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SELECTION OF THE GROUND STATE FOR NONLINEAR SCHRÖDINGER EQUATIONS

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We prove for a class of nonlinear Schrödinger systems (NLS) having two nonlinear bound states that the (generic) large time behavior is characterized by decay of the excited state, asymptotic approach to the nonlinear ground state and dispersive radiation. Our analysis elucidates the mechanism through which initial conditions which are very near the excited state branch evolve into a (nonlinear) ground state, a phenomenon known as *ground state selection*. Key steps in the analysis are the introduction of a particular linearization and the derivation of a normal form which reflects the dynamics on all time scales and yields, in particular, nonlinear master equations. Then, a novel multiple time scale dynamic stability theory is developed. Consequently, we give a detailed description of the asymptotic behavior of the two bound state NLS for all small initial data. The methods are general and can be extended to treat NLS with more than two bound states and more general nonlinearities including those of Hartree–Fock type.

Keywords: Nonlinear scattering; soliton dynamics; NLS; Gross–Pitaevskii equation; asymptotic stability; metastability.

Contents

1. Introduction and Statement of Main Results

In this paper we study the detailed dynamics of the nonlinear Schrödinger equation with a potential (NLS):

$$i\partial_t\phi = H\phi + \lambda|\phi|^2\phi\,. \tag{1.1}$$

Here, $H = -\Delta + V(x)$ is a self-adjoint operator on $L^2(\mathbb{R}^3)$ and λ is a coupling parameter, assumed real and of order one. When $V(x)$ is nonzero, Eq. (1.1) is also known as the time-dependent Gross–Pitaevskii equation.[a] We assume that $V(x)$ is a smooth potential, which decays sufficiently rapidly as $|x|$ tends to infinity (short range). Finally, we assume that the operator H has no zero energy resonance [24, 31], a condition which holds for generic V.

[a]In a similar way, one can treat nonlinear terms of the form $\lambda K[|\phi|^2]$, which can be local or nonlocal. Typical examples are: $K[|\phi|^2] = |\phi|^p$ and $K[|\phi|^2] = \mathcal{K} \star |\phi|^2$, for some convolution kernel, $\mathcal{K}$, Hartree–Fock type.

NLS is a Hamiltonian system with conserved Hamiltonian energy functional:

$$\mathbf{H}_{\rm en}[\phi] = \int |\nabla\phi(x)|^2 + V(x)|\phi(x)|^2 + \frac{\lambda}{2}|\phi(x)|^4 dx \tag{1.2}$$

and additional conserved integral

$$\mathbf{N}[\phi] = \int |\phi(x)|^2 dx\,. \tag{1.3}$$

These conserved integrals are continuous in the $H^1(\mathbb{R}^3)$ topology. An extensive discussion of the well-posedness theory can be found in [6, 22, 52]. In particular, NLS is well-posed globally in time in the space $H^1(\mathbb{R}^3)$, for initial data, ϕ_0, which is sufficiently small in H^1.

Throughout this paper we shall assume that ϕ_0 has sufficiently small H^1 norm. We shall use the notation:

$$\mathcal{E}_0 \equiv \|\phi_0\|^2_{H^1}\,. \tag{1.4}$$

For $V(x)$ which decays sufficiently rapidly as $|x|$ tends to infinity (short range potentials) the spectrum of H [35] consists of discrete spectrum, $\sigma_d(H)$, consisting of a finite number of negative point eigenvalues, and continuous spectrum, $\sigma_c(H) = [0,\infty)$. The dynamics of solutions for $\lambda = 0$ (linear Schrödinger) is very well understood. Let ψ_{j*} and E_{j*} denote bound states and bound state energies of the linear Schrödinger operator H:

$$H\psi_{j*} = E_{j*}\psi_{j*}\,, \qquad \langle\psi_{j*},\psi_{k*}\rangle_{L^2} = \delta_{jk}\,. \tag{1.5}$$

Arbitrary initial conditions in an appropriate Hilbert space, evolve as t tends to infinity into a time-quasiperiodic part consisting of a superposition of time periodic and spatially localized states with frequencies given by the eigenvalues and a dispersive or radiative part, which decays to zero as t tends to infinity in appropriate spaces, e.g. $L^p, p > 2$, $L^2(\langle x\rangle^{-\sigma}\ dx)$.

In order to be more precise, introduce P_{c*}, the orthogonal projection onto the continuous spectral subspace of H:

$$P_{c*}f = f - \sum_j \langle\psi_{j*}, f\rangle\psi_{j*}\,. \tag{1.6}$$

The solution of the linear Schrödinger equation can be expressed as

$$e^{-iHt}\phi_0 = \sum_j \langle\psi_{j*},\phi_0\rangle\psi_{j*}e^{-iE_{j*}t} + e^{-iHt}P_{c*}\phi_0\,. \tag{1.7}$$

The time decay of the continuous spectral part of the solution can be expressed, under suitable smoothness, decay and genericity assumptions on $V(x)$, in terms of *local decay estimates* [24, 31]:

$$\|\langle x\rangle^{-\sigma}e^{-iHt}P_{c*}\phi_0\|_{L^2(R^3)} \le C\langle t\rangle^{-\frac{3}{2}}\|\langle x\rangle^{\sigma}\phi_0\|_{L^2(R^3)}\,, \tag{1.8}$$

$\sigma \ge \sigma_0 > 0$, and $L^1 - L^\infty$ *decay estimates* [25, 61]

$$\|e^{-iHt}P_{c*}\phi_0\|_{L^\infty(R^3)} \le C|t|^{-\frac{3}{2}}\|\phi_0\|_{L^1(R^3)}\,. \tag{1.9}$$

For $\lambda \neq 0$ the bound states of the linear problem persist, and bifurcate from the linear states at zero amplitude into branches of *nonlinear bound states* [38]. Of interest to us is the detailed dynamics of the nonlinear problem on short, intermediate and long time scales and, in particular, the manner in which the nonlinear bound states participate in the dynamics. In [38] variational methods were used to establish the existence and orbital Lyapunov stability of bound states which are local minimizers of $\mathbf{H}_{\text{en}}$ subject to fixed $\mathbf{N}$; see also [59, 15]. This result says that initial data which is close, modulo a phase adjustment, in H^1 to the ground state remains H^1 close to a phase adjusted ground state for all time. The H^1 norm is closely related to the conserved Hamiltonian energy of the system and is insensitive to dispersive phenomena. Therefore, the detailed dynamics is not addressed by this result. For example, could the large time dynamics consist of a nonlinear ground state plus a small nonlinear excited state part? The main result of this paper implies that this cannot occur.

For the nonlinear problem, the simplest question to consider is the case where H has only one simple eigenvalue and the norm of the solution is small. The detailed dynamics was studied in [43, 44, 33, 57]. Small norm initial data are shown to evolve into an asymptotic nonlinear ground state and a radiative decaying part.

In this paper we study the multibound state variant of this question. We consider the specific case where H has two simple eigenvalues, E_{0*} and E_{1*}. The linear Schrödinger equation then has two time-periodic solutions $\psi_{0*}e^{-iE_{0*}t}$ and $\psi_{1*}e^{-iE_{1*}t}$, with $H\psi_{j*} = E_{j*}\psi_{j*},\ \psi_{j*} \in L^2$. Therefore [38], NLS has two branches of nonlinear bound states bifurcating from the zero state at the eigenvalues of H, $\Psi_{\alpha_0}e^{-iE_0 t}$ and $\Psi_{\alpha_1}e^{-iE_1 t}$, with $\Psi_{\alpha_j} \in L^2$ satisfying

$$H\Psi_{\alpha_j} + \lambda|\Psi_{\alpha_j}|^2\Psi_{\alpha_j} = E_j\Psi_{\alpha_j}\,. \tag{1.10}$$

Here, α_j denotes a coordinate along the jth nonlinear bound state branch and

$$E_j = E_{j*} + \mathcal{O}(|\alpha_j|^2)\,. \tag{1.11}$$

In contrast to the linear behavior (1.7) our main result is the following:

Theorem 1.1. *Consider NLS with $V(x)$ a short range potential supporting two bound states as described above. Furthermore, assume that the linear Schrödinger operator, H, has no zero energy resonance* [24].

(i) *Assume the initial data, $\phi(0)$, is small in the norm defined by*:

$$[\phi(0)]_X \equiv \|\langle x\rangle^\sigma \phi(0)\|_{H^k}\,, \tag{1.12}$$

where $k > 2$ and $\sigma > 0$ are sufficiently large. Let $\phi(t)$ solve the initial value problem for NLS.

Assume the (generically satisfied) nonlinear Fermi golden rule resonance condition[b]

$$\Gamma_{\omega_*} \equiv \lambda^2 \pi \left\langle \psi_{0*}\psi_{1*}^2, \delta(H-\omega_*)\psi_{0*}\psi_{1*}^2 \right\rangle > 0 \tag{1.13}$$

holds, where

$$\omega_* = 2E_{1*} - E_{0*} > 0\,. \tag{1.14}$$

Then, as $t \to \infty$

$$\phi(t) \to e^{-i\omega_j(t)} \Psi_{\alpha_j(\infty)} + e^{i\Delta t}\phi_+\,, \tag{1.15}$$

in L^2, *where either* $j = 0$ *or* $j = 1$. *The phase* ω_j *satisfies*

$$\omega_j(t) = \omega_j^\infty t + \mathcal{O}(\log t)\,. \tag{1.16}$$

Here, $\Psi_{\alpha_j(\infty)}$ *is a nonlinear bound state (Sec. 3), with frequency* $E_j(\infty)$ *near* E_{j*}. *When* $j = 0$, *the solution is asymptotic to a nonlinear ground state, while in the case* $j = 1$ *the solution is asymptotic to a nonlinear excited state.*

(ii) *More specifically, we have the following expansion of the solution* $\phi(t)$:

$$\phi(t) = e^{-i\tilde{\omega}_0(t)}\Psi_{\alpha_0(t)} + e^{-i\tilde{\omega}_1(t)}\Psi_{\alpha_1(t)} + \pi_1 e^{-i\mathcal{H}_0(\infty)t} P_c \tilde{\phi}_+(t) + R_{\rm loc}(t) + R_{\rm nloc}(t),$$

where as $t \to \infty$

$$\tilde{\omega}_j(t) - \omega_j(t) \to 0\,, \qquad \alpha_j(t) - \alpha_j(\infty) \to 0 \tag{1.17}$$

and such that for each initial state, $\phi(0)$,

$$\begin{aligned} |\alpha_0(\infty)| \cdot |\alpha_1(\infty)| \;&=\; 0\,, \\ \tilde{\phi}_+(t) &\to \phi_+ \text{ in } L^2\,, \\ \|R_{\rm loc}(\cdot,t)\|_2 \;&=\; \mathcal{O}(t^{-\frac{1}{2}}) \\ \|R_{\rm nloc}(\cdot,t)\|_\infty \;&=\; \mathcal{O}(t^{-\frac{1}{2}})\,. \end{aligned}$$

Here, $\mathcal{H}_0(\infty)$ *is a small spatially localized perturbation of the operator*

$$\sigma_3(-\Delta + V(x))$$

and $P_c = P_c(\mathcal{H}_0(\infty))$, *the projection onto its continuous spectral part. Finally,* π_1 *maps the vector* (z_1, z_2) *to* $(z_1, 0)$.

Remark 1.1. (i) Theorem 1.1 implies the absence of small norm time-quasiperiodic solutions for this class of nonlinear Schrödinger equations [41]. Intuitively, one can explain why one expects only a *pure state* in the limit $t \to \infty$ and how the condition on $\omega_* = 2E_{1*} - E_{0*}$ arises. Our intuition is based on viewing the nonlinearity as

[b]The operator $f \mapsto \delta(H-\omega_*)f$ projects f onto the generalized eigenfunction of H with generalized eigenvalue ω_*. The expression in (1.13) is finite by local decay estimates (1.8); see e.g. [47].

linear time-dependent potential; see also [41, 42]. An approximate superposition of a nonlinear ground state and excited state $\phi \sim \Psi_{\alpha_0} e^{-iE_0 t} + \Psi_{\alpha_1} e^{-iE_1 t}$ can be viewed as defining a self-consistent time-dependent potential:

$$\begin{aligned} W(x,t) &= \lambda |\Psi_{\alpha_0} e^{-iE_0 t} + \Psi_{\alpha_1} e^{-iE_1 t}|^2 , \\ &= \lambda |\Psi_{\alpha_0} + \Psi_{\alpha_1} e^{-i(E_1 - E_0)t}|^2 \\ &\sim \lambda |\Psi_{\alpha_0}|^2 + 2\lambda \cos((E_1 - E_0)t + \gamma) \Psi_{|\alpha_0|} \Psi_{|\alpha_1|} \end{aligned} \tag{1.18}$$

for $|\alpha_1| \ll |\alpha_0|$. As shown in [48], [28] and [29] data with initial conditions given by the unperturbed excited state decay exponentially on a time scale of order $\tau \sim \mathcal{O}(|\alpha_0 \alpha_1|^{-2})$ provided the forcing frequency, $E_1 - E_0 \sim E_{1*} - E_{0*} > -E_{1*}$ or $\omega_* = 2E_{1*} - E_{0*} > 0$.

(ii) Theorem 1.1 implies asymptotic stability and selection of the ground state for generic small data. Theorem 1.1(i) implies a form of asymptotic completeness.

(iii) Since we control the decay of solutions in $W^{k,\infty}$, our results imply global existence of small solutions in H^s for all s sufficiently large.

(iv) The asymptotic state where $|\alpha_1(\infty)| \neq 0$ (and therefore $|\alpha_0(\infty)| = 0$) is non-generic. This can be seen by linearization about the excited state. The linearized operator, $\mathcal{H}_1$, is a localized perturbation of an operator having embedded eigenvalues in its continuous spectrum, under our hypothesis $\omega_* \equiv 2E_{1*} - E_{0*} > 0$. The connection between embedded eigenvalues in the continuous spectrum of an appropriate linear operator and the non-persistence of localized time periodic states and between embedded eigenvalues in the continuous spectrum of an appropriate linear operator was explored first in [41, 42]. It is well known that embedded eigenvalues in the continuous spectrum are unstable to generic perturbations; see, for example, [7, 14, 47]. In this case, the embedded eigenvalues are perturbed to complex eigenvalues, with corresponding eigenstates whose evolution is exponentially *growing* with time, under the condition (7.4). The perturbation to the linear operator with embedded eigenvalues is however both non-generic (in that it comes from linearization of a Hamiltonian nonlinear term about a critical point of the energy) and breaks self-adjointness with respect to the standard L^2 inner product. A second order perturbation theory calculation shows that if $\omega_* \equiv 2E_{1*} - E_{0*} > 0$ generically the embedded eigenvalue perturbs to an exponential instability [45]. This suggests the existence of an unstable manifold of solutions for the *nonlinear* equation. The existence of such non-generic solutions of NLS with $E_1(\infty) \neq 0$ for the full nonlinear flow has recently been demonstrated [55].

(v) Theorem 1.1 is stated for the case of two nonlinear bound state branches. The technique of proof, however, can be used to consider the more general case. We expect results which are analogous to those of our main theorem, but more complicated due to the presence of: direct bound state-bound state interactions, bound state-continuum interactions **and** bound state-bound state interactions mediated by the continuum. Multimode Hamiltonian systems have been considered in the context of linear time almost periodic perturbations have been studied in [29].

1.1. *Relation to other work*

We now wish to further put our results in context. Research on nonlinear scattering in the presence of bound states has followed two related lines.

(a) Nonlinear dispersive waves in systems with defects, potentials, etc.: Our analysis centers around nonlinear bound states which bifurcate from linear bound states of the operator, H, obtained by linearizing about the zero solution. These bound states exist at all sufficiently small amplitudes (measured in any H^s norm, $s \geq 0$). The behavior of the "bifurcation diagram" for larger amplitudes depends in a detailed way on the details of the nonlinearity, the spatial dimension and the norm [38]. Such nonlinear bound states are also called nonlinear defect modes, nonlinear localized modes or nonlinear pinned modes. They are localized about or "pinned" to the support of the potential, V, and arise due to a local deviation from translation invariance or a "defect" in the homogeneous background which acts as an attractive potential well. To get a more refined picture of the dynamics than in the H^1 theory, one must consider the linearized evolution about the family of nonlinear ground states. This linearized operator has continuous and discrete spectral parts inherited from the linear bound state spectral structure. In particular, the discrete spectrum contains an eigenvalue at zero corresponding to the ground state and a pair of eigenvalues (located symmetrically about zero) corresponding to the excited state. Thus, at linear order a solution infinitesimally close to the ground state formally appears to be quasiperiodic in time — a ground state plus a small excited state oscillation. However, at higher order in perturbation theory one finds nonlinear resonant coupling of the neutral oscillatory modes to the continuum and as a result these slowly damp to zero; generically, for very large time energy splits between the ground state and dispersive parts of the solution. This mechanism for relaxation to the ground state was earlier considered for the nonlinear Klein–Gordon wave equation with a potential, where the decay of "breather-like" solutions was studied [49]. In this work, small norm solutions relax to the zero solution via resonant energy transfer out of the bound state to radiation modes and dispersive radiation of energy to infinity; the zero solution plays the role of the ground state. Results concerning special classes of initial data are considered in the work of Cuccagna [11] and Tsai and Yau [53, 54].

(b) Nonlinear dispersive translation invariant equations: A closely-related line of research focuses on the translation invariant nonlinear Schrödinger equation. Here the equation is (1.1) with V taken to be identically zero nonlinear coupling parameter $\lambda < 0$. In this case, the equation has solitary wave solutions, obtainable by minimization of $\mathcal{H}_{\rm en}[\phi]$ subject to $\mathcal{N}[\phi] = \mathcal{N}_0$. For NLS in dimension n with cubic nonlinearity replaced by the general power nonlinearity $|\phi|^{p-1}\phi$, we have that if $p < 1+\frac{4}{n}$, the foregoing variational problem has a unique (up to translation) radially symmetric ground state solution for any $\mathcal{N}_0 > 0$. In the case when $V(x) \neq 0$ is a potential supporting bound states, the small $\mathcal{N}_0$ solutions agree with the bifurcating bound states discussed above [38]. As pointed out earlier, constrained energy minimizers are H^1 orbitally Lyapunov stable [12, 59]. An interesting feature of solitary waves in the translation invariant case is the presence of spurious neutral oscillations. These are sometimes called *internal modes* [23]. To explain this, consider the linearization about a ground state solitary wave ($p < 1 + \frac{4}{n}$). Due to the underlying symmetric group of the equation (translation invariance, phase invariance, Galilean invariance, etc.) this linearization has a (generalized) zero eigenvalue of multiplicity related to the dimension of the equation's symmetry group. In the non-integrable cases ($n = 1$, $p \neq 3$ and $n \geq 2$) the linearization has additional neutral modes. These neutral modes approach zero as p approaches $1 + \frac{4}{n}$; the dimension of the zero subspace jumps by two at $p = 1 + \frac{4}{n}$, the critical case, corresponding to the larger group of symmetries and the existence of a pseudo-conformal invariant [58]. Buslaev and Perel'man [3] considered the problem in one space dimension and showed that nonlinear resonance of these "internal modes" with the continuum is responsible for their damping on long time scales and the asymptotic stability of solitons. See also the recent work of Buslaev and Sulem [4]. Their analysis was restricted to one space dimension only in their use of explicit eigenfunction expansion methods to obtain the required local energy decay estimates. Cuccagna [10, 11] extended their results to more general nonlinearities and general space dimensions. In his analysis, the required dispersive estimates are obtained by adapting K. Yajima's [61] approach in which the wave operators, which conjugate the linearized operator on its continuous spectral part to the constant coefficient "free" dispersive evolution, are shown to be bounded on $W^{k,p}$ spaces. This method was also used in [49].

Another feature, common to problems of type (a) and (b) is the use of the method of normal forms. In the context of nonlinear scattering, normal form ideas were used to obtain the local behavior in a neighborhood of a soliton in [3] and for the decaying breather-like state in [49]. In contrast to the normal form for finite dimensional Hamiltonian systems, resonant interaction with the *continuous* spectrum gives rise to a more general normal form which captures internal damping, due to energy transfer out of certain discrete modes to the continuum modes; see the discussion in the introduction to [49]. In the present work, we derive a *nonlinear master equation*, coupled equations for the renormalized (up to near identity transformations on the complex discrete mode amplitudes) discrete mode square amplitudes ("mode powers"), which governs generic dynamics on large intermediate and very long time scales. Normal forms of this type, expected to be valid for very long times, were derived and studied in the local analysis about the steady and "wobbling" kink-like solutions of discrete nonlinear wave equations in [26]. A key feature of the normal form of the current work is our analysis of its behavior on different time scales and the analysis of its transitional behavior across time scales, for general initial data.

Finally, we point out that there are many important areas of application which motivate the study of the class of models we treated in this paper. We mention two.

At the most fundamental level the Gross–Pitaevskii equation (NLS with a potential) arises as a mean field limit model governing the interaction of a very large number of weakly interacting bosons [21, 51, 30, 9]. At a macroscopic level, it has been shown that equations of this type arise as the equation governing the evolution of the envelope of the electric field of a light pulse propagating in a medium with defects. See, for example, [17, 19, 20].

2. Structure of the Proof

We now sketch our analysis. Certain notations are defined in Appendix A. Analogous with the approach introduced in [43, 44] in the one bound state case, we represent the solution in terms of the dynamics of the bound state part, described through the evolution of the *collective coordinates* $\alpha_0(t)$ and $\alpha_1(t)$, and a remainder ϕ_2, whose dynamics is controlled by a dispersive equation. In particular we have

$$\phi(t,x) = e^{-i\int_0^t E_0(s)ds - i\tilde{\Theta}(t)}\left(\Psi_{\alpha_0(t)} + \Psi_{\alpha_1(t)} + \phi_2(t,x)\right). \tag{2.1}$$

We substitute (2.1) into NLS and use the nonlinear equations (1.10) for Ψ_{α_j} to simplify. Anticipating the decay of the excited state, we center the dynamics about the ground state. We therefore obtain for $\Phi_2 \equiv (\phi_2, \overline{\phi_2})^T$ the equation:

$$i\partial_t\Phi_2 = \mathcal{H}_0(t)\Phi_2 + \mathcal{G}(t, x, \Phi_2; \partial_t\vec{\alpha}(t), \partial_t\vec{\alpha}, \partial_t\tilde{\Theta}(t)) \tag{2.2}$$

where, $\mathcal{H}_0(t)$ denotes the matrix operator which is the linearization about the time-dependent nonlinear ground state $\Psi_{\alpha_0(t)}$. The idea is that in order for $\phi_2(t,x)$ to decay dispersively to zero we must choose $\alpha_0(t)$ and $\alpha_1(t)$ to evolve in such a way as to remove all secular resonance terms from $\mathcal{G}$. Thus we require,

$$P_b(\mathcal{H}_0(t))\Phi_2(t) = 0\,, \tag{2.3}$$

where $P_b(\mathcal{H}_0)$ and $P_c = I - P_b(\mathcal{H}_0)$ denote the discrete and continuous spectral projections of $\mathcal{H}_0$; see also condition (5.13). Since the discrete subspace of $\mathcal{H}_0(t)$ is four-dimensional (consisting of a generalized null space of dimension two plus two oscillating neutral modes) (2.3) is equivalent to four orthogonality conditions implying four differential equations for α_0, α_1 and their complex conjugates. These equations are coupled to the dispersive partial differential equation for Φ_2. At this stage we have that NLS is equivalent to a dynamical system consisting of a finite dimensional part (6.23) and (6.24), governing $\vec{\alpha}_j = (\alpha_j, \overline{\alpha_j})$, $j = 0, 1$, coupled to an infinite dimensional dispersive part governing Φ_2:

$$\begin{aligned} i\partial_t\vec{\alpha} &= \mathcal{A}(t)\vec{\alpha} + \vec{F}_\alpha \\ i\partial_t\Phi_2 &= \mathcal{H}_0(t)\Phi_2 + \vec{F}_\phi\,. \end{aligned} \tag{2.4}$$

We expect $\mathcal{A}(t)$ and $\mathcal{H}_0(t)$ to have limits as $t \to \pm\infty$. Our strategy is to fix $T > 0$ arbitrarily large, and to study the dynamics on the interval $[0, T]$. In this we follow the strategy of [3, 11]. We shall rewrite (2.4) as:

$$\begin{aligned} i\partial_t\vec{\alpha} &= \mathcal{A}(T)\vec{\alpha} + (\mathcal{A}(t) - \mathcal{A}(T))\vec{\alpha} + \vec{F}_\alpha \\ i\partial_t\Phi_2 &= \mathcal{H}(T)\Phi_2 + (\mathcal{H}_0(t) - \mathcal{H}_0(T))\Phi_2 + \vec{F}_\phi \end{aligned} \tag{2.5}$$

and implement a perturbative analysis about the *time-independent* reference linear, respectively, matrix and differential, operators $\mathcal{A}(T)$ and $\mathcal{H}_0(T)$.

More specifically, we analyze the dynamics of (2.5) by using (a) the eigenvalues of $\mathcal{A}(T)$ to calculate the key resonant terms and (b) together with the dispersive estimates of $e^{-i\mathcal{H}_0(T)t}P_c(T)$ [10].

Note also that $P_c(T)\Phi_2(t) \neq \Phi_2(t)$ because $\Phi_2(t) \in$ Range $P_c(t) \neq$ Range $P_c(T)$. We therefore decompose Φ_2 as:

$$\Phi_2 = \text{disc}(t;T) + \eta\,, \qquad \eta = P_c(T)\eta \tag{2.6}$$

where $\text{disc}(t;T)$ lies in the discrete spectral subspace of $\mathcal{H}_0(T)$ and show that $\text{disc}(t;T)$ can be controlled in terms of η.

The expected generic behavior of this system is that α_1 and Φ_2 decay with a rate $t^{-\frac{1}{2}}$. This slow rate actually leads to an equation for α_1 with the character $\partial_t\alpha_1 \sim (\frac{1}{t}\rho - \partial_t\tilde{\Theta})\alpha_1+$ integrable in t; see (6.12). Thus, $\tilde{\Theta}$ is chosen to satisfy $\partial_t\tilde{\Theta} \sim \frac{1}{t}\rho$ ensuring that α_1 has a limit. In this way a logarithmic correction to the standard phase arises; see (1.16).

Next we explicitly factor out the rapid oscillations from α_1 and show that, after a near identity change of variables $(\alpha_0, \alpha_1) \mapsto (\tilde{\alpha}_0, \tilde{\beta}_1)$, that the modified ground and excited state amplitudes satisfy the system:

$$\begin{aligned} i\partial_t\tilde{\alpha}_0 &= (c_{1022} + i\Gamma_\omega)|\tilde{\beta}_1|^4\tilde{\alpha}_0 + F_\alpha[\tilde{\alpha}_0, \tilde{\beta}_1, \eta, t] \\ i\partial_t\tilde{\beta}_1 &= (c_{1121} - 2i\Gamma_\omega)|\tilde{\alpha}_0|^2|\tilde{\beta}_1|^2\tilde{\beta}_1 + F_\beta[\tilde{\alpha}_0, \tilde{\beta}_1, \eta, t]\,; \end{aligned} \tag{2.7}$$

see Proposition 7.1. It follows that a *nonlinear master equation* governs $P_j = |\tilde{\alpha}_j|^2$, the power in the jth mode:

$$\begin{aligned} \frac{dP_0}{dt} &= 2\Gamma P_1^2 P_0 + R_0(t) \\ \frac{dP_1}{dt} &= -4\Gamma P_1^2 P_0 + R_1(t)\,. \end{aligned} \tag{2.8}$$

Coupling to the dispersive part, Φ_2, is through the source terms R_0 and R_1. The expression "master equation" is used since the role played by (2.8) is analogous to the role of master equations in the quantum theory of open systems [13].

A novel multiscale Lyapunov argument is implemented in Sec. 8 characterizing the behavior of the system (2.8) coupled to that of the dispersive part on short, intermediate and long time scales. We consider the system (2.8) on three time intervals: $I_0 = [0, t_0]$ (initial phase) $I_1 = [t_0, t_1]$ (embryonic phase) and $I_2 = [t_1, \infty)$ (selection of the ground state).

For $t \geq t_0$, the terms $R_0(t)$ and $R_1(t)$ are shown (Proposition 12.1) to have the form

$$R_0(t) \sim \frac{b_0(t_0, \mathcal{E}_0)}{\langle t\rangle^2} + \rho_0(\mathcal{E}_0, t)P_0P_1^2 \tag{2.9}$$

$$R_1(t) \sim \frac{b_1(t_0, \mathcal{E}_0)}{\langle t\rangle^2} + \delta_m(t)\sqrt{P_0}P_1^m + \rho_1(\mathcal{E}_0, t)P_0P_1^2\,, \tag{2.10}$$

where

$$b_0 = \mathcal{O}(\langle t_0\rangle^{-1}), \qquad b_1 = \mathcal{O}(\langle t_0\rangle^{-\frac{1}{2}}). \tag{2.11}$$

$R_0(t)$ and $R_1(t)$ have parts which are local in time and nonlocal in time. The handling of nonlocal terms is explained in Sec. 8.

We set (Proposition 9.2)

$$Q_0 = P_0 - \frac{\tilde{b_0}}{\langle t\rangle}, \qquad Q_1 = P_1 + \frac{\tilde{b_1}}{\langle t\rangle} \tag{2.12}$$

where $\tilde{b_0}$ and $\tilde{b_1}$ are positive and satisfy (2.11) as well.

2.1. *Discussion of time scales*

Estimates (2.9), (2.10) and the definitions (2.12) imply an effective finite-dimensional reduction to a system of equations for the "effective mode powers": $Q_0(t)$ and $Q_1(t)$, whose character on different time scales dictates the full infinite dimensional dynamics, in a manner analogous to the role of a center manifold reduction of a dissipative system [5].

Initial phase – $t \in I_0 = [0, t_0]$: Here, I_0 is the maximal interval on which $Q_0(t) \leq 0$. If $t_0 = \infty$, then $P_0(t) = \mathcal{O}(\langle t\rangle^{-2})$ and the ground state decays to zero. In this case, we show in Sec. 15 that the excited state amplitude has a limit as well (which may or may not be zero). This case is non-generic.

Embryonic phase – $t \in I_1 = [t_0, t_1]$: If $t_0 < \infty$, then for $t > t_0$:

$$\begin{aligned} \frac{dQ_0}{dt} &\geq 2\Gamma' Q_0 Q_1^2 \\ \frac{dQ_1}{dt} &\leq -4\Gamma' Q_0 Q_1^2 + \mathcal{O}(\sqrt{Q_0}Q_1^m)\cdot m \geq 4\,. \end{aligned} \tag{2.13}$$

Therefore, Q_0 is monotonically increasing; *the ground state grows.* Furthermore, if Q_0 is small relative to Q_1, then

$$\frac{Q_0}{Q_1} \text{ is monotonically increasing}, \tag{2.14}$$

in fact exponentially increasing; *the ground state grows rapidly relative to the excited state.*

Selection of the ground state $t \in I_2 = [t_1, \infty)$: There exists a time $t = t_1$, $t_0 \leq t_1 < \infty$ at which the $\mathcal{O}(\sqrt{Q_0}Q_1^m)$ term in (2.13) is dominated by the leading ("dissipative") term. For $t \geq t_1$ we have

$$\begin{aligned} \frac{dQ_0}{dt} &\geq 2\Gamma' Q_0 Q_1^2 \\ \frac{dQ_1}{dt} &\leq -4\Gamma' Q_0 Q_1^2\,. \end{aligned} \tag{2.15}$$

It follows that $Q_0(t) \to Q_0(\infty) > 0$ and $Q_1(t) \to 0$ as $t \to \infty$; *the ground state is selected.*

3. Linear and Nonlinear Bound States

In this section we introduce bound states of the linear ($\lambda = 0$) and nonlinear ($\lambda \neq 0$) Schrödinger equation (1.1).

3.1. *Bound states of the unperturbed problem*

Let $H = -\Delta + V(x)$. We assume that $V(x)$ is smooth and sufficiently rapidly decaying, so that H defines a self-adjoint operator in L^2. Additionally, we assume that the spectrum of H consists of a continuous spectrum extending from 0 to positive infinity and two discrete negative eigenvalues, each of multiplicity one.

$$\sigma(H) = \{E_{0*}, E_{1*}\} \cup [0, \infty)\,. \tag{3.1}$$

Therefore, there exist eigenstates, $\psi_{j*} \in \mathcal{D}(H),\ j = 0, 1$ such that

$$H\psi_{j*} = E_{j*}\psi_{j*}\,. \tag{3.2}$$

We also introduce spectral projections onto the discrete eigenstates and continuous spectral part of H, respectively:

$$P_{j*}f \equiv \langle\psi_{j*}, f\rangle\psi_{j*}\,, \qquad j = 0, 1$$

$$P_{c*} \equiv I - P_{0*} - P_{1*}\,.$$

3.2. *Nonlinear bound states*

We seek solutions of (1.1) of the form

$$\phi = e^{-iE_j t}\Psi_{E_j}\,. \tag{3.3}$$

Substitution into (1.1) yields

$$H\Psi_{E_j} + \lambda|\Psi_{E_j}|^2\Psi_{E_j} = E_j\psi_{E_j}\,. \tag{3.4}$$

We introduce a bifurcation parameter, α_j, for the jth nonlinear bound state branch and define

$$\vec{\alpha}_j = (\alpha_j, \bar{\alpha}_j)\,. \tag{3.5}$$

Proposition 3.1 ([38]). *For each $j = 0, 1$ we have a one-parameter family, $\Psi_{\alpha_j} = \Psi_j$, of bound states depending on the complex parameter $\alpha_j = |\alpha_j|e^{i\gamma_j}$ and defined for $|\alpha_j|$ sufficiently small*:

$$\begin{aligned}\Psi_j(x) &\equiv \alpha_j\psi_j(x; |\alpha_j|^2) = e^{i\gamma_j}|\alpha_j|\psi_j(x; |\alpha_j|^2)\\ &= \alpha_j\big(\psi_{j*}(x) + \lambda|\alpha_j|^2\psi_j^{(1)}(x; |\alpha_j|^2)\big)\\ &= \alpha_j\big(\psi_{j*}(x) + \lambda|\alpha_j|^2\psi_j^{(1)}(x; 0) + \mathcal{O}(\lambda^2|\alpha_j|^4)\,.\end{aligned} \tag{3.6}$$

Here,

$$\psi_j^{(1)}(x;0) = -(H - E_{0*})^{-1}(I - P_{E_{j*}})\psi_{j*}^3 \tag{3.7}$$

$$\begin{aligned} E_j \equiv E_j(\vec{\alpha}_j) &\equiv E_{j*} + |\alpha_j|^2 E_j^{(1)}(|\alpha_j|^2) \\ &= E_{j*} + \lambda \int \psi_{0*}^4 dx\, |\alpha_j|^2 + \mathcal{O}(|\alpha_j|^4)\,. \end{aligned} \tag{3.8}$$

The mapping $\vec{\alpha}_j \mapsto (E_j(\vec{\alpha}_j), \Psi_j(\cdot;\vec{\alpha}_j))$ *is smooth.*

The proof uses standard bifurcation theory [32], which is based on the implicit function theorem. The analysis extends to the case of nonlocal nonlinearities. A variational approach can also be used to construct nonlinear bound states. Variational approaches, though more global, do not directly yield the information we require concerning smooth variation with respect to parameters.

Remark 3.1. Ψ_j depends on α_j and $\overline{\alpha_j}$. We shall compute derivatives of the nonlinear bound states Ψ_α with respect to α_j and $\overline{\alpha_j}$ and use the notation:

$$\vec{\nabla}_j = \left(\partial_{\alpha_j}, \partial_{\overline{\alpha_j}}\right) \equiv (\partial_j, \overline{\partial_j})\,. \tag{3.9}$$

In what follows we shall "modulate" these bound states. That is, we shall allow $\vec{\alpha}$ to vary with time. For convenience, we shall use the notation:

$$\begin{aligned} \Psi_j(t,x) &= \Psi_{\alpha_j(t)}(x)\,, \\ E_j(t) &= E_j(|\alpha_j(t)|^2)\,. \end{aligned}$$

4. Linearization about the Ground State

Let Ψ denote a nonlinear bound state (ground state or excited state) of (1.1); see Sec. 3. Then,

$$H\Psi + \lambda|\Psi|^2\Psi = E\,\Psi\,. \tag{4.1}$$

We first derive the linear stability problem. Let

$$\phi = (\Psi + p)\,e^{-iEt}\,, \tag{4.2}$$

where p denotes the perturbation about Ψ. Substituting (4.2) into (1.1) and neglecting all terms which are nonlinear in p and $\bar{p}$, we obtain the *linearized perturbation equation*

$$i\partial_t p = \left(H - E + 2\lambda|\Psi|^2\right) p + \lambda\Psi^2\,\bar{p}\,. \tag{4.3}$$

Since $\bar{p}$ appears explicitly in (4.3) it is natural to consider the system for

$$\vec{p} = \begin{pmatrix} p \\ \bar{p} \end{pmatrix}, \tag{4.4}$$

$$i\partial_t \vec{p} = \mathcal{H}\vec{p}\,, \tag{4.5}$$

where

$$\mathcal{H} = \sigma_3 \begin{pmatrix} H - E + 2\lambda|\Psi|^2 & \lambda\Psi^2 \\ \lambda\bar{\Psi}^2 & H - E + 2\lambda|\Psi|^2 \end{pmatrix}, \tag{4.6}$$

and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.7}$$

Later in this paper we shall refer specifically to the linearization about a "curve" of bound states $(\Psi_j(t), E_j(t))$ and will denote by $\mathcal{H}_j(t)$ the operator (4.6) with E replaced by $E_j(t)$ and Ψ replaced by $\Psi_j(t)$. Our main focus will be on the operator family

$$\mathcal{H}_0(t) = \mathcal{H}_{E_0(t),\Psi_0(t)}. \tag{4.8}$$

The nonlinear bound state Ψ is *linearly spectrally stable* if the spectrum of $\mathcal{H}$, $\sigma(\mathcal{H})$, is a subset of the real line.[c] Ψ is *linearly dynamically stable* if, in an appropriate space, all solutions of the initial value problem for (4.5) are bounded in time. That is, in some norm $e^{-i\mathcal{H}_0 t}$ is a bounded operator. Linear dynamical stability of the ground state Ψ_0 follows from [58]. For this result and the necessary stronger dispersive estimates on $e^{-i\mathcal{H}_0 t}$ [10], we require information on the discrete spectrum of $\mathcal{H}_0$ and the corresponding spectral subspaces.

Before stating these results we observe that the operator $e^{-i\mathcal{H}_0 t}$ can be expressed in terms of the operator treated explicitly in [58, 10]. To see this, express the ground state as $\Psi_0 = |\Psi_0|e^{i\gamma}$ ($|\Psi_0| > 0$), where $\gamma = \arg\alpha$ is a constant. Set $p = e^{i\gamma}q$. Then, by (4.3) we have:

$$i\partial_t q = \left(H - E_0 + 2\lambda|\Psi_0|^2\right) q + \lambda\Psi_0^2\,\bar{q}. \tag{4.9}$$

Now let $q = u + iv$, where u and v are real. Then,

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = J\tilde{H}_0 \begin{pmatrix} u \\ v \end{pmatrix}, \tag{4.10}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{H}_0 = \begin{pmatrix} H_0 - E_0 - \lambda|\Psi_0|^2 & 0 \\ 0 & H_0 - E_0 - 3\lambda|\Psi_0|^2 \end{pmatrix} \tag{4.11}$$

Note that

$$\begin{aligned} p(t) &= \pi_1\vec{p}(t) = \pi_1 e^{-i\mathcal{H}_0 t}\,\vec{p}_0 \\ &= e^{i\gamma}\left(\pi_1 e^{J\tilde{H}_0 t}\,\Re q_0 + i\pi_2 e^{J\tilde{H}_0 t}\,\Im q_0\right), \end{aligned} \tag{4.12}$$

where $\pi_1(z_1, z_2) = (z_1, 0)$ and $\pi_2(z_1, z_2) = (0, z_2)$. Therefore, we have:

[c] The spectrum of $\mathcal{H}$ has the symmetries one expects for Hamiltonian systems. The mappings $\lambda \mapsto -\lambda$ and $\lambda \mapsto \bar{\lambda}$ send points in the spectrum to points in the spectrum. Note that if ξ is an eigenvector of $\mathcal{H}$ with eigenvalue μ then $\overline{\sigma_1\xi}$ is an eigenvector of $\mathcal{H}$ with eigenvalue $-\mu$. Therefore, $\xi_{-\mu} = \overline{\sigma_1\xi_\mu}$.

Proposition 4.1. *Estimates on* $e^{-i\mathcal{H}_0 t}$ *are equivalent to those for* $e^{J\tilde{H}_0 t}$ *and are independent of* γ.

We now turn to a detailed discussion of the spectral properties of $\mathcal{H}_0$.

Proposition 4.2. *Consider* $\mathcal{H}_0$, *the linearization about the ground state. Let* α_0 *be sufficiently small.*
(i) $\sigma(\mathcal{H}_0)$ *is a subset of the real line.*
(ii) $\sigma_{\text{discrete}}(\mathcal{H}_0) = \{-\mu, 0, \mu\}$, *where* $0 < \mu < |E_0|$.
(iii) *Zero is a generalized eigenvalue of* $\mathcal{H}_0$. *The generalized null space,* $N_g(\mathcal{H}_0)$, *is given by*

$$N_g(\mathcal{H}_0) = \text{span}\left\{ \sigma_3 \begin{pmatrix} \Psi_0 \\ \overline{\Psi_0} \end{pmatrix}, \begin{pmatrix} \partial_{E_0}\Psi_0 \\ \overline{\partial_{E_0}\Psi_0} \end{pmatrix} \right\}. \tag{4.13}$$

(iv) $\pm\mu$ *are simple eigenvalues. We denote their corresponding eigenfunctions by* ξ_μ *and* $\xi_{-\mu}$. *For* $|\alpha_0|$ *small we have the expansion*:

$$\mu = E_1 - E_0 + \mathcal{O}(|\alpha_0|^2) \tag{4.14}$$

$$\xi_\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \psi_{1*} + \begin{pmatrix} |\alpha_0|^2 c_1(|\alpha_0|^2) \\ \overline{\alpha_0}^2 c_2(|\alpha_0|^2) \end{pmatrix} \tag{4.15}$$

$$\xi_{-\mu} = \overline{\sigma_1 \xi_\mu}\,, \tag{4.16}$$

where $c_1(a)$ *and* $c_2(a)$ *are real analytic functions in* a.
(v) $\sigma(\mathcal{H}_0) - \sigma_{\text{discrete}}(\mathcal{H}_0) = (-\infty, E_0] \cup [-E_0, \infty)$.

The proof of Proposition 4.2 is in Appendix B.

Note that if ω is an eigenvalue of $\mathcal{H}$:

$$\sigma_3 \mathcal{L} g \equiv \mathcal{H} g = \omega g \tag{4.17}$$

then ω is an eigenvalue of $\mathcal{H}^*$ with corresponding eigenfunction $\sigma_3 g$:

$$\mathcal{L}\sigma_3(\sigma_3 g) = \mathcal{H}^*(\sigma_3 g) = \omega \sigma_3 g\,. \tag{4.18}$$

Therefore, we have

Proposition 4.3. (i) $\sigma(\mathcal{H}_0^*) = \sigma(\mathcal{H}_0)$.
(ii)

$$N_g(\mathcal{H}_0^*) = \text{span}\left\{ \begin{pmatrix} \Psi_0 \\ \overline{\Psi_0} \end{pmatrix}, \sigma_3 \begin{pmatrix} \partial_{E_0}\Psi_0 \\ \overline{\partial_{E_0}\Psi_0} \end{pmatrix} \right\}. \tag{4.19}$$

(iii)

$$N(\mathcal{H}_0^* \mp \mu) = \text{span}\{\sigma_3 \xi_\mu, \sigma_3 \xi_{-\mu}\} \equiv \{\zeta_\mu, \zeta_{-\mu}\}\,. \tag{4.20}$$

Here,

$$\zeta_\mu = \sigma_3 \xi_\mu \qquad \text{and} \qquad \zeta_{-\mu} = -\sigma_3 \xi_{-\mu}\,, \tag{4.21}$$

where this choice of $\zeta_{-\mu}$ *is taken so that* $\langle \zeta_{-\mu}, \xi_{-\mu} \rangle = 1$ *in the* $|\alpha_0| \downarrow 0$ *limit.*

4.1. *Nondegenerate basis for the discrete subspaces of $\mathcal{H}_0$ and $\mathcal{H}_0^*$*

For fixed $E \neq E_0$, the basis of $N_g(\mathcal{H}_0^*)$ displayed in (4.19) is a natural basis due to its direct connection to the symmetries of NLS. However, this basis is degenerate and singular in the limit $E \to E_{0*}$, as we shall now see.

