

Diffusion of Power in Randomly Perturbed Hamiltonian Partial Differential Equations

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Abstract: We study the evolution of the energy (mode-power) distribution for a class of randomly perturbed Hamiltonian partial differential equations and derive *master equations* for the dynamics of the expected power in the discrete modes. In the case where the unperturbed dynamics has only discrete frequencies (finitely or infinitely many) the mode-power distribution is governed by an equation of discrete diffusion type for times of order $\mathcal{O}(\varepsilon^{-2})$. Here ε denotes the size of the random perturbation. If the unperturbed system has discrete and continuous spectrum the mode-power distribution is governed by an equation of discrete diffusion-damping type for times of order $\mathcal{O}(\varepsilon^{-2})$. The methods involve an extension of the authors' work on deterministic periodic and almost periodic perturbations, and yield new results which complement results of others, derived by probabilistic methods.

1. Introduction

The evolution of an arbitrary initial condition of linear autonomous Hamiltonian partial differential equation (Schrödinger equation),

$$i \partial_t \phi = H_0 \phi, \quad (1)$$

where H_0 is self-adjoint operator, can be studied by decomposing the initial state in terms of the eigenstates (bound modes) and generalized eigenstates (radiation or continuum modes) of H_0 . The mode amplitudes evolve independently according to a system of decoupled ordinary differential equations and the energy or power in each mode, the square of the mode amplitude, is independent of time. If the system (1) is perturbed

$$i \partial_t \phi = (H_0 + W(t)) \phi, \quad (2)$$

where $W(t)$ respects the Hamiltonian structure ($W^* = W$), then the system of ordinary differential equations typically becomes an infinite coupled system of equations, so-called coupled mode equations. If $W(t)$ has general time-dependence (periodic, almost

periodic, random,...), the solutions of the coupled mode equations can exhibit very complex behavior. Of fundamental importance is the question of how the mode-powers evolve with t . Kinetic equations, which govern their evolution are called *master equations* [25, 5] and go back to the work of Pauli [20]. A general approach to stochastic systems is presented in [17, 19, 18, 13]; see also [1, 7, 8]. Master equations have been derived in many contexts in statistical mechanics, ocean acoustics and optical wave-propagation in waveguides.

We present a theory of power evolution for (2), for a class of perturbations, $W(t)$, which are random in t . Our theory handles the case where H_0 has spectrum consisting of bound states (finitely or infinitely many discrete eigenvalues) and radiation modes (continuous spectrum). It is a natural extension of the analysis in our work on deterministic periodic, almost periodic and nonlinear systems; see, for example, [9, 11, 10, 24]. Our approach is complementary to the probabilistic approach of [7, 8, 19, 18, 13]. The model we consider is well-suited to the study of the effects of an “engineered” perturbation of the system, *e.g.* a prescribed train of light pulses incident on an atomic system, or prescribed distribution of defects encountered by waves propagating along a waveguide; see below. We also give very detailed information on the energy transfer between the subsystems governed by discrete “oscillators” and continuum “radiation field”.

In particular, we study the problem

$$i\partial_t\phi = (H_0 + \varepsilon g(t)\beta)\phi, \quad (3)$$

where ε is small, and H_0 and β are self-adjoint operators on the Hilbert space \mathcal{H} . H_0 is assumed to support finitely or infinitely many bound states. For example, $H_0 = -\Delta + V(x)$, where V is smooth and sufficiently rapidly decaying as $|x| \rightarrow \infty$. β is assumed to be bounded. $g(t)$ is a real valued function of the form of a sequence of short-lived perturbations or “defects” which are identical; see Fig. 1. Our methods can treat both the case when the “defects” are not identical and more general perturbations, *e.g.* $W(t, x) = \beta(t, x)$. For the sole purpose of simplifying the presentation we consider the separable case $W(t) = g(t)\beta(x)$, with $g(t)$ a sequence of identical short-lived perturbations, see below.

Models of the above type arise naturally in many contexts. Among them are the interaction between an atom and a train of light pulses [22, and references therein], a field of great current interest in the control of quantum systems. Such trains of localized perturbations also model sequences of localized defects along waveguides, see [15, 16], introduced by accident or design. In the context of atomic systems, the pulse forms considered in this article correspond to a sequence of identical pulses applied at random times. In the context of single frequency propagation in waveguides, the perturbation

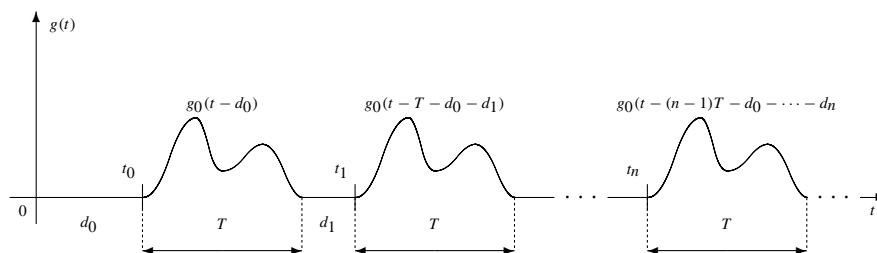


Fig. 1. Train of short lived perturbations or “defects”. The onset time for the n^{th} defect, t_n , is given by (5)

corresponds to a sequence of identical defects, occurring at random distances along the waveguide. In fact, many defects arising in fabrication of waveguides are systematic, and can be modeled in this way. As mentioned above, the methods presented in this article can be extended to treat the case where $g(t)$ is a random sequence of non-identical defects; see also Remark 3.1.

We construct $g(t)$ as follows. Start with $g_0(t)$, a fixed real-valued function with support contained in the interval $[0, T]$ and let $\{d_j\}_{j \geq 0}$ be a nonnegative sequence. Define

$$g(t) = \sum_{n=0}^{\infty} g_0(t - t_n), \text{ where} \quad (4)$$

$$\begin{aligned} t_0 &= d_0, \\ t_n &= (d_0 + T) + (d_1 + T) + \cdots + (d_{n-1} + T) + d_n, n \geq 1 \end{aligned} \quad (5)$$

denotes the onset of the n^{th} defect.

Note that, if the sequence $\{d_j\}_{j \geq 0}$ is periodic then $g(t)$ is periodic. In this case, the system (3) has already been analyzed by time-independent methods [27] or, more recently and under less restrictive hypothesis, in [9, 11]. For $\{d_j\}_{j \geq 0}$ quasiperiodic or almost periodic (see [2, 4] for a definition) the situation is more delicate. In [11] we treat a general class of almost periodic perturbations of the form:

$$W(t) = \sum_{j \geq 0} \cos(\mu_j t) \beta_j, \quad (6)$$

with appropriate ‘‘small denominator’’ hypotheses on the frequencies $\{\mu_j\}$. We leave it for a future paper [10] to consider the case of almost periodic $\{d_j\}_{j \geq 0}$ and to explore the connection with the results in [11]. We note that a particular case has already been treated in [12, Appendix E].

The model we consider is very different from the ones studied by probabilistic methods. For example, in [1] and [17] the numbers d_0, d_1, \dots , are equal to a fixed constant and $g_0(t)$ is random while in our model d_0, d_1, \dots , are random and $g_0(t)$ is fixed. Moreover, the probabilistic approach required a perturbation which is a strongly mixing stochastic process with mean zero, $\mathbb{E}(W(t)) = 0, \forall t > 0$, see [18] and also [7, 8, 13, 17, 19]. In our model $\mathbb{E}(W(t)) = \mathbb{E}(g(t))\beta$ is genuinely time dependent unless g_0 is trivial, $g_0(t) \equiv 0$. Of course one can add the mean to the deterministic part which becomes non-autonomous. The deterministic problem has now a complex evolution which is only understood in special cases, see [11, 23, 27] and references therein. Consequently it is hard if not impossible to apply the probabilistic results.

The paper is divided in two parts. The first part treats stochastic perturbations of Hamiltonian systems with discrete frequencies and the second part extends these results to the case where the unperturbed system has discrete and continuous frequencies. The stochastic perturbation is of order ε and then the vector $P(\tau) \in \ell^1$, whose components are the expected values of the squared discrete mode amplitudes (mode-powers), satisfies on time scales $t = \mathcal{O}(\varepsilon^{-2})$ or equivalently $\tau = \mathcal{O}(1)$, the master equations of diffusion or diffusion-damping type. Specifically, if H_0 has only discrete spectrum (finite or infinite) then

$$\partial_\tau P(\tau) = -BP(\tau), B \geq 0 \quad (7)$$

which has the character of a discrete diffusion equation, *i.e.*

$$\sum_k P_k(\tau) = \sum_k P_k(0), \quad \frac{d}{d\tau} P \cdot P = -\langle P, BP \rangle \leq 0. \quad (8)$$

If H_0 has both discrete and continuous spectra, then

$$\partial_\tau P(\tau) = (-B - \Gamma)P(\tau), \quad B \geq 0, \quad \Gamma = \text{diag}(\gamma_k) > 0 \quad (9)$$

for which

$$\sum_k P_k(\tau) \leq e^{-\gamma\tau} \sum_k P_k(0), \quad (10)$$

where $\gamma = \min_k \gamma_k$.

In Sects. 2 and 3 we study (3) under the hypothesis that H_0 has no continuous spectrum (i.e. no radiation modes) and in Sect. 4 we generalize to the case where H_0 has discrete and continuous spectrum. In Sect. 2 we present the main hypotheses on H_0 and $g_0(t)$ and study the effect of a single short-lived perturbation. In Sect. 3 we present our hypotheses on d_0, d_1, \dots , and analyze the effect of a train of perturbations (3-4). We show that if d_0, d_1, \dots , are independent random variables with certain distributions, see Hypothesis **(H4)** and Examples 1 and 2, diffusion occurs in the expected value for the powers of the modes. Specifically, if we start with energy in one mode, then, on a time scale of order $1/\varepsilon^2$, one can expect the energy to be distributed among all the modes. In Sect. 4 we analyze Eq. (3) under the hypothesis that H_0 has both discrete and continuous spectrum (i.e. supports both bound modes and radiation modes). We prove a result similar to the nonradiative case but now bound state-wave resonances lead to loss of power. The effect of our randomly distributed deterministic perturbation is very similar to the one induced by purely stochastic perturbations, see [1, 13, 19], but quite different from the effects of time almost periodic perturbations, see [9, 11].

Notation.

- 1) $\langle x \rangle = \sqrt{1 + x^2}$.
- 2) Fourier Transform:

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi t} g(t) dt. \quad (11)$$

- 3) We write $\zeta + c.c.$ to mean $\zeta + \bar{\zeta}$, where $\bar{\zeta}$ denotes the complex conjugate of ζ .
- 4) w' denotes the transpose of w .
- 5) $[q]$ denotes the integer part of q .

2. Short-Lived Perturbation of a System with Discrete Frequencies

In this section we consider the perturbed dynamical system

$$i\partial_t \phi(t) = H_0 \phi(t) + \varepsilon g_0(t) \beta \phi(t, x), \quad (12)$$

where H_0 has only discrete spectrum and $g_0(t)$ is a short-lived (compactly supported) function. We study the effect of this perturbation on the distribution of energy among the modes of H_0 . Here and in Sect. 4 we are extending the results in [23] to multiple bound states but under an additional assumption, see (18).

Hypotheses on H_0 , β and $g_0(t)$.

(H1) H_0 is a self adjoint operator on a Hilbert space \mathcal{H} . It has a pure point spectrum formed by the eigenvalues : $\{\lambda_j\}_{j \geq 1}$ with a complete set of orthonormal eigenvectors: $\{\psi_j\}_{j \geq 1}$:

$$H_0 \psi_j = \lambda_j \psi_j, \quad \langle \psi_i, \psi_j \rangle = \delta_{ij}. \quad (13)$$

(H2) β is a bounded self adjoint operator on \mathcal{H} and satisfies $\|\beta\| = 1$.

(H3) $g_0(t) \in L^2(\mathbb{R})$ is real valued, has compact support contained in $[0, T]$ on the positive real line and its L^1 -norm, denoted by $\|g_0\|_1$ is 1. Thus its Fourier transform has L^∞ -norm bounded by 1.

Note that one can always take $\|\beta\| = 1$ and $\|g_0\|_1 = 1$ by setting $\varepsilon = \|g_0\|_1 \cdot \|\beta\|$, thus incorporating the size of $g_0\beta$ in ε . Therefore, under assumptions (H2-H3), ε in (12) measures the actual size of the perturbation in the $L^1(\mathbb{R}, \mathcal{H})$ norm. Our results are perturbative in ε and are valid for ε sufficiently small.

By the standard contraction method one can show that (12) has an unique solution $\phi(t) \in \mathcal{H}$ for all $t \in \mathbb{R}$. Moreover, because both H_0 and $g_0(t)\beta$ are self adjoint operators, we have for all $t \in \mathbb{R}$:

$$\|\phi(t)\| = \|\phi(0)\|. \quad (14)$$

We can write $\phi(t)$ as a sum of projections onto the complete set of orthonormal eigenvectors of H_0 :

$$\phi(t, x) = \sum_j a_j(t) \psi_j(x). \quad (15)$$

By Parseval's relation

$$\sum_j |a_j(t)|^2 = \|\phi(t)\|^2 \equiv \|\phi(0)\|^2. \quad (16)$$

Now (12) can be rewritten as

$$i \partial_t a_k(t) = \lambda_k a_k(t) + \varepsilon g_0(t) \sum_j a_j(t) \langle \psi_k, \beta \psi_j \rangle, \quad k \in \{1, 2, \dots\}, \quad (17)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathcal{H} .

Hence Eq. (12) is equivalent to a weakly coupled linear system in the amplitudes: a_1, a_2, \dots , (17).

