Spatial Models, Cognitive Metrics and Majority Rule Equilibria

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Abstract

Long-standing results demonstrate that, if policy choices are defined in spaces with more than one dimension, a majority rule equilibrium fails to exist for a very general class of smooth preference profiles. However we show here that if agents perceive political similarity and difference in "city block" terms, then the dimension-by-dimension median can be a majority rule equilibrium even in spaces with an arbitrarily large number of dimensions; we provide necessary and sufficient conditions for the existence of such an equilibrium. This result is important because city block preferences accord more closely with empirical research on human perception than do many smooth preference profiles. The implication is that, if the already extensive empirical research findings on human perceptions of similarity and difference extend also to perceptions of *political* similarity and difference, then the possibility of equilibrium under majority rule re-emerges.

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1 Introduction

It is well known that for a very general class of preference profiles a Majority Rule Equilibrium (MRE) does not exist in settings with a minimally complex set of political alternatives.[2] More specifically, when voters have smooth preferences over outcomes in multidimensional choice spaces, for any given outcome there will almost always exist some majority of agents that prefers some other outcome. Normatively, these results suggest there is no privileged outcome that democratic institutions "ought" to select.[3] Substantively, though these results do *not* in themselves predict cycling, they do render the prediction of voting outcomes more difficult in complex environments.

Theorists have responded to the absence of a majority rule policy prediction in general multidimensional settings in a variety of ways. Some have used a form of institutionally structured equilibrium to identify equilibria.[4] Others have turned their attention to non-cooperative models that do generate majority-rule equilibria in n-dimensional outcome spaces. However since these models require strong assumptions about the institutional form of the decision-making process there is often a concomitant loss in generality.[5] In some models, for example, outcome prediction depends entirely on the identity of an *exogenously* determined "agenda setter," yet in many real world settings there is no unambiguous exogenous assignment of agenda setting powers.[6] Many other scholars ef-

[2] Frank DeMeyer and Charles R. Plott, 'The probability of a cyclical majority'; *Econometrica*, 38 (1970), 345-354. Charles R. Plott, 'A notion of equilibrium and its possibility under majority rule'; *American Economic Review,* 57 (1967), 787-806; Norman Schofield, 'Generic instability of majority rule'; *Review of Economic Studies*, 50 (1983), 695-705; Richard D. McKelvey, 'Intransitivities in multidimensional voting models and some implications for agenda control'; *Journal of Economic Theory*, 12 (1976) 472-482; Richard D. McKelvey and Norman Schofield, 'Structural instability of the core'; *Journal of Mathematical Economics,* 15 (1986), 179-198; Richard D. McKelvey and Norman Schofield, 'Generalized symmetry conditions at a core point'; *Econometrica,* 55 (1987), 923-933.

[3] William H. Riker, *Liberalism against populism* (San Francisco: W.H. Freeman,1982).

[4] Michael Laver and Kenneth A. Shepsle, *Making and breaking governments: cabinets and legislatures in parliamentary democracies* (New York: Cambridge University Press, 1996); Kenneth Shepsle, 'Institutional Arrangements and Equilibrium in Multidimensional Voting Models'; *American Journal of Political Science,* 23 (1979), 27-59; Kenneth A Shepsle and Barry R. Weingast, 'The Institutional Foundations of Committee Power'; *American Political Science Review,* 81 (1987), 85-104.

[5] Timothey Besley, and Stephen Coate, 'An Economic Model of Representative Democracy'; *Quarterly Journal of Economics*, 112 (1997), 85-106. Daniel Diermeier and Antonio M Merlo, 'Government turnover in parliamentary democracies'; *Journal of Economic Theory*, 94 (2000), 46-79.

[6] We note that other models that study probabilistic voting yield equilibria for

fectively sidestep the theoretical problem by assuming that the set of feasible political outcomes can be described as points in a one-dimensional space.[7]

The different trade-offs made by these different approaches are especially important in situations in which an MRE does not exist. Different modeling choices produce different predictions; yet the modeling choices themselves often rest on weak support. In contrast, in cases where there *is* a well defined equilibrium—for example in cases where the median voter result applies—the MRE can be supported for a wide class of games not only as a cooperative equilibrium but also as a non-cooperative equilibrium.[8] For this reason, knowing when there is indeed a well defined majority rule equilibrium is important for researchers in both traditions.

In this article we show that, classic non-existence results notwithstanding, there may be a well defined majority rule equilibrium even in an outcome space of many dimensions if agents employ a "city block" metric to measure similarities and differences between political outcomes. City block preferences (defined formally below) fall within the class of convex preferences. However, standard results do not apply, since these rely fundamentally on analytical techniques that assume human preferences to be smooth, while city block preferences are not smooth. Under the assumption of city block preferences, we retrieve, strengthen, and generalize, an older and largely neglected analytical result due to Rae and Taylor.[9] The Rae and Taylor result was subsequently constrained to two-dimensional spaces by Kats and Nitzan[10] and was gen-

platform selection in multidimensional settings. While related to the question examined here however these models do not study the problem of a social choice. Nevertheless in some cases they yield predictions that correspond to the MRE identified here although we note that even in these cases the predictions of probabilistic models depend on particular specifications of loss functions which are left free in the present analysis. Tse-min Lin, James M. Enelow and Han Dorussen, 'Equilibrium in multicandidate probabilistic spatial voting', *Public Choice*, 98 (1999), 59-82.

[7] Massimo Morelli, 'Party formation and policy outcomes under different electoral systems'; *Review of Economic Studies* 71 (2004), 829–53. Martin Osborne, 'Entry-deterring policy differentiation by electoral candidates'; *Mathematical Social Sciences,* 40 (2000) 41 - 62. James Snyder and Michael Ting, 'An informational rationale for political parties'; *American Journal of Political Science,* 46 (2002), 90–110.

[8] David Austen-Smith and Jeffrey S. Banks, 'Social Choice Theory, Game Theory and Positive Political Theory' *in* N.W. Polsby [Eds.] *Annual Review of Political Science*, (Palo Alto: Annual Reviews, 1998).; Hervé Moulin. 'On strategy-proofness and single-peakedness'; *Public Choice* 35 (1980),437– 55.

[9] Douglas W. Rae and Michael Taylor, 'Decision Rules and Policy Outcomes'; *British Journal of Political Science*, 1(1971), 71-90.

[10] Amoz Kats and Shmuel Nitzan, 'More on decision rules and policy outcomes'; *British Journal of Political Science*, 7 (1977), 419-422.

erated, independently, for two dimensional spaces by Wendell and Thorson.[11] McKelvey and Wendell further identified sufficient conditions for equilibrium in higher dimensional settings.[12] This line of research did not however identify necessary and sufficient conditions for the existence of an equilibrium in n-dimensional settings. In this article we identify these conditions for the first time. This in turn allows us both to characterize conditions for equilibrium and to quantify the likelihood of this.

Our results are constructive in the following sense. We not only show that an equilibrium can exist in high dimensional spaces but we also fully characterize the equilibrium whenever it exists. The set of equilibrium outcomes is in fact point valued and is equivalent to the dimension by dimension median (DDM) of agent ideal points. Thus the model generates a clear equilibrium prediction about the outcome of majority rule decision making that is closely analogous to the median voter result for one dimensional settings. This equilibrium, though identified in a social choice theoretic environment, can also be sustained in a variety of non-cooperative settings.

The substantive significance of our results clearly depends on the validity of the assumptions that underly them, specifically on whether real humans do in fact evaluate political similarity and difference using city block distances. The city block assumption is considerably less general than the assumption of smooth preferences; nonetheless there are grounds for believing that humans do in practice employ metrics that are either city block or approximate city block. Thus the city block metric is widely used by cognitive scientists who analyze human perceptions of similarity and difference in terms of distances in some cognitive space. Empirical findings in this field have led to a received wisdom that, when differences between stimuli are constructed in terms of *separable*[13] features or dimensions, spatial representations using the city block metric best fit empirical data on human perceptions of similarity and difference.[14]

[11] Richard E Wendell and Stuart J. Thorson . 'Some generalizations of social decisions under majority rule'; *Econometrica*, 42 (1974), 893-912.

[12] Richard D. McKelvey and Richard E. Wendell, 'Voting Equilibria in Multidimensional Choice Spaces'; *Mathematics of Operations Research.* 1 (1976),144–58.