Consider the basis of $N_g(\mathcal{H}_0)$ displayed in (4.13). Beginning with the first element of this basis, explicitly we have:

$$\sigma_3 \begin{pmatrix} \Psi_0 \\ \overline{\Psi_0} \end{pmatrix} = \sigma_3 \begin{pmatrix} \alpha_0 \psi_0(\cdot; |\alpha_0|^2) \\ \overline{\alpha_0} \psi_0(\cdot; |\alpha_0|^2) \end{pmatrix} = \alpha_0 \sigma_3 G_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} F_0 \,, \tag{4.22}$$

where

$$\alpha_0 = |\alpha_0| e^{i\gamma_0} \,, \qquad \mathbf{G_0} = \begin{pmatrix} e^{i\gamma_0} & 0 \\ 0 & e^{-i\gamma_0} \end{pmatrix} , \tag{4.23}$$

and

$$\begin{aligned} F_0 \equiv \psi_0(\cdot; |\alpha_0|^2) &= \psi_{0*} + |\alpha_0|^2 \chi(\cdot, |\alpha_0|^2) \\ &= \psi_{0*} + |\alpha_0|^2 \chi(\cdot, 0) + \chi_4^{(0)} \,. \end{aligned} \tag{4.24}$$

Here and subsequently we use the notation $\chi(\cdot, p)$ to denote a generic real-valued localized function of x with smooth dependence on a parameter, p, and $\chi_k^{(j)}$ is localized in x and $\mathcal{O}(|\alpha_j|^k)$.

A nonsingular element of the $N_g(\mathcal{H}_0)$ is obtained by dividing out ρ_0. We therefore define

$$\xi_{01} \equiv \sigma_3 G_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} F_0(\rho_0^2) \,. \tag{4.25}$$

We now turn to the second element of the basis displayed in (4.13). First note that

$$\begin{aligned} \frac{\partial}{\partial |\alpha_0|} \Psi_0 &= e^{i\gamma_0} \frac{\partial}{\partial |\alpha_0|} \left(|\alpha_0| \psi_0(\cdot; |\alpha_0|^2) \right) \\ &= e^{i\gamma_0} [\psi_0 + |\alpha_0|^2 \chi(\cdot; |\alpha_0|^2)] \,. \end{aligned}$$

Differentiation of (3.4) with respect to $|\alpha_0|$ yields:

$$(H - E) \partial_{|\alpha_0|} \Psi_0 + 2|\Psi_0|^2 \partial_{|\alpha_0|} \Psi_0 + \Psi_0^2 \overline{\partial_{|\alpha_0|} \Psi_0} = (\partial_{|\alpha_0|} E) \Psi_0 \,. \tag{4.26}$$

Taken together with the complex conjugate of (4.26), this yields, after multiplication by σ_3:

$$\begin{aligned} \sigma_3 \mathcal{H} G_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} F_0' &= (\partial_{|\alpha_0|} E) \, \sigma_3 G_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} |\alpha_0| \psi_0 \\ &= |\alpha_0| \xi_{01} \,. \end{aligned} \tag{4.27}$$

We define

$$\xi_{02} \equiv G_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} F_0' \,, \tag{4.28}$$

where

$$F_0' = \frac{\partial}{\partial|\alpha_0|}\left(|\alpha_0|\psi_0(\cdot;|\alpha_0|^2)\right) = \psi_{0*} + |\alpha_0|^2\chi(x,|\alpha_0|^2)\,. \tag{4.29}$$

By the above calculation ξ_{02} lies in the null space of $(\sigma_3\mathcal{H})^2$. Therefore, the pair of vectors: ξ_{01} and ξ_{02} spans $N_g(\mathcal{H}_0)$ and is nonsingular as $E_0 \to E_{0*}$. By a previous remark:

$$\zeta_{01} \equiv \sigma_3\xi_{02} \qquad \text{and} \qquad \zeta_{02} \equiv \sigma_3\xi_{01} \tag{4.30}$$

form a nonsingular basis for $\mathcal{H}_0^*$. This choice of basis will facilitate a *uniform* description of the dynamics in a neighborhood of the origin.

The above construction and Proposition 4.2 imply the following basis for the discrete subspaces of $\mathcal{H}_0$ and $\mathcal{H}_0^*$.

Proposition 4.4.

$$N_g(\mathcal{H}_0) = \text{span}\{\xi_{01},\xi_{02}\} \tag{4.31}$$

$$N_g(\mathcal{H}_0^*) = \text{span}\{\zeta_{01},\zeta_{02}\} \tag{4.32}$$

$$N(\mathcal{H}_0 \mp \mu) = \text{span}\{\xi_\mu,\xi_{-\mu}\} \tag{4.33}$$

$$N(\mathcal{H}_0^* \mp \mu) = \text{span}\{\zeta_\mu,\zeta_{-\mu}\} = \{\sigma_3\xi_\mu, -\sigma_3\xi_{-\mu}\} \tag{4.34}$$

$$\langle\zeta_a,\xi_b\rangle = C_{ab}\delta_{ab} + \mathcal{O}(|\alpha_0|^2)\,, \tag{4.35}$$

where a and b vary over the set $\{(01),(02),\mu,-\mu\}$. For α_0 small we have the expansions

$$\begin{aligned}
\xi_{01} &= \sigma_3 G_0\begin{pmatrix}1\\1\end{pmatrix}F_0 = \sigma_3 G_0\begin{pmatrix}1\\1\end{pmatrix}\left(\psi_{0*}+|\alpha_0|^2\chi_0^{(0)}\right)\\
\xi_{02} &= G_0\begin{pmatrix}1\\1\end{pmatrix}F_0' = G_0\begin{pmatrix}1\\1\end{pmatrix}\left(\psi_{0*}+|\alpha_0|^2\chi_0^{(0)}\right)\\
\xi_\mu &= \begin{pmatrix}1\\0\end{pmatrix}\left(\psi_{1*}+|\alpha_0|^2\chi_0^{(0)}\right) + \overline{\alpha_0}^2\begin{pmatrix}0\\1\end{pmatrix}\chi_0^{(0)}
\end{aligned} \tag{4.36}$$

and $\xi_{-\mu} = \overline{\sigma_1\xi_\mu}$.

Finally, we shall find it useful to note that

$$\zeta_{01} = \sigma_3\xi_{02} = \xi_{01} + |\alpha_0|^2\chi(x;|\alpha_0|^2)\,. \tag{4.37}$$

4.2. *Estimates for the linearized evolution operator*

Theorem 4.1 (Linear dynamical stability [58]). *Let*

$$\mathcal{M} \equiv N_g(\mathcal{H}_0^*)^\perp \cap \left(H^1\times H^1\right). \tag{4.38}$$

There exists $C>0$ such that for any $f \in \mathcal{M}$

$$||e^{-i\mathcal{H}_0 t} f||_{H^1} \le C||f||_{H^1} \, . \tag{4.39}$$

Theorem 4.2 (Dispersive estimates). *Let*

$$\mathcal{M}_1 \equiv [N_g(\mathcal{H}_0^*) \oplus N(H^* \mp \mu)]^\perp \tag{4.40}$$

and P_c the associated continuous spectral projection. For any $q \ge 2$ there exists $C_{1,q} > 0$ such that

$$||e^{-i\mathcal{H}_0 t} P_c f||_{L^q} \le C_{1,q}\, t^{-\frac{n}{2}+\frac{n}{q}} ||f||_{L^p} \, , \tag{4.41}$$

where $p^{-1} + q^{-1} = 1$.

The $L^1 \to L^\infty$ and, more generally, $L^p \to L^q$ estimates, are known in the self-adjoint case [25, 61, 37]. The extension to matrix Hamiltonians with non-self-adjoint off-diagonal part is more complicated. The results of [10, 18, 39, 40] cover Theorem 4.2.

Remark 4.1. We shall in later sections use the notation $\mathcal{M}(t)$ and $\mathcal{M}_1(t)$ to denote corresponding time-dependent subspaces relative to the time-dependent operator $\mathcal{H}_0^*(t)$.

Theorem 4.3 (Local decay estimate). *Let σ be sufficiently large. Let $\omega \in \{\nu \in \mathbb{R} : |\nu| > |E_0|\}$, the interior of the continuous spectrum of $\mathcal{H}_0$. Then, for $t > 0$*

$$||\langle x\rangle^{-\sigma} e^{-i\mathcal{H}_0 t} P_c \left(\mathcal{H}_0 - \omega - i0\right)^{-l} \langle x\rangle^{-\sigma}||_{\mathcal{B}(L^2)} \le C\langle t\rangle^{-\frac{3}{2}} \, , \qquad l = 0, 1 \tag{4.42}$$

$$||\langle x\rangle^{-\sigma} \tilde{\mathcal{H}}_0^k e^{-i\mathcal{H}_0 t} P_c \langle x\rangle^{-\sigma}||_{\mathcal{B}(H^{2k}, L^2)} \le C\langle t\rangle^{-\frac{3}{2}-k} \, , \tag{4.43}$$

where $\tilde{\mathcal{H}}_0 = \mathcal{H}_0 + E_0\sigma_3$. For $t < 0$, the same estimates hold with $-i0$ replaced by $+i0$ in (4.42).

We now sketch a proof, using that $\mathcal{H}_0 = H^{\rm Diag} + \epsilon^\star W$, where $\epsilon^\star$ is small and W is bounded and localized . For the propagator associated with the diagonal part, $e^{-itH^{\rm Diag}}$, we have the dispersive $L^1 \to L^\infty$ estimate with bound $t^{-\frac{3}{2}}$. This implies the bound $\langle t\rangle^{-\frac{3}{2}}$ on the propagator from $L^1 \cap L^2 \to L^2 + L^\infty$, where $\|f\|_{L^2+L^\infty} = \min\{\|f_1\|_2 + \|f_2\|_\infty, \ f = f_1 + f_2\}$. The corresponding bound for the perturbed propagator is obtained by a bootstrap argument, as follows.

Writing the DuHamel formula for $e^{-i\mathcal{H}_0 t} f$, we obtain

$$\begin{aligned}\|e^{-i\mathcal{H}_0 t} f\|_{L^2+L^\infty} &\le C\langle t\rangle^{-\frac{3}{2}} \|f\|_{L^1\cap L^2} + \epsilon^\star\, C_W \langle t\rangle^{-\frac{3}{2}} \sup_{0\le s\le t} \|e^{-i\mathcal{H}_0 s} f\|_{L^2+L^\infty} \\ &\quad + \|P_b(H^{\rm Diag}) e^{-i\mathcal{H}_0 t}\, f\|_{L^2+L^\infty} \, .\end{aligned} \tag{4.44}$$

Let $f = P_c(\mathcal{H}_0) f$. The last term of (4.44) can be rewritten as follows:

$$\begin{aligned}P_b(H^{\rm Diag}) e^{-i\mathcal{H}_0 t} f &= P_b(H^{\rm Diag})(P_c(\mathcal{H}_0) - P_c(H^{\rm Diag})) e^{-i\mathcal{H}_0 t} f \\ &= -P_b(H^{\rm Diag})(P_b(\mathcal{H}_0) - P_b(H^{\rm Diag})) e^{-i\mathcal{H}_0 t} f \, .\end{aligned} \tag{4.45}$$

Elementary perturbation theory gives that $P_b(\mathcal{H}_0) - P_b(H^{\mathrm{Diag}})$ is of order $\epsilon^\star$ times a projection onto a localized function. Therefore,

$$\|P_b(H^{\mathrm{Diag}})e^{-i\mathcal{H}_0 t}f\|_{L^2+L^\infty} \le \epsilon^\star C_W \|e^{-i\mathcal{H}_0 t}P_c(\mathcal{H}_0)f\|_{L^2+L^\infty}\,. \tag{4.46}$$

It follows that

$$\|e^{-i\mathcal{H}_0 t}f\|_{L^2+L^\infty} \le C_W\langle t\rangle^{-\frac{3}{2}}\|f\|_{L^1\cap L^2}\,. \tag{4.47}$$

Furthermore, we have the local decay estimate (4.42) with $l = 0$.

We now sketch the proof of the case (4.42) for $l = 1$ and (4.43). For ease of presentation we sketch the proof in the context of the Schrödinger operator $H_0 = -\Delta + V$. One has the local decay estimate

$$||\langle x\rangle^{-\sigma}e^{-iH_0 t}P_c(H_0)\langle x\rangle^{-\sigma}||_{\mathcal{B}(L^2)} \le C\langle t\rangle^{-\frac{3}{2}}\,, \tag{4.48}$$

where σ is positive and sufficiently large. The key to the proof is an analysis of the resolvent, $(H_0 - \lambda)^{-1}$ near $\lambda = 0$. One uses the spectral theorem

$$e^{-itH_0}P_c(H_0) = \int_0^\infty e^{-it\lambda}E'(\lambda)d\lambda \tag{4.49}$$

and an explicit expansion of $E'(\lambda)$, which can be expressed in terms of the imaginary part of the resolvent. The expansion of the resolvent is valid in the space $\mathcal{B}(L^2(\langle x\rangle^\sigma dx), L^2(\langle x\rangle^{-\sigma}dx))$ for σ sufficiently large. Time decay is arbitrarily fast for functions with spectral support away from $\lambda = 0$, while the $t^{-\frac{3}{2}}$ decay results from the behavior near $\lambda = 0$. The analogue of (4.43) is an estimate for $e^{-itH_0}H_0^kP_c(H_0)$, which follows from the expansion of [24] applied to the formula:

$$e^{-itH_0}H_0^kP_c(H_0) = \int_0^\infty e^{-it\lambda}\lambda^k E'(\lambda)d\lambda\,. \tag{4.50}$$

Remark 4.2. The $L^p \to L^q$ estimates of Theorem 4.2 are later used to estimate nonlinear terms which do not include spatially localized factors. In the case of small data, which we consider here, it is also possible to use the above weaker $L^1 \cap L^2 \to L^\infty + L^2$ estimates. Indeed, the only difference is that we need to bound the L^2 norm (not just the L^1 norm) of the nonlinearity. The L^2 norm has even faster decay, since we can bound the L^∞ norm in terms of Sobolev norms, which are controlled by the current argument.

Remark 4.3. We also point out that one can prove the $L^1 \to L^\infty$ dispersive estimate by using the $L^1 \cap L^2 \to L^\infty + L^2$ estimate and a cancellation argument which is rather involved [36].

5. Decomposition and Modulation Equations

Consider the nonlinear Schrödinger equation

$$i\partial_t\phi = H\phi + \lambda|\phi|^2\phi\,. \tag{5.1}$$

In the regime of low energy solutions we decompose the solution of NLS in the following form:

$$\phi(t) = e^{-i\Theta(t)} \left[\Psi_0(t) + \Psi_1(t) + \phi_2(t)\right]. \tag{5.2}$$

Here, $\Psi_0(t)$ and $\Psi_1(t)$ represent motion along the ground state and excited state manifolds of equilibria and ϕ_2 is a decaying correction term, lying in an appropriate *dispersive subspace*. The phase, Θ is divided into two parts:

$$\Theta(t) = \Theta_0(t) + \tilde{\Theta}(t), \tag{5.3}$$

where

$$\Theta_0(t) = \int_0^t E_0(s)ds. \tag{5.4}$$

Thus, $\partial_t\Theta_0(t) = E_0(t)$ is the modulated ground state energy, and $\tilde{\Theta}(t)$ is a "long range" logarithmic correction, which is to be derived below.

We begin by setting

$$\phi = e^{-i\Theta(t)} \left[\Psi_0(t) + \phi_1\right]. \tag{5.5}$$

Substituting (5.5) into (5.1) yields the following equation for ϕ_1:

$$\begin{aligned} i\partial_t\phi_1 &= \left(H - E_0(t)\right)\phi_1 + 2\lambda|\Psi_0(t)|^2\phi_1 + \lambda\Psi_0^2(t)\overline{\phi_1} \\ &\quad + \left(-\partial_t\tilde{\Theta} + 2\lambda|\phi_1|^2\right)\Psi_0(t) + \lambda\overline{\Psi_0(t)}\phi_1^2 + \lambda|\phi_1|^2\phi_1 - \partial_t\tilde{\Theta}(t)\phi_1 \\ &\quad - i\partial_t\Psi_0(t). \end{aligned} \tag{5.6}$$

Next, we decompose ϕ_1 into a part along the excited state manifold and a correction term:

$$\phi_1 \equiv \Psi_1(t) + \phi_2. \tag{5.7}$$

We have, using (5.6),

$$\begin{aligned} i\partial_t\phi_2 &= \left(H - E_0(t)\right)\phi_2 + 2\lambda|\Psi_0(t)|^2\phi_2 + \lambda\Psi_0^2(t)\overline{\phi_2} - \partial_t\tilde{\Theta}(t)\phi_2 \\ &\quad - \left(E_{01}(t) + \partial_t\tilde{\Theta}(t) + i\partial_t\right)\Psi_1(t) + 2\lambda|\Psi_0(t)|^2\Psi_1(t) + \lambda\Psi_0(t)^2\overline{\Psi_1(t)} \\ &\quad + \left(-\partial_t\tilde{\Theta}(t) + 2\lambda|\phi_1|^2\right)\Psi_0(t) + \lambda\overline{\Psi_0(t)}\phi_1^2 \\ &\quad + \lambda\left(|\phi_1|^2\phi_1 - |\Psi_1(t)|^2\Psi_1(t)\right) \\ &\quad - i\partial_t\Psi_0(t). \end{aligned} \tag{5.8}$$

Since equation (5.8) involves $\overline{\phi_2}$ it is natural to consider the system governing ϕ_2 and $\overline{\phi_2}$. Let

$$\Phi_2 = \begin{pmatrix} \phi_2 \\ \overline{\phi_2} \end{pmatrix} \tag{5.9}$$

and introduce the matrix linear operator:

$$\mathcal{H}_0(t) \equiv \sigma_3 \begin{pmatrix} H - E_0(t) + 2\lambda|\Psi_0(t)|^2 & \lambda\Psi_0^2(t) \\ \lambda\bar{\Psi}_0^2(t) & H - E_0(t) + 2\lambda|\Psi_0(t)|^2 \end{pmatrix} . \tag{5.10}$$

Then we have (recall that $\phi_1 = \Psi_1 + \phi_2$)

$$i\partial_t \Phi_2 = \mathcal{H}_0(t)\Phi_2 + \mathbf{F} - i\partial_t \begin{pmatrix} \Psi_0(t) \\ \text{c.c.} \end{pmatrix} - \Big((E_{01}(t) + \partial_t\tilde{\Theta}(t))\sigma_3 + i\partial_t\Big) \begin{pmatrix} \Psi_1(t) \\ \text{c.c.} \end{pmatrix} , \tag{5.11}$$

where

$$\begin{aligned} \mathbf{F} \equiv & -\partial_t\tilde{\Theta}(t)\sigma_3 \begin{pmatrix} \Psi_0 \\ \text{c.c.} \end{pmatrix} - \partial_t\tilde{\Theta}(t)\sigma_3\Phi_2 \\ & + \lambda\sigma_3 \begin{pmatrix} 2|\Psi_0|^2\Psi_1 + \Psi_0^2\overline{\Psi_1} \\ \text{c.c.} \end{pmatrix} \\ & + \lambda\sigma_3 \begin{pmatrix} 2\Psi_0|\phi_1|^2 + \overline{\Psi_0}\phi_1^2 \\ \text{c.c.} \end{pmatrix} + \lambda\sigma_3 \begin{pmatrix} |\phi_1|^2\phi_1 - |\Psi_1|^2\Psi_1 \\ \text{c.c.} \end{pmatrix} . \end{aligned} \tag{5.12}$$

5.1. *Modulation equations*

Motivated by the results of Theorem 4.2 on dispersive decay, we shall require that

$$\Phi_2(t) \in \mathcal{M}_1(t) , \tag{5.13}$$

where $\mathcal{M}_1$ is defined in (4.40). Equivalently,

$$P_c(\mathcal{H}_0(t))\Phi_2(t) = \Phi_2(t) , \tag{5.14}$$

where $P_c(\mathcal{H}_0(t))$ denotes the continuous spectral projection of $\mathcal{H}_0(t)$. By Proposition 4.3 this imposes four orthogonality conditions on $\Phi_2(t)$:

$$\langle \sigma_3\xi_a(t), \Phi_2(t)\rangle = 0 , \tag{5.15}$$

where $a \in \{(01), (02), \mu, -\mu\}$. We impose (5.15) at $t = 0$ and now derive *modulation equations* for the coordinates $\alpha_0(t)$ and $\alpha_1(t)$ ensuring that (5.15) persists for all $t \neq 0$.

To derive the modulations we first take the inner product of (5.11) with the adjoint vectors $\sigma_3\xi_a$ to obtain the identity:

$$\begin{aligned} & \left\langle \sigma_3\xi_a,\, i\partial_t \begin{pmatrix} \Psi_0 \\ \text{c.c.} \end{pmatrix} \right\rangle + \left\langle \sigma_3\xi_a,\, \big((E_{01} + \partial_t\tilde{\Theta})\sigma_3 + i\partial_t\big) \begin{pmatrix} \Psi_1 \\ \text{c.c.} \end{pmatrix} \right\rangle \\ & \quad = \langle \mathcal{H}_0^*(t)\sigma_3\xi_a, \Phi_2\rangle + \langle \sigma_3\xi_a, \mathbf{F}\rangle + i\langle \partial_t(\sigma_3\xi_a), \Phi_2\rangle - i\partial_t\langle \sigma_3\xi_a, \Phi_2\rangle \end{aligned} \tag{5.16}$$

The initial data for NLS is decomposed so that $\langle \sigma_3\xi_a(t), \Phi_2(t)\rangle = 0$ for $t = 0$. In order for this condition to persist for all time it is necessary and sufficient that the last term in (5.16) vanish, or equivalently:

Proposition 5.1. *The condition that* $P_c(\mathcal{H}_0(t))\Phi_2(t) = \Phi_2(t)$ *is equivalent to the following modulation equations for the coordinates* α_0 *and* α_1 *which specify the dynamics along the ground state and excited state manifolds of equilibria*:

$$\begin{aligned}
&\langle \sigma_3\xi_a,\, i\partial_t\vec{\Psi}_0\rangle + \langle \sigma_3\xi_a,\, \big(\sigma_3(E_{01}+\partial_t\tilde{\Theta}) + i\partial_t\big)\vec{\Psi}_1\rangle \\
&\quad = -\lambda\left\langle \xi_a, \begin{pmatrix} 2|\Psi_0|^2\Psi_1 + \Psi_0^2\overline{\Psi_1} \\ \text{c.c.} \end{pmatrix}\right\rangle \\
&\qquad + \left\langle \xi_a, \big(-\partial_t\tilde{\Theta}(t) + 2\lambda|\phi_1|^2\big)\begin{pmatrix} \Psi_0 \\ \text{c.c.} \end{pmatrix}\right\rangle \\
&\qquad + \lambda\left\langle \xi_a, \begin{pmatrix} \overline{\Psi_0}\phi_1^2 \\ c.c. \end{pmatrix}\right\rangle + \lambda\left\langle \xi_a, \begin{pmatrix} |\phi_1|^2\phi_1 - |\Psi_1|^2\Psi_1 \\ \text{c.c.} \end{pmatrix}\right\rangle \\
&\qquad - \partial_t\tilde{\Theta}(t)\langle \xi_a, \Phi_2\rangle + \; i\langle \partial_t(\sigma_3\xi_a), \Phi_2\rangle
\end{aligned} \tag{5.17}$$

where $a \in \{(01), (02), \mu, -\mu\}$ *and* $\vec{\Psi}_j \equiv (\Psi_j, \overline{\Psi_j}),\ j = 0, 1.$

Remark 5.1. (i) Note that the term $\langle \mathcal{H}_0^*(t)\sigma_3\xi_a, \Phi_2\rangle$ in (5.16) vanishes by the orthogonality constraint and because $\mathcal{H}_0^*(t)$ maps the discrete subspace into itself. It therefore does not appear in (5.17).

(ii) The last term in (5.17) is present due to the time-dependence of the eigenvectors $\xi_a(t)$. An important simplification of this "commutator term", which we require for $a = (01)$, is carried out in Appendix C.

Initial data for the system (5.17), (5.11), governing α_0, α_1 and Φ_2 are obtained as follows. Given data ϕ_0 for NLS, we find α_0 so as to minimize

$$\|\phi_0 - \Psi_{\alpha_0}\|_2\,; \tag{5.18}$$

see [44]. This α_0 is used to define the initial Hamiltonian $\mathcal{H}_0(0)$. Now decompose ϕ_0 using the biorthogonal decomposition associated with $\mathcal{H}_0(0)$. This specifies $\alpha_0(0), \alpha_1(0)$ and $\Phi_2(0)$.

5.2. *Conservation laws and a priori bounds*

In this subsection we obtain bounds on α_0, α_1 and ϕ_2 using the conservation laws of NLS, noted in the introduction.

By the L^2 conservation law: $\mathbf{N}[\phi] = \mathbf{N}[\phi_0]$ we have

$$\begin{aligned}
h(t) &\equiv |\Psi_0(t)|^2 + |\Psi_1(t)|^2 + \int |\phi_2(t,x)|^2\, dx \\
&= \mathbf{N}[\phi_0] - 2\Re\int \Psi_0\overline{\Psi_1} - 2\Re\int \Psi_0\overline{\phi_2} - \Re\int \Psi_1\overline{\phi_2}\,.
\end{aligned} \tag{5.19}$$

The last three terms in (5.19) can be estimated using the following:

- $\int \Psi_0\overline{\Psi_1} = \mathcal{O}(|\alpha_0|\,|\alpha_1|^2) + \mathcal{O}(|\alpha_0|^2\,|\alpha_1|) = \mathcal{O}(h(t)^{\frac{3}{2}})$.

- Orthogonality relation (5.15) with $a = (01)$, $\langle \sigma_3\xi_{01}, \Phi_2\rangle = 0$ or equivalently $\Re\int \Psi_0\overline{\phi_2} = 0$.
- Orthogonality relation (5.15) with $a = \mu$ or $\langle \sigma_3\xi_\mu, \Phi_2\rangle = 0$ implies $\Re\int \Psi_1\overline{\phi_2} = \mathcal{O}\big(|\alpha_1|^3 \ + \ |\alpha_1|\,|\alpha_0|^2\big)\|\phi_2\|_2 = \mathcal{O}(h(t)^2)$.

Therefore,

$$\big(1 + \mathcal{O}(h(t)^{\frac{1}{2}})\big)h(t) \le \mathbf{N}[\phi_0]\,. \tag{5.20}$$

By continuity of $h(t)$, if $\mathbf{N}[\phi_0]$ is sufficiently small

$$h(t) \le C\mathbf{N}[\phi_0]\,. \tag{5.21}$$

Furthermore, the conservation law $\mathbf{H}_{\rm en}[\phi(t)] = \mathbf{H}_{\rm en}[\phi_0]$ can be used to prove an H^1 bound on $\phi_2(t)$, provided $\|\phi_0\|_{H^1}$ is sufficiently small. In particular, substituting the decomposition (5.2) into the conserved functional $\mathbf{H}_{\rm en}[\phi(t)]$, we find after integration by parts and interpolation of the L^4 term in $\mathbf{H}_{\rm en}$:

$$\begin{aligned}\|\nabla\phi_2(t)\|_{L^2}^2 \le \mathcal{E}_0 + \int |\phi_2|(|\Delta\Psi_1| + |\Delta\Psi_2|) + \int |\nabla\Psi_1|\,|\nabla\Psi_2| + \|V\|_\infty\,\|\phi_0\|_{L^2}^2 \\ + \frac{C}{2}\,\|\phi_0\|_{L^2}\,\|\nabla\phi(t)\|_2^{\frac{3}{2}} \\ \le C\mathcal{E}_0 + \mathcal{O}(h(t)) + \mathcal{O}(h(t)^{\frac{1}{2}})\|\nabla\phi_2(t)\|_{L^2}^3\,.\end{aligned}$$

Therefore, using the bound (5.21) and continuity in time, we have that if $\|\phi_0\|_{H^1}$ is sufficiently small,

$$\int |\nabla\phi_2(t,x)|^2 dx \le C\mathcal{E}_0\,. \tag{5.22}$$

6. Toward a Normal Form — Algebraic Reductions and Frequency Analysis

We now embark on a detailed calculation leading to a form of this system, which though equivalent, is of a form to which normal form methods can be easily applied.

6.1. *Modulation equations*

Equations (5.17) are a coupled system for α_0, α_1 and their complex conjugates. It is natural to write the system as one which is nearly diagonal. This can be done by taking appropriate linear combinations of the equations in (5.17).

The equation which essentially determines α_0 can be found by adding the two equations obtained from (5.17) by setting $a = (01)$ and $a = (02)$. This gives:

$$\begin{aligned}&\langle \sigma_3(\xi_{01} + \xi_{02}),\, i\partial_t\vec{\Psi}_0\rangle + \langle \sigma_3(\xi_{01} + \xi_{02}),\, \big(\sigma_3(E_{01} + \partial_t\tilde{\Theta}) + i\partial_t\big)\vec{\Psi}_1\rangle \\ &\quad = -\lambda\left\langle \xi_{01} + \xi_{02}, \begin{pmatrix} 2|\Psi_0|^2\Psi_1 + \Psi_0^2\overline{\Psi_1} \\ \text{c.c.}\end{pmatrix}\right\rangle\end{aligned}$$

$$+\left\langle \xi_{01}+\xi_{02}, \left(-\partial_t\tilde{\Theta}(t)+2\lambda|\phi_1|^2\right)\begin{pmatrix}\Psi_0\\ \text{c.c.}\end{pmatrix}\right\rangle$$

$$+\lambda\left\langle \xi_{01}+\xi_{02}, \begin{pmatrix}\overline{\Psi_0}\phi_1^2\\ \text{c.c.}\end{pmatrix}\right\rangle+\lambda\left\langle \xi_{01}+\xi_{02}, \begin{pmatrix}|\phi_1|^2\phi_1-|\Psi_1|^2\Psi_1\\ \text{c.c.}\end{pmatrix}\right\rangle$$

$$-\partial_t\tilde{\Theta}(t)\langle\xi_{01}+\xi_{02},\Phi_2\rangle+i\langle\partial_t(\sigma_3(\xi_{01}+\xi_{02})),\Phi_2\rangle\,. \tag{6.1}$$

The difference of the $a=(01)$ and $a=(02)$ equations is the complex conjugate of the Eq. (6.1).

The equation which essentially determines α_1 is Eq. (5.17) with $a=\mu$:

$$\langle\sigma_3\xi_\mu,\, i\partial_t\vec{\Psi}_0\rangle+\langle\sigma_3\xi_\mu,\, \left(\sigma_3(E_{01}+\partial_t\tilde{\Theta})+i\partial_t\right)\vec{\Psi}_1\rangle$$

$$=-\lambda\left\langle \xi_\mu, \begin{pmatrix}2|\Psi_0|^2\Psi_1+\Psi_0^2\overline{\Psi_1}\\ \text{c.c.}\end{pmatrix}\right\rangle$$

$$+\left\langle \xi_\mu, \left(-\partial_t\tilde{\Theta}(t)+2\lambda|\phi_1|^2\right)\begin{pmatrix}\Psi_0\\ \text{c.c.}\end{pmatrix}\right\rangle$$

$$+\lambda\left\langle \xi_\mu, \begin{pmatrix}\overline{\Psi_0}\phi_1^2\\ \text{c.c.}\end{pmatrix}\right\rangle+\lambda\left\langle \xi_\mu, \begin{pmatrix}|\phi_1|^2\phi_1-|\Psi_1|^2\Psi_1\\ \text{c.c.}\end{pmatrix}\right\rangle$$

$$-\partial_t\tilde{\Theta}(t)\langle\xi_\mu,\Phi_2\rangle+i\langle\partial_t(\sigma_3\xi_\mu),\Phi_2\rangle\,. \tag{6.2}$$

The equation corresponding to $a=-\mu$ is the complex conjugate of this equation.

Since

$$\partial_t\begin{pmatrix}\Psi_k\\ \overline{\Psi_k}\end{pmatrix}=\begin{pmatrix}\partial_k\Psi_k\\ \overline{\overline{\partial_k}\Psi_k}\end{pmatrix}\partial_t\alpha_k+\begin{pmatrix}\overline{\partial_k}\Psi_k\\ \overline{\partial_k\Psi_k}\end{pmatrix}\partial_t\overline{\alpha_k}\,, \tag{6.3}$$

we see that the equations for $\vec{\alpha}_j=(\alpha_0,\overline{\alpha_0})^T,\ j=0,1$ can be expressed in the form:

$$i\mathbf{M_{00}}\partial_t\begin{pmatrix}\alpha_0\\ \overline{\alpha_0}\end{pmatrix}+i\mathbf{M_{01}}\partial_t\begin{pmatrix}\alpha_1\\ \overline{\alpha_1}\end{pmatrix}+(E_{01}+\partial_t\tilde{\Theta})\mathbf{N_{01}}\begin{pmatrix}\alpha_1\\ \overline{\alpha_1}\end{pmatrix}=\mathbf{F}_0 \tag{6.4}$$

$$i\mathbf{M_{10}}\partial_t\begin{pmatrix}\alpha_0\\ \overline{\alpha_0}\end{pmatrix}+i\mathbf{M_{11}}\partial_t\begin{pmatrix}\alpha_1\\ \overline{\alpha_1}\end{pmatrix}+(E_{01}+\partial_t\tilde{\Theta})\mathbf{N_{11}}\begin{pmatrix}\alpha_1\\ \overline{\alpha_1}\end{pmatrix}=\mathbf{F}_1\,. \tag{6.5}$$

Here, for $k=0,1$:

$$\mathbf{M_{0k}}=\mathbf{G_0}\begin{pmatrix}\left\langle\sigma_3(\xi_{01}+\xi_{02}),\begin{pmatrix}\partial_k\Psi_k\\ \overline{\overline{\partial_k}\Psi_k}\end{pmatrix}\right\rangle & \left\langle\sigma_3(\xi_{01}+\xi_{02}),\begin{pmatrix}\overline{\partial_k}\Psi_k\\ \overline{\partial_k\Psi_k}\end{pmatrix}\right\rangle\\ \left\langle\sigma_3\overline{(\xi_{01}+\xi_{02})},\begin{pmatrix}\overline{\overline{\partial_k}\Psi_k}\\ \partial_k\Psi_k\end{pmatrix}\right\rangle & \left\langle\sigma_3\overline{(\xi_{01}+\xi_{02})},\begin{pmatrix}\overline{\partial_k\Psi_k}\\ \overline{\partial_k}\Psi_k\end{pmatrix}\right\rangle\end{pmatrix} \tag{6.6}$$

$$\mathbf{N_{01}}=\mathbf{G_0}\begin{pmatrix}\left\langle\sigma_3(\xi_{01}+\xi_{02}),\begin{pmatrix}\psi_1\\ 0\end{pmatrix}\right\rangle & \left\langle\sigma_3(\xi_{01}+\xi_{02}),\begin{pmatrix}0\\ \psi_1\end{pmatrix}\right\rangle\\ \left\langle\sigma_3\overline{(\xi_{01}+\xi_{02})},\begin{pmatrix}0\\ \psi_1\end{pmatrix}\right\rangle & \left\langle\sigma_3\overline{(\xi_{01}+\xi_{02})},\begin{pmatrix}\psi_1\\ 0\end{pmatrix}\right\rangle\end{pmatrix} \tag{6.7}$$

$$\mathbf{M_{1k}} = \mathbf{G_0} \begin{pmatrix} \left\langle \sigma_3\xi_\mu, \begin{pmatrix} \partial_k\Psi_k \\ \overline{\partial_k}\Psi_k \end{pmatrix} \right\rangle & \left\langle \sigma_3\xi_\mu, \begin{pmatrix} \overline{\partial_k}\Psi_k \\ \partial_k\Psi_k \end{pmatrix} \right\rangle \\ \left\langle \sigma_3\overline{\xi_\mu}, \begin{pmatrix} \overline{\overline{\partial_k}\Psi_k} \\ \partial_k\Psi_k \end{pmatrix} \right\rangle & \left\langle \sigma_3\overline{\xi_\mu}, \begin{pmatrix} \overline{\partial_k\Psi_k} \\ \overline{\partial_k}\Psi_k \end{pmatrix} \right\rangle \end{pmatrix}, \tag{6.8}$$

$$\mathbf{N_{11}} = \mathbf{G_0} \begin{pmatrix} \left\langle \sigma_3\xi_\mu, \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} \right\rangle & \left\langle \sigma_3\xi_\mu, \begin{pmatrix} 0 \\ \psi_1 \end{pmatrix} \right\rangle \\ \left\langle \sigma_3\overline{\xi_\mu}, \begin{pmatrix} 0 \\ \psi_1 \end{pmatrix} \right\rangle & \left\langle \sigma_3\overline{\xi_\mu}, \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} \right\rangle \end{pmatrix}. \tag{6.9}$$

6.2. *Algebraic reductions and determination of $\tilde{\Theta}(t)$*

To express the modulation equations (5.17) in a tractable form we shall make use of a number of notations and relations which we now list for convenience; see also Appendix A.

$\chi_k^{(j)}$ denotes a spatially localized function of order $|\alpha_j|^k$, as $|\alpha_j| \to 0$.

$\mathcal{O}_k^{(j)}$ denotes a quantity which is of order $|\alpha_j|^k$ as $|\alpha_j| \to 0$. Both $\chi_k^{(j)}$ and $\mathcal{O}_k^{(j)}$ are invariant under the map $\alpha_j \mapsto \alpha_j e^{i\gamma}$.

$\mathcal{O}_k^{(0,1)} = \mathcal{O}_{k_1}^{(0)}\mathcal{O}_{k_2}^{(1)}$, $k = k_1 + k_2$.

$$\phi_1 = \Psi_1 + \phi_2$$

$$\alpha_0 = |\alpha_0|e^{i\gamma_0}, \qquad \overline{\alpha_0}e^{2i\gamma_0} = \alpha_0$$

$$\big(\partial_k\Psi_k, \overline{\partial_k}\Psi_k\big) = \big(\psi_{k*} + |\alpha_k|^2\chi_0^{(k)}, \alpha_k^2\chi_0^{(k)}\big), \qquad k = 0, 1$$

$$\xi_{01} + \xi_{02} = G_0 \begin{pmatrix} F_0 + F_0' \\ F_0 - F_0' \end{pmatrix} = \begin{pmatrix} 2e^{i\gamma_0}(\psi_{0*} + \chi_2^{(0)}) \\ e^{-i\gamma_0}\chi_2^{(0)} \end{pmatrix}$$

$$\xi_\mu = \begin{pmatrix} \psi_{1*} + \chi_2^{(0)} \\ \overline{\alpha_0}^2\chi_0^{(0)} \end{pmatrix}$$

$$F_0 + F_0' = 2\psi_{0*} + |\alpha_0|^2\chi(\cdot; |\alpha_0|^2) = 2(\psi_{0*} + \chi_2^{(0)})$$

$$F_0 - F_0' = |\alpha_0|^2\chi(\cdot; |\alpha_0|^2) = \chi_2^{(0)}.$$

$$\langle\psi_{0*}, \psi_0^{(1)}(\cdot; 0)\rangle = 0. \tag{6.10}$$

Using (6.10) in (6.1) we get:

$$\begin{aligned} &2i(1 + \mathcal{O}_4^{(0)})\partial_t\alpha_0 + i\alpha_0^2\mathcal{O}_0^{(0)}\partial_t\overline{\alpha_0} + \langle\sigma_3(\xi_{01} + \xi_{02}), \big(\sigma_3(E_{01} + \partial_t\tilde{\Theta}) + i\partial_t\big)\vec{\Psi}_1\rangle. \\ &= -2\big(\partial_t\tilde{\Theta}\langle F_0', \Psi_0\rangle - 2\lambda\langle F_0', |\phi_1|^2\Psi_0\rangle\big) \\ &\quad + \langle F_0 + F_0', 2\lambda|\Psi_0|^2\Psi_1 + \lambda\Psi_0^2\overline{\Psi_1} + \lambda|\phi_1|^2\phi_1 - \lambda|\Psi_1|^2\Psi_1\rangle - e^{2i\gamma_0}\langle F_0 - F_0', \text{c.c.}\rangle \\ &\quad + ie^{2i\gamma_0}\langle\partial_t[\sigma_3(\xi_{01} + \xi_{02})], \Phi_2\rangle. \end{aligned} \tag{6.11}$$

6.2.1. *Determination of* $\tilde\Theta(t)$

We anticipate that generically $|\phi_1| \sim |\alpha_1| \sim t^{-\frac{1}{2}}$ for t very large. α_0 will have a limit as $t \to \pm\infty$ if $\partial_t\alpha_0$ is integrable. We ensure this by choosing $\partial_t\tilde\Theta$ to cancel the terms which are of order t^{-1} and *non-oscillatory.* Thus, we choose $\tilde\Theta$ to satisfy

$$\partial_t\tilde\Theta\langle F_0', \Psi_0\rangle - 2\lambda\langle F_0', |\phi_1|^2\Psi_0\rangle = 0\,. \tag{6.12}$$

To leading order this gives:

$$\partial_t\tilde\Theta \sim 2\lambda\langle \psi_{0*}^2, |\phi_1|^2\rangle\,. \tag{6.13}$$

In this way, a logarithmic correction to the standard phase, $\int_0^t E_0(s)ds$ arises.

