zero-coupon bonds prices that have been deflated by Z_t^{M+1}. Equation (11) with n replaced by $M+1$ then implies that

$$\widehat{D}_n(t) = \prod_{j=n}^{M} \left(1+\delta_j L_j(t)\right). \quad (21)$$

We would like to find the market-price-of-risk process, $\eta^{M+1}(t) \in R^d$, that relates the Q-Brownian motion $W(t)$ to the the P_{M+1} Brownian motion, $W_{M+1}(t)$, so that

$$dW(t) = dW_{M+1}(t) - \eta(t)\ dt. \quad (22)$$

[9]Note that $L_1(t)$, and therefore $\mu_1(t)$, do not have any meaning for $t > T_1$.

[10]See also Glasserman, page 170.

There are a number of ways to do this but perhaps the easiest is the approach we followed with the Vasicek model when we switched to the forward measure. Equation (21) implies $\widehat{D}_M(t) = 1 + \delta_M L_M(t)$ so that

$$d\widehat{D}_M(t) = \delta_M \; dL_M(t). \tag{23}$$

We now substitute for $dL_M(t)$ in (23) using (20) evaluated at $n = M$, and then substitute for $W(t)$ using (22). Since $\widehat{D}_M(t)$ is a P_{M+1}-martingale we find that

$$\eta(t) = \sum_{j=\phi(t)}^{M} \frac{\delta_j L_j(t)\sigma_j(t)}{1 + \delta_j L_j(t)}.$$

In particular, we obtain the arbitrage-free P_{M+1}-dynamics of the forward LIBOR rates are given by

$$dL_n(t) \;=\; -\left(\sum_{j=n+1}^{M} \frac{\delta_j L_j(t)\sigma_n^T(t)\sigma_j(t)}{1 + \delta_j L_j(t)} \right) L_n(t)\; dt + L_n(t)\sigma_n(t)^T \; dW_{M+1}(t), \quad 0 \le t \le T_n, \quad n = 1, \ldots, M. \tag{24}$$

Black's Formula for Caplets

We are now in a position to derive Black's formula (see (3)) for caplet prices. If we take $n = M$ in (24), then we obtain

$$dL_M(t) \;=\; L_M(t)\sigma_M(t)^T \; dW_{M+1}(t) \tag{25}$$

implying in particular[11] that $L_M(t)$ is a P_{M+1}-martingale. If we assume that $\sigma_M(t)$ is a deterministic function, then we easily see that $L_M(t)$ is log-normally distributed. In particular, we obtain

$$\log L_M(t) \;\sim\; \mathsf{N}\left(\log(L_M(0)) - \frac{1}{2}\int_0^t ||\sigma_M(s)||^2 \; ds \;,\; \int_0^t ||\sigma_M(s)||^2 \; ds \right). \tag{26}$$

We can now obtain (3) if we let $T_M = T$ and set $\sigma^2 = \int_0^{T_M} ||\sigma_M(s)||^2 \; ds \;/\; T_M$.

Note also that there is no problem when we take $\sigma_M(t)$ to be deterministic in (25) which contrasts with the HJM framework. This is because while the numerators in the drift of (20) are quadratic in $L_j(t)$, the $1 + \delta_j L_j(t)$ term in the denominator ensures that there is no possibility of explosion in the SDE. This is a further advantage of the market model framework where we model simple LIBOR rates rather than *instantaneous* forward rates.

Remark 3 *Note that under the EMM P_{M+1}, the LIBOR rates L_n for $n < M$ are not martingales. However, if we approximate the $L_j(t)$ term with $L_j(0)$ in the drift component of (24), then we would have log-normal dynamics for $L_n(t)$, assuming that all the $\sigma_j(t)$'s are deterministic. For reasonably short maturities, this approximation is sometimes used to construct approximate analytic prices for some derivative securities.*

The Caplet Volatility Surface

For a given market caplet price, $\mathsf{Cpt}(T_M, K)$ say, we can compute the implied volatility, $\sigma(K, T_M)$, and plot it as a function of the strike, K, keeping the time-to-maturity, T_M, fixed. If the market model with a deterministic $\sigma_M(t)$ is correct then we should obtain $\sigma(K, T_M) = \sigma(T_M)$ for all K. Until approximately the mid 1990's this was more or less the case but soon afterwards a noticeable skew started to appear in the market-place. This led to the development of models where the $\sigma_j(t)$'s were themselves stochastic. Examples[12] of such models are the CEV[13] model, shifted log-normal model and mixture of log-normals model. Other approaches to capturing the skew allow for jumps in the market rates. The success of these models depend to the extent that they can price caplets, floorlets and swaptions analytically and whether or not they are reasonably straightforward to calibrate.