[13] This uses the term "separable" in the standard sense that a set of separable dimensions has the property that perceived differences between stimuli on one dimension are independent of perceived differences on other dimensions in the set.

[14] Janet Aisbett and Greg Gibbon. 'A general formulation of conceptual spaces as a meso level representation'; *Artificial Intelligence,* 133 (2001), 189-232. Fred Attneave, 'Dimensions of similarity'; *American Journal of Psychology,* 63 (1950), 546-554. Wendell R. Garner, *The Processing of Information and Structure* (Potomac, MD: Erlbaum, 1974). Roger N. Shepard, 'Toward a universal law of generalization for psychological science'; *Science*, 237 (1987), 1317-1323. Roger N. Shepard, 'In-

There is less work on cognitive distance metrics in political decision making, but such work as there is comports with the findings of cognitive scientists. Comparing the city block with the Euclidean norm (the most common alternative used in empirical modelling), Enelow, Mendell and Ramesh find across a range of diagnostic tests that models using a city block metric are better specified than those that assume Euclidean preferences.[15] Westholm sees strong *a priori* reasons to choose a city block metric when predicting Norwegian survey respondents' party thermometer scores from their multidimensional issue positions, in addition he finds that a city block specification has a better *empirical* fit than alternative linear and quadratic Euclidean specifications.[16] More recently Grynaviski and Corrigan find the city block metric better predicts voter attitudes to US Presidential candidates across a range of specifications.[17] In general, while empirical research findings are sparse, these tend to favour choice of city block over Euclidean metric. An honest summary of these results, however, is that metric assumptions for models of policy-based political decision making are under-researched and which distance metric—if any—is appropriate for modelling human political preferences remains an open question.

In what follows, we first distinguish between classes of spatial model in terms of whether or not the analyst requires cognitive assumptions about how real humans perceive political similarity and difference. If the analyst must make a cognitive assumption about how agents make political choices, we have just noted that the city block metric is a plausible candidate assumption. The central argument in the article is that, if agents do indeed think in city block terms, then the DDM (and only the DDM) emerges as a majority rule equilibrium. We also show that the likelihood the DDM is an equilibrium can be high, at least for low numbers of agents and in a low number of dimensions. We construct our main existence argument analytically, and then use simulations to show the probability of an MRE may be much higher if, as very often

tegrality versus separability of stimulus dimensions' *in* Gregory R. Lockhead and James R. Pomerantz *The Perception of Structure: Essays in Honor of Wendell R. Garner* (Washington, DC: American Psychological Association, 1991), 53-71. Peter Gärdenfors, *Conceptual spaces: the geometry of thought* (Cambridge: MIT Press, 2000).

[15] James M. Enelow, Nancy R. Mendell, Subha Ramesh, 'A Comparison of Two Distance Metrics Through Regression Diagnostics of a Model of Relative Candidate Evaluation'; *The Journal of Politics*, 50 (1988), 1057-1071.

[16] Anders Westholm, 'Distance versus direction: The illusory defeat of the proximity theory of electoral choice'; *American Political Science Review*, 91 (1997), 865-883.

[17] Jeffrey D. Grynaviski and Bryce E. Corrigan, 'Specification Issues in Proximity Models of Candidate Evaluation (with Issue Importance)'; *Political Analysis,* 14 (2006), 393-420.

happens in practice, agent ideal points are correlated across dimensions and/or if agents have different voting weights. Finally, we discuss the robustness of our results by considering implications for equilibrium of "nearly city block" preferences.

Taken together, our results imply that, if city block distances are a reasonable way to characterize how real humans compare possible political outcomes then, subject to conditions identified below, the DDM emerges as a policy prediction even in complex environments.

2 Spatial Models and Cognitive Metrics

The term "spatial model" is informally used to cover a wide class of models in which political agents have rational preferences over sets of outcomes that can be characterized as points in a space. However, although this is not always appreciated, there is an important epistemological distinction between the *analyst's mathematical description* of a set of outcomes as points in a space, and a *human agent's cognitive perception* of these same outcomes in spatial terms. This distinction leads us to distinguish two types of spatial model.

In a *weakly spatial* model, the set of outcomes, W, can be represented as a space, for example a vector space, a smooth manifold or, most commonly, a Euclidean space. Agents are assumed to have a rational (complete, reflexive and transitive) binary preference relation—$\succsim_i$—over all pairs of elements in this set. But it is not required that agents *cognitively perceive* the relative utilities of outcomes in terms of relative distances in the underlying mathematical space. Commonly, agent preferences are characterized simply by an abstract agent-specific utility function $u : W \to \mathbb{R}^1$ with the property that, for any pair of outcomes y, y', we have $y \succsim_i y' \Leftrightarrow u_i(y) \geq u_i(y')$.

In a *strongly spatial* model, the set of outcomes is characterized as being located in a space, but human agents' *evaluations* of these same outcomes are also assumed to be "spatial" in the sense that each agent locates the set of outcomes, as well as his/her most-preferred outcome (ideal point) x_i, in a cognitive space and ranks outcomes using some distance measure $d_i(.,.)$ defined over the entire space.[18] This means that, given preference relation $\succsim_i$, we have $y \succsim_i y' \Leftrightarrow d_i(x_i, y) \leq d_i(x_i, y')$. Commonly, these preferences are represented by an agent specific utility function $u : W \to \mathbb{R}^1$ with the property that for any two points y, y' we have $y \succsim_i y' \Leftrightarrow u_i(y) \geq u_i(y')$ where $u_i(.)$ is itself a composition of a

[18] A distance measure is a function $d : W \times W \to \mathbb{R}^1$ that has the properties that for points a,b, and c in W: $d(a, b) = 0 \Leftrightarrow a = b$, $d(a, b) = d(b, a)$ and $d(a, b) + d(b, c) \geq d(a, c)$.

loss function f and the distance function d_i; hence, $u_i(y) = f_i(d_i(x_i, y))$. Political theorists often assume the function f to be either a linear or quadratic function of distance from the ideal point. Psychologists and cognitive scientists, on the other hand, typically assume that the perceived similarity of two points is an exponentially decaying function of the distance between them.[19] For the questions of concern to us here, we do not need to impose any structure on f, other than that it be strictly decreasing in $d(.,.)$.

Many of the models that yield non-existence results are *weakly* spatial models that seek to characterize agent preferences in the most general and abstract form.[20] Most spatial models of party competition in the classic Downsian tradition are *strongly* spatial, in our sense, since they assume each voter to evaluate the set of potential outcomes in terms of each potential outcome's relative "closeness" to the voter's ideal point. The class of strongly spatial models extends, for essentially the same reason, to many spatial models of probabilistic voting, with or without valence parameters.[21]

Strongly spatial models of political choice require a much stronger cognitive assumption about real agents than do weakly spatial models since they require information about the perceptual geometry used (consciously or not) by agents to measure the distance between two stimuli in the assumed cognitive space. Any assumption about the cognitive metric used to measure such distances contains crucial psychological assumptions about how agents trade off perceived differences on one dimension of difference against perceived differences on another dimension. Using the Euclidean metric involves assuming that agents measure the psychological distance between two points in cognitive space using the same Pythagorean Theorem they use to measure the distance between two points in local physical space.[22] Using a city block metric assumes that

[19] Shepard, Toward a universal law of generalization for psychological science; Peter Gärdenfors, *Conceptual spaces*; but see also Keith Poole and Howard. Rosenthal, *Congress: A Political-Economic History of Roll Call Voting* (New York, Oxford University Press, 1997).

[20] McKelvey and Schofield, Structural Instability of the Core; McKelvey, Intransitivities in multidimensional voting models.

[21] Timothy Groseclose, 'A model of candidate location when one candidate has a valence advantage'; *American Journal of Political Science*, 45 (2001), 862-886; Norman Schofield, 'Valence competition and the spatial stochastic model'; *Journal of Theoretical Politics*, 15 (2003) 371-383; Norman Schofield, 'Equilibrium in the spatial valence model of politics'; *Journal of Theoretical Politics*, 16 (2004), 447-481.

