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Existence of a multicameral core

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Abstract When a single group uses majority rule to select a set of policies from an n-dimensional compact and convex set, a core generally exists if and only if $n = 1$. Finding analogous conditions for core existence when an n-dimensional action requires agreement from m groups has been an open problem. This paper provides a solution to this problem by establishing sufficient conditions for core existence and characterizing the location and dimensionality of the core for settings in which voters have Euclidean preferences. The conditions establish that a core may exist in any number of dimensions whenever $n \leq m$ as long as there is sufficient preference homogeneity within groups and heterogeneity between groups. With $m > 1$ the core is however generically empty for $n > \frac{4m-2}{3}$. These results provide a generalization of the median voter theorem and of non-existence results for contexts of concern to students of multiparty negotiation, comparative politics and international relations.

1 Introduction

A fundamental result from the spatial theory of voting is that simple majority voting over multidimensional issue spaces typically admits no political equilibrium (Plott 1967; Schofield 1978). A consequence is that theories that rely on the spatial model cannot make predictions on the outcomes of voting processes without either eliminating the complexity of the issues being voted upon or imposing procedural constraints that prevent some majority-preferred outcomes from being implemented. In practice both types of restriction are commonly applied.

The problem of determining whether or not a political equilibrium—the "core"—exists for the more general class of decision rules in which for a decision to be made

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or a bill to be passed, it must be acceptable to m groups of actors, with each group using majority rule internally, remains open.[1] This more general problem has multiple applications in the study of comparative politics and international relations. Bicameral institutions require that two houses agree simultaneously on a bill for it to pass. Corporatist institutions search for agreements that are simultaneously acceptable to business interests, labor and government. Consociational institutions designed for ethnically divided societies often require multiple majorities also: the Northern Irish assembly requires the consent of a majority of nationalists and a majority of unionists simultaneously for the passage of a bill; the Belgian constitution stipulates that modifying the limits of the linguistic regions requires a law adopted by majority vote in each linguistic group in each House; the adoption of a European constitution may require ratification by as many as 27 member states. There are multiple applications in the study of economic policy coordination and trade policy agreements that have been of long-standing concern to students of international relations (Evans et al. 1993).

Given the range of applications, it is striking that one of the dominant approaches to the study of voting, the spatial model, has almost nothing to say on the problem. Only two pieces consider the general $m \times n$ case—that is, cases with m groups in n dimensions. The first, by Cox and McKelvey (1984), provides conditions for existence and uniqueness of "ham sandwich cuts" (defined below). Cox and McKelvey remark that a necessary, but not a sufficient, condition for an outcome to be in the core—that is, to be unbeatable by any other point—in m-cameral settings with Euclidean preferences is that it lies on all such cuts (if there are any). But they do not provide conditions for the existence of a core. Most recently Bräuninger (2003) has tackled questions of direct concern to this study, independently proving a version of Proposition 3 below. This work does not, however, establish necessary, or sufficient conditions for core existence.

More has been done on the special case of two groups. Hammond and Miller (1987) establish sufficient but not necessary conditions for the existence of core points in two dimensions. Tsebelis and Money (1997) concentrate on $2 \times n$ cases. They suggest that in such cases the core may be generically empty when there are more than two dimensions although they do not provide a proof for this claim.

There also exists theoretical work for general institutional environments. This work uses the "Nakamura number," $\mathcal{N}$, to locate a maximum number of dimensions in which the existence of a core may not be guaranteed [see for example, Nakamura (1979); Schofield (1984, 1999); Strnad (1985)]. If the dimensionality of policy space is less than or equal to $\mathcal{N} - 2$, then with weak restrictions on preferences, the core is not empty (Austen-Smith and Banks 2000, Theorem 5.3). However, if the dimensionality is greater than $\mathcal{N} - 2$, then profiles exist such that voting cycles may be constructed even when preferences are constrained to be strictly convex (Austen-Smith and Banks 2000, Theorem 5.4). The Nakamura number is typically the same for unicameral and multicameral majoritarian decision rules ($\mathcal{N} = 3$) and so the results that use the Nakamura number suggest that no matter how many groups there are, a core is

[1] Note that technically the core always exists but may be the empty set. Here we use the terms core non-existence and core emptiness interchangeably.

guaranteed only in one dimension.[2] Note that this result tells us that a core cannot be *guaranteed* in dimensions greater than 1, not that the core cannot exist in higher dimensions. Related work also finds dimensionality conditions under which cores are generically empty. Austen-Smith and Banks (2000) show for example that the core is generically empty when $n > 3(|N| - 3)/2$, where $|N|$ is the total number of agents (Austen-Smith and Banks 2000, Corollary 6.2). However, this is not a tight bound. In multicameral settings with dozens or hundreds of individuals in each group, this minimum number can be extremely large indeed.

Hence these more general results say little about multicameral politics. They confirm that a core may fail to exist for $n > 1$ and that it will certainly fail to exist when n is very large. However, while from the perspective of these more general results, the multicameral setting looks much like the unicameral setting, the pessimism associated with majority rule equilibria in multidimensional settings is not justified with regard to multi-group problems. Although the Nakamura number is insensitive to the number of groups, this does not imply that core existence does not depend on the number of groups. The reason is this: present work simply places upper and lower bounds on the possibility of core existence. For settings with large numbers of voters, the range between these lower and upper bounds is very large indeed and can include all cases of substantive interest. Within this range this work is silent: we do not know under what conditions a core may or may not exist.

Hence there exists a wide range of cases of importance to the study of comparative politics and international relations in which multiple groups vote over agreements. Yet this class of cases has been largely ignored by formal theory, which, to date has focused primarily on voting in single houses.

This study responds to this problem by providing extensions to multi-group voting situations of two fundamental results from the spatial model. Specifically I find that for Euclidean preferences and $m > 1$:

- if $n > \frac{4m-2}{3}$ then the core is empty: in this case, decisive coalitions can always form involving individuals from different group that can overturn any *status quo*, provided that there is some minimal degree of heterogeneity *within* groups.
- if there are at least as many groups as there are policy dimensions, then decisive coalitions cannot always form across groups and a non-empty core may exist that lies on some central "contract space" between the groups.