BGM's Approximation for Swaption Prices

In their original paper, Brace, Gatarek and Musiela (BGM) succeeded in deriving Black's formula for caplets and thereby demonstrated its consistency with martingale pricing. Their framework did not enable them to derive

[11] Subject, as usual, to technical conditions.

[12] These are all described in Chapter 6 of the 2001 edition of Brigo and Mercurio. Rebonato's "Volatility and Correlation" devotes several chapters to modeling the smile.

[13] i.e. the constant elasticity of variance model.

Black's formula for swaptions, however. Instead they provided an analytic approximation for swaption prices that we will not describe[14] here. It is worth mentioning, however, that their approximation works well in practice and provides swaption prices that are very close to those obtained via Monte Carlo simulation.

Calibration

A good analytic approximation to swaption prices is very valuable as it allows for the possibility of calibrating the model to both cap and swaption prices simultaneously. Note that calibrating a market model amounts to selecting the parameters of the σ_j processes (or functions if they are are assumed to be deterministic functions of time). If swaption prices could only be computed or estimated via Monte-Carlo simulation then calibrating to swaption prices in this model would be very time-consuming. It is worth pointing out that caplet prices depend only on the level of volatility in the forward rates. This is clear from (26). Swaption prices, however, also depend on the *correlations* between the forward rates. So one common approach to calibration is the following:

1. Use market caplets to determine the *level* of forward rate volatility
2. Use swaption prices or historical forward rate correlations to determine the individual σ_j's subject to the caplet constraint in 1.

As with most models, calibration typically includes minimizing a (possibly weighted) sum of squares over the parameters of the processes (or functions), σ_j. This is what we do in Step 2 above although in this case we have the additional constraints of ensuring that caplets are priced correctly by the model. The ability to compute the model prices quickly is important for a successful calibration algorithm. Indeed this factor is a big influence on the choice of calibration securities in practice. As usual, however, the sum of squares is generally a non-convex function of the decision parameters and so there will typically be many local-minima, an issue which also needs to be handled carefully. We will not say any more about calibration of market models in these notes other than to point out that is a very important topic and is still a subject of ongoing research.

5 A Swap Market Model for Pricing Swaptions

Consider a *payer-swaption* that expires at time $T_n > t$ and with payments of the underlying swap taking place at times $T_{n+1}, \ldots, T_{M+1}$. Assuming a fixed rate of $\widehat{R}$ (annualized) and a notional principle of \$1, we showed in (7) that the time T_n price of the swaption is given by

$$C_{T_n} = \left(\delta \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j} \right) \left[R(T_n, T_n, T_M) - \widehat{R} \right]^+ . \tag{27}$$

This implies that the time t price of the swaption, C_t, satisfies

$$C_t = X_t \, \mathrm{E}_t^{P_x} \left[\frac{\left(\delta \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j} \right) \left[R(T_n, T_n, T_M) - \widehat{R} \right]^+}{X_{T_n}} \right] \tag{28}$$

where X_t is the time t price of the chosen numeraire security and P_x is the corresponding EMM. A particularly convenient choice of numeraire that we will adopt is the portfolio[15] consisting of δ units of each of the zero-coupon bonds maturing at times $T_{n+1}, \ldots, T_{M+1}$. Then $X_t = \delta \sum_{j=n+1}^{M+1} Z_t^{T_j}$ and we find

$$C_t = \left(\delta \sum_{j=n+1}^{M+1} Z_t^{T_j} \right) \mathrm{E}_t^{P_x} \left[\left[R(T_n, T_n, T_M) - \widehat{R} \right]^+ \right] \tag{29}$$

[14]See Chapter 9 of Cairns for a derivation.

[15]There is no difficulty taking a portfolio of securities rather than a fixed individual security as the numeraire. More generally in fact, we could take a dynamic self-financed portfolio as the numeraire security, assuming of course that it has strictly positive value at all times.

