Institutions of Proportionality and the Evolution of Groups*

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Abstract

I examine the evolution of group structures under discrete time Markov processes for settings in which institutions are designed to benefit groups in proportion to their size. I study three forms of proportionality: proportionality in the division of private goods, proportionality in decision making power over public policies, and proportionate jurisdictional power over policies that affect clubs. These institutions are studied for contexts varying along three dimensions: first as a function of within group decision making processes; second as a function of limitations on the acceptable number of groups; and third as a function of the rule governing the entry into groups. For each case I examine the set of stable states; since these are often multiple I use stochastic stability as a refinement. Outcomes, I find, are very sensitive to both institutions and context; broadly however, private and club proportionality lead to more fragmented structures with more homogenous membership while public proportionality leads to greater assimilation and greater within group heterogeneity.

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1 Introduction

I examine the evolution of group demographies under different institutional conditions. The institutional arrangements I examine—institions of proportionality—are ones that are introduced in deeply divided societies as a response to those divisions. Whereas in such contexts past scholarship has focussed on the implications of these institutions for inter-group behavior conditional on a group structure (Lijphart 1977, Horowitz 1991), I consider here the implications of these institutions for the group structure itself. Understanding such processes is relevant to addressing questions of the form: ‘are demographies likely to become more or less ethnically fragmented as a result of consociational institutions?’ or ‘are groups likely to become more or less internally homogenous?’ or ‘when are federal systems more likely to consolidate or dissolve?’ The answer depends entirely on the manner in which proportionality is introduced and it depends on the context into which it is introduced. Indeed it is possible to show that the set of equilibrium structures are disjoint under different institutions.

I focus on three forms of proportionality: proportionality in the division of private goods, proportionality in decision making power over policies affecting everyone in society and proportionate jurisdictional power over policies that may affect different subgroups.

For each of these I consider tendencies for changes in membership of groups. There are many ways to think about the dynamics of group formation. Models can emphasize leader effects or “mass” effects; they can focus on historical features, informational effects or the psychological effects that may obtain when individuals, as members of officially identified groups, are treated in a similar manner. Here I focus on one simple aspect of institutions of proportionality: the manner in which they tie the inputs that an individual can have over a political process and the benefits that may accrue to an individual from such processes to characteristics of the group in which she claims membership. In focusing on the relation between individuals and groups I also focus on individual incentives as a vehicle for demographic change, examining conditions when individuals have incentives to change their group membership and the incentives that groups have to accept such a change. While this approach comes with analytic advantages, it prevents us from studying strategic attempts at social engineering by leaders or by groups.

In the analysis I examine different ways in which the context of decision making varies: first I distinguish between societies in which decisions that are beyond the reach of political institutions are made in a hierarchical manner and those in which these decisions are made in a more egalitarian manner.
Next I distinguish between two ways in which demographies may evolve, conducting separate analysis for “fluid” situations in which identity change is relatively easy, requiring only the desire to shift between groups on the part of a given individual, and situations in which identities are more “sticky” in the sense that for an individual to shift from one group to another requires not simply a desire to do so on the part of the individual but a willingness on the part of the receiving group to recognize the individual as one of their own (see also Greenberg (1978, 1979)).

The dynamic processes studied are similar to those examined by Laver and Benoit (2003) who consider the evolution of party systems as individuals leave parties to join others in order to increase their share of a legislative pie. Like Laver and Benoit the types of changes I look for in the evolution of groups all entail single movements by individuals from one group or another, and I consider a demography “stable” when no individual is willing and able to change group membership.

The institutional settings I examine are however very different to those studied by Laver and Benoit. Whereas Laver and Benoit study majoritarian settings in which the benefits of office are allocated on the basis of majority status (in fact according to Shapley values) I examine consociational settings in which institutions are designed to minimize between group competition. And whereas Laver and Benoit assume symmetry across all actors, much of the important variation in models considered here is driven by variation in attributes between actors, either their strength of their policy preferences. In this sense while the evolutionary dynamics studied in Laver and Benoit are driven by between group differences, the dynamics here are driven more by within group differences. In addition the focus of this study is on variation across institutional settings and thus examination is conducted separately for different institutional rules as well as for different types of transition dynamics.

Although the range of possible institutions and contexts I examine is narrow, it is rich enough to generate a surprising degree of variation in the ways that group demographic structures may evolve. It turns out that it is not possible to make simple claims of the form: proportional systems are likely to lead to greater (or less) fragmentation. However,

\footnote{Other conditions for group change exist and remain to be examined—in some treatments for example transitions may occur or be prevented contrary to the desires of the individual and the receiving group, either because an individual can be ejected from a group (Drèze and Greenberg (1980)), abducted into a group, or forcibly retained in a given group; or, from the receiving groups perspective, if a group is prevented from or required to take on members.}
the variation that obtains is highly structured and gives rich insight into the effects of different types of systems in different contexts. The results, summarized in the concluding section, suggest that in systems with proportionality over private goods, all demographies are stable in egalitarian contexts, in hierarchical contexts however, demographies tends towards greater fragmentation when identities are sticky and yield no equilibrium states when identities are fluid. In contrast, under “public proportionality,” egalitarian systems tend towards a reduction in fragmentation with individuals attempting to join groups that are both large and internally diverse. This tendency towards consolidation is reduced in more hierarchical contexts where demographies may consist of a large core and a set of smaller groups. Under club proportionality, the role of hierarchy is less pronounced; in all cases, when identities are fluid there is a tendency towards greater fragmentation with groups containing like types; with less fluidity all demographies are stable.

2 Preliminaries

2.1 Dynamic Processes and Equilibrium

Our interest is in examining the evolution of demographic structures over time. For each point in time I assume that each individual in \( N = \{1, 2, \ldots, n\} \) is a member of some group, \( G_j \). The collection of groups, \( \Gamma \), is a partitioning of \( N \) in the sense that everybody is in some group (\( \bigcup_{j=1,2,\ldots,m} G_j = N \)) and no person is in two groups (\( G_j \cap G_k = \emptyset \) for \( j \neq k \)). The size of group \( j \), \( n_j = |G_j| \), is given by the number of players in the group and the relative size of group \( j \) is denoted \( \lambda_j \), where \( \lambda_j = n_j/n \).

**States.** A state is a partitioning of \( N \) defined uniquely up to the reordering of cells and of individuals within cells (thus \( \{\{1,2\},\{3,4\}\} \) is considered the same state as \( \{\{4,3\},\{1,2\}\} \)). A state, \( \Gamma \), is considered admissible if its cardinality, \( |\Gamma| \), does not exceed some exogenous institutional limit on the number of possible group, \( m \).\(^2\) The universe of all states \( \mathcal{G} \) for a given process is the set of all admissible states.

For a population of size \( n \) and a limit of \( m \leq n \) admissible groups, the number of admissible states is given by \( |\mathcal{G}| = \sum_{k=1}^{m} S(n, k) \), where \( S(n, k) \) is the Stirling number of the second kind for \( n \) and \( k \). For \( m = n \), \( |\mathcal{G}| \) is the \( n \)th Bell number (Rota 1964), a number which grows very rapidly in \( n \) (indeed Bell referred to them as “exponential numbers”). The set of partitions for \( n = 4 \) is 15. For \( n = 10 \) the Bell number is 115,975.

\(^2\)Other plausible criteria for admissability could use information on the number of individuals in each group.
Labels are useful for some particular states or types of state. For any player $i$ and partition $\Gamma$, we let $\Gamma(i)$ denote the partition in $\Gamma$ that contains $i$. We say that a state is **atomistic** if there is only one agent in each partition. We say that a state is the “**grand coalition**” if all agents are in the same partition. We say that a pair of groups, $A, B$, is an **ordered pair** with respect to $\alpha$ and write $A >_\alpha (<_\alpha) B$ if for all $a \in A$ and $b \in B$, $\alpha_a \geq (\leq) \alpha_b$; partitioning $\Gamma$ is a **complete ordered partitioning** if all pairs are ordered.

**Markov processes.** To the finite state space $\mathcal{G}$ we associate a discrete time Markov process, summarized by a matrix of time homogeneous transition probabilities, $P$. An element of $P$ gives the probability of transitioning from one element of $\mathcal{G}$ to another. Conditional on being in state $\Gamma_k$ in period $t$, $P_{\Gamma_k \Gamma_h}$ denotes the probability of being in state $h$ in period $t + 1$, $(P^2)_{\Gamma_k \Gamma_h}$ denotes the probability of being in state $h$ in period $t + 2$ and so on. Given this structure we review some standard terminology (see Young 1998). We say that state $\Gamma'$ is **accessible** from $\Gamma$ if for some $t$, $(P^t)_{\Gamma' \Gamma} > 0$; states $\Gamma$ and $\Gamma'$ **communicate** if each is accessible from the other. A set of states is a **communication class** if each pair communicates and is a **recurrent class** if it is a communication class from which no state outside the class is accessible. A state is **absorbing** if it is a singleton recurrent class. Hence, a state is absorbing if from that state it is only possible to move to that state. A process is **irreducible** if there is only one recurrent class, consisting of the entire state space. A distribution, $\mu$, is a **stationary distribution** if $\mu P = \mu$. The **asymptotic frequency distribution** of a process beginning at $\mu_0$ is $\mu_0 P^\infty$. Finally, $P$ is **ergodic** if $\mu_0 P^\infty$ does not depend on $\mu_0$; that is the starting state is not relevant for determining the long run outcome.

**Equilibrium states.** We say that a state $\Gamma$ is an **equilibrium** of the process $P$ if it is absorbing, that is if $P_{\Gamma \Gamma} = 1$.

**Example 1** Two examples of Markov processes are given in (1).

\[ P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, P^2 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, P^\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}, Q^\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix} \]

Matrix $P$, describes a process in which from state 1 it is only possible to move to state 1. State 1 is an absorbing state; it is stable in the sense that once a process reaches such a state it does not leave it. From state 2 it is possible to move to state 1 directly but it is also possible to move to state 3. From state 3 it is not possible to move to state 1 directly.
although it is possible to move indirectly through state 2. \( PP = P^2 \) shows the probability of being in each state after two periods, confirming that it is possible to move in two steps from 3 to 1 but also from 2 back to 2. The asymptotic distribution of states for this process is given by \( P^\infty \); in this case every row is identical, indicating that the process is ergodic: the ultimate distribution does not depend on where you start. For this process wherever you start you will always reach state 1 eventually and stay there from there on out. Matrix \( Q \) is a more difficult matrix. Again state 1 is a stable state, but in this case states 2 and 3 do not communicate with 1; they form a recurrent class; once inside this class the process cycles among states in the class. As can be seen from \( Q^\infty \) the process is not ergodic.

