Product Improvement in a Winner-Take-All Market

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Abstract

Mixed and pure equilibria are characterized for a duopoly model of simultaneous quality improvement followed by price competition in a market with limited consumer heterogeneity. In the second stage Bertrand equilibrium, the higher-quality firm captures the entire market, earning a profit proportional to the quality difference. If initial product qualities are not very different, then there are many equilibria of the full game. These equilibria have substantially different expected welfare, and predict a wide range of possible quality outcomes. Furthermore, it is possible to "purify" the mixed equilibria, i.e. multiple pure equilibria with approximately the same outcome distributions exist in nearby extended games of incomplete information. Finally, analogous pure and mixed equilibria exist for nearby dynamic games for which discounting is sufficiently great and quality depreciation sufficiently quick.

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1. Introduction

Product improvement is so obviously a crucial means by which firms compete. In consumer technology markets, for example, Apple stays ahead in the mobile handset market by periodically launching new versions of the iPhone, Intel remains dominant in the computer processor market with ever more powerful chips, and Microsoft holds a grip on PC operating system market with successive improvements to Windows. In consumer markets in which novelty and fashion matter, for example, McDonalds tempts consumers with innovative sandwiches, Coach stays abreast of shifting handbag fashions, and HBO updates its program lineup. In these and other retail markets, competitors gain and keep quality advantages by periodically developing new and improved versions of existing products.

[1] Some results in this paper were presented at the 27th Australasian Economic Theory Workshop at Massey University in 2009 under the title "Quality Competition and Multiple Equilibria", and at a conference in honor of James Friedman at Duke University in 2006 under the title "Quality Competition in a Winner-Take-All-Market". I thank conference participants, as well as Kyle Bagwell, Richard Gilbert, and seminar participants at Columbia University and the Toulouse School of Economics for helpful comments. The paper builds on some results in Gilbert and Riordan (2003, 2007). I am grateful to the Toulouse School of Economics also for hospitality and research support during my sabbatical in 2012-13, and to Ilton Soares for careful research assistance.

Despite the real-world importance of product improvement competition, the economics literature has only a limited understanding of how it works. The vertical differentiation literature provides a reasonably good but still incomplete understanding of markets in which competitors divide the market by offering distinctly different quality products. Following seminal work by Gabszewicz and Thisse (1979) and Shaked and Sutton (1982), a standard setting for the analysis of such markets is a two-stage game in which duopoly firms first choose their qualities and then compete on price to attract consumers who have heterogeneous preferences for quality. The literature has focused almost entirely on situations in which there is sufficient consumer heterogeneity to support an equilibrium in which the firms select different quality products and consumers choose among the alternative offers according to their willingness to pay for higher quality. The logic of the market-sharing result is that vertical product differentiation with sufficient consumer heterogeneity relaxes price competition, enabling firms to set positive margins and recover endogenous fixed costs.

I focus instead on the simpler scenario in which there is insufficient consumer heterogeneity to support contemporaneous market sharing by vertically differentiated firms. For this case, Gabszewicz and Thisse (1979) showed that equilibrium price competition has a winner-take-all outcome, in which the highest quality firm serves all consumers at a price constrained by the availability of the next available quality. What are equilibrium incentives for product improvement in this case? My main result is to demonstrate a plethora of equilibria, including potentially many in which firms asymmetrically mix over quality choices.

A symmetric mixed equilibrium is intuitive when the rival firms are positioned symmetrically *ex ante*. What if one firm starts with a small advantage? I model an asymmetric positioning of firms as an initial quality advantage of a market leader, and quality competition as product improvement game in which the leader and the laggard have symmetric opportunities to improve their respective products. In the asymmetric case, while a symmetric equilibrium fails to exist, there potentially are many asymmetric equilibria depending on the magnitude of the initial quality advantage. In particular, if the initial advantage of the leader is small, there are a large number of discrete equilibria of the product improvement game. The number of these equilibria is "pruned" as the initial leadership advantage grows until there is but a single equilibrium in pure strategies for a sufficiently large initial advantage. In this case, only the leader invests in quality improvement. Thus a market leader with a sufficiently large initial advantage remains dominant, i.e. strategic uncertainty vanishes from a winner-take-all market with a dominant firm already far enough ahead. For only a small initial advantage, however, a mixed equilibrium in which each firm wins the market with some probability arguably is a more intuitive solution than a pure equilibrium with a preordained winner.

For the case of a quadratic cost of quality improvement, I compute analytically the set of discrete equilibria for an arbitrary initial cost advantage. The number of discrete equilibria is finite and odd in the asymmetric case. If the firms are symmetric *ex ante*, then there are a twice-countable infinity of discrete equilibria. The mixed equilibria typically are less efficient than pure equilibria in which either the leading or the lagging firms invest efficiently to win the market. In particular, the mixed equilibria feature inefficiently low investments by the winning firm, and wasteful investments by the losing firm. A great variety of equilibrium outcomes are possible.

A common objection is that mixed strategies are artificial. The objection comes from a perspective that firms make definite strategic choices rather than "flip coins" to determine what products they bring to market. This objection, however, can be laid to rest with a "purification" argument along lines suggested by Bagwell and Wolinsky (2002) following the pioneering work of Harsanyi (1973). I consider a nearby game of incomplete information in which each rival is privately informed about its cost of product improvement, and demonstrate that the pure equilibrium outcomes of the nearby incomplete information game closely approximate the equilibrium outcomes of the complete information game as the degree of private information shrinks. Thus a mixed equilibrium can be interpreted as representing equilibrium uncertainty about the conduct of rivals (Aumann and Brandenburg, 1995). In other words, the potential for strategic uncertainty, and the ensuing multiplicity of possible market outcomes, is intrinsic to a winner-all-market rather than an artifact of the equilibrium concept.

Another possible objection is that a one-shot product improvement game is inapt for studying market dynamics, and, in particular, changes in market leadership. The static asymmetric product improvement game, however, lays a foundation for a dynamic model. I consider a dynamic game in which two rivals have symmetric opportunities for quality improvement at each investment date, and in which the current leader's quality advantage defines the market state in Markov perfect equilibrium. A key element of the dynamic model is that the value of product improvements depreciates over time, for example, due to imitation by a competitive fringe or to obsolescence. Also, firms discount future profits, and in the limit as firms become myopic the dynamic game repeats the static product improvement game period by period with possibly shifting leadership. A first result for the dynamic game is to demonstrate sufficient conditions for (Markov perfect) equilibrium in which market leadership persists, i.e. a firm with an initial quality advantage stays ahead by repeatedly investing in product improvement while its rival is stagnant. A second result is to demonstrate sufficient conditions, including quickly depreciating quality advantages, for a pure equilibrium in which market leadership alternates between rival firms. Finally, for sufficiently great discounting and sufficiently quick quality depreciation, there is a binary mixed equilibrium in which leadership changes stochastically. Thus equilibrium multiplicity exists for nearby dynamic games similarly as for the one-shot product improvement game. The alternating and stochastic leadership equilibria of the dynamic game are interesting because, while the market outcome is winner-take-all period-by-period, the rivals nevertheless manage to share the market over time.

The sequel is organized as follows. Section 2 characterizes the set of discrete equilibria for a one-shot winner-take-all product improvement game that condenses price competition to a reduced form profit function consistent with limited consumer heterogeneity, and computes the equilibrium set in closed form for the case of a quadratic cost of quality improvement. Section 3 demonstrates how the mixed equilibrium outcomes approximate the pure strategy equilibrium outcomes of nearby games of incomplete information. Section 4 builds a dynamic model of product improvement on the foundation of the static model, and derives sufficient conditions for pure strategy equilibria featuring persistent, alternating, or stochastic market leadership. Section 5 concludes with some discussions and interpretations, and points out promising directions for further research.

2. One-shot Product Improvement

2.1. Model

Consider the following simple game. There are two players, indexed $i \in \{1,2\}$. An action for Player i is a non-negative number, $q_i \in R^+$. The payoff of Player 1 is

$$\pi_1(q_1,q_2) = \max\{0, q_1 - q_2 + \Gamma\} - r(q_1) \tag{1}$$

and the payoff of Player 2 is

$$\pi_2(q_1,q_2) = \max\{0, q_2 - q_1 - \Gamma\} - r(q_2) \tag{2}$$

where $\Gamma \geq 0$ and

$$r(q) = \tfrac{1}{2}q^2 \tag{2}$$

The unit of measurement for quality is normalized so that $r'(1) = 1$ and $q = 1$ is the "efficient quality" that maximizes joint surplus. While the quadratic cost of quality function is assumed for simplicity and tractability, some results hold more generally for strictly increasing, convex, differentiable functions, as will be clear from the exposition.

This game has the following interpretation. There are two firms competing in a market. The firms each invest to improve the quality of their products and then choose price. Firm 1 is endowed with an initial quality advantage Γ known to both firms. For each firm, the cost of improving quality by an amount q is $r(q)$. Production costs are normalized to zero. After selecting and observing product qualities, the firms choose prices in Bertrand competition. Consumers are homogeneous, or, more generally, consumer heterogeneity is limited. Consequently, in equilibrium, all consumers purchase the higher quality product at a price that reflects the quality difference.[2] The size of the population of consumers is normalized to unity. Thus the reduced form profit function of Firm *i* is $\pi_i(q_1, q_2)$. Note that these profit functions are not quasi-concave. The function π_i might have local maximum at $q_i = 1$ and a second local maximum at $q_i = 0$.

A strategy for Firm *i* is a cumulative probability distribution over quality improvement. Each firm's quality investment is restricted without loss of generality to the compact interval $Q \in [0, \overline{q}]$, where $\overline{q} + \Gamma = r(\overline{q})$. The reason is that any quality greater then $\overline{q}$ is dominated strictly by $q_i = 0$. By construction, $\overline{q} > 1$. Thus a strategy is a function $F_i(q)$ with domain *Q* and range [0,1]. It is monotone, continuous from the right, and satisfies $F_i(\overline{q}) = 1$. The support of a strategy is

$$S_i = \{q \in Q \mid F_i(q) > 0 \text{ and } F_i(q') < F_i(q) \text{ if } q' < q\} \tag{3}$$

S_i is compact because $F_i(q)$ is right continuous and Q is bounded. Let $\overline{q}_i = \max S_i$ and $\underline{q}_i = \min S_i$.