Since the perturbation size is ε we expect, in general, that the change in energy in the k^{th} mode, $|a_k(t)|^2 - |a_k(0)|^2$, to be of order ε . However with a suitable random initial condition we can prove more subtle behavior.

Suppose that there exists an averaging procedure applicable to the amplitudes: a_1, a_2, \dots of the solutions of (12), denoted by

$$a(t) \mapsto \mathbb{E}(a(t)) \in \mathbb{C}.$$

We now state a fundamental result, applied throughout this paper, for a single defect which is compactly supported in time:

Theorem 2.1. *Assume the conditions (H1)-(H3) hold and the initial values for (12) are such that*

$$\mathbb{E} \left(a_j(0) \overline{a_k(0)} \right) = 0 \quad \text{whenever } j \neq k. \quad (18)$$

Then for all $t > \sup\{s \in \mathbb{R} \mid g_0(s) \neq 0\}$ and $k \in \{1, 2, \dots\}$ we have

$$P_k(t) - P_k(0) = \varepsilon^2 \sum_j |\alpha_{kj}|^2 |\hat{g}_0(-\Delta_{kj})|^2 (P_j(0) - P_k(0)) + \mathcal{O}(\varepsilon^3), \quad (19)$$

where

$$P_k(t) \equiv \mathbb{E} \left(|a_k(t)|^2 \right)$$

denotes the average power in the k^{th} -mode at time t , $\alpha_{kj} \equiv \langle \psi_k, \beta \psi_j \rangle$, \hat{g}_0 denotes the Fourier transform of g_0 and $\Delta_{kj} \equiv \lambda_k - \lambda_j$.

Note that (19) can be written in the form:

$$P_k(t) = T_\varepsilon P_k(0) + \mathcal{O}(\varepsilon^3), \quad (20)$$

where

$$T_\varepsilon = \mathbb{I} - \varepsilon^2 B; \quad B \geq 0, \quad (21)$$

\mathbb{I} is the identity operator (matrix) and B is given by

$$B = (b_{kj})_{1 \leq k, j}, \quad b_{kj} = \begin{cases} -|\alpha_{kj}|^2 |\hat{g}_0(-\Delta_{kj})|^2, & \text{for } j \neq k, \\ \sum_{l, l \neq k} |\alpha_{kl}|^2 |\hat{g}_0(-\Delta_{kl})|^2, & \text{for } j = k. \end{cases} \quad (22)$$

In Sect. 3 we will discuss and use the properties of B and T_ε .

Proof of Theorem 2.1. In the amplitude system, (17), we remove the fast oscillations by letting

$$a_k(t) = e^{-i\lambda_k t} A_k(t). \quad (23)$$

Note that by (16)

$$\sum_j |A_j(t)|^2 \equiv \|\phi(0)\|^2. \quad (24)$$

Now (17) becomes

$$i \partial_t A_k(t) = \varepsilon g_0(t) \sum_j \alpha_{kj} e^{i\Delta_{kj} t} A_j(t), \quad (25)$$

where

$$\Delta_{kj} \equiv \lambda_k - \lambda_j, \quad (26)$$

$$\alpha_{kj} \equiv \langle \psi_k, \beta \psi_j \rangle = \bar{\alpha}_{jk}. \quad (27)$$

The above system leads to the following one in product of amplitudes, $A_k(t)\bar{A}_l(t)$:

$$\begin{aligned} \partial_t(A_k(t)\bar{A}_l(t)) &= i\varepsilon g_0(t) \sum_j \alpha_{jl} e^{i\Delta_{jl}t} A_k(t)\bar{A}_j(t) \\ &\quad - i\varepsilon g_0(t) \sum_j \alpha_{kj} e^{i\Delta_{kj}t} A_j(t)\bar{A}_l(t). \end{aligned} \quad (28)$$

In the particular case $k = l$ we have the power equation for each mode:

$$\partial_t |A_k(t)|^2 = i\varepsilon g_0(t) \sum_j \alpha_{jk} e^{i\Delta_{jk}t} A_k(t)\bar{A}_j(t) + c.c. \quad (29)$$

Note that the sum in (29) commutes with time integral and expected value operators. This is due to (24) and the dominant convergence theorem, see for example [6]. Indeed consider

$$f_m(t) = \sum_{j=1}^m \alpha_{jk} e^{i\Delta_{jk}t} A_k(t)\bar{A}_j(t) g_0(t).$$

From (15) we have for all $t \in \mathbb{R}$,

$$\lim_{m \rightarrow \infty} f_m(t) = \langle \phi(t), \beta \psi_k \rangle a_k(t) g_0(t).$$

From (24) and the Cauchy-Schwarz inequality $|\langle a, b \rangle| \leq \|a\| \|b\|$, we have for all $t \in \mathbb{R}$,

$$|f_m(t)| \leq \|\phi(0)\|^2 |g_0(t)|. \quad (30)$$

The right-hand side of (30) is integrable and the dominant convergence theorem applies. A similar argument is valid for expected values. Therefore, from now on, we are going to commute both time integrals and expected values with summations like the one in (29).

We integrate (29) from 0 to $t > \sup\{s \in \mathbb{R} \mid g_0(s) \neq 0\}$ and integrate by parts the right-hand side. The result is:

$$\begin{aligned} |A_k(t)|^2 - |A_k(0)|^2 &= i\varepsilon \sum_j \alpha_{jk} \int_0^t g_0(s) e^{i\Delta_{jk}s} A_k(s)\bar{A}_j(s) + c.c. \\ &= -i\varepsilon \sum_j \alpha_{jk} \int_s^\infty g_0(\tau) e^{i\Delta_{jk}\tau} d\tau A_k(s)\bar{A}_j(s) \Big|_{s=0}^{s=t} + c.c. \quad (31) \\ &\quad + i\varepsilon \sum_j \alpha_{jk} \int_0^t \int_s^\infty g_0(\tau) e^{i\Delta_{jk}\tau} d\tau \partial_s (A_k\bar{A}_j)(s) ds + c.c. \end{aligned}$$

The boundary terms are

$$\begin{aligned} & -i\varepsilon \sum_j \alpha_{jk} \int_s^\infty g_0(\tau) e^{i\Delta_{jk}\tau} d\tau A_k(s) \bar{A}_j(s) \Big|_{s=0}^{s=t} + c.c. \\ & = i\varepsilon \sum_j \alpha_{jk} \hat{g}_0(-\Delta_{jk}) A_k(0) \bar{A}_j(0) + c.c., \end{aligned} \quad (32)$$

where \hat{g}_0 denotes the Fourier Transform of g_0 ; see (11). Note that upon taking the average, using (18) and the fact that $\hat{g}_0(0)$ is real, these boundary terms vanish.

Into the last term in (31) we substitute (28):

$$\begin{aligned} & i\varepsilon \sum_j \alpha_{jk} \int_0^t \int_s^\infty g_0(\tau) e^{i\Delta_{jk}\tau} d\tau \partial_s (A_k \bar{A}_j)(s) ds = \\ & = +|\varepsilon|^2 \sum_{j,p} \alpha_{jk} \alpha_{kp} \int_0^t \int_s^\infty g_0(\tau) e^{i\Delta_{jk}\tau} d\tau g_0(s) e^{i\Delta_{kp}s} A_p(s) \bar{A}_j(s) ds \\ & \quad - |\varepsilon|^2 \sum_{j,q} \alpha_{jk} \alpha_{qj} \int_0^t \int_s^\infty g_0(\tau) e^{i\Delta_{jk}\tau} d\tau g_0(s) e^{i\Delta_{qj}s} A_k(s) \bar{A}_q(s) ds. \end{aligned} \quad (33)$$

We again integrate by parts both terms in (33):

$$\begin{aligned} & i\varepsilon \sum_j \alpha_{jk} \int_0^t \int_s^\infty g_0(\tau) e^{i\Delta_{jk}\tau} d\tau \partial_s (A_k \bar{A}_j)(s) ds \\ & = -|\varepsilon|^2 \sum_{j,p} \alpha_{jk} \alpha_{kp} \int_u^\infty g_0(s) e^{i\Delta_{kp}s} \int_s^\infty g_0(\tau) e^{i\Delta_{jk}\tau} d\tau ds A_p(u) \bar{A}_j(u) \Big|_{u=0}^{u=t} \\ & \quad + |\varepsilon|^2 \sum_{j,q} \alpha_{jk} \alpha_{qj} \int_u^\infty g_0(s) e^{i\Delta_{qj}s} \int_s^\infty g_0(\tau) e^{i\Delta_{jk}\tau} d\tau ds A_k(u) \bar{A}_q(u) \Big|_{u=0}^{u=t} \\ & \quad + |\varepsilon|^2 \sum_{j,p} \alpha_{jk} \alpha_{kp} \int_0^t \int_u^\infty \int_s^\infty g_0(\tau) e^{i\Delta_{jk}\tau} d\tau g_0(s) e^{i\Delta_{kp}s} ds \partial_u (A_p \bar{A}_j)(u) du \\ & \quad + |\varepsilon|^2 \sum_{j,q} \alpha_{jk} \alpha_{qj} \int_0^t \int_u^\infty \int_s^\infty g_0(\tau) e^{i\Delta_{jk}\tau} d\tau g_0(s) e^{i\Delta_{qj}s} ds \partial_u (A_k \bar{A}_q)(u) du. \end{aligned} \quad (34)$$

Note that the boundary terms calculated at “ $u = t$ ” are zero since $t > \sup\{s \in \mathbb{R} \mid g_0(s) \neq 0\}$. Upon taking the expected value and using (18) the only boundary terms contributing are the ones for which $u = 0$ and $j = p$ in the second row of (34):

$$\begin{aligned} & \sum_j |\alpha_{kj}|^2 \int_0^\infty \int_s^\infty g_0(\tau) e^{i\Delta_{jk}\tau} d\tau g_0(s) e^{i\Delta_{kj}s} ds \mathbb{E}(|A_j(0)|^2) + c.c. \\ & = \sum_j |\alpha_{kj}|^2 \mathbb{E}(|A_j(0)|^2) \cdot 2\Re \int_0^\infty \int_s^\infty g_0(\tau) e^{i\Delta_{jk}\tau} d\tau g_0(s) e^{i\Delta_{kj}s} ds, \end{aligned} \quad (35)$$

and the ones for which $u = 0$ and $q = k$ in the third row of (34):

$$\begin{aligned} & \sum_j |\alpha_{kj}|^2 \int_0^\infty \int_s^\infty g_0(\tau) e^{i\Delta_{jk}\tau} d\tau g_0(s) e^{i\Delta_{kj}s} ds \mathbb{E}(|A_k(0)|^2) + c.c. \\ &= \sum_j |\alpha_{kj}|^2 \mathbb{E}(|A_k(0)|^2) \cdot 2\Re \int_0^\infty \int_s^\infty g_0(\tau) e^{i\Delta_{jk}\tau} d\tau g_0(s) e^{i\Delta_{kj}s} ds. \end{aligned} \quad (36)$$

To compute (35-36) we use the lemma:

Lemma 2.1. *If $g_0(t)$, $t \in \mathbb{R}$ is real valued and square integrable with compact support included in the positive real line then for all $\lambda \in \mathbb{R}$ the following identity holds:*

$$2\Re \int_0^\infty \int_s^\infty g_0(\tau) e^{i\lambda\tau} d\tau g_0(s) e^{-i\lambda s} ds = |\hat{g}_0(-\lambda)|^2.$$

Proof. For any $\lambda \in \mathbb{R}$ we have:

$$\begin{aligned} I(\lambda) &\equiv 2\Re \int_0^\infty \int_s^\infty g_0(\tau) e^{i\lambda\tau} d\tau g_0(s) e^{-i\lambda s} ds \\ &= 2 \int_0^\infty \int_s^\infty g_0(\tau) g_0(s) \cos[\lambda(\tau - s)] d\tau ds. \end{aligned}$$

As $(s, \tau) \mapsto g_0(\tau) g_0(s) \cos[\lambda(\tau - s)]$ is symmetric with respect to the diagonal $\tau = s$,

$$\begin{aligned} I(\lambda) &= \int_0^\infty \int_0^\infty g_0(\tau) g_0(s) \cos[\lambda(\tau - s)] d\tau ds \\ &= \frac{1}{2} \int_0^\infty \int_0^\infty g_0(\tau) g_0(s) \left(e^{i\lambda\tau} e^{-i\lambda s} + e^{-i\lambda\tau} e^{i\lambda s} \right) d\tau ds \\ &= |\hat{g}_0(-\lambda)|^2. \end{aligned}$$

□

Into the triple integral terms of (34) we again substitute (28). Then one can show that the 1-norm of this correction vector is dominated by $|\varepsilon|^3 \|g_0\|_1^3 \|\beta\|^3 \|\phi(0)\|^2$. Hence, it is of order $\mathcal{O}(|\varepsilon|^3)$.

Thus, after applying Lemma 2.1 to (35–36) and using (31) we arrive at the conclusion of Theorem 2.1. □

3. Diffusion of Power in Discrete Frequency (Nonradiative) Systems

In the previous section we calculated the effect of a single defect on the mode-power distribution. In this section we show how to apply this result to prove diffusion of power for the perturbed Hamiltonian system, (2), where $g(t)$ is a random function of the form (4), defined in terms of a random sequence $\{d_j\}_{j \geq 0}$. In particular, the sequence $\{d_j\}_{j \geq 0}$ will be taken to be generated by independent, identically distributed random variables. This will result in a *mixing the phases* of the complex mode amplitudes, after each defect.

We assume that **(H1-H3)** are satisfied. The following hypothesis ensures that (18) holds before each defect, thus enabling repeated application of Theorem 2.1.