Equations (6.11) and (6.12) together with Proposition C.1 imply

$$\begin{aligned}
&2i(1+\mathcal{O}_4^{(0)})\partial_t\alpha_0 + i\alpha_0^2\mathcal{O}_0^{(0)}\partial_t\overline{\alpha_0} + (E_{01}+\partial_t\tilde\Theta)\langle\xi_{01}+\xi_{02}\ ,\vec\Psi_1\rangle\\
&\quad + \langle\sigma_3(\xi_{01}+\xi_{02}), i\partial_t\vec\Psi_1\rangle\\
&\ = -\partial_t\tilde\Theta\langle\chi_0^{(0)},\phi_2\rangle + \partial_t\tilde\Theta\alpha_0^2\langle\chi_0^{(0)},\overline{\phi_2}\rangle\\
&\quad + \lambda\mathcal{O}_0^{(0)}(|\alpha_0|^2\alpha_1 + \alpha_0^2\overline{\alpha_1}) + \lambda\langle 2\psi_{0*}+\chi_0^{(2)}, |\phi_1|^2\phi_1 - |\Psi_1|^2\Psi_1\rangle\\
&\quad - \lambda\alpha_0^2\langle\chi_0^{(0)}, |\phi_1|^2\overline{\phi_1} - |\Psi_1|^2\overline{\Psi_1}\rangle - \partial_t|\alpha_0|^2\left[\langle\chi_0^{(0)},\phi_2\rangle + e^{2i\gamma_0}\langle\chi_0^{(0)},\overline{\phi_2}\rangle\right]\\
&\quad + i|\alpha_0|^2\partial_t\gamma_0\left[\langle\chi_0^{(0)},\phi_2\rangle + e^{2i\gamma_0}\langle\chi_0^{(0)},\overline{\phi_2}\rangle\right].
\end{aligned}\tag{6.14}$$

We now turn to Eq. (6.2). Using (6.10), Eq. (6.2) can be written as:

$$\begin{aligned}
&(1+\mathcal{O}_2^{(0,1)})i\partial_t\alpha_1 + (\mathcal{O}(\alpha_0^2)+\mathcal{O}(\alpha_1^2))i\partial_t\overline{\alpha_1}\\
&\quad + ((1+\mathcal{O}_2^{(0,1)})\alpha_1 + \alpha_0^2\overline{\alpha_1}\mathcal{O}_2^{(0,1)})(E_{01}+\partial_t\tilde\Theta) + \langle\sigma_3\xi_\mu, i\partial_t\vec\Psi_0\rangle\\
&\ = -\partial_t\tilde\Theta(\langle\chi,\phi_2\rangle + \alpha_0^2\langle\chi,\overline{\phi_2}\rangle) - \partial_t\tilde\Theta\mathcal{O}_0^{(0,1)}\alpha_0\\
&\quad - \lambda\mathcal{O}_2^{(0,1)}(|\alpha_0|^2\alpha_1 + \alpha_0^2\overline{\alpha_1}) + \lambda\mathcal{O}_0^{(0)}\alpha_0\langle\chi,|\phi_1|^2\rangle\\
&\quad + \lambda\overline{\alpha_0}\langle\chi,\phi_1^2\rangle + \lambda\alpha_0^3\langle\chi,\overline{\phi_1}^2\rangle\\
&\quad + \lambda\langle\chi,|\phi_1|^2\phi_1 - |\Psi_1|^2\Psi_1\rangle + \lambda\alpha_0^2\langle\chi,|\phi_1|^2\overline{\phi_1} - |\Psi_1|^2\overline{\Psi_1}\rangle\\
&\quad + i\langle\partial_t(\sigma_3\xi_\mu),\Phi_2\rangle\,.
\end{aligned}\tag{6.15}$$

We next write the systems for $\vec\alpha_0$ and $\vec\alpha_1$.

$$\begin{aligned}
&i\mathbf{M_{00}}\partial_t\vec\alpha_0 + i\mathbf{M_{01}}\partial_t\vec\alpha_1 + (E_{01}+\partial_t\tilde\Theta)\sigma_3\mathbf{N_{01}}\vec\alpha_1\\
&\ = \lambda\mathcal{O}_0^{(0)}\sigma_3\begin{pmatrix}|\alpha_0|^2\alpha_1 + \alpha_0^2\overline{\alpha_1}\\ \text{c.c.}\end{pmatrix}\\
&\quad + \lambda\sigma_3\begin{pmatrix}\langle\psi_{0*}+\chi_0^{(2)}, |\phi_1|^2\phi_1 - |\Psi_1|^2\Psi_1\rangle + \alpha_0^2\langle\chi_0^{(0)}, |\phi_1|^2\overline{\phi_1} - |\Psi_1|^2\overline{\Psi_1}\rangle\\ \text{c.c.}\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&- \partial_t\tilde{\Theta} \begin{pmatrix} \langle \chi_0^{(0)}, \phi_2\rangle + \alpha_0^2\langle \chi_0^{(0)}, \overline{\phi_2}\rangle \\ \text{c.c.} \end{pmatrix} \\
&- \partial_t|\alpha_0|^2\sigma_3 \begin{pmatrix} \langle \chi_0^{(0)}, \phi_2\rangle + e^{2i\gamma_0}\langle \chi_0^{(0)}, \overline{\phi_2}\rangle \\ \text{c.c.} \end{pmatrix} \\
&+ i|\alpha_0|^2\partial_t\gamma_0 \begin{pmatrix} \langle \chi_0^{(0)}, \phi_2\rangle + e^{2i\gamma_0}\langle \chi_0^{(0)}, \overline{\phi_2}\rangle \\ \text{c.c.} \end{pmatrix}
\end{aligned}
\tag{6.16}
$$

$$
\begin{aligned}
&i\mathbf{M_{11}}\partial_t\vec{\alpha}_1 + i\mathbf{M_{10}}\partial_t\vec{\alpha}_0 + (E_{01} + \partial_t\tilde{\Theta})\sigma_3\mathbf{N_{11}}\vec{\alpha}_1 \\
&\quad = \lambda\mathcal{O}_0^{(0,1)}\sigma_3 \begin{pmatrix} |\alpha_0|^2\alpha_1 + \alpha_0^2\overline{\alpha_1} \\ \text{c.c.} \end{pmatrix} \\
&\quad + \lambda\sigma_3 \begin{pmatrix} \langle \chi, |\phi_1|^2\phi_1 - |\Psi_1|^2\Psi_1\rangle + \alpha_0^2\langle \chi, |\phi_1|^2\overline{\phi_1} - |\Psi_1|^2\overline{\Psi_1}\rangle \\ \text{c.c.} \end{pmatrix} \\
&\quad + \lambda\sigma_3 \begin{pmatrix} \alpha_0\langle \chi, |\phi_1|^2\rangle + \overline{\alpha_0}\langle \chi, \phi_1^2\rangle + \alpha_0^3\langle \chi, \overline{\phi_1}^2\rangle \\ \text{c.c.} \end{pmatrix} \\
&\quad - \partial_t\tilde{\Theta}\,\sigma_3 \begin{pmatrix} \langle \chi, \phi_2\rangle + \alpha_0^2\langle \chi, \overline{\phi_2}\rangle \\ \text{c.c.} \end{pmatrix} + i \begin{pmatrix} \langle \partial_t(\sigma_3\xi_\mu), \Phi_2\rangle \\ \text{c.c.} \end{pmatrix}.
\end{aligned}
\tag{6.17}
$$

The matrices $\mathbf{M_{jk}}$, $\mathbf{N_{j1}}$, $j, k = 0, 1$ are displayed in Eqs. (6.6)–(6.9). For the (generic) case where we expect $|\alpha_0|$ to approach a nonzero limit and α_1 and ϕ_2 to decay to zero, we shall use the expansion:

$$
\begin{aligned}
\mathbf{M_{00}} &= \begin{pmatrix} 2\left(1 + \mathcal{O}_4^{(0)}\right) & \alpha_0^2\mathcal{O}_0^{(0)} \\ \overline{\alpha_0}^2\mathcal{O}_0^{(0)} & 2\left(1 + \mathcal{O}_4^{(0)}\right) \end{pmatrix} \\
\mathbf{M_{11}} &= \begin{pmatrix} 1 + \mathcal{O}_2^{(0,1)} & \mathcal{O}\left(\alpha_0^2\right) + \mathcal{O}\left(\alpha_1^2\right) \\ \mathcal{O}\left(\overline{\alpha_0}^2\right) + \mathcal{O}\left(\overline{\alpha_1}^2\right) & 1 + \mathcal{O}_2^{(0)} \end{pmatrix}.
\end{aligned}
\tag{6.18}
$$

The matrices $\mathbf{M_{10}}$ and $\mathbf{M_{10}}$ are higher order in α_0 and satisfy:

$$
\mathbf{M_{10}},\ \mathbf{M_{01}} = \begin{pmatrix} \mathcal{O}_2^{(0)} & \alpha_0^2\mathcal{O}_0^{(0)} \\ \overline{\alpha_0}^2\mathcal{O}_0^{(0)} & \mathcal{O}_2^{(0)} \end{pmatrix}.
\tag{6.19}
$$

The matrices $\mathbf{N_{01}}$ and $\mathbf{N_{11}}$ satisfy:

$$
\begin{aligned}
\mathbf{N_{01}} &= \begin{pmatrix} \mathcal{O}_2^{(0)} + \mathcal{O}_2^{(1)} & \alpha_0^2(\mathcal{O}_2^{(0)} + \mathcal{O}_2^{(1)}) \\ \overline{\alpha_0}^2(\mathcal{O}_2^{(0)} + \mathcal{O}_2^{(1)}) & \mathcal{O}_2^{(0)} + \mathcal{O}_2^{(1)} \end{pmatrix} \\
\mathbf{N_{11}} &= \begin{pmatrix} 1 + \mathcal{O}_2^{(0)} & \alpha_0^2\mathcal{O}_0^{(0,1)} \\ \overline{\alpha_0}^2\mathcal{O}_0^{(0,1)} & 1 + \mathcal{O}_2^{(0)} \end{pmatrix}.
\end{aligned}
$$

6.2.2. *Simplifications to Eqs.* (6.1) *and* (6.2)

(1) Since $\mathbf{M_{10}}$ and $\mathbf{M_{01}}$ are higher order in α_0 we can eliminate $\partial_t\alpha_1$ from (6.16) and $\partial_t\alpha_0$ from (6.17).

(2) Note also, that "commutator terms" with factors like $\partial_t|\alpha_0|^2$ or $\partial_t\alpha_0^2$ can be eliminated via redefinition of the near identity matrix $\mathbf{M_{00}}$ through incorporation of a higher-order correction.

(3) We can eliminate the term proportional to $|\alpha_0|^2\partial_t\gamma_0$ as follows. Consider the last two terms of (6.16). Since $\partial_t|\alpha_0|^2 = \alpha_0\partial_t\overline{\alpha_0} + \alpha_0\partial_t\overline{\alpha_0}$ we can incorporate the second to last term of (6.16) as a higher-order correction to the near identity matrix $\mathbf{M_{00}}$, $\widetilde{\mathbf{M_{00}}}$. Our goal is now to eliminate the $\partial_t\gamma_0$ from the equation. The system can now be written as:

$$i\partial_t\vec{\alpha}_0 = \widetilde{\mathbf{M_{00}}}^{-1}\left(\cdots + i\mathcal{O}(|\alpha_0|^2)\partial_t\gamma_0\right) . \tag{6.20}$$

The first component has the form:

$$\begin{aligned} &i\partial_t\alpha_0 = \text{1st component of the vector :}\\ &\widetilde{\mathbf{M_{00}}}^{-1}(\cdots) + i\mathcal{O}(|\alpha_0|^2)\partial_t\gamma_0\widetilde{\mathbf{M_{00}}}^{-1} . \end{aligned} \tag{6.21}$$

Since $\alpha_0 = |\alpha_0|e^{i\gamma_0}$ we have

$$\begin{aligned} &i\partial_t|\alpha_0| - |\alpha_0|\partial_t\gamma_0 = e^{-i\gamma_0} \times \text{1st component of the vector :}\\ &\widetilde{\mathbf{M_{00}}}^{-1}(\cdots) + i\mathcal{O}(|\alpha_0|)\ |\alpha_0|\partial_t\gamma_0\widetilde{\mathbf{M_{00}}}^{-1} . \end{aligned} \tag{6.22}$$

By taking the real part of (6.22), for $|\alpha_0|$ small, we can solve for $|\alpha_0|\partial_t\gamma_0$. This enables us to eliminate it from Eq. (6.16) as a higher-order term.

Implementation of these simplifications leads to the following

Proposition 6.1.

$$\begin{aligned} i\partial_t\vec{\alpha}_0 = \mathbf{M}^{\#}_{\mathbf{00}}\Bigg[&- (E_{01} + \partial_t\tilde{\Theta})\sigma_3\mathbf{N_{01}}\vec{\alpha}_1\\ &+ \lambda\mathcal{O}_0^{(0)}\sigma_3\begin{pmatrix} |\alpha_0|^2\alpha_1 + \alpha_0^2\overline{\alpha_1}\\ \text{c.c.}\end{pmatrix}\\ &+ \lambda\sigma_3\begin{pmatrix} \langle\psi_{0*} + \chi_0^{(2)}, |\phi_1|^2\phi_1 - |\Psi_1|^2\Psi_1\rangle + \alpha_0^2\langle\chi_0^{(0)}, |\phi_1|^2\overline{\phi_1} - |\Psi_1|^2\overline{\Psi_1}\rangle\\ \text{c.c.}\end{pmatrix}\\ &- \partial_t\tilde{\Theta}\begin{pmatrix} \langle\chi_0^{(0)}, \phi_2\rangle + \alpha_0^2\langle\chi_0^{(0)}, \overline{\phi_2}\rangle\\ \text{c.c.}\end{pmatrix}\Bigg] \end{aligned} \tag{6.23}$$

$$\begin{aligned} i\partial_t\vec{\alpha}_1 = {}&\mathbf{A}(t)\vec{\alpha}_1\\ &+ \mathbf{M}^{\#}_{\mathbf{11}}\Bigg[\lambda\sigma_3\begin{pmatrix} \langle\chi, |\phi_1|^2\phi_1 - |\Psi_1|^2\Psi_1\rangle + \alpha_0^2\langle\chi, |\phi_1|^2\overline{\phi_1} - |\Psi_1|^2\overline{\Psi_1}\rangle\\ \text{c.c.}\end{pmatrix} \end{aligned}$$

$$+\lambda\sigma_3\begin{pmatrix}\alpha_0\langle\chi,|\phi_1|^2\rangle+\overline{\alpha_0}\langle\chi,\phi_1^2\rangle+\alpha_0^3\langle\chi,\overline{\phi_1}^2\rangle\\ \text{c.c.}\end{pmatrix}$$

$$-\partial_t\tilde{\Theta}\,\sigma_3\begin{pmatrix}\langle\chi,\phi_2\rangle+\alpha_0^2\langle\chi,\overline{\phi_2}\rangle\\ \text{c.c.}\end{pmatrix}\Bigg]. \tag{6.24}$$

Here,

$$\mathbf{A}=\begin{pmatrix}E_{10}+\mathcal{O}_0^{(0,1)}|\alpha_0|^2 & \mathcal{O}_0^{(0,1)}\alpha_0^2\\ -\mathcal{O}_0^{(0,1)}\overline{\alpha_0}^2 & -E_{10}-\mathcal{O}_0^{(0,1)}|\alpha_0|^2\end{pmatrix}$$

$$\mathbf{A}_{22}=-\mathbf{A}_{11},\ \mathbf{A}_{21}=-\overline{\mathbf{A}_{12}},\ \text{and} \tag{6.25}$$

$$\phi_1=\Psi_1+\phi_2\,.$$

$\mathbf{M}_{\mathbf{00}}^{\#}$ *and* $\mathbf{M}_{\mathbf{11}}^{\#}$ *are near-identity matrices, whose deviations from the identity give rise to higher-order terms which are subordinate to the leading order behavior obtained in the analysis which follows.*

Finally, Eqs. (6.23) *and* (6.24) *are coupled to the equation for* Φ_2 *given by* (5.11).

6.3. *Peeling off the rapid oscillations of* $\boldsymbol{\alpha_1}$

Fix $T>0$ and large. We rewrite (6.24) centered about the ground state at time $t=T$:

$$i\partial_t\vec{\alpha}_1=\mathbf{A}(T)\vec{\alpha}_1+(\mathbf{A}(t)-\mathbf{A}(T))\vec{\alpha}_1+\vec{F}\,, \tag{6.26}$$

where

$$\mathbf{A}(T)=\begin{pmatrix}E_{01}(T)+\mathcal{O}_0^{(0,1)}(T)|\alpha_0(T)|^2 & \mathcal{O}_0^{(0,1)}(T)\alpha_0^2(T)\\ -\mathcal{O}_0^{(0,1)}(T)\overline{\alpha_0}^2(T) & -E_{01}(T)-\mathcal{O}_0^{(0,1)}(T)|\alpha_0(T)|^2\end{pmatrix}. \tag{6.27}$$

Since $\mathbf{A}(T)$ is a constant coefficient matrix it is a simple matter to obtain the fundamental matrix.

Proposition 6.2. *The system*

$$i\partial_t\vec{\alpha}_1=\mathbf{A}(T)\vec{\alpha}_1 \tag{6.28}$$

has a fundamental solution matrix:

$$\mathbf{X}(t)=\begin{pmatrix}c_{11}^{+} & c_{12}^{-}\\ c_{21}^{+} & c_{22}^{-}\end{pmatrix}\begin{pmatrix}e^{-i\lambda_+t} & 0\\ 0 & e^{-i\lambda_-t}\end{pmatrix}$$

$$=\begin{pmatrix}1 & \mathcal{O}_0^{(0,1)}(T)\alpha_0(T)^2E_{10}^{-1}(T)\\ \mathcal{O}_0^{(0,1)}(T)\overline{\alpha_0(T)}^2E_{10}^{-1}(T) & 1\end{pmatrix}\begin{pmatrix}e^{-i\lambda_+t} & 0\\ 0 & e^{-i\lambda_-t}\end{pmatrix}. \tag{6.29}$$

The eigenfrequencies, $\lambda_\pm(T)$ *are given by* $\lambda_+(T)$ *and* $\lambda_-(T)=-\lambda_+(T)$, *where*:

$$\lambda_+(T)\ =\ E_{10}(T)+\mathcal{O}_0^{(0,1)}|\alpha_0(T)|^2\,, \tag{6.30}$$

where $E_{10}=E_1-E_0$, *and provided* $|\alpha_0(T)|^2/E_{10}(T)$ *is sufficiently small.*

We use the fundamental matrix, $X(t)$, to define a change of variables:

$$\vec{\alpha}_1 \equiv \mathbf{X}(t)\vec{\beta} = \mathbf{X}(t)\begin{pmatrix} \beta_1 \\ \overline{\beta_1} \end{pmatrix}. \tag{6.31}$$

Therefore,

$$\alpha_1 = e^{-i\lambda_+ t}\beta_1 + \mathcal{O}_0^{(0,1)}\alpha_0^2(T)E_{10}^{-1}(T)e^{-i\lambda_- t}\overline{\beta_1}. \tag{6.32}$$

Then, $\vec{\beta}$ satisfies

$$i\partial_t\vec{\beta} = \mathbf{X}^{-1}(t)\vec{F}(\mathbf{X}(t)\vec{\beta}(t), t). \tag{6.33}$$

Note that since the linear in α_1 terms have been removed by the change of variables (6.31), $\partial_t\vec{\beta}_1 \sim \mathcal{O}(|\beta_1|^2)$.

Note that by (6.29)

$$\mathbf{X}^{-1}(t) = \begin{pmatrix} e^{i\lambda_+ t} & 0 \\ 0 & e^{i\lambda_- t} \end{pmatrix}\begin{pmatrix} 1 + \mathcal{O}_0^{(0,1)}(T)|\alpha_0(T)|^4 E_{10}^{-2}(T) & -\mathcal{O}_0^{(0,1)}(T)\alpha_0^2(T)E_{10}^{-1}(T) \\ -\mathcal{O}_0^{(0,1)}(T)\overline{\alpha_0(T)}^2 E_{10}^{-1}(T) & 1 + \mathcal{O}_0^{(0,1)}(T)|\alpha_0(T)|^2 E_{10}^{-2}(T) \end{pmatrix}. \tag{6.34}$$

Written out in detail, from (6.33) and the various definitions, we have

Proposition 6.3. *The equation for* β_1 *has the form*

$$\begin{aligned} i\partial_t\beta_1 = {} & \lambda e^{i\lambda_+ t}\langle\chi_0, |e^{-i\lambda_+ t}\beta_1\psi_{1*} + \phi_2|^2\rangle\alpha_0 \\ & + \lambda e^{i\lambda_+ t}\langle\chi_0, (e^{-i\lambda_+ t}\beta_1\psi_{1*} + \phi_2)^2\rangle\overline{\alpha_0} \\ & + \lambda e^{i\lambda_+ t}\langle\chi_0, (e^{i\lambda_+ t}\overline{\beta_1}\psi_{1*} + \overline{\phi_2})^2\rangle\alpha_0^3 \\ & + \lambda\left\langle\chi_0, \left[|e^{-i\lambda_+ t}\beta_1\psi_{1*} + \phi_2|^2(e^{-i\lambda_+ t}\beta_1\psi_{1*} + \phi_2) - e^{-i\lambda_+ t}|\beta_1|^2\beta_1\psi_{1*}^3\right]\right\rangle \\ & + \mathcal{R}_\beta, \end{aligned} \tag{6.35}$$

where

$$\mathcal{R}_\beta = \mathbf{X}^{-1}(t)\left[(\mathbf{A}(t) - \mathbf{A}(T))\mathbf{X}(t)\vec{\beta}_1 + \mathcal{R}_1\right]. \tag{6.36}$$

Remark 6.1.

$$\mathbf{X}^{-1}(t)[\mathbf{A}(t) - \mathbf{A}(T)]\mathbf{X}(t) = \text{ Real Symmetric Diagonal } \mathbf{S}(t) + e^{-2i\lambda_+ t}\mathbf{B}(t) \tag{6.37}$$

Therefore, the non-oscillatory part involving $\mathbf{S(t)}$ does not effect the evolution of $|\beta_1|^2$.

6.4. *Expansion of* ϕ_2

The next step is to get an appropriate expansion of ϕ_2, which upon substitution into (6.35) can be used to isolate the key resonant terms in the β_1 equation. The analogous steps are then repeated for the α_0 equation. Finally, a near-identity change of variables is constructed which maps the system for α_0 and β_1 to a new system (a *normal form* plus corrections) for which the dynamical behavior is more transparent.

Φ_2 solves Eq. (5.11). We shall require dispersive decay estimates for Φ_2 and these are most naturally obtained relative to a time-independent Hamiltonian. Since $\alpha_0(t)$ is expected to tend to a limit as $t \to \infty$ and since we are fixing a time interval $[0, T]$, it is natural to use as *reference Hamiltonian*, the operator $\mathcal{H}_0(T)$. We now make use of the linear spectral theory of Sec. 3 and decompose Φ_2 into a part lying in the "discrete subspace of $\mathcal{H}_0(T)$":

$$N_g(\mathcal{H}_0(T)) \oplus N(\mathcal{H}_0(T) - \mu(T)) \oplus N(\mathcal{H}_0(T) + \mu(T)) \tag{6.38}$$

and a part lying in $\mathcal{M}_1(T)$, the "dispersive subspace of $\mathcal{H}_0(T)$"; see (4.40).

Let [3, 11]

$$\Phi_2 = k + n + \eta\,, \tag{6.39}$$

where

$$k \equiv \sum_{\xi_a \in N_g(\mathcal{H}_0(T))} \langle \sigma_3 \xi_a(T), \Phi_2 \rangle \xi_a \tag{6.40}$$

$$n \equiv \sum_{\xi_b \in N(\mathcal{H}_0(T) \mp \mu(T))} \langle \sigma_3 \xi_b(T), \Phi_2 \rangle \xi_b \tag{6.41}$$

$$\eta \equiv P_c(T)\Phi_2 = \Phi_2 - k - n\,. \tag{6.42}$$

Since

$$\langle \sigma_3 \xi_a(t), \Phi_2(t) \rangle = 0\,, \qquad \xi_a \in N_g(\mathcal{H}_0(t))$$

$$\langle \sigma_3 \xi_b(t), \Phi_2(t) \rangle = 0\,, \qquad \xi_b \in N_g(\mathcal{H}_0(t) \mp \mu(t))\,,$$

$\xi(t)$ may be replaced by $\xi(t) - \xi(T)$ in the definitions of k and n. Inserting the expansion (6.39) into (6.40) and (6.41) and defining

$$p_{\text{null}}(t, T) = \sum_{\xi_a \in N_g(\mathcal{H}_0(T))} \langle \sigma_3(\xi_a(T) - \xi_a(t)), \cdot \rangle \xi_a \tag{6.43}$$

$$p_{\text{neut}}(t, T) = \sum_{\xi_b \in N(\mathcal{H}_0(T) \mp \mu(T))} \langle \sigma_3(\xi_b(T) - \xi_b(t)), \cdot \rangle \xi_a \tag{6.44}$$

we have that k and n may be expressed in terms of η as follows: Then,

$$\left[I - \begin{pmatrix} p_{\text{null}}(t,T) & p_{\text{null}}(t,T) \\ p_{\text{neut}}(t,T) & p_{\text{neut}}(t,T) \end{pmatrix} \right] \begin{pmatrix} k \\ n \end{pmatrix} = \begin{pmatrix} p_{\text{null}}(t,T)\eta \\ p_{\text{neut}}(t,T)\eta \end{pmatrix}. \tag{6.45}$$

Therefore, we have

Proposition 6.4. *There exists* $\varepsilon_0 > 0$ *such that if* $|\alpha_0(t) - \alpha_0(T)| < \varepsilon_0$ *then the relation* (6.45) *can be inverted and we have*

$$\Phi_2 = \Phi_2[\eta] = k[\eta] + n[\eta] + \eta\,, \tag{6.46}$$

where Φ_2 *is linear in* η *and continuous in the weighted* (*local decay*) *norm of* $f \mapsto \|\langle x\rangle^{-\sigma} f\|_2$.

The statement about continuity in the weighted norm follows from the spatial localization of the generalized eigenfunctions.

Remark 6.2. Note that since $|\xi_a(T) - \xi_a(t)| \le C\mathcal{E}_0^{\frac{1}{2}}|\alpha_0(T) - \alpha_0(t)|$, we have the simple estimate:

$$|k|,\ |n| \ \le\ C|\alpha_0(T) - \alpha_0(t)|\ ||\langle x\rangle^{-\sigma}\eta(t)||_2\,. \tag{6.47}$$

Anticipating that for t very large, $|\alpha_0(T) - \alpha_0(t)| \sim t^{-\frac{1}{2}}$ and $||\langle x\rangle^{-\sigma}\Phi_2(t)||_2 \sim t^{-\frac{1}{2}}$, it follows that $|k|,\ |n| \sim t^{-1}$. Thus, $|k|$ and $|n|$, are expected to decay faster than η.

Finally, η satisfies the following evolution equation obtained from (5.11) by explicitly introducing the reference Hamiltonian, $\mathcal{H}_0(T)$, and applying the projection $P_c(T)$ to the equation.

$$\begin{aligned} i\partial_t\eta &= \mathcal{H}_0(T)\eta \\ &\quad + (\mathcal{H}_0(t) - \mathcal{H}_0(T))\Phi_2[\eta] - \partial_t\tilde{\Theta}P_c(T)\sigma_3\Phi_2[\eta] \\ &\quad + (E_1(t) - E_0(t))P_c(T)\sigma_3\begin{pmatrix}\Psi_1(t)\\ \text{c.c.}\end{pmatrix} + \lambda P_c(T)\sigma_3\begin{pmatrix}2|\Psi_0|^2\Psi_1 + \Psi_0^2\overline{\Psi_1}\\ \text{c.c.}\end{pmatrix} \\ &\quad + \lambda P_c(T)\sigma_3\begin{pmatrix}2\Psi_0|\Psi_1 + \pi_1\Phi_2[\eta]|^2 + \overline{\Psi_0}(\Psi_1 + \pi_1\Phi_2[\eta])^2\\ \text{c.c.}\end{pmatrix} \\ &\quad + \lambda P_c(T)\sigma_3\begin{pmatrix}|\Psi_1 + \pi_1\Phi_2[\eta]|^2(\Psi_1 + \pi_1\Phi_2[\eta]) - |\Psi_1|^2\Psi_1\\ \text{c.c.}\end{pmatrix} \\ &\quad - iP_c(T)\begin{pmatrix}\vec{\nabla}_1\Psi_1(t)\cdot\partial_t\vec{\alpha}_1(t) + \vec{\nabla}_0\Psi_0(t)\cdot\partial_t\vec{\alpha}_0(t)\\ \text{c.c.}\end{pmatrix}. \end{aligned} \tag{6.48}$$

7. Normal Form and Master Equations

In Sec. 4 we decomposed the solution, ϕ, in terms of coordinates $\alpha_0(t)$ and $\alpha_1(t)$ along manifolds of nonlinear bound states and ϕ_2, a correction which lies in a time-dependent subspace, $\mathcal{M}_1(t)$, of continuum modes. $\phi_2(t)$ was then decomposed into its discrete (k and n) and continuous (η) components with respect to a time-*independent* Hamiltonian, $\mathcal{H}_0(T)$. We also observed that k and n are determined by and are expected to be more rapidly decaying than η (Proposition 6.4 and

Remark 6.2). Therefore, the evolution of ϕ is determined by α_0, α_1 and η. Finally, the fast oscillations of α_1 are removed by the introduction of $\beta_1 = [X(t)\vec{\alpha}_1]_1 \sim e^{-i\lambda_+ t}\alpha_1$.

We now seek a form of the system for α_0 and β_1 from which the large time dynamics can be deduced. We obtain this "normal form" by first solving for η (see (7.21), (6.48)) as a functional of α_0, β_1 and the initial data $\eta(0)$ and then substituting an appropriate expansion (see Secs. 7 and 8) into the equations for α_0 and β_1.

Proposition 7.1 (The Normal Form). *There exists a near identity change of variables*

$$\begin{pmatrix} \tilde{\alpha}_0 \\ \tilde{\beta}_1 \end{pmatrix} \equiv \begin{pmatrix} \alpha_0 \\ \beta_1 \end{pmatrix} + \begin{pmatrix} J_\alpha[\alpha_0, \beta_1, t] \\ J_\beta[\alpha_0, \beta_1, t] \end{pmatrix} \tag{7.1}$$

where

$$J_k[\alpha_0, \beta_1, t] = \mathcal{O}(|\alpha_0|^2 + |\beta_1|^2)\,, \qquad k = \alpha, \beta \tag{7.2}$$

and bounded uniformly in t, *and such that*

$$\begin{aligned} i\partial_t \tilde{\alpha}_0 &= (c_{1022} + i\Gamma_\omega)\,|\tilde{\beta}_1|^4 \tilde{\alpha}_0 + F_\alpha[\tilde{\alpha}_0, \tilde{\beta}_1, \eta, t] \\ i\partial_t \tilde{\beta}_1 &= (c_{1121} - 2i\Gamma_\omega)|\tilde{\alpha}_0|^2 |\tilde{\beta}_1|^2 \tilde{\beta}_1 + F_\beta[\tilde{\alpha}_0, \tilde{\beta}_1, \eta, t]\,. \end{aligned} \tag{7.3}$$

The properties of F_α *and* F_β *are briefly discussed in Remark* 7.2 *following Corollary* 7.1 *below and described in detail in Sec.* 8.

Furthermore,

$$\Gamma = \Gamma_{\omega_*} + \mathcal{O}(|\alpha_0(T)|^2) > 0\,,$$

where,

$$\begin{aligned} \Gamma_{\omega_*} &\equiv \lambda^2 \pi \langle \psi_{0*}\psi_{1*}^2, \delta(H - \omega_*)\psi_{0*}\psi_{1*}^2 \rangle > 0 \\ \omega_* &= 2E_{1*} - E_{0*}\,. \end{aligned} \tag{7.4}$$

The coefficients $c_{klmn} = \mathcal{O}_0^{(0,1)}$ *are real constants multiplying monomials of the form* $\alpha_0^k \bar{\alpha}_0^l \beta_1^m \bar{\beta}_1^n$.

Remark 7.1. Due to our choice of the phase correction, $\tilde{\Theta}(t)$ (see (6.12), a term of the form $c_{1011}\mathcal{O}_0^{(0,1)}|\tilde{\beta}_1|^2\tilde{\alpha}_0$, is absent from the differential equation for $\tilde{\alpha}_0$ in (7.3).

Now let

$$P_0 \equiv |\tilde{\alpha}_0|^2 \qquad \text{and} \qquad P_1 \equiv |\tilde{\beta}_1|^2 \tag{7.5}$$

denote the (renormalized) ground state and excited state powers. Then, by (7.3) we have the *Nonlinear Master Equation*:

Corollary 7.1.

$$\frac{dP_0}{dt} = 2\Gamma P_0 P_1^2 + R_0 \tag{7.6}$$

$$\frac{dP_1}{dt} = -4\Gamma P_0 P_1^2 + R_1\,, \tag{7.7}$$

where

$$\begin{aligned} R_0 &= R_0[\tilde\alpha_0, \tilde\beta_1, \eta, t] = 2\Im(\overline{\tilde\alpha_0} F_\alpha)\,,\\ R_1 &= R_1[\tilde\alpha_0, \tilde\beta_1, \eta, t] = 2\Im(\overline{\tilde\beta_1} F_\beta)\,. \end{aligned} \tag{7.8}$$

A more precise and revealing variant of Corollary 7.1 is Proposition 11.1, which is stated and proved in Sec. 8.

Remark 7.2. The terms F_α and F_β are such that R_0 and R_1 are not small perturbations of the leading order terms in (7.6), (7.7) for all $t \geq 0$. In fact there are three time intervals defined in terms of transition times t_0 and t_1 (see Sec. 8), in which we consider the system (7.6), (7.7): $I_0 = [0, t_0]$, $I_1 = [t_0, t_1]$ and $I_2 = [t_1, \infty)$. It is only for sufficiently large time, ($t \in I_2$), where R_0 and R_1 are negligible. The behavior on short ($0 \leq t \leq t_0$) and intermediate ($t_0 \leq t \leq t_1$) time scales can be very different. We go into the details of R_0 and R_1 in Sec. 8 but wish to make some remarks at this stage which indicate our approach.

If we drop the terms R_j then we have a flow, which evolves in the first quadrant of the $P_0 - P_1$ plane according to:

$$\frac{dp_0}{dt} = 2\Gamma p_0 p_1^2 \tag{7.9}$$

$$\frac{dp_1}{dt} = -4\Gamma p_0 p_1^2\,, \tag{7.10}$$

where solutions for typical data converge to the p_0 axis with a rate $\langle t\rangle^{-1}$. In order for the corrections coming from R_0 and R_1 to be small, *intuitively* it is sufficient that

$$R_j \sim \mathcal{E}_0^\rho (P_0 P_1^2 + \langle t\rangle^{-3})\,, \tag{7.11}$$

where $\mathcal{E}_0$ is small and $\rho > 0$. This is what we show for $t \geq t_1$. For the intermediate time range, $t_0 \leq t \leq t_1$, we show that the behavior is controlled by the system:

$$\begin{aligned} \frac{dP_0}{dt} &= 2\Gamma P_0 P_1^2 + \mathcal{O}(\langle t\rangle^{-3})\\ \frac{dP_1}{dt} &= -4\Gamma P_0 P_1^2 + \mathcal{O}(\sqrt{P_0} P_1^m) + \mathcal{O}(\langle t\rangle^{-3})\,, \end{aligned} \tag{7.12}$$

where $m \geq 3$. Therefore, for intermediate times we need to show:

$$\begin{aligned} R_0 &\sim \mathcal{E}_0^\rho (P_0 P_1^2 + \langle t\rangle^{-3})\\ R_1 &\sim \mathcal{E}_0^\rho (P_0 P_1^2 + \sqrt{P_0} P_1^m + \langle t\rangle^{-3})\,. \end{aligned} \tag{7.13}$$

What makes the analysis subtle is the dependence of R_j on P_0 and P_1 in a manner which is *nonlocal in time*. That is,

$$\eta \sim \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c(\mathcal{H}_0)\chi\alpha_0^{m_1}\beta_1^{m_2}\overline{\alpha_0}^{m_3}\overline{\beta_1}^{m_4}\, ds\,. \tag{7.14}$$

Local in time terms are simple to dominate by the leading terms. However, nonlocal terms require careful analysis. Note in particular, that due to the "history dependence" of such terms, being expressed as time integrals from 0 up to t, an analysis of the effect of such terms for $t \geq t_1$ requires use of estimates on other time regimes $t \leq t_0$ and $t_0 \leq t \leq t_1$ as well. Furthermore, there are no decay estimates on either P_0 or P_1 in the intermediate interval $t_0 \leq t \leq t_1$ or on the size of this interval.

The normal form of Proposition 7.1 is essentially the Poincaré–Dulac normal form which can be constructed along the lines explicitly implemented in [49]; see also [1]. We now give a detailed outline of the procedure with explicit illustrative detail of key points concerning the treatment of resonant and nonresonant terms.

Resonant terms and removal nonresonant terms: Here we illustrate, by way of a simple example, how non-resonant terms can be removed by near identity changes of variables. Consider the scalar ordinary differential equation

$$A'(t) = |A(t)|^2 e^{i\Omega t} \tag{7.15}$$

where $A(t)$ is a complex valued function. We shall introduce a change of variables $A \mapsto \tilde{A} = A + q_2(A, \bar{A}, t)$, where $q_2(A, \bar{A}, t) = \mathcal{O}(|A|^2)$ and $q_2(A, \bar{A}, t + \frac{2\pi}{\Omega}) = q_2(A, \bar{A}, t)$, which is therefore approximately the identity for $|A|$ small, and such that

$$\tilde{A}'(t) = \frac{i}{\Omega}|\tilde{A}(t)|^2\tilde{A}(t) + \frac{i}{\Omega}e^{2i\Omega t}|\tilde{A}(t)|^2\overline{\tilde{A}(t)} + E_4(\tilde{A}(t), \overline{\tilde{A}(t)}, t)\,, \tag{7.16}$$

Here, E_4 is $2\pi/\Omega$ periodic in t and

$$E_4(A(t), \tilde{A}(t), t) = \mathcal{O}(|\tilde{A}(t)|^4) \tag{7.17}$$

The change of variables can be derived by elementary means. Integration of (7.15) gives:

$$\begin{aligned} A(t) - A(0) &= \int_0^t |A(s)|^2 e^{i\Omega s} ds \\ &= \int_0^t |A(s)|^2 \frac{1}{i\Omega}\frac{d}{ds}e^{i\Omega s} ds \\ &= |A(s)|^2 \frac{1}{i\Omega}e^{i\Omega s}\Big|_0^t - \int_0^t \frac{1}{i\Omega}e^{i\Omega s}\frac{d}{ds}|A(s)|^2\, ds \\ &= |A(s)|^2 \frac{1}{i\Omega}e^{i\Omega s}\Big|_0^t - \frac{1}{i\Omega}\int_0^t e^{i\Omega s}\overline{A(s)}|A(s)|^2 e^{i\Omega s} ds \\ &\quad - \frac{1}{i\Omega}\int_0^t |A(s)|^2 A(s) ds\,. \end{aligned} \tag{7.18}$$

Define $\tilde{A}(t) = A(t) - (i\Omega)^{-1}|A(t)|^2 e^{i\Omega t}$. Then, $\tilde{A}$ satisfies the renormalized ODE, in which resonant quadratic terms have been removed. The process can be repeated; by introducing further changes of variables $\tilde{A} \mapsto \tilde{A}_1 = \tilde{A}+$ higher order in $\tilde{A}$ and period in t, non-resonant (oscillatory) cubic terms can be removed to obtain:

$$\tilde{A}_1'(t) = \frac{i}{\Omega}|\tilde{A}_1(t)|^2\tilde{A}_1(t) + ik|\tilde{A}_1(t)|^4\tilde{A}_1^(t) + \dots, \tag{7.19}$$

where k is real. That the coefficients in the first to terms of this *normal form* are purely imaginary implies that, to this order, the amplitude $|\tilde{A}_1(t)|$ is independent in time. This is the typical situation of the norm form finite dimensional Hamiltonian systems, in which resonances occur between isolated discrete frequencies.

We next examine resonances between discrete and frequencies and the continuum of frequencies, associated with the continuous spectral (dispersive) part of $\mathcal{H}_0$. These can introduce *nonconservative* terms into the normal from (via coefficients with real as well as imaginary parts), which are responsible for energy transfer between discrete modes (bound states) and radiation.

Nonconservative resonant terms and energy transfer: We explain how to find the key resonant energy transfer terms, the leading terms in (7.3). These are terms responsible for the exchange of energy among the nonlinear ground and excited states mediated by interaction with continuums modes. We focus on the β_1 equation. Analogous considerations apply to the α_0 equation.

Equation (6.35) can be written the following compact form:

$$i\partial_t\beta_1 = \sum_{p,q,r} C_{pqr}\beta_1^p\overline{\beta_1}^q e^{-i\omega_r t}\langle\chi_{pqr}, \eta\rangle + \mathcal{O}(\eta^2) + \cdots\,, \tag{7.20}$$

where C_{pqr} are of order 1, α_0 or higher order in α_0, $\omega_r \in \{\pm\lambda_\pm, \pm 2\lambda_\pm, 0\}$, and χ_{pqr} denote functions which are exponentially localized in space. The equation for α_0 has a similar structure. The equation for $\eta = \eta_0 + \eta_1 + \eta_2$, can be formally solved giving:

$$\begin{aligned}\eta &= \mathcal{O}(\eta_0) + \mathcal{O}(\eta_0^2) + \mathcal{O}(\eta^2)\\ &\quad + \sum_{p_1,q_1,r_1} D_{p_1q_1r_1}\mathbf{G}\left(\beta_1^{p_1}\overline{\beta_1}^{q_1} e^{-i\nu_{r_1}s}\,\tilde{\chi}_{p_1q_1r_1}\right).\end{aligned} \tag{7.21}$$

Here, $\mathbf{G}$ denotes the operator

$$f \mapsto \mathbf{G}f \equiv -i\int_0^t e^{-i\mathcal{H}_0(t-s)}P_c\,f(s)ds\,. \tag{7.22}$$

We insert the expansion for η, (7.21) into the terms involving inner products $\langle\chi_{pqr}, \eta\rangle$ in (7.20) and in the corresponding equation for α_0. This yields a coupled system for α_0 and β_1 which is closed up to higher order. The terms in the resulting equations are of the form

$$\sum_{p,q,r}\sum_{p_1,q_1,r_1} C_{pqr}D_{p_1q_1r_1}\beta_1^p\overline{\beta_1}^q e^{-i\omega_r t}\left\langle\chi_{pqr},\, \mathbf{G}\beta_1^{p_1}\overline{\beta_1}^{q_1} e^{-i\nu_{r_1}s}\,\tilde{\chi}_{p_1q_1r_1}\right\rangle. \tag{7.23}$$

We now use the integration by parts lemma:

Lemma 7.1.

$$\begin{aligned}\int_0^t e^{iAs} f(s)ds &= \lim_{\delta\downarrow 0}\int_0^t e^{i(A\pm i\delta)s} f(s)ds \\ &= -i(A\pm i0)^{-1}e^{iAt}f(t) + i(A\pm i0)^{-1}f(0) \\ &\quad + i(A\pm i0))^{-1}\int_0^t e^{iAs}f'(s)ds\,. \end{aligned} \tag{7.24}$$

Applying this lemma, we obtain

$$\begin{aligned} &e^{-i\omega_r t}\langle \chi_{pqr}\,\mathbf{G}\beta_1^{p_1}\overline{\beta_1}^{q_1}e^{-i\nu_{r_1}s}\,\tilde{\chi}_{p_1q_1r_1}\rangle \\ &\; = -e^{-i(\omega_r+\nu_{r_1})t}\beta_1^{p_1}\overline{\beta_1}^{q_1}\left\langle \chi_{pqr},\,(\mathcal{H}_0(T)-\nu_{r_1}\mp i0)^{-1}P_c(T)\tilde{\chi}_{p_1q_1r_1}\right\rangle \\ &\quad + \beta_1^{p_1}(0)\overline{\beta_1}^{q_1}(0)\langle \chi_{pqr},\,(\mathcal{H}_0(T)-\nu_{r_1}\mp i0)^{-1}e^{-i\mathcal{H}_0(T)t}P_c(T)\tilde{\chi}_{p_1q_1r_1}\rangle \\ &\quad + \int_0^t\left\langle \chi_{pqr},\,(\mathcal{H}_0(T)-\nu_{r_1}\mp i0)^{-1}e^{-i\mathcal{H}_0(T)(t-s)}e^{-i\nu_{r_1}s}P_c(T)\frac{d}{ds}(\beta_1^{p_1}\overline{\beta_1}^{q_1}\tilde{\chi}_{p_1q_1r_1})ds\right\rangle\,. \end{aligned} \tag{7.25}$$

We first focus on the first term in the expansion (7.25). This contributes a resonant term, which cannot be transformed by a near identity transformation to higher order if

$$\omega_r + \nu_{r_1} = 0\,. \tag{7.26}$$

Now consider such a resonant term. We find that in the β_1 equation they are of the form:

$$-|\alpha_0|^{2a_1}|\beta_1|^{2b_1}\beta_1\left\langle \vec{\chi},\,(\mathcal{H}_0-\nu_{r_1}\mp i0)^{-1}P_c\vec{\chi}\right\rangle \tag{7.27}$$

where $\nu_{r_1} = -\omega_r \in \{\pm\lambda_\pm, \pm 2\lambda_\pm, 0\}$.