Stochastically stable states. In some instances it is possible for a process to produce many equilibria. In such cases we seek refinements that allow us to select among equilibria on the basis of some notion of their likelihood. The notion we employ here is that of stochastic stability; the stochastic stability criterion requires that we expect states to obtain in the long run in the presence of small errors by players. Formally, given a process \( P \) let \( P^\varepsilon \) denote a regular perturbed Markov process if (i) \( P^\varepsilon \) is irreducible for small \( \varepsilon \), (ii) \( \lim_{\varepsilon \to 0} P^\varepsilon = P \) and (iii) \( P^\varepsilon_{\Gamma, \Gamma'} > 0 \) for \( \varepsilon > 0 \to 0 \) \( \lim_{\varepsilon \to 0} \frac{P^\varepsilon_{\Gamma, \Gamma'}}{\varepsilon (1 + \varepsilon)} < \infty \) for some scalar \( r(\Gamma, \Gamma') \) which we term the resistance associated with the transition \( \Gamma, \Gamma' \). Let \( \mu^\varepsilon \) denote the (unique) stationary distribution associated with \( P^\varepsilon \). We say that a state \( \Gamma \) is stochastically stable if \( \lim_{\varepsilon \to 0} \mu^\varepsilon_{\Gamma} > 0 \) (Young 1998).

Stage games. The transition matrices studied in this paper are generated through a simple game played in each period. In each period one individual and one group (possibly the individual’s own group, and possibly an empty group) is randomly selected. The individual then decides whether she wants to join the selected group or not (we confine attention to pure strategies). In fluid environments a decision to join is sufficient to generate a transition. In sticky environments the move has to be ratified by any non-empty receiving group using a sequential up-down vote given some exogenous ordering of members. A transition is attractive if the first condition is satisfied, acceptable if the second condition is satisfied, and potential if both are satisfied. An equilibrium can therefore also be defined as a state for which no transitions are potential. For acceptability we restrict attention here to the case where unanimity is required for accepting a member although for two of the three cases considered the main results are consistent with any rule that reflects the preferences of any subset of members.
Once a decision is made to change groups (or not), payoffs are allocated according to the demographic structure and institutions in question (described more fully below). Finally we assume that in calculating their decisions individuals take account of their stage payoffs only. This requires minimal sophistication: they need to calculate for example what the profile of policies will be if they leave a given group and join another group and not simply consider the payoffs they would achieve from membership of a given group without taking into account how the demographic change will alter those payoffs. It does not require however that agents calculate payoffs that will arise in future periods, possibly as a result of future demographic changes. Clearly we can also view each stage of the process as a sequence of simple games in which players place no value on the future and let the transitions correspond to the Nash equilibria of these games.

**Perturbed stage games.** There are many ways to generate perturbed transition matrices. For most of the analyses we generate perturbed matrices, $P^\varepsilon$, by assuming that individuals make transition-facilitating mistakes with probability $\varepsilon$. Thus for example if $\{\{1, 2\}, \{3\}\}$ is preferred by only one of players 2 and 3 to $\{\{1\}, \{2, 3\}\}$ then (in a sticky environment) a single error is required for this transition; this occurs with probability $\varepsilon$, conditional on this encounter taking place. If it is preferred by neither then two errors are required, which occurs with probability $\varepsilon^2$ (conditional on the encounter). If it is preferred by both then the transition occurs with probability $1 - \varepsilon - \varepsilon^2$ (conditional on the encounter). Other types of perturbations could be considered by treating errors of joining and errors of accepting differently or by relating the probability of errors to the utility gains or benefits that correspond to the error.

### 2.1.1 Within Group Decision Making.

I simplify within-group decision making processes considerably and assume that the outcome of group decision making depends in a very simple manner on a collection of player characteristics, given by the vector $\alpha \in \mathbb{R}_n$. For a given individual, $\alpha_i$ describes the agent’s relative influence and can be thought of as an index of the individual’s wealth, patience, leverage, aggression or some other measure of personal power. I assume that $\alpha_i$ is strictly positive, although it may be arbitrarily small.

I assume that the outcomes of group processes is given by the weighted average of the ideals of the members of a group, where the weights are given by the relative size of the $\alpha_i$ terms. Such outcomes can be sustained by either cooperative or non-cooperative solution concepts. For
distributive games such an outcome corresponds to what would obtain in the limit for example if each player had preferences over income given by \( u_i(y) = y^{\alpha_i} \) for \( \alpha_i \in (0, 1) \) and groups engaged in alternating offers bargaining under a unanimity rule (implementing the Nash bargaining solution) (see Binmore 1987); for voting games such an outcome corresponds to the results predicted by the mean voter theorem (Caplin and Nalebuff 1991), in games with probabilistic voting, in which voter weights are given by \( (\alpha_i)_{i \in G} \), or in games with efficient bargaining and transfers (Deirmeier and Merlo 1990).

2.2 Institutional Arrangements

I embed the three forms of proportionality in the context of a single model. Each agent \( i \in N \) has preferences over both public goods and private goods given by a simple quasilinear utility function with quadratic loss of the form:

\[
    u_i(y_i, \pi) = y_i - \sum_{j=1}^{n} (\tilde{\pi}_j - \pi_i)^2
\]

where \( \tilde{\pi} \in \mathbb{R}^n \), is a vector of realizations of some public policy, \( \pi_i \in \mathbb{R}^1 \) is players' preferred policy and \( y_i \) is an income allocation to \( i \). Note that in effect we consider only one dimension of policy but we allow the possibility of observing multiple and possibly diverse applications of policy in that area. For example, we could imagine that there is a single dimension of left-right policy but that different left-right policies might nonetheless be implemented with respect to education, health and social welfare. For simplicity we assume that there are exactly \( n \) equally and additively valued realizations of the policy outcome.

Now consider three approaches to institutionalizing proportionality. The first is the proportionate allocation of individual benefits to groups: the private benefits of public policy are allocated proportionately. The second provides each group with a monopoly over decision making in a share \( \lambda_j \) of issues of public concern: policy jurisdictions are allocated proportionately. The final approach provides each group with monopoly power over all issue areas but the decisions they make in their areas apply only to their own members; in other words subject jurisdictions are allocated proportionately. Abstractly, the three approaches can be seen in terms of the allocation of public goods, of private goods and of club goods and in what follows we refer to the three approaches as private proportionality, public proportionality and club proportionality. The most obvious case of club proportionality for ethnic groups is in an ethnofederalist system; but jurisdictions can be divided up in a non-geographic manner also; for example decisions with respect to aspects
of law, can be determined differently for different groups, as is the case
for family law for example for different religious groups in Israel.

We represent these three approaches formally as follows:

Under **private proportionality**, the choice of public goods is indepen-
dent of the demography, and hence we ignore it in the representation of
player utility; however, each group $j$ is provided with a share $\lambda_j = \frac{|G_j|}{N}$ of
some stock of private goods (normalized to unity). The within-group allo-
ocation of private goods to its members is determined internally by
the group. The utility to player $i$, a member of group $A$, from a given
demography under private proportionality is then given by:

$$u_i(\Gamma) = \frac{n_A}{n} \frac{\alpha_i}{\sum_A \alpha_k}$$ (3)

Under **public proportionality**, the allocations of private goods, $y_i$, is
independent of the group demography and normalized to 0 for all players;
however each group $A$ is given monopoly control over decision making
in a share $\lambda_A$ of policy areas. The utility to player $i$ from a given de-
mosophy under public proportionality is then given by:

$$u_i(\Gamma) = -\sum^m_{k=1} \frac{n_k}{n} \left( \sum_{h \in G_k} \frac{\alpha_h \pi_h}{\sum_{h \in G_k} \alpha_h} - \pi_i \right)^2$$ (4)

Under **club proportionality**, the choice of private goods is again in-
dependent of the demography outside a player’s group, but public deci-
sions (over all $n$ realizations) made by a group affect only the group’s
own members. The utility to player $i$, a member of group $A$, from a given
demography under private proportionality is then given by:

$$u_i(\Gamma) = -\left( \sum_A \frac{\alpha_k \pi_k}{\sum_A \alpha_h} - \pi_i \right)^2$$ (5)

2.2.1 Variation within and between groups

For some of the results that follow I draw on the notion of within group
homogeneity and between group heterogeneity or vice versa. Given our
use of quadratic loss functions we can describe these notions in a partic-
ularly simple way in which within group homogeneity implies between
group heterogeneity and vice versa.

Let $a(x)$ denote the mean of a vector $x$ and let $a(x)_\alpha$ denote the
weighted average of $x$ given some weighting vector $\alpha$. For compactness
we also write $\bar{x}(A)$ for the mean of the subvector of $x$ corresponding
to group $A$, $\bar{x}(A) = a((x_i)_{i \in A})$. 

9
Let the “heterogeneity” of a group $G_j$ (with respect to $x$) be given by $\rho(A) = a((x_i)_{i \in A} - \overline{x}(A))^2$; analogously define the “weighted heterogeneity” of $A$, given some weighting vector $\alpha$ as: $\rho(A)_\alpha \equiv a((x_i)_{i \in A} - a((x_i)_{i \in A})\alpha^2)_\alpha$.

The (weighted) between group heterogeneity of a coalition structure, $\rho(\Gamma)_{\lambda}$, is given by the heterogeneity of the (possibly weighted) means of each of the groups in $\Gamma$, with a weighting vector given by the relative size (or sum of weights) of the groups.

The average within group heterogeneity $a(\rho(\Gamma))_\lambda$ is then proportionate to the sum of squared distance of all players from the means of their own groups.

Usefully, for a given demographic structure $G$, the total heterogeneity $\rho(N)$ is equal to the sum of the between group heterogeneity and the within group heterogeneity. In particular:

$$\rho(N) = \frac{1}{n} \sum_{j=1}^{m} \sum_{i \in G_j} [x^*_i - \overline{x}]^2 = a(\rho(\Gamma))_{\lambda} + \rho(\Gamma)_{\lambda}$$

A consequence of this is that conditional upon a population $N$, a rise in within group heterogeneity resulting from a changing demographic structure necessarily corresponds to a decline in between group heterogeneity and vice versa.

### 2.2.2 Genericity.

Throughout I assume that player strength and preference parameters are generic in the sense that given any two non-equivalent sets of players, the averages of the parameters over these sets parameters are distinct. For example if 2 players have weights given by 1,2, then the weight of a third person cannot be exactly one of the rational numbers in the set $X = \{1, \frac{1}{2}, 2, 3\}$ although an infinity of other positive real number are admissible.

### 3 Results I: Private Proportionality

Under private proportionality the private income of player $i \in N$ is given by $y_i = \frac{\alpha_i}{n} \frac{\sum_A \alpha_k}{\sum_A \alpha_k}$. More compactly this can be written: $y_i = \frac{\alpha_i}{\sum_A \alpha_k} \frac{1}{n}$. Thus an individual's reward is her egalitarian share, $\frac{1}{n}$, multiplied by a factor $\frac{\alpha_i}{\sum_A \alpha_k}$ that reflects whether a player’s weight is above or below the average for her group.
Note that the payoffs take a very simple form in this game. In particular, an individual’s payoff is a function only of the numbers in her own group and of the strengths of the members of her own group. It does not depend on other aspects of the partitioning, and in particular on the group membership or powers of influence of individuals not in the player’s own group.