(H4) d_0, d_1, \dots are independent identically distributed random variables taking only nonnegative values and such that for any $l \in \{0, 1, \dots\}$ and $j \neq k \in \{1, 2, \dots\}$ we have

$$\mathbb{E}\left(e^{i(\lambda_j - \lambda_k)d_l}\right) = 0,$$

where $\mathbb{E}(\cdot)$ denotes the expected value.

Clearly **(H4)** requires the eigenvalues to be distinct but aside from these we claim that for any *finitely many, distinct* eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$ there exist a random variable satisfying **(H4)**.

Example 1 (finitely many bound states). Given $\lambda_1, \lambda_2, \dots, \lambda_m$ distinct choose the random variables $d_l, l = 0, 1, \dots$ to be identically distributed with distribution d :

$$d = \sum_{1 \leq j < k \leq m} d_{jk},$$

where d_{jk} are independent random variables such that the distribution of d_{jk} is either uniform on the interval $[0, 2\pi/|\lambda_j - \lambda_k|]$ or d_{jk} takes each of the values 0 and $\pi/|\lambda_j - \lambda_k|$ with probability 1/2. In any case, for any $j' \neq k' \in \{1, 2, \dots\}$

$$\mathbb{E}\left(e^{i(\lambda_{j'} - \lambda_{k'})d}\right) = \mathbb{E}\left(\prod_{1 \leq j < k} e^{i(\lambda_{j'} - \lambda_{k'})d_{jk}}\right) = \prod_{1 \leq j < k} \mathbb{E}\left(e^{i(\lambda_{j'} - \lambda_{k'})d_{jk}}\right) = 0$$

since $\mathbb{E}\left(e^{i(\lambda_{j'} - \lambda_{k'})d_{j'k'}}\right) = 0$.

(H4) does not restrict us to a system with finitely many bound states:

Example 2 (infinitely many bound states). Let the quantum harmonic oscillator in one dimension:

$$H_0 = -\frac{\hbar^2}{2}\partial_x^2 + \omega^2 x^2, \quad x \in \mathbb{R},$$

be the unperturbed Hamiltonian. Then $\lambda_n = \hbar\omega(n + 1/2)$, $n = 0, 1, 2, \dots$, see for example [14]. Note that **(H4)** holds provided that we choose $d_l, l = 0, 1, \dots$ to be identically and uniformly distributed on the interval $[0, 2\pi/(\hbar\omega)]$.

Note on degenerate eigenvalues. As discussed above **(H4)** cannot be satisfied in the case H_0 admits degenerate eigenvalues. However, at least in some cases, our theory can be applied. In general the degeneracy is a consequence of the symmetries of H_0 , i.e. the existence of a self-adjoint operator, say L , commuting with H_0 , $[L, H_0] = 0$. To recover our results it is sufficient to assume that β , the “space-like” part of the perturbation, respects the symmetry, i.e. commutes with L . One can now factor out L , i.e. work on the invariant subspaces of L where H_0 is nondegenerate. Along the lines of Example 2 one can consider the quantum harmonic oscillator in three dimensions which has a spherically symmetric Hamiltonian and degenerate eigenvalues, see for example [14]. If β is spherically symmetric then it only couples bound states with the same angular momentum. Hence the problem reduces to subsystems consisting of bound states with the same angular momentum but different energy, therefore nondegenerate. The choice we made in Example 2 will satisfy **(H4)** in each of the subsystems.

3.1. Power diffusion after a fixed (large) number of defects.

Theorem 3.1. *Consider Eq. (12) with g of the form (4). Assume **(H1-H4)** hold. Then the expected value of the power vector after passing a fixed number of perturbations “ n ” satisfies*

$$P^{(n)} = T_\varepsilon^n P(0) + \mathcal{O}(n\varepsilon^3), \quad (37)$$

where T_ε is given in (21),

$$P_k^{(n)} = \mathbb{E}(|a_k(t)|^2), \quad k = 1, 2, \dots, \quad (38)$$

$t_{n-1} + T \leq t \leq t_n$, (t ranging between the n^{th} and $(n+1)^{\text{st}}$ defects.

Proof. We will prove the theorem by induction on $n \geq 0$, the number of defects traversed. For $n = 0$ the assertion is obvious. Suppose now that for $n \geq 0$ we have

$$P^{(n)} = T_\varepsilon^n P(0) + \mathcal{O}(n\varepsilon^3). \quad (39)$$

We will show

$$P^{(n+1)} = T_\varepsilon^{n+1} P(0) + \mathcal{O}((n+1)\varepsilon^3) \quad (40)$$

by applying Theorem 2.1 to (39). In order to apply Theorem 2.1 we need to verify that (18) is satisfied before the $n+1^{\text{st}}$ defect. Specifically, we must verify that for any pair $k \neq j$,

$$\begin{aligned} \mathbb{E}(a_k(t_{n+1})\bar{a}_j(t_{n+1})) &= \mathbb{E}\left(a_k\left(nT + \sum_{k=0}^{n+1}d_k\right)\bar{a}_j\left(nT + \sum_{k=0}^{n+1}d_k\right)\right) \\ &= 0. \end{aligned} \quad (41)$$

Using the fact that d_{n+1} is independent of $d_0 + d_1 + \dots + d_n$, and **(H4)** we have:

$$\begin{aligned} \mathbb{E}\left(a_k\bar{a}_j\left(nT + \sum_{k=0}^{n+1}d_k\right)\right) &= \mathbb{E}\left(a_k\bar{a}_j\left(nT + \sum_{k=0}^n d_k\right) e^{i(\lambda_j - \lambda_k)d_{n+1}}\right) \\ &= \mathbb{E}\left(a_k\bar{a}_j\left(nT + \sum_{k=0}^n d_k\right)\right) \mathbb{E}\left(e^{i(\lambda_j - \lambda_k)d_{n+1}}\right) = 0. \end{aligned}$$

Thus (41) holds and all the hypotheses of Theorem 2.1 are now satisfied. By applying it and using (39) we have

$$\begin{aligned} P^{(n+1)} &= T_\varepsilon P^{(n)} + \mathcal{O}(\varepsilon^3) \\ &= T_\varepsilon \left(T_\varepsilon^n P(0) + \mathcal{O}(n\varepsilon^3)\right) + \mathcal{O}(\varepsilon^3) \\ &= T_\varepsilon^{n+1} P(0) + \mathcal{O}((n+1)\varepsilon^3). \end{aligned}$$

Hence (39) implies (40). This concludes the induction step and the proof of Theorem 3.1 is now complete. \square

In the next two corollaries we describe the asymptotic behavior of the vector of expected powers when the number of defects n tends to infinity. Note that after a

possible reordering of the eigenvectors ψ_1, ψ_2, \dots , of H_0 , the operator B given by (22) might look like¹:

$$B = \text{diag} [B_1, B_2, \dots, B_q, \dots], \quad (42)$$

where $B_1, B_2, \dots, B_q, \dots$ are square matrices (linear operators) of dimensions $m_1, m_2, \dots, m_q, \dots$, $1 \leq m_q \leq \infty$, $q = 1, 2, \dots$. In linear algebra terms this means that B is reducible. In terms of the dynamical system (37) generated by $T_\varepsilon = \mathbb{I} - \varepsilon^2 B$ it means that, after a possible reordering, the first m_1 bound states of H_0 are isolated from the rest. The same is valid for the next m_2 bound states, etc. To understand the evolution of the full system it is sufficient to analyze each of the isolated subsystems separately. They all evolve according to (37) with $T_\varepsilon = \mathbb{I} - \varepsilon^2 B_q$ and B_q given by (22) but the indices span only a subset of the eigenvectors ψ_1, ψ_2, \dots of H_0 . The main difference is that now B_q is irreducible. In what follows we are focusing on one such subsystem and drop the index q .

Corollary 3.1. *If the subsystem has a finite number of bound states, say m , then*

$$\lim_{n \rightarrow \infty} P^{(n)} = \begin{cases} P(0), & \text{if } n \ll \varepsilon^{-2} \\ e^{-B\tau} P(0) & \text{if } n = \tau \varepsilon^{-2}, \\ \frac{E}{m} (1, 1, \dots, 1)', & \text{if } \varepsilon^{-2} \ll n \ll |\varepsilon|^{-3} \end{cases}, \quad (43)$$

where $E = P_1(0) + P_2(0) + \dots + P_m(0)$ is the expected total power in the subsystem and it is conserved.

Proof. We use the following properties of the irreducible matrix B :

(B1) B is self adjoint and $B \geq 0$;

(B2) 0 is a simple eigenvalue for B with corresponding normalized eigenvector

$$r_0 = \frac{1}{\sqrt{m}} (1, 1, \dots, 1)'. \quad (44)$$

These properties are proved in the Appendix.

Let $\beta_0 = 0, \beta_1, \beta_2, \dots, \beta_{m-1}$ be the eigenvalues of B counting multiplicity, and let r_0, r_1, \dots, r_{m-1} be the corresponding orthonormalized eigenvectors. By (B1) and (B2) $\beta_1, \beta_2, \dots, \beta_{m-1}$ are strictly positive. Let

$$R = [r_0, r_1, \dots, r_{m-1}]$$

be the matrix whose columns are orthonormalized eigenvectors of B and let R' be its transpose. Then

$$\begin{aligned} R' B R &= \text{diag} [\beta_0, \beta_1, \beta_2, \dots, \beta_{m-1}], \\ R' R &= \mathbb{I} = R R'. \end{aligned}$$

It follows that

$$\begin{aligned} T_\varepsilon^n &= (\mathbb{I} - \varepsilon^2 B)^n = R \left[R' (\mathbb{I} - \varepsilon^2 B) R \right]^n R' \\ &= R \text{diag} \left[(1 - \varepsilon^2 \beta_0)^n, (1 - \varepsilon^2 \beta_1)^n, \dots, (1 - \varepsilon^2 \beta_{m-1})^n \right] R'. \end{aligned}$$

¹ For such a decomposition to occur it is sufficient that H_0 and β have common invariant subspaces $\mathcal{H}_1 \subset \mathcal{H}, \mathcal{H}_2 \subset \mathcal{H}, \dots, \mathcal{H}_q \subset \mathcal{H}, \dots$

We now study $\lim_{n \rightarrow \infty} T_\varepsilon^n$ for the three asymptotic regimes of (43). Note that for $0 \leq k \leq m-1$ we have:

$$\begin{aligned} \lim_{n \rightarrow \infty, \varepsilon^2 n \rightarrow 0} (1 - \varepsilon^2 \beta_k)^n &= 1, \\ \lim_{n \rightarrow \infty, \varepsilon^2 n = \tau} (1 - \varepsilon^2 \beta_k)^n &= e^{-\beta_k \tau}, \\ \lim_{n \rightarrow \infty, \varepsilon^2 n \rightarrow \infty} (1 - \varepsilon^2 \beta_k)^n &= 0, \quad \beta_k > 0, \\ \lim_{n \rightarrow \infty, \varepsilon^2 n \rightarrow \infty} (1 - \varepsilon^2 \beta_k)^n &= 1, \quad \beta_k = 0. \end{aligned} \quad (45)$$

Consequently,

$$\lim_{n \rightarrow \infty} T_\varepsilon^n = \begin{cases} R \operatorname{diag}[1, 1, \dots, 1] R' = \mathbb{I} & \text{if } n \ll \varepsilon^2 \\ R \operatorname{diag}[e^{-\beta_0 \tau}, e^{-\beta_1 \tau}, \dots, e^{-\beta_{m-1} \tau}] = e^{-B\tau} & \text{if } n = \tau \varepsilon^{-2}, \\ R \operatorname{diag}[1, 0, 0, \dots, 0] R' = \text{projection onto } r_0 & \text{if } \varepsilon^{-2} \ll n \ll |\varepsilon|^{-3} \end{cases} \quad (46)$$

where r_0 is defined in (44).

Substitution of (46) into (37) completes the proof of Corollary 3.1. \square

Corollary 3.2. *If the subsystem has an infinite number of bound states, then*

$$\lim_{n \rightarrow 0} P^{(n)} = \begin{cases} P(0), & \text{if } n \ll \varepsilon^{-2} \\ e^{-B\tau} P(0) & \text{if } n = \tau \varepsilon^{-2}. \end{cases} \quad (47)$$

For $n \gg \varepsilon^{-2}$ the limit in ℓ^2 is 0, while the limit in ℓ^1 does not exist. More precisely, although the total power in the subsystem is conserved,

$$\sum_{k=1}^{\infty} P_k^{(n)} = E, \quad \forall n \geq 0, \quad (48)$$

$\{P^{(n)}\}$ does not converge in ℓ^1 due to an energy transfer to the high modes. In particular, for any fixed $N \geq 1$:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=N}^{\infty} P_k^{(n)} &= E, \\ \lim_{n \rightarrow \infty} \sum_{k=1}^N P_k^{(n)} &= 0. \end{aligned} \quad (49)$$

We note that similar results have been obtained in [1] but for different types of random perturbation.

Corollaries 3.1 and 3.2 show that, on time scales of order $1/\varepsilon^2$, the dynamical system is equivalent with

$$\partial_\tau P(\tau) = -BP(\tau). \quad (50)$$

Moreover the definition of $-B$ in (22) together with $-B \leq 0$ and e^{-B} unitary on ℓ^1 implies that the flow (50) is very much like that of a discrete heat or diffusion equation.

In conclusion the number of defects encountered should be comparable with $1/\varepsilon^2$ to have a significant effect. Once they are numerous enough, the defects diffuse the power in the system. If the number of defects is much larger than $1/\varepsilon^2$ the power becomes uniformly distributed among the bound states.