[22] We say that a player, $i \in N$, has generalized **Euclidean preferences** over points in $\mathbb{R}^m$ if there exists an $x_i \in \mathbb{R}^m$ and a semipositive definite matrix A_i such that for all $y, y' \in \mathbb{R}^m$:$y \succeq_i y' \Leftrightarrow (x_i - y)'A(x_i - y) \leq (x_i - y')'A(x_i - y')$. Euclidean preferences (with uniform weights) obtain if $A = I_{n \times n}$.

agents sum or average distances along each dimension which, as we have seen, is what many cognitive scientists have found that humans tend to do when dimensions of perceived difference are separable.[23]

Our purpose in what follows is to build on the admittedly limited empirical research noted in the previous section, which at the very least identifies the city block metric as a strong candidate for the most appropriate cognitive assumption to make in rational choice models of political decision-making. Given this, we demonstrate that majority rule equilibria in settings in which agents have city block preferences differ significantly from those derived from models that assume strictly convex preferences.

3 Equilibrium with Uniform City Block Preference Profiles

3.1 The Model

Consider a set of agents given by $N = \{1, 2, ..., n\}$, n odd, an outcome space given by $\mathbb{R}^m$ and player specific rational preference relations over elements of $\mathbb{R}^m$, $(\succsim_i)_{i \in N}$. In this context define city block preferences as follows.

Definition 1 *An agent, $i \in N$, has "**city block preferences**" if there exists a $y_i \in \mathbb{R}^m$ and positive scalars $(\alpha_{i,j})_{j=1,2,...,m}$ such that for any $x, x' \in \mathbb{R}^m$:*

$$x \succsim_i x' \Leftrightarrow \sum_{j=1}^{m} \alpha_{i,j} |y_{i,j} - x_j| \leq \sum_{j=1}^{m} \alpha_{i,j} |y_{i,j} - x'_j|$$

The weighting parameters, $(\alpha_{i,j})_{j=1,2,...,m}$ depend on how the space itself is defined. Conditions can be imposed upon these weights if agents are seen to view the space in similar ways. In what follows we use such a condition to define a "uniform city block preference profile".

[23] It is important to emphasize that the assumption of a Euclidean cognitive metric to measure distance is quite distinct from the assumption of a quadratic loss function, despite the use of quadratic terms in each. The metric assumption is a cognitive assumption about how agents trade off differences between stimuli on distinct dimensions of difference. The loss function is a cognitive assumption about how these distances, calculated having made these trade-offs, map into agent utility. Thus it is quite consistent to apply quadratic or exponential loss functions to city block distances.

Definition 2 *A collection of agents,* $N = \{1, 2, ...n\}$*, has a"**uniform city block preference profile**" if there exists for each* $i \in N$ *a vector* $y_i \in \mathbb{R}^m$ *and positive scalars* $(\alpha_{i,j})_{j=1,2,...,m}$ *such that for any* $x, x' \in \mathbb{R}^m$*:*

$$x \succsim_i x' \Leftrightarrow \sum_{j=1}^{m} \alpha_{i,j}|y_{i,j} - x_j| \leq \sum_{j=1}^{m} \alpha_{i,j}|y_{i,j} - x'_j|$$

and for any two agents i *and* h *and any two dimensions* j *and* k*,* $\frac{\alpha_{i,j}}{\alpha_{i,k}} = \frac{\alpha_{h,j}}{\alpha_{h,k}}$.

In words, there is a uniform city block preference profile when all agents have city block preferences and all attach the same (relative) weights to the different dimensions. This is a strong assumption although we note that it is one that is made explicitly or implicitly in many strongly spatial models. In particular models in which all agents have circular, or spherical, indifference curves assume common (relative) weights.

Now, consider a strongly spatial model in which a set of agents $N = \{1, 2, ..., n\}$, n odd, has a uniform city block preference profile.

We are interested in conditions under which a point $x \in \mathbb{R}^m$ is a "**Majority Rule Equilibrium**" (MRE) point, in the sense there is no other point $x' \in \mathbb{R}^m$ such that at least $\frac{n+1}{2}$ agents strictly prefer to x. Note that if an MRE exists it is a Condorcet winner and is in the core.[24]

Without loss of generality, we label axes such that the dimension by dimension median (DDM) at the origin. We assume furthermore that agents have ideals in "general position,"[25] and in particular that only one agent is median on each dimension. Finally when we study uniform city block preference profiles we normalize such that $\alpha_{i,j} = 1$ for all agents i, and dimensions, j.

The discussion that follows focuses on the DDM (the origin under our normalization) as a candidate MRE. We do this because under city block preferences no point *other* than the DDM can be an MRE. To see this assume that x is an MRE but that x is not at the origin; that is, for some dimension j, $x_j \neq 0$. Consider now a rival policy, x' that differs from x only in that $x'_j = 0$. Clearly we then have $x \succsim_i x' \Leftrightarrow |y_{i,j} - x_j| \leq |y_{i,j} - x'_j|$ and hence $x' \succ_i x \Leftrightarrow |y_{i,j} - x'_j| < |y_{i,j} - x_j|$. Since x'_j is the median on dimension j there is a majority, including the dimension j median, for whom $|y_{i,j} - x'_j| < |y_{i,j} - x_j|$ and hence a majority that prefers x' to x.

[24] David Austen-Smith and Jeffrey S Banks, *Positive Political Theory 1.* (Ann Arbor: Michigan University Press, 2000).

[25] As opposed to some particular pathological configuration. Formally a collection of points in $\mathbb{R}^m$ is in "general position" if for any $k \leq m$, no more than m ideal points lie on any $m - 1$ dimensional affine manifold.

Hence a necessary condition for a point x to be an equilibrium is it is at the DDM, or, under our normalization, that $x = 0$.

For a point to be the DDM is not, however, a sufficient condition for it to be an MRE and indeed cases in which the DDM is not an MRE can easily be found.[26] Standard sufficiency conditions for the existence of an MRE use information on the gradient vectors of agents' utility functions around the equilibrium point to construct relations of *opposition*. We say that agent i is *opposed* to agent k in relation to an outcome at the origin if the gradient vectors, $\nabla_i(0)$ and $\nabla_k(0)$ are well defined and for some scalar $\beta < 0$, $\nabla_i(0) = \beta\nabla_k(0)$. Informally, agents are opposed if any movement that is acceptable to one is opposed by the other. But utility functions that represent city block preferences cannot be differentiated everywhere (the problem arises when an outcome on some dimension is equal to an agent's ideal on that dimension). This prevents us from using conditions such as those found by Plott or McKelvey and Schofield, at the candidate equilibrium point.[27] However symmetry conditions that are analogous to Plott's can be used.

In the next section we provide such symmetry conditions. The necessary and sufficient condition for the existence of an MRE that we identify is akin to the result for Euclidean preferences that a point, x, is an MRE if and only if for every supporting hyperplane, H of x, each closed half space of H contains a majority of voter ideals.[28] When agents have city block preferences, as we shall see, this condition is considerably less restrictive.

3.2 Results

Our main result relies on the fact that, with a uniform city block preference profile, all agents with ideal points in a given (open) orthant have similar preferences over small movements away from the origin. Given this feature, in order to locate an equilibrium, we require a language to talk about *collections of orthants* in which agent ideal points may lie. We do this as follows.

Let $\mathcal{O}$ denote the collection of 2^m open orthants of $\mathbb{R}^m$ and define $X \equiv \{-1, 1\}^m$. Clearly there is a one-to-one correspondence between the set of orthants and the elements of X: the set X contains 2^m points

[26] Consider for example a case in which five players have ideals given by $(1,1,0), (1,0,1)$ $(0,1,1), (-1,-1,-1), (-1,-1,-1)$. In this case $(1,1,1)$ is preferred by the first three agents to $(0,0,0)$.

[27] Plott, A notion of equilibrium; McKelvey and Schofield, Generalized symmetry conditions at a core point.

[28] Gary Cox and Richard McKelvey 'A Ham Sandwich Theorem for General Measures'; *Social Choice and Welfare*, 1 (1984), 75-83.

with each point lying in one of the 2^m orthants of $\mathbb{R}^m$. Specifically, let the function $f : X \to \mathcal{O}$, choose the (open) orthant in which $x \in X$ lies: $f(x) = \{z \in \mathbb{R}^m : \text{sgn}(z_j) = x_j \text{ for } j = 1, 2, ..., m\}$. And let g denote the inverse of f, that is $g : \mathcal{O} \to X$, $g(O) = \{x \in X : x_j = \text{sgn}(o_j) \text{ for all } o \in O \text{ and } j = 1, 2, ..., m\}$. With this mapping, for a collection of orthants $\mathcal{C}$ we can without ambiguity denote the corresponding subset of points in X by $X_{\mathcal{C}}$.