By providing sufficient conditions for existence or non-existence of the core in $m \times n$ cases this analysis provides results analogous to the much-used median voter theorem

[2] The Nakamura number, $\mathcal{N}$, is "the cardinality of the smallest set of winning coalitions with the property that the intersection of its members is empty" (see for example, Schofield 1978). In simple majority rule games the Nakamura number is 3, except for the special case where there are exactly four players, in which case the Nakamura number is 4. If there is a single player in a group, then, by convention, the Nakamura number is ∞. See Schofield (1978, 1984, 1985, 1999) and Austen-Smith and Banks (2000). Letting $\mathcal{N}^j$ denote the Nakamura number for a unicameral game for group j, then the Nakamura number for a multicameral game is $\mathcal{N} = \max\{\mathcal{N}^j\}_{j=1,\ldots,m}$. In general then, *no matter how many groups there are*, we can expect the Nakamura number for multicameral games to be 3 (although it may be 4 if there is some group with only four members and it may be ∞ if there is some group with only one member).

(Black 1958) and the more pessimistic results on the emptiness of the majority rule core (such as those in Plott 1967; Schofield 1984) for a more general class of games.[3]

These new results provide a basis for qualitative statements about the possibility of core existence and the form that the core takes even in high dimensions.

2 Preliminaries

2.1 Model

Let $G = \{G^j\}_{j=1,2,\ldots,m}$ denote a collection of m groups of voters. The set of all voters is given by $N = \cup_{j=1}^{m} G^j$. We assume that there is more than one voter in each group G^j and that the number of voters in each group is odd.

The set of feasible policies available to voters is a compact and convex subset, X, of $\mathbb{R}^n$. We assume that X is of full dimension.

Each individual $i \in N$, has "Euclidean" preferences representable by a utility function, $u^i : X \to \mathbb{R}^1$, given by $u^i(x) = \phi_i(||p^i - x||)$, where $||.||$ denotes the Euclidean norm, $p^i \in X$ represents an "ideal point" and ϕ_i is any strictly monotone decreasing function. Note that the set of points that individual i "prefers" to a given point x^*, is given by the convex set $\{x \in X : ||p^i - x|| < ||p^i - x^*||\}$.

Let $\mathcal{P} = X^{|N|}$ denote the set of all $|N|$-tuples of ideal points and let $p = \{p^i\}_{i \in N}$ denote a typical element of $\mathcal{P}$. Let $P(p) = \{P^j\}_{j=1,2,\ldots,m}$ denote the partitioning of p that places p^i in P^j if and only if $i \in G^j$. Hence P inherits from G the properties that there is more than one ideal point in each P^j and that $|P^j|$ is odd for all j.

We consider an aggregation rule f, that maps from $\mathcal{P}$ to the set of all complete and reflexive binary relations on X, given by

$$xf(p)y \Leftrightarrow \exists C \subseteq N \text{ s.t. } \begin{array}{ll} \text{(i)} & u_i(x) \geq u_i(y) \forall i \in C \\ \text{(ii)} & |C \cap G^j| \geq \frac{|G^j|+1}{2} \text{ for some } G^j \in G \end{array}$$

Informally, the multicameral rule states that "x is weakly preferred to y if and only if there exists a set of voters that weakly prefer x to y and that constitute a majority of some group." Note that this rule can also be written more compactly as

$$xf(p)y \Leftrightarrow \exists G^j \in G : |\{i \in G^j : u_i(x) \geq u_i(y)\}| \geq \frac{|G^j| + 1}{2}$$

This has the interpretation: "x is weakly preferred to y if and only if a majority of some group weakly prefers x to y." We emphasize however that although this statement is perhaps more intuitive, the relation between alternatives is correctly seen in this model as a function of the preferences of individuals and coalitions of individuals and not as a function of the preferences of the groups.

[3] Although the preferences considered in this paper are less general than those examined in Plott (1967) and Schofield (1984).

The asymmetric part of f is written as f', with

$$xf'(p)y \Leftrightarrow \exists C \subseteq N \text{ s.t. } \begin{array}{ll} \text{(i)} & u_i(x) > u_i(y) \forall i \in C \\ \text{(ii)} & |C \cap G^j| \geq \frac{|G^j|+1}{2} \forall G^j \in G \end{array}$$

Informally: "*x beats y* if and only if there is a coalition that strictly prefers x to y and that includes a majority in every group," or, more simply, "if and only if a majority in every group prefers x to y." Like the simple majority decision rule, the multicameral rule is *monotonic* and *proper*, but unlike the simple majority rule (with odd numbers of players) it is not *strong*.[4]

The "*core,*" C, for f in X is the set of "unbeatable" outcomes; that is, $C(f(p)) = \{x \in X : \forall y \in X, xf(p)y\}$ (Austen-Smith and Banks 2000, Definition 4.5; McKelvey and Schofield 1987, p. 924). Hence under the multicameral aggregation rule, a point, x, is in the core if it is impossible to find a coalition that contains a majority of each group that strictly prefers some other point to that point. We say that the core "exists" if $C(f(p)) \neq \varnothing$.

2.2 Notation and definitions

Co($\mathcal{B}$), Aff($\mathcal{B}$), $\overline{\mathcal{B}}$ and Int($\mathcal{B}$) are used respectively to denote the convex hull, the affine hull, the closure and the interior of a set $\mathcal{B}$. The interior of a convex set $\mathcal{B}$ relative to Aff($\mathcal{B}$) is written $ri(\mathcal{B})$. A "*hyperplane*," L, in $\mathbb{R}^n$ is an $(n-1)$-dimensional affine manifold, defined as: $L(v, a) = \{x|x.v = a\}$ for $a \in \mathbb{R}^1$ and directional vector $v \in S^{n-1}$. A set of hyperplanes is considered "independent" if their directional vectors are linearly independent. The "*open half spaces*" of hyperplane, $L(v, a)$, are the convex sets $L^+ = L^+(v, a) = \{x|v.x > a\}$ and $L^- = L^-(v, a) = \{x|v.x < a\}$.

We are especially interested in hyperplanes that divide the collection of ideal points in particular ways. To describe such planes we make use of a group-specific function $\mu^j : 2^X \to [0, 1]$ that records the share of points in P^j in a given subset $\mathcal{B}$ of X : $\mu^j(\mathcal{B}) = \sum_{\{i \in G^j : p^i \in \mathcal{B}\}} \frac{1}{|P^j|}$.

Making use of this measure, a "*median hyperplane*" for a group G^j in $\mathbb{R}^n$ is a hyperplane, M, for which $\mu^j(\overline{M^+}) \geq 0.5$ and $\mu^j(\overline{M^-}) \geq 0.5$. From the assumption that groups contain an odd number of voters it follows that there is no set $\mathcal{B}$ such that $\mu^j(\mathcal{B}) = 0.5$ for any group G^j. The exclusion of this possibility allows us to define a median hyperplane alternatively as an $(n-1)$-dimensional hyperplane, M, with $\mu^j(M^+) < 0.5$ and $\mu^j(M^-) < 0.5$. We say that a halfspace, $\overline{M^{+j}}$, is a "*majority half space*", for group G^j, if $\mu^j(\overline{M^{+j}}) > 0.5$.