Remark 3.1. The asymptotic picture described above remains valid even when the “defects” are not identical, that is when (4) is replaced by $g(t) = \sum_n g_n(t - t_n)$ with g_n real valued functions satisfying **(H3)**. In this case the coupling matrix $B^{(n)}$ for the n^{th} defect is given by (22) with g_0 replaced by g_n while the corresponding transmission matrix is $T_\varepsilon^{(n)} = \mathbb{I} - \varepsilon^2 B^{(n)}$. As in Theorem 3.1 the expected power after n defects will be $P^{(n)} = T_\varepsilon^{(n-1)} T_\varepsilon^{(n-2)} \dots T_\varepsilon^{(0)} P(0) + \mathcal{O}(n\varepsilon^3)$. Moreover, for $n \ll \varepsilon^2$ or $n \gg \varepsilon^2$ the results of Corollaries 3.1 and 3.2 hold. For $n \sim \varepsilon^2$ the limit might not exist in general. There are exceptions though. One is when g_n converges in $L^2(\mathbb{R})$ to a certain shape denoted by g_0 as $n \rightarrow \infty$. Another one is when g_n is an almost periodic sequence in which case we denote by g_0 its mean, see [2]. It would be interesting to understand what happens when the shapes g_n are random. We speculate that a diffusion matrix B can still be computed using the technique in [17].

Remark 3.2. Hypothesis **(H4)** is important. If we do not assume **(H4)** then the correction term for each defect is of size ε , since the boundary terms (32) no longer vanish. Consequently the correction term in the main result (37) is $\mathcal{O}(n\varepsilon)$ which on the “diffusion time scale” $n \sim \varepsilon^{-2}$ is very large.

Proof of Corollary 3.2. In the case of an infinite number of bound states, B has the following properties, see the Appendix:

- (B1 $_\infty$) B is a nonnegative, bounded self adjoint operator on ℓ^2 with spectral radius less than or equal to 2;
- (B2 $_\infty$) 0 is not an eigenvalue for B ;
- (B3 $_\infty$) B is a bounded operator on ℓ^1 with norm $\|B\|_1 \leq 2$;
- (B4 $_\infty$) For $|\varepsilon| \leq 1$ the operator $T_\varepsilon = (\mathbb{I} - \varepsilon^2 B)$ transforms positive vectors (i.e. all components positive) into positive vectors and conserves their ℓ^1 norm.

We are going to focus first on ℓ^2 results. Based on the spectral representation theorem, see [21], we have for any Borel measurable real function f :

$$f(B) = \int_0^2 f(s) d\mu(s). \quad (51)$$

Here $d\mu(s)$ is the spectral measure induced by B . Note that B2 $_\infty$ implies the continuity of $\mu(s)$ at zero.

Now

$$T_\varepsilon^n = (\mathbb{I} - \varepsilon^2 B)^n = \int_0^2 (1 - \varepsilon^2 s)^n d\mu(s)$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} T_\varepsilon^n &= \lim_{\varepsilon \rightarrow 0} \int_0^2 (1 - \varepsilon^2 s)^n d\mu(s) \\ &= \int_0^2 \lim_{\varepsilon \rightarrow 0} (1 - \varepsilon^2 s)^n d\mu(s). \end{aligned} \quad (52)$$

For the last equality we used the dominant convergence theorem with $|1 - \varepsilon^2 s|^n \leq 1$ for $0 \leq s \leq 2$, $|\varepsilon| \leq 1$ and $\int_0^2 1 d\mu(s) = \mathbb{I}$. Using (45), with s replacing β_k , we have that (52) becomes

$$\lim_{n \rightarrow \infty} T_\varepsilon^n = \begin{cases} \int_0^2 1 d\mu(s) = \mathbb{I} & \text{if } n \ll \varepsilon^2 \\ \int_0^2 e^{-s\tau} d\mu(s) = e^{-B\tau} & \text{if } n = \tau \varepsilon^{-2} \\ \mu(0+) - \mu(0) = 0 & \text{if } \varepsilon^{-2} \ll n \ll |\varepsilon|^{-3} \end{cases}, \quad (53)$$

where we used (51) and the continuity of $\mu(s)$ at zero.

Plugging (53) in (37) gives the required results in ℓ^2 .

For the results in ℓ^1 we use series expansions:

$$\begin{aligned} (\mathbb{I} - \varepsilon^2 B)^n &= \mathbb{I} + \binom{n}{1} \varepsilon^2 (-B) + \binom{n}{2} \varepsilon^4 (-B)^2 \\ &\quad + \dots + \binom{n}{n} \varepsilon^{2n} (-B)^n. \end{aligned} \quad (54)$$

Since $\|B\|_1 \leq 2$, (see property B3 $_\infty$), the finite series above is dominated in the ℓ^1 operator norm by:

$$1 + 2\varepsilon^2 \binom{n}{1} + (2\varepsilon^2)^2 \binom{n}{2} + \dots + (2\varepsilon^2)^n \binom{n}{n} = (1 + 2\varepsilon^2)^n \leq e^{2n\varepsilon^2}. \quad (55)$$

As $n \rightarrow \infty$ the series in (55) becomes infinite. However, as long as $n \leq \tau/\varepsilon^2$, $\tau > 0$ fixed, the sum in (55) is finite and hence that in (54) is convergent. Now for each $k = 1, 2, \dots$, the $(k+1)$ st term in the series (54) has the property:

$$\lim_{n \rightarrow \infty} \binom{n}{k} \varepsilon^{2k} (-B)^k = \begin{cases} 0 & \text{if } n \ll \varepsilon^{-1} \\ \frac{\tau^k}{k!} (-B)^k & \text{if } n = \tau \varepsilon^{-2} \end{cases}.$$

Hence by the Weierstrass criterion for absolutely convergent series we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} T_\varepsilon^n &= \lim_{n \rightarrow \infty} (\mathbb{I} - \varepsilon^2 B)^n \\ &= \begin{cases} \mathbb{I} - 0 + 0 - \dots = \mathbb{I} & \text{if } n \ll \varepsilon^{-1} \\ \mathbb{I} - \tau B + \frac{(\tau B)^2}{2!} - \frac{(\tau B)^3}{3!} + \dots = e^{-\tau B} & \text{if } n = \tau \varepsilon^{-2} \end{cases}. \end{aligned} \quad (56)$$

It remains to prove that as $n \rightarrow \infty$, $\varepsilon^2 n \rightarrow \infty$, $\{P^{(n)}\}$ does not converge in ℓ^1 . Let $P^{(0)} \in \ell^1 \cap \ell^2$ denote a vector with positive components, and consider the sequence:

$$P^{(n)} = T_\varepsilon^n P^{(0)} \in \ell^1 \cap \ell^2. \quad (57)$$

By the third part of (53), $\|P^{(n)}\|_2 \rightarrow 0$. Assume now that there exists $P \in \ell^1$ such that $\|P^{(n)} - P\|_1 = 0$. Since both ℓ^1 and ℓ^2 convergence imply convergence of each component, we deduce that $P = 0$. On the other hand, by $P^{(n)} = T_\varepsilon P^{(n-1)}$, $n=1, 2, \dots$ and property B4 $_\infty$, we deduce that $P^{(n)}$ is a positive vector for which $\|P^{(n)}\|_1 = \|P^{(0)}\|_1 \stackrel{def}{=} E > 0$ for all $n \geq 0$. Consequently P is a nonnegative vector with $\|P\|_1 = E > 0$, a contradiction. The proof of the corollary is now complete. \square

3.2. *Power diffusion after a fixed (large) time interval and a random number of defects.* As pointed out in its statement, Theorem 3.1 is valid when one measures the power vector after a fixed number of defects “ n ” regardless of the realizations of the random variables. That is after each realization of d_0, d_1, \dots the power vector is measured in between the n^{th} and the $(n + 1)^{\text{st}}$ defect. Averaging the measurements over all the realizations of d_0, d_1, d_2, \dots gives the result of Theorem 3.1. What happens if one chooses to measure the power vector at a fixed time “ t ” (i.e. a fixed distance along the fiber)? The answer is given by the next theorem:

Theorem 3.2. *Consider Eq. (12) with g of the form (4). Assume that (H1-H4) are satisfied and that all random variables d_0, d_1, \dots , have finite mean, variance and third momentum. Fix a time t , $0 \leq t \ll 1/|\varepsilon|^3$. Then the expected value of the power vector at a fixed time $P(t)$ satisfies*

$$P(t) = T_\varepsilon^n P(0) + \mathcal{O}(\max\{t\varepsilon^3, \varepsilon^{4/5}\}), \quad (58)$$

where $n = \lfloor t/(T + M) \rfloor$ denotes the integer part of $t/(T + M)$, T is the common time span of the defects and M is the mean of the identically distributed random variables d_0, d_1, \dots .

Corollary 3.3. *In this setting, the conclusions of Corollaries 3.1, 3.2 and Remark 3.1 hold with n replaced by t .*

Proof of Theorem 3.2. As before, let $P^{(k)}$ be the expected power vector after exactly “ k ” defects. Denote by N the random variable counting the number of “defects” up until the fixed time t , i.e.

$$(N - 1)T + d_0 + \dots + d_{N-1} < t \leq NT + d_0 + \dots + d_N, \quad (59)$$

and let $\delta(\varepsilon)$ denote the integer, which grows as ε decreases:

$$\begin{aligned} \tilde{\delta} &= \max \left\{ 1.39 \left(\frac{\rho}{\sigma^2(T + M)} \right)^{2/5} \varepsilon^{-6/5}, \frac{\sigma}{T + M} \sqrt{n \log(\varepsilon^{-2})} + \left(\frac{\sigma}{T + M} \right)^2 \log(\varepsilon^{-2}) \right\} \\ \delta &= \lfloor \tilde{\delta} \rfloor + 1, \end{aligned} \quad (60)$$

where M, σ^2 , respectively ρ are the mean, variance and the centered third momentum, of the identically distributed variables d_0, d_1, d_2, \dots , and n is the integer part of $t/(T + M)$. Note that for $t \sim \varepsilon^{-3}$ or smaller $\delta \ll \varepsilon^{-2}$. The choice of $\delta(\varepsilon)$ is explained below.

The proof consists of three stages:

1. $P(t) = P^{(n+\delta)} + \mathcal{O}(\varepsilon) + \mathcal{O}(\delta\varepsilon^2)$,
2. $P^{(n+\delta)} = P^{(n)} + \mathcal{O}(\delta\varepsilon^2)$,
3. $P^{(n)} = T_\varepsilon^n P(0) + \mathcal{O}(n\varepsilon^3)$,

where $n = \lfloor t/(T + M) \rfloor$. The last stage is simply Theorem 3.1.

For the second stage one applies again the previous theorem to get:

$$P^{(n+\delta)} = T_\varepsilon^\delta P^{(n)} + \mathcal{O}(\delta\varepsilon^3).$$

Now $T_\varepsilon = I - \mathcal{O}(\varepsilon^2)$ and since $\delta \ll \varepsilon^{-2}$ stage two follows.

The first stage is the trickiest. Without loss of generality we can assume that $t/(T+M)$ is an integer. Indeed, for $n = \lfloor t/(T+M) \rfloor$ we have

$$P(t) - P(n(T+d)) = \mathcal{O}(\varepsilon(T+M)) = \mathcal{O}(\varepsilon),$$

an error which is already accounted for in this stage.

Suppose first $n - \delta \leq N \leq n + \delta$, i.e. we condition the expected values to the realization of $|N - n| \leq \delta$. The condition restricts only the realizations of d_0, d_1, \dots, d_N leaving the realizations of $d_{N+1}, \dots, d_{n+\delta}$ arbitrary; see (59). Hence, as in stage two, the conditional expected values satisfy:

$$P^{(n+\delta)} = P^{(N+1)} + \mathcal{O}(\delta\varepsilon^2).$$

In addition

$$P^{(N+1)} = P(t) + \mathcal{O}(\varepsilon),$$

since there are at most 2 defects of size ε from “ t ” up until after the $(N+1)^{th}$ defect. Until now we have

$$P^{(n+\delta)} = P(t) + \mathcal{O}(\varepsilon) + \mathcal{O}(\delta\varepsilon^2), \quad |N - n| \leq \delta. \quad (61)$$

Let $p(t)$ denote the power vector

$$p(t) = (|a_1(t)|^2, |a_2(t)|^2, \dots).$$

Recall that by definition $P(t) = \mathbb{E}(p(t))$ and the total power in the system (12) is conserved, *i.e.*

$$\|p(t)\|_1 \stackrel{\text{def}}{=} \sum_k |a_k(t)|^2 \equiv \|p(0)\|_1, \quad t \in \mathbb{R}. \quad (62)$$

Moreover, by (61) and (62) we have

$$\begin{aligned} P(t) &= \mathbb{E}(p(t) : |N - n| \leq \delta) + \mathbb{E}(p(t) : |N - n| > \delta) \\ &= P^{(n+\delta)} + \mathcal{O}(\delta\varepsilon^2) + \mathcal{O}(\varepsilon) + \mathcal{O}(\|p(0)\|_1 \text{Prob}(|N - n| > \delta)). \end{aligned} \quad (63)$$

We claim that for δ given by (60)

$$\text{Prob}(|N - n| > \delta) = \mathcal{O}(\varepsilon) + \mathcal{O}(\delta\varepsilon^2). \quad (64)$$

Indeed, since $t = n(T+M)$,

$$\begin{aligned} \text{Prob}(|N - n| > \delta) &= \text{Prob}\left(\sum_{k=0}^{n+\delta} (T + d_k) \leq t\right) + \text{Prob}\left(\sum_{k=0}^{n-\delta} (T + d_k) > t\right) \\ &= \text{Prob}\left(\frac{\sum_{k=0}^{n+\delta} (T + d_k) - (n + \delta)(T + M)}{\sigma\sqrt{n + \delta}} \leq -\frac{\delta(T + M)}{\sigma\sqrt{n + \delta}}\right) \\ &\quad + \text{Prob}\left(\frac{\sum_{k=0}^{n-\delta} (T + d_k) - (n - \delta)(T + M)}{\sigma\sqrt{n - \delta}} > \frac{\delta(T + M)}{\sigma\sqrt{n - \delta}}\right). \end{aligned} \quad (65)$$

We are going to show how the choice (60) implies

$$\text{Prob} \left(\frac{\sum_{k=0}^{n-\delta} (T + d_k) - (n - \delta)(T + M)}{\sigma \sqrt{n - \delta}} > \frac{\delta(T + M)}{\sigma \sqrt{n - \delta}} \right) \leq \frac{\varepsilon}{2} + \frac{\delta \varepsilon^2}{2}. \quad (66)$$

The other half of (65):

$$\text{Prob} \left(\frac{\sum_{k=0}^{n+\delta} (T + d_k) - (n + \delta)(T + M)}{\sigma \sqrt{n + \delta}} < -\frac{\delta(T + M)}{\sigma \sqrt{n + \delta}} \right) \leq \frac{\varepsilon}{2} + \frac{\delta \varepsilon^2}{2} \quad (67)$$

is analogous.