The expected profit of Firm 1 for quality q_1 is

$$\Pi_1(q_1) = \int_0^{\min\{\overline{q}_2, q_1+\Gamma\}} (\Gamma + q_1 - q)\, dF_2(q) - r(q_1) \tag{4}$$

Similarly, the expected profit for Firm 2 is

$$\Pi_2(q_2) = \int_0^{\min\{\overline{q}_1, q_2-\Gamma\}} (q_2 - \Gamma - q)\, dF_1(q) - r(q_2). \tag{5}$$

2.2. Equilibrium

[2] Appendix A demonstrates how limited consumer heterogeneity results in a winner-take-all outcome of a vertically differentiated price game. See also Gabszewicz and Thisse (1979), Tirole (1988), and Wauthy (1996).

Equilibrium is defined in the usual manner. A pair of strategies (F_1, F_2) form a Nash equilibrium if

$$S_i \subset \arg\max_{q \in Q} \Pi_i(q) \tag{6}$$

for $i = 1,2$. The definition incorporates two key properties. First, all elements of the support of an equilibrium strategy yield the same expected profit, i.e. $\Pi_i(q_i) = \Pi_i(\overline{q}_i)$ for $q_i \in S_i$ ("Indifference"). Second, each element of the support maximizes expected profit, the necessary first-order conditions for which are stated in the next lemma.

<u>Lemma 1 ("Local Optimality")</u>: In equilibrium, $F_2(q_1 + \Gamma) = r'(q_1)$ for $q_1 \in S_1$, and $F_1(q_2 - \Gamma) = r'(q_2)$ for $q_2 \in S_2$.

<u>Proof:</u> This follows from necessary conditions for optimality. Let $F_i^+(q) = \lim_{\tilde{q} \downarrow q} F(\tilde{q})$ and $F_i^-(q) = \lim_{\tilde{q} \uparrow q} F(\tilde{q})$. In equilibrium, a necessary condition for $q_1 \in S_1$ is $F_2^+(q_1 + \Gamma) \le r'(q_1) \le F_2^-(q_1 + \Gamma)$; otherwise, Firm 1 could increase its expected profit by choosing a slightly lower or higher quality. This local optimality condition implies $F_2(q_1 + \Gamma) = r'(q_1)$ for $q_1 \in S_1$, because $F_i(q)$ is increasing and $F_i^+(q) \ge F_i(q) \ge F_i^-(q)$. Similarly, $F_1^+(q_2 - \Gamma) \le r'(q_2) \le F_1^-(q_2 - \Gamma)$ for $q_2 \in S_2$ implies $F_1(q_2 - \Gamma) = r'(q_2)$ for $q_2 \in S_2$. Q.E.D.

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These properties are important in what comes later for two related reasons. First, Local Optimality links Firm 1's equilibrium mixed strategy to Firm 2's equilibrium support, and vice versa. Thus it is possible to construct equilibrium mixed strategies from a knowledge of equilibrium supports. Second, Local Optimality and Indifference are instrumental for characterizing equilibrium supports. Thus a method for describing the equilibria of the game is to first describe equilibrium supports, and second describe corresponding probability distributions.

2.3. Usual Suspects

It is useful to begin by describing the most prominent equilibria. The obviousness of these equilibria perhaps is why the winner-take-all product improvement game has not received much attention in the literature.

If the two firms are symmetric, then there exists a symmetric equilibrium in which each firm randomizes over quality improvements ranging from 0 to 1. In symmetric equilibrium, both firms earn zero expected profits.

Proposition 1 ("Symmetry"): If $\Gamma = 0$, then there is a symmetric equilibrium in which both firms choose the strategy $F(q) = r'(q)$ with the support $S = [0,1]$.

Proof: The symmetric strategy satisfies the necessary equilibrium indifference condition, $\int_0^q (q-z)dF(z) - r(q) = 0$ for all $q \in S$. Moreover, a firm's deviation profit is lower for any $q > 1$ by the convexity of $r(q)$. Q.E.D.

A pure equilibrium has the property that both S_i are singleton sets. There always exists a pure equilibrium in which Firm 1 chooses the efficient quality, and Firm 2 declines to invest. There exists a second pure equilibrium that reverses these roles if Firm 1's initial quality advantage is not too great. In a pure equilibrium, only the designated winning firm ever earns a positive profit.

Proposition 2 ("Purity"): There exists a pure equilibrium in which Firm 1 chooses $q_1 = 1$ and Firm 2 chooses $q_2 = 0$. If $r(1) \geq \Gamma$, then there exists a second pure equilibrium in which Firm 1 chooses $q_1 = 0$ and Firm 2 chooses $q_2 = 1$. There are no other pure equilibria.

Proof: Straightforward.

These familiar-looking equilibria pose a quandary. On the one hand, the symmetric mixed equilibrium is intuitive for symmetrically positioned firms, i.e. for $\Gamma = 0$. On the other hand, the pure equilibrium in which only the leader invests is intuitive if the lead is long, i.e. for $\Gamma >> 0$. What about in between? Any initial asymmetry destroys the symmetric equilibrium, but a pure equilibrium in which one firm earns a substantial profit and the other earns none is counterintuitive if the firms are nearly symmetric, i.e. for $\Gamma > 0$ arbitrarily small. The set of discrete equilibria discussed next bridges the intuitive gap.

2.4. Discrete Equilibria

A *discrete equilibrium* has the property that supports are discrete sets. Furthermore, Local Optimality requires that finite supports are ordered in one of two ways. The elements of the supports for the two firms fit together, one on top of the other, like a zipper. The two types of discrete equilibria are distinguished by which firm is on top.

Lemma 2 ("Zipper Principle"): In a finite discrete equilibrium, $S_1 = \{q_1^1, q_1^2, ... q_1^n\}$ and $S_2 = \{q_2^1, q_2^2, ... q_2^n\}$ for a positive integer *n*. Moreover, either $q_1^j + \Gamma > q_2^j$ for $j = 1, ... n$ with $q_1^n = 1$ and $q_2^1 = 0$ ("Type A equilibrium"), or $q_2^j > q_1^j + \Gamma$ for $j = 1, ... n$ with $q_1^1 = 0$ and $q_2^n = 1$ ("Type B equilibrium").

Proof: Appendix B.

Note that pure equilibria are special cases ($n = 1$).

With knowledge of the equilibrium supports in hand, it is straightforward to use Local Optimality and the Zipper Principle to construct the equilibrium strategies for both types of equilibria.

Lemma 3: (a) $F_1(q_1^k) = r'(q_2^{k+1})$ for $k = 1, ... n-1$, $F_1(q_1^n) = 1$, and $F_2(q_2^k) = r'(q_1^k)$ for $k = 1, ... n$ in a Type A equilibrium. (b) $F_1(q_1^k) = r'(q_2^k)$ for $k = 1, ... n$, $F_2(q_2^k) = r'(q_1^{k+1})$ for $k = 1, ... n-1$, and $F_2(q_2^n) = 1$ in a Type B equilibrium.

2.5. Computation

All the needed elements are in place to compute equilibrium supports for finite discrete equilibria. Type A and Type B equilibria are treated in turn.

Consider a candidate Type A mixed equilibrium ($n \geq 2$). Let α_i^j denote the probability that Firm *i* selects $q_i^j \in S_i$, i.e. $\alpha_i^1 = F_i(q_i^1)$ and $\alpha_i^j = F_i(q_i^j) - F_i(q_i^{j-1})$ for $j = 2,..n$. Using this notation, the expected profit of Firm 1 is

$$\Pi_1(q_1^k) = \sum_{j=1}^{k} \alpha_2^j (q_1^k + \Gamma - q_2^j) - r(q_1^k) \tag{7}$$

and Local Optimality requires

$$r'(q_1^k) = \sum_{j=1}^{k} \alpha_2^j \tag{8}$$

for $k = 1,...n$. Therefore,

$$\Pi_1(q_1^k) = \sum_{j=1}^{k} [r'(q_1^j) - r'(q_1^{j-1})](q_1^k + \Gamma - q_2^j) - r(q_1^k) \tag{9}$$

for $k = 1,...n$ and $q_1^0 \equiv 0$. In a Type A equilibrium, Indifference means there is a number $\Pi > 0$ such that $\Pi_1(q_1^k) = \Pi$ for $q_1^k \in S_1$. The reason why Firm 1 earns a strictly positive profit is that $\alpha_2^1 > 0$ implies Firm 1 earns at least $\alpha_2^1 \Gamma > 0$. Given S_2, it is possible to solve for S_1 using this indifference condition. The precise value of Π is pinned down by the boundary condition $q_1^n = 1$.

Alternatively, the Type A equilibrium conditions for Firm 1 imply $q_1^n = 1$ and

$$\Pi_1(q_1^k) - \Pi_1(q_1^{k-1}) = \alpha_2^k[q_1^k + \Gamma - q_2^k] + \sum_{j=1}^{k-1} \alpha_2^j [q_1^k - q_1^{k-1}] - r(q_1^k) + r(q_1^{k-1}) \qquad (10)$$

for $k = 2,...n$. Therefore, Indifference and Local Optimality further imply

$$[r'(q_1^k) - r'(q_1^{k-1})][q_1^k + \Gamma - q_2^k] + r'(q_1^{k-1})[q_1^k - q_1^{k-1}] = r(q_1^k) - r(q_1^{k-1}) \qquad (11)$$

or, equivalently

$$r'(q_1^k)[q_1^k + \Gamma - q_2^k] + r'(q_1^{k-1})[q_2^k - q_1^{k-1} - \Gamma] = r(q_1^k) - r(q_1^{k-1}) \qquad (12)$$

for $k = 2,...n$. Given $S_2 = \{q_2^1,...q_2^n\}$, this is a first-order difference equation in q_1^k. Starting with $q_1^n = 1$, the equation can be solved recursively for values of q_1^{k-1}. For example, q_1^{n-1} solves $[1 + \Gamma - q_2^n] + r'(q_1^{n-1})[q_2^n - q_1^{n-1} - \Gamma] = r(1) - r(q_1^{n-1})$. Moreover, Firm 1 has no incentive unilaterally to deviate from this computed strategy because, given the convexity of $r(q)$, q_1^1 is by construction optimal on the interval $[0, q_2^2 - \Gamma)$, q_1^k is optimal on $[q_2^k - \Gamma, q_2^{k+1} - \Gamma)$ for $k = 2,..n-1$, and $q_1^n = 1$ is optimal on $[q_2^n - \Gamma, \bar{q}]$.