There are two cases to consider: (i) ν_{r_1} *not in* the continuous spectrum of $\mathcal{H}_0$ and (ii) ν_{r_1} *in* the continuous spectrum of $\mathcal{H}_0$.[d] If ν_{r_1} is not in the continuous spectrum of $\mathcal{H}_0$ then the inner product in (7.27) does not involve a singular limit and we get the limit

$$\left\langle \vec{\chi},\,(\mathcal{H}_0-\nu_{r_1})^{-1}P_c\vec{\chi}\right\rangle\,. \tag{7.28}$$

In this case, the coefficient of $|\alpha_0|^{2a_1}|\beta_1|^{2b_1}\beta_1$ is *real*. Such a term results only in a nonlinear distortion of the phase of β_1 and does not effect the amplitude. If ν_{r_1} is in the interior of the continuous spectrum of $\mathcal{H}_0$ then the limit is singular. We choose the plus sign $(+i0)$ if we study the evolution for $t > 0$ and the negative sign $(-i0)$

[d] In case (ii) we consider the generic case where if ν_{r_1} lies in the interior of the continuous spectrum of $\mathcal{H}_0$.

for $t < 0$. This choice is related to the condition of outgoing radiation explained below; see also [49], for example. Evaluation of this singular limit gives:

$$\left\langle\chi,\, (\mathcal{H}_0 - \nu_{r_1} \mp i0)^{-1} P_c\chi\right\rangle = \tilde{\Lambda} + i\tilde{\Gamma}\,,$$

where

$$\tilde{\Gamma} = \pi\langle\chi,\, \delta(\mathcal{H}_0 - \nu_{r_1})\chi\rangle\,, \qquad \tilde{\Lambda} = \left\langle\chi,\, \text{P.V.}\ (\mathcal{H}_0 - \nu_{r_1})^{-1}\chi\right\rangle\,.$$

Contributions to the imaginary part are therefore responsible for a change in amplitude (here damping of β_1).

Now, when does a frequency ν *lie in the continuous spectrum of* $\mathcal{H}_0$? By Proposition 4.2, we must have $\nu > -E_0 = |E_0|$ or $\nu < E_0$. By (6.30), $\lambda_\pm \sim \pm(E_1 - E_0)$. Since ν varies over the frequencies $0, \pm\lambda_\pm, \pm 2\lambda_\pm$ we find that $\nu_{r_1} = \pm 2\lambda_\pm = -\omega_r$ resonances are in the continuous spectrum and therefore are those giving rise to energy transfer, provided $2E_1 - E_0 > 0$; see (7.4). We now embark on the details.

7.1. *Expansion of* η

We expand η as follows:

$$\eta(t) = \eta_0(t) + \eta_1(t) + \eta_2(t)\,, \tag{7.29}$$

where $\eta_0(t)$ corresponds to the linear homogeneous evolution with initial data $\eta(0) = P_c(T)\Phi_2(0)$ and η_1 solves the inhomogeneous linear equation driven by α_0, α_1 and $\eta_0(t)$.

Equation for $\eta_0(t)$:

$$\begin{aligned} i\partial_t\eta_0 &= \mathcal{H}_0(T)\eta_0 \\ \eta_0(0) &= P_c(T)\Phi_2(0)\,. \end{aligned} \tag{7.30}$$

Thus,

$$\eta_0(t) = e^{-i\mathcal{H}_0(T)t} P_c(T)\Phi_2(0)\,. \tag{7.31}$$

Equation for $\eta_1(t)$:

$$\begin{aligned} i\partial_t\eta_1 &= \mathcal{H}_0(T)\eta_1 + P_c(T)[\mathcal{H}_0(t) - \mathcal{H}_0(T)]\Phi_2[\eta_0] + E_{10}(t)P_c(T)\sigma_3 \begin{pmatrix} \Psi_1 \\ \text{c.c.} \end{pmatrix} \\ &\quad + \lambda P_c(T)\sigma_3 \begin{pmatrix} 2|\Psi_0|^2\Psi_1 + \Psi_0^2\overline{\Psi_1} \\ \text{c.c.} \end{pmatrix} - \partial_t\tilde{\Theta} P_c(T)\sigma_3\Phi_2[\eta_0] \\ &\quad + \lambda P_c(T)\sigma_3 \begin{pmatrix} 2\Psi_0|\Psi_1 + \pi_1\Phi_2[\eta_0]|^2 + \overline{\Psi_0}(\Psi_1 + \pi_1\Phi_2[\eta_0])^2 \\ \text{c.c.} \end{pmatrix} \\ &\quad + \lambda P_c(T)\sigma_3 \begin{pmatrix} |\Psi_1 + \pi_1\Phi_2[\eta_0]|^2(\Psi_1 + \pi_1\Phi_2[\eta_0]) - |\Psi_1|^2\Psi_1 \\ \text{c.c.} \end{pmatrix} \\ &\quad - iP_c \begin{pmatrix} \vec{\nabla}_1\Psi_1(t)\cdot\partial_t\vec{\alpha}_1(t) + \vec{\nabla}_0\Psi_0(t)\cdot\partial_t\vec{\alpha}_0(t) \\ \text{c.c.} \end{pmatrix}. \end{aligned} \tag{7.32}$$

The initial data for η_1 is $\eta_1(0) = 0$.

Remark 7.3. A direct computation using the biorthogonal decomposition of the discrete subspaces of $\mathcal{H}_0$ and $\mathcal{H}_0^*$ of Sec. 4 yields that the second term in (6.48) is of a higher order than is explicit:

$$E_{10}(t)P_c(T)\sigma_3\begin{pmatrix}\Psi_1\\ \text{c.c.}\end{pmatrix} = \mathcal{O}(|\alpha_0|^2|\alpha_1| + |\alpha_1|^3)\big(\chi_0^{(0)} + \chi_0^{(1)} + \chi_0^{(0,1)}\big) \tag{7.33}$$

for $|\alpha_0| \ll 1$ and $|\alpha_1| \downarrow 0$.

Equation for $\eta_2(t)$:

$$\begin{aligned}
&i\partial_t\eta_2\\
&\ = \mathcal{H}_0(T)\eta_2 + P_c(T)[\mathcal{H}_0(t) - \mathcal{H}_0(T)]\Phi_2[\eta_1+\eta_2] - \partial_t\tilde{\Theta}P_c(T)\sigma_3\Phi_2[\eta_1+\eta_2]\\
&\quad + \lambda P_c(T)\sigma_3\begin{pmatrix}2\Psi_0\left[(\Psi_1 + \pi_1\Phi_2[\eta_0])\overline{\pi_1\Phi_2[\eta_1+\eta_2]} + \text{c.c.}\right]\\ \text{c.c.}\end{pmatrix}\\
&\quad + \lambda P_c(T)\sigma_3\begin{pmatrix}2\Psi_0|\pi_1\Phi_2[\eta_1+\eta_2]|^2 + 2\overline{\Psi_0}(\Psi_1 + \pi_1\Phi_2[\eta_0])\pi_1\Phi_2[\eta_1+\eta_2]\\ \text{c.c.}\end{pmatrix}\\
&\quad + \lambda P_c(T)\sigma_3\begin{pmatrix}\overline{\Psi_0}(\pi_1\Phi_2[\eta_1+\eta_2])^2\\ \text{c.c.}\end{pmatrix}\\
&\quad + \lambda P_c(T)\sigma_3\begin{pmatrix}2|\Psi_1 + \pi_1\Phi_2[\eta_0]|^2\pi_2\Phi[\eta_1+\eta_2]\\ \text{c.c.}\end{pmatrix}\\
&\quad + \lambda P_c(T)\sigma_3\begin{pmatrix}(\pi_1\Phi_2[\eta_1+\eta_2])^2(\overline{\Psi_1} + \overline{\pi_1\Phi_2[\eta_0]}) + \overline{\pi_1\Phi_2[\eta_1+\eta_2]}(\Psi_1 + \pi_1\Phi_2[\eta_0])^2\\ \text{c.c.}\end{pmatrix}\\
&\quad + \lambda P_c(T)\sigma_3\begin{pmatrix}2(\Psi_1 + \pi_1\Phi_2[\eta_0])|\pi_1\Phi_2[\eta_1+\eta_2]|^2 \ + \ |\pi_1\Phi_2[\eta_1+\eta_2]|^2\pi_1\Phi_2[\eta_1+\eta_2]\\ \text{c.c.}\end{pmatrix}.
\end{aligned} \tag{7.34}$$

We expect that $\eta_2 = \mathcal{O}(\langle t\rangle^{-1})$. Let

$$\eta_2 = \eta_{2a} + \eta_{2b}\,. \tag{7.35}$$

By construction we will show that $\eta_{2a} = \mathcal{O}(\langle t\rangle^{-1})$ and $\eta_{2b} = \mathcal{O}(\langle t\rangle^{-\frac{3}{2}})$.

$$\begin{aligned}
&i\partial_t\eta_{2a}\\
&\ = \mathcal{H}_0(T)\eta_{2a} + P_c(T)[\mathcal{H}_0(t) - \mathcal{H}_0(T)]\Phi_2[\eta_1] - \partial_t\tilde{\Theta}\sigma_3\Phi_2[\eta_1]\\
&\quad + 2\lambda P_c(T)\sigma_3\begin{pmatrix}\Psi_0(\Psi_1\pi_1\overline{\Phi_2[\eta_1]} + \overline{\Psi_1}\pi_2\Phi_2[\eta_1]) + \Psi_0|\pi_1\Phi_2[\eta_1]|^2\\ \text{c.c.}\end{pmatrix}\\
&\quad + \lambda P_c(T)\sigma_3\begin{pmatrix}2\overline{\Psi_0}\pi_1\Phi_2[\eta_1](\Psi_1 + \pi_1\Phi_2[\eta_0]) + \overline{\Psi_0}(\pi_1\Phi_2[\eta_1])^2\\ \text{c.c.}\end{pmatrix}
\end{aligned}$$

$$+\lambda P_c(T)\sigma_3 \begin{pmatrix} 2|\Psi_1+\pi_1\Phi_2[\eta_0]|^2\pi_1\Phi_2[\eta_1] + (\Psi_1+\pi_1\Phi_2[\eta_0])^2\overline{\pi_1\Phi_2[\eta_1]} \\ \text{c.c} \end{pmatrix}$$

$$+\lambda P_c(T)\sigma_3 \begin{pmatrix} 2(\Psi_1+\pi_1\Phi_2[\eta_0])\,|\pi_1\Phi_2[\eta_1]|^2 + \overline{(\Psi_1+\pi_1\Phi_2[\eta_0])}\,(\pi_1\Phi_2[\eta_1])^2 \\ \text{c.c} \end{pmatrix}$$

$$+\lambda P_c(T)\sigma_3 \begin{pmatrix} |\pi_1\Phi_2[\eta_1]|^2\pi_1\Phi_2[\eta_1] \\ \text{c.c.} \end{pmatrix}. \tag{7.36}$$

7.2. *Normal form and master equations*

Using (7.33) and explicitly inserting in (7.32) the representation

$$\alpha_1 = (X(t)\beta(t))_1 = e^{-i\lambda_+(T)t}\beta_1 + \mathcal{O}_0^{(0,1)}\alpha_0^2(T)E_{10}^{-1}(T)e^{-i\lambda_-(T)t}\overline{\beta_1} \tag{7.37}$$

gives the following equation for η_1:

$$\begin{aligned} i\partial_t\eta_1 &= \mathcal{H}_0(T)\eta_1 + P_c(T)\left(\mathcal{H}_0(t)-\mathcal{H}_0(T)\right)\Phi_2[\eta_0] - \partial_t\tilde{\Theta}P_c(T)\sigma_3\Phi_2[\eta_0] \\ &\quad + E_{10}(t)P_c(T)\sigma_3(|\tilde{\alpha}_0|^2\chi_0^{(0)} + |\tilde{\beta}_1|^2\chi_1^{(1)}) \\ &\qquad \times \begin{pmatrix} e^{-i\lambda_+(T)t}\beta_1 + \mathcal{O}_0^{(0,1)}\alpha_0^2(T)E_{10}^{-1}(T)e^{-i\lambda_- t}\overline{\beta_1} \\ \text{c.c.} \end{pmatrix} \\ &\quad + \lambda P_c(T)\sigma_3\psi_{0*}^2\psi_{1*} \begin{pmatrix} 2|\alpha_0|^2\beta_1 e^{-i\lambda_+(T)t} + \alpha_0^2\overline{\beta_1}e^{i\lambda_+(T)t} \\ \text{c.c.} \end{pmatrix} \\ &\quad + \lambda P_c(T)\sigma_3\psi_{0*}^2\psi_{1*} \begin{pmatrix} 2\alpha_0|\beta_1|^2 + \overline{\alpha_0}\beta_1^2 e^{-2i\lambda_+(T)t} \\ \text{c.c.} \end{pmatrix} + \vec{\mathcal{R}}_{\eta_1}\,, \end{aligned} \tag{7.38}$$

with initial condition $\eta_1(0) = 0$.

Substitution of η_1 into Eqs. (6.23) and (6.35) for α_0 and β_1 gives rise to terms which make explicit the resonant exchange of energy between the ground state and excited state. We next isolate the key terms in the expansion of η_1 relating to this energy exchange.

Let us begin with the β_1 equation, (6.35). Written out in greater detail we have:

$$\begin{aligned} i\partial_t\beta_1 &= 2\lambda\langle\psi_{0*},\psi_{1*}^3\rangle|\beta_1|^2\alpha_0 e^{i\lambda_+(T)t} + 2\lambda\langle\psi_{0*}\psi_{1*}^2,\pi_2\Phi_2\rangle\beta_1\alpha_0 \\ &\quad + \lambda\langle\psi_{0*},\psi_{1*}^3\rangle\beta_1^2\overline{\alpha_0}e^{-i\lambda_+(T)t} + \mathbf{2}\lambda\langle\boldsymbol{\psi_{0*}\psi_{1*}^2},\boldsymbol{\pi_1\Phi_2}\rangle(\overline{\boldsymbol{\beta_1}}\boldsymbol{\alpha_0}\mathbf{e}^{\mathbf{2i}\lambda_+\mathbf{(T)t}} + \beta_1\overline{\alpha_0}) \\ &\quad + 2\lambda\langle\psi_{0*}\psi_{1*},|\pi_1\Phi_2|^2\rangle\alpha_0 e^{i\lambda_+(T)t} + \lambda\langle\psi_{0*}\psi_{1*},(\pi_1\Phi_2)^2\rangle\overline{\alpha_0}e^{i\lambda_+(T)t} \\ &\quad + \lambda\langle\psi_{1*}^3,\pi_1\Phi_2\rangle|\beta_1|^2 e^{i\lambda_+(T)t} + \lambda\langle\psi_{1*}^2,(\pi_1\Phi_2)^2\rangle\overline{\beta_1}e^{2i\lambda_+(T)t} \\ &\quad + \lambda\langle\psi_{1*}^3,\pi_2\Phi_2\rangle\beta_1^2 e^{-i\lambda_+(T)t} + 2\lambda\langle\psi_{1*}^2,|\pi_1\Phi_2|^2\rangle\beta_1 e^{-i\lambda_+(T)t} \\ &\quad + \lambda\langle\psi_{1*},|\pi_1\Phi_2|^2\pi_1\Phi_2\rangle e^{i\lambda_+(T)t} + \mathcal{R}_\beta\,, \end{aligned} \tag{7.39}$$

where $\mathcal{R}_\beta$ is defined in (6.36).

We claim that the key term in (7.39) responsible for energy transfer is the term

$$\mathbf{2}\lambda\langle\psi_{\mathbf{0}*}\psi_{\mathbf{1}*}^{\mathbf{2}}, \pi_{\mathbf{1}}\Phi_{\mathbf{2}}\rangle\overline{\beta_{\mathbf{1}}}\alpha_{\mathbf{0}}\mathbf{e}^{\mathbf{2i}\lambda_{+}(\mathbf{T})\mathbf{t}} \tag{7.40}$$

on the second line of (7.39). To see this we decompose η_1 into "resonant" and "nonresonant" parts

$$\eta_1 = \eta_{1\mathrm{R}} + \eta_{1\mathrm{NR}}\,. \tag{7.41}$$

The part $\eta_{1\mathrm{NR}}$ gives rise to the key resonant energy transfer term in the equation for β_1 and is the solution to the initial value problem

$$\begin{aligned} &i\partial_t\eta_{1\mathrm{R}} = \mathcal{H}_0(T)\eta_{1\mathrm{R}} + \lambda P_c(T)\sigma_3\psi_{1*}^2\psi_{0*}\overline{\alpha_0}\beta_1^2 e^{-2i\lambda_+(T)t}\begin{pmatrix}1\\0\end{pmatrix}\\ &\eta_{1\mathrm{R}}(0) = 0\,. \end{aligned} \tag{7.42}$$

Solving (7.42) using DuHamel's principle we have

$$\begin{aligned} \eta_{1\mathrm{R}}(t) &= -i\lambda\int_0^t e^{-i\mathcal{H}_0(T)(t-s)}P_c(T)\sigma_3\psi_{1*}^2\psi_{0*}\overline{\alpha_0(s)}\beta_1^2(s)e^{-2i\lambda_+(T)s}\begin{pmatrix}1\\0\end{pmatrix}ds\\ &= -i\lambda e^{-i\mathcal{H}_0(T)t}\int_0^t e^{i[\mathcal{H}_0(T)-2\lambda_+(T)]s}P_c(T)\sigma_3\psi_{1*}^2\psi_{0*}\overline{\alpha_0(s)}\beta_1^2(s)\begin{pmatrix}1\\0\end{pmatrix}ds\,. \end{aligned} \tag{7.43}$$

Recall that the continuous spectrum of $\mathcal{H}_0(T)$ is given by points ω such that $|\omega| \geq |E_0|$. By Proposition 6.2, for $|\alpha_0|^2/E_{10}$ sufficiently small

$$\lambda_+(T) = E_{10}(T) + \mathcal{O}_0^{(0,1)}|\alpha_0(T)|^2\,. \tag{7.44}$$

By the hypothesis (7.4), $2E_{1*} - E_{0*} > 0$, if $|\alpha_0|$ is sufficiently small then $2\lambda_+(T)$ lies in the continuous spectrum of $\mathcal{H}_0(T)$. Therefore, (7.42) is a resonantly forced system. We expand the solution as follows. Let $\delta > 0$ and set

$$\eta_{1\mathrm{R}}^{\delta}(t) = -i\lambda e^{-i\mathcal{H}_0(T)t}\int_0^t e^{i[\mathcal{H}_0(T)-2\lambda_+(T)-i\delta]s}P_c(T)\sigma_3\psi_{1*}^2\psi_{0*}\overline{\alpha_0(s)}\beta_1^2(s)\begin{pmatrix}1\\0\end{pmatrix}ds\,. \tag{7.45}$$

Then, $\eta_{1\mathrm{R}} = \lim_{\delta\to 0}\eta_{1\mathrm{R}}^{\delta}$.

We now apply Lemma 7.1 to $\eta_{1\mathrm{R}}$ with $A = \mathcal{H}_0(T) - 2\lambda_+(T)$. The result is

Proposition 7.2. *The limit* $\lim_{\delta\to 0}\eta_{1\mathrm{R}}^{\delta} = \eta_{1\mathrm{R}}$ *exists in* $\mathcal{S}'$ *and*

$$\begin{aligned} \eta_{1\mathrm{R}}(t) &= -\lambda e^{-i\lambda_+(T)t}\overline{\alpha_0(t)}\beta_1^2(t)(\mathcal{H}_0(T) - 2\lambda_+(T) - i0)^{-1}P_c(T)\begin{pmatrix}1\\0\end{pmatrix}\psi_{1*}^2\psi_{0*}\\ &\quad + \lambda\overline{\alpha_0(0)}\beta_1^2(0)e^{-i\mathcal{H}_0(T)t}(\mathcal{H}_0(T) - 2\lambda_+(T) - i0)^{-1}P_c(T)\begin{pmatrix}1\\0\end{pmatrix}\psi_{1*}^2\psi_{0*}\\ &\quad + \lambda e^{-i\mathcal{H}_0(T)t}(\mathcal{H}_0(T) - 2\lambda_+(T) - i0)^{-1} \end{aligned}$$

$$\cdot \int_0^t e^{i[\mathcal{H}_0(T)-2\lambda_+(T)]s} P_c(T) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \psi_{1*}^2 \psi_{0*} \frac{d}{ds} \left(\overline{\alpha_0(s)} \beta_1^2(s)\right) ds$$

$$= \eta_{1\mathrm{R}a} + \eta_{1\mathrm{R}b} + \eta_{1Rc} \,. \tag{7.46}$$

We substitute (7.46) into the key term (7.40) in (7.39).

$$2\lambda\langle \psi_{0*}\psi_{1*}^2, \pi_1 \Phi_2[\eta]\rangle \overline{\beta_1} \alpha_0 e^{2i\lambda_+(T)t} = 2\lambda\langle \psi_{0*}\psi_{1*}^2, \pi_1 \eta_{1\mathrm{R}a}\rangle \overline{\beta_1} \alpha_0 e^{2i\lambda_+(T)t} + \mathbf{R} \,. \tag{7.47}$$

Here, $\mathbf{R}$ denotes rapidly dispersively decaying terms plus higher-order terms in $|\alpha_0 \beta_1|$. By Proposition 7.2 and the Plemelj formula (A.3)

$$\begin{aligned} &2\lambda\langle \psi_{0*}\psi_{1*}^2, \pi_1 \eta_{1\mathrm{R}a}\rangle \overline{\beta_1} \alpha_0 e^{2i\lambda_+(T)t} \\ &= -2\lambda^2 \left\langle \psi_{0*}\psi_{1*}^2, \pi_1 (\mathcal{H}_0(T) - 2\lambda_+(T) - i0)^{-1} P_c(T) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \psi_{0*}\psi_{1*}^2 \right\rangle |\alpha_0|^2 |\beta_1|^2 \beta_1 \\ &= -2(\Lambda + i\Gamma)|\alpha_0|^2 |\beta_1|^2 \beta_1 \,, \end{aligned} \tag{7.48}$$

where (using that $E_0 - 2E_{10} = E_0 - 2(E_1 - E_0) = \omega_* + \mathcal{O}(|\alpha_0(T)|^2)$) we have

$$\begin{aligned} \Lambda &= \Lambda_{\omega_*} \mathcal{O}_0^{(0)} \\ &= \lambda^2 \langle \psi_{0*}\psi_{1*}^2, \mathrm{P.V.}(H - \omega_*)^{-1} \psi_{0*}\psi_{1*}^2 \rangle \mathcal{O}_0^{(0)} \end{aligned} \tag{7.49}$$

$$\begin{aligned} \Gamma &= \Gamma_{\omega_*} \mathcal{O}_0^{(0)} \\ &= \lambda^2 \pi \langle \psi_{0*}\psi_{1*}^2, \delta(H - \omega_*) \psi_{0*}\psi_{1*}^2 \rangle \mathcal{O}_0^{(0)} \,. \end{aligned} \tag{7.50}$$

Recall that $\mathcal{O}_0^{(0)}$ denotes a term of the form $1 + \mathcal{O}(|\alpha_0|^2)$. Returning to the equation for β_1 we have:

$$i\partial_t \beta_1 = (\Lambda_{\omega_*} - i\Gamma_{\omega_*})|\alpha_0|^2 |\beta_1|^2 \beta_1 + \cdots \,. \tag{7.51}$$

We now seek the key terms in the α_0 equation, (6.23). Using that $\phi_1 = \Psi_1 + \phi_2$ and the representation (7.37), we have

$$\begin{aligned} i\partial_t \alpha_0 = {} & \lambda\langle \psi_{0*}^2, \psi_{1*}^2 \rangle e^{-2i\lambda_+(T)t} \beta_1^2 \overline{\alpha_0} \\ & + 2\lambda\langle \psi_{0*}^2 \psi_{1*}, \phi_2 \rangle e^{-i\lambda_+(T)t} \overline{\alpha_0} \beta_1 + \lambda\langle \psi_{0*}^2, \phi_2^2 \rangle \overline{\alpha_0} \\ & + 2\lambda\langle \psi_{0*}\psi_{1*}^2, \phi_2 \rangle |\beta_1|^2 \\ & + \lambda\langle \psi_{0*}\psi_{1*}, \phi_2^2 \rangle \overline{\beta_1} e^{i\lambda_+(T)t} + \lambda\langle \boldsymbol{\psi_{0*}\psi_{1*}^2}, \overline{\boldsymbol{\phi_2}} \rangle \boldsymbol{\beta_1^2 \mathrm{e}^{-2i\lambda_+(T)t}} \\ & + 2\lambda\langle \psi_{0*}\psi_{1*}, |\phi_2|^2 \rangle \beta_1 e^{-i\lambda_+(T)t} + \lambda\langle \psi_{0*}, |\phi_2|^2 \phi_2 \rangle \\ & + \mathcal{R}_{\alpha_0} \,. \end{aligned} \tag{7.52}$$

We first focus on the key resonant term in (7.52) which is responsible for the system settling onto the nonlinear ground state. We claim this term is:

$$\lambda\langle \boldsymbol{\psi_{0*}\psi_{1*}^2}, \overline{\boldsymbol{\phi_2}} \rangle \boldsymbol{\beta_1^2 \mathrm{e}^{-2i\lambda_+(T)t}} \,. \tag{7.53}$$

In analogy with the previous calculation, we have

$$\lambda\langle\psi_{0*}\psi_{1*}^2, \overline{\phi_2}\rangle\beta_1^2 e^{-2i\lambda_+(T)t} = \lambda\langle\psi_{0*}\psi_{1*}^2, \pi_2\eta_{1Ra}\rangle\beta_1^2 e^{-2i\lambda_+(T)t} + \mathbf{R}\,, \tag{7.54}$$

where $\mathbf{R}$ is as above. Therefore, applying Proposition 7.2 we get

$$\lambda\langle\psi_{0*}\psi_{1*}^2, \pi_2\Phi_2\rangle\beta_1^2 e^{-2i\lambda_+(T)t} = (-\Lambda_{\omega_*} + i\Gamma_{\omega_*})|\beta_1|^4\alpha_0 + \mathbf{R}\,. \tag{7.55}$$

The expression $\mathbf{R}$ denotes terms which are higher order, oscillatory type and dispersively decaying. See also Remark 7.2.

In summary, we have the following system for α_0 and β_1:

Proposition 7.3.

$$\begin{aligned} i\partial_t\alpha_0 &= (-\Lambda_{\omega_*} + i\Gamma_{\omega_*})|\beta_1|^4\alpha_0 + \mathbf{R} \\ i\partial_t\beta_1 &= 2(\Lambda_{\omega_*} - i\Gamma_{\omega_*})|\alpha_0|^2|\beta_1|^2\beta_1 + \mathbf{R}\,. \end{aligned} \tag{7.56}$$

The proof of Proposition 7.1 follows by constructing an appropriate near-identity change of variables, transforming (7.56) to (7.3). This is implemented as in [49].

8. Stability Analysis on Different Time Scales — Overview

In Corollary 7.1 we obtained coupled power equations or nonlinear master equations governing the (renormalized) ground state and excited state square amplitudes:

$$P_0 = |\tilde{\alpha}_0|^2 \qquad \text{and} \qquad P_1 = |\tilde{\beta}_1|^2\,. \tag{8.1}$$

If we neglect the correction terms R_0 and R_1, in (7.6) and (7.7) we obtain the simpler autonomous system of differential equations:

$$\frac{dp_0}{dt} = 2\Gamma p_0 p_1^2 \tag{8.2}$$

$$\frac{dp_1}{dt} = -4\Gamma p_0 p_1^2\,. \tag{8.3}$$

Note that this system is exactly solvable. Addition of twice (8.2) to (8.3) yields that along any solution trajectory:

$$2p_0(t) + p_1(t) = 2p_0(0) + p_1(0)\,. \tag{8.4}$$

This relation can be used to eliminate p_1 from (8.2) or p_0 from (8.3). $p_0(t)$ and $p_1(t)$ are thus obtained by quadrature. The dynamics of this finite dimensional reduced system anticipates that an initial state, arbitrarily close to but not exactly on the excited state branch, with energy distributed among the ground state and excited state, will evolve to a state with an increased ground state energy and no energy in the excited state. While not strictly correct, since there are nongeneric data giving rise to solutions which converge to the excited state [55], this captures the generic very large time dynamics. The correction terms R_0 and R_1 in (7.6) and (7.7) lead to different transient behaviors which may be quite different from that suggested by the system (8.2) and (8.3). However, we show that eventually ($t \geq t_1$), this system dominates. Moreover, a large class of data, for which the system (8.2) and (8.3) controls the behavior is that for which $P_0(0) > P_1(0)$ and sufficiently small initial dispersive part.

Before embarking on the details we give a brief overview of the strategy. Using the change of variables $(\alpha, \beta) \mapsto (\tilde{\alpha}, \tilde{\beta})$ of Proposition 7.1 we have transformed away all *local in time* nonresonant terms. This introduces contributions to F_α and F_β, and therefore contributions to R_0 and R_1 in Eqs. (7.6) and (7.7), which are of two types:

(i) *local in time* terms depending on $\tilde{\alpha}_0$ and $\tilde{\beta}_1$, which can be absorbed by the leading terms in (7.6) and (7.7), with a small correction to the coefficient Γ and are of order $b_0\langle t\rangle^{-2}$; see Proposition 9.1 below.
(ii) *nonlocal in time* functions of $\tilde{\alpha}_0$ and $\tilde{\beta}_1$ defined in terms of $\eta = \eta[\eta_0, \eta_1, \eta_2]$ in F_α and F_β. These contribute terms to (7.3) with the same (anticipated) time-decay rate as the leading order terms in (7.3). Correspondingly, there are *nonlocal in time* functions of $\tilde{\alpha}_0$ and $\tilde{\beta}_1$ which contribute to R_j in Eqs. (7.6) and (7.7) which are of the same (anticipated) decay rate as the leading order terms in (7.6) and (7.7). The goal is to control these nonlocal terms, to the extent possible, by the leading order terms. However, due to the different behaviors of $\tilde{\alpha}_0$ and $\tilde{\beta}_1$ on different time scales the argument is somewhat tricky and we now explain our strategy.

Let

$$t_0 + 1 \equiv \sup_{\tau \geq 0} \left\{ \tau : \ 0 \leq \tau' \leq \tau, \ P_0(\tau') \leq \frac{5}{2} \frac{[\eta_0]_X + \mathcal{E}_0}{\langle \tau - 1\rangle \langle \tau' \rangle^2} \right\} . \tag{8.5}$$

Propositions 11.2–12.1 and 13.1 will justify this choice, by implying the inequalities (8.7). If $t_0 < \infty$, then we have the bound:

$$P_0(t) \leq \frac{5}{2} \frac{[\eta_0]_X + \mathcal{E}_0}{\langle t_0\rangle \langle t\rangle^2} , \qquad 0 \leq t \leq t_0 . \tag{8.6}$$

Consider the system for $P_0(t)$ and $P_1(t)$, (7.6) and (7.7). Decomposing R_j into local and nonlocal in time parts we have:

$$\begin{aligned} R_j &= R_j^{\mathrm{l}}(P_0, P_1, \eta_0, \eta) + \int_0^t K(t,s) r_j^{\mathrm{nl}}(s) ds \\ &= R_j^{\mathrm{l}}(P_0, P_1, \eta_0, \eta) + \int_0^{t_0} K(t,s) r_j^{\mathrm{nl}}(s) ds \\ &\quad + \int_{t_0}^t K(t,s) r_j^{\mathrm{nl}}(s) ds . \end{aligned}$$

The terms r_j^{nl} arise from nonlocal in time functions of $\tilde{\alpha}_0$ and $\tilde{\beta}_1$ (see (ii) above); explicit expressions of this type are analyzed in Sec. 11. For the local in time contributions we have the estimates:

$$
\begin{aligned}
R_0^{\rm l}(P_0, P_1, \eta_0, \eta) &\le \mathcal{O}(\mathcal{E}_0^{\rho_0})\Big(P_0P_1^2 + \frac{b_0}{\langle t\rangle^2}\Big)\\
R_1^{\rm l}(P_0, P_1, \eta_0, \eta) &\le \mathcal{O}(\mathcal{E}_0^{\rho_1})\Big(P_0P_1^2 + \frac{b_1}{\langle t\rangle^2} + \mathcal{O}\big(\sqrt{P_0}P_1^m\big)\Big)\,,
\end{aligned}
\tag{8.7}
$$

where $\mathcal{E}_0$ is the initial total energy and $\rho_j > 0$. Therefore, for $\mathcal{E}_0$ small

$$
\frac{dP_0}{dt} \ge 2\Gamma' P_0P_1^2 - \frac{b_0}{\langle t\rangle^2} + \int_{t_0}^t K_0(t,s)r_0^{\rm nl}(s)ds
$$

$$
\frac{dP_1}{dt} \le -4\Gamma' P_0P_1^2 + \frac{b_1}{\langle t\rangle^2} + \int_{t_0}^t K_1(t,s)r_1^{\rm nl}(s)ds + \mathcal{O}\big(\sqrt{P_0}P_1^m\big)\,,
$$

with $\Gamma' \sim \Gamma$. The reverse inequalities hold with Γ' replaced by $\Gamma'' = \Gamma + o(\Gamma)$.

Next (Proposition 9.2) we introduce the auxiliary quantities

$$
Q_0 = P_0 - k_0\langle t\rangle^{-1} \qquad \text{and} \qquad Q_1 = P_1 + k_1\langle t\rangle^{-1}\,, \tag{8.8}
$$

where $k_0(b_0, b_1, \mathcal{E}_0)$ and $k_1(b_0, b_1, \mathcal{E}_0)$ are chosen appropriately and derive equations of the form

$$
\begin{aligned}
\frac{dQ_0}{dt} &\ge 2\Gamma'' Q_0Q_1^2 + \int_{t_0}^t K_0(t,s)r_0^{\rm nl}(s)ds\\
\frac{dQ_1}{dt} &\le -4\Gamma'' Q_0Q_1^2 + \int_{t_0}^t K_1(t,s)r_1^{\rm nl}(s)ds + \mathcal{O}\big(\sqrt{Q_0}Q_1^m\big)\,,
\end{aligned}
\tag{8.9}
$$

where $\Gamma'' \sim \Gamma' \sim \Gamma$.

We then proceed with the following continuity argument. At $t = t_0$,

$$
\begin{aligned}
\frac{dQ_0(t_0)}{dt} &\ge 2\Gamma'' Q_0(t_0)Q_1^2(t_0)\,,\\
\frac{dQ_1(t_0)}{dt} &\le -4\Gamma'' Q_0(t_0)Q_1^2(t_0) + \mathcal{O}\big(\sqrt{Q_0(t_0)}Q_1^m(t_0)\big)\,.
\end{aligned}
\tag{8.10}
$$

Therefore, by continuity, the following inequalities hold for some time interval $t_0 \le t \le t_{0,1}$, with Γ'' replaced by $\frac{\Gamma''}{2}$:

$$
\frac{dQ_0}{dt} \ge \Gamma'' Q_0Q_1^2\,, \qquad \frac{dQ_1}{dt} \le -2\Gamma'' Q_0Q_1^2 + \mathcal{O}\big(\sqrt{Q_0}Q_1^m\big)\,. \tag{8.11}
$$

Let $t^* \equiv \sup\{t \ge t_0 : \text{inequalities (8.11) hold }\}$. We show that

$$
\int_{t_0}^t K_1(t,s)r_1^{\rm nl}(s)ds \le \mathcal{O}(\mathcal{E}_0^\rho)\left(Q_0Q_1^2 + \frac{b_j}{\langle t\rangle^2}\right) \tag{8.12}
$$

and therefore, up to renormalization of Q_j (adding higher-order terms of order $\mathcal{E}_0^\rho k_j\langle t\rangle^{-1}$ to the definition of Q_j)). Use of this estimate in (8.9) implies (8.11), for $\mathcal{E}_0$ sufficiently small. The argument can be repeated and therefore, $t^* = T$.

9. Finite Dimensional Reduction and Its Analysis on Different Time Scales

We now begin our study of the generic case, where $t_0 < \infty$ and the solution converges to the nonlinear ground state family as $t \to \infty$. The following three propositions concern the various time scales which enter the analysis. The first is a basic result, a normal form, which is the point of departure for our analysis on all time scales.

Proposition 9.1. *Let $m \geq 4$. Let*

$$b_0 = \langle t_0 \rangle^{-1} \left([\eta_0]_X + c_* \mathcal{E}_0^2 \right) , \tag{9.1}$$

$$b_1 = \langle t_0 \rangle^{-\frac{1}{2}} \left([\eta_0]_X^{\frac{2}{3}} + d_* \mathcal{E}_0^2 \right) , \tag{9.2}$$

for some order one constants c_* and d_*.

If for some t_0, positive and finite,

$$P_0(t_0) \geq \frac{3b_0\left(t_0, [\eta_0]_X\right)}{\langle t_0 \rangle} , \tag{9.3}$$

then for $t \geq t_0$

$$\frac{dP_0}{dt} \geq 2(1-\delta_1)\Gamma P_0 P_1^2 - \frac{b_0}{\langle t \rangle^2} + J_0 \tag{9.4}$$

$$\frac{dP_1}{dt} \leq -4(1-\delta_1)\Gamma P_0 P_1^2 + \mathcal{O}(\sqrt{P_0} P_1^m) + \frac{b_1}{\langle t \rangle^2} + J_1 , \tag{9.5}$$

where J_0 and J_1 are nonlocal in time terms, which have the form:

$$J_j = \int_{t_0}^{t} K(t,s) r_j^{\mathrm{nl}}(s) ds . \tag{9.6}$$

The terms encompassed in J_j are derived and estimated in the coming sections.

Remark 9.1. The reverse inequalities of (9.4) and (9.5) hold as well with a different constant $\delta_2 \sim \delta_1$.

The proof of Proposition 9.1 will be given following the estimates on the remainder terms, R_i (Proposition 11.1).

Proposition 9.2. *Assume that t_0 is positive and finite as in Proposition 9.1. Then, there exist $k_0 = k_0(b_0, b_1, \mathcal{E}_0)$ and $k_1 = k_1(b_0, b_1, \mathcal{E}_0)$, such that for $t \geq t_0$ the auxiliary functions*

$$Q_0(t) \equiv P_0(t) - \frac{k_0}{\langle t \rangle} , \qquad Q_1(t) \equiv P_1(t) + \frac{k_1}{\langle t \rangle} \tag{9.7}$$

satisfy

$$\begin{aligned} \frac{dQ_0}{dt} &\geq 2\Gamma' Q_0 Q_1^2 + J_0 + \frac{\mathcal{E}_0^2 c_*}{\langle t_0 \rangle \langle t \rangle^2} \\ \frac{dQ_1}{dt} &\leq -4\Gamma' Q_0 Q_1^2 + \mathcal{O}(\sqrt{Q_0} Q_1^m) + J_1 - \frac{\mathcal{E}_0^2 d_*}{\langle t_0 \rangle^{\frac{1}{2}} \langle t \rangle^2} , \end{aligned} \tag{9.8}$$

for some positive free constants c_* and d_*. In particular, $Q_0(t)$ is monotonically increasing for $t \geq t_0$.

The next result shows that c_* and d_* can be chosen to control the terms J_j.

Proposition 9.3 (Monotonicity of Q_0 for $\tau \geq t_0$). *There exist c_* and d_* of order one, such that for $t \geq t_0$.*

$$\begin{aligned} \frac{dQ_0}{dt} &\geq 2\Gamma' Q_0 Q_1^2 \\ \frac{dQ_1}{dt} &\leq -4\Gamma' Q_0 Q_1^2 + \mathcal{O}(\sqrt{Q_0} Q_1^m)\,. \end{aligned} \tag{9.9}$$

The above three propositions are established in the next two sections. We complete the current section by working out the consequences of the finite dimensional reduction (9.9).

The next proposition, shows that even if Q_0 is very small at some stage, it will eventually become large relative to Q_1 and the $\mathcal{O}(\sqrt{Q_0} Q_1^m)$ term in (9.8) will become negligible.

Proposition 9.4. *Assume $t \geq t_0$ and suppose for some $r > 0$ that*

$$\frac{Q_0}{Q_1} \leq \mathcal{E}_0^r\,. \tag{9.10}$$

Then,

$$\frac{Q_0(t)}{Q_1(t)} \text{ is increasing for } t \geq t_0 \tag{9.11}$$

and there exists t_1 such that

$$\frac{Q_0(t_1)}{Q_1(t_1)} = \mathcal{E}_0^r\,. \tag{9.12}$$

Furthermore, for $t \geq t_1$:

$$\begin{aligned} \frac{dQ_0}{dt} &\geq 2\Gamma' Q_0 Q_1^2 \\ \frac{dQ_1}{dt} &\leq -4\Gamma' Q_0 Q_1^2\,. \end{aligned} \tag{9.13}$$

Finally, for $t \geq t_1$:

$$Q_1(t) \leq \frac{Q_1(t_1)}{1 + 4\Gamma'' Q_1(t_1)(\inf_{[t_1,T]} |Q_0|) \cdot (t - t_1)}\,. \tag{9.14}$$

Proof of Proposition 9.2. Define

$$Q_0 \equiv P_0 - k_0 \langle t \rangle^{-1}\,, \qquad Q_1 \equiv P_1 + k_1 \langle t \rangle^{-1}\,, \tag{9.15}$$

where $k_0, k_1 > 0$ are to be appropriately chosen. Note that

$$Q_0(t) \leq P_0(t) \qquad \text{and} \qquad Q_1(t) \geq P_1(t)\,. \tag{9.16}$$

Using (9.4) and (9.5) and some estimation we deduce a simplified system for Q_0 and Q_1. We calculate, omitting the terms J_0 and J_1, which are carried along passively.

We begin with Q_0.

$$\begin{aligned}\frac{dQ_0}{dt} &= \frac{dP_0}{dt} + \frac{k_0}{\langle t\rangle^2}\\ &\geq 2\Gamma P_0 P_1^2 - \frac{b_0}{\langle t\rangle^2} + \frac{k_0}{\langle t\rangle^2}\\ &= 2\Gamma\left(Q_0 + \frac{k_0}{\langle t\rangle}\right)\left(Q_1 - \frac{k_1}{\langle t\rangle}\right)^2 - \frac{b_0}{\langle t\rangle^2} + \frac{k_0}{\langle t\rangle^2}\\ &\geq 2\Gamma Q_0 Q_1^2 - 4\Gamma\frac{k_1}{\langle t\rangle}Q_0Q_1 + 2\Gamma\frac{k_1^2}{\langle t\rangle^2}\\ &\quad + \frac{k_0 - b_0}{\langle t\rangle^2}\,. \end{aligned} \tag{9.17}$$

We estimate the second term on the right-hand side as follows. For any $s > 0$,

$$-4\Gamma\frac{k_1}{\langle t\rangle}Q_0Q_1 \geq -2s\Gamma Q_0Q_1^2 - 2\Gamma Q_0\frac{k_1^2}{s\langle t\rangle^2}. \tag{9.18}$$

Therefore,

$$\frac{dQ_0}{dt} \geq 2\Gamma(1-s)Q_0Q_1^2 + \left(2\Gamma k_1^2 Q_0\left(1 - \frac{1}{s}\right) + k_0 - b_0\right)\frac{1}{\langle t\rangle^2}\,. \tag{9.19}$$

Now set $s = \frac{1}{10}$ and assume

$$k_1 \geq b_1\,. \tag{9.20}$$

Then, using that $k_1 = 10b_1$, we have

$$\frac{dQ_0}{dt} \geq 2\cdot\frac{9}{10}\Gamma Q_0Q_1^2 + \left(k_0 - b_0 - 18\Gamma b_1^2 Q_0\right)\frac{1}{\langle t\rangle^2}\,. \tag{9.21}$$

If

$$k_0 \equiv b_0 + 18\Gamma b_1^2 \sup_{t_0\leq t\leq T} Q_0 + \frac{\mathcal{E}_0^2 c_*}{\langle t_0\rangle}\,, \tag{9.22}$$

we have by (9.21) and (9.28)

$$\frac{dQ_0}{dt} \geq 2\cdot\frac{9}{10}\Gamma Q_0Q_1^2 + \frac{\mathcal{E}_0^2 c_*}{\langle t_0\rangle\langle t\rangle^2}\,. \tag{9.23}$$

In particular, Q_0 is increasing for $t \geq t_0$.