Depending on the size of n one has either:

$$\frac{0.8\rho}{\sigma^3 \sqrt{n - \delta}} \leq \frac{\delta \varepsilon^2}{2} \quad (68)$$

or:

$$\frac{0.8\rho}{\sigma^3 \sqrt{n - \delta}} > \frac{\delta \varepsilon^2}{2}. \quad (69)$$

If (68) holds, which corresponds to large n , we use the central limit theorem with the Van Beek rate of convergence, see [6]:

$$\begin{aligned} & \text{Prob} \left(\frac{\sum_{k=0}^{n-\delta} (T + d_k) - (n - \delta)(T + M)}{\sigma \sqrt{n - \delta}} > \frac{\delta(T + M)}{\sigma \sqrt{n - \delta}} \right) \\ & \leq \frac{1}{\sqrt{2\pi}} \int_{\frac{\delta(T+M)}{\sigma \sqrt{n-\delta}}}^{\infty} e^{-x^2/2} dx + \frac{0.8\rho}{\sigma^3 \sqrt{n - \delta}}. \end{aligned}$$

This together with (68), the inequality

$$\frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{-x^2/2} dx \leq \frac{e^{-a^2/2}}{2},$$

and the fact that $\delta \geq \frac{\sigma}{(T+M)} \sqrt{n \log \varepsilon^{-2}}$ implies $\frac{\delta(T+M)}{\sigma \sqrt{n-\delta}} \geq 2 \ln \varepsilon^{-1}$, proves (66) for the case (68). If (69) holds then we apply the Chebyshev inequality:

$$\text{Prob} \left(\frac{\sum_{k=0}^{n-\delta} (T + d_k) - (n - \delta)(T + M)}{\sigma \sqrt{n - \delta}} > \frac{\delta(T + M)}{\sigma \sqrt{n - \delta}} \right) \leq \frac{\sigma^2(n - \delta)}{\delta^2(T + M)^2} \leq \frac{\delta \varepsilon^2}{2},$$

where the latter inequality follows from (69) and

$$\delta \leq 1.39 \left(\frac{\rho}{\sigma^2(T + M)} \right)^{2/5} \varepsilon^{-6/5}.$$

From (65), (66) and (67) we get relation (64). The latter plugged into (63) proves the first stage.

Finally, the three stages imply Theorem 3.2 provided that both ε and $\delta \varepsilon^2$ are dominated by $C \max\{n\varepsilon^3, \varepsilon^{3/4}\}$, for an appropriate constant $C > 0$. This follows directly from $\varepsilon \leq 1$ and (60). The proof is now complete. \square

4. Diffusion of Power in Systems with Discrete and Continuous Spectrum

Thusfar we have considered systems with Hamiltonian, H_0 , having only discrete spectrum. We now extend our analysis to the case where H_0 has both discrete and continuous spectrum. Continuous spectrum is associated with radiative behavior and this is manifested in a *dissipative* correction to the operator (21), entering at $\mathcal{O}(\varepsilon^2)$. Therefore, the dynamics on time scales $n \sim \varepsilon^{-2}$ is characterized by diffusion of energy among the discrete modes **and** radiative damping due to coupling of bound modes to the “heat bath” of radiation modes.

The hypotheses on the unperturbed Hamiltonian H_0 are similar to those in [11]. There is one exception though, the singular local decay estimates are replaced by a condition appropriate for perturbations with continuous spectral components, see Hypothesis **(H7’)** below. For convenience we list here and label all the hypotheses we use:

(H1’) H_0 is self-adjoint on the Hilbert space \mathcal{H} . The norm, respectively scalar product, on \mathcal{H} are denoted by $\|\cdot\|$, respectively $\langle \cdot, \cdot \rangle$.

(H2’) The spectrum of H_0 is assumed to consist of an absolutely continuous part, $\sigma_{\text{cont}}(H_0)$, with associated spectral projection, \mathbf{P}_c , spectral measure $dm(\xi)$ and a discrete part formed by isolated eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$ (counting multiplicity) with an orthonormalized set of eigenvectors $\psi_1, \psi_2, \dots, \psi_m$, i.e. for $k, j = 1, \dots, m$,

$$H_0\psi_k = \lambda_k\psi_k, \quad \langle \psi_k, \psi_j \rangle = \delta_{kj},$$

where δ_{kj} is the Kronecker-delta symbol.

(H3’) Local decay estimates on e^{-iH_0t} . There exist self-adjoint “weights”, w_-, w_+ , number $r_1 > 1$ and a constant \mathcal{C} such that

- (i) w_+ is defined on a dense subspace of \mathcal{H} and on which $w_+ \geq cI, c > 0$,
- (ii) w_- is bounded, i.e. $w_- \in \mathcal{L}(\mathcal{H})$, such that $\text{Range}(w_-) \subseteq \text{Domain}(w_+)$,
- (iii) $w_+ w_- \mathbf{P}_c = \mathbf{P}_c$ and $\mathbf{P}_c = \mathbf{P}_c w_- w_+$ on the domain of w_+ and for all $f \in \mathcal{H}$ satisfying $w_+ f \in \mathcal{H}$ we have

$$\|w_- e^{-iH_0t} \mathbf{P}_c f\| \leq \mathcal{C} \langle t \rangle^{-r_1} \|w_+ f\|, \quad t \in \mathbb{R}.$$

The hypotheses on the perturbation are similar to the ones used in the previous sections for discrete systems, namely:

(H4’) β is a bounded self adjoint operator on \mathcal{H} and satisfies $\|\beta\| = 1$. In addition we suppose that β is “localized”, i.e. $w_+ \beta$ and $w_+ \beta w_+$ are bounded on \mathcal{H} , respectively on $\text{Domain}(w_+)$.

(H5’) $g_0(t) \in L^2(\mathbb{R})$ is real valued, has compact support contained in $[0, T]$ on the positive real line and its L^1 -norm, denoted by $\|g_0\|_1$ is 1. Therefore its Fourier transform, \hat{g}_0 is smooth and $\|\hat{g}_0\|_\infty \leq 1$.

(H6’) d_0, d_1, \dots are independent identically distributed random variables taking only nonnegative values, with finite mean, M , and such that for any $l \in \{0, 1, \dots\}$ and $j \neq k \in \{1, 2, \dots, m\}$ we have

$$\mathbb{E} \left(e^{i(\lambda_j - \lambda_k)d_l} \right) = 0,$$

where $\mathbb{E}(\cdot)$ denotes the expected value.

Define the common characteristic (moment generating) function for the random variables $d_0 + T, d_1 + T, \dots$,

$$\rho(\xi) \equiv \mathbb{E} \left(e^{-i\xi(d_0+T)} \right) = \mathbb{E} \left(e^{-i\xi(d_1+T)} \right) = \dots \quad (70)$$

Note that ρ is a continuous function on \mathbb{R} bounded by 1. Then (H6') is equivalent to

$$\rho(\lambda_k - \lambda_j) = 0$$

for all $j \neq k \in \{1, 2, \dots, m\}$.

We require an additional local decay estimate:

(H7') There exists the number $r_2 > 2$ such that for all $f \in \mathcal{H}$ satisfying $w_+ f \in \mathcal{H}$ and all $\lambda_k, \lambda_j, k, j = 1, \dots, m$ we have:

$$\|w_- e^{-iH_0 t} \rho(H_0 - \lambda_k) \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_j - H_0) \mathbf{P}_c f\| \leq \frac{C \|g_0\|_1^2}{\langle t \rangle^{r_2}} \|w_+ f\|, \quad t \in \mathbb{R}.$$

Here \hat{g}_0 denotes the Fourier Transform, see (11), and the operators $\rho(H_0 - \lambda) \mathbf{P}_c$, $\hat{g}_0(\lambda - H_0) \mathbf{P}_c$ are defined via the spectral theorem:

$$\begin{aligned} \rho(H_0 - \lambda) \mathbf{P}_c &= \int_{\sigma_{\text{cont}}(H_0)} \rho(\xi - \lambda) dm(\xi) \\ &= e^{-i(H_0 - \lambda)T} \mathbb{E} \left(e^{-i(H_0 - \lambda)d_l} \right), \quad l = 1, 2, \dots, \end{aligned} \quad (71)$$

$$\begin{aligned} \hat{g}_0(\lambda - H_0) \mathbf{P}_c &= \int_{\sigma_{\text{cont}}(H_0)} \hat{g}_0(\lambda - \xi) dm(\xi) \\ &= \int_0^T g_0(t) e^{-i(\lambda - H_0)t} \mathbf{P}_c dt, \end{aligned} \quad (72)$$

where $dm(\xi)$ is the absolutely continuous part of the spectral measure of H_0 .

Remark 4.1. Conditions implying (H7'). If $H_0 = -\Delta + V(x)$ is a Schrödinger operator with potential, $V(x)$, which decays sufficiently rapidly as x tends to infinity, then either

$$\mathbb{E} \left(e^{i\lambda_j d_l} \right) = 0, \quad l = 0, 1, \dots \text{ and } j = 1, 2, \dots, m \quad (73)$$

or

$$\hat{g}_0(\lambda_j) = 0, \quad j = 1, \dots, m \quad (74)$$

imply **(H7')**, provided the mean and variance of the random variables d_0, d_1, \dots , are finite. Note that (73) is equivalent to adding the threshold, $\lambda_0 = 0$, of the continuous spectrum to the set of eigenvalues $\{\lambda_k : k = 1, 2, \dots, m\}$ for which **(H6')** must hold. Hypothesis (74) means that the perturbation should not induce a resonant coupling between the bound states and the threshold generalized eigenfunction associated with $\lambda_0 = 0$.

In analogy with the case of discrete spectrum, we write the solution of (2) in the form

$$\phi(t, x) = \sum_{j=1}^m a_j(t) \psi_j(x) + \mathbf{P}_c \phi(t, x).$$

Recall that the expected power vector $P(t)$ is defined as the column vector

$$P(t) = (\mathbb{E}(\bar{a}_1 a_1(t)), \mathbb{E}(\bar{a}_2 a_2(t)), \dots, \mathbb{E}(\bar{a}_m a_m(t))).$$

We denote by

$$P^{(n)} = P(t), \quad t_{n-1} + T \leq t < t_n$$

the expected power vector after $n \geq 1$ defects (note that $P(t)$ is constant on the above intervals).

We will show that the change in the power vector induced by each defect can be expressed in terms of a power transmission matrix

$$\begin{aligned} \mathcal{T}_\varepsilon &= T_{disc,\varepsilon} - \varepsilon^2 \text{diag}[\gamma_1, \gamma_2, \dots, \gamma_m] \\ &= \mathbb{I} - \varepsilon^2 B - \varepsilon^2 \text{diag}[\gamma_1, \gamma_2, \dots, \gamma_m]. \end{aligned} \quad (75)$$

Recall that $T_{disc,\varepsilon} = T_\varepsilon = \mathbb{I} - \varepsilon^2 B$, displayed in (21–22), is the power transmission matrix for systems governed by discrete spectrum. Each damping coefficient $\gamma_k > 0$, $k = 1, 2, \dots, m$ results from the interaction between the corresponding bound state and the radiation field. In contrast to the results in [11], there are no contributions from bound state - bound state interactions mediated by the continuous spectrum; these terms cancel out by stochastic averaging.

Remark 4.2. For sufficiently small ε we have:

$$\|\mathcal{T}_\varepsilon\|_1 = 1 - \varepsilon^2 \min\{\gamma_1, \gamma_2, \dots, \gamma_m\} < 1. \quad (76)$$

The damping coefficients are given by:

$$\begin{aligned} \gamma_k &= \lim_{\eta \searrow 0} \left\| \hat{g}_0(H_0 - \lambda_k) \sqrt{\mathbb{I} - |\rho(H_0 - \lambda_k - i\eta)|^2} (\mathbb{I} - \rho(H_0 - \lambda_k - i\eta))^{-1} \mathbf{P}_c[\beta\psi_k] \right\|^2 \\ &> 0, \end{aligned} \quad (77)$$

for all $k = 1, 2, \dots, m$. Here the operators which are functions of H_0 are defined via the spectral theorem and \mathbb{I} is the identity on \mathcal{H} .

Remark 4.3. XYZ. When the pulses are not identical, see Remark 3.1, one can still prove the existence of $\gamma_k \geq 0$. However, we can recover formula (77) only in the cases when the shapes g_n converge to a fixed one denoted by g_0 or form an almost periodic sequence with mean g_0 .

