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Optimal Control and Hedging of Operations in the Presence of Financial Markets

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We consider the problem of dynamically hedging the profits of a corporation when these profits are correlated with returns in the financial markets. In particular, we consider the general problem of simultaneously optimizing over both the operating policy and the hedging strategy of the corporation. We discuss how different informational assumptions give rise to different types of hedging and solution techniques. Finally, we solve some problems commonly encountered in operations management to demonstrate the methodology.

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1. Introduction. In this paper, we propose a framework for modelling the operations of a nonfinancial corporation that also trades in the financial markets. The corporation must simultaneously choose an optimal operating policy and an optimal trading strategy in the financial markets. In practice, it has long been observed that nonfinancial corporations do, in fact, hedge using financial markets. The goal of this paper then is to describe a method by which operations *and* hedging might be conducted.

One immediate difficulty that arises when modelling this problem is that most of the operations literature assumes that corporations are risk neutral. Indeed, this is supported by the famous work of Modigliani and Miller [24] who argue that in a frictionless world there is no need for corporations to hedge as shareholders can do so themselves. While this argument has some merit, we do, of course, live in a world with many frictions. These frictions include the costs of financial distress, taxes, and agency costs, as well as frictions in the capital markets. As a result, it is often the case that corporations should and do hedge. Once this is recognized, it is no longer plausible to assume that corporations are always risk neutral.

In this paper, we therefore consider the problem of dynamically hedging the profits of a *risk-averse* corporation when these profits are correlated with returns in the financial markets. The central modelling insight is to view the operations and facilities of the corporation as an asset in the corporation's portfolio. This view enables us to pose the problem as one of financial hedging in incomplete markets, a problem that has been studied extensively in the recent literature in mathematical finance, e.g., Schweizer [31]. Though we pose the problem as one of hedging in incomplete markets, we also have the added complexity of simultaneously seeking to choose an optimal operating policy. As a result, we also have some control over the type of asset to be hedged. This is a distinguishing feature of this paper that is generally not found in the mathematical finance literature. We also discuss how different informational assumptions give rise to different types of hedging and solution techniques. In particular, the class of feasible hedging strategies that are available to the corporation will depend on whether or not the corporation can observe the evolution of all relevant state variables.

To maintain tractability, we will assume that the corporation has a mean-variance objective function. While this of course is somewhat restrictive, it is often used in practice and can serve as a useful first approximation. The techniques that we use are based on the mean-variance analysis of Schweizer [30], and the martingale approach of Cox and Huang [8] and Karatzas et al. [19]. While we are aware of the very recent progress that has been made towards solving hedging problems for more general utility functions and general price processes (e.g., Bertsimas et al. [2], Delbaen et al. [10], Gouriéroux et al. [16], Laurent and Pham [22], Lim [23], Pham et al. [28], Schweizer [31]), we have not attempted to apply this work here. Indeed, much of this literature is concerned with issues regarding existence and uniqueness of solutions and does not lend itself easily to the computation of such solutions. Moreover, the purpose of this paper is simply to highlight the modelling framework and demonstrate that it can be used to solve some interesting problems in operations management.

We will see that our framework appears to be most useful when the operational control is a scalar or vector of scalars as opposed to a dynamic control policy. In the latter case, we can still use the framework to solve the problem numerically (see Appendix B), but it is not at all clear that this provides any improvement beyond the standard Hamilton-Jacobi-Bellman (HJB) approach.

The remainder of this paper is organized as follows. In §2, we formulate the problem under two different informational assumptions and show how to solve the problem in each case. In §§3 and 4, we demonstrate the methodology by solving two problems from operations management, including the so-called *newsboy problem*. We finally conclude and discuss future research directions in §5.

2. Model and problem formulation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with two independent standard Brownian motions, $B_{1,t}$ and $B_{2,t}$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ the usual filtration generated by (B_1, B_2) where T is a fixed-time horizon. We also define a subfiltration $\widetilde{\mathbb{F}}$ of $\mathbb{F}$ that will represent the evolution of the observable information in the model. When all relevant information is observable, then we will have $\widetilde{\mathbb{F}} = \mathbb{F}$.

The financial market that we consider consists of a risk-free cash account and a risky stock. Without lost of generality, we will assume throughout that the risk-free interest rate, r, is identically zero. The time t stock price, X_t, satisfies the stochastic differential equation

$$\mathrm{d}X_t = \mu_t X_t \,\mathrm{d}t + \sigma_t X_t \,\mathrm{d}B_{1,t}, \tag{1}$$

where μ_t and σ_t are assumed to be bounded adapted processes. We define the set Θ of self-financing trading strategies to be the collection of $\widetilde{\mathbb{F}}$-predictable processes $(\theta_t)_{0\le t\le T}$ satisfying

$$\mathbb{E}\left[\int_0^T \theta_t^2 X_t^2 \,\mathrm{d}t\right] < \infty.$$

We interpret θ_t as the number of shares in the stock held at time t. Given $(\theta_t)_{0\le t\le T}$, the self-financing condition then implicitly defines the position in the cash account for all $t \in [0, T]$. Recalling that $r \equiv 0$, the gain process, $G_t(\theta)$, associated with a trading strategy, $\theta \in \Theta$, is defined to be

$$G_t(\theta) := \int_0^t \theta_t \,\mathrm{d}X_t \quad \text{for all } t \in [0, T].$$

We consider the problem of a risk-averse nonfinancial corporation that earns a terminal payoff, H_T, that may be interpreted as the profits that are earned from operating in $[0, T]$. Of course H_T will depend on the operating policy, γ, that is adopted in this interval. In this context, γ represents a generic operational policy that may in principle be anything from a scalar control to a complex state-dependent control policy. We will usually write $H_T^{(\gamma)}$ to signify the dependence of the operating profits on γ. The set of $\widetilde{\mathbb{F}}$-predictable admissible policies is denoted by Γ and we will assume that for $\gamma \in \Gamma$, the payoff $H_T^{(\gamma)}$ is an $\mathcal{F}_T$-measurable random variable that satisfies $H_T^{(\gamma)} \in \mathcal{L}^p(\mathbb{P})$ for some $p > 2$.

In addition to the nonfinancial operations, the corporation is able to trade in the financial market by employing a self-financing trading strategy, $\theta \in \Theta$. Therefore, for a given initial wealth, W_0, and for a given strategy, $(\gamma, \theta) \in \Gamma \times \Theta$, the corporation's time T wealth is given by $W_0 + H_T^{(\gamma)} + G_T(\theta)$. (Due to the possibility of unlimited borrowing, we may assume that Θ is independent of the operating policy, γ.)

We assume that the corporation has a quadratic utility function, $u(\cdot)$, defined over terminal wealth so that $u(w) = w - lw^2$ where l is a positive constant. Then, the corporation's problem is to solve

$$\max_{(\gamma,\theta)\in\Gamma\times\Theta} \mathbb{E}\big[u\big(W_0 + H_T^{(\gamma)} + G_T(\theta)\big)\big]. \tag{2}$$

An important potential weakness in our problem formulation can be seen in (2) where operational profits and trading profits are treated identically in the corporation's utility function. In practice, this is certainly not the case. For example, suppose the corporation has a choice over two possible random variables, Y_1 and Y_2, which represent terminal profits. If Y_1 and Y_2 are identically distributed, then in our problem formulation, the corporation should be indifferent between the two. Suppose now, however, that Y_1 is strongly positively correlated with the financial market and that Y_2 is independent of the financial market. Then, for diversification reasons, shareholders (and therefore the corporation) would prefer Y_2 to Y_1. A simple argument based on the capital asset pricing model (CAPM) makes this clear, but the reasoning applies more generally.

A solution $(H_T^{(\gamma^*)}, G_T^*)$ to (2), where $H_T^{(\gamma^*)} + G_T^*$ is highly correlated with the financial markets due to a significant reduction in operational activity, might therefore be unappealing. This would be true in particular if the solution was intended to be implemented on a permanent basis, i.e., if after each period of length T the corporation intended to implement the same optimal operating and hedging strategies. However, there are many reasons for why such a significant reduction in operations might make sense on a *temporary* basis. We also mention that, in many contexts, we would *not* expect the optimal solution to (2) to necessarily result in a reduction in operations. We mention finally that any problems associated with a solution resulting in a significant reduction in operational activity could be overcome by working instead with an *equivalent martingale measure* (EMM), $\mathbb{Q}$, rather than the physical measure, $\mathbb{P}$. The cumulative trading gains process is a martingale under any such measure, $\mathbb{Q}$, and so the only motivation for trading would be to hedge operational cash flows. On the other hand, it is more difficult to interpret the quadratic objective function when we use an EMM, $\mathbb{Q}$.

Let us now comment on the relationships between X_t, B_2, and γ. We have assumed that X represents some financial market, e.g., an equity index, an exchange rate, or possibly an economic index. As such, we would not expect X to depend in any way on the operating policy, γ, of an individual corporation. It is possible that an argument could be made for introducing some dependency if the corporation was extremely large and the hedging security, X, was, for example, an index reflecting the performance of the industry in which the corporation operates. However, we do not have that situation in mind in this paper, and so we assume that γ has no impact upon X. Similarly, B_2 represents nonfinancial or idiosyncratic noise. It is firm specific and might represent that part of the market demand for a particular good that is *not* explained by the current state of the economy or financial market. Or it might simply represent the uncertain quality of a product to be purchased or manufactured at some future date. Because of this interpretation, we make the natural assumption that μ_t and σ_t are adapted to the filtration generated by B_1 only. Finally, the manner in which $H_T^{(\gamma)}$ then depends on X and B_2 will depend on the application in question. We will give some examples in §§3 and 4.

We now conclude this section with two observations. First, we mention that the models we consider in this paper could be extended to the case where B_1 and B_2 are correlated *multidimensional* Brownian motions. Second, it is easy to see that the utility function, $u(\cdot)$, satisfies

$$u(w) = \frac{1}{4l} - l\left(\frac{1}{2l} - w\right)^2.$$

Therefore, in order to avoid trivial solutions in (2), we assume that the initial wealth, W_0, satisfies $W_0 < 1/(2l)$. Of course, for the same reason, we should also expect that $W_0 + V_0^{(\gamma)} < 1/(2l)$ where $V_0^{(\gamma)}$ is some *intrinsic* initial value of $H_T^{(\gamma)}$. This is discussed in more detail in §2.4.

2.1. Informational assumptions. As stated above, the filtration $\tilde{\mathbb{F}}$ represents the evolution of observable information in the model, while $\mathbb{F}$ is the usual filtration generated by (B_1, B_2).

In this paper, we will consider two scenarios regarding observable information. In the first, we will assume that the corporation can only observe the evolution of $B_{1,t}$. The interpretation, then, is that the corporation can observe the financial market but cannot observe the evolution of the nonfinancial noise, $B_{2,t}$. This might occur, for example, if $B_{2,t}$ represents the unobserved quality of a supplier's goods, or the unobserved quality of a competitor's products or services. Depending on the context, many other interpretations are possible. In this scenario, $\tilde{\mathbb{F}} := (\mathcal{F}_t^{\mathrm{F}})_{0 \le t \le T}$, which in turn is defined to be the usual filtration generated by $B_{1,t}$. Such a model will be referred to as an *incomplete information* (II) model.

In the second scenario, we assume that the corporation is able to observe the evolution of $B_{1,t}$ and $B_{2,t}$. In this case, $\tilde{\mathbb{F}} = \mathbb{F}$. We will refer to this model as a *complete information* (CI) model. When considering the CI model, it will be necessary to assume that the mean-variance trade-off, μ_t/σ_t, is bounded and deterministic. This assumption is standard in the hedging literature despite its obvious shortcomings. Fortunately, it is satisfied by geometric Brownian motion, the canonical model for modelling financial price processes. We can also argue that, from the perspective of a nonfinancial corporation, this assumption may not present any serious difficulty. In particular, a corporation might be quite happy to approximately hedge financial risks by *assuming* that the hedging instrument, X_t, has a deterministic mean-variance trade-off, even if this is only approximately true.