We use g to define the following family of subsets of $\mathbb{R}^m$ that will prove central to our analysis.

Definition 3 *A set of points is a "**rugged halfspace**" if it is the closure of a collection, $\mathcal{C}$, of 2^{m-1} (open) orthants that has the property that there exists a vector $v \in \mathbb{R}^m$ such that for every orthant $O \in \mathcal{C}$, $v.g(O) > 0$.*

It is easy to see that the set of rugged halfspaces is non-empty (any v for which $v.x \neq 0$ for all $x \in X$ induces a rugged halfspace, including all vectors that have a zero in all but one entry); further for every rugged halfspace with "normal" vector v, there exists an opposite rugged halfspace with normal $-v$; finally rugged halfspaces are *half*spaces in the sense that the union of two opposite rugged halfspaces is the set $\mathbb{R}^m$.

Other features of rugged halfspaces will prove important.

First, since a rugged halfspace is the closure of a collection of (open) orthants, it includes all boundary points of the orthants; in particular, since 0 is in the boundary of each orthant it is contained in every rugged halfspace. More generally if a point x is in a rugged halfspace, so is any point x' that differs from x only in that for some dimension j, $x'_j = 0$ but $x_j \neq 0$.

Second, rugged halfspaces differ from the standard notion of a halfspace generated by a hyperplane through the origin. Conventionally, such a halfspace is given by a set of points $Z \subset \mathbb{R}^m$ for which there exists a normal vector $v \in \mathbb{R}^m$ with $v.z \geq 0$ for all $z \in Z$. However whereas in $\mathbb{R}^1$ and $\mathbb{R}^2$, all rugged halfspaces are also halfspaces in this standard sense, this is not generally the case. In $\mathbb{R}^3$ the vectors $X_{\mathcal{C}} = \{\{1, 1, 1\}, \{-1, 1, 1\}, \{1, -1, 1\}, \{1, 1, -1\}\}$ induce a rugged halfspace (with a normal given by $v = \{1, 1, 1\}$) but the rugged halfspace, being blocky, cannot be described as the half space lying above a hyperplane. This rugged half space is illustrated in Figure 1.

We now state and prove our main result:

Proposition 1 *With a uniform city block preference profile and an odd number of voters, an MRE exists (and corresponds to the dimension by dimension median) if and only if every rugged halfspace contains a majority of ideal points.*

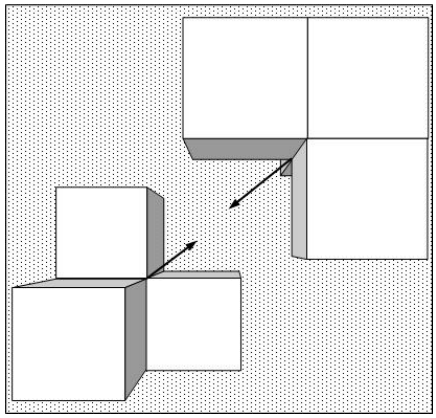

Figure 1: Illustration of two rugged half spaces in $\mathbb{R}^3$. With $v = \{1, 1, 1\}$ the upper right rugged halfspace is generated by the set of elements of X with $x.v > 0$; the lower left rugged half space is generated by the vectors in X for which $x.(-v) > 0$.

Proof. See Appendix. ■

A crucial feature of Proposition 1 is that conditions for equilibrium do not depend on non-generic distributions of ideal points. Informally a set of preference profiles, A is generic, if *whether or not* a given profile, P, is in A, some small perturbation of P, P' *will* be in A; that is the set A is open and dense in the set of admissible profiles. The set of distributions of ideal points that do *not* satisfy radial symmetry is generic, and so an equilibrium that depends on radial symmetry holds only for preferences that are *not* generic. In contrast, conditions identified in Proposition 1 are much weaker. We can fully characterize conditions for MRE existence using only information about the distribution of agent ideal points *across orthants*. Since the set of profiles that do not satisfy the conditions in Proposition 1 is not generic, the non-existence of an MRE is not generic under uniform city block preference profiles. Informally, under city block preferences, if an MRE exists, then it will continue to exist even if locations of ideal points are subjected to small shocks.

The intuition behind Proposition 1 is illustrated in Figure 2. Figure 2 shows the positions of three voters with ideal points at A, B, and C, all elements of $\mathbb{R}^2$. The dimension-by-dimension median of the voter positions is marked as DDM – re-scaled without loss of generality to the origin of the space. The dashed axes thus contain the median voter on each dimension.

The dashed circles in Figure 2 show Euclidean indifference curves for each voter with respect to the DDM. The three lens-shaped intersections of these indifference curves show the "winset" of the DDM—the set of points in the policy space that are majority preferred to the DDM. The DDM will lose in a majority vote to any point in this set and therefore is not an equilibrium outcome. The non-existence results show that the winset of any arbitrary point in this space, and the DDM in particular, is "generically" non-empty under majority rule in multidimensional policy spaces.

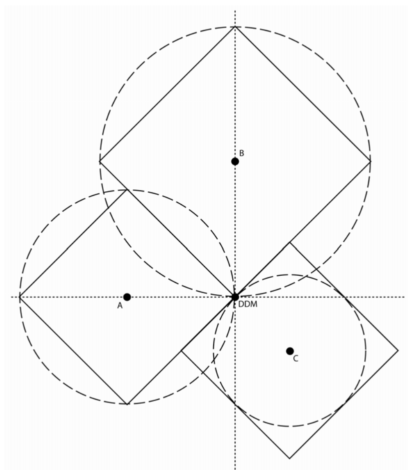

Figure 2: Majority rule equilibrium at the dimension-by-dimension median with city block preferences

The rotated squares in the figure show the city block indifference curves for each voter with respect to the DDM. These curves do not intersect and thus the DDM has an empty winset. It is therefore a majority rule equilibrium. In terms of Proposition 1, it can be seen that each rugged halfspace contains the ideal points of two of the three voters; a movement in any direction will be opposed by some such pair of voters.

Intuitively the difference between the two cases arises because of two features of uniform city block preference profiles. The first is that the indifference curves of agents that are median on some dimension are kinked at the origin. To generate an improvement in policy for an agent with such a kink is as difficult as it is to generate an improvement for two non-median agents with ideals in distinct (but not opposite) orthants—in Figure 2 for example agents A and B are willing to accept only one quarter of all possible directions of movement from the origin whereas agent C will accept one half. With strictly convex preferences, agents that are median on one dimension do not hold such privileged positions.

The second feature of city block preferences is that all agents with ideal points in the interior of a given orthant have indifference curves of the same slope (a) as each other and (b) as agents in opposite orthants. An implication of (a) is that the condition for opposition with another agent is relatively weak, all that is required is that there is an agent in the opposite orthant. An implication of (b) is that a small change in the location of an agent's ideal point—consider for example voter C in the figure—does not induce a change in shape of the voter's indifference curve at the equilibrium point. This local invariance of the shape of indifference curves to the location of ideal points explains why "radial symmetry" exceptions to non-existence results (identified for example by Plott) require less demanding conditions. In this important sense, the class of preference profiles (within the set of uniform city block preference profiles) for which the DDM has a non-empty winset is not non-generic.

3.3 Qualitative Implications

Proposition 1 has four implications of interest.