3.1 Egalitarian Private Systems

As a benchmark consider a situation in which all bargaining weights are uniform. In this case, it is evident that \( y_i = \frac{1}{n} \) for all players. But in this case, payoffs are independent of the demographic structure \( \Gamma \). Hence, although, formally, goods are distributed via group membership, the proportionality and equidivision rules make the form of \( \Gamma \) irrelevant: all demographies are stable. This is the case whether demographies are fluid or sticky. This, though simple, is the first result. Under private proportionality the interesting dynamics take place within inegalitarian structures. We examine these next.

3.2 Inegalitarian Private Systems

Under private proportionality any interesting distinction between demographies depends on some notion of possible inequalities within groups. Making use of the strength indices, \( \alpha \), described above, we refer to individual \( i \) whose weight is below (above) the average weight of group \( A \) as a “weak” (“strong”) relative to \( A \). Clearly payoffs of players who are weak in their groups are below \( \frac{1}{n} \) while payoffs of strong players exceed \( \frac{1}{n} \).

To characterize equilibrium in this game more generally, we consider two cases, the case with no binding constraint on the number or size of groups (\( m \geq n \)) and the case with such a constraint (\( m < n \)).

Proposition 2 For any stable state \( \Gamma \), \( |\Gamma| = \min(m, n) \).

The proposition follows immediately from the observation that from our genericity condition there is at least one loser in every group with more than one member. If new groups are permissible such losers have an incentive to join them. For the \( m \geq n \) case the proposition implies the following corollary.

Corollary 3 If \( m \geq n \) then a state is an equilibrium of a sticky process if and only if it is the atomistic structure.
The corollary implies both that stable demographies exist in $m \geq n$ cases for sticky structures and that they are highly egalitarian—since in the atomistic structure all players receive the same payoff.

The properties of stable demographies are more complex in situations where there is a limit on either the number or the minimum size of groups $m < n$. If there are less permissible groups than there are players and so singletons can not always be formed, then the properties of stable structures will be different. Weak members of groups will not always be able to separate from groups in order to claim their share of the pie unless they can find another group willing to accept them. In the absence of such groups we may observe in-group heterogeneity and multiple stable demographies that are not payoff equivalent.

Individuals prefer to join groups that are weak relative to them. Wanting to find themselves in groups in which their bargaining power is strong, they find a change desirable only if it improves their relative status. More formally:

**Lemma 4 (Attractiveness)** [Inegalitarian, Private] A shift from group $A$ to group $B$ is desirable for player $i$ if and only if the average strength of group $B$ members, augmented by $i$’s membership, is lower than the average strength of group $A$, including $i$’s membership

In fluid structures transitions depend only on the preferences of would-be joiners. From Lemma 4 we can then work out whether or not equilibria exist under fluid processes. As shown next, they do not.

**Proposition 5** No states are stable under fluid hierarchical processes.

For sticky structures, which we will focus on from here on out, a change is acceptable if and only if the incoming member would become a weak member of the new group. It turns out that the acceptability of a member is uniform across members of the receiving group and so the internal decision-making rule for admission is unimportant: if one member of a group supports a new entrant, all members do. We state and prove these claims next.

**Lemma 6 (Acceptability)** [Inegalitarian, Private] Individual $i \in A$ will be acceptable to an arbitrary member of group $B \neq A$ if and only if she is weak relative to $B$.

The intuition for this is the following: the distributive rule employed allocates a share of the pie to a group as a function only of the group’s size but not of the characteristics of its members; hence whenever weak
members join, their take from the group is smaller than the value they bring to the group, leaving a surplus from which everyone else in the group can benefit. All members receive a share of this surplus (even if they are weak), in proportion to their strength.

These conditions for attractiveness and acceptability provide the basis for the necessary and sufficient conditions for stability. These are given next.

**Proposition 7** [Inegalitarian, Private] A demographic structure, $\Gamma$, is stable under a sticky process if and only if $|\Gamma| = \min(m, n)$ and for each weak player there is no group in which she would be less weak (but still weak).

Under a sticky structure, every weak player is in the group in which she is relatively least weak. Intuitively this implies a homogenizing effect, suggesting that individuals should cluster in groups with individuals of similar strength. This intuition is partly correct. The following proposition provides conditions under which such clustering by size induces stability, providing conditions for when movements are impossible between two groups that are ordered with respect to $\alpha$.

**Proposition 8** No movement is possible between groups $A$ and $B$ if $A <_\alpha B$, and $n_A - 1 \leq n_B$.

One immediate corollary is of this proposition is the following:

**Corollary 9** Equilibria exist; in particular any partitioning that is completely ordered on $\alpha$ is stable if for each $j$, $G_j >_\alpha G_{j-1}$ implies $n_j \geq n_{j-1} - 1$.

**Example 10** Given a process with $n = 4$ and $m = 2$, and labelling players such that $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ Corollary 9 implies that $\{1\}, \{2, 3, 4\}$ and $\{1, 2\}, \{3, 4\}$ are both stable states irrespective of the cardinal value of the player weights. The ordered partition $\{\{1, 2, 3\}, \{4\}\}$ is not however guaranteed to be stable.

The condition in Proposition 8, applied to all pairs of groups, provides a sufficient condition for a state to be stable. For $n > 3$ however the condition is not necessary as the next example indicates.

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3This result clearly stands in contrast to situations in which inter-group relations affect shares; for example to one in which the group's share is a function of the number of “high quality” members.
Figure 1: An example of an equilibrium with high within group heterogeneity.

Example 11 Consider a process with two groups and four players of size $\alpha_1 = 1$, $\alpha_2 = 5 - \varepsilon$, $\alpha_3 = 5 + \varepsilon$ and $\alpha_4 = 6$, for small $\varepsilon$. In this game the most inequalitarian partitioning $G = \{\{1, 4\}, \{2, 3\}\}$ is stable. This equilibrium is illustrated in Figure 1. Weak as he is in negotiating with the strongest player (receiving $\frac{2}{7}$), the smallest player is even weaker negotiating with two large players (receiving $\frac{3}{11}$).

The example shows that equilibria are not always characterized by ordered partitions but to a minimal extent however, all stable partitionings are at least partially rankable. In particular we can show the following:

Proposition 12 In every stable demography, the mean strength of the group containing the $j^{th}$ weakest player is no larger than the mean strength of the $j^{th}$ weakest group.

Attractiveness of Equilibrium. We have already established in Corollary 9 that equilibrium always obtains for sticky Markov processes. In addition however we can show that such processes will always lead to some equilibrium state. We establish this property next.

Lemmas 4 and 6 describe the conditions for when individual movements will be potential. They also give insights into the paths that demographies may take as they evolve towards a stable state. For any feasible transition we can show that only weak members of groups will shift and only then to groups in which they will remain as weak members. Once they move, the group to which they move will be on average weaker than the group from which they moved; the group they leave behind will, as a result be stronger on average. These regularities suggest that in the dynamic process we should observe a general movement
of weak players from stronger to weaker groups. However these general trends notwithstanding, the dynamic process can hold a number of important non-monotonicities.

It is possible to find demographies from which movements may take place from relatively strong to relatively weak groups or from weak to strong groups. It is also possible that movements are potential both from group \( A \) to \( B \) and from group \( B \) to \( A \). These features raise the prospect that the dynamic process may not eventually result in the attainment of a stable demography: they raise the concern that stable demographies, while retentive, may not be attractive. In fact, this is not the case. Since the state space is finite, all paths either lead to cycles or to steady states; we now demonstrate that they invariably lead to a steady state by showing that they cannot produce cycles.

**Proposition 13** All recurrent classes contain only one state. Furthermore the average weight of the weakest group declines monotonically during this process.

### 3.3 Complete characterization of equilibrium for the private goods case, with \( n = 4, m = 2 \).

We now characterize equilibria and stochastic dynamics in the \( n = 4, m = 2 \) case assuming a sticky structure and inegalitarian within group politics characterized by generic weighting vectors normalized such that \( \sum \alpha_i = 1 \).

The universe of states, \( G \), contains just 8 elements which we label as follows.

\[
\begin{align*}
\Gamma_I &= \{ \{1,2,3,4\}, \emptyset \} \\
\Gamma_{II} &= \{ \{1\}, \{2,3,4\} \} \\
\Gamma_{III} &= \{ \{2\}, \{1,3,4\} \} \\
\Gamma_{IV} &= \{ \{3\}, \{1,2,4\} \} \\
\Gamma_V &= \{ \{4\}, \{1,2,3\} \} \\
\Gamma_{VI} &= \{ \{1,2\}, \{3,4\} \} \\
\Gamma_{VII} &= \{ \{1,4\}, \{2,3\} \} \\
\Gamma_{VIII} &= \{ \{1,3\}, \{2,4\} \} 
\end{align*}
\]

The question is which of these states are stable and which of these are "likely." Using ordinal information alone we have from Proposition 8 and its corollary that \( \Gamma_{II} \) and \( \Gamma_{VI} \) are stable states, from Proposition 2 that \( \Gamma_I \) is unstable (since the transition \( \Gamma_I \to \Gamma_{II} \) is feasible) and from Proposition 7 that \( \Gamma_{III} \) is unstable since a switch by 1 to transition to \( \Gamma_{VI} \) is potential. Our propositions do not tell us however whether or not \( \Gamma_{IV}, \Gamma_V, \Gamma_{VII} \) or \( \Gamma_{VIII} \) are equilibria.
Indeed a full examination of all possible structures confirms that no propositions could establish further which states are necessarily stable or not using ordinal information alone; each of the remaining states may or may not be stable depending on the exact value of \( \alpha \).

**Proposition 14** For games with \( n = 4 \) and \( m = 2 \), and \( \alpha \) normalized such that \( \sum \alpha = 1 \) the equilibrium states are:

A. \( \Gamma_{II}, \Gamma_{IV}, \Gamma_{V}, \) and \( \Gamma_{VI} \) if \( \alpha_1 > \frac{2-5\alpha_4}{3} \)

B. \( \Gamma_{II}, \Gamma_{V}, \) and \( \Gamma_{VI} \) if \( \alpha_1 < \frac{2-5\alpha_4}{3} \) and \( \alpha_1 > \frac{2-5\alpha_3}{3} \) and \( \alpha_1 > 2\alpha_2 - \alpha_3 \)

C. \( \Gamma_{II}, \Gamma_{V}, \Gamma_{VI} \) and \( \Gamma_{VIII} \) if \( \alpha_1 < \frac{2-5\alpha_3}{3} \) and \( \alpha_1 > \frac{2-5\alpha_4}{3} \) and \( \alpha_1 < 2\alpha_2 - \alpha_3 \)

D. \( \Gamma_{II}, \Gamma_{VI}, \Gamma_{VII} \) and \( \Gamma_{VIII} \) if \( \alpha_1 < \frac{2-5\alpha_3}{3} \) and \( \alpha_1 < \frac{2-5\alpha_4}{3} \).