Jamshidian (1997) developed a term structure framework where at any time t the current term structure was given in terms of the forward swap rates, $R(t, T_i, T_M)$ for $i = \phi(t), \ldots, M$. In particular, he showed that it was possible to assume that the P_x-dynamics of $R(t, T_n, T_M)$ satisfy

$$dR(t, T_n, T_M) \;=\; R(t, T_n, T_M)\sigma(t)^T \; dW^x(t) \tag{30}$$

where $\sigma(t)$ is a deterministic vector of volatilities. This implies that the forward swap rate is log-normally distributed so we can obtain[16] Black's formula for swaption prices (8).

Remark 4 *When we model swap rates directly as in (30) we say that we have a* swap *market model. This contrasts with the* LIBOR *market models of Section 4.*

Remark 5 *The advantage of Black's swaption formula is that it is elegant and exact, whereas the BGM formula is cumbersome and only an approximation. However, the BGM approximation is consistent with Black's formulae for caplets and caps whereas Black's swaption formula is not. Indeed, it may be shown[17] that if forward LIBOR rates have deterministic volatilities then it it is not possible for swap rates to also have deterministic volatilities. Therefore Black's formulae for caplets and swaptions cannot both hold within the same model. That said, within the LIBOR market framework with deterministic volatilities, it can be argued that forward swap rates are* approximately *log-normally distributed.*

---

6 Monte-Carlo Simulation

While it is possible to price many commonly traded derivative securities such as caps, floors and swaptions in the market model framework, it is in general necessary to use Monte Carlo methods to price other securities. Indeed, if our market model has stochastic volatility functions then it will typically be necessary to also use Monte Carlo methods to price even caps, floors and swaptions.

The typical approach is to use some discretization scheme such as the Euler scheme when performing the Monte Carlo simulation. This does not create too much of a computational burden as we will only need to simulate the SDE's describing the forward LIBOR dynamics for a *finite* number of maturities. This contrasts with the HJM framework where we had infinitely many maturities which meant it was practically infeasible to use a very fine discretization. This in turn prompted the development of the discrete-time HJM framework with the resulting discrete-time arbitrage-free restriction on the drift.

It is also possible to develop discrete-time arbitrage-free market models in a manner that is analogous to our discrete-time HJM development. As described above, however, the need to do so is not as urgent as it is practically feasible to simulate the market model SDE's on a sufficiently fine grid and this is what is typically done in practice.

Nonetheless, Glasserman's *Monte Carlo Methods for Financial Engineering* describes how to build discrete-time arbitrage-free market models. It turns out to be inconvenient to choose the LIBOR rates as the fundamental variables that we choose to discretize. Instead it is more convenient to directly model deflated bond prices as discrete-time Q-martingales[18] and to define LIBOR rates in terms of these bond prices. Other choices of discretization variable are also possible. As usual, we can choose to simulate under any EMM that we prefer and all of the usual variance reduction techniques may be employed.

---

[16] Of course we need to reinterpret σ in (8) in terms of the deterministic function $\sigma(t)$ in (30).
[17] This is done by applying Itô's Lemma to the forward swap rate given in (5).
[18] This ensures the discrete-time model is+ arbitrage-free.

7 Hedging and the Greeks

For the swap market model with deterministic volatilities, it is straightforward to construct self-financing hedging strategies for swaptions (and therefore caplets and floorlets as they are simply one-period swaptions). But these hedging strategies rely on the assumption that the volatilities are deterministic. In practice, users of both the LIBOR and swap market models will want to hedge against changes in volatility. How they hedge will typically be very model dependent and ir probably as much of an art as a science. It also requires a very good understanding of the strengths and weaknesses of the model or models that they are using to hedge.

8 Pricing Bermudan Swaptions

Perhaps the most commonly traded exotic interest rate derivative is the Bermudan swaption. We already saw how to price these options using lattice models earlier in the course but they are considerably more difficult to handle in the context of market models. This is because market models are *high-dimensional*, with a separate state variable for each forward rate. Lattice or finite difference methods do not work well in high-dimensions due to the so-called *curse of dimensionality* and so other solution techniques are required. In the earlier part of this decade simulation methods based on *cross-path* regressions were developed and these have proved extremely successful at generating good approximate prices to Bermudan / American options. We will first describe the cross-path regression technique which is used to generate a good feasible exercise strategy. Because it results in a feasible exercise strategy, the fair value of this strategy constitutes a lower bound on the true price of the Bermudan option. The better the exercise strategy the closer the corresponding lower bound will be to the true option price.