A first implication of the proposition is that with uniform city block preferences *an equilibrium always exists if there are only three agents.* This follows from the fact that with only three agents, a majority of ideal points (two) necessarily lie in every rugged halfspace.[29]

A second implication is that with uniform city block preferences *an equilibrium always exists if there are only two dimensions.* This result, given also by Rae and Taylor and Wendell and Thorson follows directly from the proposition since with only two dimensions a majority of ideals necessarily lies in each rugged halfspace.[30] In two dimensions the only rugged halfspaces are composed of the pairs of adjacent orthants either above or below the horizontal axis or to the left or the right of the vertical axis. The origin is defined however such that there is a majority on or

[29] To see this let x, y and z denote the ideals of three players and assume that x and only x lies in a rugged halfspace, $\mathcal{H}$, with normal, v. Without loss of generality assume that $v_j \geq 0$ for $j = 1, 2, ...m$. Define y' by $y'_j = \begin{cases} sgn(y_j) \text{ if } y_j \neq 0 \\ 1 \text{ if } y_j = 0 \end{cases}$. Since y lies in clos$(f(y'))$ but not in $\mathcal{H}$ we have $y'.v < 0$, or $\sum_{\{j:sgn(y_j)=-1\}} v_j > \sum_{\{j:sgn(y_j)\geq 0\}} v_j$. Defining z' in the same way we have $\sum_{\{j:sgn(z_j)=-1\}} v_j > \sum_{\{j:sgn(z_j)\geq 0\}} v_j$. But with only three players we have $sgn(y_j) = -1$ implies $sgn(z_j) \geq 0$ and hence $\sum_{\{j:sgn(z_j)\geq 0\}} v_j \geq \sum_{\{j:sgn(y_j)=-1\}} v_j$. Together these inequalities imply $\sum_{\{j:sgn(z_j)=-1\}} v_j > \sum_{\{j:sgn(y_j)\geq 0\}} v_j$. But by the same logic, if $sgn(z_j) = -1$ then $sgn(y_j) \geq 0$ and so $\sum_{\{j:sgn(y_j)\geq 0\}} v_j \geq \sum_{\{j:sgn(z_j)=-1\}} v_j$, contradicting the last inequality

[30] Rae and Taylor, Decision Rules and Policy Outcomes; Wendell and Thorson, Some generalizations of social decisions under majority rule.

to one side of each axis.[31]

A third implication is that Proposition 1 generalizes Plott's results (conditional upon our restriction to city block preferences) on radial symmetry since the conditions of the proposition are implied by Plott's proposition *whenever this can be applied*. To see this, consider a situation in which one agent has an ideal at the DDM, and all other agents can be divided into pairs that are *opposed* in the sense described above. Assuming a smooth loss function, all agents other than the one with $y_i = 0$ have well defined gradients that are proportional to $[sgn(y_{i,1}), sgn(y_{i,2}), ..., sgn(y_{i,m})]$. Hence two agents are opposed if their ideals lie in opposite orthants. With all remaining agents divided into pairs with ideals in opposite orthants Plott's conditions are satisfied and we necessarily have a majority in every rugged halfspace. One implication of this is that the conditions for symmetry under city block preferences—that players can be paired such that they have ideals in opposite orthants—are considerably less onerous under city block preferences than they are with strictly convex preferences.

A fourth implication is that even if Plott conditions do not hold and there is no agent ideal point at the MRE, there may still exist classes of cases where an MRE exists at the DDM, in any number of dimensions. For an example, consider a case with 3 voters in three dimensions with ideals with a corresponding sign ordering, s, (with $s_{i,j} = sgn(y_{i,j})$) given by:

$$s = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

This class of sign orderings can be readily extended to any odd number of dimensions, m, with m voters such that, for each i, $y_{i,i} = 0$, and, for $k \in 1, 2, ...m - 1$, $y_{i,(i+k)(\text{mod}(m))} < 0$ if k is odd and $y_{i,(i+k)(\text{mod}(m))} > 0$ if k is even. To see that this set of sign patterns satisfies the condition of the Proposition note that for each pair of neighboring points, y_i and $y_{(i+1)(\text{mod}(m))}$, one belongs in one rugged half space and the other belongs in the opposite rugged half space.[32] This guarantees that there is a majority in every rugged half space.[33] For $n > m$ voters such classes

[31] But as shown in Wendell and Thorson (1974) and Kats and Nitzan (1977) this existence result in two dimensions does not extend to higher dimensions.

[32] To see this let k be given by $k = (i + 1)(\text{mod}(m))$. Define $\tilde{y}_i$ and $\tilde{y}_k$ for $j = 1, 2, ..., m$ by $\tilde{y}_{i,j} = \begin{cases} sgn(y_{i,j}) \text{ if } y_{i,j} \neq 0 \\ -1 \text{ if } y_{i,j} = 0 \end{cases}$ and $\tilde{y}_{k,j} = \begin{cases} sgn(y_{k,j}) \text{ if } y_{k,j} \neq 0 \\ 1 \text{ if } y_{k,j} = 0 \end{cases}$. Note that $y_i \in$closure$(f(\tilde{y}_i))$ and $y_k \in$closure$(f(\tilde{y}_k))$ but that $\tilde{y}_i = -\tilde{y}_k$. Hence $\tilde{y}_i.v > (<)0$ implies $\tilde{y}_k.v < (>)0$, and so y_i and y_k lie in opposite halfspaces.

[33] Assume to the contrary that a rugged halfspace $\mathcal{H}(v)$ contained only a minority

can obtain if the remaining $n-m$ voters can be divided into pairs with ideals in opposite orthants.

3.4 Probability of Equilibrium

Having established the *possibility* of majority rule voting equilibria under city block preferences in spaces of any dimension, we now move on to consider the *probability* of these. To generate a baseline estimate of the frequency of MRE existence we employ simulation techniques to estimate the likelihood of MRE existence given a random distribution of ideal points, as the number of voters and dimensions increases.[34] The results are reported in Table 1.

		m					
		1	2	3	4	5	6
n	1	1.00	1.00	1.00	1.00	1.00	1.00
	3	1.00	1.00	1.00	1.00	1.00	1.00
	5	1.00	1.00	0.66	0.31	0.11	0.78
	7	1.00	1.00	0.46	0.14	0.03	0.00
	9	1.00	1.00	0.39	0.06	0.01	0.00
	11	1.00	1.00	0.34	0.03	0.00	0.00
	13	1.00	1.00	0.31	0.02	0.00	0.00
	15	1.00	1.00	0.25	0.01	0.00	0.00

Table 1. The estimated probability of observing an MRE in m dimensions when n players have ideals drawn from a multivariate normal distribution with a diagonal variance co-variance matrix. Estimates are based on 500 independent draws of n person polities. For each estimate, p, the standard error is given by $\sqrt{\frac{p(1-p)}{500}}$.

The results in Table 1 confirm that an MRE exists with probability 1 for dimensions 1 and 2 and for committees with 1 and 3 voters, but the probability of MRE existence declines rapidly thereafter, reaching close to 0 with eleven voters in five dimensions. Hence although even in high dimensional space there is no *generic* absence of an MRE, the existence of an equilibrium point may be very *unlikely*, at least if agent ideal points are distributed randomly.

of ideal points, then a majority of ideals would lie in the interior of $\mathcal{H}(-v)$. But such a majority would contain two (neighboring) points that are not in opposite halfspaces.

[34] In the simulations presented in Table 1, players have ideals drawn independently from a multivariate normal distribution with $\mu = 0$ and $\Sigma_{m\times m} = I_{m\times m}$. For each draw the DDM is subjected to 5000 alternatives within 0.001 of the DDM on each dimension.

We close this section by considering two important ways in which distributions of the ideal points of human agents are likely to be *non-random* in ways that increase the likelihood of equilibrium existence in higher dimensional settings.

The first has to do with correlations of agents' positions on different dimensions – so that an agent's position on stem cell research, for example, can be predicted from her position on gay marriage. If positions on issues are correlated in this way it becomes more likely that pairs of voters have ideals located in opposing orthants of the space. For this reason, configurations of ideal points yielding majority rule equilibria under city block preferences are more likely with correlated ideals than they are with random ideals. This is not true for example of Euclidean preferences where correlations short of perfect correlation have no qualitative implications for equilibrium existence. Note that non-independence of ideals across dimensions is unrelated to the question of whether players have separable preferences; the former relates to the location of ideal points, the latter to the shape of indifference curves given a particular ideal point.

To assess the importance of this effect we recalculated Table 1 under the assumption that agents have ideal points drawn from a multivariate normal distribution with $m \times m$ variance covariance matrix 1 on the diagonal and .9 off the diagonal. otherwise. In this case the probability of a MRE rises substantially in all cases in which the MRE was not already guaranteed. In the $n = 7$, $m = 6$ case for example the probability rises from approximately 0 to approximately .5; the probability eventually declines however as m and n continue to rise, and we estimate the probability of an MRE at 6% for $m = 6$ and $n = 15$.