**Figure 2:** Taxonomy of equilibrium states for \( n = 4 \), \( m = 2 \).

Figure 2 illustrates the set of equilibria identified by Proposition 14. We see that each of the four states \( \Gamma_{IV}, \Gamma_{V}, \Gamma_{VII} \) and \( \Gamma_{VIII} \) may be sustained as equilibria for some values of \( \alpha \) but not others.

The proposition can be used to characterize the degree of within group heterogeneity that can be sustained in \( m = 2 \), \( n = 4 \) cases. The greatest value that the sum of squared errors can take (\( SSE \)), is for the \( \alpha \) vector (close to) \((0, 0, 0, 1)\), with \( SSE = \frac{3}{4} \). For such cases within group heterogeneity is minimized at state \( \Gamma_{V} \) which is indeed an equilibrium in this case. Indeed it can be shown that \( \Gamma_{V} \) is an equilibrium whenever \( SSE \) exceeds .11. We see here that a necessary and sufficient condition for the high within group heterogeneity equilibrium that we examined in Example 11 is that \( \alpha_1 < \frac{2-5\alpha_4}{3} \), a condition that can only be satisfied for \( \alpha_4 < \frac{2}{5} \). From the Proposition however we can see that although this state has the highest within group heterogeneity among possible states,
it occurs only when the total amount of heterogeneity is low. The total sum of squares and the within cluster sum of squares are maximized for group $D$ cases for the $\alpha$ vector $(0, \frac{1}{5}, \frac{2}{5}, \frac{2}{5})$, which admits a total sum of squared errors of only $\frac{5^2}{20} + \frac{1^2}{20} + \frac{3^2}{20} + \frac{3^2}{20} = \frac{11}{100}$ and a total within group heterogeneity of $\frac{10}{100}$. $\Gamma_{IV}$ is also a heterogenous case, but this too only arises when overall heterogeneity is low. The most heterogenous case for which $\Gamma_{IV}$ can be sustained is for $\alpha$ vector close to $(0, 0, \frac{2}{5}, \frac{3}{5})$, which yields a within group heterogeneity (for $\Gamma_{IV}$) of $\frac{24}{100}$. Thus whenever the most heterogenous within-group states are equilibrium overall within-group heterogeneity is low.

Finally, we consider stochastic stability. The question of interest is whether for a given process, the equilibria with the greatest within group heterogeneity can be sustained even as stochastically stable equilibria. The answer is yes and we illustrate with the case already discussed in Example 11.

For the case given in Example 11, the perturbed transition matrix for this case, $P^\varepsilon$ is given by:

$$P^\varepsilon = \begin{bmatrix}
\frac{7-3\varepsilon}{8} & \varepsilon & \varepsilon & \varepsilon & 0 & 0 & 0 \\
\frac{\varepsilon}{8} & \frac{8-4\varepsilon}{8} & 0 & 0 & \varepsilon & \varepsilon & \varepsilon \\
\frac{\varepsilon}{8} & \frac{\varepsilon}{8} & 0 & \frac{7-2\varepsilon-\varepsilon^3}{8} & 0 & \varepsilon & \varepsilon \\
\frac{\varepsilon}{8} & \frac{\varepsilon}{8} & 0 & 0 & \frac{7-2\varepsilon-\varepsilon^3}{8} & \varepsilon & \varepsilon \\
0 & \frac{\varepsilon}{8} & \frac{\varepsilon}{8} & \frac{\varepsilon}{8} & \frac{8-2\varepsilon-2\varepsilon^2}{8} & 0 & 0 \\
0 & \frac{\varepsilon}{8} & \frac{\varepsilon}{8} & \frac{\varepsilon}{8} & \frac{\varepsilon}{8} & \frac{8-2\varepsilon-\varepsilon^3}{8} & 0 \\
0 & \frac{\varepsilon}{8} & \frac{\varepsilon}{8} & \frac{\varepsilon}{8} & \frac{\varepsilon}{8} & \frac{\varepsilon}{8} & 0 & \frac{8-3\varepsilon-\varepsilon^2}{8}
\end{bmatrix}$$

This perturbation arises when, with probability $\varepsilon$ an individual accepts a change to the status quo that is against her interest. Some types of moves require a single error for— for example $P^\varepsilon_{13}$ requires only that player 2 mistakenly decides to leave the grand coalition. Others, such as $P^\varepsilon_{37}$ require the compounding of two errors, in this case, player 3 mistakenly choosing to leave $\{1, 3, 4\}$ to join player 2, and player 2 mistakenly accepting him. Other moves require 3 errors, for example the move $P^\varepsilon_{72}$, which involves 4 mistakenly choosing to leave and 2 and 3 mistakenly accepting him. Thus the three element path from the grand coalition, to one in which 2 is on her own, is then joined by 3 and finally by 4 requires a substantial compounding of errors and occurs with probability $\frac{\varepsilon}{8} \cdot \frac{\varepsilon}{8} \cdot \frac{\varepsilon}{8} = \frac{\varepsilon^3}{8}$.

Note that this is a regular perturbed matrix of $P$ for which $0 < \lim_{\varepsilon \to 0} \frac{P^\varepsilon_{IV}}{\varepsilon} < \infty$ for precisely the resistances given in the matrix $r$.
\[
\begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
3 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\
3 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 2 & 2 & 0 & 0 & 0 \\
0 & 3 & 2 & 2 & 1 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

For any \( P^\varepsilon \) we can find the corresponding stationary distribution, \( \mu^\varepsilon \) by solving \( \mu^\varepsilon P^\varepsilon = \mu^\varepsilon \) (computationally this can be done by locating the normalized eigenvector corresponding to the transpose of \( P^\varepsilon \) given an eigenvalue of 1). A graph showing the evolution of \( \mu^\varepsilon \) is given in Figure 3. The figure demonstrates that State \( \Gamma_{VI} \) is the unique stochastically stable equilibrium. Notably however with moderate errors, state \( \Gamma_{VI} \) also occurs with high frequency.

![Figure 3](image-url)

**Figure 3:** The horizontal axis gives the probability that a given player will make an error, \( \varepsilon \in [0.5, 0) \). On the vertical we show the proportion of time spent at each state for any given \( \varepsilon \). It can be seen that as \( \varepsilon \) vanishes the share of time spent in state \( VII \ (\Gamma_{VII}) \) tends to 1.

We can also demonstrate analytically that state \( \Gamma_{VII} \) is the unique SSE in this case by calculating the stochastic potential associated with each state (Young 1998, Theorem 3.1). These are given by 3 for \( \Gamma_{VII} \) and 4 for all other equilibria. The lower stochastic potential for \( \Gamma_{VII} \) arises in fact because given this equilibrium, leaving state \( \Gamma_{VII} \) is difficult since the pair \( \{1, 4\} \) do not want new entrants and no other group wants to welcome 4; any transitions other than \( \Gamma_{VII} \to \Gamma_V \), thus requires more
than one error; in addition any subsequent movement from \( \Gamma_V \) to any state other than \( \Gamma_{VIII} \) requires further errors.

We have then that this outcome can be sustained as a unique stochastically stable state and so stochastic stability can not be relied upon to select internally homogenous coalition structures. A full characterization of stochastic stability is still required for these games.

4 Results II: Public Proportionality

4.1 Egalitarian Public Systems

In the case of public proportionality we have that utility depends on the preferences of all players in all groups. Not only does a player care about which other player is in her group, she also cares about which other players are in other groups, since each of these affects the aggregate policy outcome.

Whereas in the case of private goods we identified incentives to fragment, in this case we find incentives to consolidate.

Our first proposition shows that complete integration is an equilibrium in this game in egalitarian settings. Furthermore, all other equilibrium approximate this equilibrium; while it is not the case that full integration is the unique equilibrium, any equilibrium other than the grand coalition is similar to the grand coalition in the sense that each group contains an internally diverse preference structure similar to that of the grand coalition, and payoffs at every equilibrium approximate payoffs at the grand coalition. These features are summarized in the next proposition.

**Proposition 15** In an egalitarian polity (whether fluid or sticky): (i) the grand coalition is stable (and efficient); (ii) shifts are potential if and only if they reduce between group heterogeneity; and (iii) all demographic changes shift payoffs towards those payoffs that are provided by the grand coalition.

**Corollary 16 (No Cycling)** All recurrent classes contain only one state.

**Proof.** Follows from part (ii) of Proposition 15 □

The proof of the Proposition is given in the appendix but some of the intuition of the proof is worth noting. That the grand coalition is an equilibrium follows from the concavity of player utility functions. With concave preferences all players prefer the policy outcome that corresponds to the average of the ideals of all players to the average of the
utility arising from their own ideal and the utility arising from the average of all other ideals. The same logic can be used to show not simply that the grand coalition is an equilibrium, but also that no equilibrium can contain singletons.

From this we can calculate the maximum number of groups in any stable structure:

**Proposition 17** For any stable structure $\Gamma$, $|\Gamma| \leq \sum_{k=2}^{m} S(n, k)$, where $S(n, k)$ is the Stirling number of the second kind for $n, k$.

More generally, the condition for player $a$ to be willing to switch from group $A$ to group $B$ in this setting is (for $n_A > 1$):

$$
-n_A (\bar{\pi}_A - \pi^*_i)^2 - n_B (\bar{\pi}_B - \pi^*_i)^2 < - (n_A - 1) \left( \frac{n_A \pi_A - \pi^*_i}{n_A - 1} - \pi^*_i \right)^2 - (n_B + 1) \left( \frac{n_B \pi_B + \pi^*_i}{n_B + 1} - \pi^*_i \right)^2
$$

Or, more compactly:

$$
\frac{(\bar{\pi}_A - \pi^*_i)^2}{(\bar{\pi}_B - \pi^*_i)^2} < \frac{n_B}{n_B + 1} \frac{n_A - 1}{n_A}
$$

(6)

This condition is more likely to hold if $i$ is relatively centrist with respect to his own group and relatively extreme with respect to another group. It is also most likely to hold when groups are large, in which case the right hand side rises to unity and all that is required for a switch is that some player prefers the mean position of the other group less than the mean position of her own group. This results in a narrowing of the distance between the groups and hence the relationship between individual motivations to switch and the global narrowing of between group heterogeneity. Finally we find that in this game the incentives of players are strongly aligned: in the egalitarian case the condition for a switch for $i$ from group $j$ to group $h$ to be acceptable to some third player $m$ is in fact independent of $m$’s own ideal, depending only on the smoothing effects of $j$’s shift. An implication of this last feature is that every shift is Pareto improving.

Although it is possible to have equilibria other than the grand coalition, the proposition suggests that any such equilibria are similar in terms of payoffs to all players to what would obtain under the grand coalition. More precisely, any equilibrium must be characterized by low between-group and high within-group heterogeneity. In particular, as shown in the next proposition, all equilibrium coalitional structures are “cross cutting” and the ordered partitions that we saw to be equilibria in the Private goods case are never equilibria in the Public goods case.