Of course, intermediate cases are possible where $B_{2,t}$ is not fully observable, but some information over and beyond what can be inferred from the financial markets is available. We will not consider these cases in this paper.

When considering the II model in this paper, we will simplify its solution considerably by making a *complete financial markets* assumption. In particular, we will assume that any suitably integrable contingent claim that is $\mathcal{F}_T^{\mathrm{F}}$ measurable is attainable by a self-financing trading strategy, $\theta \in \Theta$. In words, it states that if we ignore the nonfinancial noise, B_2, then markets are complete. Such an assumption has been made before in other contexts. For example, Smith and Nau [33] make a similar assumption for pricing real options in a discrete time framework. They call their market a *partially complete market*, whereas we will use the term complete financial market. Collin Dufresne and Hugonnier [7] also make a complete financial markets assumption in the context of pricing and hedging contingent claims whose payoffs depend on nonmarket events. We should point out that in the model with only two Brownian motions that we have described, this assumption is in fact redundant. However, as our model generalizes easily to problems with multiple Brownian motions, we observe that the complete financial markets assumption for these more general models amounts to assuming that if there are m traded securities, then there are less than or equal to m Brownian motions driving their price processes. $H_T^{(\gamma)}$ will then depend on these m Brownian motions as well as other sources of nonfinancial noise.

We can again argue that such an assumption is not particularly restrictive from the perspective of a nonfinancial corporation that is happy to approximately hedge its financial risks. Moreover, given the ability to trade continuously in various derivative securities, the assumption of complete financial markets becomes ever easier to justify. Finally, we can even imagine the situation where the corporation purchases from a financial intermediary a derivative security with payoff, D_T. In such a situation, it is immaterial to the corporation whether or not financial markets are complete. All that is required by the corporation to choose its optimal hedging strategy is knowledge of the intermediary's state price density process. (A state price density process is a process whose value at a particular state and time may be interpreted as the price, per unit probability, of a security that pays \$1 in that state and time. See Duffie [11] for further details.) We also note that should such a scenario prevail, then it would enable us to consider other price processes such as jump diffusions, for example, when solving II models.

The benefit of the complete financial markets assumption is that it will allow us to easily identify an appropriate state price density (SPD), $\pi_{F,t}$, for pricing claims that only depend on B_1. Since our hedging gains in the II model are only allowed to depend on B_1, we can use $\pi_{F,t}$ to find the optimal hedging strategy. We now describe how to do this.

2.2. The incomplete information solution. As stated earlier, the market in this model is incomplete and the operational payoff $H_T^{(\gamma)}$ is not attainable using a self-financing trading strategy. In the absence of arbitrage, which we naturally assume, this implies that there are infinitely many SPDs. However, our completeness of financial markets assumption implies that all of these SPDs price financial risks, that is, risks that depend only on B_1 identically. Therefore, we can choose any such SPD, $\pi_{F,t}$, knowing that $\mathbb{E}[\pi_{F,T}C_T]$ is the unique arbitrage-free price of any $\mathcal{F}_T^{\mathrm{F}}$-measurable claim that pays C_T at time T. Given the stock price process (1), we can choose $\pi_{F,t}=\exp(-(1/2)\int_0^t \eta_s^2\,\mathrm{d}s-\int_0^t \eta_s\,\mathrm{d}B_{1,s})$ where $\eta_s:=\mu_s/\sigma_s$. Note that this choice assumes the *market price of risk* for B_2 is identically zero. However, this choice is merely for convenience and other choices would work equally well. We will now solve the II model.

The solution procedure will be to find the optimal trading gain, $G_T(\theta)$, for a fixed operating policy, γ, and to then optimize over γ. (We do not explicitly recognize the dependence of G_T on γ.) So, suppose now that $\gamma\in\Gamma$ is fixed. Then, the corporation's hedging problem is

$$\max_{G_T}\ \mathbb{E}\big[H_T^{(\gamma)}+G_T-l\big(H_T^{(\gamma)}+G_T\big)^2\big] \tag{3}$$

$$\text{subject to}\ \ \mathbb{E}[\pi_{F,T}G_T]=W_0 \tag{4}$$

$$\text{and}\ \ G_T\ \text{is}\ \mathcal{F}_T^{\mathrm{F}}\text{-measurable.}$$

The constraint in (4) is the budget constraint. This specifies that the time 0 value of the hedging gain, G_T, must equal W_0, the initial cash that is available for the corporation's hedging strategy. This problem is solved by first taking conditional expectations with respect to $\mathcal{F}_T^{\mathrm{F}}$ in (3) and using the $\mathcal{F}_T^{\mathrm{F}}$ measurability of G_T to obtain the equivalent formulation

$$\max_{G_T}\ \mathbb{E}\big[\mathbb{E}\big[H_T^{(\gamma)}-lH_T^{(\gamma)^2}\mid\mathcal{F}_T^{\mathrm{F}}\big]+G_T-lG_T^2-2lG_T\mathbb{E}\big[H_T^{(\gamma)}\mid\mathcal{F}_T^{\mathrm{F}}\big]\big] \tag{5}$$

$$\text{subject to}\ \ \mathbb{E}[\pi_{F,T}G_T]=W_0. \tag{6}$$

Now the problem is easily solved (see, for example, Duffie [11]) using a single Lagrange multiplier, λ, for the constraint (6). We no longer need to explicitly impose the $\mathcal{F}_T^{\mathrm{F}}$ measurability of G_T since this constraint will now be automatically satisfied. The first-order condition, which is also sufficient given the concavity of the objective in (5), is

$$1-2lG_T-2l\mathbb{E}\big[H_T^{(\gamma)}\mid\mathcal{F}_T^{\mathrm{F}}\big]=\lambda\pi_{F,T}. \tag{7}$$

We solve (7) for G_T and then substitute for G_T in (6) to obtain

$$\lambda^* = \frac{1 - 2l\mathbb{E}\big[\pi_{\mathrm{F},T}\mathbb{E}\big[W_0 + H_T^{(\gamma)} \mid \mathcal{F}_T^{\mathrm{F}}\big]\big]}{\mathbb{E}[\pi_{\mathrm{F},T}^2]}. \tag{8}$$

Finally, combining (7) and (8), we obtain

$$G_T^* = \frac{1}{2l} - \mathbb{E}\big[H_T^{(\gamma)} \mid \mathcal{F}_T^{\mathrm{F}}\big] - \frac{\lambda^*}{2l}\pi_{\mathrm{F},T} \tag{9}$$

so that the optimal hedging wealth, G_T^*, is $\mathcal{F}_T^{\mathrm{F}}$-measurable as desired.

To find the optimal operating policy, γ, we must now solve $\max_\gamma \mathbb{E}[u(H_T^{(\gamma)} + G_T^*)]$. We can easily rewrite this optimization problem as

$$\max_{\gamma\in\Gamma}\left\{\frac{1}{4l} - \frac{l}{\mathbb{E}[\pi_{\mathrm{F},T}^2]}\mathbb{E}\left[\left(\frac{1}{2l} - W_0 - \widehat{\mathbb{E}}\big[H_T^{(\gamma)}\big]\right)^2 + \mathbb{E}[\pi_{\mathrm{F},T}^2]\mathbb{E}\big[\big(H_T^{(\gamma)} - \mathbb{E}\big[H_T^{(\gamma)} \mid \mathcal{F}_T^{\mathrm{F}}\big]\big)^2\big]\right]\right\}, \tag{10}$$

where $\widehat{\mathbb{E}}[H_T^{(\gamma)}] := \mathbb{E}[\pi_{\mathrm{F},T} H_T^{(\gamma)}]$.

At this point, we make the observation that while this methodology solves the II problem, it might also serve as a good approximation for the CI problem when we cannot solve the latter explicitly. We would expect the II approximation to work well for CI problems when the financial noise, B_1, has a significant impact on $H_T^{(\gamma)}$. Even when B_1 has little or no impact on $H_T^{(\gamma)}$, however, the II approximation is often useful as it will hedge, in some sense, against an *average* value of $H_T^{(\gamma)}$.

We also remark that the methods we have used in this section can, in principle, be applied to more general utility functions, $u(\cdot)$. The issue then is one of tractability and whether or not we can compute $\mathbb{E}[u(H_T^{(\gamma)} + G_T) \mid \mathcal{F}_T^{\mathrm{F}}]$ explicitly and then solve for G_T^* so that optimizing over γ is then possible. In the case of the newsboy problem of §3, for example, this would be less of an issue since then the optimal γ is a scalar for which we could solve numerically.

2.3. The complete information solution. To solve the complete information problem, we will use the results on mean-variance hedging that were obtained by Schweizer [30]. This work in turn was motivated by Duffie and Richardson [12] and Föllmer and Schweizer [14].

Throughout this section, we will assume that the mean-variance trade-off of the traded stock, $\eta_t := \mu_t/\sigma_t$, is a bounded and deterministic function. Again, we will initially fix the operating strategy, $\gamma \in \Gamma$, and then solve the hedging problem for this fixed γ. The first step towards solving (2) for a fixed γ is to find an appropriate optimality condition for the hedging strategy, θ. Since $\mathcal{L}^2(\mathbb{P})$ is a Hilbert space equipped with the inner product $(X, Y) = \mathbb{E}[XY]$ and since $G_T(\Theta) := \{G_T(\theta) \mid \theta \in \Theta\}$ is a closed (see, for example, Monat and Stricker [25]) linear subspace of $\mathcal{L}^2(\mathbb{P})$, we can use the projection theorem to characterize the optimal trading strategy $\theta^* \in \Theta$ and obtain

$$\mathbb{E}\left[\left(H_T^{(\gamma)} + G_T(\theta^*) - \frac{1}{2l}\right)G_T(\theta)\right] = 0 \quad \text{for all } \theta \in \Theta. \tag{11}$$

The solution to (11) can be found using the *minimum equivalent martingale measure* (MEMM), $\widehat{\mathbb{P}} \approx \mathbb{P}$, defined by

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} := \exp\left(-\int_0^T \eta_t \, dB_{1,t} - \frac{1}{2}\int_0^T \eta_t^2 \, dt\right). \tag{12}$$

The minimal equivalent martingale measure was originally introduced by Föllmer and Schweizer [14] in the context of local risk minimization. Since then it has been recognized that for mean-variance hedging problems, the appropriate martingale measure is in fact the *variance optimal equivalent martingale measure* (VOEMM) (Delbaen and Schachermayer [9]). However, the two measures coincide when the mean-variance trade-off is deterministic, as we have assumed here. (In general, it is easy to find the MEMM, but establishing the existence of the VOEMM and actually computing it when it is known to exist are much harder. See the survey by Schweizer [31] for further details.)

Girsanov's theorem then implies that under $\widehat{\mathbb{P}}$, both X and B_2 are square-integrable martingales. Because of its importance for our analysis, we summarize in the following theorem the key result of Schweizer [30]. We will use $\widehat{\mathbb{E}}[\cdot]$ to denote expectation under $\widehat{\mathbb{P}}$. (Since the CI and II methods both use the same equivalent martingale measure (EMM), $\widehat{\mathbb{P}}$, there is no ambiguity in our notation. As an earlier observation pointed out, however, there are other EMMs that work equally well for the II model. No such flexibility exists for the CI model.)