A second source of structure in the set of ideal points that can increase the probability of equilibrium under city block preferences arises in weighted voting games, such as those played in a legislature by well-disciplined political parties. In such games, each party has weight that reflects a situation in which a subset of voters belong to the same party and behave in a disciplined manner, "as if" they share the same ideal point. Two things happen in this setting. First, a large number of agents coalesce into a small number of disciplined parties with different voting weights; the number of effective agents is reduced and the above arguments apply. Second, weighted voting changes the probability that a large voting weight is located at the DDM and hence is contained in every rugged halfspace. This is because the probability that a party will be median on some dimension is strongly conditioned by that party's weight, as well as by its position in the ordering of parties on the dimension in question.

Trivially, a party with a majority weight will be median on every dimension and that party's ideal point will be a majority rule equilibrium in a space of any dimension under any conceivable assumption about preferences. Extending work on "dominant players" in majority voting games with non-uniform weights define a "system-dominant" party in a weighted voting game with four or more parties as a party which does not itself control a majority, but which can form a majority with any other single party, even the smallest.[35] This implies no other two-party coalition can form a majority.[36] Given Proposition 1 this further implies that, a sufficient condition for an equilibrium under a uniform city block preference profile is that a system dominant party is at the dimension-by-dimension median (which, we argue below, is "likely") and at least one party other than the system dominant party is in each rugged halfspace—a distribution that could obtain for example if any two non dominant parties had ideals in opposite orthants.

Since it can form a winning coalition with any other party, it is easy to see that a system dominant party occupies the median position on every dimension for which it is not at one of the two extreme positions in the ordering. Thus, for example, in an eight-party system with random party positions, the probability a system dominant party is median on an arbitrary dimension is 0.75. More generally the probability that a system-dominant party is at the DDM of an n-party m-dimensional space is $\left[\frac{n-2}{n}\right]^m$.[37] If we combine this with the Downsian intuition in relation to majority voting in legislatures that the largest party is *less* likely than others to be at an extreme position on salient policy dimensions, then this probability will increase in real-world policy spaces. More generally in weighted voting games, the probability a largest party with a random ideal point is at the median on an arbitrary dimension, and hence at the DDM, is greater than the probability an arbitrary voter is

[35] Bezalel Peleg, 'Coalition formation in simple games with dominant players'; *International Journal of Game Theory,* 1 (1981), 11 - 13; Ezra Einy, 'On connected coalitions in dominated simple games'; *International Journal of Game Theory,* 2 (1985), 103-125.; Adrian M. A. van Deemen, 'Dominant players and minimum size coalitions'; *European Journal of Political Research*, 17 (1989), 313-332.

[36] The largest two-party rival to a winning coalition between the largest and smallest parties is a coalition between the second and third largest parties, which must be losing since it is in the complement of the winning coalition between largest and smallest parties. Some non-cooperative game theorists refer to a system dominant party as an "apex player".

[37] More generally still, we note that the probability *any* party in a weighted majority rule voting game is median on an arbitrary policy dimension is its Shapley value in the game. The Shapley value is the probability a party is pivotal in a random ordering; in this context we consider the probability a party is pivotal (median) in a random policy ordering of parties on an arbitrary dimension.

at the DDM in an unweighted voting game with random ideal points. In this sense the baseline probability of a majority rule equilibrium under city block preferences increases in weighted voting games.

Note that *none* of the above enhancements in the probability of majority rule equilibria applies under the classical non-existence results with smooth preferences, since such equilibria generically do not obtain.

4 Non uniform city block preference profiles

Proposition 1 depends not only on agents having city block preferences but also on them having a *uniform* city block preference profile. When agents differ in the weights they attach to each dimension, conditions for equilibrium are more severe. In this case a non-median voter i's gradient vector, evaluated at $y_j = 0$ is proportional to:

$$[\alpha_{i,1} sgn(x_{i,1}), \alpha_{i,2} sgn(x_{i,2}), ..., \alpha_{i,3} sgn(x_{i,m})]$$

In the case where there is a single satiated voter at the DDM, we can apply Plott's theorem for necessary conditions for MRE existence. In this instance since each voter's gradient vector depends on the m scalars, $(\alpha_{i,j})_{j=1,2,...,m}$, the space of all agents' gradients is given by $\mathbb{R}^m$. However, the set of profiles of α vectors in $\mathbb{R}^m$ that do not satisfy Plott's conditions is open and dense and hence we return to a problem of generic non-existence of equilibria (with Plott points). The essential problem in this setting is that agents with ideals in opposite orthants are *not* necessarily opposed, in the sense of having opposing gradients.

Recall we noted equilibrium is more likely under city block preferences for two reasons. First, agents median on some dimension have kinked indifference curves and accept, as a result, changes in the status quo in fewer directions than agents with smooth indifference curves (at the equilibrium point). Second, parallel indifference curves render relations of opposition much more common. Relaxing the uniformity constraint on agent weights removes the second but not the first of these effects. The result is that, while equilibrium is much less likely with non-uniform than with uniform city block preferences, it is not generically absent in dimensions greater than 1 and can obtain in situations where multiple agents are median on different dimensions.

Given a random distribution of ideals, the chances of an MRE obtaining with 3 agents in $\mathbb{R}^2$ is $\frac{1}{9}$.[38] For an example of such a situation in $\mathbb{R}^3$, consider a case of three agents with ideal points that have a

[38] For 1/3 of the cases one player is at the DDM and no MRE exists; for the remaining 2/3 only 1/6 of have weights that guarantee an MRE.

sign structure given by: $s_1 = (0, 1, -1)$, $s_2 = (-1, 0, 1), s_3 = (1, -1, 0)$. Assume further that each agent i places weights on loss in these dimensions given by $(\alpha_{i,j})_{j=1,2,3}$ with $\frac{\alpha_{1,1}}{\alpha_{1,2}} > \frac{\alpha_{3,1}}{\alpha_{3,2}} > \frac{\alpha_{2,1}}{\alpha_{2,2}}$, $\frac{\alpha_{2,2}}{\alpha_{2,3}} > \frac{\alpha_{1,2}}{\alpha_{1,3}} > \frac{\alpha_{3,2}}{\alpha_{3,3}}$, and $\frac{\alpha_{3,3}}{\alpha_{3,1}} > \frac{\alpha_{2,3}}{\alpha_{2,1}} > \frac{\alpha_{1,3}}{\alpha_{1,1}}$.[39] In this case the origin is an MRE.[40]

Such possibilities exist then in two and three dimensions. They are however rare. And the problem of MRE existence becomes considerably more severe with either more agents or more dimensions. We can identify a stable MRE with 5 agents in 3 dimensions, but the conditions for this are very restrictive and our simulation work suggests that they obtain for a random polity with probability close to 0. Finally, it is easy to see analytically that there is generically no MRE in four dimensions even with only three agents. To see this note that in such cases there will always be some pair of agents for whom there are two dimensions on which neither of the two are median. (In particular, one of three agents is necessarily median on two dimensions, hence with ideals in general position, neither of the remaining two agents is median on those two dimensions). With random weights this pair (a majority) will find some trade-off mutually beneficial on these two dimensions.

The importance of these results lies in the fact that, if non-uniform weights are empirically appropriate, then we may discount our results based on the assumption of uniform weights outright; in the combined set of uniform and non-uniform preference profiles, the set of non-uniform profiles is generic.

5 Robustness

We have shown that strong results found for uniform city block preferences are substantially weakened for non-uniform city block preferences. Furthermore we know from classic results that, for strictly convex preferences that are arbitrarily close to city block preferences, no MRE exists. These results suggest a sharp discontinuity between city block preferences and preferences "close" to city block. Under conditions described in Proposition 1, the set of points that beat an equilibrium status quo is empty when preferences are city block but non-empty when they are not. This is of concern since a reliable prediction should not depend on a strong assumption about preferences. To counter this concern we now consider whether metrics that are "close" to city block produce outcomes

[39] These inequalities ensure that each player cares (relatively) more about the dimension that she is median on than do the other players. This makes it difficult for two players to strike a deal involving changes in either of the (two) dimensions on which they are median.

[40] A proof of this claim is available from the authors.

that are close to the outcomes produced by city block.