**Proposition 18** No complete ordered partitioning is stable.
4.2 Complete characterization of equilibrium for the public goods case, with \( n = 4, m = 2 \).

We know from Proposition 15 that in the four person case no partitions involving singletons are equilibria and from Proposition 18 that no ordered partitions are equilibria. These two propositions then rule out twelve of fifteen possible structures leaving only three:
\[
\{\{1, 2, 3, 4\}\}, \{\{1, 3\}, \{2, 4\}\}, \{\{1, 4\}, \{2, 3\}\}.
\]

The first of these, the grand coalition, we know to be stable. The only remaining question from Proposition 15 is whether the other two, non-ordered pairs, can be equilibrium structures. The answer is that they can, but in the spirit of the proposition, only if they produce low between group homogeneity.

To study these two cases consider a state containing two pairs, \( L \) and \( H \) with \( \bar{\pi}_L = 0 \) and \( \bar{\pi}_H = 1 \). An individual in group \( H \) will switch if:
\[
(1 - \pi^*_i) \frac{(\pi^*_i)}{2} < \frac{N_L + 1}{N_L} \frac{N_H - 1}{N_H} \quad \text{and will stay put if} \quad \left(\frac{\pi^*_i}{1 - \pi^*_i}\right)^2 \leq \frac{N_L + 1}{N_L} \frac{N_H}{N_H - 1}.
\]

Consider some player \( h \) in \( H \) with \( \pi^*_i \in \max\{\pi^*_k \mid k \in H\} \). For this individual \( \pi^*_h > 1 \) and so the condition not to switch is
\[
\pi^*_h \leq \frac{1 - \sqrt{N_L + 1 \frac{N_H}{N_H - 1}}}{N_L} \quad \text{(7)}
\]

The right hand side of (7) tends to infinity as the number in each group increases. For \( N_L = N_H = 2 \) it takes a minimum of approximately 2.4; stability then requires that the highest player of group \( H \) has an ideal of at least 2.4 and the lowest of less than –0.4.

Consider next some player \( l \) in \( L \) with \( \pi^*_l < \bar{\pi}_L = 0 \). By the same reasoning, in this case for a movement not to be desirable, we need
\[
\frac{(\pi^*_l)}{2} \geq \frac{N_H}{N_H + 1} \frac{N_L - 1}{N_L} \quad \text{or:}
\]
\[
\pi^*_l \leq \frac{1}{1 - \sqrt{N_L + 1 \frac{N_H + 1}{N_H}}} \quad \text{(8)}
\]

The RHS is negative but takes on its highest value in situations with fewer players in each group; in the case with two two-player groups the RHS takes the value \( \frac{1}{1 - \sqrt{3}} \simeq -1.4 \).

Hence a stable four person game in which there is not complete integration requires heterogeneity at least as large as would obtain if players in one group have ideals \(-1.4, 1.4\), and players in the other group have ideals \(-1.4, 2.4\). Graphically the overlap implied by these ideals is shown
Figure 4: This figure shows the minimal degree of cross cuttingness of two group in equilibrium under public proportionality.

in Figure 4. Note the within group heterogeneity is so large relative to the between group heterogeneity that some members in each actually prefer the policies set by the other group to those set by their own group; they remain in their own group to balance the policy preferences of their other group members.

This implies that for stability we require that the share of total heterogeneity accounted for by within group heterogeneity is about \( \frac{2}{4 - \sqrt{3}} > 88\% \). Hence even in the two group two player case—which admits the maximum rigidity—any equilibrium in which some between group heterogeneity persists, this within group heterogeneity can not account for more than 12\% of total heterogeneity.\(^4\)

Finally we consider the question of stochastic stability. In this process the structures with trios and singletons act as bridges: a change from these to any of the equilibrium state is achievable with zero resistance. If resistance is only incurred by those leaving equilibrium structures (in fluid systems), then each of the three states has a stochastic potential of 2. If however resistance also arises from the requirement of acceptance (in sticky systems), then paired structures are stochastically stable states. A move away from a paired situation is resisted both by the

\(^4\)Note that in this 2 group case, \textit{minimal} total heterogeneity is given by:

\[
\rho(N) = \frac{(\frac{1}{\sqrt{3}} - 5)^2 + (\frac{1}{\sqrt{3}} - 5)^2 + (\sqrt{3} - 5)^2}{4}.
\]

This can be decomposed into

within group heterogeneity, given by: \( \rho_w = \frac{(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}})^2 + (\sqrt{3} - \frac{1}{\sqrt{3}} - 1)^2 + (\sqrt{3} - \frac{2}{\sqrt{3}} - 1)^2}{4} \)

and

between group heterogeneity, given by:

\( \rho_b = \frac{1}{2} \left( \frac{1}{2} \right)^2 + \frac{1}{2} \left( \frac{1}{2} \right)^2. \)

To see that that 88\% figure is the \textit{minimum} share note that our normalization of \( \bar{\pi}_h = 0, \bar{\pi}_j = 1 \), implies that any \( \varepsilon \) increase in the position of the player with the highest value of \( \pi_i^* \), say \( \bar{\pi}_4 \) corresponds to a fall in the value of the other member of group \( j \), say \( \pi_2^* \) leaving \( \bar{\pi}_j \) and \( \bar{\pi} \) unchanged. Since \( \pi_4^* > \bar{\pi} \) and \( \pi_2^* < \bar{\pi} \) this produces a rise in \( \rho \) with no change in \( \rho_b \). Hence we have a decline in \( \rho_b/\rho \) and a corresponding rise in \( \rho_w/\rho \).
mover and by the accepting pair, for a total resistance of 3. In contrast a move from the grand coalition requires resistance only from the mover. In a case with two pair equilibria, the stochastic potential of each pair is 4, while the stochastic potential of the grand coalition is 6.

The fact that the grand coalition is not selected as the unique stochastically stable equilibrium (or in some cases not selected at all) is surprising given the fact that it is the unique Pareto optimal structure. It arises in part because of the lack of sophistication of the agents. An intuition from sub-game perfection would suggest that if one player in a pair left to form a triple the other member would soon follow to everyone’s benefit. Such farsightedness is not admitted here however.

4.3 Inegalitarian Public Systems

In the case in which players are likely to vary in their ability to affect the outcome of decisions made within their groups, the characterization of the equilibrium is more involved.

The condition for $i$ to want to shift from $A$ to $B$ is:

$$(n_B + 1) \left( \frac{\sum B \alpha_k \pi_k + \alpha_i \pi_i}{\sum B \alpha_k + \alpha_i} - \pi^*_i \right)^2 + (n_A - 1) \left( \frac{\sum A \alpha_k \pi_k - \alpha_i \pi_i}{\sum A \alpha_k - \alpha_i} - \pi^*_i \right)^2 < n_B \left( \frac{\sum B \alpha_k \pi_k}{\sum B \alpha_k} - \pi_i \right)^2 + n_A \left( \frac{\sum A \alpha_k \pi_k}{\sum A \alpha_k} - \pi_i \right)^2$$

With some manipulation this simplifies to:

$$\frac{(\bar{\pi}_h - \pi^*_i)^2}{(\bar{\pi}_j - \pi^*_i)^2} < \frac{\Psi_A}{\Psi_B} \text{ for } \Psi_B > 0$$

$$\frac{(\bar{\pi}_h - \pi^*_i)^2}{(\bar{\pi}_j - \pi^*_i)^2} > \frac{\Psi_A}{\Psi_B} \text{ for } \Psi_B < 0$$

where $\Psi_A = n_A - (n_A - 1) \left( \frac{\sum A \alpha_k}{\sum A \alpha_k - \alpha_i} \right)^2$ and $\Psi_B = (n_B + 1) \left( \frac{\sum B \alpha_k}{\sum B \alpha_k + \alpha_i} \right)^2 - n_B$.

The trade-off then depends on the distance between a player’s ideal and the policy of each of the two groups. The comparative statics depend on the sign of $\Psi_B$—a measure of a player’s weakness in group $B$. This renders interpretation difficult. Nonetheless a number of regularities can be inferred.
• A bandwagonning effect obtains: a switch is more likely if $B$ is large and $A$ is small. To see this note that both $\Psi_A$ and $\Psi_B$ are decreasing in $n_A$.

• Avoiding weakness. If a player is strong in the receiving group $B$ such that $\alpha_i > \left(\sqrt{n_B + n_B^2} - n_B\right) \bar{\alpha}_B$ and hence $\Psi_B < 0$, but weak in his own group $A$, $\alpha_i < \left(n_A - \sqrt{n_A^2 - n_A}\right) \bar{\alpha}_A$, such that $\Psi_A > 0$, then a switch is always desirable, independent of the policy positions of the two groups. Conversely if a player is weak in the receiving group, but strong in his own group, then a switch is never desirable.

• A Polarization effect can obtain if a player is especially weak in both groups; in the sense that $\Psi_A > 0$, and $\Psi_B > 0$. For reasonably large $n$ these conditions obtain when $\alpha_i < \frac{\bar{\alpha}_A}{2}, \frac{\bar{\alpha}_B}{2}$. In this case a player will prefer to be in the group that is like him—in shifting groups he has little direct impact on policy in any area, one way or the other, but by teaming up with like types he can at least make sure that his interests get represented indirectly through the assignment of power to his group. This effect is especially strong if the player is strong in the like group relative to the unlike group.

• A Homogenizing effect obtains for individuals that are strong in both groups, $\Psi_A < 0$, $\Psi_B < 0$. In this case the condition is more likely to hold when $\left(\bar{\pi}_B - \pi_i\right)^2$ is large relative to $\left(\bar{\pi}_B - \pi_i\right)^2$—the more extreme a player is in a group (relative to his position in his own group) the more likely he is to join that group.

Hence a player always weighs two features—the gain from joining a like group by increasing the number of issue areas controlled by like types, and the gain from joining an unlike group that arises from exerting a moderating influence on the group. Relatively influential players are driven more by the latter condition, weaker players by the former. All players however are attracted by large groups.

Note that the condition for strength here is always satisfied for players with average strength or above. The strength of the condition depends on the size of the groups. $\Psi_B < 0$ is always satisfied for players with $\alpha_i > \frac{\bar{\alpha}_B}{2}$ and does not depend greatly on $n_B$. Condition $\Psi_A < 0$ is more demanding for low $n_A$ but tends to the condition $\alpha_i > \frac{\bar{\alpha}_A}{2}$ as $n_A$ increases. Importantly, it follows that there are always strong players for whom $n_A > \frac{\bar{\alpha}_A}{2}$. Furthermore, letting group $A$ denote the weaker group, there are always players for whom switching is optimal if and only if between group inequality can be sufficiently reduced by the switch.
Such strong players might not switch if they are already having a moderating effect on their own group.

These results suggest that in inegalitarian polities, an equilibrium is likely to be characterized by groups that contain multiple influential players with diverse preferences, alongside weak players with centrist preferences, and satellite groups of like minded weak players with extreme preferences. These equilibrium partitions have however not yet been studied.