THEOREM 1 (SCHWEIZER [30]). *For any $\mathcal{F}_T$-measurable claim $H_T^{(\gamma)} \in \mathcal{L}^p(\mathbb{P})$ for some $p > 2$, there is a hedging strategy, $\vartheta^{(\gamma)}$, and a process, $\delta^{(\gamma)} \in \mathcal{L}^2(\mathbb{P})$, such that $H_T^{(\gamma)}$ admits the decomposition*

$$H_T^{(\gamma)} = V_0^{(\gamma)} + \int_0^T \vartheta_t^{(\gamma)}\, \mathrm{d}X_t + \int_0^T \delta_t^{(\gamma)}\, \mathrm{d}B_{2,t},$$

where $V_0^{(\gamma)} := \widehat{\mathbb{E}}[H_T^{(\gamma)}]$. In addition, the optimal strategy, θ^*, that solves (11) is given by $\theta^* = \Phi(G_t^*)$ where G_t^* solves the stochastic differential equation (SDE)

$$\mathrm{d}G_t^* = -\Phi(G_t^*)\mathrm{d}X_t,$$

where $G_0^* = 0$, $\Phi(G_t^*) = \vartheta_t^{(\gamma)} + \mu_t/(\sigma_t^2 X_t)(V_t^{(\gamma)} + G_t^* + W_0 - 1/(2l))$, and $V_t^{(\gamma)}$ is the intrinsic value process defined by

$$V_t^{(\gamma)} := \widehat{\mathbb{E}}\big[H_T^{(\gamma)} \mid \mathcal{F}_t\big] = V_0^{(\gamma)} + \int_0^t \vartheta_s^{(\gamma)}\, \mathrm{d}X_s + \int_0^t \delta_s^{(\gamma)}\, \mathrm{d}B_{2,s}. \tag{13}$$

The decomposition in (13) is known as the Galtchouk-Kunita-Watanabe (GKW) decomposition (e.g., Jacod [18]) of $V_t^{(\gamma)}$ under $\widehat{\mathbb{P}}$ with respect to X. This result then solves the optimal hedging problem when the operating policy, γ, is fixed. In order to now optimize over γ, we first need to compute the expected utility as a function of γ, given the optimal strategy θ^* defined in Theorem 1 above. We then define

$$\Pi_t^{(\gamma)} := \frac{1}{4l} - l\,\mathbb{E}\left[\left(\frac{1}{2l} - V_t^{(\gamma)} - G_t^* - W_0\right)^2\right]$$

and we note that $\Pi_T^{(\gamma)}$ is the optimal expected utility for the fixed control, γ. The following result characterizes the dynamics of $\Pi_t^{(\gamma)}$. It is not presented by Schweizer [30], but the proof is very similar to the proof of Theorem 1 in Schweizer [30], and we present it in Appendix A only for the sake of completion. Similar expressions may be found in Pham [27].

THEOREM 2. *Let us define the auxiliary process $A_t^{(\gamma)} := \mathbb{E}[(1/(2l) - V_t^{(\gamma)} - G_t^* - W_0)^2]$ so that $\Pi_t^{(\gamma)} = 1/(4l) - lA_t^{(\gamma)}$. Then, $A_t^{(\gamma)}$ is given by*

$$A_t^{(\gamma)} = \exp\left(-\int_0^t \eta_s^2\, \mathrm{d}s\right)\left[\left(\frac{1}{2l} - V_0^{(\gamma)} - W_0\right)^2 + \int_0^t \exp\left(\int_0^s \eta_\tau^2\, \mathrm{d}\tau\right)\mathbb{E}\big[\delta_s^{(\gamma)2}\big]\,\mathrm{d}s\right]. \tag{14}$$

Theorem 2 implies that $A_t^{(\gamma)}$ is an increasing function of $\mathbb{E}[\delta_t^{(\gamma)^2}]$, and we would therefore like to have $\mathbb{E}[\delta_t^{(\gamma)^2}]$ as small as possible. In the event that $\mathbb{E}[\delta_t^{(\gamma)^2}]$ is identically zero and that $V_0^{(\gamma)} + W_0 = 1/(2l)$, then we can achieve the target wealth, $1/(2l)$, almost surely.

Theorem 2 also implies that we obtain the following control problem for the complete information problem:

$$\max_{\gamma\in\Gamma}\left\{\frac{1}{4l} - lA_T^{(\gamma)}\right\} \quad \text{or equivalently,} \quad \min_{\gamma\in\Gamma} A_T^{(\gamma)}. \tag{15}$$

In general, one of the main difficulties with solving complete information problems is that of finding the GKW decomposition in (13). In particular, we need to compute $V_0^{(\gamma)} = \widehat{\mathbb{E}}[H_T^{(\gamma)}]$ and $\delta_t^{(\gamma)}$ in order to solve (15). Finding $\widehat{\mathbb{E}}[H_T^{(\gamma)} \mid \mathcal{F}_t]$ should be sufficient as we can then use Itô's lemma to find $V_0^{(\gamma)}$ and $\delta_t^{(\gamma)}$. If we cannot compute $\widehat{\mathbb{E}}[H_T^{(\gamma)} \mid \mathcal{F}_t]$ directly, however, then we might still be able to compute it if the problem is Markovian (e.g., Pham et al. [28]) by representing $V_t = \widehat{\mathbb{E}}[H_T^{(\gamma)} \mid \mathcal{F}_t]$ as a solution to a particular partial differential equation. For problems involving generic *stochastic* operating policies, γ_t, finding an analytic expression for the GKW decomposition of $H_T^{(\gamma)}$ appears to be very difficult. Appendix B details how the optimization problem in (15) can be solved numerically in such circumstances. Nor is it clear that Theorems 1 and 2 and the GKW approach then provide a computational advantage over the standard Hamilton-Jacobi-Bellman approach to the problem. On the other hand, Theorems 1 and 2 and the GKW approach are useful when the operating policy, γ, is a scalar or vector of scalars. In this event, it is often easy to find the GKW decomposition of $H_T^{(\gamma)}$ which may then be used to solve for the optimal γ^*. The examples we consider in §§3 and 4 fall under this category.

Before proceeding further, we will briefly discuss the relation of the II solution to the CI solution. It is natural to conjecture that the II solution might be found from the CI solution by simply projecting the latter onto an appropriate subspace. Föllmer and Schweizer [14] show that this is indeed true in the context of *local* risk minimization problems and for certain types of market incompleteness. However, for the mean-variance problems considered here, we do not know of such a result.

2.4. The value of information. In an operations context, it may be the case that the corporation only has access to imperfect information, but by paying a cost, K, it might be able to obtain access to all relevant information. The cost K might be the cost associated with monitoring the operations and quality of a supplier, for example, or it might be the cost associated with doing a market survey to understand customer preferences and their demand functions. In the proposition below, we determine whether or not a corporation should pay this fixed cost when there is no flexibility in the operating strategy.

Proposition 1. *Suppose that the stock price process, X_t, has a deterministic mean-variance tradeoff. Then, for a fixed control, γ, and payoff, $H_T^{(\gamma)}$, with decomposition (13), the corporation would be better off paying a cost, K, to obtain perfect information if and only if*

$$\mathbb{E}[\pi_{\mathrm{F},T}^2]\mathbb{E}\big[\big(H_T^{(\gamma)}-\mathbb{E}\big[H_T^{(\gamma)}\mid \mathcal{F}_T^{\mathrm{F}}\big]\big)^2\big]-\int_0^T \mathbb{E}[\pi_{\mathrm{F},t}^2]\mathbb{E}\big[\delta_t^{(\gamma)^2}\big]\,\mathrm{d}t \ge K\left(\frac{1}{l}+K-2\big(V_0^{(\gamma)}+W_0\big)\right). \tag{16}$$

The proof follows directly from (10) and (15) together with the identities $\mathbb{E}[\pi_{\mathrm{F},t}^2]=\exp(\int_0^t \eta_s^2\,\mathrm{d}s)$ and $\mathbb{E}[\pi_{\mathrm{F},T}H_T^{(\gamma)}]=V_0^{(\gamma)}$. Interpreting $V_0^{(\gamma)}$ as the time zero value of $H_T^{(\gamma)}$, we see that assuming $(V_0^{(\gamma)}+W_0)<1/(2l)$ is equivalent to making a particular type of nonsatiation assumption, and that under this assumption, we see that the right-hand side in (16) is nonnegative for K. In this case, we see that if $T\downarrow 0$, then the condition in (16) is violated, reflecting the fact that extra information is of little value if there is not sufficient time available to use it. Similarly, and as we would expect, if $H_T^{(\gamma)}$ is $\mathcal{F}_T^{\mathrm{F}}$ measurable, then the left-hand side in (16) is 0 and there is no point in paying a positive cost, K, for information that has no value, assuming again that $V_0^{(\gamma)}+W_0<1/(2l)$.

In deciding whether or not to purchase access to information in practice, we would ideally like to compare the expected utility of the CI model evaluated at γ_{CI}^* with the expected utility of the II model evaluated at γ_{II}^*. This can certainly be done on a case-by-case basis when these quantities are computable. On the other hand, Proposition 1 compares the two models evaluated at the same policy, γ. It is clear, however, that if $\gamma=\gamma_{II}^*$ and the condition in (16) is satisfied, then it will also be worthwhile to pay K for the extra information when there is freedom to choose the operating policy.

3. The newsboy problem. The first example we consider is the so-called newsboy problem. At time $t=0$, a newsboy selects the number, I, of newspapers that he will order to satisfy a stochastic demand, D_T, that is revealed at time $T>0$. The net profit that the newsboy will then collect at time T is

$$H_T^{(I)}(D_T):=R\min\{D_T,I\}+r(I-D_T)^+-b(D_T-I)^+-pI,$$

where R and p are the unit retail and purchasing prices of a newspaper, respectively, r is the salvage value of each unsold unit, and b is the per unit penalty cost of unsatisfied demand.

To avoid trivial solutions, we assume that $R>p>r$. It is straightforward to show that $H_T^{(I)}(D_T)$ may also be expressed as

$$H_T^{(I)}(D_T)=(R-r)D_T+(r-p)I-(R+b-r)(D_T-I)^+. \tag{17}$$

We assume that the demand, D_T, is a function of the time T value of a financial index, X_T, and some other noise, B_{2T}, where B_2 is a Brownian motion independent of X. For expositional purposes, we will consider a simple demand function that is commonly used in the operations literature. In particular, we assume

$$D_T(X_T,B_{2,T}):=\zeta+\beta\ln(X_T)+\alpha B_{2T},$$

where $\zeta>0$, β, and α are fixed parameters.

We will assume that the index, X_t, is a geometric Brownian motion so that $X_t=X_0\exp((\mu-\frac{1}{2}\sigma^2)t+\sigma B_{1t})$. Under this assumption, D_T is normally distributed with mean $\mu_{D_T}:=\tilde{\zeta}+\beta(\mu-\frac{1}{2}\sigma^2)T$ and variance $\sigma_{D_T}^2:=(\beta^2\sigma^2+\alpha^2)T$, where $\tilde{\zeta}:=\zeta+\beta\ln(X_0)$.

Despite its simplicity, the newsboy problem is a rich modelling device that has been studied extensively in the operations literature (see the recent surveys by Silver et al. [32] and Khouja [20] for details). This is due to the wide range of practical applications that share this simple structure. These applications include, for example, retailers in the apparel industry who need to determine stock levels, manufacturers and services who need to determine operating capacity, or hotels and airlines who need to manage booking and overbooking policies. In the standard version of the problem, the newsboy is a risk-neutral agent that maximizes the expected value of $H_T^{(I)}(D_T)$ in (17). In this case, it is well known that the optimal solution, I^*, solves the fractile equation

$$\Phi_{D_T}(I^*)=\frac{R+b-p}{R+b-r},$$

where $\Phi_{D_T}(\cdot)$ is the cumulative probability distribution of D_T.

In this example, we will assume that the newsboy is risk averse and, in particular, that he is a mean-variance optimizer.

The risk-averse newsboy has received some attention in the operations literature. Eeckhoudt et al. [13] discuss the solution for a general concave utility function and show that the risk-averse newsboy orders less than his risk-neutral counterpart. The particular case of a mean-variance utility function is considered by Chen and Federgruen [4]. Similarly to Eeckhoudt et al. [13], Chen and Federgruen show that when the newsboy's terminal wealth is based on net profits, as in (17) above, then the mean-variance optimal ordering level is always smaller than the risk-neutral solution. On the other hand, if the newsboy's terminal wealth is based on costs rather than profits, then the mean-variance optimal solution can, depending on the demand distribution, be larger or smaller than the risk-neutral solution. Lau [21] studies the newsboy problem using a mean-standard deviation formulation as well as a performance criterion that maximizes the probability of achieving a given terminal wealth.