We now turn to Q_1; we have

$$\begin{aligned}\frac{dQ_1}{dt} &= \frac{dP_1}{dt} - \frac{k_1}{\langle t\rangle^2}\\ &\leq -4\Gamma P_0P_1^2 + \frac{b_1}{\langle t\rangle^2} - \frac{k_1}{\langle t\rangle^2} + \mathcal{O}(\sqrt{P_0}P_1^m)\end{aligned}$$

$$= -4\Gamma\left(Q_0 + \frac{k_0}{\langle t\rangle}\right)\left(Q_1 - \frac{k_1}{\langle t\rangle}\right)^2 + \frac{b_1 - k_1}{\langle t\rangle^2} + \mathcal{O}(\sqrt{P_0}P_1^m)$$

$$\leq -4\Gamma Q_0\left(Q_1 - \frac{k_1}{\langle t\rangle}\right)^2 + \frac{b_1 - k_1}{\langle t\rangle^2} + \mathcal{O}(\sqrt{P_0}P_1^m)$$

$$= -4\Gamma Q_0 Q_1^2 + 8\Gamma\frac{k_1}{\langle t\rangle}Q_0 Q_1 - 4\Gamma Q_0\frac{k_1^2}{\langle t\rangle^2} + \frac{b_1 - k_1}{\langle t\rangle^2} + \mathcal{O}(\sqrt{P_0}P_1^m)\,. \tag{9.24}$$

The second term on the right-hand side is estimated as follows. For any $r > 0$ we have, since $2ab \leq ra^2 + r^{-1}b^2$, that

$$8\Gamma\frac{k_1}{\langle t\rangle}Q_0 Q_1 \leq \frac{4\Gamma k_1 r Q_0}{\langle t\rangle^2} + \frac{4\Gamma k_1 Q_0 Q_1^2}{r}\,. \tag{9.25}$$

Therefore,

$$\frac{dQ_1}{dt} \leq -4\Gamma\left(1 - \frac{k_1}{r}\right)Q_0 Q_1^2 + 4\Gamma k_1 Q_0\frac{r - k_1}{\langle t\rangle^2} + \frac{b_1 - k_1}{\langle t\rangle^2} + \mathcal{O}\big(\sqrt{P_0}P_1^m\big)\,. \tag{9.26}$$

Let

$$r \equiv 20k_1 \qquad \text{and} \qquad b_1 \equiv k_1(1 - 76\Gamma k_1 Q_0) - \frac{\mathcal{E}_0^2 d_*}{\langle t_0\rangle^{\frac{1}{2}}}\,, \tag{9.27}$$

which is consistent with the constraint (9.20). Then,

$$\frac{dQ_1}{dt} \leq -4\cdot\frac{19}{20}\Gamma Q_0 Q_1^2 - \frac{\mathcal{E}_0^2 d_*}{\langle t_0\rangle^{\frac{1}{2}}\langle t\rangle^2} + \mathcal{O}(\sqrt{P_0}P_1^m)\,. \tag{9.28}$$

By (9.8), Q_0 is increasing for $t \geq t_0$. Therefore,

$$Q_0(t_0) \geq \frac{1}{100}\frac{k_0}{\langle t_0\rangle} \Rightarrow$$

$$P_0(t) \leq Q_0(t) + \frac{k_0}{\langle t\rangle} = Q_0(t) + 100\frac{k_0}{100\langle t\rangle}$$

$$\leq Q_0(t) + 100 Q_0(t_0) \leq 101 Q_0(t)\,.$$

Also, by definition $P_1 \leq Q_1$ and so

$$\frac{dQ_1}{dt} \leq -4\cdot\frac{19}{20}\Gamma Q_0 Q_1^2 + \mathcal{O}(\sqrt{Q_0}Q_1^m)\,. \tag{9.29}$$

This completes the proof of Proposition 9.2. □

Proof of Proposition 9.4.

$$\frac{d}{dt}\left(\frac{Q_0}{Q_1}\right) = \frac{\dot{Q}_0 Q_1 - \dot{Q}_1 Q_0}{Q_1^2} \geq \frac{1}{Q_1^2}\left\{\Gamma Q_0 Q_1^2 Q_1 + 4\Gamma Q_0^2 Q_1^2 - C Q_0^{3/2} Q_1^m\right\}$$

$$\geq \frac{1}{Q_1^2}\left\{2\Gamma Q_0\left[Q_1^3 - C Q_0^{1/2} Q_1^{-1/2} Q_1^{m+1/2}\right] + 4\Gamma Q_0^2 Q_1^2\right\}.$$

$\frac{Q_0}{Q_1} \le \mathcal{E}_0^r$ then implies that

$$\frac{d}{dt}\left(\frac{Q_0}{Q_1}\right) \ge \frac{1}{Q_1^2}\{2\Gamma Q_0\big[Q_1^3 - \mathcal{E}_0^{r/2}Q_1^{m+1/2}\big] + 4\Gamma Q_0^2 Q_1^2\}$$
$$\ge \Gamma\left(\frac{Q_0}{Q_1}\right)Q_1^2 + 4\Gamma Q_1^2\left(\frac{Q_0}{Q_1}\right)^2$$
$$\ge \Gamma\left(\frac{Q_0}{Q_1}\right)Q_1^2\,, \tag{9.30}$$

for $m + \frac{1}{2} > 3$ and $|Q_1| \le \mathcal{E}_0 \ll 1$. Hence

$$\left.\frac{Q_0}{Q_1}\right|_t \ge \left.\frac{Q_0}{Q_1}\right|_{t_0} \exp\left(\Gamma\int_{t_0}^t Q_1^2(s)ds\right).$$

Since $Q_1 > 0$, either $Q_1 \downarrow 0$ or $\frac{Q_0}{Q_1}$ grows exponentially with $t \ge t_0$. In either case, there exists $t_1 \ge t_0$, such that for $t = t_1$, $\left.\frac{Q_0}{Q_1}\right|_{t_1} = \mathcal{E}_0^r$. Now whenever,

$$\frac{Q_0}{Q_1} \ge \mathcal{E}_0^r\,, \tag{9.31}$$

we have

$$\frac{dQ_1}{dt} \le -4\Gamma Q_0 Q_1^2 + +\mathcal{O}(\sqrt{Q_0}Q_1^m)$$
$$\le -4\Gamma Q_0 Q_1^2 + Q_0 Q_1^{m-1/2}\mathcal{E}_0^{-r/2}$$
$$\le -4\Gamma Q_0\big[Q_1^2 - C\mathcal{E}_0^{-r/2}\mathcal{E}_0^{m-5/2}Q_1^2\big]$$
$$\le -4\Gamma Q_0 Q_1^2 \tag{9.32}$$

for $m > \frac{5}{2} + \frac{r}{2}$. Therefore, by (9.31) we have $\left.\frac{dQ_1}{dt}\right|_{t_1} < 0$. Since Q_0 is increasing for $t \ge t_1 \ge t_0$, the inequality $\frac{Q_0}{Q_1} \ge \mathcal{E}_0^r$ persists and (9.32) holds for all $t \ge t_1$. Hence

$$\frac{dQ_1}{dt} \le -4\Gamma' Q_0 Q_1^2 \qquad \text{and} \qquad \frac{dQ_0}{dt} \ge 2\Gamma' Q_0 Q_1^2\,. \tag{9.33}$$

Finally, for $t \ge t_1$

$$\frac{dQ_1}{dt} \le -4\frac{19}{20}\Gamma\left(\inf_{[t_0,T]} Q_0\right)Q_1^2\,. \tag{9.34}$$

Solving this scalar inequality

$$Q_1(t) \le \frac{Q_1(t_1)}{1 + 4\Gamma'' Q_1(t_1)|Q_0(t_1)|(t - t_1)}\,. \tag{9.35}$$

Since $P_1 \le Q_1$, $P_1(t)$ decays to zero like $\langle t - t_1\rangle^{-1}$. This completes the proof of Proposition 9.4. ◻

10. Decomposition and Estimation of the Dispersion

In this section, we revisit the decomposition of the dispersive part, η, which satisfies Eq. (6.48). Here, we decompose η in a manner suitable for consideration of the solution on the various time scales.

Proposition 10.1.

$$\begin{aligned}\eta(t) &= \eta_0(t) + e_0(t) + \eta_b(t) \\ &= e^{i\sigma_3[\int_t^T (E_0(t')-E_0(T))dt' + \tilde\Theta(t)]}\,(\tilde\eta_0(t) + \tilde e_0(t) + \tilde\eta_b(t))\,. \end{aligned} \tag{10.1}$$

The three terms can be described as follows:

(i) $\eta_0(t)$ *is a dispersive wave generated by the data* $\eta(0) = \eta_{\rm in}$:

$$(i\partial_t - \mathcal{H}_0)\tilde\eta_0 = 0\,, \qquad \eta(0) = \eta_{\rm in}\,, \tag{10.2}$$

(ii) $e_0(t)$ *is driven principally by* $\eta_0(t)$:

$$(i\partial_t - \mathcal{H}_0)\tilde e_0 = P_c S^{(0)}[e_0, \eta_b; \eta_0]\,, \qquad e_0(0) = 0\,, \tag{10.3}$$

and

(iii) $\eta_b(t)$, *which is driven by bound state dynamics*:

$$(i\partial_t - \mathcal{H}_0)\tilde\eta_b + P_c S^{(b)}[e_0, \eta_b; \eta_0]\,, \qquad \eta_b(0) = 0\,. \tag{10.4}$$

We display expressions for $S^{(0)}$ *and* $S^{(b)}$ *with the detail required in our analysis. We let* χ *denote a generic exponentially localized function of position*, x.

$$\begin{aligned} &e^{i[\int_t^T (E_0(t')-E_0(T))dt' + \tilde\Theta(t)]}\,S^{(0)} \\ &\quad\equiv\ S_0^{(0)} + S_1^{(0)} e_0 + S_2^{(0)} e_0^2 + e_0^3 \\ &\quad\sim\ \big(\alpha_0^2(t) - \alpha_0^2(T)\big)\chi\eta_0 + \eta_0^3 + \alpha_0\alpha_1\chi\eta_0 + \alpha_0\chi\eta_0^2 \\ &\qquad + \big(\alpha_0\alpha_1 + \alpha_1^2\big)\chi e_0 + (\alpha_0 + \alpha_1)\chi(\eta_0 + \eta_b)e_0 + \eta_0\eta_b e_0 \\ &\qquad + \big(\alpha_0^2(t) - \alpha_0^2(T)\big)\chi e_0 + (\alpha_0 + \alpha_1)\chi e_0^2 + (\eta_0 + \eta_b)e_0^2 \\ &\qquad + e_0^3 \end{aligned} \tag{10.5}$$

$$\begin{aligned} &e^{i[\int_t^T (E_0(t')-E_0(T))dt' + \tilde\Theta(t)]}\,S^{(b)} \\ &\quad\equiv\ S_0^{(b)} + S_1^{(b)}\eta_b + S_2^{(b)}\eta_b^2 + \eta_b^3 \\ &\quad\sim\ \big(\alpha_0^2(t) - \alpha_0^2(T)\big)\chi\eta_b \\ &\qquad + iP_c(T)\begin{pmatrix} \vec\nabla_1\Psi_1(t)\cdot\partial_t\vec\alpha_1(t) + \vec\nabla_0\Psi_0(t)\cdot\partial_t\vec\alpha_0(t) \\ \text{c.c.} \end{pmatrix} \\ &\qquad + \big(E_1(t) - E_0(t)\big)P_c(T)\sigma_3\begin{pmatrix}\Psi_1(t) \\ \text{c.c.}\end{pmatrix} \end{aligned}$$

$$+\left(|\alpha_0|^2|\alpha_1| + |\alpha_0||\alpha_1|^2\right)\chi$$
$$+\alpha_0\alpha_1\chi\eta_b + (\alpha_0+\alpha_1)\,\chi\eta_b^2 + \left(\eta_0+e_0\right)\eta_b^2$$
$$+|\alpha_0|\chi\eta_0\eta_b + (\eta_0^2+e_0^2)\eta_b + \eta_b^3\,. \tag{10.6}$$

Remark 10.1. Due to the decay of $E_0(t) - E_0(T)$ and $\partial_t\tilde{\Theta}(t)$, integration by parts implies that contributions from the phase factors multiplying $S^{(0)}$ and $S^{(b)}$ contribute at higher and negligible order. Also note that η_0, e_0 and η_b satisfy the same energy (H^s) and dispersive(L^p, local decay) estimates as $\tilde{\eta}_0, \tilde{e}_0$ and $\tilde{\eta}_b$, since these functions differ only by a time-dependent phase. These properties will be used repeatedly below and in Secs. 11–13.

By (5.22) we have the following H^1 bounds on e_0 and η_b in terms of one another:

$$\|e_0(t)\|_{H^1} \le \mathcal{E}_0 + \|\eta_0\|_{H^1} + \|\eta_b(t)\|_{H^1} \tag{10.7}$$

$$\|\eta_b(t)\|_{H^1} \le \mathcal{E}_0 + \|\eta_0\|_{H^1} + \|e_0(t)\|_{H^1}\,. \tag{10.8}$$

Using DuHamel's formula, both the e_0 and η_b equations can be written as equivalent integral equations:

$$e_0(t) = -i\int_0^t e^{-i\mathcal{H}_0(t-s)}P_cS^{(0)}(s)ds \tag{10.9}$$

$$\eta_b(t) = -i\int_0^t e^{-i\mathcal{H}_0(t-s)}P_cS^{(b)}(s)ds\,. \tag{10.10}$$

Therefore, in both cases we must estimate an expression of the form:

$$w(t) = \int_0^t e^{-i\mathcal{H}_0(t-s)}P_cS(s)ds\,. \tag{10.11}$$

For estimation, we shall require the following class of dispersive estimates:

Proposition 10.2. (i) *Let* $2 \le p < \infty$, $p', q \ge 2, q'$ *and* s *be related by*:

$$3\left(\frac{1}{q} - \frac{1}{p}\right) = s\,, \qquad p^{-1} + (p')^{-1} = 1\,, \qquad q^{-1} + (q')^{-1} = 1\,. \tag{10.12}$$

Then,

$$\|e^{-i\mathcal{H}_0t}P_cf\|_p \le |t|^{-3(\frac{1}{2}-\frac{1}{q})}\langle t\rangle^{-3(\frac{1}{q}-\frac{1}{p})}\left(\|f\|_{p'} + \|\partial^s f\|_{q'}\right). \tag{10.13}$$

(ii) *Assume* $q \ge 2$ *and* $s > 3/q$. *Then*,

$$\|e^{-i\mathcal{H}_0t}P_cf\|_\infty \le |t|^{-3(\frac{1}{2}-\frac{1}{q})}\langle t\rangle^{-\frac{3}{q}}\left(\|f\|_1 + \|\partial^s f\|_{q'}\right). \tag{10.14}$$

Proof of Proposition 10.2. We use the classical Sobolev inequality for functions defined on $\mathbb{R}^3$:

$$\|f\|_p \le C\,\|\partial^s f\|_q\,, \qquad 3(q^{-1} - p^{-1}) = s \tag{10.15}$$

and the $L^p - L^{p'}$ estimate (Theorem 4.2)

$$\|e^{-i\mathcal{H}_0 t}P_c f\|_p \ \leq\ C_{1,p}\ t^{-\frac{3}{2}+\frac{3}{p}}\|f\|_{p'},\ p^{-1}+(p')^{-1}=1\,. \tag{10.16}$$

For $|t|\geq 1$, we use (10.16). For $|t|\leq 1$, we have by (10.15) that

$$\begin{aligned}\|e^{-i\mathcal{H}_0 t}P_c f\|_p &\leq \|\partial^s e^{-i\mathcal{H}_0 t}P_c f\|_q\\ &\leq C\,\|\mathcal{H}_0^{\frac{s}{2}}e^{-i\mathcal{H}_0 t}P_c f\|_q\\ &\leq C\,|t|^{-3(\frac{1}{2}-\frac{1}{q})}\ \|D^s f\|_{q'}\,, \qquad (10.17)\\ &\quad q^{-1}+(q')^{-1}=1\,.\end{aligned}$$

From estimates (10.16) and (10.17) we obtain (10.13). Estimate (10.14) is obtained similarly. This completes the proof of Proposition 10.2. □

We now apply Proposition 10.2 with $q=4$ and $s=1>3/4$ to the integral equation (10.11) and obtain the bound:

$$\|w(t)\|_\infty \leq \int_0^t \frac{1}{|t-s|^{\frac{3}{4}}}\frac{1}{\langle t-s\rangle^{\frac{3}{4}}}\big(\|S(s)\|_1+\|\partial S(s)\|_{\frac{4}{3}}\big)ds\,. \tag{10.18}$$

We shall use this bound as the first step in estimating $\|e_0(t)\|_\infty$ and $\|\eta_b(t)\|_\infty$.

More specifically, our estimation strategy seeks a closed system of inequalities for the following:
norms of e_0: **1.** $[e_0]_{\infty,\frac{3}{2}}$, **2.** $[\partial e_0]_{L^2_{\rm loc};\frac{3}{2}}$, **3.** $[e_0]_{H^1;0}$
norms of η_b: **4.** $[\eta_b]_{\infty;0}$, **5.** $[\eta_b]_{H^1;0}$,

in terms of norms of the initial data $[\phi(0)]_X$.

10.1. *Estimation of* $\|e_0\|_\infty$

By (10.18), we have

$$\|e_0(t)\|_\infty \leq \int_0^t \frac{1}{|t-s|^{\frac{3}{4}}}\frac{1}{\langle t-s\rangle^{\frac{3}{4}}}\big(\|S^{(0)}(s)\|_1+\|\partial S^{(0)}(s)\|_{\frac{4}{3}}\big)ds\,, \tag{10.19}$$

so it sufficies to bound $\|S^{(0)}\|_1$ and $\|\partial S^{(0)}\|_{\frac{4}{3}}$.

<u>$\|S^{(0)}\|_1$:</u> For any $t\geq 0$,

$$\begin{aligned}\|S_0^{(0)}\|_1 &\leq C\langle s\rangle^{-\frac{3}{2}}\big(\mathcal{E}_0[\eta_0(0)]_X+\sqrt{\mathcal{E}_0}[\eta_0(0)]_X^2\big)\leq C(\mathcal{E}_0)D(\eta(0))\langle s\rangle^{-\frac{3}{2}}\\ \|S_1^{(0)}e_0\|_1 &\leq C\langle s\rangle^{-\frac{3}{2}}[e_0]_{\infty;\frac{3}{2}}\big(\mathcal{E}_0+\mathcal{E}_0^{\frac{1}{2}}[\eta_b]_{\infty;0}+\mathcal{E}_0^{\frac{1}{2}}[\eta_0]_{\infty;0}+[\eta_0]^2_{\infty;\frac{3}{2}}\big)\\ &\quad +C\langle s\rangle^{-\frac{3}{2}}[e_0]_{\infty;\frac{3}{2}}[\eta_0]_{\infty;\frac{3}{2}}[\eta_b]_{\infty;0}\leq C\langle s\rangle^{-\frac{3}{2}} \qquad (10.20)\\ \|S_2^{(0)}e_0^2\|_1 &\leq C\langle s\rangle^{-3}[e_0]^2_{\infty;\frac{3}{2}}\big(C\sqrt{\mathcal{E}_0}+[\eta_0]_{\infty;\frac{3}{2}}+[\eta_b]_{\infty;0}\big)\\ \|e_0^3\|_1 &\leq C\langle s\rangle^{-\frac{3}{2}}[e_0]_{\infty;\frac{3}{2}}[e_0]^2_{H^1;0}\,.\end{aligned}$$

$\underline{\|\partial S^{(0)}\|_{\frac{4}{3}}}$**:** For any $t \geq 0$,

$$
\begin{aligned}
\|\partial S_0^{(0)}\|_{\frac{4}{3}} &\leq C\langle s\rangle^{-\frac{3}{2}}\big(\mathcal{E}_0[\partial\eta_0]_{\infty;\frac{3}{2}} + \mathcal{E}_0^{\frac{1}{2}}[\eta_0]_{\infty;\frac{3}{2}}[\eta_0]_{H^1;0} + \mathcal{E}_0^{\frac{1}{4}}[\eta_0]^{\frac{3}{2}}_{\infty,\frac{3}{2}}\big) \\
\|\partial S_1^{(0)}e_0\|_{\frac{4}{3}} &\leq C\langle s\rangle^{-\frac{3}{2}}\big(\mathcal{E}_0[\eta_0]_{\infty;\frac{3}{2}} + \mathcal{E}_0[e_0]_{L^2_{\rm loc};\frac{3}{2}}\big) + C\langle s\rangle^{-\frac{3}{2}}\big(\mathcal{E}_0^{\frac{1}{2}}[\eta_b]_{\infty;0}[\eta_0]_{\infty;\frac{3}{2}} \\
&\quad + \mathcal{E}_0^{\frac{1}{2}}[e_0]_{\infty,\frac{3}{2}}[\eta_b]_{H^1;0} + \mathcal{E}_0^{\frac{1}{2}}[\eta_b]_{\infty;0}[\partial e_0]_{L^2_{\rm loc};\frac{3}{2}}\big) \\
\|\partial S_2^{(0)}e_0^2\|_{\frac{4}{3}} &\leq C\langle s\rangle^{-\frac{3}{2}}[e_0]_{\infty;\frac{3}{2}}\big(\mathcal{E}_0^{\frac{1}{2}}[e_0]_{\infty;\frac{3}{2}} + \mathcal{E}_0^{\frac{1}{2}}[e_0]_{H^1;0}\big) \qquad (10.21)\\
&\quad + C\langle s\rangle^{-\frac{3}{2}}[e_0]_{\infty;\frac{3}{2}}\big([\eta_0]_{H^1;0} + [\eta_b]_{H^1;0}\big)[e_0]_{H^1;0} \\
&\quad + [e_0]^{\frac{3}{2}}_{\infty;\frac{3}{2}}[e_0]^{\frac{1}{2}}_{L^2;0}\big([\eta_0]_{H^1;0} + [\eta_b]_{H^1;0}\big) \\
\|\partial e_0^3\|_{\frac{4}{3}} &\leq C\langle s\rangle^{-\frac{3}{2}}[e_0]_{\infty;\frac{3}{2}}[e_0]^2_{H^1;0}\,.
\end{aligned}
$$

Using the bounds (10.20) and (10.21) in (10.19) implies

Proposition 10.3.

$$
\begin{aligned}
\|e_0(t)\|_\infty \leq \langle t\rangle^{-\frac{3}{2}}&\big(\mathcal{C}(\mathcal{E}_0, [\eta_0(0)]_X) + \mathcal{E}_0^{\frac{1}{2}}[\eta_b(t)]_{\infty;0} + [\eta_0(0)]_X[\eta_b(t)]_{\infty;0}[e_0]_{\infty;\frac{3}{2}}\big) \\
&+ \langle t\rangle^{-\frac{3}{2}}\big(\mathcal{E}_0^{\frac{1}{2}} + [\eta_0(0)]_X + [e_0(t)]^2_{H^1;0} + [\eta_b(t)]_{\infty;\frac{3}{2}}\big)\big([e_0]_{\infty;\frac{3}{2}} + [e_0]^2_{\infty;\frac{3}{2}}\big) \\
&+ \langle t\rangle^{-\frac{3}{2}}\big([e_0]_{\infty;\frac{3}{2}}[e_0]^2_{H^1;0} + \mathcal{E}_0[e_0]_{L^2_{\rm loc};\frac{3}{2}} + \mathcal{E}_0^{\frac{1}{2}}[e_0]^2_{\infty;\frac{3}{2}}\big)\,. \qquad (10.22)
\end{aligned}
$$

10.2. *Estimation of* $[\partial e_0(t)]_{L^2_{\rm loc};\frac{3}{2}}$

$$
\begin{aligned}
\|\partial e_0(t)\|_{L^2_{\rm loc}} &\sim \|\chi\partial e_0(t)\|_2 \,(\chi \text{ localized}) \\
&\sim \|\chi\langle\mathcal{H}_0\rangle^{\frac{1}{2}}e_0(t)\|_2\big(\partial\langle\mathcal{H}_0\rangle^{-\frac{1}{2}} \in \mathcal{B}(L^2)\big) \\
&\leq \int_0^t \|\chi\langle\mathcal{H}_0\rangle^{\frac{1}{2}}e^{-i\mathcal{H}_0(t-s)}P_cS^{(0)}(s)\|_2 ds\,(\text{by (10.9)})\,. \qquad (10.23)
\end{aligned}
$$

For the purpose of continuing the estimation, we regard $S^{(0)}$ as consisting of terms of two types: (i) terms having spatially localized (exponentially) functions of x as a factor, coming from Ψ_j, $j = 0, 1$, and (ii) terms like e_0^3 or $\eta_b\eta_0 e_0$ and others which are not of this type. It is convenient to refer to such terms as $S^{(0)}_{\rm LOC}$ and $S^{(0)}_{\rm NLOC}$ below. From (10.23) we have:

$$
\begin{aligned}
\|\partial e_0(t)\|_{L^2_{\rm loc}} &\leq \int_0^t \|\chi\langle\mathcal{H}_0\rangle^{\frac{1}{2}}e^{-i\mathcal{H}_0(t-s)}P_cS^{(0)}_{\rm LOC}(s)\|_2 ds \\
&\quad + \left(\int_0^{t-1} + \int_{t-1}^t\right)\|\chi\langle\mathcal{H}_0\rangle^{\frac{1}{2}}e^{-i\mathcal{H}_0(t-s)}P_cS^{(0)}_{\rm NLOC}(s)\|_2 ds \\
&\leq \int_0^t \frac{C}{\langle t-s\rangle^{\frac{3}{2}}}\|\langle x\rangle^\sigma S^{(0)}_{\rm LOC}\|ds
\end{aligned}
$$

$$+\int_0^{t-1}\|e^{-i\mathcal{H}_0(t-s)}P_c\langle\mathcal{H}_0\rangle^{\frac{1}{2}}S^{(0)}_{\rm NLOC}(s)\|_\infty ds$$

$$+\int_{t-1}^{t}\|\langle\mathcal{H}_0\rangle^{\frac{1}{2}}e^{-i\mathcal{H}_0(t-s)}P_cS^{(0)}_{\rm NLOC}(s)\|_2 ds$$

$$\le\int 0_0^t\frac{C}{\langle t-s\rangle^{\frac{3}{2}}}\|\langle x\rangle^\sigma S^{(0)}_{\rm LOC}\|ds$$

$$+C\int_0^{t-1}\frac{1}{|t-s|^{\frac{3}{2}}}\|\partial S^{(0)}_{\rm NLOC}(s)\|_\infty ds+\int_{t-1}^{t}\|\partial S^{(0)}_{\rm NLOC}(s)\|_2 ds$$

$$=\mathbf{A}+\mathbf{B}+\mathbf{C}\,. \qquad (10.24)$$

We estimate the integrands of **B** and **C**; that of **A** is similar.

$$|\text{integrand of }\mathbf{B}|\le C\langle t\rangle^{-\frac{3}{2}}\left([e_0]_{\infty;\frac{3}{2}}+[\eta_0]_{\infty;\frac{3}{2}}\right)\left([e_0]^2_{H^1;0}+[\eta_b]^2_{H^1;0}+[\eta_0]^2_{H^1;0}\right) \qquad (10.25)$$

$$|\text{integrand of }\mathbf{C}|\le\langle t\rangle^{-\frac{3}{2}}\Big([e_0]^2_{\infty;\frac{3}{2}}\left([e_0]_{H^1;0}+[\eta_b]_{H^1;0}\right)$$

$$+[e_0]_{\infty;\frac{3}{2}}[\eta_0]_{\infty;\frac{3}{2}}[\eta_b]_{H^1;0}$$

$$+[\eta_0]_{W^{1,\infty};\frac{3}{2}}\left([\eta_b]_{\infty;0}[e_0]_{H^1;0}+[\eta_0]_{\infty;\frac{3}{2}}[\eta_b]_{H^1;0}+[e_0]_{\infty;\frac{3}{2}}[e_0]_{H^1;0}\right)$$

$$+[\eta_0]^2_{W^{1,\infty};\frac{3}{2}}[e_0]_{H^1;0}\Big)\,.$$

Substituting of these bounds into (10.24), we have:

Proposition 10.4.

$$\|\partial e_0(t)\|_{L^2_{\rm loc};\frac{3}{2}}\le C\langle t\rangle^{-\frac{3}{2}}\left([e_0]_{\infty;\frac{3}{2}}+[\eta_0(0)]_X\right)\left([e_0]_{\infty;\frac{3}{2}}+[e_0]_{H^1;0}\right). \qquad (10.26)$$

10.3. *Estimation of* $[e_0(t)]_{H^1;0}$

$$\|e_0(t)\|_{H^1}\sim\|\mathcal{H}_0^{\frac{1}{2}}e_0(t)\|_2\le\int_0^t\|\partial S^{(0)}(s)\|_2 ds \qquad (10.27)$$

$$\|\partial S^{(0)}_0\|_2\ \le C\langle s\rangle^{-\frac{3}{2}}\Big(\mathcal{E}_0[\eta_0]_{W^{1,\infty};\frac{3}{2}}+\mathcal{E}_0^{\frac{1}{2}}[\eta_0]^2_{\infty;\frac{3}{2}}$$

$$+[\eta_0]_{H^1;0}[\eta_0]^2_{\infty;\frac{3}{2}}+\mathcal{E}_0[\eta_0]_{W^{1,\infty};\frac{3}{2}}\Big)$$

$$\|\partial S^{(1)}_0 e_0\|_2\le C\langle s\rangle^{-\frac{3}{2}}\cdots$$

$$\|\partial S^{(2)}_0 e_0^2\|_2\ \le C\langle s\rangle^{-\frac{3}{2}}\left(\mathcal{E}_0^{\frac{1}{2}}[e_0]^2_{\infty;\frac{3}{2}}+\mathcal{E}_0^{\frac{1}{2}}[e_0]_{\infty;\frac{3}{2}}[\eta_0]_{H^1;0}\right)$$

$$+C\langle s\rangle^{-\frac{3}{2}}[e_0]^2_{\infty;\frac{3}{2}}\left([\eta_0]_{H^1;0}+[\eta_b]_{H^1;0}\right)$$

$$+ C\langle s\rangle^{-\frac{3}{2}}[e_0]_{\infty;\frac{3}{2}}[\eta_0]_{W^{1,\infty};\frac{3}{2}}\left([\eta_0]_{2;0} + [\eta_b]_{2;0}\right)$$

$$\|\partial e_0^3\|_2 \le C\langle s\rangle^{-3}[e_0]^2_{\infty;\frac{3}{2}}[e_0]_{H^1;0}\,.$$

Proposition 10.5. *For $t \ge 0$,*

$$\|e_0(t)\|_{H^1} \le \mathcal{C}(\mathcal{E}_0, [\eta_0(0)]_X) + [e_0]_{\infty;\frac{3}{2}}\big([\eta_0(0)]_X + [e_0]_{H^1;0}\big)\,. \tag{10.28}$$

We now turn to the estimation of $\|\eta_b(t)\|_\infty$ and $\|\eta_b(t)\|_{H^1}$. By (10.18) we have

$$\|\eta_b(t)\|_\infty \le \int_0^t \frac{1}{|t-s|^{\frac{3}{4}}}\frac{1}{\langle t-s\rangle^{\frac{3}{4}}}\big(\|S^{(b)}(s)\|_1 + \|\partial S^{(b)}(s)\|_{\frac{4}{3}}\big)ds\,, \tag{10.29}$$

where $S^{(b)}$ is given by (10.6).

Proposition 10.6. *In terms of $Q_j, j = 0,1$ we have*

$$S^{(b)} = S_0^{(b)} + S_1^{(b)}\eta_b + S_2^{(b)}\eta_b^2 + \eta_b^3 \tag{10.30}$$

where

$$S_0^{(b)} \sim Q_0^{\frac{1}{2}}Q_1\chi + Q_0Q_1^{\frac{1}{2}}\chi$$

$$S_1^{(b)}\eta_b \sim Q_0^{\frac{1}{2}}Q_1^{\frac{1}{2}}\chi\eta_b + (\eta_0^2 + e_0^2)\eta_b + Q_0^{\frac{1}{2}}\chi\eta_0\eta_b$$

$$S_2^{(b)}\eta_b^2 \sim \big(Q_0^{\frac{1}{2}} + Q_1^{\frac{1}{2}}\big)\chi\eta_b^2 + (\eta_0 + e_0)\,\eta_b^2\,.$$

We now proceed with estimates of $S^{(b)}$ in L^1 and in $W^{1,\frac{4}{3}}$.
Beginning with $\|S^{(b)}\|_1$ we have:

$$\begin{aligned}
&\|S^{(b)}\|_1 \\
&\le C\big(Q_0^{\frac{1}{2}}Q_1 + Q_0Q_1^{\frac{1}{2}} + Q_0^{\frac{1}{2}}Q_1^{\frac{1}{2}}\|\eta_b\|_\infty + \|\eta_b\|_\infty(\|\eta_0\|_2^2 + \|e_0\|_2^2)\big) \\
&+ C\big(Q_0^{\frac{1}{2}}\|\eta_0\|_\infty\|\eta_b\|_2 + \big(Q_0^{\frac{1}{2}} + Q_1^{\frac{1}{2}}\big)\|\eta_b\|_\infty^2 + \|\eta_b\|_\infty(\|\eta_0\|\|\eta_b\|_2 + \|e_0\|_2\|\eta_b\|_2)\big) \\
&+ C\|\eta_b\|_\infty\|\eta_b\|_2^2\,.
\end{aligned} \tag{10.31}$$

We now turn to $\|\partial S^{(b)}\|_{\frac{4}{3}}$:

$$\begin{aligned}
&\|\partial S^{(b)}\|_{\frac{4}{3}} \\
&\le C\big(Q_0^{\frac{1}{2}}Q_1 + Q_0Q_1^{\frac{1}{2}} + Q_0^{\frac{1}{2}}Q_1^{\frac{1}{2}}\|\eta_b\|_{H^1}\big) \\
&+ C\big(\|(\eta_0^2 + e_0^2)\partial\eta_b\|_{\frac{4}{3}} + \|\eta_0\partial\eta_0\eta_b\|_{\frac{4}{3}} + Q_0^{\frac{1}{2}}\|\eta_b\|_2(\|\eta_0\|_\infty + \|\partial\eta_0\|_\infty)\big) \\
&+ C(Q_0^{\frac{1}{2}}\|\eta_0\|_\infty\|\partial\eta_b\|_2 + \|(\eta_0 + e_0)\eta_b\partial\eta_b\|_{\frac{4}{3}} + \|(\partial\eta_0 + \partial e_0)\eta_b^2\|_{\frac{4}{3}} + \|\eta_b^2\partial\eta_b\|_{\frac{4}{3}})\,.
\end{aligned} \tag{10.32}$$

To further estimate $\|\partial S^{(b)}\|_{\frac{4}{3}}$ we use the following estimates of individual terms:

$$
\begin{aligned}
\|\eta_0^2\partial\eta_b\|_{\frac{4}{3}} &\le \|\eta_0\|_\infty\|\eta_0\|_2^{\frac{1}{2}}\|\eta_b\|_{H^1}\\
\|e_0^2\partial\eta_b\|_{\frac{4}{3}} &\le \|\eta_0\|_\infty^{\frac{3}{2}}\|e_0\|_2^{\frac{1}{2}}\|\eta_b\|_{H^1}\\
\|\eta_0\partial\eta_0\eta_b\|_{\frac{4}{3}} &\le \|\partial\eta_0\|_\infty\|\eta_0\|_\infty^{\frac{1}{2}}\|\eta_0\|_2^{\frac{1}{2}}\|\eta_b\|_2\\
\|\eta_0\eta_b\partial\eta_b\|_{\frac{4}{3}} &\le \|\eta_b\|_\infty\|\eta_0\|_\infty^{\frac{1}{2}}\|\eta_0\|_2^{\frac{1}{2}}\|\eta_b\|_{H^1}\\
\|e_0\eta_b\partial\eta_b\|_{\frac{4}{3}} &\le \|\eta_b\|_\infty\|e_0\|_\infty^{\frac{1}{2}}\|e_0\|_2^{\frac{1}{2}}\|\eta_b\|_{H^1}\\
\|\partial\eta_0\eta_b^2\|_{\frac{4}{3}} &\le \|\eta_b\|_\infty\|\partial\eta_0\|_\infty^{\frac{1}{2}}\|\partial\eta_0\|_4\|\eta_b\|_2\\
\|\partial e_0\eta_b^2\|_{\frac{4}{3}} &\le \|e_0\|_{H^1}\|\eta_b\|_\infty^{\frac{3}{2}}\|\eta_b\|_2^{\frac{1}{2}}\\
\|\eta_b^2\partial\eta_b\|_{\frac{4}{3}} &\le \|\eta_b\|_\infty\|\partial\eta_b\|_2\|\eta_b\|_2\,.
\end{aligned}
\tag{10.33}
$$

Recall that $\|\eta_b(t)\|_{H^1}$ can be estimated in terms of $\|e_0(t)\|_{H^1}$; see (10.8).

Since η_b is driven by the bound state amplitudes (Q_0 and Q_1), which have different behaviors on the intervals I_j, we now estimate $\eta_b(t)$ separately on $I_0 = [0,t_0]$, $I_1 = [t_0,t_1]$ and $I_2 = [t_1,\infty)$.

We now introduce appropriate norms on different time scales. Define

$$
\begin{aligned}
M_{I_0}(t) \equiv{}& \sup_{0\le t\wedge t_0}\langle t\rangle^{\frac{3}{2}}\|e_0(t)\|_\infty + \sup_{0\le t\wedge t_0}\langle t\rangle\|\eta_b(t)\|_\infty + \sup_{0\le t\wedge t_0}\|e_0(t)\|_{H^1}\\
&+ \sup_{0\le t\wedge t_0}\langle t\rangle^{\frac{3}{2}}\|\partial e_0(t)\|_{L^2_{\rm loc}} + \sup_{0\le t\wedge t_0}\langle t\rangle^2|Q_0(t)| + \sup_{0\le t\wedge t_0}|Q_1(t)|
\end{aligned}
\tag{10.34}
$$

$$
\begin{aligned}
M_{I_1}(t) \equiv{}& \sup_{t_0\le t\wedge t_1}\langle t\rangle^{\frac{3}{2}}\|e_0(t)\|_\infty + \sup_{t_0\le t\wedge t_1}\|\eta_b(t)\|_\infty + \sup_{t_0\le t\wedge t_1}\|e_0(t)\|_{H^1}\\
&+ \sup_{t_0\le t\wedge t_1}\langle t\rangle^{\frac{3}{2}}\|\partial e_0(t)\|_{L^2_{\rm loc}} + \sup_{t_0\le t\wedge t_1}|Q_0(t)| + \sup_{t_0\le t\wedge t_1}|Q_1(t)|
\end{aligned}
\tag{10.35}
$$

$$
\begin{aligned}
M_{I_2}(t) \equiv{}& \sup_{t_1\le t\wedge T}\langle t\rangle^{\frac{3}{2}}\|e_0(t)\|_\infty + \sup_{t_1\le t\wedge T}\langle t-t_1\rangle^{\frac{1}{2}}\|\eta_b(t)\|_\infty + \sup_{t_1\le t\wedge T}\|e_0(t)\|_{H^1}\\
&+ \sup_{t_1\le t\wedge T}\langle t\rangle^{\frac{3}{2}}\|\partial e_0(t)\|_{L^2_{\rm loc}} + \sup_{t_1\le t\wedge T}|Q_0(t)|\\
&+ \sup_{t_1\le t\wedge T}\langle t-t_1\rangle\Gamma'|Q_0(t_1)||Q_1(t_1)||Q_1(t)|\,.
\end{aligned}
\tag{10.36}
$$

Remark 10.2. By Propositions 10.3–10.5, the e_0 contributions to the norms M_{I_k} are controlled in terms of the initial conditions. Therefore, to control M_{I_k}, it suffices to bound Q_j and η_b.

The above estimates can be used together with the bounds on Q_j of Sec. 9 to obtain the following three propositions, which give bounds for η_b on the intervals I_0, I_1 and I_2.

Proposition 10.7. *($\eta_b(t)$ for $t \in I_0$) Assume $t \in I_0 = [0, t_0]$, i.e. $Q_0(t) \leq C([\eta_0]_X + \mathcal{E}_0)\langle t\rangle^{-2}$ for $t \in [0, t_0]$. Then, for $\|\phi(0)\|_X$ sufficiently small*

$$\|\eta_b(t)\|_\infty \leq \mathcal{C}(\|\phi(0)\|_X, M_{I_0}(t_0))\langle t\rangle^{-1} . \tag{10.37}$$

Proposition 10.8. *($\eta_b(t)$ for $t \in I_1$) Assume $t \in I_1 = [t_0, t_1]$. Then, for $\|\phi(0)\|_X$ sufficiently small*

$$\|\eta_b(t)\|_\infty \leq \mathcal{C}\left(\|\phi(0)\|_X, M_{I_0}(t_0), M_{I_1}(t_1)\right) . \tag{10.38}$$

Proposition 10.9. *($\eta_b(t)$ for $t \in I_2$) Assume $t \in I_2 = [t_1, T]$. Then, for $\|\phi(0)\|_X$ sufficiently small*

$$\|\eta_b(t)\|_\infty \leq \mathcal{C}(\|\phi(0)\|_X, M_{I_0}(t_0), M_{I_1}(t_1), M_{I_2}(T))\langle t - t_1\rangle^{-\frac{1}{2}} . \tag{10.39}$$

In our estimates of Sec. 11, we shall use the following result to estimate the size of correction terms in the system for Q_0 and Q_1 for $t \geq t_0$, where Q_0 is monotonically increasing.

Proposition 10.10. *Let $\zeta = \zeta(x, t)$, $x \in \mathbb{R}^3$, $t \geq t_0$, with $\zeta(x, t_0) = 0$ satisfy the following dispersive equation*

$$i\partial_t\zeta = \mathcal{H}_0\zeta + P_c(S_1(t) + S_2(t)\zeta + \chi\zeta^2 + \zeta^3) , \tag{10.40}$$

where for all $k \geq 0$ and $j = 1, 2$:

$$\|S_j(t)\|_{H^k} = \mathcal{O}\big(\mathcal{C}([\phi(0)]_X\ Q_0^{\frac{1}{2}}(t)\big)$$

$$\partial_t Q_0 \geq 0 \text{ for } t \geq t_0, \text{ and } Q_0 \leq \mathcal{E}_0 . \tag{10.41}$$

Suppose $\|\zeta(t)\|_{H^1} \leq \mathcal{C}([\phi(0)]_X, M_{I_1}(t_1), M_{I_2}(T))$ for all $t_0 \leq t \leq T$, where $[\phi(0)]_X$ is sufficiently small. Then,

$$\|\zeta(t)\|_\infty \leq C\big([\phi(0)]_X, M_{I_{1(t_1)}}, M_{I_2}(T)\big)Q_0(t)^{1/2} .$$

Corollary 10.1. *Let $p > 6$. $\mathcal{E}_0^{\bar{m}} = \sup_{t' \leq t} \epsilon_1(t')$, then*

$$\|\zeta\|_p \leq cQ_0^{1/2}(t) \int_0^t \frac{\epsilon_1(s)}{\langle t - s\rangle^{3/2 - 3/p}} ds .$$

Corollary 10.2. *Let $p > 6$.*

$$|\langle\chi, \zeta\rangle| \leq cQ_0^{1/2}(t) \int_0^t \frac{\epsilon_1(s)}{\langle t - s\rangle^{3/2 - 3/p}}\, ds .$$

which follows by using

$$\|\chi e^{-iHt} P_c\bar{\chi} g\|_2 \leq c\langle t\rangle^{-3/2}\|g\|_2 .$$

Corollary 10.3.

$$\|\partial^k\zeta\|_\infty \leq CQ_0^{1/2}(t) \int_0^t \frac{\epsilon_1(s)}{\langle t - s\rangle^{3/2}}\, ds .$$

Proof. This result follows from applying ∂^k to the equation for ζ and estimating, as above, in L^p for any α and $p > 6$. By the Sobolev inequality this implies control derivatives in L^∞. □

Remark 10.3. In our applications of Proposition 10.10 and its corollaries, $\epsilon_1(s)$ will be given by the source terms depending on Q_0 and $Q_1^{\frac{1}{2}}$. For $t > t_1$, $Q_1 = \mathcal{O}(\langle t - t_1\rangle^{-1})$. Therefore, since the lowest order term in Q_1 contributing to $\epsilon_1(s)$ is $\mathcal{O}(Q_1)$ (see Eq. (7.32)), it follows that for $t > t_1$

$$\|\eta_b\|_{W^{k,\infty}} = \mathcal{O}(\langle t - t_1\rangle^{-\frac{1}{2}}) + \mathcal{O}(\langle t - t_1\rangle^{-1}) + \mathcal{O}(\langle t - t_1\rangle^{-\frac{3}{2}})\,. \tag{10.42}$$

Since $\eta = \mathcal{O}(\eta_b) + \mathcal{O}(\eta_0)$, the conclusion of the main theorem for large t, $t > t_1$, follows. Namely, for

$$\|\eta\|_{W^{k,\infty}} = \mathcal{O}(\langle t - t_1\rangle^{-\frac{1}{2}}) + e^{-i\mathcal{H}_0 t} P_c[\eta(0) + \mathcal{E}_0^2]\,, \tag{10.43}$$

where the non-free wave part is coming from spatially-localized source terms.