5 Results III: Club Proportionality

5.1 Equilibrium

In the third case we examine a player’s utility depends on the policy preferences of the members of her own group but not the preferences of members of other groups.

In this setting, the condition for player $i$ to be willing to switch from group $A$ to group $B$ is:

$$-\left(\frac{\sum_B \alpha_k \bar{\pi}_k + \alpha_i \pi_i}{\sum_B \alpha_k + \alpha_i} - \bar{\pi}_i\right)^2 > - (\bar{\pi}_A - \pi_i)^2$$

or, equivalently:

$$\left(\frac{\pi_B - \pi_i}{\pi_A - \pi_i}\right)^2 < \left(1 + \frac{\alpha_i}{\sum_B \alpha_k}\right)^2$$

Clearly a player always has an incentive to switch to an empty group whenever this is feasible, giving rise to analogues results to those from our study of private goods.

**Proposition 19** For any stable state $\Gamma$, $|\Gamma| = \min(m, n)$.

**Corollary 20** If $m \geq n$ then a state is an equilibrium if and only if it is the atomistic structure.

What of acceptability under club proportionality? It is immediate from our genericity condition that any new entrant into a group will result in a change in policy for a group which will in turn produce at least one loser. Thus, under the unanimity rule considered here there is always opposition to new entrants and so we expect no transitions in sticky polities with club proportionality.

**Proposition 21** All states are stable under sticky processes.
In consequence, we focus here on on fluid polities for which a transition requires only that there exists an individual willing to change groups. For such contexts a necessary and sufficient condition for stability is that for every pair of groups $A$ and $B$ for all $i$ in group $A$:

$$\frac{(\pi_B - \pi_i)^2}{(\pi_A - \pi_i)^2} \geq \left(1 + \frac{\alpha_i}{n_B \alpha_B}\right)^2 > 1$$  \hspace{1cm} (11)$$

This is worth comparing to the condition derived for the public goods case:

$$\frac{(\bar{\pi}_A - \pi_i)^2}{(\bar{\pi}_B - \pi_i)^2} \geq \frac{n_B}{n_B + 1} > 1$$  \hspace{1cm} (12)

Strikingly, this comparison tells us that there is no overlap between public goods equilibria (with equal weights) and club goods equilibria. In the public goods case fully ordered partitions were equilibria whereas in the club good case condition 11 tells us that only ordered groups are stable: each player should be strictly closer to the center of his own group than to the center of all other groups. In particular a player cannot be precisely on the border of two groups because in such a situation he would have an incentive to change groups (with the immediate result that the policies of that group would be closer to him next period that the policies of his present group).

5.2 Complete characterization of equilibrium for the club goods (equal weights) case, $n = 4, m = 2$.

Consider a game with four equally weighted players. Normalize the positions such that $\pi_1 = 0$, and $\pi_4 = 1$ and $\pi_1 < \pi_2 < \pi_3 < \pi_4$. From Condition 11 and Proposition 19 we have that of 8 states there are only 3 candidate equilibria: the 3 two-group ordered partitions, $\Gamma_1 = \{\{1\}, \{2, 3, 4\}\}$, $\Gamma_2 = \{\{1, 2\}, \{3, 4\}\}$ and $\Gamma_3 = \{\{1, 2, 3\}, \{4\}\}$. Note that player 2 is decisive over $\Gamma_1$ and $\Gamma_2$. He will prefer $\Gamma_2$ if and only if

$$\pi_2 - \frac{\pi_2}{2} < \frac{\pi_2 + \pi_3 + 1}{3} - \pi_2,$$

or equivalently, if and only if:

$$\pi_3 < \frac{7}{2} \pi_2 - 1$$  \hspace{1cm} (13)

Furthermore among the set of ordered partitions, $\Gamma_2$ is the only possible deviation from $\Gamma_1$. Hence $\Gamma_1$ is an equilibrium if and only if Condition 13 holds. Similarly 3 is decisive over $\Gamma_3$ and $\Gamma_2$ and prefers 3 if and only if

$$\pi_3 - \frac{\pi_3 + 1}{2} < \frac{\pi_3 + 1}{3} - \pi_3,$$

or equivalently:
\[ \pi_3 < \frac{2}{7}\pi_2 + \frac{3}{7} \quad (14) \]

Since among the set of ordered partitions, \( \Gamma_2 \) is the only possible deviation from \( \Gamma_3 \), \( \Gamma_3 \) is an equilibrium if and only if Condition 14 holds. If neither condition holds then since \( \Gamma_1 \) and \( \Gamma_3 \) are the only ordered deviations from \( \Gamma_2 \), \( \Gamma_2 \) is an equilibrium. Finally we note that it is possible that two equilibria exist if and only if \( \pi_3 < \frac{2}{7}\pi_2 - 1 \) and \( \pi_3 < \frac{2}{7}\pi_2 + \frac{3}{7} \). Given the restriction that \( \pi_2 < \pi_3 \) there is only a narrow range in which both of these conditions hold simultaneously. For these cases the two centrist groups are sufficiently similar that they prefer to club together with a third player rather than divide and make alliances with the extremes. These equilibria are illustrated in Figure 5.

Stochastic stability is required only for the case where two equilibria obtain. In this case both states are stochastically stable as a result of the symmetry of the situation; the stochastic potential of each state is 2.

Figure 5: The set of equilibrium states for \( 0 = \pi_1 < \pi_2 < \pi_3 < \pi_4 = 1 \) combinations.
5.3 Equilibrium and Centroidal Voronoi Tessellations

The necessary condition for equilibrium given in Equation 11 has a useful geometric interpretation. Given a set of \( m \) distinct generating points, \( x = \{x\}^m_{i=1} \) a Voronoi tessellation on \( X \subset \mathbb{R}^1 \) is a partitioning of \( X \) into \( m \) regions \( \{V\}^m_{i=1} \) such that each point \( y \in X \) is in region \( V_i \) if and only if \( y \) is closer to \( x_i \) than to any other point in \( X \). Given a measure function \( f \) over \( X \), a centroidal Voronoi tessellation is a tessellation with the property that the mean of each partition (under \( f \)) is its generating point.

It can be seen from equation 11 that every equilibrium partition must be a Centroidal Voronoi tessellation on \( \mathbb{R}^1 \) with generating points given by the policies of each group and the player ideals of each group contained within the corresponding Voronoi cell of the tessellation. However, as the following example demonstrates, not every centroidal tessellation is an equilibrium.

Example 22 Consider a game with \( n = 4 \), \( m = 2 \), equal weights and preference parameters given by \( \pi_1 = -2 \), \( \pi_2 = 2 \), \( \pi_3 = 3 \), \( \pi_4 = 4 \), \( \pi_5 = 5 \). Restricting attention only to ordered non-empty groups we have 4 possible partitions, \( (\Gamma_1)_{i=1}^4 \), such that \( j \in G_1(\Gamma_i) \) if and only if \( i \leq j \). Thus \( \Gamma_1 = \{\{1\},\{2,3,4,5\}\} \) generates a centroidal tessellation with generating points \(-2\) and \(3.5\). \( \Gamma_2 = \{\{1,2\},\{3,4,5\}\} \) also generates a centroidal Voronoi tessellation with generators \(0\) and \(4\). Player 2 is decisive over these two states, but she is closer to \(3.5\) than to \(0\), hence although it is a centroidal Voronoi tessellation, \( \Gamma_2 \) is not an equilibrium state.

The reason why one of the centroidal Voronoi tessellations in Example 22 is not an equilibrium is because in state \( \Gamma_2 \) player 2 compares not the present policy she receives in Group 1 (0) to what is produced in the Group 2 (4), but rather what would be achieved in Group 2 were she to join. In effect, she anticipates her influence.

There are two circumstances in which we can ignore this effect; first if the size of all groups are large such that individual influence is small; or second if players are not sufficiently strategic to take their future influence into account. Let us call such a condition a “diminished expected marginal effect condition” (DEMEC).

Under DEMEC we have from a modified form of Equation 11 for which equilibrium states correspond exactly to the centroidal Voronoi tessellations of \( \Gamma \) (or, for discrete distributions, \( k \)-means clusters). Much is known about clusters of these forms and so it is worth considering
properties of equilibrium under these assumptions.\footnote{This problem is closely related to questions examined in two other areas. One is the study of Tiebout public goods. Tiebout equilibria are used in settings in which “the consumer-voter may be viewed as picking that community which best satisfies his preference pattern for public goods.” If it is the case that the choices of a given community in turn depends on the membership of a community, then the problem is that studied by Caplin and Nalebuff (1992). Voronoi diagrams have a number of other applications in politics. Most obviously they can be used to illustrate the set of voters that support a given candidate in multicandidate elections, assuming sincere voting. They can also be used to study districts and the location of public goods (Simeone et al). For the study of ethnic politics they can also be used to link types to “stereotypes”: given a collection of stereotypes (Voronoi points) a Voronoi diagram classifies individuals according to which stereotype they most closely resemble (for a recent study see Fryer and Jackson 2002).}

In older work on the subject Fisher (1958) demonstrated the existence of $k$-means clusters and showed that they are always ordered partitionings. It is also well known that for any number $m$ the partitioning of a space into $m$ sets that minimizes within group variance is given by a centroidal Voronoi tesselations (for a review see Du et al 1999), as such we have that the state that minimizes within group heterogeneity is an equilibrium of our system. This establishes existence but it also has striking normative properties, standing in stark contrast to the result on public goods, given in Proposition 15; there we found that demographic evolution minimized rather than maximized between group heterogeneity. Finally with demec we can draw on a large body of work that describes the evolution of partitions towards equilibrium.

MacQueen (1964, 1967) shows that a simple algorithm can be used to identify centroidal Voronoi tesselations. We can provide an easy interpretation of MacQueen’s algorithm as follows. Start with a set of $k$ leaders that dictate policy for their group. Then randomly select an adherent and let the adherent join the group whose policies he prefers. Then redefine the policy for this group based on the updated membership. Select another individual and repeat. As the size of the population increases the sum of within group variance that can arise from allocation of cells to the ensuing set of Voronoi points converges. A stronger result due to Pollard establishes that the Voronoi points themselves converge to the minimizing set as the population increases.

Lloyd (1982) proposes a faster algorithm that is also consistent with models of adaptive play. Begin with any demography and identify the set of political outcomes associated with this demography (under our genericity condition we take this set to be composed of $k$ distinct outcomes). Then let all individuals simultaneously select the groups whose policies they prefer, producing a new demography. Identify the new set of political outcomes. Then repeat the process, letting all individuals re-
select their preferred group again. The process converges to a centroidal tesselation. Du et al (2006) show that if the measure of types over the space is positive and smooth then the process is globally convergent in the one dimensional case.6

6 Conclusion

Our inquiry reveals tendencies for how demographic profiles may evolve under different institutions of proportionality and under different assumptions of the conditions under which group change is feasible. The context we examine abstracts greatly from many important features of demographic change. The analysis very deliberately abstracts from between group competition; the between group allocation is treated as governed entirely by a set of rules; in practice however these rules are themselves a subject of contestation. Beyond this, other absent features include the possibility of elite manipulations, the existence of affective ties to groups, and limitations on group formation that might derive from physiological or other sticky attributes (Posner 2005, Chandra and Boulet 2003).