The following theorem is a generalization of our previous result on the effect of a single defect on the mode-power distribution, adapted to the case where the Hamiltonian has both discrete and continuous spectrum:

Theorem 4.1. *Consider the Schrödinger equation*

$$i \partial_t \phi = H_0 \phi + g(t) \beta \phi, \quad (78)$$

where $g(t)$ is a random function, defined in terms of $g_0(t)$, given by (4). Assume that hypotheses **(H1'–H7')** hold. Consider initial conditions for (2) such that $w_+ \mathbf{P}_c \phi_0 \in \mathcal{H}$. Then there exists an $\varepsilon_0 > 0$ such that whenever $|\varepsilon| \leq \varepsilon_0$ the solution of (2) satisfy:

$$P^{(n+1)} = \mathcal{T}_\varepsilon P^{(n)} + \mathcal{O}(\varepsilon^3) + \mathcal{O}\left(\frac{\varepsilon}{\langle nT \rangle^r}\right), \quad n = 0, 1, 2, \dots, \quad (79)$$

where the matrix \mathcal{T}_ε is given in (75) and $r = \min\{r_1, r_2 - 1\} > 1$.

By applying this theorem successively we get the change over $n \geq 1$ defects:

$$P^{(n)} = \mathcal{T}_\varepsilon^n P(0) + \sum_{k=0}^{n-1} \mathcal{T}_\varepsilon^k \left(\mathcal{O}(\varepsilon^3) + \mathcal{O}\left(\frac{\varepsilon}{((n-k)T)^r}\right) \right). \quad (80)$$

Using $\|\mathcal{T}_\varepsilon\|_1 < 1$ and

$$\sum_{n=1}^{\infty} (nT)^{-r} < \infty$$

we can conclude that the last correction term in (80) is of order $\mathcal{O}(\varepsilon)$.² As for the other correction term we have two ways in computing its size. The first is based on $\|\mathcal{T}_\varepsilon^k\|_1 < 1$, and gives

$$\sum_{k=0}^{n-1} \mathcal{T}_\varepsilon^k \mathcal{O}(\varepsilon^3) = \mathcal{O}(n\varepsilon^3).$$

The second is based on

$$\sum_{k=0}^{n-1} \|\mathcal{T}_\varepsilon^k\|_1 \leq (1 - \|\mathcal{T}_\varepsilon\|_1)^{-1} \leq \frac{1}{\gamma\varepsilon^2},$$

where $\gamma = \min\{\gamma_1, \gamma_2, \dots, \gamma_m\}$, and gives

$$\sum_{k=0}^{n-1} \mathcal{T}_\varepsilon^k \mathcal{O}(\varepsilon^3) = \mathcal{O}(\varepsilon\gamma^{-1}).$$

We have proved the following theorem:

Theorem 4.2. *Under the assumptions of Theorem 4.1, the expected power vector after n defects, $n = 1, 2, \dots$, satisfies:*

$$P^{(n)} = \mathcal{T}_\varepsilon^n P(0) + \mathcal{O}\left(\min(\varepsilon\gamma^{-1}, n\varepsilon^3)\right) + \mathcal{O}(\varepsilon).$$

Here, \mathcal{T}_ε is the diffusion/damping power transmission matrix given in (75).

Moreover, the argument we used in the proof of Theorem 3.2 now gives

Theorem 4.3. *Under the assumptions of Theorem 4.1, the expected power vector at a fixed time t , $0 \leq t < \infty$ satisfies:*

$$P(t) = \mathcal{T}_\varepsilon^n P(0) + \mathcal{O}(\varepsilon^{4/5}). \quad (81)$$

Here, n is the integer part of $t/(T + M)$, T is the common time span of the defects and M is the mean of the identically distributed random variables d_0, d_1, \dots .

The nicer form of the correction term in (81) compared to (58) is due to the fact that $\min(t\varepsilon^3, \varepsilon/\gamma)$ is now dominated by $\mathcal{O}(\varepsilon^{4/5})$.

² One can actually show that $\sum_{k=0}^{n-1} \mathcal{T}_\varepsilon^k \mathcal{O}\left(\frac{\varepsilon}{((n-k)T)^r}\right) = \mathcal{O}\left(\min\left\{\varepsilon, \frac{(nT)^{-r}}{\varepsilon\gamma}\right\}\right)$. However, as $n \rightarrow \infty$ the other correction term dominates and the result of Theorem 4.2 cannot be improved.

In analogy with Corollary 3.1 we have, in the present context, the following limiting behavior:

Corollary 4.1. *Under the assumption of Theorem 4.1 the following holds:*

$$\lim_{t \rightarrow \infty} P(t) = \begin{cases} P(0), & \text{if } t \ll \varepsilon^{-2} \\ e^{-(B+\Gamma)\tau} P(0) & \text{if } t = \tau \varepsilon^{-2} \\ 0, & \text{if } t \gg \varepsilon^{-2}, \varepsilon \rightarrow 0 \end{cases}, \quad (82)$$

where B is displayed in (22) and

$$\Gamma = \text{diag} [\gamma_1, \gamma_2, \dots, \gamma_m] > 0.$$

Proof. Since $\mathcal{T}_\varepsilon = \mathbb{I} - \varepsilon^2(B + \Gamma)$ and $B + \Gamma$ is self adjoint with

$$B + \Gamma \geq \min\{\gamma_k : k = 1, 2, \dots, m\} > 0$$

we have

$$\lim_{n \rightarrow \infty} \mathcal{T}_\varepsilon^{(n)} = \begin{cases} \mathbb{I}, & \text{if } n \ll \varepsilon^{-2} \\ e^{-(B+\Gamma)\tau} & \text{if } n = \tau \varepsilon^{-2} \\ 0 & \text{if } n \gg \varepsilon^{-2}, \varepsilon \rightarrow 0 \end{cases}. \quad (83)$$

This follows from writing \mathcal{T}_ε in the basis which diagonalizes $B + \Gamma$ and using the fact that all eigenvalues of $B + \Gamma$ are strictly positive, see the proof of Corollary 3.1.

Clearly, (83) and Theorem 4.3 imply the conclusion of the corollary. \square

Note that on time scales of order $1/\varepsilon^2$ the dynamical system is now equivalent to:

$$\partial_\tau P(\tau) = (-B - \Gamma)P(\tau),$$

where $-B$ is a diffusion operator, see the discussion after relation (50), while $-\Gamma$ is a damping operator.

It remains to prove Theorem 4.1.

Proof of Theorem 4.1. Consider one realization of the random variables d_0, d_1, \dots . For this realization the system (2) is linear, Hamiltonian and deterministic. It is well known that such systems have a unique solution, $\phi(t)$, defined for all $t \geq 0$ and continuously differentiable with respect to t . Moreover

$$\|\phi(t)\| \equiv \|\phi_0\|. \quad (84)$$

We decompose the solution in its projections onto the bound states and continuous spectrum of the unperturbed Hamiltonian:

$$\phi(t, x) = \sum_{j=1}^m a_j(t) \psi_j + \mathbf{P}_c \phi(t) = \phi_b(t) + \phi_d(t), \quad (85)$$

where ϕ_b and ϕ_d are, respectively, the bound and dispersive parts of ϕ :

$$\begin{aligned} \phi_b(t) &= \sum_{j=1}^m a_j(t) \psi_j, \\ \phi_d(t) &= \mathbf{P}_c \phi(t), \end{aligned} \quad (86)$$

and

$$\langle \phi_b(t), \phi_d(t) \rangle \equiv 0. \quad (87)$$

Note that (84) and (87) imply

$$\|\phi_b(t)\| \leq \|\phi_0\|, \quad \|\phi_d(t)\| \leq \|\phi_0\|, \quad (88)$$

for all $t \geq 0$. Consequently,

$$|a_k(t)| \leq \|\phi_0\|, \quad (89)$$

for all $t \geq 0$.

By inserting (85) into (2) and projecting the later onto the bound states and continuous spectrum we get the coupled system:

$$i\partial_t a_k(t) = \lambda_k a_k(t) + \varepsilon g(t) \langle \psi_k, \beta \phi_b(t) \rangle + \varepsilon g(t) \langle \psi_k, \beta \phi_d(t) \rangle, \quad (90)$$

$$i\partial_t \phi_d(t) = H_0 \phi_d(t) + \varepsilon g(t) \mathbf{P}_c \beta \phi_d(t) + \varepsilon g(t) \mathbf{P}_c \beta \phi_b(t), \quad (91)$$

where $k = 1, 2, \dots, m$. Duhamel's principle applied to (91) yields

$$\begin{aligned} \phi_d(t) &= e^{-iH_0 t} \phi_d(0) - i\varepsilon \int_0^t g(s) e^{-iH_0(t-s)} \mathbf{P}_c \beta \phi_d(s) ds \\ &\quad - i\varepsilon \int_0^t g(s) e^{-iH_0(t-s)} \mathbf{P}_c \beta \phi_b(s) ds. \end{aligned} \quad (92)$$

In a manner analogous to the one in [3] we are going to isolate ϕ_d in (92). Consider the following two operators acting on $C(\mathbb{R}^+, \text{Domain}(w_+))$ respectively $C(\mathbb{R}^+, \mathcal{H})$, the space of continuous functions on positive real numbers with values in $\text{Domain}(w_+)$ respectively \mathcal{H} :

$$K^+[f](t) = \int_0^t g(s) w_- e^{-iH_0(t-s)} \mathbf{P}_c \beta w_+ f(s) ds, \quad (93)$$

$$K[f](t) = \int_0^t g(s) w_- e^{-iH_0(t-s)} \mathbf{P}_c \beta f(s) ds. \quad (94)$$

Then, by applying the w_- operator on both sides of (92) we get:

$$w_- \phi_d(t) = w_- e^{-iH_0 t} \phi_d(0) - i\varepsilon K^+[w_- \phi_d](t) - i\varepsilon K[\phi_b](t). \quad (95)$$

On $C(\mathbb{R}^+, \mathcal{H})$ we introduce the family of norms depending on $\alpha \geq 0$:

$$\|f\|_\alpha = \sup_{t \geq 0} \langle t \rangle^\alpha \|f(t)\| \quad (96)$$

and define the operator norm:

$$\|\mathcal{A}\|_\alpha = \sup_{\|f\|_\alpha \leq 1} \|\mathcal{A}f\|_\alpha. \quad (97)$$

The local decay hypothesis **(H3')** together with **(H4')** and **(H5')** imply:

Lemma 4.1. *If $0 \leq \alpha \leq r_1$ then there exists a constant C_α such that*

$$\begin{aligned} \|K^+\|_\alpha &\leq C_\alpha, \\ \|K\|_\alpha &\leq C_\alpha. \end{aligned}$$

Proof of Lemma 4.1. Fix α , $0 \leq \alpha \leq r_1$ and $f \in C(\mathbb{R}_+, \text{Domain}(w_+))$ such that $\|f\|_\alpha \leq 1$. Then

$$\begin{aligned} \langle t \rangle^\alpha \|K^+[f](t)\| &= \langle t \rangle^\alpha \left| \int_0^t g(s) w_- e^{-iH_0(t-s)} \mathbf{P}_c \beta w_+ f(s) ds \right| \\ &\leq \langle t \rangle^\alpha \int_0^t |g(s)| \|w_- e^{-iH_0(t-s)} \mathbf{P}_c w_-\| \cdot \|w_+ \beta w_+\| \cdot \|f(s)\| ds \\ &\leq \langle t \rangle^\alpha C \|w_+ \beta w_+\| \int_0^t \frac{|g(s)|}{\langle t-s \rangle^{r_1}} \|f(s)\| ds, \end{aligned}$$

where we used **(H3')**. Furthermore, from $\|f\|_\alpha \leq 1$ and $\|w_+ \beta w_+\|$ bounded, we have

$$\begin{aligned} \langle t \rangle^\alpha \|K^+[f](t)\| &\leq C \langle t \rangle^\alpha \int_0^t \frac{|g(s)|}{\langle t-s \rangle^{r_1} \langle s \rangle^\alpha} \langle s \rangle^\alpha \|f(s)\| ds \\ &\leq C \langle t \rangle^\alpha \|f\|_\alpha \int_0^t \frac{|g(s)|}{\langle t-s \rangle^{r_1} \langle s \rangle^\alpha} ds \\ &\leq C \langle t \rangle^\alpha \sum_{\{j: \tilde{t}_j < t\}} \int_{\tilde{t}_j}^{\min(t, \tilde{t}_j + T)} \frac{|g(s)|}{\langle t-s \rangle^{r_1} \langle s \rangle^\alpha} ds. \end{aligned}$$

By the mean value theorem

$$\int_{\tilde{t}_j}^{\min(t, \tilde{t}_j + T)} \frac{|g(s)|}{\langle t-s \rangle^{r_1} \langle s \rangle^\alpha} ds = \langle t - \tilde{t}_j \rangle^{-r_1} \langle \tilde{t}_j \rangle^{-\alpha} \|g_0\|_1,$$

for some

$$t_j \leq \tilde{t}_j \leq \min(t, t_j + T). \quad (98)$$

Hence

$$\langle t \rangle^\alpha \|K^+[f](t)\| \leq C \langle t \rangle^\alpha \sum_{\{j: \tilde{t}_j < t\}} \langle t - \tilde{t}_j \rangle^{-r_1} \langle \tilde{t}_j \rangle^{-\alpha}. \quad (99)$$