[4] A set of voters, $C \subseteq N$ is *decisive* if for every profile of preferences and for all pairs x, y if every member of C strictly prefers x to y then x is strictly preferred to y. Under the multicameral rule examined here, a set of voters $C \subseteq N$, is *decisive* if $|C \cap G^j| \geq \frac{|G^j|+1}{2}$ for all $G^j \in G$. It can be seen immediately that the set of decisive coalitions under the multicameral rule is *monotonic* (if C is decisive so is any superset of C) and *proper* (if C is decisive then $N \backslash C$ is not). A rule is *strong* if for every coalition, C, either C or its complement is decisive. Note that if C contains a majority of some but not all groups then so too does its complement and hence neither coalition is decisive under the multicameral rule.

A "*ham sandwich cut*" is a plane that is median to n groups. A ham sandwich cut through the collection $P(p)$ leaves half the points in each group on either side of it. Hence H is a ham sandwich cut if and only if $\mu^j(H^+) < 0.5$ and $\mu^j(H^-) < 0.5$ for all j. Existence of a ham sandwich cut for n groups in n dimensions is guaranteed by the generalized ham sandwich theorem for discrete distributions (Cox and McKelvey 1984; Hill 1988).[5]

3 Core existence

We make use of four general features that follow from our assumptions regarding group sizes and individual preferences.

The first, given in Lemma 1, is that if a point x may be beaten by any point, x^*, then it may be beaten by all points in the interior of $Co(\{x, x^*\})$. This follows directly from the fact that players have convex preferences.

Lemma 1 *If $x^* f'(p) x$, then $z f'(p) x \forall z \in Co(\{x, x^*\})$.*

The second, following from the fact that there is an odd number of voters in each group, guarantees that for each direction there is only one median plane for each group. A consequence of this result is that for any $v \in S^{n-1}$ we can unambiguously refer to a median plane for a group, G^j, by its directional vector, writing $M_v^j = M^j(v, a_j(v))$.

Lemma 2 *For each group and for each direction there exists a unique median hyperplane.*

The third feature, given in Lemma 3, is that whenever points do not lie on a median plane we can readily identify other points that majorities prefer to them. In particular an orthogonal projection of a given point onto a median plane of group G^j is preferred by a majority of voters in G^j. One implication, illustrated in Fig. 1, is that any core point must lie on all ham sandwich cuts.

Lemma 3 *If a point, x, does not lie on some median hyperplane of a group then some point on that hyperplane is preferred by some majority of voters in that group.*

Lemmas 1 and 3 together imply that given a median hyperplane $M(v, a)$, and point x, with $a < x.v$, a majority prefers not only some point on $M(v, a')$ to x but also some point on any parallel hyperplane, $L(v, a')$, between x and M (with $a < a' < x.v$).

The final feature, given in Lemma 4, is that if we know that a majority prefers one point to another then we can infer information about the location of a majority half space.

Lemma 4 *For two distinct points x and x', choose $v \in S^{n-1}$ and $\varepsilon > 0$, such that $x' = x + \varepsilon v$. If a majority of group j prefers x' to x then $M_v^j \subset L^+(v, x.v)$ and $\mu^j(L^+(v, x.v)) > 0.5$.*

[5] The original ham sandwich theorem for continuous measures derives from Steinhaus (1945).

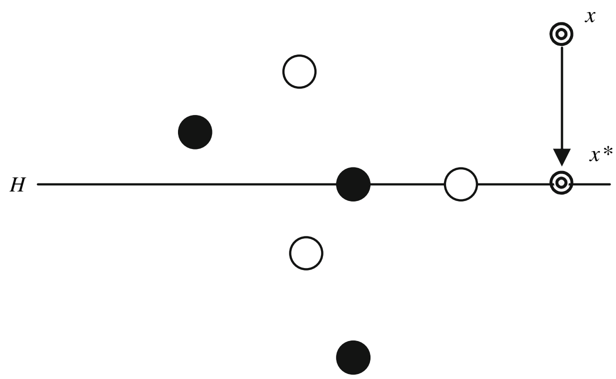


Fig. 1 The line H is median to the set of *black points* and to the set of *white points*. The point x lies off of H and so can be beaten by x^*, an orthogonal projection of x onto H

Our first major result states that no matter how many dimensions there are, a core may exist provided there are as many groups. The proof of the proposition is constructive and helps us to locate particular core points for a given distribution of voter ideal points.

Proposition 1 *Assume* $m = n$ *and consider some collection of ideal points* $p \in \mathcal{P}$. *Let* $H(u, a)$ *denote a ham sandwich cut through* $P(p)$ *and define the set of convex sets* $\mathcal{B} = \{\mathcal{B}^j\}_{j=1,2,\ldots,m}$ *where* $\mathcal{B}^j = Co(P^j) \cap H$. *Then* $C(f(p)) \neq \varnothing$ *if* $\mathcal{B}$ *is strongly separable on* H.

This Proposition makes use of a notion of strong separability—a strengthening of the idea of separable sets used in Cox and McKelvey (1984). We say that a collection of m sets $\mathcal{B} = \{\mathcal{B}^j\}_{j=1,2,\ldots,m}$ in $\mathbb{R}^n$ is "*separable*" if for all $\mathcal{B}^j$ in $\mathcal{B}$, there exists some $(n-1)$-hyperplane, L^j, with $\mathcal{B}^j \subseteq L^{j+}$ and $\mathcal{B}\backslash\mathcal{B}^j \subseteq L^{j-}$. We say that a collection of m sets, $\mathcal{B} = \{\mathcal{B}^j\}_{j=1,2,\ldots,m}$ in $\mathbb{R}^n$ is "*strongly separable*" if for all $\mathcal{B}^j$ in $\mathcal{B}$, there exists an $(n-1)$-hyperplane, L^j, with each sub-collection of $m-1$ hyperplanes in $\{L^j\}_{j=1,2,\ldots,m}$ independent, such that:

1. $\mathcal{B}^j \subseteq L^{j+}$ and $\mathcal{B}\backslash\mathcal{B}^j \subseteq L^{j-}$
2. $\text{Int}\left(\bigcap_{j=1}^{m} \overline{L^{j+}}\right) \neq \varnothing$.

Similarly we say that m sets, $\mathcal{B} = \{\mathcal{B}^j\}_{j=1,2,\ldots,m}$ are strongly separable on an affine set, H, if the intersections of the sets with H are non-empty and are strongly separable on the affine space spanned by H.