8.1 Computing Lower Bounds using Cross-Path Regressions

The general[19] Bermudan option pricing problem at time $t = 0$ is to compute

$$V_0 := \sup_{\tau \in \mathcal{T}} \mathrm{E}_0^{\mathbb{Q}} \left[\frac{h_\tau}{B_\tau} \right] \tag{31}$$

where $\mathcal{T} = \{0 \leq t_1, \ldots, t_n = T\}$ is the set of possible exercise dates, B_t is the value of the cash account at time t and $h_t = h(X_t)$ is the payoff function if the option is exercised at time t. X_t represents the time t (vector) value of the *state* variables in the model. In the case of a Bermudan swaption in the LIBOR market model, for example, X_t would represent the time t value of the various forward LIBOR rates. In theory (31) is easily solved using value[20] iteration. In particular, we would obtain

$$\begin{aligned} V_T &= h(X_T) \quad \text{and} \\ V_t &= \max\left(h(X_t),\ \mathrm{E}_t^{\mathbb{Q}} \left[\frac{B_t}{B_{t+1}} V_{t+1}(X_{t+1}) \right] \right). \end{aligned}$$

The price of the option is then given by $V_0(X_0)$ where X_0 is the initial state vector. As an alternative to value iteration we could use *Q-value iteration*. If the Q-value function is defined to be the value of the option conditional on it not being exercised today, i.e. the *continuation value* of the option, then we also have

$$Q_t(X_t) = \mathrm{E}_t^{\mathbb{Q}} \left[\frac{B_t}{B_{t+1}} V_{t+1}(X_{t+1}) \right]. \tag{32}$$

[19] We will specialize to the Bermudan *swaption* later.

[20] The term "value iteration" refers to the dynamic programming approach of computing the value function iteratively by working backwards in time. This is how we compute the price of American options in the binomial model.

The value of the option at time $t+1$ is then

$$V_{t+1}(X_{t+1}) = \max(h(X_{t+1}), Q_{t+1}(X_{t+1})) \tag{33}$$

so that if we substitute (33) into (32) we obtain

$$Q_t(X_t) = \mathrm{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_{t+1}} \max\left(h(X_{t+1}), Q_{t+1}(X_{t+1})\right)\right]. \tag{34}$$

Equation (34) clearly gives a natural analog to value iteration, namely *Q-value iteration*. If X_t is high dimensional, then both value iteration and Q-value iteration are not feasible in practice. However, we could perform an *approximate* and efficient version of Q-value iteration, and we now describe how to do this using cross-path regressions.

The first step is to choose a set of *basis functions*, $\phi_1(X), \ldots, \phi_m(X)$. These basis functions define the *linear architecture* that will be used to approximate the Q-value functions. In particular, we will approximate $Q_t(X_t)$ with

$$\widetilde{Q}_t(X_t) \;=\; r_t^1\phi_1(X_t) \;+\; \ldots \;+\; r_t^m\phi_m(X_t)$$

where $r_t := (r_t^1, \ldots, r_t^m)$ is a vector of time t parameters that is determined by the algorithm which proceeds as follows:

Cross-Path Regression Algorithm for Approximate Q-Value Iteration

> **generate** N independent paths of the state vector, X_t for $t = 1, ..., T$
> **set** $\widetilde{Q}_T(X_T^i) = 0$ for all $i = 1$ to N
> **for** $t = T-1$ Down to 1
>   Estimate $r_t = (r_t^1, \ldots, r_t^m)$
>   **set** $\widetilde{Q}_t(X_t^i) \;=\; \sum_k r_t^k \phi_k(X_t^i)$ for all i
> **end for**
> **set** $\widetilde{V}_0(X_0) = \max\Big(h(X_0),\ \widetilde{Q}_0(X_0)\Big)$

Two steps require further explanation.

1. First, we estimate r_t by regressing $\alpha \max\Big(h(X_{t+1}),\ \widetilde{Q}(X_{t+1})\Big)$ on $(\phi_1(X_t), \ldots, \phi_m(X_t))$ where $\alpha = B_t/B_{t+1}$ is the discount factor for moving from $t+1$ to t. We have N observations for this regression and N is usually taken to be somewhere between $10,000$ and $50,000$, though it will of course depend on the problem at hand.
2. Second, since all N paths have the same starting point, X_0, we can estimate $\widetilde{Q}_0(X_0)$ by averaging and discounting $\widetilde{Q}_1(\cdot)$ evaluated at the N successor points of X_0.