Despite the simplicity of the model however it has generated a rich array of propositions regarding how group structures might respond to institutional rules. These results are summarized in Figure 6.

The overall message is that very distinct patterns emerge from each of the three types of proportionality considered, with outcomes ranging from a wide multiplicity of equilibria, to the absence of equilibria, from strong tendencies to fragmentation, to stagnation, to strong pressures for assimilation. Most strikingly the different systems also have dramatic implications for not just the number and size of groups but their internal composition with in some cases a strong tendency towards coalitions of like types and others a tendency to join with unlike types in a bid to balance.

Much of the dynamics examined here warrant further and deeper exploration; equilibrium has not been fully characterized in the case of public goods and hierarchical structures, the dynamics of the club case have not been explored for cases in which the Demec condition does not hold. Furthermore, in many cases it seems plausible that results could be found with weaker assumptions on preferences and institutional structures. Finally this theoretical work has been developed entirely without

6In addition Du et al provide approximations for the size of each group at convergence.
Figure 6: Summary of analysis: Demographic Evolution as a function of institutions and social context

reference to actual cases. Yet there is a wide field of application, and great scope for generating empirical tests of the prepositions developed here, either for the study of the evolution of ethnic demographics, for changes in party systems, or for population movements in federal systems.
7 Appendix (Proofs)

Lemma 4

Proof. Group $B$ is attractive to $i \in A$ iff:

$|A| \frac{\alpha_i}{\sum_{k \in A} \alpha_k} < \frac{|B| + 1}{|B| + 1} \frac{\alpha_i}{\sum_{k \in B} \alpha_k + \alpha_i}$

$\Leftrightarrow \frac{\sum_{k \in B} \alpha_k + \alpha_i}{|B| + 1} < \frac{\sum_{k \in A} \alpha_k}{|A|} \Leftrightarrow \bar{\alpha}(B \cup \{i\}) < \bar{\alpha}(A)$. ■

Proposition 5.

Proof. With $m > 1$ any equilibrium must contain at least two non-empty groups which we label $A$ and $B$. From our genericity assumption, either $\bar{\alpha}(A) < \bar{\alpha}(B)$ or $\bar{\alpha}(A) > \bar{\alpha}(B)$. Assume without loss of generality that $\bar{\alpha}(A) > \bar{\alpha}(B)$. There exists some player $i \in A$ with $\alpha_i \leq \bar{\alpha}(A)$. But then $\alpha_i \leq \bar{\alpha}(A)$ and $\bar{\alpha}(B) < \bar{\alpha}(A)$ which implies $\bar{\alpha}(B \cup \{i\}) < \bar{\alpha}(A)$. To see this assume to the contrary that $\bar{\alpha}(B \cup \{i\}) \geq \bar{\alpha}(A)$ then $\bar{\alpha}(B \cup \{i\}) \geq \bar{\alpha}(A) > \bar{\alpha}(B)$ so $\bar{\alpha}(B \cup \{i\}) > \bar{\alpha}(B)$ and so $\alpha_i > \bar{\alpha}(B)$. However, we would then also have $\bar{\alpha}(B \cup \{i\}) \geq \bar{\alpha}(A) \geq \alpha_i$ and so $\bar{\alpha}(B \cup \{i\}) \geq \alpha_i$ and so $\alpha_i \leq \bar{\alpha}(B)$, a contradiction. With $\bar{\alpha}(B \cup \{i\}) < \bar{\alpha}(A)$ Lemma 4 implies that a move is desirable and hence, under a fluid process, potential. ■

Lemma 6

Proof. The condition for a member $h$ of group $B$ to be willing to accept a new member $i$ into her group is given by:

$\frac{|B|}{N} \frac{\alpha_h}{\sum_{k \in B} \alpha_k} < \frac{|B| + 1}{|B| + 1} \frac{\alpha_h}{\sum_{k \in B} \alpha_k + \alpha_i}$

$\Leftrightarrow \frac{\sum_{k \in B} \alpha_k}{|B|} < \frac{\alpha_i}{\sum_{k \in B} \alpha_k} \Leftrightarrow \frac{\alpha_i}{\bar{\alpha}(B)}$

This condition depends only on the weight of player $i$ and a population statistic of group $B$, and hence the same condition applies for all incumbent members of group $B$. ■

Proposition 7.

Proof. The conditions for stability follow directly from lemmas 6 and 4: $\alpha_i < \bar{\alpha}(B)$ and $\bar{\alpha}(B \cup \{i\}) < \bar{\alpha}(A)$. The latter condition implies in particular that $\frac{\bar{\alpha}(B \cup \{i\})}{\alpha(B \cup \{i\})} > \frac{\bar{\alpha}(B)}{\bar{\alpha}(A)}$. ■

Proposition 9
Proof. Assume $G_2 >_a G_1$. Clearly since all members of $G_2$ are stronger than all members from $G_2$ to $G_1$ is possible. We focus on a movement of some player $i$ from group $G_1$ to $G_2$. Normalize weights such that $\sum_{G_1 \cup G_2} \alpha_k = 1$ and define $\gamma = \sum_{G_1 \setminus \{i\}} \alpha_k$. Note that the average weight of group 1 is $\bar{\alpha} = \frac{\gamma + \alpha_i}{n_1}$, the average weight of group 2 is $\bar{\alpha} = \frac{1 - \gamma - \alpha_i}{n_2}$. Since the movement is acceptable it must be that $\alpha_i < \frac{1 - \gamma - \alpha_i}{n_2}$.

Since player $i$ wants to shift we have:

$$\frac{\alpha_i}{\gamma + \alpha_i} n_1 < \frac{\alpha_i}{1 - \gamma} (n_2 + 1)$$

Which yields Condition 15:

$$\frac{n_1}{n_2 + 1} (1 - \gamma) - \gamma < \alpha_i$$

(a). In addition we have $\alpha_i < \frac{\gamma}{n_1 - 1}$, which follows from the fact that $i$ is weak in his own group. Together with Condition 15 this implies $\alpha_i < \frac{n_1 - 1}{n_2 + 1} (1 - \gamma) - \gamma < \frac{n_1 - 1}{n_2 + 1}$ or equivalently: $\gamma > \frac{n_1 - 1}{n_2 + 1}$.

(b). We also have from the fact that the partition is ordered that $\frac{\gamma}{n_1 - 1} < \frac{1 - \gamma - \alpha_i}{n_2}$, that is, the average of the other members in $G_1$ are of lower weight than the average of $G_2$. Reorganizing this condition gives: $\alpha_i < 1 - \gamma - \frac{n_2}{n_1 - 1}$. Then combining this with Condition 15 gives $\frac{n_1}{n_2 + 1} (1 - \gamma) - \gamma < 1 - \gamma - \frac{2n_2}{n_1 - 1}$. This condition can never be satisfied for $n_2 = n_1 - 1$. We then focus on the cases $n_2 \geq n_1$ for these the condition is equivalent to $\gamma < \frac{n_1 - 1}{n_1 + n_2}$. But this condition is incompatible with the analogous conditions from (b). Hence the condition can not be satisfied for $n_2 \geq n_1 - 1$. ■

Proposition 12

Proof. Label the players such that $\alpha_i < \alpha_{i+1}$ and the groups such that $\bar{\alpha}(j) < \bar{\alpha}(j+1)$ for all $i < n$ and $j < |\Gamma|$. Assume first that $\Gamma(1) > 1$, that is, that the weakest player is not in the weakest group. Since 1 is acceptable to all groups, she will move if there is any group such that $\bar{\alpha}(j) < \bar{\alpha}(\Gamma(1))$. But there is such a group, in particular, $\bar{\alpha}(1) < \bar{\alpha}(\Gamma(1))$. Hence in a stable demography the weakest player must be in the weakest group. We now show that in any equilibrium, for any $s$ if player $s$ is in one of the smallest $s$ groups, then player $s + 1$ is a member of one of the smallest $s + 1$ groups. Given that players $1, 2, ..., s$ are in the smallest $s$ groups, player $s + 1$ is acceptable at least to all groups larger than $s$. Furthermore, if $\Gamma(s + 1) > s + 1$ then group $s + 1$ is desirable to player $s + 1$ since $\bar{\alpha}(s + 1) < \bar{\alpha}(\Gamma(s + 1))$. The proposition follows by induction. ■
Proposition 13:
Proof. We establish the first part of the Proposition by showing that no cycles are possible. Assume that over the course of some cycle from time 1 to time $T > 1$ exactly $k$ groups are involved in changes in membership. Let $\bar{\alpha}(t) \in \mathbb{R}^k$ denote the vector of average strengths of these $k$ groups in time $t$. Without loss of generality, let the groups be such that $k$ is the weakest group: $\bar{\alpha}_k \in \min(\bar{\alpha}^1)$. In any cycle we have $\bar{\alpha}_1 = \bar{\alpha}_T$ and so in particular, $\min(\bar{\alpha}^T) = \bar{\alpha}_1$.

Consider a move by $i \in G_h$ to $G_j$. From Lemma 4 we have that $\bar{\alpha}(G_j \cup \{i\}) < \bar{\alpha}_h$, or equivalently $\bar{\alpha}_j^{t+1} < \bar{\alpha}_h^t$. To be acceptable to $G_j$ we must have that $i$ is weak relative to $j$ which gives $\bar{\alpha}_j^{t+1} < \bar{\alpha}_j^t$. Hence $\bar{\alpha}_j^{t+1}$ is strictly weaker than the weakest of $\bar{\alpha}_j^t$ and $\bar{\alpha}_h^t$. This has two implications:

(i) If in any period $t$ a player $i \in G_h$ moves to $G_j$ then $\bar{\alpha}_j^{t+1} < \min(\bar{\alpha}_j^t, \bar{\alpha}_h^t)$.

(ii) In any two consecutive periods $\min(\bar{\alpha}^t) \leq \min(\bar{\alpha}^{t+1})$.

Let $s$ denote the first period in which group $k$ is involved in a demographic change and let $j$ denote the group with which it is involved. From (i) we have $\min(\bar{\alpha}^{t+1}) \leq \min(\bar{\alpha}_j^{t+1}, \bar{\alpha}_k^{t+1}) < \min(\bar{\alpha}_j^t, \bar{\alpha}_k^t) \leq \bar{\alpha}_1$. Then from (ii) we have $\min(\bar{\alpha}^T) < \min(\bar{\alpha}^1)$ contradicting the fact that $\bar{\alpha}^T = \bar{\alpha}^1$. ■

Proposition 14
Proof. We make use of the following nested structure of the conditions.