Figure 2 illustrates the notions of separability and strong separability, showing how strong separability on $\mathbb{R}^n$ is indeed a stronger condition than separability.[6]

To prove the proposition we require two further Lemmas. The first, Lemma 5, illustrated in Fig. 3, establishes that when a $(n-1)$-hyperplane cuts through an

[6] Strong separability on $\mathbb{R}^n$ implies separability but separabilty does not imply strong separability. Also, whereas for any $m > 1$ and $n > 1$, some arrangement of m groups may be separable in $\mathbb{R}^n$, the independence condition implies that m groups may only be strongly separable in $\mathbb{R}^n$ if $m \leq n + 1$.

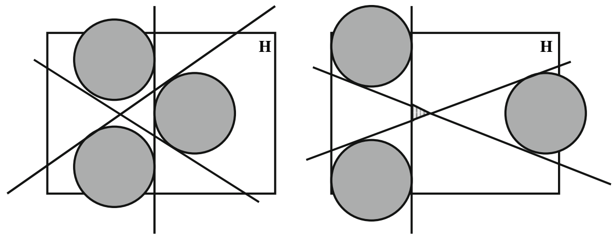
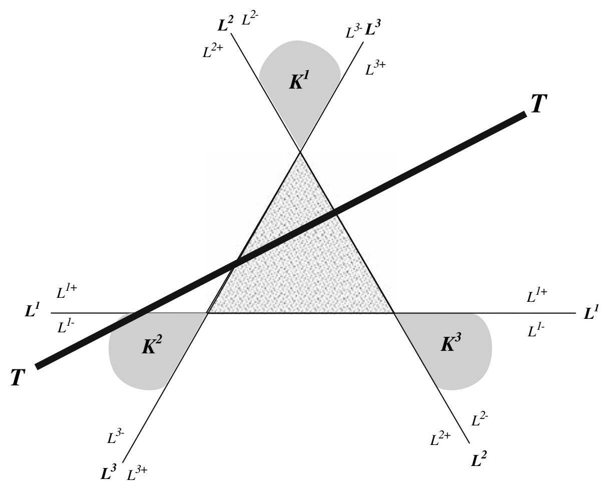


Fig. 2 The *panels* above show three discs on a plane H. The first collection of discs is separable but not strongly separable on H. The second is strongly separable on H: the hatched area in the *second panel* shows a non-empty set $\bigcap_{j=1}^{3} \overline{L^{j+}} \cap H$

Fig. 3 The hyperplane (*line*) T intersects with the interior of Δ. As a result, one of the three cones induced by Δ lies completely on either side of T, K_1 on one side and K_3 on another. K_2 does not lie completely on one side or the other of T



n-dimensional simplex in $\mathbb{R}^n$, it places one region of the space—an affine cone with a vertex of the simplex as its vertex—entirely on one side of the plane and another such region entirely on the other (note that although the term "simplex" is often used to describe sets of dimension $(n-1)$ in $\mathbb{R}^n$, our concern in this lemma is with full dimensional simplices).

Lemma 5 *Consider some non-empty n dimensional simplex Δ in $\mathbb{R}^n$ bounded by an arrangement of $n+1$ hyperplanes $\{L^j\}_{j=1,2,\ldots,n+1}$ and label the halfspaces of these hyperplanes such that $\Delta = \bigcap_{j=1}^{n+1} \overline{L^{j+}}$. Let $\mathcal{K}$ denote the collection of sets $\{K^j\}_{j=1,2,\ldots,n+1}$ where $K^j = \bigcap_{k\in\{1,2,\ldots,n+1\}\setminus j} L^{k-} \cap L^{j+}$. Then for any hyperplane L such that $L \cap \mathrm{Int}(\Delta) \neq \varnothing$, at least one element of $\mathcal{K}$ lies in L^+ and at least one other lies in L^-.*

Proof See Appendix. □

The second, Lemma 6, establishes that the intersection of two median planes of a group intersects with the convex hull of the ideal points of a group (see Fig. 4).

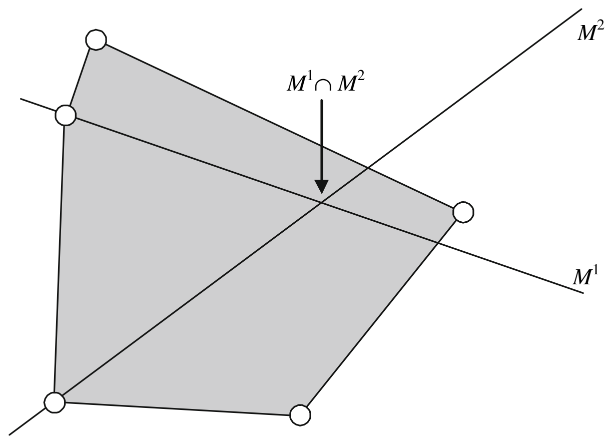


Fig. 4 The intersection of two Median Hyperplanes for a given group intersects with the convex hull of the ideal points of the voters in that group

Lemma 6 *Let M^1 and M^2 be two distinct median hyperplanes for a group G^j. Then $M^1 \cap M^2 \cap Co(P^j)$ is not empty.*

Proof Follows from Lemma 1 in Cox and McKelvey (1984), the compactness of X, and Lemma 2. □

With these two lemmas we turn to the proof of the main proposition.

Proof Note that H spans an affine space of dimension $n-1$. Furthermore, since H is median for all groups, $\mathcal{B}^j$ is non-empty for all j. Choose some arrangement of $(n-2)$-dimensional hyperplanes $\mathcal{L} = \{L^j\}_{j=1,\ldots,n}$ in H such that for all j, $\mathcal{B}^j \subseteq L^{j+}$ and $\mathcal{B} \backslash \mathcal{B}^j \subseteq L^{j-}$ and $\Delta = \bigcap_{j=1}^n \overline{L^{j+}} \cap H$ has a non-empty interior relative to H, where L^{j+} (L^{j-}) denotes the open upper (lower) half space of L^j relative to H. Existence of such an arrangement is guaranteed by the strong separability of $\mathcal{B}$ on H. Note that, by construction, $\mathcal{B}^j \subset \bigcap_{k \in \{1,2,\ldots,n\} \backslash j} L^{k-} \cap L^{j+} \cap H$. Finally, note that the set Δ is an $(n-1)$-dimensional simplex whose vertices are given by the n points $\{\bigcap_{k \in \{1,2,\ldots,n\} \backslash j} L^k \cap H\}_{j=1,\ldots,n}$. We now show in particular that $x^* \in \text{ri}(\Delta)$ implies $x^* \in C(f(p))$.