A few papers have considered the possibility of using financial markets to hedge the newsboy's risk exposure. Anvari [1] studies the newsboy problem using the capital asset pricing model (CAPM) while Chung [5] provides an alternative derivation of Anvari's results. In this formulation, the newsboy borrows capital at time $t=0$ to invest in two independent projects, namely the newspaper operation and the financial market. The investment in the financial market generates random returns at time T that are assumed to be correlated with the newspaper demand. Under the assumption that the expected return of this portfolio can be modelled using the one-period CAPM, Anvari [1] computes the optimal inventory strategy in the case of a normally distributed newspaper demand. Depending on the sign of the covariance between the newspaper demand and the financial market returns, the optimal inventory level can be larger or smaller than the traditional risk-neutral solution. Gaur and Seshadri [15] also consider a single-period model where the newspaper demand is modelled as a linear function of the price of a financial asset, and where hedging strategies are restricted to combinations of long positions on a European call option and short-sale positions on the underlying financial asset.

The mean-variance analysis of the newsboy problem that we consider here extends the current literature in two directions. First, we allow the newsboy to trade continuously in the financial market in the interval $[0, T]$ in order to hedge the operational risk. The only restriction we place on the hedging strategy is the natural restriction that it be self-financing. This contrasts with the static single-period models that have been studied to date. Second, we solve the problem for two different filtrations that model possible evolutions of the newsboy's information.

We now solve the CI and II newsboy problems and, for comparison purposes, we also solve two alternative versions of the problem. In the first alternative, the *static-financial-hedging* model, the newsboy uses a simple "buy-and-hold" trading strategy so that he only trades once in the financial markets, at $t=0$ (e.g., as in Anvari's [1] model). In the second alternative, the *no-financial-hedging* model, there is no possibility of trading in the financial markets. We note that this no-hedging model is consistent with most of the operations literature.

3.1. Complete information solution. Since the newsboy has complete information, the optimal ordering strategy is obtained by solving (15). That is, we solve

$$\min_I \left\{ \left(\frac{1}{2l} - V_0^{(I)}\right)^2 + \int_0^T \exp(\eta^2 t)\mathbb{E}\big[\delta_t^{(I)^2}\big]\,\mathrm{d}t \right\}, \tag{18}$$

where as before, $\eta = \mu/\sigma$, $V_0^{(I)} = \widehat{\mathbb{E}}[H_T^{(I)}]$, and $\delta_t^{(I)}$ is defined by (13). $V_0^{(I)}$ and $\delta_t^{(I)}$ are given by the following theorem which we prove in Appendix C.

Theorem 3. (a) *Let the process $H_t^{(I)}$ be defined by replacing D_T with D_t in the definition of $H_T^{(I)}$. Then, $V_t^{(I)} = \widehat{\mathbb{E}}[H_T^{(I)} \mid \mathcal{F}_t]$ satisfies*

$$V_t^{(I)} = H_t^{(I)} - \frac{1}{2}\beta\sigma^2 \int_t^T [R-r-(R+b-r)(1-\widehat{\Phi}_{D_s|\mathcal{F}_t}(I))]\,\mathrm{d}s - \frac{1}{2}(R+b-r)(\beta^2\sigma^2+\alpha^2)\int_t^T \hat{\phi}_{D_s|\mathcal{F}_t}(I)\,\mathrm{d}s, \tag{19}$$

for all $t \in [0, T]$ almost surely, where $\hat{\phi}_{D_s|\mathcal{F}_t}(\cdot)$ is the density function associated with the CDF, $\widehat{\Phi}_{D_s|\mathcal{F}_t}(\cdot)$, of D_s given $\mathcal{F}_t$.

(b) *In addition, $\delta_t^{(I)}$ is defined (up to a set of $\mathrm{d}\widehat{\mathbb{P}} \times \mathrm{d}t$ measure 0) as*

$$\delta_t^{(I)} = \alpha\left[(R-r) - (R+b-r)\left(\mathbf{1}(D_t \ge I) + \frac{1}{2}\int_t^T \left(\frac{I-D_t}{s-t} - \frac{1}{2}\beta\sigma^2\right)\hat{\phi}_{D_s|\mathcal{F}_t}(I)\,\mathrm{d}s\right)\right]. \tag{20}$$

We can solve for the optimal ordering quantity, I^*, by first substituting for $V_0^{(I)}$ and $\mathbb{E}[\delta_t^{(I)^2}]$ in (18), and then solving numerically for the optimal I. Results are presented in §3.5.

3.2. Incomplete information solution. When the newsboy has incomplete information, the optimization problem is given by (10). Since X_t is a geometric Brownian motion, we have $\mathbb{E}[\pi^2_{\mathrm{F},T}] = \exp(\eta^2 T)$, and so (10) is equivalent to

$$\min_I \left\{ \left(\frac{1}{2l} - V_0^{(I)}\right)^2 + \exp(\eta^2 T)\mathbb{E}\left[\left(H_T^{(I)} - \mathbb{E}\left[H_T^{(I)} \mid \mathcal{F}_T^{\mathrm{F}}\right]\right)^2\right]\right\}.$$

In §3.1 we saw how to compute $V_0^{(I)}$, so now we will focus on the computation of $\mathbb{E}[(H_T^{(I)} - \mathbb{E}[H_T^{(I)} \mid \mathcal{F}_T^{\mathrm{F}}])^2]$. The following result, whose proof is omitted, will be useful for computations in our setting.

Lemma 3.1. *Let Z be a normally distributed random variable with mean μ_Z, variance σ_Z^2, and CDF $\Phi_Z(z)$. Then,*

$$\mathbb{E}[(Z-I)^+] = \frac{\sigma_Z}{\sqrt{2\pi}} \exp\left(-\frac{(\mu_Z - I)^2}{2\sigma_Z^2}\right) + (\mu_Z - I)(1 - \Phi_Z(I)).$$

We first observe that

$$H_T^{(I)} - \mathbb{E}\left[H_T^{(I)} \mid \mathcal{F}_T^{\mathrm{F}}\right] = \alpha(R-r)B_{2T} - (R+b-r)\left((D_T - I)^+ - \mathbb{E}\left[(D_T - I)^+ \mid \mathcal{F}_T^{\mathrm{F}}\right]\right).$$

Then, conditional on $\mathcal{F}_T^{\mathrm{F}}$, D_T is normally distributed with mean $\zeta + \beta\ln(X_T)$ and variance $\alpha^2 T$. Letting $\Phi_{D_T \mid \mathcal{F}_T^{\mathrm{F}}}(\cdot)$ be the corresponding CDF, we then have by Lemma 3.1

$$\begin{aligned} G(I, \ln(X_T)) &:= \mathbb{E}\left[(D_T - I)^+ \mid \mathcal{F}_T^{\mathrm{F}}\right] \\ &= \frac{\alpha^2 T}{\sqrt{2\pi}} \exp\left(-\frac{(\zeta + \beta\ln(X_T) - I)^2}{2\alpha^2 T}\right) + (\zeta + \beta\ln(X_T) - I)\left(1 - \Phi_{D_T\mid\mathcal{F}_T^{\mathrm{F}}}(I)\right). \end{aligned}$$

Combining this result and the definition of D_T, we have that

$$H_T^{(I)} - \mathbb{E}\left[H_T^{(I)} \mid \mathcal{F}_T^{\mathrm{F}}\right] = \alpha(R-r)B_{2T} - (R+b-r)((\zeta + \beta\ln(X_T) + \alpha B_{2T} - I)^+ - G(I, \ln(X_T))).$$

Finally, since B_{2T} and $\ln(X_T)$ are independent normally distributed random variables, we can easily compute $\mathbb{E}[(H_T^{(I)} - \mathbb{E}[H_T^{(I)} \mid \mathcal{F}_T^{\mathrm{F}}])^2]$ by numerical integration.

3.3. Static-financial-hedging solution. Suppose now that the newsboy is able to trade in the financial market only at time $t = 0$. Under this "buy-and-hold" constraint, the trading gain takes the simple form $G_T = \theta(X_T - X_0)$, where θ represents the number of shares of the risky asset purchased at time $t = 0$. In this setting, the newsboy strategy is characterized by the pair (I^*, θ^*) that solves

$$\min_{\theta, I} \mathbb{E}\left[\left(\frac{1}{2l} - \theta(X_T - X_0) - H_T^{(I)}\right)^2\right]. \tag{21}$$

Note that we have implicitly assumed in (21) that unlimited borrowing is available. There is no inconsistency here, however, as the same assumption is also made implicitly when we admit self-financing trading strategies in the complete and incomplete information problems.

To solve (21), we first compute $\theta_*^{(I)} := \arg\max_\theta\{\phi^{(I)}(\theta)\}$ where $\phi^{(I)}(\cdot)$ is defined consistently with (21). From the concavity of the objective on θ, it follows that $\theta_*^{(I)}$ is the unique solution to the first-order optimality condition, that is,

$$\begin{aligned} \theta_*^{(I)} &= \mathbb{E}\left[\left(\frac{1}{2l} - H_T^{(I)}\right)(X_T - X_0)\right](\mathbb{E}[(X_T - X_0)^2])^{-1} \\ &= \frac{X_0(2l)^{-1}(\exp(\mu T) - 1) - \mathbb{E}\left[X_T H_T^{(I)}\right] + X_0\mathbb{E}\left[H_T^{(I)}\right]}{X_0^2(\exp((2\mu + \sigma^2)T) - 2\exp(\mu T) + 1)}, \end{aligned}$$

where the second equality uses the fact that X_t is a (μ, σ)-geometric Brownian motion. Substituting $\theta_*^{(I)}$ into the optimization problem (21), the optimal inventory policy I^* can be found by solving

$$\min_I \mathbb{E}\left[\left(\frac{1}{2l} - H_T^{(I)}\right)^2\right] - \frac{\left(X_0(2l)^{-1}(\exp(\mu T) - 1) - \mathbb{E}\left[X_T H_T^{(I)}\right] + X_0\mathbb{E}\left[H_T^{(I)}\right]\right)^2}{X_0^2(\exp((2\mu + \sigma^2)T) - 2\exp(\mu T) + 1)}. \tag{22}$$

In order to solve this optimization problem, we need to compute first $\mathbb{E}[H_T^{(I)}]$, $\mathbb{E}[H_T^{(I)^2}]$, and $\mathbb{E}[X_T H_T^{(I)}]$. From (17) and Lemma 3.1, it is straightforward to show that

$$\begin{aligned}\mathbb{E}\big[H_T^{(I)}\big] &= (R-r)\mu_{D_T} + (r-p)I - (R+b-r)\,\mathbb{E}[(D_T-I)^+]\\ &= (R-r)\mu_{D_T} + (r-p)I - (R+b-r)\bigg(\frac{\sigma_{D_T}}{\sqrt{2\pi}}\exp\bigg(-\frac{(\mu_{D_T}-I)^2}{2\sigma_{D_T}^2}\bigg) + (\mu_{D_T}-I)(1-\Phi_{D_T}(I))\bigg),\end{aligned}$$

where $\Phi_{D_T}(\cdot)$ is the CDF of D_T. We can also write $H_T^{(I)^2}$ as

$$\begin{aligned}H_T^{(I)^2} &= (R-r)^2 D_T^2 + (r-p)^2 I^2 + 2(R-r)(r-p)ID_T\\ &\quad + (R+b-r)(D_T-I)^+((b+r-R)D_T - (b+R+r-2p)I).\end{aligned}$$