We use simulation techniques to examine this matter for the class of strongly spatial preferences based on a family of Minkowski distance functions.[41] Letting p denote the Minkowski exponent, we consider the sequence $p_n = 1 + \frac{1}{2}^{n-1}$. The first element in this sequence, p_1 corresponds to the Euclidean case ($p_1 = 2$) and the city block metric ($p = 1$) obtains as the limit of the sequence ($\lim_{n \to \infty} p_n = 1$).We examine settings in which the preferences of three players are given by the Minkowski norm and player ideal points are selected randomly from unit hypercubes of dimensions 2, 3 and 4. For each element in p_n we estimate the measure of the set of points in the hypercube that are majority preferred to the DDM. We know from Proposition 1 that in the three player case, the measure of the set of majority preferred points is 0 in the limit case. The question now is whether the measure approaches 0 as p approaches 1.

The results in Figure 3 suggest that it does: the measure of the set of points that beat the DDM approaches 0 as the Minkowski exponent approaches unity: when preferences are nearly city block the DDM is "nearly" in equilibrium.

In some contexts it is reasonable to question whether it is helpful to talk of equilibria that "nearly" exist. In some multistage games, for example, small initial policy changes can lead to larger subsequent changes. However, a substantial literature suggests that, in some settings, points that can be beaten only by a small set of similar points, we call these "$\varepsilon-$equilibrium points," may be reasonable policy predictions. The arguments in this literature focus on constraints on the set of feasible outcomes, or on various constraints on human behavior and cognition.

The set of feasible outcomes may well be a finite subset of the real space within which agent preferences are described. For example, actual policy outputs may be denominated in discrete units (dollars, space shuttle launches, legalization or not of gay marriage, invasion or not of a foreign country) rather than being points in a real space. In such settings, for arbitrarily small deviations from city block preferences, no majority preferred feasible alternative will exist to the DDM, which in our terms will be an $\varepsilon-$equilibrium arising from the *exogenously* finite nature of the set of feasible outcomes. Other restrictions on the set of

[41] The Minowski distance of order p between two points, x and y is given by $d(x, y|p) = (\sum_{i=1}^{m} |x_i - y_i|^p)^{\frac{1}{p}}$ for $p \geq 1$. The Euclidean and city block metrics are special cases with $p = 2$ and $p = 1$ respecitvely. The $\infty-$norm distance between two stimuli is the distance between them on the dimension on which they are farthest apart.

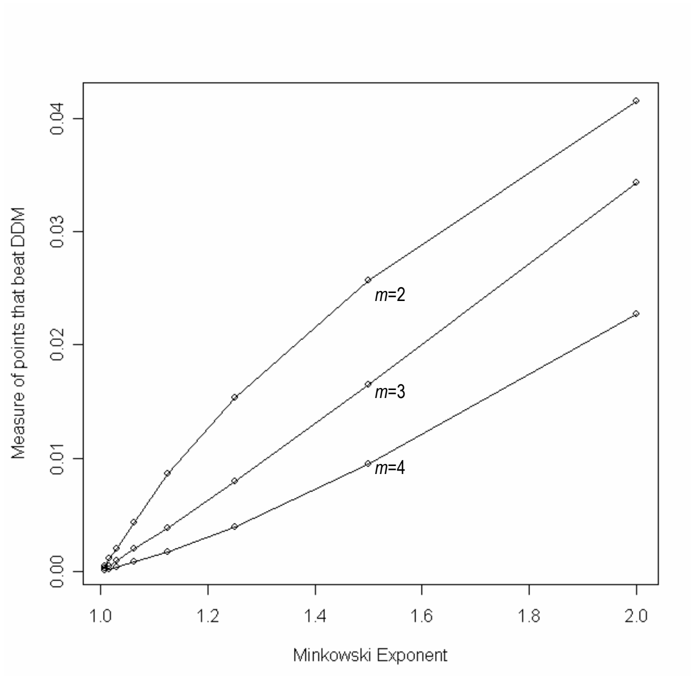

Figure 3: The figure shows the measure of the set of points that are majority preferred to the DDM when ideal points of a three member committee are drawn randomly from a hypercube of dimensions $2-4$. Results for each point are based on simulations with 2000 rival policies in each of 2000 randomly drawn polities. Results reported for Minkowski metrics $\theta(n) = 1 + \frac{1}{2}^{n-1}$ for $n \in \{1, 2, ...8\}$.

feasible outcomes may arise *endogenously* within the process of political competition itself. Thus Koehler, for example, considers the possibility that proposals are only made from the set of ideal points of committee members.[42] Laver and Shepsle model government formation in terms of a finite set of strategically credible policy proposals, comprising only those proposals that can be underwritten by an allocation of policy jurisdictions between cabinet ministers with known ideal points.[43] Other rationales for restricting theoretical attention to a finite set of potential outcomes arise from restrictions on player cognition. McKelvey, for example, suggested that real humans cannot make arbitrarily fine distinctions between policy positions, inducing *de facto* indifference over

[42] David H. Koehler, 'Convergence and Restricted Preference Maximizing under Simple Majority Rule: Results from a Computer Simulation of Committee Choice in Two-Dimensional Space'; American Political Science Review, 95 (2001), 155-67.

[43] Laver and Shepsle, *Making and Breaking Governments*.

a set of theoretically distinct outcomes.[44] Other behavioral arguments emphasize transaction costs arising from changes in the *status quo.* Classically these costs are measured on a utility scale [45] although other work considers "minimal spatial distances" that are required as a result of transaction costs.[46]

While much of the work noted above aims to determine the minimum ε for which an equilibrium exists,[47] or to consider how the set of stable points varies with changes in ε,[48] the underlying rationale has clear implications for our results. If for some family of preferences the measure of the set of points that beats the DDM is monotonic and continuous in preferences around the city block norm, then, for *arbitrarily small* ε, if the DDM is an equilibrium point for city block preferences it is also an ε-equilibrium for some non-city block preferences that are arbitrarily close to city block preferences. If there are barriers to trivial changes in policy, then the implication for our results is that the empirical relevance of Proposition 1 depends not on whether agents have city block preferences but rather on how closely their preferences approximate city block.[49]

6 Conclusion

For a very general class of smooth preference profiles, long-standing results show that a majority rule equilibrium fails to exist whenever policy choices are defined in a space of dimension greater than one. However, we have shown here that if agents have a uniform city block preference profile the non-existence of equilibrium is not generic: an equilibrium is possible in spaces with an arbitrarily large number of dimensions and "likely" in situations with relatively few dimensions and agents. These results extend in part to non-uniform city block preference profiles, but

[44] McKelvey, Intransitivities in multidimensional voting models.

[45] Lloyd S. Shapley and Martin Shubik, 'Quasi-cores in a monetary economy with nonconvex preferences'; *Econometrica*, 34 (1966), 805-827.

[46] Which of these notions of distance is used matters substantively: a small deviation from city block may produce a set of points that beat the DDM that, though small in measure, includes points that are nonethless distant from the DDM. As an example consider two ideals point in $\mathbb{R}^2$, $(1,0)$ and $(0,1)$. Now consider the set $P = \{(x,x) : x \in (0,1)\}$. With Minkowski exponent $p = 1$, no point in P is preferred by both players to the origin. But for any $p \in (1,2)$, all points in P are preferred to the origin.

[47] Thomas Bräuninger, 'Stability in Spatial Voting Games with Restricted Preference Maximizing'; *Journal of Theoretical Politics,* 19 (2007) 173-191.

[48] Koehler, Convergence adn Restricted Preference Maximizing.

[49] Note that a similar argument can be made for small deviations from Plott conditions.

only for spaces of dimensionality no greater than three and only under more restrictive conditions. Hence, city block preference profiles, though "close" to a smooth preference profile for which generic non-existence of equilibrium obtains, are different in crucial ways.

These differences allow analysts to predict an outcome—the dimension-by-dimension median—that is robust across a range of institutional settings. In addition they provide the building blocks for richer non-cooperative models. In many such models, equilibria of voting games (for example inside legislative parties, or houses of a legislature) are often required as inputs to some more complex process under investigation. Given the difficulty in predicting voting outcomes in policy spaces of more than one dimension, such models have often restricted their analysis, and resulting intuitions, to sparse and unrealistic one dimensional settings. Our results provide a rationale—and conditions—for using an analog of the median voter result as part of a larger model capable of analyzing higher dimensional settings.