Assume to the contrary that for some $x^* \in \text{ri}(\Delta)$ we have $x^* \notin C(f(p))$ and in particular that there exists some point x' with $x' f' x^*$. The rival point x' may be written $x' = x^* + \varepsilon v$ with $\varepsilon > 0$ and $v \in S^{n-1}$. We know from Lemma 3 that, since x' is strictly preferred to x^* by a majority in all groups and H is a median plane for all groups, v and u must be linearly independent. Consider now the $(n-1)$-hyperplane, L_v, with directional vector v that passes through x^*. For each j let M_v^j denote the unique median hyperplane for group j in direction v. Existence of these hyperplanes is guaranteed by Lemma 2. Since x' beats x^* there lies a majority of ideal points from every group inside the open half space, L_v^+ of L_v and there exist corresponding majority halfspaces $\overline{M_v^{j+}}$ for each group G^j which are strict subsets of L_v^+ (Lemma 4). Now, from Lemma 6, for each group G^j, there exists some point w^j in $M_v^j \cap H$ with

$w^j \in \text{Co}(P^j)$. Since such a point is in both H and $\text{Co}(P^j)$ it lies in $\mathcal{B}^j$. Furthermore, since w^j is in $\overline{M_v^{j+}}$ and $\overline{M_v^{j+}}$ is a strict subset of L_v^+ then w^j is not in $\overline{L_v^-}$. Hence for every group j, some point in $\mathcal{B}^j$ does not lie in $\overline{L_v^-}$. We have then:

(∗) for no j, is $\mathcal{B}^j \subset \overline{L_v^-} \cap H$.

Now since $L_v \cap \text{ri}(\Delta) \neq \varnothing$ and v and u are linearly independent, $L_v \cap H$ is an $n-2$ dimensional hyperplane that intersects with $\text{ri}(\Delta)$. Applying Lemma 5 to the $(n-1)$-dimensional affine subspace H we have:

(∗∗) for some j, $\mathcal{B}^j \subset \overline{L_v^-} \cap H$.

But (∗∗) contradicts (∗) and hence no x' beats x^*. □

Proposition 1 provides good news for the possibility of core existence in high dimensions. Significantly, the condition of the Proposition requires neither separability of P nor strong separability of P on $\mathbb{R}^n$ (strong separability on an affine subspace of $\mathbb{R}^n$ is sufficient). And the Proposition is constructive insofar as it locates core points in the relative interior of Δ.

Figure 5 illustrates the proposition. The figure shows two three member groups in $\mathbb{R}^2$. The first panel demonstrates that $m = n$ in not sufficient for core existence: if the separability condition of the proposition is not fulfilled for any ham sandwich cut a core may fail to exist. The second panel emphasizes that the condition in Proposition 1, although a sufficient condition, is not a necessary and sufficient condition. The condition may fail to hold but a core point may still exist. The final panel illustrates the sufficiency condition. In this case the strong separability condition is satisfied for the unique ham sandwich cut and a core is guaranteed.

While Preposition 1 is constructive it does not fully characterize the core. Propositions 2 and 3 make further steps towards such a characterization.

Proposition 2 *The core is convex.*

Proposition 3 *For $m = n$ if a core exists then it is of dimensionality of no more than $n-1$.*

Proof From Lemma 3, if a core exists, all points in the core lie on every $(n-1)$-dimensional ham sandwich cut. From the ham sandwich theorem at least one such cut exists. Finally, the intersection of these cuts is of at most dimensionality $n-1$. □

In the $m = n$ case, then, the core is never full dimensional. In such cases when the conditions of Proposition 1 are satisfied there is always at least one "dimension of agreement" (orthogonal to the ham sandwich cut) between groups. In addition when the core exists there is also a "zone of disagreement" which lies on a ham sandwich cut: when this core is of dimension $n-1$ (as for example in the cases identified in Proposition 1) it is as a subset of the affine hull of the ideal points of a particular collection of players—one player from each group. Small changes in the locations of these ideals alter the location of the core.

I now turn to consider the case in which there are more groups than dimensions. From the definition of the multicameral aggregation rule it is easy to see that if a core

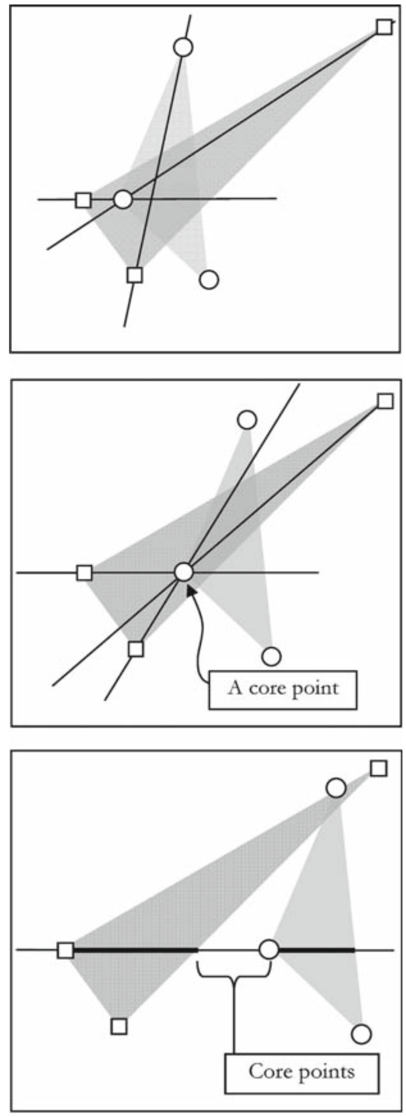


Fig. 5 A situation with $m = n = 2$. The ideal points of one group are marked with circles and those of the other with squares. Ham sandwich cuts are marked with lines. The separability condition from Proposition 1 is not satisfied in the first two panels. In the first no point lies on all ham sandwich cuts and so the core is empty. In the second there is a core point even though the separability condition is not fulfilled. In the third case the intersections of the ham sandwich cut and the convex hulls of player ideals are marked with thick lines. These sets are strongly separable on the ham sandwich cut and so the condition of Proposition 1 is fulfilled and hence there is a non-empty core

exists for some subcollection of groups then a core exists for the full collection of group. Hence, with $m > n$, if the conditions of Proposition 1 are satisfied for any subcollection of n groups, then a core also exists for the $m \times n$ case. Proposition 4 continues by showing, more constructively, that a "full dimensional" core may exist with $m > n$ under conditions that may be satisfied *even if* no core exists among any collection of n groups. To do so, Proposition 4 makes use of the idea of the "*yolk*" of a group, $\mathcal{Y}^j$, defined as the smallest hypersphere that intersects with all median hyperplanes of a collection of points P^j (Feld et al. 1988).