Since $I \geq 0$, we can show

$$\mathbb{E}[D_T(D_T-I)^+] = \int_0^\infty \bigg[1-\Phi_{D_T}\bigg(\frac{I+\sqrt{I^2+4x}}{2}\bigg)\bigg]\,\mathrm{d}x,$$

so that by Lemma 3.1

$$\begin{aligned}\mathbb{E}\big[H_T^{(I)^2}\big] &= (R-r)^2\big(\mu_{D_T}^2+\sigma_{D_T}^2\big) + (r-p)^2 I^2 + 2(R-r)(r-p)I\mu_{D_T}\\ &\quad + (R+b-r)(b+r-R)\int_0^\infty\bigg[1-\Phi_{D_T}\bigg(\frac{I+\sqrt{I^2+4x}}{2}\bigg)\bigg]\,\mathrm{d}x\\ &\quad - (R+b-r)(b+R+r-2p)I\bigg(\frac{\sigma_{D_T}}{\sqrt{2\pi}}\exp\bigg(-\frac{(\mu_{D_T}-I)^2}{2\sigma_{D_T}^2}\bigg) + (\mu_{D_T}-I)(1-\Phi_{D_T}(I))\bigg).\end{aligned}$$

Finally, it remains to compute the covariance term

$$\mathbb{E}\big[X_T H_T^{(I)}\big] = (R-r)\mathbb{E}[X_T D_T] + (r-p)I\mathbb{E}[X_T] - (R+b-r)\mathbb{E}[X_T(D_T-I)^+].$$

Using the definition of $D_T = \zeta + \beta\ln(X_T) + \alpha B_{2T}$, it follows that

$$\begin{aligned}\mathbb{E}[X_T D_T] &= \mathbb{E}[X_T(\zeta+\alpha B_{2T})] + \beta\mathbb{E}[X_T\ln(X_T)]\\ &= \zeta\mathbb{E}[X_T] + \beta\mathbb{E}[X_T\ln(X_T)]\\ &= \zeta X_0\exp(\mu T) + \beta\bigg(X_0\ln(X_0) + X_0\bigg(\mu+\frac{1}{2}\sigma^2\bigg)T\exp(\mu T)\bigg).\end{aligned}$$

On the other hand, Lemma 3.1 and some straightforward manipulations imply

$$\begin{aligned}\mathbb{E}[X_T(D_T-I)^+] &= \mathbb{E}[X_T(\zeta+\beta\ln(X_T)+\alpha B_{2T}-I)^+]\\ &= X_0\exp(\mu T)\mathbb{E}[(D_T+\beta\sigma^2 T-I)^+]\\ &= X_0\exp(\mu T)\frac{\sigma_{D_T}}{\sqrt{2\pi}}\exp\bigg(-\frac{(\mu_{D_T}+\beta\sigma^2T-I)^2}{2\sigma_{D_T}^2}\bigg)\\ &\quad + X_0\exp(\mu T)(\mu_{D_T}+\beta\sigma^2T-I)(1-\Phi_{D_T}(I-\beta\sigma^2T)).\end{aligned}$$

Again, we can now solve the optimization problem (22) numerically and compute the optimal ordering level, I^*, and static-hedging strategy $\theta^* = \theta_*^{(I^*)}$.

3.4. No-financial-hedging solution. When no financial hedging is allowed, the problem is

$$\min_I \mathbb{E}\bigg[\bigg(\frac{1}{2l} - H_T^{(I)}\bigg)^2\bigg] \equiv \max_I\bigg\{\frac{1}{l}\mathbb{E}\big[H_T^{(I)}\big] - \mathbb{E}\big[H_T^{(I)^2}\big]\bigg\}.$$

The solution to this problem is a a special case of the analysis of the static-hedging model of the previous section. Again, we can now solve numerically for the optimal ordering level, I^*.

3.5. Numerical results. To conclude the newsboy example, we will present some computational results that compare the four alternative modes of operation. The parameter values that we use are:

$$R=\$1{,}000,\quad r=\$200,\quad b=\$300,\quad p=\$500,\quad T=2 \text{ years},\quad l=10^{-6},\quad \text{and}$$
$$\zeta=200,\quad \alpha=100\quad (\text{Table 1 only}),\quad \beta=100,\quad X_0=1.0,\quad \mu=0.1\quad (\text{Table 2 only}),\quad \sigma=0.2.$$

In Table 1, we compare the root mean squared error (RMSE) and the optimal ordering level, I^*, as a function of the market price of risk, $\eta=\mu/\sigma$. The RMSE is defined by

$$\text{RMSE} := \mathbb{E}\left[\left(\frac{1}{2l}-H_T^{(I^*)}-G_T(\theta^*)\right)^2\right]^{1/2},$$

with the understanding that $G_T\equiv 0$ when there is no hedging allowed. As we can see, the RMSE decreases with η for all four modes of operation. As we expect, the CI solution has the smallest RMSE while the no-hedging solution has the largest. For small values of η, the four solutions are quite similar, but as η increases, the CI and II solutions outperform the static-hedging and no-hedging solutions by a considerable margin. An important observation that follows from the results in Table 1, specifically comparing the static-hedging and II solutions, is that to fully take advantage of the financial markets we need to use dynamic hedging strategies. Of course, if the static hedge was allowed to buy and sell derivatives at $t=0$ in addition to the underlying security, then its performance could improve considerably, e.g., Haugh and Lo [17].

In terms of the optimal ordering level, I^*, the no-hedging and static-hedging solutions order more than the CI and II solutions, and the amount increases with η. In contrast, the ordering level for the CI and II solutions decrease with η.

Table 2 displays the RMSE and I^* as a function of α. Recall that α is a measure of the dependency between the operational profit, $H_T^{(I^*)}$, and the nonfinancial noise, B_{2t}. In general, both the RMSE and I^* increase with α. For small values of α, the CI and II solutions are similar, but as α increases, the advantage of the CI model becomes more pronounced as we would expect. It is interesting to observe that in this setting, the complete information newsboy orders more than his incomplete information counterpart.

We would not draw any hard conclusions from these results, however, since no attempt was made to fit the parameters to real data. Indeed, had we done this, we would often expect the ordering quantity, I^*, to be higher for the CI and II solutions than for the no-hedging solution. As mentioned earlier, we would also expect this to occur if we worked with an EMM, $\mathbb{Q}$, rather than the physical measure, $\mathbb{P}$, thereby removing any profit motive for trading in the financial markets.

4. A production example. In this example, we consider the case of a simple production facility that manufactures a single product between times 0 and T for retail at time T. The purpose of this example is to demonstrate that the class of problems which are amenable to our analysis is not restricted to the class of newsboy problems. As a result, we do not focus here on the strengths or weaknesses of the model, but, instead, we concentrate on solving the two associated hedging problems.

TABLE 1. RMSE and optimal ordering level, I^*, as a function of the market price of risk, $\eta=\mu/\sigma$.

	RMSE				Optimal ordering level I^*			
η	No hedging	Static	Incomplete	Complete	No hedging	Static	Incomplete	Complete
0.0	468,244	468,037	461,523	458,068	288	289	266	274
0.2	464,468	452,965	443,593	440,376	296	291	265	273
0.4	460,696	418,987	394,555	391,414	305	295	260	271
0.6	456,926	380,974	325,142	321,994	314	300	252	265
0.8	453,160	346,430	248,909	245,818	322	307	236	253
1.0	449,397	317,417	177,911	175,123	331	315	210	229
1.2	445,637	293,706	120,179	117,570	339	323	179	190
1.4	441,881	274,435	78,519	75,327	348	333	147	138
1.6	438,129	258,705	53,423	47,767	356	343	123	91
1.8	434,380	245,760	41,927	33,172	365	353	114	60
2.0	430,635	235,002	38,205	27,651	373	364	114	45

Table 2. RMSE and optimal ordering level, I^*, as a function of α.

	RMSE				Optimal ordering level I^*			
α	No hedging	Static	Incomplete	Complete	No hedging	Static	Incomplete	Complete
50	424,719	367,922	336,563	335,819	267	254	235	238
70	437,673	380,016	346,045	344,522	284	271	245	250
90	451,572	393,034	356,245	353,696	301	289	253	262
110	466,220	406,833	367,000	363,221	318	305	260	274
130	481,505	421,310	378,212	373,027	334	322	266	285
150	497,345	436,386	389,808	383,077	350	338	272	295
170	513,677	451,992	401,736	393,342	366	353	276	304
190	530,447	468,072	413,951	403,786	381	368	279	314
210	547,610	484,573	426,414	414,400	396	383	281	323
230	565,159	501,451	439,101	425,166	411	397	284	332
250	582,963	518,667	451,979	436,055	426	411	285	339

As before, we assume that there are two independent Brownian motions, $B_{1,t}$ and $B_{2,t}$, and a stock price process that is a function of $B_{1,t}$ only. In particular, we will assume that the stock price, X_t, again follows a geometric Brownian motion so that

$$dX_t = \mu X_t dt + \sigma X_t dB_{1,t}.$$

The amount of inventory, I_t, held by the company at time t satisfies

$$dI_t = (aX_t + b)dt. \tag{23}$$

Again, and without loss of generality, we assume that the risk-free interest rate is zero. We also assume that $I_0 = W_0 = 0$. The scalars a and b in (23) are controls that are chosen by the manufacturer at date $t = 0$. In principle, we could formulate the problem where the drift of I_t was controlled stochastically, but it would then be necessary to solve this problem numerically (see Appendix B).

At time T, the I_T units of the product are sold at a per unit price of $R(X_T, B_{2,T}, I_T)$ where we assume

$$R(X_T, B_{2,T}, I_T) = R_0 + \alpha B_{2,T} + \beta X_T - I_T. \tag{24}$$

The per unit production cost is constant and equal to C_0. This leads to a time T profit

$$H_T^{(a,b)} = \left[A + \alpha B_{2,T} + \beta X_T - I_T^{(a,b)}\right] I_T^{(a,b)}, \tag{25}$$

where $A := R_0 - C_0$.

Remarks. (i) In this example, we have assumed that all of the time T inventory, I_T, is sold. This assumption is commonly made in the operations literature and the random sale price, $R(X_T, B_{2,T}, I_T)$, then represents the *average* or *clearance* price at which the inventory is sold. This assumption reflects the situation where all excess inventory is ultimately sold, either through discounts or for its salvage value. Note that we have modelled the (inverse) market demand as a linear function of I where the intercept, $A + \alpha B_{2,T} + \beta X_T$, is a random variable that depends on the financial market as well as on the nonfinancial noise, $B_{2,T}$.

(ii) There are two different types of financial hedging that take place in this example. The first type is the hedging that takes place when the manufacturer trades in the financial markets, as we have discussed throughout the paper. The second type is the hedging that implicitly takes place when the manufacturer adjusts the rate of production in such a way that it depends on the state of the financial market. This rate of production depends on the control, a, as may be seen from (23). In practice, most manufacturers base their operating decisions in part on the prevailing economic conditions and we have captured this effect in (23).

We now proceed to solve the hedging and operations problems for the CI and II models. In order to compare their solutions, we again compute the solution for the case where there is no possibility of trading in the financial markets. We also mention at this point that the computations required for this example are straightforward and require little more than evaluating a large number of integrals. For the sake of brevity, we omit most of the details here.