Remark 6 *Obviously many more details are required to fully specify the algorithm. In particular, the parameter values N and m, and the basis functions need to be chosen. Specific implementation details can also vary. For example, if the option is out-of-the-money at some time t and in some simulated state X_t, then you could choose to omit this sample from the regression at time t. In practice the selection of basis functions is vital.*

In practice, it is quite common for an alternative estimate, $\underline{V}_0$, of V_0 to be obtained by simulating the exercise strategy that is defined implicity by the sequence of Q-value function approximations. That is, we define $\widetilde{\tau} = \min\{t \in \mathcal{T} : \widetilde{Q}_t \le h_t\}$ and

$$\underline{V}_0 = \mathrm{E}_0^{\mathbb{Q}}\left[\frac{h_{\widetilde{\tau}}}{B_{\widetilde{\tau}}}\right].$$

$\underline{V}_0$ is then an unbiased lower bound on the true value of the option as it is the price that corresponds to a feasible exercise strategy.

Exercise 4 *Given how the lower bound, $\underline{V}_0$, is computed, can you guess why we prefer to do an approximate Q-value iteration instead of an approximate value-iteration?*

These algorithms[21] have performed surprisingly well on realistic high-dimensional problems and there has also been considerable theoretical work explaining why this is so. The quality of $\underline{V}_0$, for example, can be explained in part by noting that exercise errors are never made as long as $Q_t(\cdot)$ and $\widetilde{Q}_t(\cdot)$ lie on the *same* side of the optimal exercise boundary. This means in particular, that it is possible to have large errors in $\widetilde{Q}_t(\cdot)$ that do not impact the quality of $\underline{V}_0$.

8.2 Back to Bermudan Swaptions

We can now use the algorithm of Section 8.1 to price Bermudan swaptions in the LIBOR market model. But first, we will define again the Bermudan swaption contract.

Definition 2 *Let $\mathcal{T} = \{0 \leq t_1, \ldots, t_{n-1}\}$ be the set of possible exercise dates. Then the holder of a Bermudan swaption has the right to enter at any time $t \in \mathcal{T}$ into an interest-rate swap with fixed rate K that expires with last payment at time $t_n = T$. If the swaption is exercised at time t_l, then the first reset point is t_l.*

A *payer* Bermudan swaption is a a swaption where the holder of the option will pay the fixed rate if the option is exercised. A *receiver* Bermudan swaption, on the other hand, will receive the fixed rate if the option is exercised. Note, for example, that a $2-8$ Bermudan swaption is an option where the first exercise date occurs in 2 years[22] time and, regardless of when (if ever) the option is exercised, the underlying swap will expire in $2+8=10$ years time.
In order to price the Bermudan swaption, we need to specify the parameters for the cross-path regression algorithm and in particular, the basis functions. We could, for example, select

$$L_j(t_i), \ldots, L_{n-1}(t_i), S_n(t_i), S_n(t_i)^2 \text{ and } S_n(t_i)^3$$

as our time t_i basis functions where

$$S_n(t_i) := \frac{1 - Z_i^n}{\delta \sum_{j=i+1}^{n} Z_i^j}.$$

is the underlying swap rate at time t_i. Note that under this specification, the number of basis functions decreases by one as you move from one period to the next. Once we specify the number of paths, N, to simulate we can apply the cross-path regressions algorithm and obtain a lower bound on the price of the Bermudan swaption. the quality of this lower bound, will depend mainly on the chosen set of basis functions.

Appendix: Computing Upper Bounds to Bermudan Option Prices

While the cross-path regression algorithm has been very successful in practice, an important weakness is the inability to determine how far the solution, i.e. estimated option price, is from the true option price in any given problem. Dual-based methods have recently been developed[23] for evaluating any approximate solution by using it to construct an upper bound on the option price. This is of value since we can then compute lower and upper bounds on the optimal price of the Bermudan option and therefore determine how far the approximate solution is from optimality. We now describe these dual based methods.