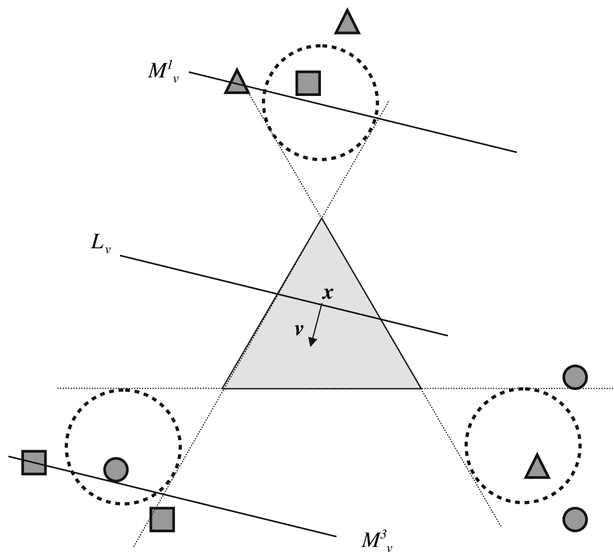


Fig. 6 Illustration of Proposition 4. There are three groups in $\mathbb{R}^2$, the *triangles*, the *circles*, and the *squares*. No bicameral core exists between any two groups but a tricameral core exists because the yolks of the three groups are strongly separable. Any movement from a point such as x in Δ is always opposed by a majority in at least one group. In this case two triangles oppose the move as seen from the fact that M_v^1 lies in L_v^-

Proposition 4 *Assume $m = n+1$ and consider some $p \in \mathcal{P}$. Let $\mathcal{Y} = \{\mathcal{Y}^j\}_{j=1,2,\ldots,n+1}$ denote a collection of $n+1$ yolks corresponding to the collection $P(p)$. If $\mathcal{Y}$ is strongly separable on $\mathbb{R}^n$ by some collection of $n+1$ hyperplanes $\mathcal{L} = \{L^1, \ldots, L^{n+1}\}$, with each collection of n hyperplanes independent, such that $\mathcal{Y}^j \subset \bigcap_{k\in\{1,2,\ldots,n+1\}\setminus j} L^{k-} \cap L^{j+}$ for $j = 1, 2, \ldots, n+1$ and $\Delta = \bigcap_{j=1}^{n+1} \overline{L^{j+}} \neq \emptyset$, then every point x in the interior of Δ is in the core.*

Proof See Appendix. □

Figure 6, which illustrates Proposition 4 also shows the yolks of three groups in $\mathbb{R}^2$. In this example it can be shown that there is no bicameral core between any pair of the groups. The yolks of the three groups are strongly separable however and all points in the (shaded) simplex "between" the yolks are in the core. The intuition is that a move from any core point in direction v is necessarily a move away from a median hyperplane with normal v for at least one group.

From the preceding propositions we know that in some situations a core may exist and know when it may lie on an $(n-1)$-dimensional space or on an n-dimensional space. The proofs of the sufficient conditions for core existence have been constructive insofar as they have identified core points. And these points have been "centrist." The following proposition places upper bounds on the size of the core and confirms that in some sense *all* points in the core are centrist.

Proposition 5 *No point outside the convex hull of the yolks of all groups is in the core.*

Proof Note that the convex hull of $\mathcal{Y} = \{\mathcal{Y}^j\}_{j=1,2,\ldots,n+1}$ is compact. Consider some point x not in this convex hull. By the separating hyperplane theorem there exists some plane, $L_v = L(v, a)$ with $v.v = 1$ such that $x \in L_v^-$ and $\mathcal{Y} \subset L_v^+$. We now show that $x^* = x + (a - x.v)v$, (that is, the projection of x onto L_v), is preferred by a majority in each group to x.

Consider an arbitrary group j and let $M(v, a^j)$ denote the median plane for group j with directional vector v (existence of this median planes is guaranteed by Lemma 2). From the definition of the yolk there exists some $y^j \in \mathcal{Y}^j$ with $y^j.v = a^j$. However, since for all points $y \in \mathcal{Y}$, $y.v > a$ it follows that $a^j > a$. Let $x^{*j} = x + (a^j - x.v)v$ denote the orthogonal projection of x onto $M(v, a^j)$. Since $x^{*j} \in M(v, a^j)$, we know that x^{*j} is preferred by a majority in group j to x (Lemma 3). But since $a^j > a > x.v$ it is easy to check that x^* is a convex combination of x and x^{*j} (specifically, $x^* = \frac{a^j - a}{a^j - x.v}x + \frac{a - x.v}{a^j - x.v}x^{*j}$) and hence (from Lemma 1) that a majority of voters in group j prefers x^* to x. □

4 Non-existence of the core

We now have sufficient conditions for core existence in the $m \geq n$ case. Core points have been identified and a bound has been placed on the size and location of the set of core points. The sufficient conditions produced for the $m \geq n$ cases for core existence provide a counter to the pessimism that is associated with core existence in more than one dimension.

However, I conclude this article on a pessimistic note. The emptiness of the core in higher dimensions, familiar from the study of majority rule equilibrium, resurfaces in general $m \times n$ settings: the core is *generically* empty whenever the number of dimensions sufficiently exceeds the number of groups. In particular, generically, with $m > 1$, the core is empty if $n > \frac{4m-2}{3}$. This result is stated and proved as the final proposition, Proposition 6. The proposition implies that for $m < 5$, the core is empty whenever there are more dimensions than groups. The upper bound is somewhat looser for higher numbers of groups—hence for example, if $m = 7$ the proposition guarantees the emptiness of the core only if $n \geq 9$.

To establish genericity we demonstrate that the core is empty in sufficiently high dimensions whenever ideal points are in "general position." A collection of points in $\mathbb{R}^n$ is in "general position" if for any $k \leq n$, no more than k ideal points lie on any $k - 1$ dimensional affine manifold. Such profiles are generic under two distinct notions of genericity. The first common definition of genericity (used for example in Austen-Smith and Banks 2000; Saari 1997) is that a set is generic if it is open and dense. Since the set of profiles of ideal points that are *not* in general position is open (if p is in general position, then all profiles arbitrarily close to p are also in general position) and dense (if p is not in general position, then some other profile arbitrarily close to p is), the set of $|N|$-tupiles of ideal points in general position is generic and results that hold for ideals in general position, hold generically (within the class of Euclidean preference profiles).[7] Under another, related definition, a set is generic if its

[7] See Theorems 1 and 2 in Pontriagin (1952) for proofs of these openness and denseness claims, respectively.