Similarly, still supposing a Type A equilibrium, it is possible to solve for S_2 given S_1 by imposing the break-even condition $\Pi_2(q_2^k) = \Pi_2(0) = 0$ and the boundary condition $q_2^1 = 0$. We have

$$\Pi_2(q_2^k) = \sum_{j=1}^{k-1} \alpha_1^j (q_2^k - \Gamma - q_1^j) - r(q_2^k) \qquad (13)$$

.

Therefore, $\Pi_2(q_2^k) - \Pi_2(q_2^{k-1}) = 0$ implies (for $k = 2,...n$)

$$\alpha_1^{k-1}(q_2^k - \Gamma - q_1^{k-1}) + \sum_{j=1}^{k-2} \alpha_1^j (q_2^k - q_2^{k-1}) = r(q_2^k) - r(q_2^{k-1}) \qquad (14)$$

Local Optimality then implies

$$r'(q_2^k)(q_2^k - \Gamma - q_1^{k-1}) + r'(q_2^{k-1})(q_1^{k-1} + \Gamma - q_2^{k-1}) = r(q_2^k) - r(q_2^{k-1}) \qquad (15)$$

Given S_1 and $q_2^1 = 0$, this equation is solved recursively for q_2^k. Firm 2 has no incentive to deviate from this computed strategy by the convexity of $r(q)$.

Summarizing, the computation of a Type A mixed equilibrium amounts to using Indifference, Local Optimality, and Discreteness to find (S_1, S_2). These necessary conditions lead to a pair of difference equations and boundary conditions. The equilibrium supports are the solution of this system, and Local Optimality determines the corresponding choice probabilities. Sufficiency of this solution for equilibrium follows by verifying that neither firm has an incentive to deviate at the solution.

The computation of a Type B equilibrium is similar, except that there is an additional necessary. The relevant difference equations are

$$r'(q_2^k)[q_2^k - \Gamma - q_1^k] + r'(q_2^{k-1})[q_1^k - q_2^{k-1} + \Gamma] = r(q_2^k) - r(q_2^{k-1}) \tag{16}$$

and

$$r'(q_1^k)(q_1^k + \Gamma - q_2^{k-1}) + r'(q_1^{k-1})(q_2^{k-1} - \Gamma - q_1^{k-1}) = r(q_1^k) - r(q_1^{k-1}) \tag{17}$$

The additional No-Leapfrogging condition assures that the more efficient Firm 1 does not have an incentive to choose $q_1 = 1$ and win with probability one. This is the most profitable deviation that guarantees victory for Firm 1, so any higher quality deviation cannot be profitable either. Since Firm 1 earns zero profit in a Type B equilibrium, No-Leapfrogging requires the profits from the most profitable deviation to be non-positive:

$$\Gamma + \sum_{k=2}^{n} (q_2^k - q_2^{k-1}) r'(q_1^k) \le r(1) \tag{17}$$

Applying these results to quadratic $r(q)$, a Type A mixed equilibrium with n points of support satisfies the difference equations

$$q_1^k(q_1^k+\Gamma-q_2^k)+q_1^{k-1}(q_2^k-q_1^{k-1}-\Gamma)=\tfrac{1}{2}(q_1^k)^2-\tfrac{1}{2}(q_1^{k-1})^2 \tag{18}$$

and

$$q_2^k(q_2^k-\Gamma-q_1^{k-1})+q_2^{k-1}(q_1^{k-1}+\Gamma-q_2^{k-1})=\tfrac{1}{2}(q_2^k)^2-\tfrac{1}{2}(q_2^{k-1})^2 \tag{19}$$

for $k=2,...n$, and the boundary conditions $q_1^n=1$ and $q_2^1=0$. The difference equations simplify to

$$q_2^k-\Gamma=\tfrac{1}{2}\left(q_1^k+q_1^{k-1}\right) \tag{20}$$

and

$$q_1^{k-1}+\Gamma=\tfrac{1}{2}\left(q_2^k+q_2^{k-1}\right) \tag{21}$$

for $k=2,...n$.

The above difference equations can be manipulated to imply a simple second-order linear difference equation for Firm 2:

$$q_2^{k+1}-2q_2^k+q_2^{k-1}=0 \tag{22}$$

for $k=2,...n-1$. The two boundary conditions are $q_2^1=0$ and

$$q_2^n=\tfrac{2}{3}\left(1+\Gamma\right)+\tfrac{1}{3}q_2^{n-1} \tag{23}$$

Equilibrium strategies are constructed by solving this difference equation in closed form. A similar approach yields closed form solutions for Type B equilibria. The next proposition states the results, and Appendix C completes the proof.

Proposition 3 (Computation): (a) Type A discrete equilibrium with $n=\#S_1=\#S_2$ exists if and only if

$$n\le 1+\frac{1}{2\Gamma} \tag{24}$$

The elements of the equilibrium supports are:

$$q_1^k = \frac{2k-1}{2n-1}(1+\Gamma) - \Gamma \tag{25}$$

$$q_2^k = \frac{2(k-1)}{2n-1}(1+\Gamma) \tag{25}$$

for $k = 1,...n$. The equilibrium strategies are:

$$F_1(q_1^k) = \begin{cases} \frac{2k}{2n-1}(1+\Gamma) & \text{for } k = 1,...n-1 \\ 1 & \text{for } k = n \end{cases} \tag{26}$$

$$F_2(q_2^k) = \tfrac{2k-1}{2n-1}(1+\Gamma) - \Gamma \tag{27}$$

(b) Type B discrete equilibrium with $n = \#S_1 = \#S_2$ exists if and only if

$$n \le \frac{1}{2\Gamma} \tag{28}$$

The elements of the equilibrium supports are:

$$q_1^k = \left[\frac{2(k-1)}{2n-1}\right](1-\Gamma) \tag{29}$$

and

$$q_2^k = \left[\frac{2k-1}{2n-1}\right](1-\Gamma) + \Gamma \tag{30}$$

for $k = 1,...n$. The equilibrium strategies are:

$$F_2(q_2^k) = \begin{cases} \frac{2k}{2n-1}(1-\Gamma) & \text{for } k = 1,...n-1 \\ 1 & \text{for } k = n \end{cases} \tag{31}$$

$$F_1(q_1^k) = \tfrac{2k-1}{2n-1}(1-\Gamma) + \Gamma \tag{32}$$

The number of Type B equilibria is exactly one fewer than Type A equilibria. Consequently, the total number of discrete equilibria when $\Gamma > 0$ is finite and odd. The reason for fewer Type B equilibria is No Leapfrogging, i.e. Firm 1 must not have an incentive to invest efficiently and win the market with probability one.

Equilibrium choice frequencies are constructed by differencing the equilibrium strategies. For Type A equilibria:

$$\alpha_1^k = \begin{cases} \frac{2}{2n-1}(1+\Gamma) & \text{for } k = 1,\ldots n-1 \\ 1 - \frac{2(n-1)}{2n-1}\left(1+\Gamma\right) & \text{for } k = n \end{cases} \tag{33}$$

$$\alpha_2^k = \begin{cases} \frac{1}{2n-1}(1+\Gamma) - \Gamma & \text{for } k = 1 \\ \frac{2}{2n-1}(1+\Gamma) & \text{for } k = 2,\ldots n \end{cases} \tag{34}$$

for $k = 1,\ldots n$. Notice that the frequencies are almost uniform. The only exceptions to uniformity for Type A equilibrium is that Firm 1 chooses $q_1^n = 1$ and Firm 2 chooses $q_2^1 = 0$ with half the frequency of other choices. Similarly, for Type B equilibria:

$$\alpha_2^k = \begin{cases} \frac{2}{2n-1}(1-\Gamma) & \text{for } k = 1,\ldots n-1 \\ 1 - \frac{2(n-1)}{2n-1}\left(1-\Gamma\right) & \text{for } k = n \end{cases} \tag{35}$$

$$\alpha_1^k = \begin{cases} \frac{1}{2n-1}(1-\Gamma) + \Gamma & \text{for } k = 1 \\ \frac{2}{2n-1}(1-\Gamma) & \text{for } k = 2,\ldots n \end{cases} \tag{36}$$

for $k = 1,\ldots n$. The departures from uniformity for Type B equilibria are for $q_1^1 = 0$ and $q_2^n = 1$.

As $\Gamma \to 0$, there is a twice-countable infinity of discrete equilibria. Furthermore, as $n \to \infty$, the equilibrium supports for both firms are packed into the unit interval, and the asymmetric equilibrium strategies of the two firms converge to the symmetric equilibrium strategy of Proposition 1. Thus, for small $\Gamma > 0$, a discrete mixed strategy equilibrium, especially one with maximal n, approximates the continuous mixed strategy equilibrium for the symmetric case. The approximation is closer and closer with higher values of n. This conclusion resolves the quandary posed earlier, because, with selection of the maximal n mixed equilibrium, a small initial asymmetry results in only a small difference in conduct. By virtue of this convergence property, a mixed equilibrium seems more intuitive than a pure equilibrium when $\Gamma > 0$ is small.