4.1. Complete information solution. The first step in finding the solution to the complete information model is to determine the intrinsic value process, $V_t = \widehat{\mathbb{E}}[H_T^{(a,b)} \mid \mathcal{F}_t]$, which is straightforward given the representation (25). It is given by

$$V_t^{(a,b)} = H_t^{(a,b)} + (T-t)\left[(aX_t+b)(A+\alpha B_{2,t}-2I_t)+\beta bX_t+\frac{2a^2X_t^2}{\sigma^2}\right] - (T-t)^2(b^2+2abX_t)+X_t^2\left(\frac{e^{\sigma^2(T-t)}-1}{\sigma^2}\right)\left(\beta a-\frac{2a^2}{\sigma^2}\right). \tag{26}$$

The next step is to determine the GKW decomposition of V_t under $\widehat{\mathbb{P}}$ with respect to X. In particular, we can apply Itô's lemma to our expression for $V_t^{(a,b)}$ in (26) to write

$$V_t^{(a,b)} = V_0^{(a,b)} + \int_0^t \vartheta_s^{(a,b)}\,\mathrm{d}X_s + \int_0^t \delta_s^{(a,b)}\,\mathrm{d}B_{2,s}. \tag{27}$$

The only terms that we need from (27) to solve the manufacturer's hedging problem are $\delta_t^{(a,b)}$ and $V_0^{(a,b)}$. They are easily found and are given by

$$\delta_t^{(a,b)} = \alpha\big[I_t^{(a,b)}+(T-t)(aX_t+b)\big] = \alpha\left[a\int_0^T X_{s\wedge t}\,\mathrm{d}s+bT\right] \tag{28}$$

$$V_0^{(a,b)} = T\left[(aX_0+b)A+\beta bX_0+\frac{2a^2X_0^2}{\sigma^2}\right]-T^2(b^2+2abX_0)+X_0^2\left(\frac{e^{\sigma^2T}-1}{\sigma^2}\right)\left(\beta a-\frac{2a^2}{\sigma^2}\right).$$

At this point, we now substitute for V_0 and δ_s in (14) and solve the optimization program in (15). That is, we minimize $A_T^{(a,b)}$ with respect to (a,b).

We now show that this optimization problem is well posed. Since $\delta_t^{(a,b)^2}$ and $V_0^{(a,b)}$ are easily seen to be quadratic functions of a and b, it follows that minimizing $A_T^{(a,b)}$ with respect to γ reduces to optimizing a polynomial of degree four. Now observe that

$$\mathbb{E}\big[\delta_t^{(a,b)^2}\big] = \alpha^2[a^2\mathrm{Var}(Y_t)+(a\mathbb{E}[Y_t]+bT)^2] \quad \text{with } Y_t \triangleq \int_0^T X_{s\wedge t}\,\mathrm{d}s. \tag{29}$$

Since the first summand in (29) goes to infinity with $|a|$, we can clearly limit a to a compact set in R. Given this restriction on a, a similar argument now implies that we can also restrict b to a compact set. Since our problem can therefore be stated as minimizing a continuous function over a compact set, a solution is guaranteed and our problem is well posed. Similar arguments may be used to guarantee that the incomplete information and no-hedging problems are also well posed. In each case, we use standard numerical software to find the optimal values (a^*, b^*).

4.2. Incomplete information solution. The II hedging problem is to solve

$$V_T^{(a,b)} = \min_{G_T}\mathbb{E}\left[\left(H_T^{(a,b)}+G_T-\frac{1}{2l}\right)^2\right] \quad \text{subject to } \mathbb{E}[\pi_{\mathrm{F},T}G_T]=0 \tag{30}$$

and the $\mathcal{F}_T^{\mathrm{F}}$ measurability of G_T. As before, $\pi_{\mathrm{F},T}$ is the SPD that we use for pricing $\mathcal{F}_T^{\mathrm{F}}$-measurable contingent claims, and G_T is the gain from trading in the financial market. Following the steps of §2.2, we find that

$$G_T^* = \frac{1}{2l}-\mathbb{E}\big[H_T^{(a,b)}\mid\mathcal{F}_T^{\mathrm{F}}\big]+\exp(-\eta^2T)\left[\mathbb{E}\big[\pi_{\mathrm{F},T}\mathbb{E}\big[H_T^{(a,b)}\mid\mathcal{F}_T^{\mathrm{F}}\big]\big]-\frac{1}{2l}\right]\pi_{\mathrm{F},T}. \tag{31}$$

We then substitute for G_T^* in (30) to get the following optimization problem

$$\min_{a,b}\left\{\left(\frac{1}{2l}-V_0^{(a,b)}\right)^2+\exp(\eta^2T)\mathbb{E}\big[\big(H_T^{(a,b)}-\mathbb{E}\big[H_T^{(a,b)}\mid\mathcal{F}_T^{\mathrm{F}}\big]\big)^2\big]\right\}. \tag{32}$$

Furthermore, by noticing that $\mathbb{E}[I_t\mid\mathcal{F}_T^{\mathrm{F}}]=I_t$ for all $t\in[0,T]$, it follows that

$$\mathbb{E}\big[H_T^{(a,b)}\mid\mathcal{F}_T^{\mathrm{F}}\big] = AI_T+\beta X_TI_T-I_T^2.$$

Therefore, the expectation on the second summand in (32) becomes

$$\begin{aligned}\mathbb{E}\big[\big(H_T^{(a,b)}-\mathbb{E}\big[H_T^{(a,b)}\mid\mathcal{F}_T^{\mathrm{F}}\big]\big)^2\big] &= \alpha^2\mathbb{E}[(B_{2,T}I_T)^2]=\alpha^2T\mathbb{E}[I_T^2]\\ &= 2a^2X_0^2\alpha^2T\left(\frac{\exp((2\mu+\sigma^2)T)-1}{(\mu+\sigma^2)(2\mu+\sigma^2)}\right)\\ &\quad+2aX_0\alpha^2T\left(\frac{\exp(\mu T)-1}{\mu}\right)\left(bT-\frac{aX_0}{\mu+\sigma^2}\right)+b^2\alpha^2T^3.\end{aligned}$$

Finally, we determine the optimal values of a and b solving (32) numerically. As before with the complete information model, this is a simple optimization problem involving a polynomial of degree four.

4.3. No-financial-hedging solution. For comparison purposes, we also compute the optimal solution to the manufacturer's problem when there is no possibility of hedging directly using the financial markets. In this case, the problem is simply

$$\min_{a,\,b} \mathbb{E}\left[\left(H_T^{(a,\,b)} - \frac{1}{2l}\right)^2\right], \tag{33}$$

which is again easily solved explicitly since $H_T^{(a,\,b)}$ is a quadratic function of a and b.

4.4. Numerical results. In this section, we describe two numerical experiments that are merely intended to demonstrate the methodology at work. In particular, we do not claim that the model or the model parameters are representative problems that arise in practice. They do demonstrate, however, that it is often possible for the optimal complete and incomplete objective functions to be very close, as we claimed earlier.

The following parameter values were used for both experiments:

$$R_0 = \$1{,}000, \quad C_0 = \$500, \quad 1/(2l) = 750{,}000, \quad \beta = 10, \quad X_0 = \$100, \quad \text{and} \quad T = 1 \text{ year.}$$

Varying the market price of risk. In the first experiment, we set $\alpha = 100$ and allow the market price of risk, $\eta = \mu/\sigma$, to vary between 0 and 2. At each value of η, we solve the complete information, incomplete information, and no-hedging problems. The root-mean-squared errors are plotted against η in Figure 1. As expected, the complete information solution is superior to the incomplete information solution which, in turn, is superior to the no-hedging solution.

An interesting feature of this experiment (also present in the newsboy example) is that as η increases from 0, a nonhedging motivation is introduced for trading in the financial markets. That is, when $\eta = 0$, the only motivation for trading is to reduce variance since expected returns from trading are identically 0. However, as η increases, then expected returns are no longer 0, and the manufacturer may prefer to trade in the markets to increase his expected profit. As a result, he may simultaneously choose to produce a smaller quantity of the product. As discussed in §2, this would generally be an unattractive solution if it was to be used on a repeated basis. However, such a solution might be very appropriate on a one-off basis if, for whatever reason, it is very important to attain the target level of profits, $1/(2l)$, at time T. This, for example, might be due to the availability of an attractive investment opportunity at time T that requires a substantial initial investment, W. Due possibly to various market frictions, it may be the case that the manufacturer cannot or does not wish to raise part of W in the capital markets. In such a scenario, it may well be appropriate to trade in the financial markets between dates 0 and T in an attempt to raise the required outlay of W internally.

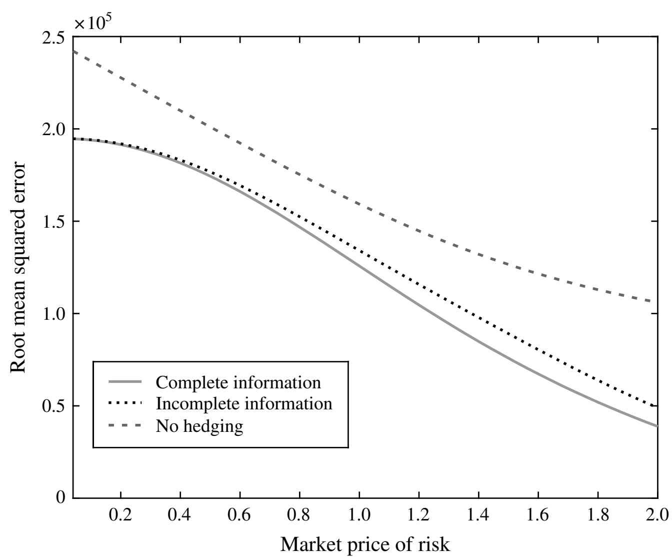

Figure 1. Varying the market price of risk.

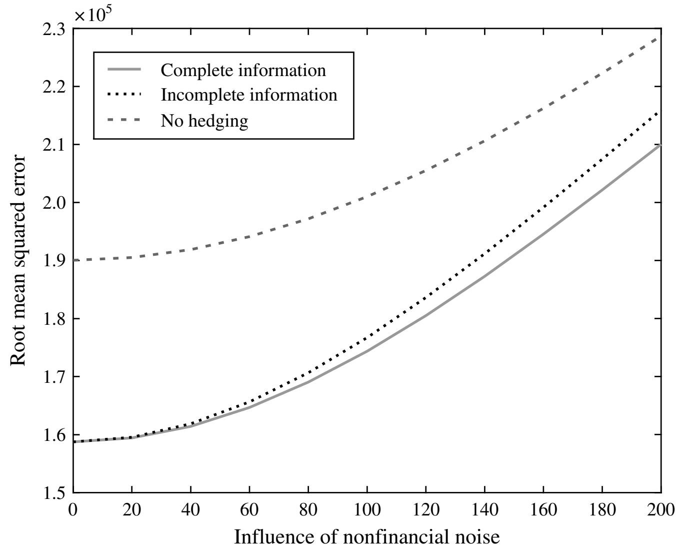

FIGURE 2. Varying the influence of nonfinancial noise.

Varying the influence of nonfinancial noise. In the second experiment, we set $\mu = 0.1$, $\sigma = 0.2$ and allow α to vary between 0 and 200. Note that in varying α, we are varying the impact of the nonfinancial noise, $B_{2,t}$ on the production process in (23). Again, we solve the complete information, incomplete information, and no-hedging problems for each value of α. The root mean squared errors are plotted against α in Figure 2 and once again, the complete information solution is superior to the incomplete information solution which, in turn, is superior to the no-hedging solution.

5. Conclusions and further research. In this paper, we have studied the problem of dynamically hedging the operating profits of a nonfinancial corporation. The central modelling insight was to view the operations and facilities of the corporation as an asset in the corporation's portfolio. Taking this view enabled us to pose the problem as one of financial hedging in incomplete markets. This problem has been studied extensively in the recent literature in mathematical finance. Using this modelling framework, we demonstrated how to solve the more general problem of simultaneously optimizing over both the operating and hedging policies of the corporation. Different informational assumptions regarding whether or not the operational state variables were observable gave rise to different solution techniques. Finally, we solved some simple but commonly encountered problems that arise in operations and supply chain management.

There are a number of possible future research directions. First, it would be interesting to generalize the results of this paper to more general price processes and utility functions. To some extent, this may already be possible using some of the recent results from mathematical finance that generalize from mean-variance hedging to hedging with more general utility functions. Similarly, there is also recent work that explores dropping the requirement that the mean-variance trade-off be deterministic in the complete information model. We expect that some of these recent developments, together with future research in mathematical finance, will enable us to solve ever more complex and realistic problems. Other hedging techniques such as *local risk minimization* (e.g., Schweizer [31]) might also prove useful in this regard.

Second, it would be interesting to apply the mean-variance hedging methodology of this paper to other problems in operations and supply chain management. This methodology is not restricted to the two sample problems that we solved in this paper; they only served to demonstrate that solving such problems is indeed feasible. Future research should examine other classes of problems. In particular, we are currently exploring the possibility of applying these techniques to problems where corporations do not act in isolation, but instead compete with, and write contracts with, one another.