11. Beginning of Proof of Proposition 9.1

The key to Proposition 9.1 is the following more detailed version of Corollary 7.1:

Proposition 11.1. *Let $t \ge 0$. The equations for P_0, P_1 can be written in the following form*

$$\begin{aligned} \frac{dP_0}{dt} &= 2\Gamma P_0 P_1^2 + R_0^{(0)}[\eta_0] + R_1^{(0)}[\eta_b] + R_2^{(0)}[P_0, P_1] \\ \frac{dP_1}{dt} &= -4\Gamma P_0 P_1^2 + R_0^{(1)}[\eta_0] + R_1^{(1)}[\eta_b] + R_2^{(1)}[P_0, P_1] \end{aligned} \tag{11.1}$$

where (i) $R_0^{(i)}[\eta_0]$ *are η_0-dependent terms only both local and nonlocal in time*, *and may also depend on $\tilde{\alpha}_0, \tilde{\beta}_1$.*
(ii) η_b *is the bound state driven part of the dispersion*; *see Sec.* 5.
(iii) $R_2^{(0)}$ *depends only on P_0, P_1, t, but not on η*; *it is formally linear in P_0*, *of high order in P_1*, *and contains both local and nonlocal in time terms.*
(iv) $R_2^{(1)}$ *depends only on P_0, P_1, t, but not on η. It is of high order in P_1*, *local and nonlocal terms included*, *but has terms which are linear in $\tilde{\alpha}_0 \sim \sqrt{P_0}$.*

The proof uses repeated application of near-identity transformations of the variables $\tilde{\alpha}_0, \tilde{\beta}_1$, derivable by integration by parts (see the discussion of resonant and nonresonant terms in Sec. 7) and the decomposition of η given in (10.1).

Remark 11.1. As noted in Sec. 10 additional terms which arise, when replacing η_0, e_0 and η_b by $\tilde{\eta}_0, \tilde{e}_0$ and $\tilde{\eta}_b$, are treated perturbatively. This is the case because integration by parts of these terms gives rise to an extra factor from the derivative of the phase which is $\mathcal{O}(E_0(t) - E_0(T)) + \mathcal{O}(\partial_t\tilde{\Theta}) = \mathcal{O}(t^{-1})$. Throughout Secs. 11–13 we shall use this fact as well as the fact that dispersive and energy estimates of these sets of functions are the same because they differ by a purely time-varying phase.

The proof is long so we break it up into three parts, which are presented in three different sections. The following is an overview.

Part 1: The terms $R_0^{(0)}[\eta_0]$ and $R_0^{(1)}[\eta_0]$ are forcing terms in the ODE dynamics, which are driven by the dispersive part of the initial conditions. They are studied and estimated in this section; Proposition 11.2.

Part 2: The terms $R_1^{(0)}[\eta_b]$ and $R_1^{(1)}[\eta_b]$ are studied and estimated in Sec. 13; Proposition 13.1.

Part 3: The terms $R_2^{(0)}[P_0, P_1]$ and $R_2^{(1)}(P_0, P_1)$ are studied and estimated in Sec. 12; Proposition 12.1.

In Parts 1–3, we require estimates for all $t \geq 0$. On $I_0 = \{t : 0 \leq t \leq t_0\}$ we use the *a priori* bound on $P_0(t)$, implied by the definition of t_0; see Eq. (8.5). For $t > t_0$, we use the monotonicity property Q.

Monotonicity property *Q*

$$Q_0 \text{ and } \frac{Q_0}{Q_1} \text{ are monotonically increasing}\,, \tag{11.2}$$

where Q_0 and Q_1 are the *modified* bound state energies related to P_0 and P_1 (see (9.7), (9.1), (9.2)). This monotonicity property is shown to hold at $t = t_0$ and is then shown to continue for all time, t, by a continuity argument; see Sec. 14. Since there are many terms, we focus on those which are most problematic, namely, those which are *nonlocal* in time and of slowest time-decay rate. These calculations are very lengthy and before embarking on them we present a calculation, related to the normal form discussion in Sec. 7, and which is repeated in order to exploit rapid oscillations in time.

Expansion of oscillatory integrals, resonances and improved time decay
In deriving and estimating the terms $R_j^{(i)}$ in (11.1), we must frequently expand and/or estimate terms of the form:

$$\left\langle \chi_1, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \chi_2(s) e^{i\Omega_1 s} ds \right\rangle . \tag{11.3}$$

Here, χ_1 and $\chi_2(s)$ are localized functions of x, χ_1 is independent of s and $\chi_2(s)$ depends on s through its dependence on $\alpha_0(s), \beta_1(s)$ or $\eta_0(s)$.

Recall that $\mathcal{H}_0$, defined in (4.6), is of the form

$$\mathcal{H}_0 = \tilde{\mathcal{H}}_0 + E_0 \sigma_3 \tag{11.4}$$

and $\tilde{\mathcal{H}}_0 = \sigma_3(-\Delta)$ plus a matrix potential which decays to zero rapidly as $|x|$ tends to infinity. In (11.3) we would like to integrate by parts, exploiting the oscillation of frequency E_0. However, "peeling off" these oscillations is a little tricky because σ_3 does not commute with $\tilde{\mathcal{H}}_0$. We handle this as follows, using Theorem 4.3.

Recall that

$$\begin{aligned} Z\mathcal{H}_0 W &= \sigma_3\left(-\Delta - E_0\right), \\ Ze^{-i\mathcal{H}_0 t P_c} W &= e^{-i\sigma_3(-\Delta - E_0)t} \end{aligned} \tag{11.5}$$

where W denotes the wave operator

$$\begin{aligned} W &= \lim_{t\to\infty} e^{-i\mathcal{H}_0 t} e^{-i\sigma_3(-\Delta-E_0)t}, \\ Z &= W^{-1}. \end{aligned} \tag{11.6}$$

By (11.5), we can rewrite (11.3) as

$$\begin{aligned} &\left\langle \chi_1, e^{-i\mathcal{H}_0 t} \int_0^t W e^{i\sigma_3(-\Delta-E_0)s} Z P_c e^{i\Omega_1 s} \chi_2(s) ds \right\rangle \\ &= \left\langle \chi_1, e^{-i\mathcal{H}_0 t} \int_0^t W e^{-i\sigma_3(-\Delta-E_0)s} Z P_c e^{i\Omega_1 s} \chi_2(s) ds \right\rangle \\ &= \left\langle \chi_1, e^{-i\mathcal{H}_0 t} \int_0^t W e^{i\sigma_3 \Delta s} Z \cdot W e^{i\sigma_3 E_0 s} Z P_c e^{i\Omega_1 s} \chi_2(s) ds \right\rangle \\ &= \left\langle \chi_1, e^{-i\mathcal{H}_0 t} \int_0^t W e^{i\sigma_3 \Delta s} Z \cdot W e^{i(\sigma_3 E_0 + I\Omega_1)s} Z P_c \chi_2(s) ds \right\rangle \\ &= \left\langle \chi_1, e^{-i\mathcal{H}_0 t} \int_0^t W e^{i\sigma_3 \Delta s} Z \cdot (-i) \frac{d}{ds} W (\sigma_3 E_0 + I\Omega_1)^{-1} e^{i(\sigma_3 E_0 + I\Omega_1)s} Z P_c \chi_2(s) ds \right\rangle \\ &\sim -\left\langle \chi_1\ , e^{-i\mathcal{H}_0 t} \int_0^t W \sigma_3 \Delta e^{i\sigma_3 \Delta s} Z \cdot W (\sigma_3 E_0 + I\Omega_1)^{-1} e^{i(\sigma_3 E_0 + I\Omega_1)s} Z P_c \chi_2(s) ds \right\rangle \\ &\sim -\left\langle \chi_1\ , \int_0^t e^{i\mathcal{H}_0(t-s)} \tilde{\mathcal{H}}_0 (\sigma_3 E_0 + I\Omega_1)^{-1} e^{i\Omega_1 s} P_c \chi_2(s) ds \right\rangle. \end{aligned} \tag{11.7}$$

In the previous string of equations we have used the notation $f \sim g$ to mean equality up to terms which are local in time. Note that $\sigma_3 E_0 + I\Omega_1$ is invertible since its determinant is $\Omega_1^2 + E_0^2$. We can therefore carry out this procedure any finite number of times to arrange, up to local in time terms, an expression which involves an operator of the form $\chi \exp(i\mathcal{H}_0(t-s))\tilde{\mathcal{H}}_0^k \chi$, $k \geq 1$, where χ is spatially localized. Therefore, the enhanced local decay estimate (4.43) of Theorem 4.3 applies. We shall use these observations, together with the detailed dependence of $\chi_2(s)$ on on α_0, β_1 etc. to control certain terms in $R_j^{(i)}$.

Part 1: Estimation of $\left|\boldsymbol{R_0^{(0)}}[\boldsymbol{\eta_0}]\right|$ and $\left|\boldsymbol{R_0^{(1)}}[\boldsymbol{\eta_0}]\right|$

Proposition 11.2. *Assume either $t \leq t_0$ or monotonicity property Q on $[t_0, t_0 + \delta_*]$. Then,*

$$\begin{aligned} \left|R_0^{(0)}[\eta_0]\right|, \left|R_0^{(1)}[\eta_0]\right| &\leq b_i\left(t_0, [\eta_0]_X\right) \langle t\rangle^{-2} + \mathcal{O}(\mathcal{E}_0) 2\Gamma P_0 P_1^2 \\ b_0(t_0, [\eta_0]_X) &= \mathcal{O}(\langle t_0\rangle^{-1} [\eta_0]_X) \\ b_1(t_0, [\eta_0]_X) &= \mathcal{O}(\langle t_0\rangle^{-\frac{1}{2}} [\eta_0]_X^{2/3}). \end{aligned} \tag{11.8}$$

Proof of Proposition 11.2.

Proof of estimate (11.8) *for* $R_0^{(0)}[\eta_0]$: The key terms are those in the P_0 equation, which are decaying most slowly with t. These are linear in η_0, since $\|\eta_0\|_\infty = \mathcal{O}(t^{-3/2})$. We focus on the most difficult terms. These are *nonlocal* in $\eta_0(t)$. Recall Eq. (7.52) for α_0 and that the equation for P_0, (7.6), is derived from the $\tilde{\alpha}_0$ equation (related to (7.52) by a near-identity change of variables) by multiplication by $\tilde{\alpha}_0$ and taking the imaginary part of the $\partial_t \alpha_0$ equation.

We consider the following representative "most problematic" terms in $R_0^{(0)}[\eta_0]$, whose estimation introduces the necessary methods for treating them all:

$$(T1)\ \ \mathcal{O}(\tilde{\alpha}_0)\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \Psi_0(s)\Psi_1(s)\overline{\eta}_0(s)ds \right\rangle \mathcal{O}\big(\overline{\tilde{\alpha}_0}\tilde{\beta}_1 e^{-i\lambda_+ t} + |\tilde{\beta}_1|^2\big)\,, \tag{11.9}$$

(also with $\Psi_0\Psi_1\overline{\eta}_0$ replaced by $\Psi_0\bar{\Psi}_1\eta_0, \bar{\Psi}_0\Psi_1\eta_0$)

$$(T2)\mathcal{O}(\tilde{\alpha}_0)\mathcal{O}(\tilde{\alpha}_0\tilde{\beta}_1)\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \eta_b^2 \eta_0 \right\rangle ds\,. \tag{11.10}$$

Estimation of $T1$: Since the time-integral is bounded by $\mathcal{O}(\mathcal{E}_0)\langle t\rangle^{-3/2}\ [\eta_0]_X$, by the Cauchy–Schwarz inequality the second term in (11.9) is bounded as follows:

$$\left|\mathcal{O}(\tilde{\alpha}_0)\mathcal{O}(|\tilde{\beta}_1|^2)\left\langle \chi, \int_0^t \cdots \right\rangle\right| \le \mathcal{O}(\mathcal{E}_0)P_0P_1^2 + \mathcal{O}(\mathcal{E}_0)\langle t\rangle^{-3}\|\eta_0\|_X^2\,. \tag{11.11}$$

We now control the first term in (11.9). $\mathcal{O}(\overline{\tilde{\alpha}_0}\tilde{\beta}_1)\mathcal{O}(\tilde{\alpha}_0)$. We argue that the key contribution from this term which must be bounded is of the form:

$$\mathcal{O}(\tilde{\alpha}_0)\overline{\tilde{\alpha}_0}\tilde{\beta}_1 e^{-i\lambda_+ t}\frac{\partial}{\partial t}\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \Psi_0(s)\Psi_1(s)\overline{\eta}_0(s)ds \right\rangle\,. \tag{11.12}$$

To see this, consider the term in the $\tilde{\alpha}_0$ equation which corresponds to the first term in (11.9):

$$\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \Psi_0(s)\Psi_1(s)\overline{\eta}_0(s)ds \right\rangle \mathcal{O}\left(\overline{\tilde{\alpha}_0}\tilde{\beta}_1 e^{-i\lambda_+ t}\right)\,. \tag{11.13}$$

Next we integrate with respect to t, and integrate by parts, making use of the oscillatory exponential factor. The result is a boundary term, which can be subsumed in the definition of $\tilde{\alpha}_0$, by a near identity transformation, followed by a time-integral 0 to t. The latter contributes terms to the P_0 equation (which has been modified due to the slight redefinition of $\tilde{\alpha}_0$) of the following type:

$$\mathcal{O}(\tilde{\alpha}_0)\mathcal{O}(\tilde{\beta}_1\partial_t\tilde{\alpha}_0 + \tilde{\alpha}_0\partial_t\tilde{\beta}_1)\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \Psi_0\Psi_1\overline{\eta}_0 ds \right\rangle \tag{11.14}$$

$$\mathcal{O}(\tilde{\alpha}_0)\mathcal{O}(\overline{\tilde{\alpha}_0}\tilde{\beta}_1)\frac{\partial}{\partial t}\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \Psi_0\Psi_1\overline{\eta}_0 ds \right\rangle\,. \tag{11.15}$$

Since $\partial_t \tilde{\alpha}_0 = \mathcal{O}(|\tilde{\alpha}_0|^2 |\tilde{\beta}_1|)$ and $\partial_t \tilde{\beta}_1 = \mathcal{O}(|\tilde{\alpha}_0||\tilde{\beta}_1|^2)$ (11.14) is bounded by

$$\mathcal{O}(\mathcal{E}_0)\mathcal{O}\big(\tilde{\alpha}_0^3 \tilde{\beta}_1^2 \langle t\rangle^{-\frac{3}{2}} [\eta_0]_X\big)\,, \tag{11.16}$$

where we have used

$$\|\eta_0\|_\infty \le C[\eta_0]_X \ \langle t\rangle^{-3/2}\,. \tag{11.17}$$

By the Cauchy–Schwarz inequality this is bounded by $\mathcal{O}(\mathcal{E}_0)(2\Gamma P_0 P_1^2 + [\eta_0]_X^2 \langle t\rangle^{-3})$. Therefore, the contribution from this first term satisfies the estimate (11.8).

Obtaining a bound on (11.15) is more involved. We use the local decay estimate of Theorem 4.3:

$$\|\chi e^{-i\mathcal{H}_0 t} P_c \tilde{\mathcal{H}}_0 f\|_2 \le \langle t\rangle^{-5/2} \|\partial^k \langle x\rangle^\sigma f\|_2\,. \tag{11.18}$$

The expression (11.15) bounded by:

$$\begin{aligned}
&\mathcal{O}(\tilde{\alpha}_0)\mathcal{O}(\overline{\tilde{\alpha}_0}\tilde{\beta}_1)\left(\langle \chi, \Psi_0\Psi_1\eta_0\rangle - i\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \tilde{\mathcal{H}}_0 \Psi_0 \Psi_1 \bar{\eta}_0 ds\rangle\right)\\
&\quad \le \mathcal{O}(\mathcal{E}_0)\left(\Gamma Q_0 Q_1^2 + \langle t\rangle^{-3}\right)\\
&\qquad + \mathcal{O}(\tilde{\alpha}_0^2 \tilde{\beta}_1)[\eta_0]_X \int_0^t \frac{ds}{\langle t-s\rangle^{5/2}} \frac{1}{\langle s\rangle^{\frac{3}{2}}} |\tilde{\alpha}_0||\tilde{\beta}_1|\,.
\end{aligned} \tag{11.19}$$

The latter integral requires detailed estimation on different time scales.

To estimate the last integral, we split the range of integration into three regions:

$$\begin{aligned}
I_0 &\equiv \{s : 0 \le s \le t_0\}\\
I_1 &\equiv \{s : t_0 < s \le t_1\}\\
I_2 &\equiv \{s : t_1 < s \le T\}\,.
\end{aligned} \tag{11.20}$$

Estimate on I_0: Assume $t \le t_0$. Recall that by (8.5), $|\tilde{\alpha}_0(s)| \le \mathcal{O}(\mathcal{E}_0)\langle s\rangle^{-1}$. Using this we have

$$\begin{aligned}
&\mathcal{O}(\tilde{\alpha}_0^2\tilde{\beta}_1)[\eta_0]_X \int_0^t \frac{1}{\langle t-s\rangle^{5/2}} \frac{1}{\langle s\rangle^{\frac{3}{2}}} |\tilde{\alpha}_0|\ |\tilde{\beta}_1| ds\\
&\quad = \mathcal{O}(|\tilde{\alpha}_0|^{\frac{3}{2}})\big(4\Gamma \tilde{\alpha}_0^{\frac{1}{2}} \tilde{\beta}_1\big)\left(\frac{[\eta_0]_X}{4\Gamma} \int_0^t \frac{1}{\langle t-s\rangle^{5/2}} \frac{1}{\langle s\rangle^{\frac{3}{2}}} |\tilde{\alpha}_0|\ |\tilde{\beta}_1| ds\right)\\
&\quad \le \mathcal{O}(\mathcal{E}_0^{\frac{3}{4}})\big[2\Gamma P_0 P_1^2 + [\eta_0]_X^{4/3} \langle t\rangle^{-\frac{5}{2}\cdot\frac{4}{3}}\big],
\end{aligned}$$

where we have used that $ab \le a^4/4 + 3b^{\frac{4}{3}}/4$.

Estimate on I_1: Let t be such that $t_0 \le t \le t_1$. We break the integral into an integral over $[0, t_0]$ plus an integral over $[t_0, t]$. Recall the definitions of $Q_j(t),\ j = 0, 1$ in terms of $|\tilde{\alpha}_j(t)$ displayed in (9.7). Using that

- for $s \in I_0$, $|\tilde{\alpha}_0(s)| \le \mathcal{O}(\mathcal{E}_0)\langle s\rangle^{-1}$ and for
- $s \in I_1$, Q_0 is increasing and $Q_0(s) \le \mathcal{E}_0^r Q_1(s)$,

we have

$$\mathcal{O}(|\tilde\alpha_0(t)|^2|\tilde\beta_1(t)|)[\eta_0]_X \left(\int_0^{t_0}+\int_{t_0}^t\right)\frac{1}{\langle t-s\rangle^{5/2}\langle s\rangle^{3/2}}|\tilde\beta_1(s)|\;|\tilde\alpha_0(s)|ds$$

$$\le I_0 \text{ type bound} + \mathcal{O}(|\tilde\alpha_0|^2\;|\tilde\beta_1|)\;[\eta_0]_X\int_{t_0}^t\frac{|\tilde\beta_1(s)||\tilde\alpha_0(s)|}{\langle t-s\rangle^{5/2}\langle s\rangle^{3/2}}ds$$

$$\le I_0 \text{ type bound} + \mathcal{O}(|\tilde\alpha_0|^3|\tilde\beta_1|\|\tilde\beta_1\|_\infty)[\eta_0]_X\int_{t_0}^t\frac{1}{\langle t-s\rangle^{5/2}\langle s\rangle^{3/2}}ds$$

$$= I_0 \text{ type bound} + \mathcal{O}\big(\mathcal{E}_0^{\frac12}|\tilde\alpha_0|\cdot|\tilde\alpha_0|\cdot|\tilde\alpha_0||\tilde\beta_1|\big)[\eta_0]_X\langle t\rangle^{-\frac32}ds$$

$$= I_0 \text{ type bound} + \mathcal{O}\big(\mathcal{E}_0\cdot Q_0^{\frac12}\cdot\mathcal{E}_0^{\frac{r}{2}}Q_1\big)[\eta_0]_X\langle t\rangle^{-\frac32}$$

$$= I_0 \text{ type bound} + \mathcal{O}\big(\mathcal{E}_0^{\frac{r+2}{2}}\big)Q_0^{\frac12}Q_1\langle t\rangle^{-\frac32}$$

$$= I_0 \text{ type bound} + \mathcal{O}\big(\mathcal{E}_0^{\frac{r+2}{2}}\big)(Q_0Q_1^2+\langle t\rangle^{-3})\,, \tag{11.21}$$

which is a bound of the type in (11.8).

Estimate on I_2:

$$\mathcal{O}(\tilde\alpha_0^2\tilde\beta_1(t))[\eta_0]_X\int_0^t\frac{|\tilde\alpha_0||\tilde\beta_1|}{\langle t-s\rangle^{5/2}\langle s\rangle^{3/2}}ds$$

$$= \mathcal{O}(\tilde\alpha_0^2\tilde\beta_1(t)[\eta_0]_X)\left(\int_0^{t_0}+\int_{t_0}^{t_1}+\int_{t_1}^t\right)\frac{|\tilde\alpha_0||\tilde\beta_1|}{\langle t-s\rangle^{5/2}\langle s\rangle^{3/2}}\;ds$$

$$= I_0\,\&\,I_1 \text{ type bounds} + \mathcal{O}(\tilde\alpha_0^2\tilde\beta_1(t))[\eta_0]_X\int_{t_1}^t\frac{|\tilde\alpha_0||\tilde\beta_1|}{\langle t-s\rangle^{5/2}\langle s\rangle^{3/2}}ds\,. \tag{11.22}$$

The latter integral must be treated differently from the previous terms. For $t\in I_1$, we used that Q_0 monotonically increasing and bounded by a small constant times Q_1 to treat terms perturbatively. On I_2, Q_0 dominates Q_1 (which decays) and we must use a different argument. We return to the expression from which the last term in (11.22) is derived:

$$\mathcal{O}(\tilde\alpha_0)\mathcal{O}(\overline{\tilde\alpha_0}\tilde\beta_1)\left(\langle\chi,\Psi_0\Psi_1\eta_0\rangle - i\left\langle\chi,\int_0^t e^{-i\mathcal{H}_0(t-s)}P_c\tilde{\mathcal{H}}_0\Psi_0\Psi_1\bar\eta_0ds\right\rangle\right). \tag{11.23}$$

We need to expand and estimate the time integral:

$$\int_{t_1}^t e^{-i\mathcal{H}_0(t-s)}P_c\tilde{\mathcal{H}}_0\tilde\alpha_0(s)\tilde\beta_1(s)e^{-i\lambda_+s}\bar\eta_0(s)ds$$

$$= e^{-i\mathcal{H}_0t}\int_{t_1}^t e^{i(\mathcal{H}_0-\lambda_+)s}P_c\mathcal{H}_0\tilde\alpha_0(s)\tilde\beta_1(s)\bar\eta_0(s)ds \tag{11.24}$$

which we do using integration by parts. We carry this out, then take the inner product of the result with a localized function, χ, and then finally multiply by

$\mathcal{O}(|\tilde{\alpha}_0|^2\ |\tilde{\beta}_1|)$. The result is of order

$$|\tilde{\alpha}_0|^3|\tilde{\beta}_1|^2\|\eta_0(t)\|_\infty + |\tilde{\alpha}_0|^3|\tilde{\beta}_1|\left|\left\langle\chi, e^{-i\mathcal{H}_0 t}\tilde{\mathcal{H}}_0\chi\right\rangle\right|$$
$$+\,\mathcal{O}(|\tilde{\alpha}_0|^2|\tilde{\beta}_1|)\int_{t_1}^{t} e^{-i\mathcal{H}_0(t-s)}P_c(\mathcal{H}_0-\lambda_+)^{-1}\tilde{\mathcal{H}}_0\chi e^{-i\lambda_+ s}\frac{d}{ds}\big(\tilde{\alpha}_0(s)\tilde{\beta}_1(s)\bar{\eta}_0(s)\big)ds\,. \tag{11.25}$$

The first two terms in (11.25) are bounded as follows:

$$\begin{aligned}
|\tilde{\alpha}_0|^3|\tilde{\beta}_1|^2\|\eta_0(t)\|_\infty &= \mathcal{O}(\mathcal{E}_0)(|\tilde{\alpha}_0||\tilde{\beta}_1|^2)\|\eta_0(t)\|_\infty\\
&\le \mathcal{O}(\mathcal{E}_0)\left(Q_0Q_1^2 + \|\eta_0(t)\|_\infty^2\right)\\
&\le \mathcal{O}(\mathcal{E}_0)\left(Q_0Q_1^2 + \langle t\rangle^{-3}\right)\\
|\tilde{\alpha}_0|^3\ |\tilde{\beta}_1|\left|\left\langle\chi, e^{-i\mathcal{H}_0 t}\tilde{\mathcal{H}}_0\chi\right\rangle\right| &\le C|\tilde{\alpha}_0|^{5/2}(|\tilde{\alpha}_0|^{1/2}|\tilde{\beta}_1|)\langle t\rangle^{-5/2}\\
&\le \mathcal{O}(\mathcal{E}_0^{5/2})\big(Q_0Q_1^2 + \langle t\rangle^{-\frac{5}{2}\cdot\frac{4}{3}}\big)\\
&\le \mathcal{O}(\mathcal{E}_0^{5/2})\big(Q_0Q_1^2 + \langle t\rangle^{-3}\big)\,.
\end{aligned} \tag{11.26}$$

We now turn to the nonlocal term, in (11.25), which we denote $\mathcal{I}_1$:

$$\mathcal{I}_1 \sim \mathcal{O}(|\tilde{\alpha}_0|^2|\tilde{\beta}_1|)\int_{t_1}^{t} e^{-i\mathcal{H}_0(t-s)}P_c(\mathcal{H}_0-\lambda_+)^{-1}\tilde{\mathcal{H}}_0\chi e^{-i\lambda_+ s}$$
$$\times\big(\partial_s\tilde{\alpha}_0(s)\tilde{\beta}_1(s)\bar{\eta}_0(s) + \tilde{\alpha}_0(s)\partial_s\tilde{\beta}_1(s)\bar{\eta}_0(s) + \tilde{\alpha}_0(s)\tilde{\beta}_1(s)\partial_s\bar{\eta}_0(s)\big)\,. \tag{11.27}$$

Using that

$$\partial_t\tilde{\alpha}_0 \sim \mathcal{O}(\tilde{\beta}_1^2\tilde{\alpha}_0)\,,\quad \partial_t\tilde{\beta}_1 \sim \mathcal{O}(\tilde{\alpha}_0\tilde{\beta}_1^2)\,,\quad \text{and}\quad \partial_t\eta_0 = \big(-i\mathcal{H}_0 + \mathcal{O}(\langle t\rangle^{-1})\big)\eta_0\,, \tag{11.28}$$

(see also Remark 11.1) we have

$$\begin{aligned}
\mathcal{I}_1 &\sim \mathcal{O}(|\tilde{\alpha}_0|^2|\tilde{\beta}_1|)\int_{t_1}^{t} e^{-i\mathcal{H}_0(t-s)}P_c(\mathcal{H}_0-\lambda_+)^{-1}\tilde{\mathcal{H}}_0\chi e^{-i\lambda_+ s}\\
&\quad\times\big(\tilde{\alpha}_0\tilde{\beta}_1^3\bar{\eta}_0 + \tilde{\alpha}_0^2\tilde{\beta}_1^2\bar{\eta}_0(s) + \tilde{\alpha}_0(s)\tilde{\beta}_1(s)\tilde{\mathcal{H}}_0\bar{\eta}_0(s)\big)\\
&= \mathcal{I}_{1a} + \mathcal{I}_{1b} + \mathcal{I}_{1c}\,.
\end{aligned} \tag{11.29}$$

Each of the three terms $\mathcal{I}_{1j},\ j = a, b, c$ satisfies a bound of the form:

$$|\mathcal{I}_{1j}| \le C\mathcal{E}_0^2\ [\eta_0(0)]_X\frac{1}{\langle t_0\rangle}\frac{1}{\langle t-t_1\rangle^2}\,. \tag{11.30}$$

We illustrate this by estimating $\mathcal{I}_{1a}$; the other two terms are estimated similarly.

Using that $|\tilde\alpha_0|,\ |\tilde\beta_1| = \mathcal{O}(\mathcal{E}_0^{\frac12})$ we have

$$|\mathcal{I}_{1a}| \le |\tilde\alpha_0|^2|\tilde\beta_1| \int_{t_1}^t \frac{1}{\langle t-s\rangle^{\frac52}} |\tilde\beta_1|^3|\tilde\alpha_0||\eta_0|ds$$

$$\le \mathcal{E}_0^2[\eta_0(0)]_X \left(\sup_{t\ge t_1}\langle t-t_1\rangle^{\frac12}|\tilde\beta_1(t)|\right)^3 \int_{t_1}^t \frac{1}{\langle t-s\rangle^{\frac52}}\frac{1}{\langle s-t_1\rangle^{\frac32}}\frac{1}{\langle s\rangle^{\frac32}}\,.$$

Separate estimation of the contributions from the intervals $[t_1, \frac12(t+t_1)]$ and $[\frac12(t+t_1), t]$ yields the bound:

$$|\mathcal{I}_{1a}| \le C\mathcal{E}_0^2[\eta_0(0)]_X \left(\sup_{t\ge t_1}\langle t-t_1\rangle^{\frac12}|\tilde\beta_1(t)|\right)^3 \frac{1}{\langle t_0\rangle}\frac{1}{\langle t-t_1\rangle^2}\,. \tag{11.31}$$

<u>*Estimation of* $T2$</u>: Consider the term

$$\mathrm{T2} = \mathcal{O}(\tilde\alpha_0(t))\mathcal{O}(\tilde\alpha_0(t)\tilde\beta_1(t)) \left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)}P_c\eta_b^2(s)\eta_0(s)\right\rangle ds\,. \tag{11.32}$$

<u>$t\in I_1 \equiv [t_0,t_1]$</u>: For $t\in I_1$,

$$|\tilde\alpha_0|^2|\tilde\beta_1| \le c\mathcal{E}_0^{\frac{r}{2}}|\tilde\alpha_0||\tilde\beta_1|^2;\ \text{Proposition 9.4}\,. \tag{11.33}$$

Therefore,

$$\begin{aligned}|\mathrm{T2}| &\le c\mathcal{E}_0^{\frac{r}{2}}|\tilde\alpha_0||\tilde\beta_1|^2 \left|\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)}P_c\eta_b^2\eta_0 ds\right\rangle\right|\\
&\le c\mathcal{E}_0^{\frac{r}{2}}|\tilde\alpha_0||\tilde\beta_1|^2\langle t\rangle^{-3/2}\left(\sup_{0\le s\le t}\langle s\rangle^{\frac32}\|\eta_b^2(s)\eta_0(s)\|_{W^{k,2}\cap L^1}\right)\\
&\le c\mathcal{E}_0^{\frac{r}{2}}|\tilde\alpha_0||\tilde\beta_1|^2\langle t\rangle^{-3/2}\|\eta_b\|^2_{W^{k,2}}[\eta_0]_X\\
&\le c\mathcal{E}_0^{\rho}[\eta_0]_X|\tilde\alpha_0||\tilde\beta_1|^2\langle t\rangle^{-3/2}\\
&\le c\mathcal{E}_0^{\rho}[\eta_0]_X\left(P_0P_1^2 + \langle t\rangle^{-3}\right).\end{aligned}$$

<u>$t\in I_2\equiv[t_1,\infty)$</u>: To see the relevant terms for $t>t_1$, we integrate by parts and obtain, besides easily estimable local terms and terms with faster time decay,

$$O(\tilde\alpha_0^2\tilde\beta_1)\left\langle\chi, \int_0^t e^{-i\mathcal{H}_0(t-s)}\tilde{\mathcal{H}}_0P_c\eta_b^2\eta_0 ds\right\rangle \qquad t>t_1\,. \tag{11.34}$$

Consider the contribution to the integral in (11.34) coming from $s\in[0,t_0]$:

$$\mathcal{O}\big(\tilde\alpha_0^2(t)\beta_1(t)\big)\left\langle\chi, \int_0^{t_0} e^{-i\mathcal{H}_0(t-s)}\tilde{\mathcal{H}}_0P_c\eta_b^2\eta_0 ds\right\rangle \qquad t>t_1>t_0\,. \tag{11.35}$$

Consider, the inner product

$$R(t) = \left\langle\chi, \int_0^{t_0} e^{-i\mathcal{H}_0(t-s)}\tilde{\mathcal{H}}_0P_c\eta_b^2\eta_0 ds\right\rangle\,. \tag{11.36}$$

We have

$$|(11.35)| \leq C|\tilde{\alpha}_0(t)|^2|\tilde{\beta}_1(t)||R(t)| \sim CQ_0(t)Q_1^{\frac{1}{2}}(t)|R(t)| \tag{11.37}$$

and therefore it suffices to prove that

$$|R(t)| \leq CQ_1^{\frac{3}{2}}(t)\,. \tag{11.38}$$

To prove (11.38), recall that for $t \geq t_1$

$$\begin{aligned} \frac{dQ_0}{dt} &\asymp \Gamma Q_0 Q_1^2 \\ \frac{dQ_1}{dt} &\asymp -2\Gamma Q_0 Q_1^2\,, \end{aligned} \tag{11.39}$$

and

$$\mathcal{E}_0^r Q_1(t_1) = Q_0(t_1)\,. \tag{11.40}$$

Therefore,

$$Q_1(t) \asymp \frac{2Q_1(t_1)}{1 + \Gamma \int_{t_1}^t Q_0(s)ds}\,. \tag{11.41}$$

Therefore, to establish (11.38) we need:

$$|R(t)| \leq C\left(\frac{2Q_1(t_1)}{1 + \Gamma \int_{t_1}^t Q_0(s)ds}\right)^{\frac{3}{2}}\,. \tag{11.42}$$

We consider two cases

Case 1: $\Gamma \sup_{s\in[t_1,t]} |Q_0(s)||t - t_1| \leq 1$, and

Case 2: $\Gamma \sup_{s\in[t_1,t]} |Q_0(s)||t - t_1| \geq 1$, where it suffices to prove

$$|R(t)| \leq \left(\frac{3Q_1(t_1)}{\Gamma \int_{t_1}^t Q_0(s)ds}\right)^{\frac{3}{2}}\,. \tag{11.43}$$

In Case 2 we prove the bound on $R(t)$, (11.38), while in Case 1 we prove that the expression (11.34) of order $\langle t - t_1\rangle^{-\frac{3}{2}}\langle t - t_0\rangle^{-\frac{3}{2}}$. We first handle Case 2, the bound (11.43). From (11.39) we have

$$\begin{aligned} \frac{d}{dt}(Q_0 + 2Q_1) &\asymp 0\,, \qquad t \geq t_1 \\ Q_0(t) + 2Q_1(t) &\asymp Q_0(t_1) + 2Q_1(t_1) \\ &= (2 + \mathcal{E}_0^r)Q_1(t_1)\,. \end{aligned} \tag{11.44}$$

Therefore, since

$$Q_1(t) \asymp \frac{2Q_1(t_1)}{1 + \Gamma \int_{t_1}^t Q_0(s)ds} \tag{11.45}$$

we have, as $t \to \infty$

$$\begin{aligned} Q_0(t) &\asymp (2+\mathcal{E}_0^r)Q_1(t_1) - 2Q_1(t) \\ &\to (2+\mathcal{E}_0^r)Q_1(t_1)\,. \end{aligned} \tag{11.46}$$

Therefore, for $t > t_1$ ($t - t_1$ large enough)

$$(1+\mathcal{E}_0^r)Q_1(t_1) \le Q_0(t) \le (2+\mathcal{E}_0^r)Q_1(t_1) \tag{11.47}$$

and therefore

$$\begin{aligned} \frac{Q_1(t_1)}{1+\Gamma\int_{t_1}^t Q_0(s)ds} &\ge \frac{Q_1(t_1)}{1+\Gamma(2+\mathcal{E}_0^r)|t-t_1|Q_1(t_1)} \\ &\ge \frac{1}{3}\frac{1}{\Gamma|t-t_1|} \ge \frac{1}{3}\frac{1}{\Gamma|t-t_0|}\,. \end{aligned} \tag{11.48}$$

On the other hand,

$$\begin{aligned} |R(t)| &\le C\left\langle \chi, \int_0^{t_0} \frac{1}{\langle t-s\rangle^{\frac{3}{2}}}\|\tilde{\mathcal{H}}_0\eta_b^2(s)\eta_0(s)\|_{L^1}ds\right\rangle \\ &\le C\sup_{\tau\in[0,t_0]}\|\eta_b(\tau)\|_{H^2}^2[\eta_0]_X\int_0^{t_0}\frac{1}{\langle t-s\rangle^{\frac{3}{2}}\langle s\rangle^{\frac{3}{2}}}ds \\ &\le C\mathcal{E}_0^2[\eta_0]_X\frac{1}{\langle t-t_0\rangle^{\frac{3}{2}}}\,. \end{aligned} \tag{11.49}$$

The bounds (11.49) and (11.48) imply (11.43).

We now turn to Case 1. In this case, $\Gamma\sup_{s\in[t_1,t]}Q_0(s)|t-t_1| \le 1$,

$$Q_0(t) \le \frac{1}{\Gamma}\frac{1}{|t-t_1|} \tag{11.50}$$

and therefore by (11.34) and (11.50)

$$|(11.34)| \le \mathcal{E}_0^2[\eta_0]_X\sup_{s\in[t_1,t]}\left(\langle s-t_1\rangle^{\frac{1}{2}}Q_1(s)\right)\frac{1}{|t-t_1|}\frac{1}{\langle t-t_1\rangle^{\frac{1}{2}}}\frac{1}{\langle t-t_0\rangle^{\frac{3}{2}}}\,. \tag{11.51}$$

We begin by noting that by Proposition 10.10 and its corollaries, for $p > 6$

$$\begin{aligned} \|\eta_b\|_{W^{k,p}} &\le \mathcal{O}(\mathcal{E}_0)\langle t\rangle^{-1}\,, \qquad t \le t_0 \\ \|\eta_b\|_{W^{k,p}} &\le \mathcal{O}(\mathcal{E}_0^\rho)Q_0(t)^{1/2}\,, \qquad t_0 < t \le t_1 \\ \|\eta_b\|_\infty &= \mathcal{O}(\langle t-t_1\rangle^{-\frac{1}{2}})\,, \qquad t \ge t_1\,. \end{aligned} \tag{11.52}$$

For $t < t_1$, the previous arguments with the known estimates on η_b, and the facts that $|\tilde{\alpha}_0(t)| \le \mathcal{O}(\mathcal{E}_0)/\langle t\rangle$ on I_0 and $Q_0(t) \le \mathcal{E}_0^r Q_1$ on I_1 imply the necessary bounds. Collecting all these, we have

$$\mathcal{O}(\tilde{\alpha}_0^2\tilde{\beta}_1)\left|\left\langle \chi, \int_0^t e^{i\mathcal{H}_0(t-s)}P_c\tilde{\mathcal{H}}_0\eta_b^2\eta_0 ds\right\rangle\right| \le \mathcal{O}(\mathcal{E}_0)[2\Gamma Q_0Q_1^2 + \langle t_0\rangle^{-1}\langle t\rangle^{-2}]\,.$$

Proof of estimate (11.8) *for* $R_0^{(1)}$: The key terms to consider in the P_1 equation are the slowest decaying nonlocal terms; see (7.39) and (7.32). Since b_1 enters as b_1^2 in (9.22), we need to bound $R_0^{(1)}$ by $\mathcal{O}(\mathcal{E}_0)[\langle t_0\rangle^{-1/2}\langle t\rangle^{-2} + 2\Gamma P_0 P_1^2]$ for $t > t_0$.

The slowest decaying nonlocal term in the β_1 equation arises from the balance:

$$i\partial_t\beta_1 \sim \lambda\langle\psi_{1*}^3, \pi_1\Phi_2\rangle|\beta_1|^2 e^{i\lambda_+(T)t}\,. \tag{11.53}$$

Since the P_1 equation is obtain by multiplying by $\overline{\beta_1}$ and taking the imaginary part, we must estimate:

$$\lambda\langle\psi_{1*}^3, \pi_1\Phi_2\rangle\ |\beta_1|^2\overline{\beta_1}e^{i\lambda_+(T)t}\,, \tag{11.54}$$

the leading order nonlocal part of which is

$$\sim |\tilde\beta_1|^2\overline{\tilde\beta_1}\left\langle\chi, \int_0^t e^{-i\mathcal{H}_0(t-s)}P_c\Psi_0\Psi_1\eta_0 ds\right\rangle e^{i\lambda_+ t}$$

$$\sim |\tilde\beta_1|^2\overline{\tilde\beta_1}\left\langle\chi, \int_0^t e^{-i\mathcal{H}_0(t-s)}P_c\tilde\alpha_0\tilde\beta_1 e^{-i\lambda_+ s}\chi\eta_0(s) ds\right\rangle e^{i\lambda_+ t}\,.$$

Integrating by parts (using the oscillatory factor $e^{i\lambda_+ t}$) and removing the non-resonant local in terms by near identity transformations the key term is

$$|\tilde\beta_1|^2\overline{\beta_1}\left\langle\chi, \int_0^t e^{-i\mathcal{H}_0(t-s)}\tilde{\mathcal{H}}_0 P_c\tilde\alpha_0\tilde\beta_1\chi e^{i\lambda_+ s}\eta_0(s) ds\right\rangle e^{i\lambda_+ t}\,. \tag{11.55}$$

Suppose $t > t_1$. Since we have no factor of $\tilde\alpha_0$ outside the integral (local in time $\alpha_0(t)$ factor), the estimate for $I_0 = \{0 < s < t_0\}$ is the critical part and requires the $\mathcal{O}(\langle t\rangle^{-1})$ bound on I_0 and Theorem 4.3:

$$\begin{aligned}
&\left|\left\langle\chi, \int_0^t e^{-i\mathcal{H}_0(t-s)}\tilde{\mathcal{H}}_0 P_c\tilde\alpha_0\tilde\beta_1\chi\eta_0 e^{-i\lambda_+ s} ds\right\rangle\right| \\
&\quad\le C[\eta_0]_X\int_0^{t_0}\frac{ds}{\langle t-s\rangle^{5/2}\langle s\rangle^{3/2}}|\tilde\alpha_0(s)||\tilde\beta_1(s)| + \left(\int_{t_0}^{t_1} + \int_{t_1}^{t}\right)\cdots ds \\
&\quad\le \mathcal{O}(\mathcal{E}_0)[\eta_0]_X\int_0^{t_0}\frac{ds}{\langle t-s\rangle^{5/2}\langle s\rangle\langle s\rangle^{3/2}} \\
&\qquad + [\eta_0]_X\int_{t_0}^{t_1}\frac{ds}{\langle t-s\rangle^{5/2}\langle s\rangle^{3/2}}|\tilde\alpha_0(t_1)||\tilde\beta_1(s)| + \int_{t_1}^{t}\cdots ds \\
&\quad\le \mathcal{O}(\mathcal{E}_0)\big[\langle t\rangle^{-5/2}[\eta_0]_X + \mathcal{O}(\mathcal{E}_0)[\eta_0]_X|\tilde\alpha_0(t)|\langle t\rangle^{-3/2} \\
&\qquad + [\eta_0]_X|\tilde\alpha_0(t)|\langle t-t_1\rangle^{-\frac{1}{2}}\langle t\rangle^{-\frac{3}{2}}\big]\,.
\end{aligned}$$

Note that we can extract the factor $|\tilde\alpha(t)|$ from the integral for $s \ge t_0$ since in this range $Q_0(t)$ is monotonically increasing and $Q_0(t) \sim P_0(t) \sim |\tilde\alpha_0(t)|$ with correction terms which are rapidly decaying in time and which are therefore dominated by the first term. Multiplication by $\mathcal{O}(|\tilde{\beta}_1|^3)$ prefactor gives the bound

$$C(\mathcal{E}_0, [\eta_0]_X)\left[\left(\sup_{s\geq t_1}\langle s-t_1\rangle^{\frac{1}{2}}|\tilde{\beta}_1(s)|\right)^3\langle t\rangle^{-5/2}\langle t-t_1\rangle^{-3/2}+Q_0Q_1^2\right]. \tag{11.56}$$

This completes the proof of Proposition 11.2. □

12. Local and Nonlocal ODE Terms: $R_2^{(j)}(P_0, P_1)$ of Proposition 11.1

In this section we prove estimates on the terms $R_2^{(0)}(P_0,P_1)$ and $R_2^{(1)}(P_0,P_1)$ of Proposition 11.1.