3. Purification

It might be tempting to dismiss the mixed equilibria as artificial, but that would be a mistake. The main properties of the winner-take-all quality competition model are robust to introducing a small amount of incomplete information, and its mixed equilibria are limits of outcome distributions of pure equilibria of an appropriate sequence of incomplete information models. The general idea that a mixed equilibrium can be "purified" in this way stems from Harsanyi (1973), and the specific purification used here is patterned after Bagwell and Wolinsky (2002).[3]

Consider for simplicity the symmetric model, and extend it to a nearby incomplete information model as follows. The cost of quality improvement for Firm i in the extended model is $r(q,t_i) = \delta c(t_i)q + \frac{1}{2}q^2$, where $c(t)$ is a continuously decreasing function on $[0,1]$ with $c(0)=1$ and $c(1)=0$, and δ is a small positive constant. Firm i's type t_i is an independent draw from a standard uniform distribution, and each firm privately learns its type before choosing quality. In other respects, quality competition is the same as before.

[3] The reader who needs no further convincing of the merits of mixed equilibria can skip to the next section with no loss of continuity.

Bayes-Nash equilibria of the extended model with δ sufficiently small are constructed as follows. For any positive integer *n*, let $\{s_1^1,...s_1^n\}$ and $\{s_2^1,...s_2^n\}$ be two strictly increasing sequences with $s_2^n = 1$, $s_1^1 = 0$, and $s_2^k > s_1^k$, and define quantity sequences

$$\hat{q}_1^k(t) = s_2^k - \delta c(t) \quad \text{for } k \in \{1,...n\} \tag{37}$$

and

$$\hat{q}_2^k(t) = s_1^k - \delta c(t) \quad \text{for } k \in \{2...n\} \tag{38}$$

Using these functions, and letting $s_2^0 \equiv 0$ and $\hat{q}_2^1 \equiv 0$, further define the corresponding profit sequences

$$\hat{\pi}_1^k(t_1) = \tfrac{1}{2}\hat{q}_1^k(t_1)^2 - \sum_{j=1}^{k}\int_{s_2^{j-1}}^{s_2^j} \hat{q}_2^j(t)\,dt \quad \text{for } k \in \{1,...n\} \tag{39}$$

and

$$\hat{\pi}_2^k(t_2) = \begin{cases} 0 & \text{for } k = 1 \\ \tfrac{1}{2}\hat{q}_2^k(t_2)^2 - \sum_{j=2}^{k}\int_{s_1^{j-1}}^{s_1^j} \hat{q}_1^{j-1}(t)\,dt & \text{for } k \in \{2,...n\} \end{cases} \tag{40}$$

Now require the s_i^k to satisfy the following indifference conditions:

$$\hat{\pi}_1^k(s_1^k) = \hat{\pi}_1^{k-1}(s_1^k) \quad \text{for } k \in \{2,...n\} \tag{41}$$

$$\hat{\pi}_2^2(s_2^1) = 0\,; \tag{42}$$

$$\hat{\pi}_2^{k+1}(s_2^k) = \hat{\pi}_2^k(s_2^k) \quad \text{for } k \in \{2,...n-1\} \tag{43}$$

Consider the following strategies:

$$q_1(t) = \begin{cases} \hat{q}_1^k(t) & \text{if} \quad s_1^{k+1} > t \geq s_1^k \text{ and } k \in \{1,...n-1\} \\ \hat{q}_1^n(t) & \text{if} \quad t \geq s_1^n \end{cases} \tag{44}$$

$$q_2(t) = \begin{cases} 0 & \text{if} \quad s_2^1 \geq t \\ \hat{q}_2^k(t) & \text{if} \quad s_2^k \geq t > s_2^{k-1} \quad \text{and } k \in \{2,...n\} \end{cases} \tag{45}$$

The distribution of outcomes resulting from these strategies converges to the Type A equilibrium corresponding to *n* of the original game as $\delta \to 0$. Furthermore, it follows from $\hat{q}_1^k(t) \to s_2^k$ and $\hat{q}_2^k(t) \to s_1^k$ that

$$\hat{\pi}_1^k(t_1) \to \tfrac{1}{2}\left(s_2^k\right)^2 - \sum_{j=1}^{k}\left(s_2^j - s_2^{j-1}\right)s_1^j \qquad \text{for } k \in \{1,...n\} \tag{46}$$

and

$$\hat{\pi}_2^k(t) \to \tfrac{1}{2}\left(s_1^k\right)^2 - \sum_{j=2}^{k}\left(s_1^j - s_1^{j-1}\right)s_2^{j-1} \qquad \text{for } k \in \{2,...n\} \tag{47}$$

Therefore, in the limit, the indifference conditions become

$$\tfrac{1}{2}(s_2^k)^2 - \sum_{j=1}^{k}\left(s_2^j - s_2^{j-1}\right)s_1^j = \tfrac{1}{2}(s_2^{k-1})^2 - \sum_{j=1}^{k-1}\left(s_2^j - s_2^{j-1}\right)s_1^j \tag{48}$$

and

$$\tfrac{1}{2}\left(s_1^k\right)^2 - \sum_{j=2}^{k}\left(s_1^j - s_1^{j-1}\right)s_2^{j-1} = 0 \tag{49}$$

for $k \in \{2,...n\}$. The limiting system of difference equation is solved by $s_1^k = \dfrac{2(k-1)}{2n-1}$ and $s_2^k = \dfrac{2k-1}{2n-1}$, whence

$$\hat{q}_1^k \to \frac{2k-1}{2n-1} \tag{50}$$

and

$$\hat{q}_2^k \to \frac{2(k-1)}{2n-1} \tag{51}$$

Furthermore, the probabilities of these outcomes are the same as for the corresponding mixed equilibrium pf the complete information game.

It remains to argue that $q_1(t)$ and $q_2(t)$ are equilibrium strategies of the extended game for δ sufficiently small. First, there exist $\{s_1^1,...s_1^n\}$ and $\{s_2^1,...s_2^n\}$ satisfying the requisite conditions. More specifically, there exists a solution at $\delta = 0$ as shown above, and the solution is differentiable at $\delta = 0$, implying a nearby solution for δ sufficiently small. Second, Firm 1's profit function if Firm 2 follows $q_2(t)$ is

$$\pi_1(q,t_1) = s_2^1 q + \sum_{k=2}^{n} \int_{s_2^{k-1}}^{s_2^k} \max\{q - \hat{q}_2^k(t), 0\} dt - r(q,t_1) \quad (52)$$

The $\hat{q}_1^k(t_1)$ are the local maxima of this function, and $\hat{\pi}_1^k(t_1)$ are the corresponding profit levels. Third, by construction, $\hat{\pi}_1^k(s_1^k) = \hat{\pi}_1^{k-1}(s_1^k)$ and $\frac{d\hat{\pi}_1^k(t)}{dt} > \frac{d\hat{\pi}_1^{k-1}(t)}{dt}$. It follows that $q_1(t)$ is a best response. $q_2(t)$ is a best response for Firm 2 by a similar argument.

In conclusion, the Type A mixed equilibria of the original game with $\Gamma = 0$ approximate the outcome distribution of pure equilibria of nearby games of incomplete information.[4] Indeed the mixed strategies can be interpreted as capturing equilibrium beliefs about rivals' investments in quality improvement (Aumann and Brandenburger, 1995). Thus strategic uncertainty and multiplicity of equilibria can be viewed as intrinsic to winner-take-all quality competition games, and do not hinge on randomized actions.

4. Dynamics

4.1. Introduction

[4] A similar constructive argument is apparent for $\Gamma > 0$ and for Type B mixed equilibria.

A reasonable criticism is that a static product improvement game is a poor vehicle for studying market leadership, and therefore is best for analyzing symmetrically positioned firms. Furthermore, since the symmetric mixed equilibrium is intuitive and already well understood, an enumeration of additional less intuitive asymmetric mixed equilibria adds little to useful economic understanding. The static asymmetric product improvement game, however, lays a foundation for a dynamic game in which two rivals have symmetric opportunities for quality improvement at each investment date, and in which leadership evolves over time. I demonstrate below that various equilibria of the static game have close analogs in nearby dynamic models.

4.2. Model

Consider the following dynamic model that builds on the static product improvement model. Product improvements are made by each of two firms at an infinite sequence of dates $t = 1, \dots \infty$. The cost of quality improvement at any date is quadratic, i.e. $r(q) = \frac{1}{2}q^2$. Future profits are discounted by a factor of δ, and qualities depreciate each period by a factor of γ. At the start of each date, the state of the market is defined as the identity of the leader and the length of lead Γ, and Markov perfect equilibrium is the solution concept.

The discount and depreciation factors are the key dynamic elements of the model. δ can be interpreted as indicating the length of the innovation cycle, and γ as indicating the degree of intellectual property protection or the degree to which improvements become obsolete. In order to emphasize comparisons with the one-shot product improvement model, I focus mostly on markets for which δ and γ are small, e.g. innovation cycles are long and intellectual property protection is weak.