Finally, it would also be interesting to solve problems where risk aversion is modelled through constraints rather than through a risk-averse utility function in the objective. Such formulations have a number of advantages (e.g., Caldentey and Haugh [3]) and would also be more appropriate in many operations contexts.

We have barely discussed in this paper the issue of corporate preferences. Unfortunately, it is not clear what an appropriate utility function for a corporation is, and exactly what types of risks should be hedged. Indeed, there does not appear to be universal agreement among economists regarding these questions. However, as others have pointed out (e.g., Duffie and Richardson [12]), using a mean-variance objective should often be a good first approximation. Moreover, as our ability to solve more complex dynamic hedging problems grows, then so too might our understanding of corporate preferences and the particular risks they should hedge.

Appendix A. Proof of Theorem 2. For notational convenience, we will suppress the dependence of the various terms on γ. Define the process $N_t := (V_t + G_t^* + W_0 - 1/(2l))^2$. Using Itô's lemma, it follows that

$$\mathrm{d}N_t = 2\left(V_t + G_t^* + W_0 - \frac{1}{2l}\right)(\mathrm{d}V_t + \mathrm{d}G_t^*) + \mathrm{d}\langle V + G^*, V + G^*\rangle_t,$$

where $\langle X, X\rangle_t$ is the quadratic variation process (e.g., Protter [29]) associated with X_t. Using the definition of V_t and G_t^*, we then obtain

$$N_t = N_0 + 2\int_0^t \left(V_s + G_s^* + W_0 - \frac{1}{2l}\right)\left(\delta_s\,\mathrm{d}B_{2,s} - \eta_s\left(V_s + G_s^* + W_0 - \frac{1}{2l}\right)\mathrm{d}B_{1,s}\right) + \int_0^t \left(\delta_t^2 - \eta_s^2\left(V_s + G_s^* + W_0 - \frac{1}{2l}\right)^2\right)\mathrm{d}s.$$

Now, taking expectations, cancelling all martingales terms, and using Fubini's theorem together with the deterministic mean-variance assumption, we obtain

$$A_t = \mathbb{E}[N_t] = \mathbb{E}[N_0] + \int_0^t (\mathbb{E}[\delta_s^2] - \eta_s^2 A_s)\,\mathrm{d}s,$$

which immediately implies the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}A_t + \eta_t^2 A_t = \mathbb{E}[\delta_t^2].$$

Finally, we use the integrating factor $\exp(\int_0^t \eta_s^2\,\mathrm{d}s)$ and the boundary condition $A_0 = (V_0(\gamma) + W_0 - 1/(2l))^2$ to obtain the desired result. □

Appendix B. In this appendix, we study the solution of the optimization problem in (15) for the special case in which the operational payoff, $H_T^{(\gamma)}$, is Markovian in the following sense: We assume there exists a twice-continuously differentiable function, $f(x, y)$, and a controlled operational process, Y_t, such that

$$\begin{aligned} H_T^{(\gamma)} &= f(X_T, Y_T), \qquad \text{where } X \text{ and } Y \text{ satisfy the SDEs} \\ \mathrm{d}X_t &= \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}B_{1,t}, \quad \text{and} \\ \mathrm{d}Y_t &= \alpha(X_t, Y_t; \gamma)\mathrm{d}t + \beta(X_t, Y_t; \gamma)\mathrm{d}B_{2,t}, \end{aligned}$$

where we assume that α, β, and γ satisfy sufficient regularity conditions that guarantee the existence and uniqueness of solutions to the SDEs and control problem. (See, for example, Øksendal [26].)

Many well-known operational problems can be modelled with such a formulation. For example, in a capacity-planning problem, we continuously adjust capacity, Y_t, in $[0, T]$ to meet a demand that is realized at time T. The profit function is $H_T^{(\gamma)} = f(X_T, Y_T)$ and it depends on available capacity and the state of the financial markets/economy, X_T. More generally, Y_t would have dimension greater than 1 with some components representing uncontrollable operational noise.

In this case, we know that there exists a tracking function $V(t, x, y)$ defined as $V(t, X_t, Y_t) := \widehat{\mathbb{E}}[H_T^{(\gamma)} \mid \mathcal{F}_t]$ which satisfies

$$\widehat{\mathcal{G}}^{(\gamma)} V(t, x, y) = 0, \qquad V(T, x, y) = f(x, y),$$

where $\widehat{\mathcal{G}}^{(\gamma)}$ is the infinitesimal generator of (X_t, Y_t) at time t under $\widehat{\mathbb{P}}$ when $\gamma_t = \gamma$. It is defined by

$$\widehat{\mathcal{G}}^{(\gamma)} h(t, x, y) \triangleq h_t(t, x, y) + \alpha(x, y; \gamma)h_y(t, x, y) + \frac{1}{2}(\beta^2(x, y; \gamma)h_{yy}(t, x, y) + x^2\sigma^2 h_{xx}(t, x, y)),$$

for any test function, $h(t, x, y) \in C^{1,2,2}$. (It is the martingale property of $V(t, X_t, Y_t)$ that implies $\widehat{\mathcal{G}}^{(\gamma)} V(t, x, y) = 0$. Note also that since X_t is a $\widehat{\mathbb{P}}$ martingale, there is no h_x term in the definition of $\widehat{\mathcal{G}}^{(\gamma)}$.)

The function V also defines the Galtchouk-Kunita-Watanabe (GKW) decomposition of $H_T^{(\gamma)}$ in equation (13) as follows:

$$H_T^{(\gamma)} = \underbrace{V(0, X_0, Y_0)}_{V_0^{(\gamma)}} + \int_0^T \underbrace{V_x(t, X_t, Y_t)}_{\vartheta_t^{(\gamma)}} \,\mathrm{d}X_t + \int_0^T \underbrace{\beta(X_t, Y_t; \gamma_t) V_y(t, X_t, Y_t)}_{\delta_t^{(\gamma)}} \,\mathrm{d}B_{2,t}. \tag{B.1}$$

Based on this decomposition and the identity $V_0^{(\gamma)} = \hat{\mathbb{E}}[f(X_T, Y_T)]$, the problem of selecting the optimal operational strategy γ reduces (see (14)) to

$$V^*(X_0, Y_0) = \inf_{\gamma\in\Gamma}\left\{\left(\frac{1}{2l} - W_0 - \hat{\mathbb{E}}_0[f(X_T, Y_T)]\right)^2 + \int_0^T \exp(\eta^2 t)\mathbb{E}_0[(\beta(X_t, Y_t; \gamma_t)V_y(t, X_t, Y_t))^2]\,\mathrm{d}t\right\} \tag{B.2}$$

$$\text{subject to} \quad \hat{\mathcal{G}}^{(\gamma)} V(t, x, y) = 0, \quad V(T, x, y) = f(x, y) \tag{B.3}$$

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}B_{1,t} \tag{B.4}$$

$$\mathrm{d}Y_t = \alpha(X_t, Y_t; \gamma_t)\mathrm{d}t + \beta(X_t, Y_t; \gamma_t)\,\mathrm{d}B_{2,t}. \tag{B.5}$$

This is a nonstandard control problem, as its objective (B.2) depends on the tracking process $V(t, x, y)$ that solves the backward PDE in (B.3). In addition, the presence of a quadratic term on $\hat{\mathbb{E}}[f(X_T, Y_T)]$ in the objective prevents us from applying directly the principle of dynamic programming, at least in its elementary form. Fortunately, we can handle this problem using the following modified version of (B.2)–(B.5):

$$V(X_0, Y_0, \psi) = \inf_{\gamma\in\Gamma}\left\{\int_0^T \exp(\eta^2 t)\mathbb{E}[(\beta(X_t, Y_t; \gamma_t)V_y(t, X_t, Y_t))^2]\,\mathrm{d}t\right\} + \left(\frac{1}{2l} - W_0 - \psi\right)^2 \tag{B.6}$$

$$\text{subject to} \quad \hat{\mathcal{G}}^{(\gamma)} V(t, x, y) = 0, \quad V(T, x, y) = f(x, y), \tag{B.7}$$

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}B_{1,t} \tag{B.8}$$

$$\mathrm{d}Y_t = \alpha(X_t, Y_t; \gamma_t)\,\mathrm{d}t + \beta(X_t, Y_t; \gamma_t)\,\mathrm{d}B_{2,t} \tag{B.9}$$

$$\hat{\mathbb{E}}[f(X_T, Y_T)] = \psi. \tag{B.10}$$

Since this modified optimization problem (B.6)–(B.10) is more constrained that the original problem (B.2)–(B.5), it follows that $V(X_0, Y_0, \psi) \geq V^*(X_0, Y_0)$ for every $\psi \in \mathbb{R}$. This inequality uses the convention $V(X_0, Y_0, \psi) = \infty$ if the modified problem is infeasible. We define $\Psi \triangleq \{\psi \in \mathbb{R}\colon\ V(X_0, Y_0, \psi) < \infty\}$ to be the corresponding feasible set.

If the original problem (B.2)–(B.5) admits an optimal control γ^*, then letting $\psi^* \triangleq \hat{\mathbb{E}}[f(X_T, Y_T(\gamma^*))]$ (where $Y_t(\gamma^*)$ is the resulting Y_t process under γ^*), it follows that $V(X_0, Y_0, \psi^*) = V^*(X_0, Y_0)$. Therefore, under the assumption that the original problem admits a solution γ^*, we can compute $V^*(X_0, Y_0)$ solving the auxiliary problem

$$V^*(X_0, Y_0) = \min_{\psi\in\Psi} V(X_0, Y_0, \psi).$$

In order to compute $V(X_0, Y_0, \psi)$, we relax constraint (B.10) to obtain the following control problem that we can solve using dynamic programming:

$$V(X_0, Y_0, \psi, \lambda) = \inf_{\gamma\in\Gamma}\left\{\int_0^T \exp(\eta^2 t)\mathbb{E}[(\beta(X_t, Y_t; \gamma_t)V_y(t, X_t, Y_t))^2]\,\mathrm{d}t + \lambda\left(\hat{\mathbb{E}}[f(X_T, Y_T)] - \psi\right)\right\} + \left(\frac{1}{2l} - W_0 - \psi\right)^2 \tag{B.11}$$

$$\text{subject to} \quad \hat{\mathcal{G}}^{(\gamma)} V(t, x, y) = 0, \quad V(T, x, y) = f(x, y), \tag{B.12}$$

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}B_{1,t} \tag{B.13}$$

$$\mathrm{d}Y_t = \alpha(X_t, Y_t; \gamma_t)\,\mathrm{d}t + \beta(X_t, Y_t; \gamma_t)\,\mathrm{d}B_{2,t}. \tag{B.14}$$

Note that the problem formulated in (B.11) to (B.14) can be solved numerically by working backwards from time T. The distinguishing feature of this particular control problem is the presence of the PDE in (B.12) that is simultaneously solved backwards in time.

Solving our original control problem then amounts to solving for the optimal pair, $(\psi^*, \lambda(\psi^*))$. This can be done by means of an iterative procedure, though, in general, this would be very time consuming. Note that there is a trade-off between solving the problem using this method, i.e., the GKW approach, and a direct approach based on the HJB equation. The latter approach needs to solve for both the trading strategy and the operating strategy, whereas the former approach only needs to solve for the operating strategy. Moreover, after substituting for the optimal trading strategy (in terms of the value function and its partial derivatives), the HJB-based PDE might be very complex and difficult to solve (even numerically). On the other hand, the GKW approach has the added complexity of needing to find $(\psi^*, \lambda(\psi^*))$ as well as numerically solving the PDE in (B.12). This would appear to be very computationally demanding and we expect that it would be unlikely to offer an advantage over the direct HJB approach. However, to make a conclusive statement regarding the merits of one approach over the other would require more problem-specific information.