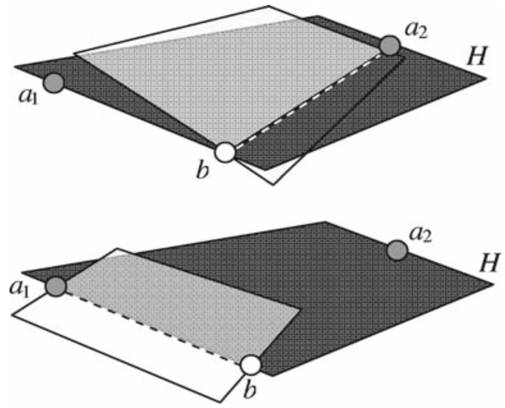


Fig. 7 H is a ham sandwich cut containing two points in A and one in B. Intersections of cuts in the neighborhood of H establish that core points must lie on the affine hull of $\{a_1, b\}$ and in the affine hull of $\{a_2, b\}$ and must therefore lie on $b \in B$

complement has zero (Lebesgue) measure; again under this definition existing results establish that the set of profiles in general position is generic (see Keerthi et al. 1992).

The "general position" assumption is important for this result. It is always possible for a core to exist in any number of dimensions when ideal points are not in general position, but in these cases the core results are always vulnerable to arbitrarily small perturbations in the location of ideals. As an example consider a situation with two three member groups in n dimensions, $n > 2$. Let the ideals of the first player in each group be given by $(-1, -1, \dots, -1)$, the ideal of the second by $(0, 0, \dots, 0)$ and the ideal of the third by $(1, 1, \dots, 1)$. It is clear in this case that the origin is a core point under majority rule for each group and also in the bicameral game. Yet as is known from Plott's results, a small perturbation of the ideals of the players into general position is sufficient to upset the existence of the unicameral core, and also, from Proposition 6, the bicameral core.

Proposition 6 *Assume $m \geq 2$ and $n > \frac{4m-2}{3}$. If ideal points are in general position, then the core is empty.*

The strategy of proof is to show that under the conditions of the proposition not all ham sandwich cuts intersect. The key step is to identify sets of ham sandwich cuts that intersect on lower dimensional sets and to show that these lower dimensional sets do not themselves have a positive intersection. The logic is demonstrated in Fig. 7 for the $n = 3$, $m = 2$ case. In the example shown in the figure we exploit the fact that $n > m$ to identify a ham sandwich cut that contains two points, a_1 and a_2 from group A (the shaded group) and one point, b, from group B (the white group). The ham sandwich cut, H, is the affine hull of $\{a_1, a_2, b\}$ and is marked in the figure. Slight perturbations of the ham sandwich cut yields other ham sandwich cuts that include $\{a_1, b\}$ but exclude a_2, and cuts that include $\{a_2, b\}$ but exclude a_1. The intersection of all of these cuts is the point $b \in B$. Thus if there is a core point it is b. In the same way a cut H' can be identified that contains two points in B and one point, a_3, in A and by the same reasoning we establish that the core must be a_3. But since $a_3 \neq b$ the core is empty. The proof is directly analogous for higher dimensional cases.

5 Conclusions

In this paper I have provided sufficient conditions for core non-emptiness under multicameral decision rules in multidimensional environments. By providing sufficient conditions for the generic emptiness of the core, I also provide necessary conditions for the existence of core points. These results give grounds for optimism regarding core non-emptiness in this important class of decision rules. The sufficient conditions for core non-emptiness used in Proposition 1 are also intuitively appealing—an equilibrium obtains when there is more disagreement between groups than within groups. In the absence of strong separability of groups, coalitions can form across groups and overturn any status quo. The results, while demonstrating the possibility of stable policy predictions in these settings, also provide grounds for caution for students of international relations and comparative politics who analyze decision making between multiple internally fragmented groups: in sufficiently complex decision making settings, multiple veto points generated by institutional arrangements such as consociational or bicameralism are *not* sufficient to guarantee an equilibrium.

The findings generalize fundamental results from the spatial model of voting, but there is room for further generalization along these lines. First, there remains indeterminacy about the possibility of core existence for cases where $m < n \leq \frac{4m-2}{3}$.[8] Core existence in these cases requires the positive intersection of an infinity of ham sandwich cuts; as yet no results rule out such a possibility, even for points in general position. Second, conditional upon $n \leq m$ I have identified sufficient but not necessary and sufficient conditions for core existence. Without such conditions we are hampered in quantifying the likelihood of observing a core and in locating all elements of the core when we turn to data. Third, the results in this paper have been developed only for the class of Euclidean preferences. While this class of preferences is consistent with those used in much theoretical and empirical research, more is needed to confirm that the results hold for more general classes of preferences.

Beyond such generalizations, however, the results from the model open up possibilities for more applied work. I close by highlighting two.

First, consider the link between relations between groups on the one hand and the salience of individual actors within groups on the other—a link of interest for example for the study of the impacts of foreign affairs on domestic politics. We have seen that in $n \times n$ settings, the core, if it exists, lies on the affine hull of the ideal points of n voters: one voter from each group. Hence the model identifies particular voters *within* groups that have a privileged position that is endogenous to *inter*-group interactions. In fact, these voters play a role similar to that of the median in one dimensional settings: they are median along the dimension on which groups most agree, a dimension orthogonal to the core.

Second, the identification of the core may provide a tool to solve extensive-form games designed to study more highly specified political interactions. This is the role that the median voter theorem has played in the modeling of policy choice: it has provided a building block for more complex games. The emptiness of the core in

[8] This interval contains positive integers only for $m \geq 5$.

higher-dimensional settings has limited the scope of this work, leading to frustration with the core as a solution concept. As a fix, scholars, hoping to make progress, either assume that political contention takes place over a single dimension or ignore the heterogeneity between groups. Both responses remove a key feature of multi-group politics—the possibility for coalitions to form *across* groups. Establishing that a non-empty core may exist for multicameral decision rules means that for students of multi-group politics, such frustration—as well as common responses to it—may no longer be justified.