For an arbitrary supermartingale, π_t, the value of an American option, V_0, satisfies

[21] They were introduced originally by Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (2001). But see also Carriere (1996).
[22] Or months, depending on the context.
[23] The dual methods were introduced independently by Haugh and Kogan (2004) and Rogers (2002).

$$
\begin{aligned}
V_0 = \sup_{\tau \in \mathcal{T}} \mathrm{E}_0^{\mathcal{Q}}\left[\frac{h_\tau}{B_\tau}\right] &= \sup_{\tau \in \mathcal{T}} \mathrm{E}_0^{\mathcal{Q}}\left[\frac{h_\tau}{B_\tau} - \pi_\tau + \pi_\tau\right] \\
&\leq \sup_{\tau \in \mathcal{T}} \mathrm{E}_0^{\mathcal{Q}}\left[\frac{h_\tau}{B_\tau} - \pi_\tau\right] + \sup_{\tau \in \mathcal{T}} \mathrm{E}_0^{\mathcal{Q}}\left[\pi_\tau\right] \\
&\leq \sup_{\tau \in \mathcal{T}} \mathrm{E}_0^{\mathcal{Q}}\left[\frac{h_\tau}{B_\tau} - \pi_\tau\right] + \pi_0 \\
&\leq \mathrm{E}_0^{\mathcal{Q}}\left[\max_{t \in \mathcal{T}}\left(\frac{h_t}{B_t} - \pi_t\right)\right] + \pi_0 \qquad (35)
\end{aligned}
$$

where the second inequality follows from the optional sampling theorem for supermartingales. Taking the infimum over all supermartingales, π_t, on the right hand side of (35) implies

$$
V_0 \;\leq\; U_0 \;:=\; \inf_{\pi}\; \mathrm{E}_0^{\mathcal{Q}}\left[\max_{t \in \mathcal{T}}\left(\frac{h_t}{B_t} - \pi_t\right)\right] + \pi_0 \qquad (36)
$$

On the other hand, it is a known fact that the process V_t/B_t is itself a supermartingale, which implies

$$
U_0 \leq \mathrm{E}_0^{\mathcal{Q}}\left[\max_{t \in \mathcal{T}}\left(h_t/B_t - V_t/B_t\right)\right] + V_0.
$$

Since $V_t \geq h_t$ for all t, we conclude that $U_0 \leq V_0$. Therefore, $V_0 = U_0$, and equality is attained when $\pi_t = V_t/B_t$.

This shows that an upper bound on the price of the American option can be constructed simply by evaluating the right-hand-side of (35) for a given supermartingale, π_t. In particular, if such a supermartingale satisfies $\pi_t \geq h_t/B_t$, the option price V_0 is bounded above by π_0.

When the supermartingale π_t in (35) coincides with the discounted option value process, V_t/B_t, the upper bound on the right-hand-side of (35) equals the true price of the American option. This suggests that a tight upper bound can be obtained by using an accurate approximation, $\widetilde{V}_t$, to define π_t. One possibility[24] is to define π_t as a martingale:

$$
\begin{aligned}
\pi_0 &= \widetilde{V}_0 \qquad (37) \\
\pi_{t+1} &= \pi_t + \frac{\widetilde{V}_{t+1}}{B_{t+1}} - \frac{\widetilde{V}_t}{B_t} - \mathrm{E}_t\left[\frac{\widetilde{V}_{t+1}}{B_{t+1}} - \frac{\widetilde{V}_t}{B_t}\right]. \qquad (38)
\end{aligned}
$$

Let $\overline{V}_0$ denote the upper bound we get from (35) corresponding to our choice of supermartingale in (37) and (38). Then it is easy to see that the upper bound is explicitly given by

$$
\overline{V}_0 = \widetilde{V}_0 + \mathrm{E}_0^{\mathcal{Q}}\left[\max_{t \in \mathcal{T}}\left(\frac{h_t}{B_t} - \frac{\widetilde{V}_t}{B_t} + \sum_{j=1}^{t} \mathrm{E}_{j-1}^{\mathcal{Q}}\left[\frac{\widetilde{V}_j}{B_j} - \frac{\widetilde{V}_{j-1}}{B_{j-1}}\right]\right)\right]. \qquad (39)
$$

As may be seen from (39), obtaining an accurate estimate of $\overline{V}_0$ is computationally demanding. First, a number of sample paths must be simulated to estimate the outermost expectation on the right-hand-side of (39). While this number need not be large in practice, we also need to accurately estimate a conditional expectation at each time period along each simulated path. This requires some effort and clearly variance reduction methods would be useful in this context. Low discrepancy sequences are also very useful for this task. Variations and extensions of these algorithms have also been developed recently and are a subject of ongoing research.[25]

[24] See Haugh and Kogan (2002) and Andersen and Broadie (2002) for further comments related to the choice of π_t.

[25] See Chapter 8 of Glasserman (2003) for a detailed treatment of Monte Carlo methods for pricing American options.