It is useful at the start to consider the benchmark of a "perpetual monopoly" starting with a quality $\Gamma \geq 0$ at some initial date.[5] An investment of $r(q) = \frac{1}{2}q^2$ at some initial date yields a sequence of returns $\{q, \gamma q, \gamma^2 q, \dots.\}$ with a present discounted value of $\frac{q}{1-\delta\gamma}$. Therefore, the value of an optimal investment in any period is

$$M(\delta\gamma) \equiv \max_{q\geq 0}\left\{\tfrac{1}{1-\delta\gamma}q - r(q)\right\} \equiv \frac{1}{2(1-\delta\gamma)^2} \tag{53}$$

Since a perpetual monopolist has the opportunity to make this profit-maximizing investment each and every period, the value of the perpetual monopoly with initial quality Γ is

$$\frac{\Gamma}{1-\delta\gamma} + \frac{M(\delta\gamma)}{1-\delta} \equiv \frac{\Gamma}{1-\delta\gamma} + \frac{1}{2(1-\delta)(1-\delta\gamma)^2} \tag{54}$$

Note that, as $\delta \to 0$ the value of a perpetual monopoly converges to $\Gamma + \frac{1}{2}$, thus tracking the Type A pure equilibrium of the one-shot model. The above formula also shows how discounting and depreciation have distinct effects on the returns from quality improvement. The two dynamic elements interact multiplicatively, however, to determine the evolution of the market. Profit-maximizing quality improves by $\frac{1}{1-\delta\gamma}$ each period, and total quality converges over time to $\frac{1}{(1-\gamma)(1-\delta\gamma)}$. If $\Gamma < \frac{1}{(1-\gamma)(1-\delta\gamma)}$ initially,

[5] The maintained implicit assumption is that there is a competitively supplied alternative whose value is normalized to zero each period. In particular, the depreciation of quality can be interpreted to mean that the competitive market imitates quality improvements at rate γ.

then total quality grows over time, reaching $\frac{1}{(1-\gamma)(1-\delta\gamma)}$ asymptotically. Conversely, if $\Gamma > \frac{1}{(1-\gamma)(1-\delta\gamma)}$ then the asymptote is approached from above.

I now return to the dynamic duopoly game. I examine below three different kinds of equilibria, focusing on models appropriately close to the one-shot product improvement model. My purpose is to show that the multiple equilibria of the one-shot model have close analogs in the dynamic model. The possible equilibrium outcomes I consider are market dominance by one of the firms, alternating leadership whereby duopolist share the market dynamically, and stochastic leadership due to strategic uncertainty over quality improvements.

4.3. Market Dominance

An economics literature on technology adoption in dynamic Bertrand models examines conditions such that Markov perfect equilibria feature market dominance or leapfrogging along the equilibrium path (Riordan and Salant, 1994; Giovannetti, 2001; Iskhahov, Rust, and Schjerning, 2013). These models assume that state-of-the-art production technology evolves exogenously, and focus on the incentives of the market leader and laggard to incur the fixed costs of adopting the state-of-the-art technology. My analysis of dynamic product improvement similarly examines conditions for market dominance and leapfrogging, but my model is different because the state-of-the-art evolves endogenously depending on how firms invest over time.

First, I consider the conditions for a duopoly equilibrium in which only the market leader invests in quality improvement. Suppose that Firm 1 has an arbitrary initial advantage $\Gamma \geq 0$. In the equilibrium characterized below, the initial leader is effectively a perpetual monopolist along the equilibrium path, investing a constant amount independent of the length of the lead, while the lagging firm invests 0. Since a perpetual monopoly is more profitable, the laggard must be sufficiently myopic to acquiesce, which means δ cannot be too large.

Proposition 4 ("Market Dominance"): If $\delta \le \frac{1}{2}$ in the dynamic game, then there is an equilibrium such that the leader invests $\frac{1}{1-\delta\gamma}$ and the laggard invests 0 for subgames with $\Gamma > 0$, and the firms play a symmetric mixed strategy for subgames with $\Gamma = 0$.

Proof: See Appendix D.

The result establishes that the pure equilibrium of the one-shot game in which the leader wins approximates conduct in the dynamic market dominance equilibrium if firms are sufficiently myopic or quality leadership is sufficiently transitory. Indeed, as $\delta \to 0$ or $\gamma \to 0$, equilibrium conduct is the same as the Type A pure equilibrium on the static model. Note that the existence result is independent of the depreciation factor γ.

The market dominance equilibrium fails, however, if the initial quality advantage is small and the discount factor is sufficiently above $\frac{1}{2}$. The reason is that the laggard wants to displace the leader as a perpetual monopolist. A lower discount factor controls this incentive to leapfrog by reducing the value of the perpetual monopoly. The condition on δ can be interpreted to mean that market dominance equilibrium exists if the innovation cycle is sufficiently long, so the returns to an overtaking laggard are pushed forward into the future.

4.4. Alternating Leadership

Next, I consider the possibility of an equilibrium in which the two firms alternate winning the market. That is, in each period, the lagging firm captures the market, and thus over time the two firms share the market. I refer to the firm that is about to overtake the lead as the "current laggard" and to its rival as the "current leader".

In the equilibrium described below, the depreciation factor (γ) must be sufficiently low to keep the market lead from becoming so long that the leader necessarily is dominant. As long as the lead length stays within bounds, the current laggard retains an incentive to invest sufficiently more than the current leader to overtake the lead. Consequently, the two firms comfortably share the market over time, taking turns at winning the market.

Equilibrium encompasses a description of actions for all subgames, including those never reached on the equilibrium path. The alternating leadership equilibrium proscribes that, should an excessively long lead occur out of equilibrium, say by some accident, the current laggard would adopt a simple "wait it out" strategy, allowing the leader's quality advantage to depreciate until it is again profitable for the laggard to resume positive investments in product improvement. Thus, while leadership alternates strictly on the equilibrium path, there might be short periods of market dominance at out-of-equilibrium subgames, the brevity of which depends on the length of the accidental lead and on the speed at which the leaders' advantage depreciates.

Proposition 5 (“Alternating Leadership”): If γ is sufficiently small in the dynamic game, then there is an equilibrium such that, for some critical $\hat{\Gamma} > 0$: (a) the current leader invests $\frac{\delta\gamma}{1-\delta^2\gamma^2}$ and the current laggard invests $\frac{1}{1-\delta^2\gamma^2}$ for all subgames with $0 < \Gamma \leq \hat{\Gamma}$; (b) the current leader invests $\frac{1}{1-\delta^2\gamma^2}$ and the laggard invests 0 for all subgames with $\Gamma > \hat{\Gamma}$; (c) Firm 1 invests $\frac{1}{1-\delta^2\gamma^2}$ and Firm 2 invests invests $\frac{\delta\gamma}{1-\delta^2\gamma^2}$ for the $\Gamma = 0$ subgame.

Proof: See Appendix D.

Alternating leadership equilibrium has the interesting property that the firms invest in product even during the period when they do not make any sales. This is because the investments have a future return. This result addresses the concern expressed by Gabszewicz and Thisse (1979) for a one-shot Bertrand model that a lagging firm has no reason to participate in a winner-take-all market other than to provide price discipline. In the alternating leadership equilibrium of the dynamic model, both leader and laggard each period invest with an eye toward future returns to leadership.

4.5. Stochastic Leadership

There also exist mixed equilibria pf the dynamic game in which market leadership evolves stochastically. This is clear for myopic dynamic model. If $\delta = 0$ and, $\gamma > 0$ then the rival firms are myopic but begin each period in asymmetric positions. In this case, each subgame of the dynamic model inherits the multiple of equilibria of the static model, and the market evolves in various alternative ways depending on the period-by-period realization of Γ and equilibrium selection.[6] If γ is sufficiently large, then eventually $\Gamma > 0$, at which point the market leader becomes a perpetual monopolist, but the there are many possible equilibrium paths to the market dominance outcome. While it is reasonable to expect that similar multiplicity of equilibria and variety of leadership dynamics also exist for sufficiently small $\delta > 0$, a full consideration of the equilibrium set for the non-myopic model is complex and beyond the scope of this paper. Instead, I consider binary mixed strategies for the dynamic model that are close analogs of the binary Type A equilibrium strategies of the one-shot model when both δ and γ are small.[7]

For simplicity, I restrict attention to the case of γ sufficiently small that it is unprofitable for even a perpetual monopolist to maintain $\Gamma \geq \frac{1}{2}$. To be more precise, if the monopolist invests q, then the market evolves according to

$$\Gamma' = \gamma(q + \Gamma).$$

Therefore to maintain $\Gamma = \frac{1}{2}$, investment must be at least equal to

$$q = \frac{1-\gamma}{2\gamma}$$

in which case steady state profits are

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[6] Alternatively, if $\gamma = 0$ and $\delta > 0$, then both firms care about the future but cannot influence it, and begin each period in a symmetric position. In this case, repetitions of the symmetric mixed strategy equilibrium of the one-shot game results in a stationary distribution of market outcomes, although many asymmetric industry evolutions are possible depending on equilibrium outcomes.

[7] While I only consider binary mixed strategy equilibrium, it is an eminently reasonable conjecture that similar analyses would demonstrate discrete mixed equilibria with an arbitrary number points of support for δ and γ sufficiently small.

$$\frac{1}{2\gamma}-\frac{1}{2}\left(\frac{1-\gamma}{2\gamma}\right)^2=\frac{1}{2\gamma}\left[1-\frac{(1-\gamma)^2}{4\gamma}\right]$$

The value function for this strategy is negative if γ sufficiently small. Under such conditions, obviously it also would be unprofitable for a duopolist to maintain a lead of $\Gamma \geq \frac{1}{2}$ that foreclosed its rival. Consequently, if δ and γ are sufficiently small, Γ remains remains within the bounds required for equilibrium binary mixed actions.

The next proposition takes a straightforward comparative dynamics approach to examining equilibrium binary mixed actions in the neighborhood of the myopic case ($\delta = 0$). To interpret the proposition, it is helpful to keep in mind that the myopic case has the leader investing $q_1^1 > 0$ with probability α_1^1 and $q_1^2 = 1 > q_1^1$ otherwise, and the laggard investing $q_2^2 < 1$ with probability α_2^2 and $q_2^1 = 0 < q_2^2$ otherwise. The proposition shows the departures from this benchmark for infinitesimally positive δ.