6 Appendix

6.1 Lemmas

Proofs of Lemmas 1, 2, 3 and 4 and of Propositions 2 and 6 are available at http://www.columbia.edu/~mh2245/papers1/scw2008/

Lemma 7 *Consider some non-empty n-dimensional simplex* Δ *in* $\mathbb{R}^n$ *bounded by an arrangement of* $n+1$ *hyperplanes* $\{L^j\}_{j=1,2,\ldots,n+1}$ *and label the halfspaces of these hyperplanes such that* $\Delta = \bigcap_{j=1}^{n+1} \overline{L^{j+}}$. *Let* $\mathcal{K}$ *denote the collection of sets* $\{K^j\}_{j=1,2,\ldots,n+1}$ *where* $K^j = \bigcap_{k\in\{1,2,\ldots,n+1\}\setminus j} L^{k-} \cap L^{j+}$. *Then for any hyperplane* L *such that* $L \cap \text{Int}(\Delta) \neq \varnothing$, *at least one element of* $\mathcal{K}$ *lies in* L^+ *and at least one other lies in* L^-.

Proof Consider the set $\mathcal{Q} = \{q^j\}_{j=1,2,\ldots,n+1}$ with typical element $q^j = \{\bigcap_{k\in\{1,2,\ldots,n+1\}\setminus j} L^k\}_{j=1,2,\ldots,n+1}$. The points in $\mathcal{Q}$ are the vertices of Δ and are affinely independent. Every point y in $\mathbb{R}^n$ can be uniquely written $y = \sum_{j=1}^{n+1} \lambda_j q^j$, subject to $\sum \lambda_j = 1$. (the scalars $\{\lambda_j\}$ are the *barycentric coordinates* of y with respect to the vertex set $\mathcal{Q}$). The $n+1$ hyperplanes $\{L^j\}_{j=1,2,\ldots,n+1}$ divide $\mathbb{R}^n$ into $2^{n+1}-1$ open regions determined by the signs of their barycentric coordinates. Points in the interior of Δ are strict convex combinations of $\{q^j\}_{j=1,2,\ldots,n+1}$ and hence have $\lambda_j > 0$ for all j. Points in K^j have $\lambda_j > 0$ and $\lambda_i < 0$ for $i \in \{1, 2, \ldots, n+1\}\setminus j$.

Now, consider some hyperplane $L(u, a)$ that intersects the interior of Δ. Assume, without loss of generality, that $a = 0$. Define $v_j = q^j.u \in \mathbb{R}^1$ and note that $\min\{v_j\} \neq \max\{v_j\}$ as otherwise the points in $\mathcal{Q}$ would be affinely dependent (v_j is a measure of how far a given vertex q^j is from $L(u, a)$).

A point x with barycentric coordinates $\{\lambda^j\}$ is in L if $x.u = 0$, or equivalently, $\sum_{j=1}^{n+1} \lambda_j q^j.u = \sum_{j=1}^{n+1} \lambda_j v_j = 0$. Similarly a point with barycentric coordinates $\{\lambda^j\}$ is in L^+ if $\sum_{j=1}^{n+1} \lambda_j v_j > 0$ and is in L^- if $\sum_{j=1}^{n+1} \lambda_j v_j < 0$.

If $x \in L\cap\text{Int}(\Delta)$ then the barycentric coordinates of x satisfy $\sum_{j=1}^{n+1} \lambda_j v_j = 0$ and $\lambda_j > 0$ for all j. Since not all $v_i = 0$ this implies that some $v_i > 0$ and some $v_j < 0$. Assume without loss of generality that $v_1 \in \min\{v_j\} < 0$ and $v_{n+1} \in \max\{v_j\} > 0$.

We now demonstrate that K^1 and K^{n+1} lie in different halfspaces of $L(u, a)$.

First substitute $(1 - \sum_{j=2}^{n+1} \lambda_j)$ for λ_1 to give $\sum_{j=1}^{n+1} \lambda_j v_j = v_1 + \lambda_2(v_2 - v_1) + \cdots + \lambda_{n+1}(v_{n+1} - v_1)$. Now consider a point in K^1. For such a point $\lambda_2, \lambda_3, \ldots, \lambda_{n+1}$ are all negative. Note however that since all the coefficients on $\lambda_2, \lambda_3, \ldots, \lambda_{n+1}$ are non-negative, and v_1 is negative, for any point in K^1, $\sum_{j=1}^{n+1} \lambda_j v_j = v_1 + \lambda_2(v_2 - v_1) + \cdots + \lambda_{n+1}(v_{n+1} - v_1)$ is negative and so all points in K^1 lie in L^-.

Next, substituting $(1 - \sum_{j=1}^{n} \lambda_j)$ for λ_{n+1} we have $\sum_{j=1}^{n+1} \lambda_j v_j = \lambda_1(v_1 - v_{n+1}) + \lambda_2(v_2 - v_{n+1}) + \cdots + v_{n+1}$. Now consider a point in K^{n+1}. For such a point, $\lambda_1, \ldots, \lambda_n$ are all negative however. Since each of the coefficients $(v_j - v_{n+1})$ is also negative and v_{n+1} is positive, for such a point $\sum_{j=1}^{n+1} \lambda_j v_j > 0$ and so all points in K^{n+1} lie in L^+.

Hence all points in K^{n+1} lie in L^+ and so K^1 and K^{n+1} lie on different sides of L. □

Proof of Proposition 4 (by contradiction) Let the collection $\mathcal{Y}$ be as in the statement of the proposition. Under these conditions Δ is a non-empty n-dimensional simplex in $\mathbb{R}^n$, with vertices given by the $n+1$ affinely independent points $\{\bigcap_{k \in \{1,2,\ldots,n+1\} \setminus j} L^k\}_{j=1,2,\ldots,n+1}$. From Lemma 5 we have that for any arbitrary directional vector v and plane L_v with $L_v \cap \mathrm{Int}(\Delta) \neq \varnothing$:

(∗) for some G^j, $\mathcal{Y}^j \subset \overline{L_v^-}$

Now, choose some point x that is in the interior of Δ but that may be beaten by some rival point $x' = x + \varepsilon v$ where $\varepsilon > 0$ and $v \in S^{n-1}$. Consider now the plane L_v with directional vector v that contains x. Since x' beats x there lies a majority of every group inside the open half space, L_v^+ of L_v and there exist corresponding majority halfspaces $\overline{M_v^{j+}}$ which are strict subsets of L_v^+ (Lemmas 2 and 4). Now, from the definition of the yolk for each group G^j, there exists some point w^j in M_v^j with $w^j \in \mathcal{Y}^j$. Since w^j is in $\overline{M_v^{j+}}$ and $\overline{M_v^{j+}}$ is a subset of L_v^+ then w^j is not in $\overline{L_v^-}$. Since for every group j, some point in $\mathcal{Y}^j$ does not lie in $\overline{L_v^-}$ we have:

(∗∗) for no G^j, is $\mathcal{Y}^j \subset \overline{L_v^-}$

But (∗∗) contradicts (∗) and hence no x' beats x^*. □

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