Proposition 12.1. *Assume either $t\leq t_0$ or monotonicity property Q on $[t_0, t_0+\delta_*]$. Then, for $m\geq 4$,*

(i) $|R_2^{(0)}(P_0,P_1)|\leq \mathcal{O}(\mathcal{E}_0^\rho)\Gamma P_0P_1^2+\mathcal{O}(\langle t\rangle^{-3})(P_0,P_1\ll 1)$
(ii) $|R_2^{(1)}(P_0,P_1)|\leq \mathcal{O}(\mathcal{E}_0^\rho)\Gamma P_0P_1^2+\mathcal{O}(\alpha_0)P_1^{2m}+\mathcal{O}(\langle t\rangle^{-3})$, $|\alpha_0|\sim\sqrt{P_0}$.

12.1. *Proof of part (1) of Proposition 12.1*

The most problematic terms are nonlocal, slowest decaying. The terms which are linear in η in the α_0 equation contribute the slowest terms; these are nonlocal in time, t.

We have to consider terms arising in Eq. (7.52) of the type:

$$\langle\chi,\eta\rangle e^{-i\lambda_+t}\overline{\alpha_0}\beta_1,\ \langle\chi,\eta\rangle|\beta_1|^2,\ \langle\chi,\eta\rangle\beta_1^2e^{-2i\lambda_+t}, \tag{12.1}$$

where $\phi_2\sim\eta$ and $\eta(t)=\eta_0(t)+e_0(t)+\eta_b(t)$; see (10.1).

As calculated earlier, the last term is resonant and its contribution is $+2\Gamma P_0P_1^2$ to the P_0 equation; see (7.55).

The leading ODE terms in η are the source terms of the type in (7.32):

$$|\Psi_0|^2\Psi_1,\Psi_0^2\overline{\Psi_1},\Psi_0|\Psi_1|^2,\overline{\Psi_0}\Psi_1^2. \tag{12.2}$$

Recall that the P_0 equation is obtained from the $\tilde{\alpha}_0$ equation by multiplication by $\overline{\tilde{\alpha}_0}$ and taking the imaginary part.

Solving for η_b and plugging the source term contributions into the $\langle\chi,\eta\rangle$ terms in the P_0 equation gives, apart from the resonant term, terms of the type

$$\overline{\alpha_0}\mathcal{O}(\bar{\alpha}_0\beta_1e^{i\lambda_+t}+|\beta_1|^2)\Big\langle\chi,\int_0^t e^{-i\mathcal{H}_0(t-s)}P_c\big(|\Psi_0|^2\Psi_1 + \Psi_0^2\bar{\Psi}_1+\Psi_0|\Psi_1|^2+\bar{\Psi}_0\Psi_1^2\big)ds\Big\rangle. \tag{12.3}$$

For $t>t_1$ the slowest terms are $\mathcal{O}(|\Psi_0|^2\Psi_1+\Psi_0^2\bar{\Psi}_1)$ source terms. For $t_0<t<t_1$ the problematic terms are $\mathcal{O}(\Psi_0|\Psi_1|^2+\bar{\Psi}_0\Psi_1^2)$ source terms, since on this interval

β_1 does not necessarily decay. Since $\Psi_1 \sim e^{-i\lambda_+ t}\beta_1$ + higher order terms, we can integrate by parts the $\mathcal{O}(|\Psi_0|^2\Psi_1 + \Psi_0^2\bar{\Psi}_1)$ terms in (12.3) to get arbitrary order β_1^m terms, which are nonlocal. The remaining terms are local and its lowest order term is of the order

$$\overline{\alpha_0}\mathcal{O}(\overline{\alpha_0}e^{i\lambda_+ t}\beta_1 + |\beta_1|^2)\left\langle \chi, \mathcal{O}(|\alpha_0|^2\beta_1 e^{i\lambda_+ t} + \alpha_0^2\bar{\beta}_1 e^{-i\lambda_+ t})\right\rangle . \tag{12.4}$$

The nonresonant local terms can be transformed to higher order by a near identity change of variables giving

$$\mathcal{O}(|\alpha_0|^3\beta_1^{2m}e^{i\Omega t}),\ \Omega > 0 \text{ and } \ m \text{ arbitrarily large}\,,$$

while resonant terms are of the type already derived but of higher order (bounded by $\mathcal{O}(\mathcal{E}_0)Q_0Q_1^2)$). We are therefore left to consider the following nonlocal terms

$$\bar{\alpha}_0\mathcal{O}\big(\bar{\alpha}_0\beta_1 e^{-i\lambda_+ t} + |\beta_1|^2\big)\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)}P_c\alpha_0\beta_1^{2m}e^{i\Omega_m s}\tilde{\chi}ds\right\rangle . \tag{12.5}$$

Consider the first term in (12.5). Due to the oscillatory factor $e^{-i\lambda_+ t}$ in the $\mathcal{O}(\bar{\alpha}_0\beta_1 e^{-i\lambda_+ t})$ term we can, by a near-identity transformation, in the α_0 equation, remove this term in exchange for one higher order

$$\mathcal{O}(|\alpha_0|^2\beta_1^3)\left\langle \chi, \int_0^t \cdots\right\rangle + \mathcal{O}(\bar{\alpha}_0^2\beta_1 e^{i\Omega s})\partial_t\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)}P_c\cdots\right\rangle$$
$$+ \text{higher order} + \text{local terms}\,,$$

and by further near identity transformations, we are left with the following slowest term

$$\mathcal{O}(|\alpha_0|^2\beta_1)\partial_t^m\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)}P_c\cdots ds\right\rangle . \tag{12.6}$$

All other terms are of order $o(\mathcal{E}_0)\mathcal{O}(P_0P_1^2)$, $o(\mathcal{E}_0)\langle t\rangle^{-3}$ or higher order, for $\mathcal{E}_0$ small.

$$\partial_t\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)}P_c\ \alpha_0\beta_1^{2m}\tilde{\chi}ds\right\rangle$$
$$= \left\langle \chi, \alpha_0(t)\beta_1^{2m}(t)\tilde{\chi}\right\rangle - \left\langle \chi, i\int_0^t e^{-i\mathcal{H}_0(t-s)}\tilde{\mathcal{H}}_0P_c\tilde{\chi}\alpha_0\beta_1^{2m}e^{i\Omega s}ds\right\rangle .$$

The first term is local and will contribute $o(1)\mathcal{O}(P_0P_1^2)$. It remains to estimate the nonlocal term. Note that

$$\int_0^t e^{-i\mathcal{H}_0(t-s)}e^{i\omega s}\tilde{\chi}\alpha_0\beta_1^{2m}ds = e^{-i\bar{\omega}t}\int_0^t e^{-i(\mathcal{H}_0-\bar{\omega})(t-s)}e^{i(\omega+\bar{\omega})s}\tilde{\chi}\alpha_0\beta_1^{2m}ds\,.$$

Consider now the second term in (12.5). Using the oscillatory factor $e^{-i\bar{\omega}t}$, one can transform the $\mathcal{O}(\bar{\alpha}_0|\beta_1|^2)\langle\chi, \int\cdots\rangle$ to higher order.

We now turn to the second term in (12.5). First, let us consider the case where $t \geq t_1$:

$$\mathcal{O}(\bar{\alpha}_0|\beta_1|^2)\partial_t^m\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)}\tilde{\chi}\alpha_0\beta_1^{2m}ds\right\rangle . \tag{12.7}$$

We consider (12.7) for $t \geq t_1$. For this we require the following:

Lemma 12.1. *For $t > t_1$,*

$$Q_1(t) \geq Q_1(t_1)\left[1 + 4(\Gamma' + \delta)Q_1(t_1)Q_0(t_1)(t - t_1)\right]^{-1} . \tag{12.8}$$

Proof of Lemma 12.1. For $t \geq t_1$

$$\frac{dQ_1}{dt} \geq -4(\Gamma' + \delta)Q_0 Q_1^2 . \tag{12.9}$$

Therefore,

$$-\frac{dQ_1^{-1}}{dt} \geq -4(\Gamma' + \delta)Q_0(t) \geq -4(\Gamma' + \delta)Q_0(t_1) \text{ or } \frac{dQ_1^{-1}}{dt} \leq 4(\Gamma' + \delta)Q_0(t_1) \tag{12.10}$$

since $Q_0(t_1) \leq Q_0(t)$ ($Q_0 \uparrow$ for $t > t_0$). Integrating from t to t_1, we get

$$Q_1(t)^{-1} - Q_1(t_1)^{-1} \leq 4(\Gamma' + \delta)Q_0(t_1)(t - t_1) , \tag{12.11}$$

which is equivalent (12.8). This completes the proof of Lemma 12.1. □

Estimation of (12.7) *for $t \geq t_1$*: Carrying out the differentiation in (12.7) we find that it suffices to bound terms of the type:

$$\begin{aligned} &\mathcal{O}\big(\alpha_0|\beta_1|^2\big)\langle\chi, P_c\chi\rangle\alpha_0\beta_1^{2m} \\ &\quad = \mathcal{O}(\mathcal{E}_0)Q_0Q_1^2\mathcal{O}(\alpha_0|\beta_1|^2)\left\langle\chi, \int_0^t e^{-i\mathcal{H}_0(t-s)}P_c\tilde{\mathcal{H}}_0^m\tilde{\chi}\alpha_0\beta_1^{2m}ds\right\rangle . \end{aligned} \tag{12.12}$$

We now consider (12.12), which we break into the sum of three integrals:

$$\mathcal{O}(\bar{\alpha}_0|\beta_1|^2)\left\langle\chi, \left(\int_0^{t_0} + \int_{t_0}^{t_1} + \int_{t_1}^{t}\right) e^{-i\mathcal{H}_0(t-s)}P_c\tilde{\mathcal{H}}_0^k\tilde{\chi}\alpha_0\beta_1^{2m}ds\right\rangle . \tag{12.13}$$

$\int_{t_1}^t$: By the local decay estimate for $e^{-i\mathcal{H}_0 t}\mathcal{H}_0^k P_c$ of Theorem 4.3, the integral in (12.13) is bounded above by:

$$\int_{t_1}^t \frac{ds}{\langle t - s\rangle^k}|\alpha_0\beta_1^{2m}| . \tag{12.14}$$

This in turn is bounded above by

$$Q_0(t)^{1/2}\int_{t_1}^t \frac{ds}{\langle t - s\rangle^k}Q_1^m(s) , \tag{12.15}$$

since Q_0 is increasing for $t \geq t_0$.

We now aim to further bound (12.15) by "extracting" powers of $Q_1(t)$ from under the integral. Recall that for $s \geq t_1$

$$\begin{aligned} Q_1(s) &\leq \frac{Q_1(t_1)}{1 + 2\Gamma Q_0(t_1)Q_1(t_1)|s - t_1|} \\ &= \frac{1}{2\Gamma Q_0(t_1)}\left[\frac{Q_1(t_1)Q_0(t_1)2\Gamma}{1 + 2\Gamma Q_0(t_1)Q_1(t_1)|s - t_1|}\right] \\ &\leq \min\{Q_1(t_1), 2\Gamma^{-1}Q_0^{-1}(t_1)\langle s - t_1\rangle^{-1}\} \end{aligned} \tag{12.16}$$

and

$$Q_0^{-1}(t_1) \equiv Q_1(t_1)^{-1}\mathcal{E}_0^{-r}\,, \qquad \text{and} \qquad Q_1 \le \mathcal{E}_0\,. \tag{12.17}$$

We write

$$\begin{aligned} Q_1^m(s) &= Q_1^{m-\bar{k}}(s)Q_1^{\bar{k}}(s) \le Q_1^{m-\bar{k}}(s)\frac{1}{\langle s-t_1\rangle^{\bar{k}}}\frac{1}{(2\Gamma Q_0(t_1))^{\bar{k}}} \\ &= Q_1^{m-\bar{k}}(s)\frac{1}{\langle s-t_1\rangle^{\bar{k}}}\frac{\mathcal{E}_0^{-r\bar{k}}}{(2\Gamma Q_1(t_1))^{\bar{k}}} \\ &\le Q_1^{m-2\bar{k}}(s)\mathcal{E}_0^{-r\bar{k}}\langle s-t_1\rangle^{-\bar{k}}\,, \end{aligned} \tag{12.18}$$

where we have used that $Q_1(t_1) \ge Q_1(s)$.

We consider separately the cases $t \ge 2t_1$ and $t_1 \le t \le 2t_1$.

<u>$t \ge 2t_1$</u>: Take $\bar{k} = 2$ and $m > 2(r+2)$. Then, the integral in (12.15) is bounded by

$$\mathcal{E}_0^\rho Q_0(t)^{\frac{1}{2}}\langle t\rangle^{-\min(k,\bar{k})},\ \rho > 0\,. \tag{12.19}$$

This implies, for k sufficiently large,

$$\left|\mathcal{O}(\alpha_0|\beta_1|^2)\left\langle \chi, \int_0^t \cdots \right\rangle\right| \le \mathcal{O}(\mathcal{E}_0^\rho)\big(Q_0Q_1^2 + \langle t\rangle^{-4}\big)\cdot \rho > 0\,. \tag{12.20}$$

<u>$t_1 \le t < 2t_1$</u>: Let $t = t_1 + 2M, M = (t-t_1)/2$ and rewrite the integral as

$$\int_{t_1}^t \frac{ds}{\langle t-s\rangle^k}|\alpha_0\beta_1^{2m}| = \left(\int_{t_1}^{t_1+M} + \int_{t_1+M}^{t_1+2M}\right)\frac{ds}{\langle t-s\rangle^k}|\alpha_0\beta_1^{2m}|\,. \tag{12.21}$$

In a manner similar to the previous estimate, using that Q_1 is decreasing we have, by (12.18):

$$\begin{aligned} &\int_{t_1}^{t_1+M} \frac{ds}{\langle t-s\rangle^k}Q_1(s)^{m/2}\langle s-t_1\rangle^{-\bar{k}} \\ &\quad\le c\langle t-(t_1+M)\rangle^{-k}Q_1(t_1)^{m/2} \\ &\quad\le c\langle t-t_1\rangle^{-k}Q_1(t_1)Q_1(t_1)^{\frac{m}{2}-1} \\ &\quad\le c\langle t-t_1\rangle^{-k}\left[1+4(\Gamma'+\delta)Q_1(t_1)Q_0(t_1)|t-t_1|\right]Q_1(t)Q_1(t_1)^{m/2-1} \\ &\quad\le \mathcal{O}(\mathcal{E}_0)Q_1(t) \end{aligned}$$

for $k \ge 1$. Hence

$$\mathcal{O}(\alpha_0|\beta_1|^2)Q_0^{1/2}\int_{t_1}^{t_1+M} \cdots \le \mathcal{O}(\mathcal{E}_0)Q_0Q_1^2\,. \tag{12.22}$$

Furthermore, by (12.18) the integral over $[t_1+M, t_1+2M]$ is bounded by

$$\int_{t_1+M}^{t_1+2M} \frac{ds}{\langle t-s\rangle^k\langle s-t_1\rangle^{\bar{k}}}Q_1(s)^{m/2} \le cQ_1(t_1)^{m/2}\langle M\rangle^{-\bar{k}} \le cQ_1(t_1)^{m/2}\langle t-t_1\rangle^{-\bar{k}} \tag{12.23}$$

and, as above, using the upper bound (12.8) for $Q_1(t_1)$ in terms of $Q_1(t)$ we have

$$\int_{t_1+M}^{t_1+2M} \cdots \leq \mathcal{O}(\mathcal{E}_0) Q_1(t) \tag{12.24}$$

for $\bar{k} \geq 1$. Therefore, for all $t > t_1$ the nonlocal (and local) ODE terms in the Q_0 equation are bounded by

$$\mathcal{O}(\mathcal{E}_0)[Q_0 Q_1^2 + \langle t \rangle^{-3}]\,, \tag{12.25}$$

provided we control the integral $\int_0^{t_1} \cdots\ ds$.

$\underline{\int_0^{t_1}}$: Consider

$$I(t) \equiv Q_0^{1/2} Q_1 \int_0^{t_1} \frac{ds}{\langle t-s \rangle^k} Q_0^{1/2} Q_1^{m-1} Q_1 ds\,. \tag{12.26}$$

By the mean value theorem, $Q_1(s) = Q_1(s) - Q_1(t) + Q_1(t) = \dot{Q}_1(\bar{s})|t-s| + Q_1(t)$, where $s \leq \bar{s} \leq t$. Then,

$$I(t) \leq Q_0^{\frac{1}{2}}(t) Q_1^2(t) \int_0^{t_1} \frac{ds}{\langle t-s \rangle^k} Q_0^{\frac{1}{2}}(s) Q_1^{m-1}(s) ds \\ + I_1(t)\,, \tag{12.27}$$

where

$$I_1(t) \equiv c Q_0^{\frac{1}{2}}(t) Q_1(t) \int_0^{t_1} \frac{ds}{\langle t-s \rangle^{k-1}} Q_0^{\frac{1}{2}}(s) Q_1^{m-1}(s) Q_0^{\frac{1}{2}}(\bar{s}) Q_1(\bar{s})\,, \tag{12.28}$$

where we have used

$$\dot{Q}_1 = \mathcal{O}\left(Q_0^{\frac{1}{2}} Q_1\right) + \text{h.o.t.} \tag{12.29}$$

If $\bar{s} \leq t_0$, then using that $Q_0^{\frac{1}{2}}(\bar{s}) \leq \mathcal{E}_0^\rho \langle \bar{s} \rangle^{-1} \leq \mathcal{E}_0^\rho \langle s \rangle^{-1}$, we have the bound

$$I_1(t) \leq \mathcal{O}(\mathcal{E}_0^\rho) Q_0^{\frac{1}{2}}(t) Q_1(t) \langle t \rangle^{-2} \leq \mathcal{O}(\mathcal{E}_0^\rho)\left(Q_0(t) Q_1^2(t) + \langle t \rangle^{-4}\right). \tag{12.30}$$

If $\bar{s} > t_0$, then since $Q_0(s)$ is monotonically increasing for $s \geq t_0$, and we have

$$I_1(t) \leq \frac{Q_0(t) Q_1(t)}{\langle t - t_1 \rangle} \int_0^{t_1} \frac{ds}{\langle t-s \rangle^{k-1}} Q_0^{\frac{1}{2}}(s) Q_1^{m-1}(s) Q_1(\bar{s})\,. \tag{12.31}$$

We now expand the latter factor of Q_1 in the integrand using the mean value theorem. Specifically, there exists s' with $t_0 < \bar{s} \leq s' \leq t_1$ such that:

$$\begin{aligned} Q_1(\bar{s}) &= Q_1(t_1) + \dot{Q}_1(s')(s' - t_1) \\ &= Q_1(t_1) + \mathcal{O}\big(Q_0^{\frac{1}{2}}(s') Q_1(s')\big)|s' - t_1| \\ &\leq Q_1(t_1) + \mathcal{O}\big(Q_0^{\frac{1}{2}}(t_1) Q_1(s')\big)|\bar{s} - t_1|\,, \end{aligned} \tag{12.32}$$

where the last inequality follows by monotonicity of Q_0.

Substitution into integrand in (12.32) gives:

$$I_1(t) \le c\mathcal{E}_0^\rho \frac{Q_0(t)Q_1(t)}{\langle t-t_1\rangle} Q_1^{\frac{1}{2}}(t_1) \int_0^{t_1} \frac{ds}{\langle t-s\rangle^{k-3}} Q_0^{\frac{1}{2}}(s) Q_1^{m-1}(s)\,, \tag{12.33}$$

where we have used (12.17) to replace $Q_0(t_1)$ by $Q_1(t_1)$. A higher-order term (one proportional to $Q_1(t_1)$) is subsumed by the constant, c.

From (12.8) we have

$$Q_1(t_1) \le (1 + 4(\Gamma' + \delta)Q_1(t_1)Q_0(t_1)(t-t_1))\, Q_1(t)\,. \tag{12.34}$$

Case 1: $\mathbf{4(\Gamma' + \delta)Q_1(t_1)Q_0(t_1)(t-t_1) \le 100}$. Here, $Q_1(t_1) \le 101Q(t)$, and therefore

$$I_1(t) \le c\mathcal{E}_0^\rho \frac{Q_0(t)Q_1^{\frac{3}{2}}(t)}{\langle t-t_1\rangle} \int_0^{t_1} \frac{ds}{\langle t-s\rangle^{k-3}} Q_0^{\frac{1}{2}}(s) Q_1^{m-1}(s)\,. \tag{12.35}$$

We now split the integral in (12.35) as $\int_0^{t_1} = \int_0^{t_0} + \int_{t_0}^{t_1}$. Using the $\langle s\rangle^{-1}$ decay of $Q_0^{\frac{1}{2}}$ for $s \le t_0$ we have

$$\left| \int_0^{t_0} \cdot \right| \le \mathcal{E}_0^\rho \langle s\rangle^{-1}\,, \tag{12.36}$$

and using the monotonicity of Q_0 for $t_1 \ge s \ge t_0$ and the relation $Q_0(t_1) = \mathcal{E}_0^r Q_1(t_1) \le 101\mathcal{E}_0^r Q_1(t)$ we have

$$\left| \int_{t_0}^{t_1} \cdot\, ds \right| \le \mathcal{E}_0^\rho Q_0^{\frac{1}{2}}(t_1) = \sqrt{101}\mathcal{E}_0^{\rho - \frac{r}{2}} Q_1^{\frac{1}{2}}(t)\,. \tag{12.37}$$

Therefore, choosing m sufficiently large, we get

$$I_1(t) \le c' \left(\mathcal{E}_0^{\rho_1} Q_0(t) Q_1^{\frac{3}{2}}(t)\langle t\rangle^{-1} + \mathcal{E}_0^{\rho_2} Q_0(t) Q_1^2(t) \right) \le c'' \mathcal{E}_0^{\rho_3} (Q_0 Q_1^2 + \langle t\rangle^{-4})\,, \tag{12.38}$$

where $\rho_3 = \min\{\rho_1, \rho_2\}$.

Case 2: $\mathbf{4(\Gamma' + \delta)Q_1(t_1)Q_0(t_1)(t-t_1) \ge 100}$. Then, (12.8) and monotonicity of Q_0 for $t \ge t_0$ implies

$$\frac{1}{\langle t-t_1\rangle} \le 4(\Gamma' + \delta) Q_0(t) Q_1(t)\,. \tag{12.39}$$

This gives

$$I_1(t) \le C\mathcal{E}_0^\rho Q_0(t) Q_1^2(t)\,. \tag{12.40}$$

We now turn to (12.7) for $t \in [t_0, t_1]$. It suffices to estimate the nonlocal terms

$$\mathcal{O}(\alpha_0|\beta_1|^2) \left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \tilde{\mathcal{H}}_0^m \tilde{\chi} \alpha_0 \beta_1^{2m} ds \right\rangle \tag{12.41}$$

for $t_0 \le t \le t_1$. In this region Q_0 and (Q_0/Q_1) are increasing functions. Also $Q_0(t) \le \mathcal{E}_0^r Q_1(t)$. The main difficulty is the need to pull a factor of $Q_1(t)$ out of the nonlocal term. To this end we use the following proposition:

Proposition 12.2. *There exists a constant $\delta > 0$ such that for $t \geq t_0$*

$$2\Gamma \int_{t_0}^{t} Q_0 Q_1^m ds \leq \delta Q_0^{\frac{1}{2}}(t) Q_1(t)^{m-2} + Q_0(t) Q_1^{m-2}(t) + \text{h.o.t.} \tag{12.42}$$

Corollary 12.1.

$$\int_{t_0}^{t} \frac{ds}{\langle t-s\rangle^{3/2}} Q_0^{1/2}(s) Q_1^{m-1}(s) \leq C(1+\delta) Q_0^{\frac{3}{8}} Q_1^{m-\frac{1}{2}} . \tag{12.43}$$

The corollary follows from Proposition 12.2 by the Hölder's inequality.

Proof of Proposition 12.2. Recall that

$$\frac{dQ_0}{dt} = 2\Gamma Q_0 Q_1^2 + R_0 , \qquad \frac{dQ_1}{dt} \leq -4\Gamma Q_0 Q_1^2 + \mathcal{O}(Q_0^{\frac{1}{2}} Q_1^m) . \tag{12.44}$$

Claim.

$$\frac{1}{Q_0^{\frac{1}{2}}} \int_{t_0}^{t} R_0 ds \leq \mathcal{O}(\mathcal{E}_0^\rho) \left[1 + \int_{t_0}^{t} Q_0(s) Q_1^{2m}(s) ds \right] . \tag{12.45}$$

Proof of Claim. The leading order term of R_0, in the variable α_0, which is nonlocal, is a term of the form (12.5). From this term we have after integration by parts to obtain $e^{-i\mathcal{H}_0(t-s)}\mathcal{H}_0$ from (12.5),

$$\begin{aligned}
\int_{t_0}^{t} R_0 &\leq Q_0^{\frac{1}{2}}(t) \int_{t_0}^{t} Q_1(t') \int_{t_0}^{t'} \frac{ds}{\langle t-s\rangle^{5/2}} Q_0^{1/2} Q_1^m ds dt' \\
&\leq Q_0^{\frac{1}{2}}(t) \int_{t_0}^{t} Q_1(t') dt' \left[\int_{t_0}^{t'} \frac{ds}{\langle t-s\rangle^{4-\delta}} + \frac{1}{\langle t-t'\rangle^{1+\delta'}} \int_{t_0}^{t'} Q_0 Q_1^{2m} ds \right] \\
&\leq c Q_0^{\frac{1}{2}}(t) \int_{t_0}^{t'} Q_1(t') dt' \Big[\langle t-t'\rangle^{-3+\delta'} + \langle t-t_0\rangle^{-3+\delta'} \\
&\qquad + \langle t-t'\rangle^{-1-\delta'} \int_{t_0}^{t'} Q_0 Q_1^{2m} ds \Big] \\
&\leq c Q_0^{\frac{1}{2}}(t) \left[\mathcal{O}(\mathcal{E}_0) \left(1 + \int_{t_0}^{t} Q_0 Q_1^{2m} ds \right) \right] ,
\end{aligned}$$

thus proving the claim. □

We first rewrite the right-hand side of (12.42) and use the differential equation for Q_0 and the above claim to integrate by parts:

$$\begin{aligned}
2\Gamma &\int_{t_0}^{t} Q_0 Q_1^m ds \\
&= \int_{t_0}^{t} 2\Gamma Q_0 Q_1^2 Q_1^{m-2} ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_{t_0}^{t} Q_1^{m-2} \frac{d}{ds}\left[Q_0 - \int_{t_0}^{s} R_0\right] ds \\
&= Q_1^{m-2}(t')\left[Q_0 - \int_{t_0}^{s} R_0\right]\Big|_{t_0}^{t} - \int_{t_0}^{t} \frac{d}{ds} Q^{m-2}(s)\left[Q_0(s) - \int_{t_0}^{s} R_0\right] ds \\
&= Q_1^{m-2}(t)Q_0(t) - Q_1^{m-2}(t_0)Q_0(t_0) - Q_1^{m-2}(t)\int_{t_0}^{t} R_0(s)ds \\
&\quad + (m-2)\int_{t_0}^{t} dt' Q_1^{m-1}(t')Q_0(t')\big[4\Gamma Q_0(t') + \mathcal{O}\big(Q_0^{\frac{1}{2}}(t')Q_1^{m-2}(t')\big)\big] \\
&\quad + (m-2)\int_{t_0}^{t} dt' Q_1^{m-1}(t')\big[4\Gamma Q_0(t') + \mathcal{O}\big(Q_0^{\frac{1}{2}}(t')Q_1^{m-2}(t')\big)\big]\left(\int_{t_0}^{t'} R_0(s)ds\right).
\end{aligned}
$$

The first term on the right-hand side is of the form we want (the right-hand side of (12.42)). The second term is negative so we can drop it. The third term is bounded by $\mathcal{O}(\mathcal{E}_0)Q_1^{m-2}(t)Q_0^{\frac{1}{2}}(t)$ by the claim above, plus $\mathcal{O}(\int_{t_0}^{t} Q_0 Q_1^m ds)$, which is of the above form and can be absorbed by the left hand by smallness of $\mathcal{E}_0$, where we used $t_0 < s < t_1$, $Q_0 \le \mathcal{E}_0^r Q_1$. This completes the proof of Proposition 12.2. □

Using the proposition and its corollary it is easy to bound, for $t \ge t_0$,

$$
\mathcal{O}(\alpha_0|\beta_1|^2 + |\alpha_0|^2\beta_1)\partial_t^k \left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \tilde{\chi}\alpha_0\beta_1^{2m} ds \right\rangle \tag{12.46}
$$

by $\mathcal{O}(\mathcal{E}_0^\rho)\left(Q_0 Q_1^2 + \langle t\rangle^{-3}\right) + \text{h.o.t.}$

It remains to consider $t \le t_0$. In this case we only know that $|\tilde{\alpha}_0| \le k_0\langle t\rangle^{-1}$.

The first term of (12.5)

$$
\bar{\alpha}_0 \mathcal{O}\left(\bar{\alpha}_0\beta_1 e^{-i\lambda_+ t}\right)\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \alpha_0\beta_1^{2m}\tilde{\chi} ds\right\rangle \tag{12.47}
$$

is bounded above by $\mathcal{O}(\langle t\rangle^{-3})$, due to dispersive estimates and the decay of the decay of Q_0 for $t \le t_0$.

In the second term of (12.5),

$$
\begin{aligned}
&\mathcal{O}(\alpha_0|\beta_1|^2)\partial_t^k\left\langle \chi, \int_0^t \cdots\right\rangle \\
&\quad \sim \mathcal{O}\big(Q_0^{\frac{1}{2}}(t)Q_1(t)\big)\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \tilde{\mathcal{H}}_0^k \tilde{\chi}\tilde{\alpha}_0\tilde{\beta}_1^{2m} e^{i\Omega s} ds\right\rangle, \ \Omega \ne 0
\end{aligned} \tag{12.48}
$$

we need to pull $Q_0^{\frac{1}{2}}(t)Q_1(t)$ out of the nonlocal (integral) term.

By dispersive estimates, we have the bound

$$
\mathcal{O}\big(Q_0^{\frac{1}{2}}(t)Q_1(t)\big)\int_0^t \frac{1}{\langle t-s\rangle^{\frac{3}{2}+k}} |\tilde{\alpha}_0(s)| Q_1^m(s) ds\,. \tag{12.49}
$$

To pull the Q_1 term we proceed as earlier. By the mean value theorem, there exists $\bar{s}$, with $s \le \bar{s} \le t \le t_0$ such that

$$\begin{aligned} Q_1(s) &= Q_1(t) + Q_1(s) - Q_1(t) = Q_1(t) - \dot{Q}_1(\bar{s})(t-s) \\ &= Q_1(t) + \mathcal{O}\big(Q_0^{\frac{1}{2}}(\bar{s})Q_1(\bar{s})\big)(t-s) \\ &= Q_1(t) + \mathcal{O}\big(\mathcal{E}_0^\rho \langle \bar{s}\rangle^{-1}\langle t-s\rangle\big)\,, \end{aligned}$$

where we have used that $\dot{Q}_1 \sim Q_0^{\frac{1}{2}} Q_1$ for $t \le t_0$. Therefore, the expression in (12.49) satisfies a bound of order

$$\begin{aligned} & Q_0^{\frac{1}{2}}(t)Q_1^2(t)\int_0^t \frac{1}{\langle t-s\rangle^{\frac{3}{2}+k}}|\tilde{\alpha}_0(s)|Q_1^{m-1}(s)ds \\ &\quad + Q_0^{\frac{1}{2}}(t)Q_1(t)\int_0^t \frac{1}{\langle t-s\rangle^{\frac{1}{2}+k}}Q_0^{\frac{1}{2}}(s)\mathcal{O}\big(Q_0^{\frac{1}{2}}(\bar{s})Q_1^{m'}(\bar{s})\big)Q_1^{m-1}(s)ds \\ &= Q_0^{\frac{1}{2}}(t)Q_1^2(t)\int_0^t \frac{1}{\langle t-s\rangle^{\frac{3}{2}+k}}Q_0^{\frac{1}{2}}(s)Q_1^{m-1}(s)ds \\ &\quad + \mathcal{E}_0^\rho Q_0^{\frac{1}{2}}(t)Q_1(t)\frac{1}{\langle t\rangle^2} \end{aligned} \tag{12.50}$$

where the last term, which is bounded by $\mathcal{E}_0^\rho(Q_0Q_1^2 + \langle t\rangle^{-4})$, is obtained using the decay of $Q_0(s)$ for $s \le t_0$.

It remains to estimate the second to last term in (12.50). Estimating the convolution, using the $\langle s\rangle^{-1}$ decay of $Q_0(s)$, we obtain the bound $\mathcal{O}\big(Q_0^{\frac{1}{2}}(t)Q_1^2(t)\big)\langle t\rangle^{-1}$ which is not bounded by the desired $\mathcal{O}\big(\mathcal{E}_0^\rho(Q_0Q_1^2 + \langle t\rangle^{-3})\big)$.

We will obtain the desired bound by turning to an earlier expression, derivable from (12.48). The expression we must consider is

$$\mathcal{O}\big(Q_0^{\frac{1}{2}}(t)Q_1^2(t)\big)\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)}P_c\tilde{\mathcal{H}}_0^k\tilde{\chi}\tilde{\alpha}_0(s)\tilde{\beta}_1^{2m-2}(s)e^{i\Omega s}ds\right\rangle,\ \Omega \ne 0\,. \tag{12.51}$$

We will show that this term is of order $\mathcal{E}^\rho Q_0(t)Q_1^2(t)\langle t\rangle^{-3/2}$, which implies the desired bound. We proceed as follows. First, by Eq. (7.3) of Proposition 7.1 the equation for $\tilde{\alpha}_0$ may be rewritten as:

$$i\partial_t\tilde{\alpha}_0(t) = (c+i\Gamma_\omega)|\tilde{\beta}_1(T)|^4\tilde{\alpha}_0(t) + (c+i\Gamma_\omega)\big(|\tilde{\beta}_1(t)|^4 - |\tilde{\beta}_1(T)|^4\big)\tilde{\alpha}_0(t) + F_\alpha\,. \tag{12.52}$$

Introducing $\tilde{\alpha}_0^\#(t)$ via the equation

$$\tilde{\alpha}_0(t) \equiv e^{-it(c+i\Gamma_\omega)|\tilde{\beta}_1(T)|^4 t}\tilde{\alpha}_0^\#(t)\,, \tag{12.53}$$

we have

$$i\partial_t\tilde{\alpha}_0^\#(t) = (c+i\Gamma_\omega)\big(|\tilde{\beta}_1(t)|^4 - |\tilde{\beta}_1(T)|^4\big)\tilde{\alpha}_0^\#(t) + e^{it(c+i\Gamma_\omega)|\tilde{\beta}_1(T)|^4}F_\alpha\,. \tag{12.54}$$

The integral in (12.51) can be written, keeping only leading order terms, as

$$\begin{aligned}
&\left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \tilde{\mathcal{H}}_0^k \tilde{\chi} \tilde{\alpha}_0^{\#}(s) \tilde{\beta}_1^{2m-2}(s) e^{i\tilde{\Omega} s} ds \right\rangle \\
&\quad = \left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \tilde{\mathcal{H}}_0^k \tilde{\chi}_1 \dot{\tilde{\alpha}}_0^{\#}(s) \tilde{\beta}_1^{2m-2}(s) e^{i\tilde{\Omega} s} ds + \cdots \right\rangle \\
&\quad = \left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \tilde{\mathcal{H}}_0^k \tilde{\chi}_1 \left(|\tilde{\beta}_1(t)|^4 - |\tilde{\beta}_1(T)|^4 \right) \tilde{\alpha}_0^{\#}(s) \tilde{\beta}_1^{2m-2}(s) e^{i\tilde{\Omega} s} ds \right\rangle \\
&\quad \le \mathcal{O}(\mathcal{E}_0^2) \int_0^t \frac{ds}{\langle t-s \rangle^{\frac{3}{2}+k} \langle s \rangle} \left(|\tilde{\beta}_1(t)|^4 - |\tilde{\beta}_1(T)|^4 \right) + \cdots . \qquad (12.55)
\end{aligned}$$

It suffices to show

Proposition 12.3.

$$|P_1(t) - P_1(s)| \le \frac{C}{\langle s \rangle^{\frac{1}{2}}}, \qquad 0 \le s \le t. \qquad (12.56)$$

For the proof we turn to the P_1 equation (7.7):

$$\begin{aligned}
\frac{dP_1}{dt} &= -4\Gamma P_0 P_1^2 + R_1 \\
R_1 &= R_1[\tilde{\alpha}_0, \tilde{\beta}_1, \eta, t] = 2\Im(\overline{\tilde{\beta}_1} F_\beta) .
\end{aligned} \qquad (12.57)$$

The key term in $\Im(\overline{\tilde{\beta}_1} F_\beta)$ is the form $\tilde{\beta}_1^m \tilde{\alpha}_0^{\#} e^{i\Omega t}, \Omega \neq 0$. We have, after two integrations by parts,

$$\begin{aligned}
&|P_1(t) - P_1(s)| \le \int_s^t \tilde{\beta}_1^m(s) \tilde{\alpha}_0^{\#}(s') e^{i\Omega s'} ds' + \cdots \\
&\quad = \tilde{\beta}_1^m(s) \tilde{\alpha}_0^{\#}(s') \frac{1}{i\Omega} e^{i\Omega s'} \Big|_s^t - \int_s^t \frac{d}{ds'} \left(\tilde{\beta}_1^m(s') \tilde{\alpha}_0^{\#}(s') \right) \frac{1}{i\Omega} e^{i\Omega s'} ds' + \cdots \\
&\quad = \mathcal{O}\left(\frac{1}{\langle s \rangle} \right) + \mathcal{O}\left(\frac{\log s}{\langle s \rangle} \right) + \mathcal{O}(\mathcal{E}_0) \int_s^t \left| \left(P_1(t) - P_1(s') \right)^2 \tilde{\alpha}_0^{\#}(s') e^{i\tilde{\Omega} s'} ds' \right| \\
&\quad \le \mathcal{O}\left(\frac{1}{\langle s \rangle} \right) + \mathcal{O}\left(\frac{\log s}{\langle s \rangle} \right) + \mathcal{O}(\mathcal{E}_0) \sup_{s''} \langle s'' \rangle^{\frac{1}{2}} |P(t) - P(s'')| \int_s^t \frac{1}{\langle s' \rangle^2} ds' \\
&\quad = \mathcal{O}\left(\frac{1}{\langle s \rangle} \right) + \mathcal{O}\left(\frac{\log s}{\langle s \rangle} \right) + \mathcal{O}\left(\frac{\mathcal{E}_0}{\langle s \rangle} \right) \sup_{s''} \langle s'' \rangle^{\frac{1}{2}} |P(t) - P(s'')| . \qquad (12.58)
\end{aligned}$$

Multiplication by $\langle s \rangle^{\frac{1}{2}}$ and taking the supremum over $s \le t$ implies Proposition 12.3 and therewith the proof of part (a) of Proposition 12.1.

12.2. *Proof of part (2) of Proposition 12.1*

The proof is similar to that of part (1) but simpler, since we can allow for the nonlocal terms to be controlled, in addition, by terms of order $Q_0^{\frac{1}{2}}Q_1^m$.

The leading contributions are again nonlocal, linear η_b contributions:

From (7.39) we read

$$\begin{aligned} i\partial_t\beta_1 \sim \cdots &+ 2\lambda\langle\chi, \bar{\Psi}_1\eta_b + \Psi_1\overline{\eta_b}\rangle e^{i\lambda_+ t}\alpha_0 \\ &+ \lambda e^{i\lambda_+ t}2\langle\chi, \Psi_1\eta_b\rangle\bar{\alpha}_0 \\ &+ \lambda e^{i\lambda_+ t}\langle\chi, \Psi_1^2\overline{\eta_b} + 2|\Psi_1|^2\eta_b\rangle + \text{h.o.t.} \\ &+ X^{-1}(t)(A(t) - A(T))X(t)\beta_1\,. \end{aligned} \tag{12.59}$$

The first two terms on the right-hand side of (12.59) are easily seen to be of order $O(\alpha_0)P_1^{2m}$ or $\mathcal{O}(\mathcal{E}_0^\rho)P_0P_1^2$, by integration by parts over the ODE source terms in η_b, as before.

The third term contributes to the P_1 equations, after normal form transformations and remaining resonant terms:

$$\Im\bar{\beta}_1 e^{i\lambda_+ t}\partial_t^n\left\langle\chi, |\beta_1|^2\int_0^t e^{-i\mathcal{H}_0(t-s)}P_c\alpha_0|\beta_1|^2\tilde{\chi}ds\right\rangle$$

and higher-order/similar terms.

The leading term, after integration by parts of the integral term (note that $H^{-1}P_c$ is bounded):

$$\begin{aligned} &\left|\Im\bar{\beta}_1 e^{i\lambda_+ t}|\beta_1|^2\left\langle\chi, \int_0^t e^{-i\mathcal{H}_0(t-s)}P_c\mathcal{H}_0^n\tilde{\alpha}_0\tilde{\beta}_1^{2m}\tilde{\chi}ds\right\rangle\right| \\ &\quad\le c|\beta_1|^3\int_0^t\frac{ds}{\langle t-s\rangle^k}Q_0^{1/2}(s)Q_1^m\,. \end{aligned}$$

To this end we repeat the argument of part (1) to estimate the above integral by

$$Q_0^{\frac{1}{2}}(t)Q_1^{m-1}(t) + \mathcal{O}(t^{-3}) + \text{h.o.t.(ODE)}\,.$$

The main new type of term we need to control comes from the last term on the term $\mathcal{R}_\beta$ in (6.35) and (6.36), coming from the difference $\mathbf{A}(t) - \mathbf{A}(T)$.

This term contributes to the P_1 equation terms like

$$\Im\left(|\beta_1|^2\left[\alpha_0^2(t) - \alpha_0^2(T)\right]\bar{\alpha}_0^2(t)\right), \Im\left[e^{i\omega t}\beta_1^2(P_0(t) - P_0(T))\mathcal{O}(\alpha_0^2(T))\right]. \tag{12.60}$$

The term with phase $e^{i\omega t}$ can be integrated by parts, and gives higher-order terms.

The term without a phase requires the estimate of

$$\alpha_0(t)^2 - \alpha_0(T)^2 = 2\int_t^T\alpha_0\frac{d\alpha_0}{ds}ds\,.$$

Using that

$$\frac{d\alpha_0}{ds} = c\bar{\alpha}_0\beta_1^2 e^{-2i\lambda_+ t} + \text{h.o.t.}$$

we can repeatedly integrate by parts to get

$$\alpha_0^2(t) - \alpha_0^2(T) = \text{local terms} + \int_t^T \mathcal{O}(\alpha_0^2)\beta_1^{2m}ds + \text{h.o.t.}$$

where local terms $= \mathcal{O}(\bar{\alpha}_0 P_1)$ and higher.