Proposition 6: Assume δ and γ are sufficiently small that there exists an equilibrium in which firms play binary mixed strategies for all reached Γ.[8] In the neighborhood of $\delta = 0$, the equilibrium strategies depart from the Type A equilibrium binary mixed strategies of the one-shot game according to

[8] This simplifying assumption avoids the need explicitly to consider equilibrium actions at unreached subgames. See Proposition 8 for details on how a complete equilibrium strategy might be constructed for δ and γ sufficiently small by specifying a "wait it out" strategy for the current laggard at out-of-equilibrium subgames with excessively long leads,

$$
\begin{bmatrix} dq_1^2 \\ dq_1^1 \\ dq_2^2 \\ dq_2^1 \\ d\alpha_1^1 \\ d\alpha_2^2 \end{bmatrix}_{\delta=0} = \begin{bmatrix} \frac{1}{81}\gamma\left[9-8\gamma(5+\Gamma-4\Gamma^2)\right] \\ \frac{1}{81}\gamma(2\Gamma-1)\left[8\gamma(1+\Gamma)-3\right] \\ \frac{2}{81}\gamma(1+\Gamma)\left[8\gamma(1+\Gamma)-3\right] \\ 0 \\ \frac{1}{243}(1+\Gamma)\left[9-6\gamma(\Gamma-2)+8\gamma^2(2\Gamma+7\Gamma^2-5)\right] \\ \frac{1}{243}(1+\Gamma)\left[9+6\gamma(1+\Gamma)-8\gamma^2(1+\Gamma)^2\right] \end{bmatrix} d\delta \qquad (55)
$$

Proof: See Appendix D.

How do equilibrium strategies depart from the one-shot binary equilibrium strategies when δ and γ are small? Evaluating proposition's result at $\gamma = 0$ gives

$$
\begin{bmatrix} dq_1^2 \\ dq_1^1 \\ dq_2^2 \\ dq_2^1 \\ d\alpha_1^1 \\ d\alpha_2^2 \end{bmatrix}_{\delta=0,\gamma=0} = \begin{bmatrix} 0 \\ \frac{1}{9} \\ \frac{1}{9} \\ 0 \\ \frac{1}{9} \\ -\frac{1}{9} \end{bmatrix} d\delta \qquad (56)
$$

Thus caring only slightly about the future leads Firm 1 to choose a higher value of q_1^1 with higher probability, and leads Firm 2 to choose a higher value of q_2^2 with lower probability. Thus there is not a clear interpretation about whether a small degree of foresightedness makes the rivals more or less aggressive.

How does a slightly higher value of γ affect these conclusions? Taking derivative the strategy components again with respect to γ, evaluated at $\gamma = 0$ and holding Γ constant, gives

$$\begin{bmatrix} d^2 q_1^2 \\ d^2 q_1^1 \\ d^2 q_2^2 \\ d^2 q_2^1 \\ d^2 \alpha_1^1 \\ d^2 \alpha_2^2 \end{bmatrix}_{\delta=0,\gamma=0} = \begin{bmatrix} \frac{1}{9} \\ \frac{1}{27} \\ \frac{2}{27} \\ 0 \\ -\frac{2}{27}\Gamma \\ -\frac{2}{27}\Gamma \end{bmatrix} d\delta d\gamma \tag{57}$$

Thus slightly higher γ clearly causes Firm 1 to be more aggressive, while the interpretation of Firm 2's response is muddy.

6. Conclusion

A standard winner-take-all duopoly product improvement game has multiple equilibria. These include pure equilibria in which one or the other firm invests and wins the market, and, if firms are symmetric *ex ante*, both a symmetric mixed equilibrium and a twice-countable infinity of asymmetric discrete mixed equilibria. If one of the firms has an *ex ante* advantage, then the symmetric equilibrium vanishes and the number of asymmetric equilibria is pruned, but, unless the initial asymmetry is large, asymmetric mixed equilibria still exist. For a small asymmetry, a mixed equilbrium arguably is more intuitive then is a pure equilibrium. Furthermore, the mixed equilbria are approximations of pure equilibria for nearby games of incomplete information, and therefore cannot be dismissed easily as artificial. Finally, the multiple equilibria of the one-shot product improvement game have close analogs in nearby dynamic product improvement games in which firms have repeated opportunities for product improvement.

Previous analyses of vertical product differentiation usually assume sufficient consumer heterogeneity to support a pure equilibrium in a two stage game with simultaneous quality choice followed by Bertrand-Nash price competition (e.g. Choi and Shin, 1992; Motta, 1993, Shaked and Sutton, 1982, Tirole, 1988; Wauthy, 1996). When consumer heterogeneity is limited, however, price competition a winner-take-all character (Gabszewicz and Thisse, 1979; Wauthy, 1996), and, assuming a convex increasing fixed cost of quality improvement therefore have multiple mixed equilibria as in the the case of homogeneous consumers. Furthermore, it seems a reasonable conjecture that multiple equilibria similarly exist when consumer heterogeneity is sufficient to support contemporaneous market sharing.[9] Exploring the range of equilibria under broader assumptions about consumer heterogeneity is a promising topic for future research.

Simultaneous entry models in which firms make a simple in-or-out decision are known to have multiple equilibria with different welfare consequences (Cabral, 2004; Dixit and Shapiro, 1986; Vettas, 2000). The model studied here might be reinterpreted as a richer entry model in which firms choose quality as part of the entry decision, i.e. quality is an endogenous sunk cost of entry (Sutton, 1991). The interesting implication from my results is that equilibria of entry games proliferate with the richer action space if potential entrants are not too different initially. Analysis of entry models in which product selection is part of the entry decision is a tantalizing topic for further research.[10]

Finally, multiple equilibria create problems for economic policy, because of the accompanying difficulty in predicting policy outcomes. Economic regulation might play a role of eliminating unwanted equilibria. For example, Gilbert and Riordan (2003, 2007) show how access regulation can reduce the number of equilibria in a quality

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[9] Wang and Yang (2001) studied mixed equilibria for a model with costless quality, and sufficient consumer heterogeneity that both firms remain active. Due to the different modeling assumptions, the mixed equilibria studied by Wang and Yang (2001) have a different character from those studied here. In particular, the assumption of an endogenous fixed cost of quality is critical for my results.

[10] Sound theory on this topic could provide a basis extending empirical entry models that typically postulate a simple in-or-out action set. Attention usually is restricted to pure strategies, allowing inference about underlying profit functions from the number of firms entering the market (e.g. Bresnahan and Reiss, 1991; Berry, 1992). Recent research allows for mixed strategies, trying to draw inferences about equilibrium selection probabilities from the empirical frequency of different entry outcomes (Berry and Tamer, 2006). The inference problem seems more difficult when endogenous quality substantially increases the number of mixed equilibria under consideration.

investment model, while Besanko, Doraszelski, and Kryukov (2012) examine how rules against predation can eliminate bad equilibria.

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Appendix A: Winner-take-all market

The following demand model exemplifies how limited consumer heterogeneity results in a winner-take-all market. A type θ consumer enjoys net utility $(1+\theta)q - p$ from consuming a good of quality q purchased at price p, and the population of consumers is distributed exponentially, i.e. $\theta \sim F(\theta) \equiv 1 - e^{-\lambda\theta}$.

There are two firms, indexed $i \in \{1,2\}$, with given qualities $q_1 > q_2$, who set prices p_i in Bertrand-Nash equilibrium. At an interior solution, such that the firms share the market, the marginal consumer is

$$\hat{\theta} = \frac{p_1 - p_2}{q_1 - q_2} - 1 . \tag{58}$$

Firm 1 maximizes $p_1 e^{-\lambda\hat{\theta}}$ and has a dominant strategy

$$p_1 = \frac{q_1 - q_2}{\lambda} . \tag{59}$$

Firm 2 maximizes $p_2[1 - e^{-\lambda\hat{\theta}}]$ and sets a price satisfying the first-order condition

$$\left[1 - e^{-\lambda\hat{\theta}}\right] - \frac{\lambda p_2}{q_1 - q_2} e^{-\lambda\hat{\theta}} \tag{60}$$

and the second-order condition

$$\left[\frac{-2\lambda}{q_1 - q_2} - \frac{\lambda^2 p_2}{(q_1 - q_2)^2}\right] e^{-\lambda\hat{\theta}} < 0 . \tag{61}$$

At $p_2 = 0$ the left-hand-side of Firm 2's first-order-condition is equal to $1 - e^{-(1-\lambda)}$, which is positive if $\lambda < 1$. Therefore, $p_2 > 0$ is a best response if $\lambda < 1$, and the two firms share the market. If $\lambda \geq 1$, then an interior solution fails to exist. The equilibrium is a corner solution, in which Firm 1 sets $p_1 = q_1 - q_2$, Firm 2 sets $p_2 = 0$, and Firm 1 wins the entire market.

Appendix B: Zipper Principle

Proof of Lemma 2: Assuming S_1 and S_2 are finite discrete sets, the lemma is proved in four steps. Step 1 establishes that S_1 and S_2 have the same cardinality. Therefore the supports of equilibrium strategies are $S_1 = \{q_1^1,...q_1^n\}$ and $S_2 = \{q_2^1,...q_2^n\}$, for some positive integer *n*. Moreover, without loss of generality, the elements of these sets are strictly monotone, i.e. $q_i^k < q_i^{k+1}$ for $k = 1,...n-1$. Step 2 establishes that in equilibrium consumers are never indifferent between the two products. Step 3 establishes the zipper-like ordering of equilibrium supports, and Step 4 establishes the boundary conditions for each of the two types of equilibria.

Step 1: $\# S_1 = \# S_2$.

If $\# S_1 > \# S_2$, then $F_2(q_1 + \Gamma) = F_2(q_1' + \Gamma)$ for some $q_1, q_1' \in S_1$, $q_1 \neq q_1'$, in which case $r'(q_1) \neq r'(q_1')$ violates Local Optimality.

Step 2: If $q_1 \in S_1$ and $q_2 \in S_2$, then $q_1 + \Gamma \neq q_2$.

Suppose to the contrary that $q_1 + \Gamma = q_2$ and $F_2(q_1 + \Gamma) = r'(q_1)$ as required by Local Optimality. If $q_1 > 0$, then S_2 discrete implies $F_2(q_1 + \Gamma - \varepsilon) < r'(q_1 - \varepsilon)$ for $\varepsilon > 0$ sufficiently small, and Firm 1 could increase its profit by investing slightly less. If $q_1 = 0$, then $0 = F_1(q_2 - \Gamma - \varepsilon) < r'(q_2 - \varepsilon)$ for $\varepsilon > 0$ sufficiently small implies Firm 2 could increase its profit by investing slightly less.