We claim that

$$\sum_{\{j: \tilde{t}_j < t\}} \langle t - \tilde{t}_j \rangle^{-r_1} \langle \tilde{t}_j \rangle^{-\alpha} \leq D_\alpha \langle t \rangle^{-\alpha} \quad (100)$$

for some constant D_α independent of t . This is a consequence of the fact that we are computing the convolution of two power-like sequences. For a more detailed proof we decompose the sum into two, first running for $\tilde{t}_j \leq t/2$ and the second for $t/2 < \tilde{t}_j \leq t$. For the former we have:

$$\begin{aligned} \sum_{\{j: \tilde{t}_j < t/2\}} \langle t - \tilde{t}_j \rangle^{-r_1} \langle \tilde{t}_j \rangle^{-\alpha} &\leq \left\langle \frac{t}{2} \right\rangle^{-r_1} \sum_{\{j: \tilde{t}_j < t/2\}} \langle \tilde{t}_j \rangle^{-\alpha} \\ &\leq \left\langle \frac{t}{2} \right\rangle^{-r_1} \sum_{\{j: jT < t/2\}} \langle jT \rangle^{-\alpha} \\ &\leq \left\langle \frac{t}{2} \right\rangle^{-r_1} D_\alpha \left\langle \frac{t}{2} \right\rangle^{\max(0, 1-\alpha)} \leq D_\alpha \langle t \rangle^{-\alpha}, \end{aligned} \quad (101)$$

since $r_1 > \max(1, \alpha)$ and $\tilde{t}_j \geq t_j \geq (j-1)T$, see **(H3')**, the hypotheses of this lemma, respectively (98) and (5). The remaining part of the sum is treated similarly:

$$\begin{aligned} \sum_{\{j:t/2 < \tilde{t}_j \leq t\}} \langle t - \tilde{t}_j \rangle^{-r_1} \langle \tilde{t}_j \rangle^{-\alpha} &\leq \left\langle \frac{t}{2} \right\rangle^{-\alpha} \sum_{\{j:t/2 < \tilde{t}_j \leq t/2\}} \langle t - \tilde{t}_j \rangle^{-r_1} \\ &\leq \left\langle \frac{t}{2} \right\rangle^{-\alpha} \sum_{\{k:kT < t/2\}} \langle kT \rangle^{-r_1} \\ &\leq \left\langle \frac{t}{2} \right\rangle^{-\alpha} D \leq D_\alpha \langle t \rangle^{-\alpha}, \end{aligned} \quad (102)$$

since $r_1 > 1$ and $t - \tilde{t}_j \geq kT$, where k is such that $t_{k+j} = \max\{t_p : t_p \leq t\}$, see (98) and (5).

Now (101) and (102) imply (100) which replaced in (99) proves the required estimate for the K^+ operator. For the K operator the argument is completely analogous. \square

We are going to use Lemma 4.1 for $\alpha = 0$ and $\alpha = r_1$. For C_0 and C_{r_1} defined in the lemma, let

$$C_K = \max \{C_0, C_{r_1}\}.$$

Then, for ε such that $C_K \varepsilon < 1$, the inverse operator $(I - i\varepsilon K^+)^{-1}$ exists and it is bounded in the norms (97) for $\alpha = 0$ and $\alpha = r_1$. Then (95) implies:

$$\begin{aligned} w_- \phi_d(t) &= (I - i\varepsilon K^+)^{-1} \left[w_- e^{-iH_0 t} \phi_d(0) \right] (t) - i\varepsilon (I - i\varepsilon K^+)^{-1} K[\phi_b](t) \\ &= \mathcal{O}(\langle t \rangle^{-r_1} \|w_+ \phi_d(0)\|) - i\varepsilon K[\phi_b] + \mathcal{O}(\varepsilon^2 \|K[\phi_b]\|). \end{aligned} \quad (103)$$

Thus we have expressed the dispersive part, $\phi_d(t)$ as a functional of the bound state part, $\phi_b(t)$. Substitution of (103) into (90) gives, for $k = 1, 2, \dots$:

$$\begin{aligned} \partial_t a_k(t) &= -i\lambda_k a_k(t) - i\varepsilon g(t) \sum_{j=1}^m a_j(t) \langle \psi_k, \beta \psi_j \rangle \\ &\quad - \varepsilon^2 g(t) \langle w_+ \beta \psi_k, K[\phi_b](t) \rangle \\ &\quad + \varepsilon g(t) \left(\mathcal{O}(\|w_+ \phi_d(0)\| \langle t \rangle^{-r_1}) + \mathcal{O}(\varepsilon^2 \|K[\phi_b]\|) \right) \quad k = 1, 2, \dots, m. \end{aligned} \quad (104)$$

In particular (104) implies

$$\bar{a}_k(t_n) = e^{i\lambda_k(t_n - t_l)} \bar{a}_k(t_l) + \varepsilon \sum_{p=l}^{n-1} e^{i\lambda_k(t_n - t_p)} D_p(d_0, d_1, \dots, d_p), \quad (105)$$

for all $k = 0, 1, \dots, m$, $n \geq 2$ and $l < n$. Here each constant D_p depends on the realization of d_0, d_1, \dots, d_p and does not depend on the realization of any other random variable. In addition all D_p are uniformly bounded by a constant depending only on C_K above and the initial condition $\phi(0)$. Hence

$$a_k(t_n) = e^{-i\lambda_k(t_n - t_l)} a_k(t_l) + \mathcal{O}(\varepsilon |n - l|) \quad (106)$$

for all $n, l = 0, 1, 2, \dots$, and $k = 1, 2, \dots, m$.

We multiply both sides of (104) with \bar{a}_k , then add the resulting equation to its complex conjugate. Then we integrate from t_n to $t_n + T$ and obtain for $k = 1, 2, \dots, m$

$$\bar{a}_k a_k(t_n + T) - \bar{a}_k a_k(t_n) = \mathbb{R}_1 + R_2 + R_3, \quad (107)$$

where

$$R_1 = -i\varepsilon \sum_{j=1}^m \langle \psi_k, \beta \psi_j \rangle \int_{t_n}^{t_n+T} g(t) \bar{a}_k(t) a_j(t) dt + c.c., \quad (108)$$

$$R_2 = -\varepsilon^2 \langle w_+ \beta \psi_k, \int_{t_n}^{t_n+T} g(t) \bar{a}_k(t) K[\phi_b](t) dt \rangle + c.c., \quad (109)$$

$$R_3 = \mathcal{O}(\varepsilon \langle t_n \rangle^{-r_1}) + \mathcal{O}(\varepsilon^3). \quad (110)$$

If we neglect the R_2 and R_3 in (107) we are left with R_1 , which is precisely the expression associated with the power transfer in systems with discrete spectrum; see Sect. 2. Moreover R_3 has norm asserted in (79). So, it remains to show of R_2 that

$$\begin{aligned} & \mathbb{E} \left(\langle w_+ \beta \psi_k, \int_{t_n}^{t_n+T} g(t) \bar{a}_k(t) K[\phi_b](t) dt \rangle + c.c. \right) \\ &= \gamma_k P_k^n + \mathcal{O}(\langle nT \rangle^{-r}) + \mathcal{O}(\varepsilon), \end{aligned} \quad (111)$$

where γ_k is given by (77) and $r = \min\{r_1, r_2 - 1\} > 1$.

We use integration by parts. Let

$$\tilde{K}[\phi_b](t) \equiv \int_{t_n+T}^t g(s) e^{-i\lambda_k(s-t_n)} K[\phi_b](s) ds, \quad t_n \leq t \leq t_n + T, \quad (112)$$

and note that $K[\phi_b](t_n + T) = 0$. Lemma 4.1 together with

$$g(s) = g_0(s - t_n), \quad t_n \leq s \leq t_n + T, \quad (113)$$

imply the existence of a constant C with the property:

$$\|\tilde{K}[\phi_b](t)\| \leq C \|g_0\|_1^2 = C, \quad (114)$$

uniformly in $t_n \leq t \leq t_n + T$. Define

$$A_k(t) = a_k(t) e^{i\lambda_k(t-t_n)}, \quad (115)$$

for $k = 1, 2, \dots, m$. Note that

$$A_k(t_n) = a_k(t_n). \quad (116)$$

From (104) we have

$$|\partial_t A_k(s)| \leq C |\varepsilon| |g_0(s - t_n)| \quad (117)$$

for some constant C independent of s and $t_n \leq s \leq t_n + T$. Now

$$\begin{aligned} \int_{t_n}^{t_n+T} g(t) \bar{a}_k(t) K[\phi_b](t) dt &= \int_{t_n}^{t_n+T} \bar{A}_k(t) \partial_t \tilde{K}[\phi_b](t) dt \\ &= -\bar{a}_k(t_n) \tilde{K}[\phi_b](t_n) - \int_{t_n}^{t_n+T} \partial_t \bar{A}_k(t) \tilde{K}[\phi_b](t) dt \quad (118) \\ &= \bar{a}_k(t_n) \int_{t_n}^{t_n+T} g(t) e^{i\lambda_k(t-t_n)} K[\phi_b](t) dt + \mathcal{O}(\varepsilon). \end{aligned}$$

To further rewrite (118) we note that for $t_n \leq t \leq t_n + T$,

$$\begin{aligned} K[\phi_b](t) &= \sum_{j=1}^m \int_{t_n}^t a_j(s) g(s) w_- e^{-iH_0(t-s)} \mathbf{P}_c \beta \psi_j ds \\ &\quad + \sum_{l=0}^{n-1} \sum_{j=1}^m \int_{t_l}^{t_l+T} a_j(s) g(s) w_- e^{-iH_0(t-s)} \mathbf{P}_c \beta \psi_j ds. \quad (119) \end{aligned}$$

An integration by parts similar to the one above and use of (113) leads to:

$$\begin{aligned} &\int_{t_l}^{t_l+T} a_j(s) g(s) w_- e^{-iH_0(t-s)} \mathbf{P}_c \beta \psi_j ds \\ &= a_j(t_l) \int_{t_l}^{t_l+T} g(s) e^{-i\lambda_j(s-t_l)} w_- e^{-iH_0(t-s)} \mathbf{P}_c \beta \psi_j ds + \mathcal{O}\left(\frac{\varepsilon}{\langle t-t_l-T \rangle^{r_1}}\right) \\ &= a_j(t_l) w_- \hat{g}_0(\lambda_j - H_0) e^{-iH_0(t-t_l)} \mathbf{P}_c \beta \psi_j + \mathcal{O}\left(\frac{\varepsilon}{\langle t-t_l-T \rangle^{r_1}}\right), \quad (120) \end{aligned}$$

and

$$\begin{aligned} &\int_{t_n}^t a_j(s) g(s) w_- e^{-iH_0(t-s)} \mathbf{P}_c \beta \psi_j ds \\ &= a_j(t_n) \int_{t_n}^t g(s) e^{-i\lambda_j(s-t_n)} w_- e^{-iH_0(t-s)} \mathbf{P}_c \beta \psi_j ds + \mathcal{O}(\varepsilon). \quad (121) \end{aligned}$$

By plugging (120–121) in (119) we get

$$\begin{aligned} K[\phi_b](t) &= \sum_{j=1}^m a_j(t_n) \int_{t_n}^t g(s) e^{-i\lambda_j(s-t_n)} w_- e^{-iH_0(t-s)} \mathbf{P}_c \beta \psi_j ds \\ &\quad + \sum_{j=1}^m \sum_{l=0}^{n-1} a_j(t_l) w_- \hat{g}_0(\lambda_j - H_0) e^{-iH_0(t-t_l)} \mathbf{P}_c \beta \psi_j + \mathcal{O}(\varepsilon), \quad (122) \end{aligned}$$

where to estimate the error we used the fact that the series $\sum_l \langle t-t_l-T \rangle^{-r_1}$ is convergent and uniformly bounded in t .

We now substitute (122) into the right-hand side of (118) and obtain

$$\begin{aligned}
& \int_{t_n}^{t_n+T} g(t) \bar{a}_k(t) K[\phi_b](t) dt = \mathcal{O}(\varepsilon) \\
& + \sum_{j=1}^m \bar{a}_k(t_n) a_j(t_n) \int_{t_n}^{t_n+T} g(t) e^{i\lambda_k(t-t_n)} \int_{t_n}^t g(s) e^{-i\lambda_j(s-t_n)} w_- e^{-iH_0(t-s)} \mathbf{P}_c \beta \psi_j ds dt \\
& + \sum_{j=1}^m \sum_{l=0}^{n-1} \bar{a}_k(t_n) a_j(t_l) w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_j - H_0) e^{-iH_0(t_n-t_l)} \mathbf{P}_c \beta \psi_j. \quad (123)
\end{aligned}$$

Based on (105) we can replace $\bar{a}_k(t_n) a_j(t_l)$ in (123) with

$$\begin{aligned}
\bar{a}_k(t_n) a_j(t_l) &= e^{i\lambda_k(t_n-t_l)} \bar{a}_k(t_l) a_j(t_l) + \text{error}(l, j), \\
\text{error}(l, j) &= \varepsilon \sum_{p=l}^{n-1} e^{i\lambda_k(t_n-t_p)} D_p w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_j - H_0) e^{-iH_0(t_n-t_l)} \mathbf{P}_c \beta \psi_j. \quad (124)
\end{aligned}$$

Taking into account that $t_n - t_{n-1} = d_n + T$ and the fact that $t_{n-1} - t_l$, D_p , $l \leq p \leq n-1$ do not depend on d_n , the expected value of the error can be rewritten as

$$\begin{aligned}
\mathbb{E}(\text{error}(l, j)) &= \\
&= \varepsilon \sum_{p=l}^{n-1} \mathbb{E} \left(w_- e^{i(\lambda_k - H_0)(t_n - t_{n-1})} e^{i\lambda_k(t_{n-1} - t_p)} \right. \\
&\quad \times D_p \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_j - H_0) e^{-iH_0(t_{n-1} - t_l)} \mathbf{P}_c \beta \psi_j \left. \right) \\
&= \varepsilon \sum_{p=l}^{n-1} w_- \rho(H_0 - \lambda_k) \mathbb{E} \left(e^{i\lambda_k(t_{n-1} - t_p)} \right. \\
&\quad \times D_p \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_j - H_0) e^{-iH_0(t_{n-1} - t_l)} \mathbf{P}_c \beta \psi_j \left. \right) \\
&= \varepsilon \sum_{p=l}^{n-1} \mathbb{E} \left(e^{i\lambda_k(t_{n-1} - t_p)} \right. \\
&\quad \times D_p w_- \rho(H_0 - \lambda_k) \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_j - H_0) e^{-iH_0(t_{n-1} - t_l)} \mathbf{P}_c \beta \psi_j \left. \right). \quad (125)
\end{aligned}$$