The significance of our results depends ultimately on the extent to which real humans do in fact act as if they employ a city block metric in political settings, and do so uniformly. Experimental results suggest that, once endowed with a city block metric, voters choose the DDM.[50] But whether (or when) they have such preferences in the first place is an open empirical question. A long tradition of experimental research in cognitive science suggests real humans use city block metrics when analyzing similarities and differences between non-political objects. A much more limited and recent research program in political science finds the city block norm provides a model fit better then, or at least as good as, the more commonly used Euclidean norm when predicting political choices in multidimensional attitude spaces. But we know of no more general research that establishes whether or not people act as if they employ city block metrics when evaluating similarities and differences between potential political outcomes. We have shown however that the implications of such research are important. If cognitive science results on human perceptions of similarity and difference do indeed extend to perceptions of *political* stimuli, then the possibility of equilibrium under majority rule can provide a solid basis for policy predictions in multidimensional settings that are invariant across a wide variety of different institutional assumptions.

[50] Peter Halfpenny and Michael Taylor, 'An Experimental Study of Individual and Collective Decision-Making'; *British Journal of Political Science*, 3 (1973), 425-444.

7 Appendix: Proof of Proposition 1

Restatement of Proposition 1: *With a uniform city block preference profile and an odd number of agents, an MRE exists (and corresponds to the dimension by dimension median) if and only if a majority of ideal points lies in every rugged halfspace.*

Proof. (i) *Necessity*: To see that an MRE exists only if a majority of ideal points lies in every rugged halfspace assume, contrary to the claim, that 0 is an MRE but there exists some rugged half space $\mathcal{H}_{\mathcal{C}}$—corresponding to a collection $\mathcal{C}$ of orthants—that does not contain a majority of agent ideal points. Let $X_{\mathcal{C}}$ denote the elements of X corresponding to the orthants in $\mathcal{C}$ and let $X' = X \setminus X_{\mathcal{C}}$ denote the elements of X not in $X_{\mathcal{C}}$.

Note that since $\mathcal{H}_{\mathcal{C}}$ is a rugged halfspace, there exists an (arbitrarily short) vector v with $x.v > 0$ for each x in $X_{\mathcal{C}}$. Note furthermore, that for all $x' \in X'$, $x'.v < 0$ and hence $x'.(-v) > 0$.

We aim to show that all agents with ideal points in $\mathbb{R}^m \backslash \mathcal{H}_{\mathcal{C}}$ prefer $-v$ to 0; since a majority of agents have ideals in $\mathbb{R}^m \backslash \mathcal{H}_{\mathcal{C}}$, we have then that 0 is not an MRE.

Consider an arbitrary agent i with ideal point y in $\mathbb{R}^m \backslash \mathcal{H}_{\mathcal{C}}$ and let y' denote the element of X given by:

$$y' = \left(y'_j\right)_{j=1,2,\ldots,m} \text{ where } y'_j = \begin{cases} sgn(y_j) \text{ if } y_j \neq 0 \\ sgn(v_j) \text{ if } y_j = 0 \text{ and } v_j \neq 0 \\ 1 \text{ if } y_j = 0 \text{ and } v_j = 0 \end{cases}$$

Since y is an element of the closure of $f(y')$ but lies outside of $\mathcal{H}_{\mathcal{C}}$, we have that y' is in X' and in particular that $y'.v < 0$. For sufficiently short v (such that for $y_j \neq 0$, $\text{sgn}(y_j \pm v_j)$=$\text{sgn}(y_j)$), we have:

$$\begin{aligned} -v &\precsim_i 0 \\ &\Leftrightarrow \sum_{j=1}^{m} |y_j + v_j| \geq \sum_{j=1}^{m} |y_j| \\ &\Leftrightarrow \sum_{\{j:y_j\neq 0\}} sgn(y_j)(y_j + v_j) + \sum_{\{j:y_j=0\}} |v_j| \geq \sum_{\{j:y_j\neq 0\}} sgn(y_j)y_j \\ &\Leftrightarrow \sum_{\{j:y_j\neq 0\}} sgn(y_j)v_j + \sum_{\{j:y_j=0\}} |v_j| \geq 0 \end{aligned}$$

But since, by construction, $sgn(y_j) = y'_j$ whenever $y_j \neq 0$, and $|v_j| = sgn(v_j)v_j = y'_j v_j$ whenever $y_j = 0$, we have $-v \precsim_i 0 \Leftrightarrow y'.v \geq 0$, and conversely: $-v \succ_i 0 \Leftrightarrow y'.v < 0$. Since by construction, $y'.v < 0$ we then have that $-v$ is preferred to 0 for all i with ideals in $\mathbb{R}^m \backslash \mathcal{H}_{\mathcal{C}}$.

(ii) *Sufficiency*: To see that an MRE exists if a majority of ideal points lies in every rugged halfspace assume, contrary to the claim, that a majority lies in every rugged halfspace but that a movement $v \neq 0$ is preferred by some majority. Assume without loss of generality that for no $x \in X$, $x.v = 0$ (to see that this is without loss of generality note that since each agent's preferred-to set is open, the intersection of these preferred to sets is also open and hence the set of points that beat 0 is open; hence if $x.v = 0$ we can freely select a v' arbitrarily close to v that also beats 0 but for which $x.v \neq 0$). Since for no $x \in X$, $x.v = 0$, there exists a collection of 2^{m-1} orthants, $\mathcal{C}$, and a corresponding subset of X, $X_{\mathcal{C}}$ for which $x.v < 0$ for all $x \in X_{\mathcal{C}}$. Since for this collection $x.(-v) > 0$, the closure of the union of the elements of $\mathcal{C}$ (or, equivalently, the union of the closure of these orthants), $\mathcal{H}_{\mathcal{C}}$, constitutes a rugged half space and hence, by assumption contains a majority of elements.

We now show that since $x.v < 0$, $v \prec_i 0$ for all i with ideals in the closure of each orthant in $\mathcal{C}$. From a similar argument to that given in part (i), for sufficiently small v we have:

$$\begin{aligned} & v \succsim_i 0 \\ & \quad \Leftrightarrow \sum_{j=1}^{m} |y_j - v_j| \leq \sum_{j=1}^{m} |y_j| \\ & \quad \Leftrightarrow \sum_{\{j: y_j \neq 0\}} sgn(y_j)(y_j - v_j) + \sum_{\{j: y_j = 0\}} |v_j| \leq \sum_{\{j: y_j \neq 0\}} sgn(y_j) y_j \\ & \quad \Leftrightarrow \sum_{\{j: y_j \neq 0\}} sgn(y_j) v_j - \sum_{\{j: y_j = 0\}} |v_j| \geq 0 \end{aligned}$$

If $y \in clos(f(x))$, then we have $\text{sgn}(y_j) = x_j$ whenever $\text{sgn}(y_j) \neq 0$; in cases where $\text{sgn}(y_j) = 0$, we have $|v_j| = x_j v_j$ if $\text{sgn}(x_j) = \text{sgn}(v_j)$ and $|v_j| = -x_j v_j$ otherwise. Hence

$$\sum_{\{j: y_j \neq 0\}} sgn(y_j) v_j - \sum_{\{j: y_j = 0\}} |v_j| = x.v - 2 \sum_{\{j: y_j = 0, sgn(x_j) = sgn(v_j)\}} x_j v_j$$

and so we have:

$$\begin{aligned} & v \succsim_i 0 \\ & \quad \Leftrightarrow x.v - 2 \sum_{\{j: y_j = 0, sgn(x_j) = sgn(v_j)\}} x_j v_j \geq 0 \\ & \quad \Leftrightarrow x.v \geq 2 \sum_{\{j: y_j = 0, sgn(x_j) = sgn(v_j)\}} x_j v_j \end{aligned}$$

and hence, the converse:

$$v \prec_i 0 \Leftrightarrow x.v < 2 \sum_{\{j: y_j = 0, sgn(x_j) = sgn(v_j)\}} x_j v_j$$

Since the term $2 \sum_{\{j: y_j = 0, sgn(x_j) = sgn(v_j)\}} x_j v_j$ is positive we have $x.v < 0 \Rightarrow x.v < 2 \sum_{\{j: y_j = 0, sgn(x_j) = sgn(v_j)\}} x_j v_j$ and hence $x.v < 0 \Rightarrow v \prec_i 0$.

But this implies that every agent i with ideal in the rugged half space $\mathcal{H}_{\mathcal{C}}$, opposes the movement in direction v. Since this group constitutes a majority, 0 cannot be beaten by v. This establishes sufficiency. ■