Clearly, then

$$|\beta_1|^2\bar{\alpha}_0^2(t)\,[\text{local terms} + \text{h.o.t.}] \leq \mathcal{O}(\mathcal{E}_0)[P_0P_1^2 + \langle t\rangle^{-3}]\,.$$

So, we need to estimate

$$|\beta_1|^2\bar{\alpha}_0^2(t)\int_t^T \mathcal{O}(\alpha_0^2)\beta_1^{2m}ds \text{ by } \mathcal{O}(\mathcal{E}_0^\rho)Q_0Q_1^2 + \text{h.o.t.} \tag{12.61}$$

For $t \geq t_1$, $|\beta_1|^2 \leq Q_1$ is monotonic decreasing. Hence

$$\left|\int_t^T \mathcal{O}(\alpha_0^2)\beta_1^{2m}ds\right| \leq |\beta_1(t)|^2\int_t^T Q_0Q_1^{m-1}ds\,.$$

For $t > t_1, Q_1 \downarrow$ is bounded by

$$\begin{aligned} Q_1(t) &\leq Q_1(t_1)(1 + 2\Gamma Q_0(t_1)Q_1(t_1)(t - t_1))^{-1} \leq c\langle t - t_1\rangle^{-1}Q_0(t_1)^{-1} \\ &= cQ_1(t_1)^{-1}\mathcal{E}_0^{-r}\langle t - t_1\rangle^{-1} \end{aligned}$$

since $Q_0(t_1) = \mathcal{E}_0^r Q_1(t_1)$.

So,

$$\left[Q_1(t)^{1+r}Q_1(t)\right]^k \leq \mathcal{O}(\mathcal{E}_0)\langle t - t_1\rangle^{-k}\,.$$

Hence, for $k > 1$, or for $m > 2 + r + 1 = r + 3$,

$$\int_t^T Q_0Q_1^{m-1} \leq \mathcal{O}(\mathcal{E}_0)$$

so

$$\left|\int_t^T \mathcal{O}(\alpha_0^2)\beta_1^{2m}\right| \leq \mathcal{O}(\mathcal{E}_0)Q_0Q_1^2.$$

If $t > t_0, t < t_1, Q_0(t) \leq \mathcal{E}_0^r Q_1(t)$; therefore

$$\begin{aligned} \int_t^T Q_0Q_1^{m-1}ds = \int_t^T Q_0Q_1^2Q_1^{m-3}ds &= \left(\frac{1}{2\Gamma}Q_0(s) + R_0(s)\right)Q_1^{m-3}(s)\Big|_{s=t}^T \\ &\quad - \int_t^T \left(\frac{1}{2\Gamma}Q_0 + \int_{t_0}^s R_0(s)ds\right)Q_1^{m-4}(m-3)\frac{dQ_1}{ds}ds\,. \end{aligned}$$

The first term on the right-hand side is local and high order. The second term on the right-hand side is bounded by

$$c\int_t^{t_1} Q_0^2 Q_1^{m-2} ds + \int_{t_1}^T (\cdots) + \text{h.o.t.} \le \mathcal{O}(\mathcal{E}_0) Q_1^2 + \mathcal{O}(\mathcal{E}_0^\rho)\langle t\rangle^{-3} + c\int_t^{t_1} Q_0^2 Q_1^{m-2} ds$$

$$\le \mathcal{O}(\mathcal{E}_0)[Q_1^2 + \langle t\rangle^{-3}] + c\mathcal{E}_0^r \int_t^{t_1} Q_0 Q_1^{m-1} ds\,.$$

Hence,

$$\int_t^T Q_0 Q_1^{m-1} ds \le \mathcal{O}(\mathcal{E}_0)[Q_1^2 + \langle t\rangle^{-3}] + \mathcal{O}(\mathcal{E}_0)\int_t^T Q_0 Q_1^{m-1} ds \tag{12.62}$$

which implies that

$$\int_t^T Q_0 Q_1^{m-1} ds \le \mathcal{O}(\mathcal{E}_0)[Q_1^2 + \langle t\rangle^{-3}] \tag{12.63}$$

for all $t \ge t_0$.

For $0 < t < t_0$, we need to estimate

$$|\beta_1|^2 \bar\alpha_0^2(t)\int_t^{t_0} O(\alpha_0^2)\beta_1^{2m} ds\,.$$

Using that for $t < t_0, |\alpha_0(t)|^2 \le k_0\langle t\rangle^{-1} \le \mathcal{E}_0\langle t_0\rangle^{-1}\langle t\rangle^{-1}$ the above expression is bounded by

$$\mathcal{O}(\mathcal{E}_0)k_0(t_0)\langle t\rangle^{-1}\left(\ln\frac{t_0}{\langle t\rangle}\right)k_0(t_0) \le \mathcal{O}(\mathcal{E}_0)k_0(t_0)\langle t\rangle^{-1}\left(\frac{\langle t\rangle}{\langle t_0\rangle}\ln\frac{t_0}{\langle t\rangle}\right)\langle t\rangle^{-1}$$

$$\le \mathcal{O}(\mathcal{E}_0)k_0(t_0)\langle t\rangle^{-2}\,.$$

This completes the proof of Proposition 12.1.

13. $R_1^{(j)}(\eta_b)$ terms of Proposition 11.1

Proposition 13.1. *Assume either $t \le t_0$ or monotonicity property $\boldsymbol{Q}$ on $[t_0, t_0+\delta_*]$. Then, the terms $R_1^{(i)}[\eta_b]$, $i = 0, 1$ in* (11.1) *satisfy the estimates*:

$$\left|R_1^{(i)}[\eta_b]\right| \le \mathcal{O}(\mathcal{E}_0)[2\Gamma Q_0 Q_1^2 + \langle t_0\rangle^{-1}\langle t\rangle^{-2}(\text{Poly}[P_0(0), P_1(0), P_0(T), P_1(T)])] \tag{13.1}$$

where Poly$[\cdots]$ *stands for polynomial in the bracketed variables.*

Proof. The contributions to $R_1^{(i)}(\eta_b)$ comes from linear and nonlinear terms in η_b in the P_i equations.

Consider first the nonlinear contributions:

In the P_0 equations we have terms like (7.41)

$$\bar\alpha_0^2\langle\chi, \eta_b^2\rangle\,, \qquad \bar\alpha_0\bar\beta_1 e^{i\lambda_+ t}\langle\chi, \eta_b^2\rangle\,, \qquad e^{i\lambda_+ t}\bar\alpha_0\beta_1\langle\chi, |\eta_b|^2\rangle\,, \qquad \langle\chi, |\eta_b|^2\eta_b\rangle\bar\alpha_0\,. \tag{13.2}$$

Since for $t \in I_0, I_2$ we have time decay of either α_0 or β_1 respectively, the main contribution is when $t \in I_1 = [t_0, t_1]$.

For $t \in I_2$, the bound on η_b we have is

$$\|\eta_b(t)\|_{H^k} \le c Q_0^{1/2}(t) \int_0^t \frac{ds}{\langle t-s\rangle^{3/2}} Q_1^m(s)\,.$$

The second and third terms can be integrated by parts leaving terms of the type

$$\mathcal{O}(\beta_1^2)\mathcal{O}(\bar{\alpha}_0)\partial_t^m \langle \chi, |\eta_b|^2 + \eta_b^2\rangle\,.$$

We also need to integrate by parts the two other terms. For this we need to pull out a phase factor from the leading nonlocal.

Pulling a phase as in the proof of Proposition 12.1 we are left with estimating term of the type

$$O(\alpha_0^2 + \alpha_0\beta_1)\langle \chi, \eta_b \partial_t^k \eta_b\rangle + \langle \chi, |\eta_b|^2 \partial_t^k \eta_b\rangle O(\alpha_0)\,.$$

As in the estimates of Proposition 10.3 , for $t \in I_1$,

$$\|\partial_t^k \eta_b\|_{H^s} \le C Q_0^{1/2}(t) \int_0^t \frac{ds}{\langle t-s\rangle^{k'}} Q_1^m(s)$$

with k' large for k large.

To this end we use the following.

Proposition 13.2. *For $t \in I_1$:*

$$\int_{t_0}^t \frac{ds}{\langle t-s\rangle^{k'}} Q_1^{\bar{m}}(s) \le \mathcal{O}(\mathcal{E}_0) Q_1^2(t)\,.$$

Proof of Proposition 13.2.

$$\int_{t_0}^t \frac{ds}{\langle t-s\rangle^{k'}} Q_1^{\bar{m}}(s) = Q_1(t)\int_{t_0}^t \frac{ds}{\langle t-s\rangle^{k'}} Q_1^{\bar{m}-1} + \int_{t_0}^t \frac{ds}{\langle t-s\rangle^{k'}} (Q_1(s) - Q_1(t)) Q_1^{\bar{m}-1}\,.$$

The second term on the right-hand side can be estimated, using that $\dot{Q}_1 = -\Gamma Q_0 Q_1^2 + \mathcal{O}(Q_0^{\frac{1}{2}} Q_1^m)$ and monotonicity of Q_0, as follows:

$$\begin{aligned}
&(s \le \xi \le t)\\
&\le C \int_{t_0}^t \frac{ds}{\langle t-s\rangle^{k'-1}} \dot{Q}_1(\xi) Q_1^{\bar{m}-1}(s) \le C Q_0(t) \int_{t_0}^t \frac{ds}{\langle t-s\rangle^{k'-1}} Q_1^2(\xi) Q_1^{\bar{m}-1}(s)\\
&\quad + C Q_0^{1/2}(t) \int_{t_0}^t \frac{ds}{\langle t-s\rangle^{k'-1}} Q_1^m(\xi) Q_1^{\bar{m}-1}(s)\\
&\le \mathcal{O}(\mathcal{E}_0) Q_1(t) \int_{t_0}^t \frac{ds}{\langle t-s\rangle^{k'-1}} Q_1^2(\xi) Q_1^{\bar{m}-1}\\
&\quad + \mathcal{O}(\mathcal{E}_0) Q_1^{1/2}(t) \int_{t_0}^t \frac{ds}{\langle t-s\rangle^{k'-1}} Q_1^{m-1}(\xi) Q_1^{\bar{m}-1}\,.
\end{aligned}$$

Repeating this argument, we have

$$\int_{t_0}^{t} \frac{ds}{\langle t-s\rangle^{k'}} Q_1^{\bar{m}}(s) \le \mathcal{O}(\mathcal{E}_0) Q_0^j(t) \int_{t_0}^{t} \frac{ds}{\langle t-s\rangle^{k'-2j}} Q_1^{\bar{m}-2j}$$

$$\le \mathcal{O}(\mathcal{E}_0) Q_1^j(t)$$

for $k' - 2j > 1$, which proves the Proposition 13.2. □

This Proposition together with the estimate

$$\|\eta_b\|_{W^{k,\infty}} \le \mathcal{O}(\mathcal{E}_0) Q_0^{1/2} \tag{13.3}$$

implies that

$$R_1^{(0)} \le \mathcal{O}(\mathcal{E}_0) \left[Q_0 Q_1^2 + \text{higher order terms}\right]. \tag{13.4}$$

The estimates of $R_1^{(1)}$ are similar.

It remains to estimate the linear η_b terms in the P_0, P_1 equations.

The leading order source term of η_b was estimated in Proposition 12.1. It remains to estimate the higher-order corrections.

To this end we need to estimate terms of the following type appearing in the P_0 equation, (11.1):

$$\bar{\alpha}_0(|\beta_1|^2 + \alpha_0\beta_1) \left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c \psi_0 \eta_b^2 ds \right\rangle \text{ and}$$

$$\left(|\beta_1|^2\alpha_0 + \bar{\alpha}_0\alpha_0\beta_1\right) \left\langle \chi, \int_0^t e^{-i\mathcal{H}_0(t-s)} P_c |\eta_b|^2 \eta_b ds \right\rangle,$$

and similar terms in the P_1 equation.

Again we focus on $t \in I_1$. Since for $0 \le s \le t_0$ α_0 and η_b are of order $\frac{\mathcal{O}(\mathcal{E}_0)}{\langle s\rangle}$, clearly these contributions are of order

$$\mathcal{O}(\mathcal{E}_0)\left[Q_0 Q_1^2 + \langle t_0\rangle^{-3}\right],$$

so it remains to estimate the s-integrals above on I_1.

But on $I_1, \|\eta_b\| \le \mathcal{O}(\mathcal{E}_0) Q_0^{\frac{1}{2}}(t) \le \mathcal{O}(\mathcal{E}_0) Q_1^{\frac{1}{2}}(t)$ and since Q_0 is monotonic increasing on I_1, the above nonlocal terms are bounded by

$$\mathcal{O}(\mathcal{E}_0) Q_0^{\frac{3}{2}}(t).$$

So,

$$\mathcal{O}(|\beta_1|^2\bar{\alpha}_0)\mathcal{O}(\mathcal{E}_0) Q_0^{\frac{3}{2}}(t) \le \mathcal{O}(\mathcal{E}_0) Q_1^2 Q_0$$

$$O(|\alpha_0|^2\beta_1)\mathcal{O}(\mathcal{E}_0) Q_0^{\frac{3}{2}}(t) \le \mathcal{O}(\mathcal{E}_0) Q_1^2 Q_0$$

since $Q_0 \le \mathcal{O}(\mathcal{E}_0) Q_1$ on I_1.

14. Bootstrapping It All

We assume that $t_0 < \infty$, where t_0 is given by (8.5). Consider the equations for P_0 and P_1, (11.1) displayed in Proposition 11.1. Explicit in (11.1) are terms which

(i) are driven by the dispersive part of the initial data: $R_0^0[\eta_0]$ and $R_0^1[\eta_0]$
(ii) encompass interactions of the two bound states and dispersive waves: $R_1^0[\eta_b]$ and
(iii) encompass interactions between bound states: $R_2^0[P_0, P_1]$.

By Proposition 11.2 the $R_0^j[\eta_0],\ j = 0, 1$ terms satisfy:

$$R_0^j[\eta_0] = \mathcal{O}\left(b_j(t_0, [\eta_0]_X, \mathcal{E}_0)\right)\langle t\rangle^{-2} + \mathcal{O}(\mathcal{E}_0)2\Gamma P_0 P_1^2\,. \tag{14.1}$$

Therefore, by Proposition 9.2 and its proof (see Eq. (9.22)), it is natural to introduce the functions:

$$Q_0(t) = P_0 - \frac{k_0}{\langle t\rangle}\,, \qquad Q_1(t) = P_1 + \frac{k_1}{\langle t\rangle} \tag{14.2}$$

where

$$k_0 = b_0 + \mathcal{O}(\mathcal{E}_0)b_1^2\,, \qquad k_1 = 10 b_1 \tag{14.3}$$

$$b_0 = \langle t_0\rangle^{-1}\left([\eta_0]_X + c_*\mathcal{E}_0^2\right), \tag{14.4}$$

$$b_1 = \langle t_0\rangle^{-\frac{1}{2}}\left([\eta_0]_X^{\frac{2}{3}} + d_*\mathcal{E}_0^2\right), \tag{14.5}$$

where the constants c_* and d_* are to be chosen. We find, for any $m \geq 4$ and all $t \geq 0$:

$$\begin{aligned} \frac{dQ_0}{dt} &\geq 2\Gamma Q_0 Q_1^2 + R_1^{0,\#}[\eta_b] + R_2^{0,\#}[Q_0, Q_1] + \frac{\mathcal{E}_0^2 c_*}{\langle t_0\rangle\langle t\rangle^2} \\ \frac{dQ_1}{dt} &\leq -4\Gamma Q_0 Q_1^2 + R_1^{1,\#}[\eta_b] + R_2^{1,\#}[Q_0, Q_1] + \mathcal{O}(\mathcal{E}_0; m)\sqrt{Q_0}Q_1^m - \frac{\mathcal{E}_0^2 d_*}{\langle t_0\rangle^{\frac{1}{2}}\langle t\rangle^2}\,. \end{aligned} \tag{14.6}$$

The analogous reverse inequalities hold as well with slightly different constants.

By the definition of t_0, $Q_0(t_0) > 0$. Furthermore, using the energy estimate on the bound state amplitudes and (14.2) of Sec. 5.2, we have

$$Q_0(t) + Q_1(t) \leq C\mathcal{E}_0\,, \qquad t > 0\,. \tag{14.7}$$

We now introduce a set of norms. The norm of $q(t) \equiv (Q_0(t), Q_1(t), \eta_b(t))$ is defined as

$$\|q(t)\|_{\mathbf{Y}} = |Q_0|_{y_0(t)} + |Q_1|_{y_1(t)} + \|\eta_b\|_{y_2(t)}\,. \tag{14.8}$$

The norm, $\|q(t)\|_{\mathbf{Y}}$, encodes all the estimates for Q_0, Q_1 and η_b in the intervals I_0, I_1 and I_2 through the following:

$$|Q_0|_{y_0(t)} \equiv \sup_{0\leq s\leq t}|Q_0(s)| + \sup_{0\leq s\leq \min\{t,t_0\}}\langle t_0\rangle\langle s\rangle|Q_0(s)| \tag{14.9}$$

$$|Q_1|_{y_1(t)} \equiv \sup_{0\le s\le t} |Q_1(s)| + \sup_{t_1\le s\le t} |s-t_1| Q_1(s)\Gamma' Q_0(t_1) Q_1(t_1) \tag{14.10}$$

$$\|\eta_b\|_{y_2(t)} \equiv \sup_{0\le s\le \min\{t,t_0\}} \langle s\rangle \|\eta_b(s)\|_{W^{k,\infty}} + \sup_{t_0\le s\le \min\{t,t_1\}} \|\eta_b(s)\|_{W^{k,\infty}} + \sup_{t_1\le s\le t} \langle s-t_1\rangle^{\frac{1}{2}} \|\eta_b(s)\|_{W^{k,\infty}} . \tag{14.11}$$

In these definitions we use the convention that terms for which the s-range is empty are set to zero.

By the H^1 *a priori* bounds

$$\sup_{0\le s\le t} |Q_0(s)| \le \mathcal{E}_0(1+\mathcal{E}_0^2\|\eta_b\|_\infty) \tag{14.12}$$

and by definition of t_0, (8.5), for $t_0 < \infty$,

$$\sup_{0\le s\le t_0} \langle t_0\rangle\langle s\rangle |Q_0(s)| \le \frac{1}{2}\big(\mathcal{E}_0 + [\eta_0]_X\big) . \tag{14.13}$$

In terms of these norms, we have bounds on $R_i^{j,\#}$. By Proposition 13.1

$$|R_1^{0,\#}[\eta_b]| + |R_1^{1,\#}[\eta_b]| \le \mathcal{O}(\mathcal{E}_0^\rho)\|q(t)\|_Y^{l_1} Q_0 Q_1^2 + C\frac{\mathcal{E}_0^\rho \|q(t)\|_{\mathbf{Y}}^{l_2}}{\langle t_0\rangle\langle t\rangle^2} . \tag{14.14}$$

By Proposition 12.1

$$|R_2^{0,\#}[Q_0,Q_1]| \le \mathcal{O}(\mathcal{E}_0^\rho) Q_0 Q_1^2 + C\ \frac{\mathcal{E}_0^\rho + \|q(t)\|_{\mathbf{Y}}^l}{\langle t_0\rangle\langle t\rangle^2} \tag{14.15}$$

$$|R_2^{1,\#}[Q_0,Q_1]| \le \mathcal{O}(\mathcal{E}_0^\rho) Q_0 Q_1^2 + C\frac{\mathcal{E}_0^\rho + \|q(t)\|_{\mathbf{Y}}^l}{\langle t_0\rangle^{\frac{1}{2}}\langle t\rangle^2} \tag{14.16}$$

where $l \ge 2$.

By the definition of I_0, $(0 \le t \le t_0)$ and Propositions 11.2, 12.1 and 13.1, we have estimates (14.14)–(14.16). Therefore, for an appropriate choice of c_* and d_* we have for $0 \le t \le t_0$

$$\begin{aligned} \frac{dQ_0}{dt} &\ge 2\Gamma' Q_0 Q_1^2 + \frac{c_*}{2}\frac{\mathcal{E}_0^2}{\langle t_0\rangle\langle t\rangle^2} \\ \frac{dQ_1}{dt} &\le -4\Gamma' Q_0 Q_1^2 + \mathcal{O}(\mathcal{E}_0; m)\sqrt{Q_0}Q_1^m - \frac{d_*}{2}\frac{\mathcal{E}_0^2}{\langle t_0\rangle^{\frac{1}{2}}\langle t\rangle^2} , \end{aligned} \tag{14.17}$$

where $m \ge 4$. Note that by definition of I_0, $Q_0(t) < 0$ for $t \in I_0$.

By continuity, (14.17) holds for $t_0 \le t \le t_0 + \delta$, for some $\delta > 0$. It follows, using that $Q_0(t_0) > 0$ and Propositions 9.2–9.4, that (11.2) (**Monotonicity Property Q**) holds on $t_0 \le t \le t_0 + \delta$. Therefore, by Propositions 11.2, 12.1 and 13.1 the terms J_0 and J_1 in (9.8) both satisfy the bound

$$|J_k| \le \frac{\mathcal{O}(\mathcal{E}_0^{2+\rho})}{\langle t_0\rangle\langle t\rangle^2} + \mathcal{E}_0 Q_0 Q_1^2 , \qquad \rho > 0 , \qquad k = 0, 1. \tag{14.18}$$

Therefore, for $\mathcal{E}_0$ sufficiently small (14.17) holds with c_*, d_* replaced by $c_*/2, d_*/2$.

Define

$$T_* = \sup\{t \geq t_0 :\ (14.17) \text{ holds for some } c_* > 0 \text{ and } d_* > 0\}\,. \tag{14.19}$$

For $t \in [0, T_*)$, $\|q(t)\|_{\mathbf{Y}}$ is small. We claim that $T_* = \infty$. Suppose $t_0 \leq T_* < \infty$. Then, for $t \in [t_0, T_*)$ we have, by (14.17), (14.18) and the above argument using Propositions 9.2–9.4, that the monotonicity property (11.2) holds at $t = T_*$ and slightly beyond. Thus, the *a priori* bounds on J_0, J_1 of Propositions 11.2, 12.1 and 13.1, the η_b-bounds of Propositions 10.7–10.9, and the smallness of Q_0 and Q_1 imply persistence of the inequalities (14.17), with perhaps a slightly smaller choice of positive constants c_* and d_*. This implies that $T_* = \infty$.

15. Nongeneric Behavior

Recall that t_0 is defined by (8.5) and consider the case where $t_0 = \infty$. We would like to show that

$$P_0(t) \to 0 \text{ as } t \to \infty$$

$$P_1(t) \qquad \text{has a limit}\,.$$

The following is a consequence of the definition of t_0.

Proposition 15.1. *Assume $t_0 = \infty$. Then, $P_0(t) = \mathcal{O}\left([\eta_0]_X \langle t\rangle^{-2}\right)$. Therefore, $\alpha_0 \to 0$ and the ground state decays.*

Proposition 15.2. *Let $t_0 = \infty$. Then, β_1 has a limit as $t \to \infty$.*

States with this (nongeneric) behavior were constructed in [55].

Proof of Proposition 15.2. Equation (7.7), together with the above estimate

$$P_0(t) = \mathcal{O}(\overline{\eta_0})\langle t\rangle^{-2} \tag{15.1}$$

implies

$$\begin{aligned}\frac{dP_1}{dt} &= (-4\Gamma P_0 P_1^2 + \mathcal{O}(\overline{\eta_0})\langle t\rangle^{-3/2})(1 + \mathcal{O}(P_0) + \mathcal{O}(P_1)) \\ &\quad + \Re(ce^{i\lambda_+ t}|\beta_1|^2\overline{\beta_1}\alpha_0) + \text{h.o.t.}\end{aligned} \tag{15.2}$$

To show that P_1 has a limit, we show that $\int_0^t \partial_s P_2(s)ds$ has a limit. All terms other than the $\mathcal{O}(\alpha_0)$ term, on the right-hand side are absolutely integrable since $P_0 = \mathcal{O}(\langle t\rangle^{-2})$.

It is left to integrate the $\mathcal{O}(|\beta_1|^2\bar{\beta}_1\alpha_0)$ term. For T given, let $\beta_T^2 \equiv \beta_1(T)^2$. Then, Eq. (7.52) reads

$$2i\partial_t\alpha_0 = \lambda\langle\psi_{0*}, \Psi_1(t)^2\rangle\overline{\alpha_0} + \text{integrable in } t\,. \tag{15.3}$$

Using the expression for $\Psi_1(t) = \alpha_1\psi_1(\cdot, |\alpha_1|^2)$:

$$\begin{aligned} 2i\partial_t\alpha_0 &= \lambda\langle\psi_{0*}, \alpha_1(t)^2\psi_{1*}^2\rangle\bar{\alpha}_0 + \text{h.o.t.} \\ &= \lambda\langle\psi_{0*}, \psi_{1*}^2\rangle e^{-2i\lambda_+ t}\beta_1(t)^2\bar{\alpha}_0(t) + O(\alpha_0^2(T)\bar{\beta}_1) + \text{h.o.t.} \\ &\equiv \tilde{\lambda}e^{-2i\lambda_+ t}\beta_T^2\bar{\alpha}_0(t) + \tilde{\lambda}e^{-2i\lambda_+ t}[\beta_1^2(t) - \beta_1^2(T)]\bar{\alpha}_0(t) + O\left(\frac{1}{T^2}\right) \\ &\quad + \text{h.o.t.} \end{aligned} \tag{15.4}$$

where $\tilde{\lambda} \equiv \lambda\langle\psi_{0*}\psi_{1*}^2\rangle$ and $\beta_T^2 \equiv \beta_1^2(T)$.

Solving the homogeneous part of (15.4):

$$2i\partial_t\hat{\alpha}_0 = \tilde{\lambda}e^{-2i\lambda_+ t}\beta_T^2\overline{\hat{\alpha}}(t) \tag{15.5}$$

we have, using the Ansatz

$$\alpha_0(t) = A(t)e^{i(\lambda-a)t} + B(t)e^{i(\lambda+a)t}$$

with

$$\dot{A} \sim \tilde{\lambda}e^{-2i\lambda_+ t}[\beta_1^2(t) - \beta_1^2(T)]\bar{A}e^{i\theta(t)} + \text{h.o.t.}$$

and a similar equation for $B(t)$.

We have

$$\frac{dP_1}{dt} = -4\Gamma P_0 P_1^2 + \Re\big(ce^{i\theta_A t}|\beta_1|^2\bar{\beta}_1 A + e^{i\theta_B t}|\beta_1|^2\bar{\beta}_1 B\big) + \text{h.o.t.}\,,$$

$\theta_A, \theta_B \neq 0$. Integration of the above equation, integration by parts (twice) of A and B, implies

$$\begin{aligned} \langle t\rangle^{\frac{1}{2}}|P_1(t) - P_1(T)| &\leq C\langle t\rangle^{\frac{1}{2}}\int_t^T |\beta_1|^2\bar{\beta}_1\langle t'\rangle^{-1}\langle t'\rangle[\beta_1^2(t') - \beta_1^2(T)]^2\bar{A}(t')dt' + \text{h.o.t.}\langle t\rangle^{\frac{1}{2}} \\ &\leq C\mathcal{E}_0^m\left(\int_t^T\langle t'\rangle^{-\frac{1}{2}-1}dt'\right)\left(\sup_{0\leq t'\leq T}\langle t^{\frac{1}{2}}\rangle|P_1(t') - P_1(T)|\right)^2 + \sup_t\langle t\rangle^{\frac{1}{2}}\ \text{h.o.t.} \\ &\Rightarrow |P_1(t) - P_1(T)| \leq C\langle t\rangle^{-\frac{1}{2}} \end{aligned}$$

which implies integrability of $\dot{A}(t)$ and limit of $P_1(t)$.

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Appendix A. Notation

$\Re z$ and $\Im z$ denote the real and imaginary parts of a complex number z.
$\overline{z}$ denotes the complex conjugate of z.
$\langle x\rangle = \sqrt{1 + |x|^2}$, $t \geq 0$.

P_{c*} — projection onto the continous spectral part of the self-adjoint operator, H.
$\langle f, g\rangle = \int \bar{f} g$.

$$\begin{pmatrix} a \\ \text{c.c.} \end{pmatrix} = \begin{pmatrix} a \\ \bar{a} \end{pmatrix} . \tag{A.1}$$

For $j = 1, 2$, let $\pi_j : \mathbb{C}^2 \to \mathbb{C}^1$ be defined by:

$$\begin{gathered} \pi_1 \begin{pmatrix} z \\ w \end{pmatrix} = z \,, \qquad \pi_2 \begin{pmatrix} z \\ w \end{pmatrix} = w \\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \end{gathered} \tag{A.2}$$

Plemelj identities

$$(x \mp i0)^{-1} = \text{P.V.}\, x^{-1} \pm i\pi\delta(x) \,. \tag{A.3}$$

$\vec{\nabla}_j = \left(\partial_{\alpha_j}, \partial_{\overline{\alpha_j}}\right)$.
$\chi(x, p)$ denotes the real-valued localized function of x which depends smoothly on a parameter, p.
$\chi_k^{(j)}$ denotes a spatially localized function of order $|\alpha_j|^k$, as $|\alpha_j| \to 0$.
$\mathcal{O}_k^{(j)}$ denotes a quantity which is of order $|\alpha_j|^k$ as $|\alpha_j| \to 0$. Both $\chi_k^{(j)}$ and $\mathcal{O}_k^{(j)}$ are invariant under the map $\alpha_j \mapsto \alpha_j e^{i\gamma}$.
$\mathcal{O}_k^{(0,1)} = \mathcal{O}_{k_1}^{(0)} \mathcal{O}_{k_2}^{(1)}$, $k = k_1 + k_2$.

Appendix B. Proof of Proposition 4.2

Parts (i), (iii) and (v) of Proposition 4.2 follow from [58]. We now prove parts (ii) and (iv) by a perturbation argument about the case $\alpha_0 = 0$.

Consider the eigenvalue problem $\mathcal{H}_0 \vec{f} = \mu \vec{f}$. Since α_0 is assumed small it is natural to make explicit the leading order and perturbation terms. Thus we have

$$\mathcal{H}_0 \vec{f} = \sigma_3 \left[(H - E_{0*}) - E_0^{(1)} |\alpha_0|^2 I + \psi_0^2 \begin{pmatrix} 2|\alpha_0|^2 & \alpha_0^2 \\ \overline{\alpha_0}^2 & 2|\alpha_0|^2 \end{pmatrix} \right] \vec{f} = \mu \vec{f} \tag{B.1}$$

Recall that $E_0^{(1)}$ and ψ_0 are defined in Proposition 3.1.

The zeroth order problem ($\alpha_0 = 0$) is

$$\sigma_3 (H - E_{0*}) \vec{f}_0 = \mu_0 \vec{f}_0 \,, \tag{B.2}$$

which has two linearly-independent solutions:

$$\mu_0 = E_{*1} - E_{0*},\ \vec{f}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \psi_{1*} \tag{B.3}$$

$$\mu_0 = -(E_{*1} - E_{0*}),\ \overline{\sigma_1 \vec{f}_0} \,. \tag{B.4}$$

We develop the perturbation theory of (B.3). That of the second is completely analogous.

For α_0 and small we define the perturbations about the zeroth order eigenstates via:

$$\vec{f} = \vec{f}_0 + \vec{f}_1 \tag{B.5}$$

$$\mu = E_{1*} - E_{0*} + \mu_1 \,. \tag{B.6}$$

Substitution into (B.1) yields:

$$\begin{aligned}
&[\sigma_3(H - E_{0*}) - (E_{1*} - E_{0*})I]\vec{f}_1 \\
&\quad = |\alpha_0|^2 E_0^{(1)} \sigma_3 \vec{f}_0 - \lambda\psi_0^2\sigma_3 \begin{pmatrix} 2|\alpha_0|^2 & \alpha_0^2 \\ \overline{\alpha_0}^2 & 2|\alpha_0|^2 \end{pmatrix} \vec{f}_0 \ + \mu_1 \vec{f}_0 \\
&\quad\quad + |\alpha_0|^2 E_0^{(1)} \sigma_3 \vec{f}_1 - \lambda\psi_0^2\sigma_3 \begin{pmatrix} 2|\alpha_0|^2 & \alpha_0^2 \\ \overline{\alpha_0}^2 & 2|\alpha_0|^2 \end{pmatrix} \vec{f}_1 \ + \mu_1 \vec{f}_1 \,.
\end{aligned} \tag{B.7}$$

We consider, individually, the first and second equations of the system (B.7), governing $f_{1j} = \pi_j \vec{f}_1$, $j = 1, 2$. The first component of (B.7) is:

$$\begin{aligned}
(H - E_{1*})\, f_{11} &= |\alpha_0|^2 \big(E_0^{(1)} - 2\lambda\psi_0^2\big)\psi_{1*} + \mu_1\psi_{1*} \\
&\quad + |\alpha_0|^2 \big(E_0^{(1)} - 2\lambda\psi_0^2\big) f_{11} - \lambda\alpha_0^2\psi_0^2 f_{12} + \mu_1 f_{11} \,.
\end{aligned} \tag{B.8}$$

Let $\nu_* = 2E_{0*} - E_{1*}$. The second component of (B.7) is:

$$\begin{aligned}
(H - \nu_*) f_{12} &= -\lambda\overline{\alpha_0}^2 \psi_0^2 \psi_{1*} + |\alpha_0|^2 E_0^{(1)} f_{12} \\
&\quad - \lambda\psi_0^2 \left(\overline{\alpha_0}^2 f_{11} + 2|\alpha_0|^2 f_{12}\right) - \mu_1 f_{12} \,.
\end{aligned} \tag{B.9}$$

We wish to make the dependence of $\vec{f}_1$ on α_0 and $\overline{\alpha_0}$ explicit. Define

$$\mu_1 = |\alpha_0|^2 \tilde{\mu}_1 \,, \qquad f_{11} = |\alpha_0|^2 \tilde{f}_{11} \,, \qquad f_{12} = \overline{\alpha_0}^2 \tilde{f}_{12} \,. \tag{B.10}$$

Equations (B.8) and (B.9) reduce to the following system for $\tilde{f}_{11}$ and $\tilde{f}_{12}$:

$$\begin{aligned}
(H - E_{1*})\tilde{f}_{11} &= \big(E_0^{(1)} - 2\lambda\psi_0^2\big)\psi_{1*} + \tilde{\mu}_1\psi_{1*} \\
&\quad + |\alpha_0|^2 \big(E_0^{(1)} - 2\lambda\psi_0^2\big)\tilde{f}_{11} - \lambda|\alpha_0|^2\psi_0^2 \tilde{f}_{12} + |\alpha_0|^2 \tilde{\mu}_1 \tilde{f}_{11}
\end{aligned} \tag{B.11}$$

$$\begin{aligned}
(H - \nu_*)\tilde{f}_{12} &= -\lambda\psi_0^2\psi_{1*} + |\alpha_0|^2 E_0^{(1)} \tilde{f}_{12} \\
&\quad - \lambda|\alpha_0|^2\psi_0^2 \big(2\tilde{f}_{11} + 2\tilde{f}_{12}\big) - |\alpha_0|^2 \tilde{\mu}_1 \tilde{f}_{12} \,.
\end{aligned} \tag{B.12}$$

We seek a solution to the system (B.11), (B.12):

$$|\alpha_0|^2 \mapsto \big(\tilde{f}_{11}(|\alpha_0|^2), \tilde{f}_{12}(|\alpha_0|^2), \tilde{\mu}(|\alpha_0|^2)\big) \in L^2 \times L^2 \times \mathbb{R} \tag{B.13}$$

defined in a neighborhood of $\alpha_0 = 0$.

For $\alpha_0 = 0$ the system (B.11), (B.12) reduces to:

$$(H - E_{1*})\tilde{f}_{11}^0 = \big(E_0^{(1)}(0) - 2\lambda\psi_{0*}^2\big)\psi_{1*} + \tilde{\mu}_1\psi_{1*} \tag{B.14}$$

$$(H - \nu_*)\tilde{f}_{12}^0 = -\lambda\psi_{0*}^2\psi_{1*} \,. \tag{B.15}$$

Note that $\nu_* < 0$ is not in the spectrum of H. Therefore, (B.15) is solvable for $\tilde{f}_{12}^0$ and we have:

$$\tilde{f}_{12}^0 = -\lambda(H - \nu_*)^{-1}\psi_{0*}^2\psi_{1*}\,. \tag{B.16}$$

Since $(H - E_{1*})\psi_{1*} = 0$, (B.14) is solvable if and only if its right-hand side is orthogonal to ψ_{1*}. This determines $\mu_1(0)$:

$$\tilde{\mu}_1(0) = -E_0^{(1)}(0) + 2\lambda\langle\psi_{0*}^2, \psi_{1*}^2\rangle\,, \tag{B.17}$$

and now (B.14) can be solved for $\tilde{f}_{11}^0$.

To solve in a neighborhood of $\alpha_0 = 0$ we proceed as follows. Rewrite (B.12) as

$$(H + |\alpha_0|^2 W_{12} - \nu_*)\tilde{f}_{12} = -\lambda\psi_0^2\psi_{1*} - \lambda|\alpha_0|^2\psi_0^2\tilde{f}_{11}\,, \tag{B.18}$$

where W_{12} is a multiplication operator defined by

$$W_{12}(|\alpha_0|^2, \tilde{\mu}_1) = -E_0^{(1)} + 2\lambda\psi_0^2 + \tilde{\mu}_1\,. \tag{B.19}$$

For $\tilde{\mu}_1$ in a fixed compact set and α_0 sufficiently small, the operator $H + |\alpha_0|^2 W_{12} - \nu_*$ has a bounded inverse, $B(|\alpha_0|^2)$. Thus,

$$\begin{aligned}\tilde{f}_{12} &\equiv \tilde{f}_{12}[\tilde{f}_{11}, |\alpha_0|^2] \\ &= -\lambda B(|\alpha_0|^2)\psi_0^2\psi_{1*} - \lambda|\alpha_0|^2 B(|\alpha_0|^2)\psi_0^2\tilde{f}_{11}\,.\end{aligned} \tag{B.20}$$

Substitution of (B.20) into (B.11) yields the following closed equation for $\tilde{f}_{11}$:

$$(H - E_{1*})\tilde{f}_{11} = \left(E_0^{(1)} - 2\lambda\psi_0^2\right)\psi_{1*} + \tilde{\mu}_1\psi_{1*} + |\alpha_0|^2 W_{11}\tilde{f}_{11}\,, \tag{B.21}$$

where the operator W_{11} is defined by

$$W_{11}(|\alpha_0|^2, \tilde{\mu}_1) = (E_0^{(1)} - 2\lambda\psi_0^2) + \tilde{\mu}_1 + \psi_0^2\tilde{f}_{12}[\cdot, |\alpha_0|^2]\,. \tag{B.22}$$

Setting the inner product of the right-hand side of (B.21) equal to zero, gives the solvability condition for (B.21):

$$\tilde{\mu}_1 = 2\lambda\langle\psi_0^2, \psi_{1*}^2\rangle - E_0^{(1)} - |\alpha_0|^2\langle\psi_{1*}, W_{11}\tilde{f}_{11}\rangle\,. \tag{B.23}$$

The system (B.21), (B.23) is of the form:

$$F(\tilde{f}_{11}, \tilde{\mu}_1, s) = 0\,, \tag{B.24}$$

with the solution $\tilde{f}_{11} = \tilde{f}_{11}^0, \tilde{\mu}_1 = \tilde{\mu}_1(0), s = 0$ defined by (B.17). Furthermore, the Jacobian of $F(\tilde{f}_{11}, \tilde{\mu}_1, s)$ with respect to $(\tilde{f}_{11}, \tilde{\mu}_1)$ evaluated at $(\tilde{f}_{11}^0, \tilde{\mu}_1(0), 0)$ is given by:

$$\begin{pmatrix} H - E_{1*} & -\psi_{1*} \\ 0 & I \end{pmatrix} \tag{B.25}$$

which maps $H^2 \times \mathbb{R}$ one to one and onto $L^2 \times \{\langle g, \psi_{1*}\rangle : g \in L^2\}$. Therefore, by the implicit function theorem [32], we have a real analytic curve of solutions $s \mapsto (\tilde{f}_{11}(s), \tilde{\mu}_1(s), s)$, defined in a neighborhood of $s = 0$ and coinciding with $(\tilde{f}_{11}^0, \tilde{\mu}_1(0), 0)$ for $s = 0$. The family of solutions we seek is obtained by restriction to $s = |\alpha_0|^2 \geq 0$. This completes the proof of Proposition 4.2.

Appendix C. A Commutator Term

In this section we record a calculation of a "commutator term" appearing in the modulation equations of Sec. 5.

Proposition C.1.

$$i\langle\partial_t(\sigma_3\xi_{01}),\Phi_2\rangle = i\partial_t(|\alpha_0|^2)\left\langle F_0'G_0\begin{pmatrix}1\\1\end{pmatrix},\Phi_2\right\rangle + \partial_t\gamma_0\ |\alpha_0|^2\left\langle\chi G_0\begin{pmatrix}1\\1\end{pmatrix},\Phi_2\right\rangle \tag{C.1}$$

$$i\langle\partial_t(\sigma_3\xi_{02}),\Phi_2\rangle = i\partial_t(|\alpha_0|^2)\left\langle F_0''\sigma_3G_0\begin{pmatrix}1\\1\end{pmatrix},\Phi_2\right\rangle + (\partial_t\gamma_0)|\alpha_0|^2\left\langle\chi\sigma_3G_0\begin{pmatrix}1\\1\end{pmatrix},\Phi_2\right\rangle . \tag{C.2}$$

Proof. By direct computation from (4.23)

$$\partial_tG_0(t) = i(\partial_t\gamma_0)\sigma_3G_0(t)\,. \tag{C.3}$$

Note also that by (4.37)

$$\sigma_3G_0\begin{pmatrix}1\\1\end{pmatrix}F_0 = \xi_{01} = 2\zeta_{01} + |\alpha_0|^2\sigma_3G_0\begin{pmatrix}1\\1\end{pmatrix}\chi(x;|\alpha_0|^2)$$

$$G_0\begin{pmatrix}1\\1\end{pmatrix}F_0' = \xi_{02} = \frac{1}{2}\zeta_{02} + |\alpha_0|^2G_0\begin{pmatrix}1\\1\end{pmatrix}\chi(x;|\alpha_0|^2)\,.$$

Using these relations we have for $j=1$ that

$$\begin{aligned}\partial_t(\sigma_3\xi_{01}(t)) &= \partial_t(|\alpha_0|^2)G_0\begin{pmatrix}1\\1\end{pmatrix}F_0'(|\alpha_0|^2)\\ &\quad + i(\partial_t\gamma_0)\sigma_3G_0\begin{pmatrix}1\\1\end{pmatrix}F_0(|\alpha_0|^2)\\ &= \partial_t(|\alpha_0|^2)G_0\begin{pmatrix}1\\1\end{pmatrix}F_0'(|\alpha_0|^2)\\ &\quad + 2i(\partial_t\gamma_0)\zeta_{01}(t) + (\partial_t\gamma_0)\ |\alpha_0|^2\chi(x;|\alpha_0|^2)G_0\begin{pmatrix}1\\1\end{pmatrix}.\end{aligned}$$

Substitution into the inner product $\langle\partial_t(\sigma_3\xi_{01}(t)),\Phi_2\rangle$ and using the constraint $\langle\zeta_{01}(t),\Phi_2\rangle = 0$ yields the result for $j=1$.

For $j=2$

$$\partial_t(\sigma_3\xi_{02}(t)) = \partial_t(|\alpha_0|^2)\sigma_3G_0\begin{pmatrix}1\\1\end{pmatrix}F_0''(|\alpha_0|^2)$$

$$+ i(\partial_t\gamma_0)G_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} F_0'(|\alpha_0|^2)$$

$$= \partial_t(|\alpha_0|^2)\sigma_3 G_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} F_0'(|\alpha_0|^2)$$

$$+ \frac{i}{2}(\partial_t\gamma_0)\zeta_{02}(t) + \partial_t\gamma_0|\alpha_0|^2\chi(x;|\alpha_0|^2)\sigma_3 G_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Substitution into the inner product $\langle \partial_t(\sigma_3\xi_{02}(t)), \Phi_2\rangle$ and using the constraint $\langle \zeta_{02}(t), \Phi_2\rangle = 0$ yields the result for $j = 2$. This completes the proof of Proposition C.1. □

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