Step 3: Either $q_1^k + \Gamma > q_2^k$ for $k = 1,...n$, or $q_1^k + \Gamma < q_2^k$ for $k = 1,...n$.

Either $q_1^{k+1} + \Gamma > q_2^{l+1} > q_2^l > q_1^k + \Gamma$ or $q_2^{l+1} - \Gamma > q_1^{k+1} > q_1^k > q_2^l - \Gamma$ violates Local Optimality.

Step 4: Either $q_1^n = 1$ and $q_2^1 = 0$, or $q_1^1 = 0$ and $q_2^n = 1$.

If $q_1^k + \Gamma > q_2^k$ for $k = 1,...n$, then $F_2(q_1^n + \Gamma) = 1$ and Local Optimality require $q_1^n = 1$. Similarly, $F_1(q_2^1 - \Gamma) = 0$ requires $q_2^1 = 0$. A converse argument applies if $q_2^k > q_1^k + \Gamma$.

Q.E.D.

Appendix C: Computation

Proof of Proposition 3: (a) Solving $q_2^{k+1} - 2q_2^k + q_2^{k-1} = 0$ recursively, and using $q_2^1 = 0$, gives $q_2^k = (k-1)q_2^2$. Therefore, the second boundary condition becomes

$$q_2^2 = \frac{2(1+\Gamma)}{2n-1} \tag{62}$$

and the solution for $k = 3,...n$ is

$$q_2^k = (k-1)\left[\frac{2(1+\Gamma)}{2n-1}\right]. \tag{63}$$

Obviously this is an increasing sequence. Moreover,

$$q_2^k = (k-1)\left[\frac{2(1+\Gamma)}{2n-1}\right] \tag{64}$$

satisfies $q_2^n < 1+\Gamma$ for all $n \geq 2$. Applying these results, $q_1^k + \Gamma = \frac{1}{2}\left(q_2^{k+1} + q_2^k\right)$ for $k = 1, \ldots n-1$ implies

$$q_1^1 = \frac{1+\Gamma}{2n-1} - \Gamma \,. \tag{65}$$

Therefore, $q_1^1 \geq 0$ requires

$$n \leq 1 + \frac{1}{2\Gamma} \tag{66}$$

i.e. for $\Gamma > 0$, n must be sufficiently small. This same condition also implies $q_2^n \leq 1$. Local Optimality determines the equilibrium strategies from the equilibrium supports.

(b) Type B mixed equilibrium with n points of support must satisfy the following conditions:

$$q_2^k(q_2^k - \Gamma - q_1^k) + q_2^{k-1}(q_1^k - q_2^{k-1} + \Gamma) = \tfrac{1}{2}(q_2^k)^2 - \tfrac{1}{2}(q_2^{k-1})^2 \tag{67}$$

and

$$q_1^k(q_1^k + \Gamma - q_2^{k-1}) + q_1^{k-1}(q_2^{k-1} - \Gamma - q_1^{k-1}) = \tfrac{1}{2}(q_1^k)^2 - \tfrac{1}{2}(q_1^{k-1})^2 \tag{68}$$

for $k = 2, \ldots n$, $q_1^1 = 0$, and $q_2^n = 1$. The difference equations simplify to

$$q_1^{k-1} + \Gamma = \tfrac{1}{2}\left(q_2^k + q_2^{k-1}\right) \tag{69}$$

and

$$q_2^k - \Gamma = \tfrac{1}{2}\left(q_1^k + q_1^{k-1}\right) \tag{70}$$

for $k = 2, \ldots n$. Solving these difference equations gives:

$$q_1^k = \left[\frac{2(k-1)}{2n-1}\right](1-\Gamma) \tag{71}$$

and

$$q_2^k = \left[\frac{2k-1}{2n-1}\right]\left(1-\Gamma\right)+\Gamma \qquad (72)$$

for $k = 1,...n$. Notice that the solution automatically satisfies the necessary conditions $q_2^k - \Gamma > q_1^k$.

Additionally, the Type B equilibrium must satisfy No-Leapfrogging. In the quadratic case, this condition is

$$\Gamma + \sum_{k=2}^{n}(q_2^k - q_2^{k-1})q_1^k \leq \tfrac{1}{2} \qquad (73)$$

Substituting the solution, No Leapfrogging requires:

$$\Gamma + (1-\Gamma)^2\left[\tfrac{4}{(2n-1)^2}\sum_{k=1}^{n-1} k\right] \leq \tfrac{1}{2} \qquad (74)$$

or

$$n \leq \frac{1}{2\Gamma} \ . \qquad (75)$$

Q.E.D.

Appendix D: Dynamic Model

Proof of Proposition 4: The market dominance equilibrium requires the current laggard to lack any incentive ever to leapfrog the leader. Suppose the laggard optimistically believes he can gain a perpetual monopoly by choosing $q = \Gamma + \dfrac{1}{1-\delta\gamma}$ in the first period to "match" the quality of the incumbent. Then the entrant expects a value of

$$-\frac{1}{2}\left(\Gamma + \frac{1}{1-\delta\gamma}\right)^2 + \frac{\delta M(\delta\gamma)}{1-\delta} = -\frac{1}{2}\left(\Gamma + \frac{1}{1-\delta\gamma}\right)^2 + \frac{\delta}{2(1-\delta)(1-\delta\gamma)^2} \qquad (76)$$

which is negative if

$$\Gamma > \sqrt{\frac{\delta}{(1-\delta)(1-\delta\gamma)^2}} - \frac{1}{1-\delta\gamma} \equiv \frac{1}{1-\delta\gamma}\left[\sqrt{\frac{\delta}{1-\delta}} - 1\right] \tag{77}$$

Therefore, the laggard's optimistic overtaking strategy is unprofitable for any $\Gamma > 0$ if $\delta \le \frac{1}{2}$.

Next consider a subgame with $\Gamma = 0$. I construct a symmetric mixed strategy in which each firm chooses a quality less than or equal to $q \in [0, \bar{q}]$ with a smooth cumulative probability $F(q)$. Assuming the lower bound of the support of $F(q)$ is 0, the profit of the firm from choosing $q \in [0, \bar{q}]$ is

$$\pi(q) \equiv \int_0^q \left[\tfrac{q-z}{1-\delta\gamma} + \tfrac{\delta}{1-\delta} M(\delta\gamma)\right] dF(z) - r(q) \tag{78}$$

and the equilibrium indifference condition requires

$$\pi'(q) \equiv \tfrac{F(q)}{1-\delta\gamma} + \tfrac{\delta}{1-\delta} M(\delta\gamma) F'(q) - q = 0 \tag{79}$$

The first-order linear differential equation is rewritten as

$$F'(q) + \tfrac{1-\delta}{\delta(1-\delta\gamma)M(\delta\gamma)} F(q) = \tfrac{1-\delta}{\delta M(\delta\gamma)} q \tag{80}$$

with a solution

$$F(q) = (1-\delta\gamma)q + \tfrac{\delta}{2(1-\delta)}\left[e^{-\frac{2(1-\delta)(1-\delta\gamma)}{\delta} q} - 1\right] \tag{81}$$

for $q \in [0, \bar{q}]$ with $F(\bar{q}) = 1$. Since $\bar{q} > \frac{1}{1-\delta\gamma}$ if $\delta > 0$, neither firm has an incentive to choose a higher quality. Q.E.D.

Proof of Proposition 5: Suppose the current laggard invests $\frac{1}{1-\delta^2\gamma^2}$, and the current leader invests $\frac{\delta\gamma}{1-\delta^2\gamma^2}$. First, these actions are interior optima given alternating leadership. The difference between the two actions is $\frac{1}{1+\delta\gamma}$, so the laggard overtakes a current lead of Γ if $\frac{1}{1+\delta\gamma} \geq \Gamma$. Thus these alternating leadership strategies are candidates for equilibrium only for those subgames. Second, the market state evolves according to $\Gamma' = \gamma[-\Gamma + \frac{1}{1+\delta\gamma}]$, and

$$\Gamma^* = \frac{\gamma}{(1+\gamma)(1+\delta\gamma)} \tag{82}$$

is the steady state lead although the identity of the leader changes each period. If $\frac{1}{1+\delta\gamma} \geq \Gamma > \Gamma^*$, then Γ declines and asymptotically approaches Γ^*. If $\Gamma^* > \Gamma$, then the asymptote is approached from below. Therefore, in the long run, each period the laggard overcomes the initial advantage Γ^*.

The value functions for these alternating leadership strategies are calculated as follows. First,

$$V^-(\Gamma^*) = \frac{\Gamma^*}{\gamma} - r\left(\frac{1}{1-\delta^2\gamma^2}\right) + \delta V^+(\Gamma^*) \tag{83}$$

and

$$V^+(\Gamma^*) = -r\left(\frac{\delta\gamma}{1-\delta^2\gamma^2}\right) + \delta V^-(\Gamma^*) \tag{84}$$

respectively describe the value functions of the current laggard and the current leader evaluated at Γ^*. Solving these two equations yields closed form expressions

$$V^-(\Gamma^*) = \frac{1}{1-\delta^2}\left[\frac{\Gamma^*}{\gamma} - r\left(\frac{1}{1-\delta^2\gamma^2}\right) - \delta r\left(\frac{\delta\gamma}{1-\delta^2\gamma^2}\right)\right] \tag{85}$$

and

$$V^+(\Gamma^*) = \frac{1}{1-\delta^2}\left[\delta\frac{\Gamma^*}{\gamma} - r\left(\frac{\delta\gamma}{1-\delta^2\gamma^2}\right) - \delta r\left(\frac{1}{1-\delta^2\gamma^2}\right)\right] \tag{86}$$

Both expressions are positive if γ is sufficiently small, and $V^{-}(\Gamma^{*}) > V^{+}(\Gamma^{*})$ if $\delta < 1$. Second, since the quality investments are independent of lead length, and the depreciating profits from deviations from the steady state lead length are capitalized, the value functions for departures from the steady state are

$$V^{-}(\Gamma) = V^{-}(\Gamma^{*}) - \frac{1}{1-\delta^{2}\gamma^{2}}(\Gamma - \Gamma^{*}) \tag{87}$$

and

$$V^{+}(\Gamma) = V^{+}(\Gamma^{*}) + \frac{\delta\gamma}{1-\delta^{2}\gamma^{2}}(\Gamma - \Gamma^{*}) \tag{88}$$

$V^{-}(\Gamma)$ is decreasing in Γ and $V^{+}(\Gamma)$ is increasing in Γ.