If we are willing to accept approximate solutions, however, then it is possible that the GKW approach can provide a distinct advantage. For example, we can eliminate $(\psi, \lambda(\psi))$ from the problem formulation if we approximate the quadratic expression in $\widehat{\mathbb{E}}_0[f(X_T, Y_T)]$ that appears in the objective function with a linear expression in $\widehat{\mathbb{E}}_0[f(X_T, Y_T)]$. Moreover, it is possible that in some circumstances the GKW decomposition (B.1) will already be available to us or easy to compute analytically. We would then no longer need to solve numerically the PDE in (B.12), and so the GKW approach would become even more attractive.

Appendix C. We now prove Theorem 3. This is quite straightforward and the main technical difficulty arises from the nonapplicability of the standard form of Itô's lemma due to the nondifferentiability of the newsboy profit at $D_t = I$. In the proof, we will suppress the explicit dependence of the various processes on I.

Proof of Theorem 3. (a) First, recall that the process $H_t(D_t)$ was defined by replacing D_T with D_t in equation (17). The newsboy profit is not differentiable at $D_t = I$ and so the standard form of Itô's lemma does not apply. We address this problem by using a perturbation of H_t, which is C^1 everywhere and C^2 almost everywhere. Specifically, following Chung and Williams [6], we define for $\epsilon > 0$

$$H_t^\epsilon(D_t) := (R-r)D_t + (r-p)I - (R+b-r)f_\epsilon(D_t),$$

where

$$f_\epsilon(x) = \begin{cases} 0 & \text{if } x \le I-\epsilon \\ (x-I+\epsilon)^2/4\epsilon & \text{if } I-\epsilon \le x \le I+\epsilon \\ x-I & \text{if } x \ge I+\epsilon. \end{cases}$$

Note that $f_\epsilon(x)$ converges to $(x-I)^+$ pointwise as $\epsilon \downarrow 0$. Given the differentiability of $H_t^\epsilon(D_t)$ we can apply Itô's lemma to obtain

$$H_T^\epsilon(D_T) = H_t^\epsilon(D_t) + \int_t^T (R-r-(R+b-r)f_\epsilon'(D_s))\,\mathrm{d}D_s - (R+b-r)(\beta^2\sigma^2+\alpha^2)\int_t^T \frac{\mathbf{1}(|D_s - I| \le \epsilon)}{4\epsilon}\,\mathrm{d}s,$$

where $\mathbf{1}(E)$ is the indicator function of the event E, $f_\epsilon'(x)$ is the first derivative of $f_\epsilon(x)$ with respect to x, and

$$\mathrm{d}D_s = \beta\frac{\mathrm{d}X_s}{X_s} - \frac{1}{2}\beta\sigma^2\,\mathrm{d}s + \alpha\,\mathrm{d}B_{2s}.$$

Since $H_t^\epsilon(D_t)$ is $\mathcal{F}_t$ measurable and X_t and B_{2t} are martingales under $\widehat{\mathbb{P}}$, we can write the following decomposition for $V_t^\epsilon := \widehat{\mathbb{E}}[H_T^\epsilon(D_T) \mid \mathcal{F}_t]$

$$\begin{aligned} V_t^\epsilon = H_t^\epsilon - \frac{1}{2}\beta\sigma^2 \int_t^T (R-r-(R+b-r)\widehat{\mathbb{E}}[f_\epsilon'(D_s) \mid \mathcal{F}_t])\,\mathrm{d}s \\ - (R+b-r)(\beta^2\sigma^2+\alpha^2)\int_t^T \frac{1}{4\epsilon}\widehat{\mathbb{E}}[\mathbf{1}(|D_s - I| \le \epsilon) \mid \mathcal{F}_t]\,\mathrm{d}s. \end{aligned} \tag{C.1}$$

To compute the conditional expectation inside the integrals, we note that

$$D_s = \tilde{\zeta} + \beta\left(\mu - \frac{1}{2}\sigma^2\right)s + \beta\sigma B_{1t} + \alpha B_{2t} + \beta\sigma(B_{1s} - B_{1t}) + \alpha(B_{2s} - B_{2t}) \quad \text{for } s \ge t.$$

In addition, under $\widehat{\mathbb{P}}$, the conditional distributions of $(B_{1s} - B_{1t})$ and $(B_{2s} - B_{2t})$ given $\mathcal{F}_t$ are normal with mean $-\eta(s-t)$ and 0, respectively, and common variance, $s-t$. It follows then that, under $\widehat{\mathbb{P}}$, the conditional distribution of D_s given $\mathcal{F}_t$ is also normal with mean $D_t - \frac{1}{2}\beta\sigma^2(s-t)$ and variance $(\beta^2\sigma^2+\alpha^2)(s-t)$. We denote by $\widehat{\Phi}_{D_s|\mathcal{F}_t}(\cdot)$ the corresponding cumulative probability distribution. Then, we can rewrite the decomposition of V_t^ϵ as follows:

$$V_t^\epsilon = H_t^\epsilon - \frac{1}{2}\beta\sigma^2 \int_t^T [R - r - (R+b-r)\widehat{\mathbb{E}}[f_\epsilon'(D_s) \mid \mathcal{F}_t]]\,\mathrm{d}s - \frac{(R+b-r)(\beta^2\sigma^2+\alpha^2)}{2} \int_t^T \frac{\widehat{\Phi}_{D_s|\mathcal{F}_t}(I+\epsilon) - \widehat{\Phi}_{D_s|\mathcal{F}_t}(I-\epsilon)}{2\epsilon}\,\mathrm{d}s. \tag{C.2}$$

We now use (C.2) to show that (19) of Theorem 3 holds for a fixed value of t. First, it is clear that $\lim_{\epsilon\downarrow 0} H_t^\epsilon(x) = H_t(x)$ pointwise in x. Second, we see from the definition of V_t^ϵ that

$$\lim_{\epsilon\downarrow 0} V_t^\epsilon = \lim_{\epsilon\downarrow 0} \widehat{\mathbb{E}}[H_T^\epsilon(D_T) \mid \mathcal{F}_t] = \widehat{\mathbb{E}}[H_T(D_T) \mid \mathcal{F}_t] = V_t,$$

where the second equality follows from the dominated convergence theorem and since $\lim_{\epsilon\downarrow 0} H_T^\epsilon(x) = H_T(x)$ pointwise in x. Third, we need to show that

$$\lim_{\epsilon\downarrow 0} \int_t^T \widehat{\mathbb{E}}[f_\epsilon'(D_s) \mid \mathcal{F}_t]\,\mathrm{d}s = \int_t^T \left(1 - \widehat{\Phi}_{D_s|\mathcal{F}_t}(I)\right)\mathrm{d}s. \tag{C.3}$$

Towards this end, observe that

$$|\widehat{\mathbb{E}}[f_\epsilon'(D_s) \mid \mathcal{F}_t]| = \left|1 - \widehat{\Phi}_{D_s|\mathcal{F}_t}(I+\epsilon) + \widehat{\mathbb{E}}\left[\frac{(D_s - I + \epsilon)}{2\epsilon}\mathbf{1}(|D_s - I| \le \epsilon) \mid \mathcal{F}_t\right]\right| \le 1 + \mathrm{P}(|D_s - I| \le \epsilon) \le 2.$$

Since $\widehat{\mathbb{E}}[f_\epsilon'(D_s) \mid \mathcal{F}_t] \to 1 - \widehat{\Phi}_{D_s|\mathcal{F}_t}(I)$ almost surely, the dominated convergence theorem now applies immediately to give (C.3). Finally, we must show

$$\lim_{\epsilon\downarrow 0} \int_t^T \left[\frac{\widehat{\Phi}_{D_s|\mathcal{F}_t}(I+\epsilon) - \widehat{\Phi}_{D_s|\mathcal{F}_t}(I-\epsilon)}{2\epsilon}\right]\mathrm{d}s = \int_t^T \hat{\phi}_{D_s|\mathcal{F}_t}(I)\,\mathrm{d}s. \tag{C.4}$$

However, observe, first, that

$$0 \le f_\epsilon(s) := \frac{\widehat{\Phi}_{D_s|\mathcal{F}_t}(I+\epsilon) - \widehat{\Phi}_{D_s|\mathcal{F}_t}(I-\epsilon)}{2\epsilon} \le \int_{I-\epsilon}^{I+\epsilon} \frac{\mathrm{d}x}{2\epsilon c\sqrt{s-t}} = \frac{1}{c\sqrt{s-t}} =: h(s)$$

for some constant, c. Now, since $\int_t^T h(s)\,\mathrm{d}s < \infty$, we can again apply the dominated convergence theorem to obtain (C.4).

We have therefore shown that, for a fixed t, (19) holds almost surely. The countability of the rationals then implies (19) holds almost surely for all rational $t \in [0, T]$. The continuity of V_t and the processes on the right-hand side of (19) now establish that (19) holds for all $t \in [0, T]$ almost surely.

(b) Let

$$V_T = V_0 + \int_0^t \vartheta_s\,\mathrm{d}X_s + \int_0^t \delta_s\,\mathrm{d}B_{2,s} \qquad \text{and} \qquad V_T^\epsilon = V_0^\epsilon + \int_0^t \vartheta_s^\epsilon\,\mathrm{d}X_s + \int_0^t \delta_s^\epsilon\,\mathrm{d}B_{2,s}$$

be the GKW compositions, respectively, of V_T and V_T^ϵ with respect to X under $\widehat{\mathbb{P}}$. (It is easily checked that both V_T^ϵ and V_T are square $\widehat{\mathbb{P}}$-integrable martingales, as the GKW decomposition requires.) While we know from part (a) that $V_T^\epsilon \to V_T$ almost surely as $\epsilon \to 0$ and that $V_0^\epsilon \to V_0$, it is easy to check that convergence also takes place in $L^2(\widehat{\mathbb{P}})$. Let $L^2(X)$ denote the set of $\mathcal{F}_t$-adapted processes, ϕ, with a finite norm

$$\|\phi\| := \widehat{\mathbb{E}}\left[\left(\int_0^T \phi_s\,\mathrm{d}X_s\right)^2\right] = \int_0^T \widehat{\mathbb{E}}[\phi_s^2]\,\mathrm{d}s < \infty.$$

$L^2(B_2)$ is similarly defined. We can easily find δ_s^ϵ as it is given (uniquely as an element of $L^2(B_2)$) by $\partial V_s^\epsilon/\partial B_{2,s}$. Now let δ_s^* denote the process defined by the right-hand side of (20). It may be checked that for ϵ sufficiently small, both δ_s^* and δ_s^ϵ are uniformly bounded functions in (s, D_s) and that δ_s^ϵ converges pointwise to δ_s^*. It therefore follows immediately by the Itô isometry that δ_s^ϵ converges to δ_s^* in $L^2(B_2)$ and that, equivalently, $\int_0^T \delta_s^\epsilon\,\mathrm{d}B_{2,s}$ converges in $L^2(\widehat{\mathbb{P}})$ to $\int_0^T \delta_s^*\,\mathrm{d}B_{2,s}$. This in turn, however, implies that $\int_0^T \vartheta_s^\epsilon\,\mathrm{d}X_s$ also converges in $L^2(\widehat{\mathbb{P}})$. But the Itô isometry implies that the linear subspace $\{\int_0^T \phi_s\,\mathrm{d}X_s\colon \widehat{\mathbb{E}}[\int_0^T \|\phi_s\|^2\,\mathrm{d}s] < \infty\}$ is closed in $L^2(\widehat{\mathbb{P}})$, implying in particular that $\lim_{\epsilon\to 0} \int_0^T \vartheta_s^\epsilon\,\mathrm{d}X_s \to \int_0^T \vartheta_s^*\,\mathrm{d}X_s$ in $L^2(\widehat{\mathbb{P}})$ for some adapted process ϑ_s^*. But then the uniqueness property of the GKW decomposition implies that $\vartheta_s^* = \vartheta_s$ in $L^2(X)$ and $\delta_s^* = \delta_s$ in $L^2(B_2)$. □

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