A Claim. (i) $\alpha_1 < \frac{2-5\alpha_4}{3}$ implies $\alpha_1 < \frac{2-5\alpha_3}{3}$ (ii) $\alpha_1 < \frac{2-5\alpha_3}{3}$ → $\alpha_0 > \frac{\alpha_1+\alpha_2}{2}$ (iii) $\alpha_1 > \frac{2-5\alpha_3}{3} \rightarrow \alpha_1 > \frac{2-5\alpha_4}{3}$

(i) The first part is immediate: $\alpha_1 < \frac{2-5\alpha_4}{3}$ implies $\alpha_1 < \frac{2-5\alpha_3}{3}$ since $\alpha_3 < \alpha_4$. For (ii) we need to rule out both $\alpha_1 < \frac{2-5\alpha_3}{3}$ and $\alpha_2 < \frac{\alpha_1+\alpha_2}{2}$. Note that with $\alpha_1 < \frac{2-5\alpha_4}{3}$, and $\alpha_3 < \alpha_4$, we have $\alpha_2 < \frac{\alpha_1+\alpha_2}{2} < \frac{\frac{2-5\alpha_4}{3}+\alpha_4}{2} = \frac{1-\alpha_4}{3}$, hence $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < \frac{2-5\alpha_4}{3} + \frac{1-\alpha_4}{3} + 2\alpha_4 = 1$. (iii) is implied by (i), (ii) implies that $\alpha_2 < \frac{\alpha_1+\alpha_2}{2} \rightarrow \alpha_1 > \frac{2-5\alpha_3}{3}$

B. We then proceed to establish conditions for equilibrium for each state. Given the nested nature of these conditions we have the following claims. Thus, we have $\Gamma_V$ is an equilibrium if and only if $\alpha_1 > \frac{2-5\alpha_4}{3}$.

To check this claim note the admissible transitions from $\Gamma_V$ involve $P_{V,I}$, $P_{V,VI}$, $P_{V,VI}$, $P_{V,VIII}$. Now: $P_{V,I} = 0$, since it is opposed by receivers 1, 2 and 3; $P_{V,VI} = 0$ since it is opposed by mover 3, $P_{V,VIII} = 0$ since it is opposed by mover 2; everything then depends on $P_{V,VI}$; this transition is always acceptable to receiver 4 but is attractive to 1 if and only if: $\bar{\alpha}({1,4}) < \bar{\alpha}({1,2,3})$, or equivalently if and only if: $\frac{\alpha_1+\alpha_4}{2} < \frac{1-\alpha_4}{3} \Leftrightarrow \alpha_1 < \frac{2-5\alpha_4}{3}$. Thus $\Gamma_V$ is an equilibrium if and only if
\( \alpha_1 > \frac{2-5\alpha_4}{3} \). Since, from (i) \( \alpha_1 < \frac{2-5\alpha_4}{3} \) implies \( \alpha_1 < \frac{2-5\alpha_3}{3} \), we have that \( \Gamma_V \) is an equilibrium for cases \( A, B, \) and \( C \) only. The proof follows the same method for each of the other three states \( \Gamma_{VI}, \Gamma_{VII} \) and \( \Gamma_{VIII} \). ■

**Proposition 15.**

**Proof.** (i) To check that complete integration is an equilibrium, we show that for an arbitrary player, \( i \), contributing to the decisions of the committee of the whole, \( N \), on every realization of the policy choice is preferred to monopoly power over one realization of policy choice and no input over all others.

Setting \( \sum_{h \in N} \alpha_h = 1 \), we need to show:

\[
- \left( \sum_{h \in N} \alpha_h \pi_h - \pi_i^* \right)^2 > 0 - (n - 1) \left( \sum_{h \in N \setminus \{i\}} \frac{\alpha_h \pi_h}{1 - \alpha_i} - \pi_i^* \right)^2
\]

However this inequality we can see is equivalent to:

\[
- \left( \sum_{h \in N} \alpha_h \pi_h - \pi_i^* \right)^2 > -\frac{n - 1}{(1 - \alpha_i)^2} \left( \sum_{h \in N} \alpha_h \pi_h - \pi_i^* \right)^2
\]

Which reduces to the simple quadratic: \( \alpha_i^2 - 2\alpha_i + \frac{1}{n} < 0 \). Since we have \( \alpha_i \leq 1 \), we can determine that this inequality holds for any \( \alpha_i > 1 - \sqrt{\frac{n-1}{n}} \approx \frac{1}{2n} \).

In other words, in a minimally egalitarian setting in which all individuals have a bargaining strength of at least half of what the average bargaining strength is, then the completely consolidated demography is an equilibrium.

(ii) The previous logic shows not simply that the grand coalition is an equilibrium, but also that no equilibrium can contain singletons. Hence to examine the more general case of a switch we consider the condition in which for two groups \( A, B \) with \( n_A, n_B > 1 \), some player \( i \in A \) is willing to switch to group \( B \):

\[
-(n_B + 1) \left( \sum_{k} \frac{\pi_k + \pi_i^*}{n_B + 1} - \pi_i^* \right)^2 - (n_A - 1) \left( \sum_{k} \frac{\pi_k - \pi_i^*}{n_A - 1} - \pi_i^* \right)^2
\]

\[
> -n_B \left( \frac{\sum_k \pi_k - \pi_i^*}{n_B} \right)^2 - n_A \left( \frac{\sum_k \pi_k - \pi_i^*}{n_A} \right)^2
\]

For \( \bar{\pi}_A \neq \pi_i^* \) and \( n_A, n_B > 1 \), this condition reduces to the following inequality:

\[
\frac{(\bar{\pi}_B - \pi_i^*)^2}{(\bar{\pi}_A - \pi_i^*)^2} > \frac{n_A}{n_A - 1} \frac{n_B + 1}{n_B} \quad (16)
\]
Between group heterogeneity is given by:

\[ \rho(G|\lambda) = \lambda_B \left( \sum \frac{\pi_k}{n_B} - \pi \right)^2 + \lambda_A \left( \sum \frac{\pi_k}{n_A} - \pi \right)^2 + \sum_{G_z \in \Gamma(A \cup B)} \lambda_z (\bar{\pi}_z - \pi)^2 \]

This between group heterogeneity declines due to a shift by \( i \) from group \( A \) to \( B \) iff:

\[
\frac{n_B + 1}{n} \left( \frac{\sum_{k \in B} \pi_k + \pi_i}{n_B + 1} - \pi \right)^2 - \frac{n_B}{n} \left( \frac{\sum_{k \in B} \pi_k}{n_B} - \pi \right)^2 < \frac{n_A}{n} \left( \frac{\sum_{k \in A} \pi_k}{n_A} - \pi \right)^2 - \frac{n_A - 1}{n} \left( \frac{\sum_{k \in A} \pi_k - \pi_i}{n_A - 1} - \pi \right)^2
\]

This condition reduces to:

\[
\frac{n_B}{n_B + 1} \left( \bar{\pi}_B^2 - \frac{\pi_i^2}{n_B} - 2\bar{\pi}_B\pi_i \right) > \frac{n_A}{n_A - 1} \left( \bar{\pi}_A^2 - 2\bar{\pi}_A\pi_i + \frac{\pi_i^2}{n_A} \right)
\]

which in turn reduces to:

\[
\frac{(\bar{\pi}_B - \pi_i)^2}{(\bar{\pi}_A - \pi_i)^2} > \frac{n_B + 1}{n_B} \frac{n_A}{n_A - 1}
\]

But this is exactly Equation 16, the condition for player \( i \) to want to move to group \( B \). Hence: a player will be willing to move to from group \( A \) to group \( B \) if and only if such a move will reduce between group heterogeneity.

The final feature we need to check is whether a player that is willing to move will be accepted to the new group. If so then all moves that reduce between group heterogeneity are potential.

In the egalitarian case the condition for a switch for \( i \) from group \( A \) to group \( B \) to be acceptable to some third player \( s \) is:

\[
-(n_B + 1) \left( \frac{\sum_{k \in B} \pi_k + \pi_i}{n_B + 1} - \pi_s \right)^2 - n_A \left( \frac{\sum_{k \in A} \pi_k}{n_A} - \pi_s \right)^2
\]

\[
> -n_B \left( \frac{\sum_{k \in B} \pi_k}{n_B} - \pi_s \right)^2 - (n_A + 1) \left( \frac{\sum_{k \in A} \pi_k + \pi_i}{n_A + 1} - \pi_s \right)^2
\]

Multiplying out we find that all terms with \( \pi_s \) cancel. In other words the condition for a switch to be preferable is the same for all players. If
Proposition 18

Proof. Consider two groups, A and B, part of a stable complete ordered partitioning. Assume without loss of generality that \( \bar{\pi}_A < \bar{\pi}_B \) and set \( \bar{\pi}_A = -1, \bar{\pi}_B = 1 \). Consider some player \( i \) in A with \( \pi_i^* \in \min\{(\pi_k)_{k \in A}\} < -1 \). This player will not wish to change only if:

\[
\frac{(1+\pi_i^*)^2}{(1-\pi_i^*)^2} \geq \frac{n_B}{n_B+1} \cdot \frac{n_A-1}{n_A}.
\]

The LHS is decreasing in \( \pi_i^* \) and takes on its highest values for low \( \pi_i^* \). Now, given any \( n_A \) with \( \bar{\pi}_A = -1 \), let \( \pi_{LB}^* \) denote the minimum value taken by players in Group B, given our ordering, \( \pi_j^* < \pi_{LB}^* \) for all \( j \) in A and so \( \frac{(n_A-1)\pi_{LB}^* + \pi_i^*}{n_B+1} > -1 \). This gives a lower bound on \( \pi_i^* : \pi_i^* > -(n_A-1)\pi_{LB}^* \). \( \pi_i^* > -n_A - (n_A-1)\pi_{LB}^* \). Since \( \pi_{LB}^* \leq 1 \), the lowest bound is given by \( \pi_i^* > 1 - 2n_A \). Hence for any \( n_A \) the greatest possible value for \( \pi_i^* \) is \( \frac{(1+\pi_i^*)^2}{(1-\pi_i^*)^2} \) = \( \frac{(1-n_A)^2}{(2n_A)^2} \) and the condition is:

\[
\frac{n_A-1}{n_A} \geq \frac{n_B}{n_B+1} \quad \text{or} \quad n_A \geq 1 + n_B.
\]

By a similar logic for a player \( j \) from B with \( \pi_j^* \in \max\{(\pi_k)_{k \in B}\} > 1 \) we have \( \pi_j^* < 2n_B - 1 \). Given the condition for this player not to want to change, \( \frac{(1-\pi_j^*)^2}{(1+\pi_j^*)^2} < \frac{n_A-1}{n_A+1} \cdot \frac{n_B}{n_B-1} \), we have \( \frac{(n_B-1)^2}{(n_B)^2} < \frac{n_A}{n_A+1} \cdot \frac{n_B}{n_B-1} \) and so \( \frac{n_B-1}{n_B} < \frac{n_A}{n_A+1} \) or \( n_B < 1 + n_A \). \( \blacksquare \)
References