If Γ were too high, then the current laggard would consider the option of not investing, allowing the current leader's quality advantage to depreciate, and resuming its current laggard status next period (assuming γ is sufficiently small). The value of this "wait-it-out" strategy would be $V^{-}(\delta[\Gamma + \frac{\delta\gamma}{1-\delta^{2}\gamma^{2}}])$. Note that this function also decreases in Γ, but at a slower rate than $V^{-}(\Gamma)$. Consequently, there is a unique intersection.

Now define $\hat{\Gamma}$ by $V^{-}(\hat{\Gamma}) = V^{-}(\delta[\hat{\Gamma} + \frac{\delta\gamma}{1-\delta^{2}\gamma^{2}}])$. Since $V^{-}(\frac{1}{1+\delta\gamma}) < 0$ for γ sufficiently small, it follows that $\frac{1}{1+\delta\gamma} > \hat{\Gamma} > \Gamma^{*}$. Since $V^{-}(\Gamma) < 0$ for $\Gamma > \hat{\Gamma}$, an alternating leadership equilibrium is possible only for subgames with $\Gamma \leq \hat{\Gamma}$. For subgames with $\Gamma > \hat{\Gamma}$, the market dominance actions are specified as equilibrium actions. These are optimal actions given an expected resumption of alternating leadership.

Subgames with $\Gamma > \hat{\Gamma}$ are never reached in an alternating leadership equilibrium if Γ initially is not too large and γ is sufficiently small. For these strategies to form an equilibrium, the current leader must have no incentive to deviate to create or maintain a lead greater than $\hat{\Gamma}$ and foreclose the rival in the current period and possibly future periods. Such deviations are prohibitively costly if γ is sufficiently small. The current laggard similarly would find it prohibitively costly to defect by enough to foreclose its rival in the next period.

Finally, it is straightforward that neither firm has anything to gain by defecting to $\Gamma = 0$. Firm 2 would be disadvantaged by its inferior position in this subgame. Firm 1 either would incur an additional cost for no immediate gain and the same position next period if it is the current leader, or incur a cost to delay the same profit stream if it's the current laggard.[11] Q.E.D.

Proof of Proposition 6: The multiple equilibria of the static model carry over immediately to the myopic ($\delta = 0$) dynamic model. In a Type A mixed equilibrium with $n = 2$, Firm 1 chooses

$$q_1^1 = \frac{1}{3} - \frac{2}{3}\Gamma \tag{89}$$

with probability

$$\alpha_1^1 = \frac{2}{3}(1 + \Gamma) \tag{90}$$

and $q_1^2 = 1$ with the remaining probability, while Firm 2 chooses

$$q_2^2 = \frac{2}{3}(1 + \Gamma) \tag{91}$$

[11] Alternatively, similarly to the previous proposition, the firms might play a symmetric mixed strategy when $\Gamma = 0$. The $\Gamma = 0$ subgame is never reached in equilibrium, except possibly as an initial condition at the start of the game, so the key element to the proof in any case is to show that neither firm is tempted to defect to the $\Gamma = 0$ subgame.

with probability

$$\alpha_2^2 = \frac{2}{3}(1+\Gamma) \tag{92}$$

and $q_2^1 = 0$ with remaining probability.

If firms play equilibrium binary mixed strategies in the myopic dynamic game, then the identity of the leader and length of the lead evolves stochastically, with the length of the lead staying less than $\frac{1}{2}$ if γ is sufficiently small, and the two firms playing the same strategies depending on who is the current leader over and over at all reached subgames. Then the myopic value function for leading firm is

$$V_1(\Gamma) = \frac{1}{18} + \frac{1}{9}\Gamma - \frac{4}{9}\Gamma^2 \tag{93}$$

while the myopic value function of the laggard is simply $V_2(\Gamma) = 0$. For use below, the first derivative of $V_1(\Gamma)$ is

$$v_1(\Gamma) = \frac{1}{9} - \frac{8}{9}\Gamma \tag{94}$$

Now for $\delta > 0$, let $V(\Gamma)$ denote a differentiable value function for a firm with lead $\Gamma \in (-\frac{1}{2}, \frac{1}{2})$. Assuming δ and γ are low enough to support the play binary mixed actions, and designating Firm 1 the leader, Local Optimality at an interior solution requires the following (rearranged) Bellman conditions:

$$q_1^2 = 1 + \delta\gamma\{\alpha_2^1 V'(\gamma[\Gamma + q_1^2 - q_2^1]) + \alpha_2^2 V'(\gamma[\Gamma + q_1^2 - q_2^2])\} \tag{95}$$

$$q_1^1 = \alpha_2^1 + \delta\gamma\{\alpha_2^1 V'(\gamma[\Gamma + q_1^1 - q_2^1]) + \alpha_2^2 V'(\gamma[\Gamma + q_1^1 - q_2^2])\} \tag{96}$$

$$q_2^2 = \alpha_1^1 + \delta\gamma\{\alpha_1^1 V'(\gamma[-\Gamma + q_2^2 - q_1^1]) + \alpha_1^2 V'(\gamma[-\Gamma + q_2^2 - q_1^2])\} \tag{97}$$

$$q_2^1 = \delta\gamma\{\alpha_1^1 V'(\gamma[-\Gamma + q_2^1 - q_1^1]) + \alpha_1^2 V'(\gamma[-\Gamma + q_2^1 - q_1^2])\} \tag{98}$$

Indifference requires

$$\begin{aligned}&\alpha_2^1\{\Gamma+q_1^1-q_2^1+\delta V(\gamma[\Gamma+q_1^1-q_2^1])\}+\alpha_2^2\{\delta V(\gamma[\Gamma+q_1^1-q_2^2])\}-\tfrac{1}{2}(q_1^1)^2\\&=\alpha_2^1\{\Gamma+q_1^2-q_2^1+\delta V(\gamma[\Gamma+q_1^2-q_2^1])\}+\alpha_2^2\{\Gamma+q_1^2-q_2^2+\delta V(\gamma[\Gamma+q_1^2-q_2^2])\}\\&-\tfrac{1}{2}(q_1^2)^2\end{aligned} \quad (99)$$

and

$$\begin{aligned}&\alpha_1^1\{\delta V(\gamma[-\Gamma+q_2^1-q_1^1])\}+\alpha_1^2\{\delta V(\gamma[-\Gamma+q_2^1-q_1^2])\}-\tfrac{1}{2}(q_2^1)^2\\&=\alpha_1^1\{-\Gamma+q_2^2-q_1^1+\delta V(\gamma[-\Gamma+q_2^2-q_1^1])\}+\alpha_1^2\{\delta V(\gamma[-\Gamma+q_2^2-q_2^1])\}-\tfrac{1}{2}(q_2^2)^2\end{aligned} \quad (100)$$

Given the value function, since $\alpha_1^2 \equiv 1-\alpha 1 \quad 1$ and $\alpha_2^1 \equiv 1-\alpha_2^2$, these are six equations in six unknowns $(q_1^2, q_1^1, q_2^2, q_2^1, \alpha_1^1, \alpha_2^2)$.

In general, the value function $V(Z)$ is determined by a complicated Bellman equation, but is a simple quadratic equation for the $\delta = 0$ limiting case; $V(Z) = V_1(Z)$ if $Z \geq 0$ and $V(Z) = 0$ if $Z < 0$. Taking total derivatives of the above six equations, treating γ and Γ as constants, letting $\delta \to 0$, and rearranging gives the following system:

$$dq_1^2 = \gamma\{\alpha_2^1 v_1(\gamma[\Gamma+q_1^2-q_2^1])+\alpha_2^2 v_1(\gamma[\Gamma+q_1^2-q_2^2])\}d\delta \quad (101)$$

$$dq_1^1 + d\alpha_2^2 = \gamma\{\alpha_2^1 v_1(\gamma[\Gamma+q_1^1-q_2^1])\}d\delta \quad (102)$$

$$dq_2^2 - d\alpha_1^1 = \gamma\{\alpha_1^1 v_1(\gamma[-\Gamma+q_2^2-q_1^1])\}d\delta \quad (103)$$

$$dq_2^1 = 0 \quad (104)$$

$$\begin{aligned}&\{\alpha_2^1-q_1^1\}dq_1^1+\{\alpha_2^2\}dq_2^2-\{\Gamma+q_1^1-q_2^2\}d\alpha_2^2\\&=\{\alpha_2^1 V_1(\gamma[\Gamma+q_1^2-q_2^1])+\alpha_2^2 V_1(\gamma[\Gamma+q_1^2-q_2^2])-\alpha_2^1 V_1(\gamma[\Gamma+q_1^1-q_2^1])\}d\delta\end{aligned} \quad (105)$$

$$\begin{aligned}&\{\alpha_1^1\}dq_1^1-\{\alpha_1^1-q_2^2\}dq_2^2-\{q_2^1\}dq_2^1-\{-\Gamma+q_2^2-q_1^1\}d\alpha_1^1\\&=\{\alpha_1^1 V_1(\gamma[-\Gamma+q_2^2-q_1^1])\}d\delta\end{aligned} \quad (106)$$

Substituting the values of $(q_1^2, q_1^1, q_2^2, q_2^1, \alpha_1^1, \alpha_2^2)$ at $\delta = 0$, substituting for V_1 and v_1, and solving the system gives the result. Q.E.D.