By applying the \mathcal{H} norm to (125), commuting the norm with both summation and expected value and using **(H7')** we get:

$$\|\mathbb{E}(\text{error}(l, j))\| \leq |\varepsilon| \frac{\mathcal{C}(n-l)}{\langle t_{n-1} - t_l \rangle^{r_2}} \leq C |\varepsilon| \langle (n-l)T \rangle^{1-r_2}. \quad (126)$$

Since $r_2 > 2$ the summation over l and j of all the errors will have an $\mathcal{O}(\varepsilon)$ size. By this argument (123) becomes:

$$\begin{aligned}
& \mathbb{E} \left(\left\langle w_+ \beta \psi_k, \int_{t_n}^{t_n+T} g(t) \bar{a}_k(t) K[\phi_b](t) dt \right\rangle + c.c. \right) = \sum_{j=1}^m \mathbb{E}(\bar{a}_k(t_n) a_j(t_n)) \\
& \cdot \left\langle w_+ \beta \psi_k, \int_{t_n}^{t_n+T} g(t) e^{i\lambda_k(t-t_n)} \int_{t_n}^t g(s) e^{-i\lambda_j(s-t_n)} w_- e^{-iH_0(t-s)} \mathbf{P}_c \beta \psi_j ds dt \right\rangle + c.c. \\
& + \sum_{j=1}^m \sum_{l=0}^{n-1} \mathbb{E}(\bar{a}_k(t_l) a_j(t_l)) \\
& \cdot \mathbb{E} \left(\left\langle w_+ \beta \psi_k, w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_j - H_0) e^{i(\lambda_k - H_0)(t_n - t_l)} \mathbf{P}_c \beta \psi_j \right\rangle \right) + c.c. \\
& + \mathcal{O}(\varepsilon). \tag{127}
\end{aligned}$$

But **(H6')** and the technique used to prove (41) imply

$$\mathbb{E}(\bar{a}_k(t_l) a_j(t_l)) = \begin{cases} P_k^{(l)} & \text{for } k = j \\ 0 & \text{for } k \neq j \end{cases}.$$

Moreover, an argument similar to the one we used in (124–126) allows us to replace $P^{(l)}$ by $P^{(n)}$ in (127) and incur an $\mathcal{O}(\varepsilon)$ total error. Then, (127) becomes

$$\begin{aligned}
& \mathbb{E} \left(\left\langle w_+ \beta \psi_k, \int_{t_n}^{t_n+T} g(t) \bar{a}_k(t) K[\phi_b](t) dt \right\rangle + c.c. \right) = \\
& = P_k^{(n)} \left\langle w_+ \beta \psi_k, w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \mathbf{P}_c \beta \psi_k \right\rangle \\
& + P_k^{(n)} \left\langle w_+ \beta \psi_k, w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \mathbb{E} \left(\sum_{l=0}^{n-1} e^{i(\lambda_k - H_0)(t_n - t_l)} \mathbf{P}_c \right) \beta \psi_k \right\rangle \\
& + c.c. \\
& + \mathcal{O}(\varepsilon). \tag{128}
\end{aligned}$$

We claim that

$$\begin{aligned}
\gamma_k^n & \stackrel{def}{=} \left\langle w_+ \beta \psi_k, w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \mathbf{P}_c \beta \psi_k \right\rangle \\
& + \left\langle w_+ \beta \psi_k, w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \mathbb{E} \left(\sum_{l=0}^{n-1} e^{i(\lambda_k - H_0)(t_n - t_l)} \mathbf{P}_c \right) \beta \psi_k \right\rangle + c.c. \\
& = \gamma_k + \mathcal{O}((nT)^{1-r_2}), \tag{129}
\end{aligned}$$

where γ_k is given in (77). Equation (129) replaced in (128) gives (111) which finishes the proof of this theorem.

To prove (129) we first find a simpler expression for the expected value operator involved. Since $\{d_j\}_{j \geq 0}$ are independent, identically distributed with common characteristic function, $\rho(\xi)$, using the definition of t_n , $n \geq 0$, see (5) and the spectral

resolution of the operator H_0 , see (71), we have:

$$\begin{aligned}
\mathbb{E} \left(e^{i(\lambda_k - H_0)(t_n - t_l)} \mathbf{P}_c \right) &= \int_{\sigma_{\text{cont}}(H_0)} \mathbb{E}(e^{i(\lambda_k - \xi)(t_n - t_l)}) dm(\xi) \\
&= \int_{\sigma_{\text{cont}}(H_0)} \mathbb{E}(e^{i(\lambda_k - \xi) \sum_{j=l}^{n-1} (d_j + T)}) dm(\xi) \\
&= \int_{\sigma_{\text{cont}}(H_0)} \prod_{j=l}^{n-1} \mathbb{E}(e^{i(\lambda_k - \xi)(d_j + T)}) dm(\xi) \\
&= \int_{\sigma_{\text{cont}}(H_0)} \rho^{n-l} (\xi - \lambda_k) dm(\xi) = \rho^{n-l} (H_0 - \lambda_k) \mathbf{P}_c. \quad (130)
\end{aligned}$$

Hence

$$\begin{aligned}
&w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \mathbb{E} \left(\sum_{l=0}^{n-1} e^{i(\lambda_k - H_0)(t_n - t_l)} \mathbf{P}_c \right) \beta \\
&= w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \sum_{j=1}^n \rho^j (H_0 - \lambda_k) \mathbf{P}_c \beta. \quad (131)
\end{aligned}$$

But each operator term in (131) has its \mathcal{H} -norm dominated by:

$$\begin{aligned}
&\|w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \rho^j (H_0 - \lambda_k) \mathbf{P}_c \beta\| \\
&= \|w_- \rho(H_0 - \lambda_k) \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \mathbb{E}(e^{-i(H_0 - \lambda_k)(t_{j-1} - t_0)} \mathbf{P}_c \beta)\| \\
&\leq \frac{C}{\langle t_{j-1} - t_0 \rangle^{r_2}} \|w_+ \beta\| \leq \langle (j-1)T \rangle^{-r_2}.
\end{aligned}$$

Now $r_2 > 2$ implies that the sequence $1/\langle jT \rangle^{r_2}$ is summable, and, by the dominant convergence theorem, there exists:

$$\begin{aligned}
\tilde{\gamma}_k &= \langle w_+ \beta \psi_k, w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \mathbf{P}_c \beta \psi_k \rangle \\
&\quad + \sum_{j=1}^{\infty} \langle w_+ \beta \psi_k, w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \rho^j (H_0 - \lambda_k) \mathbf{P}_c \beta \psi_k \rangle + c.c. \\
&= \lim_{n \rightarrow \infty} \gamma_k^n.
\end{aligned}$$

Moreover

$$\begin{aligned}
|\tilde{\gamma}_k - \gamma_k^n| &= \sum_{j=n+1}^{\infty} \langle w_+ \beta \psi_k, w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \rho^j (H_0 - \lambda_k) \mathbf{P}_c \beta \psi_k \rangle + c.c. \\
&\leq 2C \sum_{j=n}^{\infty} \langle jT \rangle^{-r_2} \leq D \langle nT \rangle^{1-r_2}. \quad (132)
\end{aligned}$$

Consider now, for $\eta > 0$,

$$\begin{aligned} \gamma_k^\eta &= \langle w_+ \beta \psi_k, w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \mathbf{P}_c \beta \psi_k \rangle \\ &+ \sum_{j=1}^{\infty} \langle w_+ \beta \psi_k, w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \rho^j(H_0 - \lambda_k - i\eta) \mathbf{P}_c \beta \psi_k \rangle \\ &+ c.c. \end{aligned} \quad (133)$$

On one hand

$$\rho^j(H_0 - \lambda_k - i\eta) \mathbf{P}_c = \mathbb{E}(e^{-\eta(t_j - t_0)} e^{-i(H_0 - \lambda_k)(t_j - t_0)} \mathbf{P}_c) \quad (134)$$

and, by the dominant convergence theorem, for all $j \geq 1$,

$$\lim_{\eta \searrow 0} \rho^j(H_0 - \lambda_k - i\eta) \mathbf{P}_c = \rho^j(H_0 - \lambda_k) \mathbf{P}_c.$$

On the other hand the series (133) is dominated uniformly in η by a summable series, because:

$$\begin{aligned} &\|w_- \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \rho^j(H_0 - \lambda_k - i\eta) \mathbf{P}_c \beta\| \\ &= \left\| \int_0^T \int_0^T du ds g_0(s+u) g_0(u) \mathbb{E} \left(e^{-\eta(t_j - t_0)} w_- e^{-i(H_0 - \lambda_k)(t_j - t_0 - s)} \mathbf{P}_c \beta \right) \right\| \\ &\leq \frac{C e^{-\eta j T}}{(t_j - t_0 - T)^{r_1}} \|g_0\|_1 \|w_+ \beta\| \leq \langle (j-1)T \rangle^{-r_1}. \end{aligned}$$

Here we used **(H3')**, $\|g_0\|_1 = 1$ and $\|w_+ \beta\|$ bounded. Therefore, by the Weierstrass criterion:

$$\lim_{\eta \searrow 0} \gamma_k^\eta = \tilde{\gamma}_k. \quad (135)$$

In addition (134) implies

$$\begin{aligned} \|\rho(H_0 - \lambda_k - i\eta) \mathbf{P}_c\| &\leq \mathbb{E} \left(e^{-\eta(t_1 - t_0)} \|e^{-i(H_0 - \lambda_k)(t_1 - t_0)} \mathbf{P}_c\| \right) \\ &\leq e^{-\eta T} < 1. \end{aligned}$$

This makes $(\mathbb{I} - \rho(H_0 - \lambda_k - i\eta)) \mathbf{P}_c$ invertible and given by the Neumann series:

$$(\mathbb{I} - \rho(H_0 - \lambda_k - i\eta))^{-1} \mathbf{P}_c = \sum_{j=0}^{\infty} \rho^j(H_0 - \lambda_k - i\eta) \mathbf{P}_c. \quad (136)$$

Plugging (136) in (133) we have

$$\begin{aligned} \gamma_k^\eta &= \langle \beta \psi_k, \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \mathbf{P}_c \beta \psi_k \rangle \\ &+ \left\langle \beta \psi_k, \hat{g}_0(H_0 - \lambda_k) \hat{g}_0(\lambda_k - H_0) \rho(H_0 - \lambda_k - i\eta) (\mathbb{I} - \rho(H_0 - \lambda_k - i\eta))^{-1} \right. \\ &\quad \left. \times \mathbf{P}_c \beta \psi_k \right\rangle + c.c. \end{aligned}$$

A simple inner product manipulation shows that:

$$\gamma_k^\eta = \left\| \hat{g}_0(H_0 - \lambda_k) \sqrt{\mathbb{I} - |\rho(H_0 - \lambda_k - i\eta)|^2} (\mathbb{I} - \rho(H_0 - \lambda_k - i\eta))^{-1} \mathbf{P}_c [\beta \psi_k] \right\|^2.$$

Hence

$$\tilde{\gamma}_k = \lim_{\eta \searrow 0} \gamma_k^\eta = \gamma_k, \quad (137)$$

see also (135) and (77).

Finally, (137) and (132) give the claim (129). The theorem is now completely proven.

□

5. Appendix: Properties of the Power Transmission Matrix

In this section we prove the properties of the matrix (linear operator) B we used in Corollaries 3.1 and 3.2. Recall that B is given by (22) and is irreducible, see the discussion before Corollary 3.2. We note that (22) implies in particular that for all $i, j = 1, 2, \dots, i \neq j$,

$$b_{ii} \geq 0; \quad b_{ij} \leq 0; \quad b_{ii} = - \sum_{k, k \neq i} b_{ik}. \quad (138)$$

Lemma 5.1. *If the dimension of B is finite, say m , then B is a nonnegative, self adjoint matrix having 0 as a simple eigenvalue with corresponding normalized eigenvector:*

$$r_0 = \frac{1}{\sqrt{m}}(1, 1, \dots, 1)'$$

Proof. The symmetry of B follows directly from (22). The fact that it is nonnegative follows from the identity:

$$X^* B X = \sum_{i, j=1}^m b_{ij} \bar{X}_i X_j = \sum_{i, j, i < j} |b_{ij}| \cdot |X_i - X_j|^2, \quad (139)$$

where we used (138). The latter and a direct calculation show $B r_0 = 0$, hence r_0 is an eigenvector corresponding to the eigenvalue 0.

To prove that 0 is a simple eigenvalue we use the irreducibility of B . Recall that irreducibility is equivalent to the strong connectivity of the directed graph $G(B)$ associated to B , see for example [26, pp.19–20]. Let $X = (X_1, X_2, \dots, X_m)'$ be an arbitrary 0-eigenvector for B . Then (139) becomes:

$$0 = \sum_{i, j, i < j} |b_{ij}| \cdot |X_i - X_j|^2.$$

Clearly $X_i = X_j$ whenever $b_{ij} \neq 0$. In terms of graphs this translates to $X_i = X_j$ whenever i, j are connected by a path of length 1 in the directed graph $G(B)$. By induction on the length of the path we get that $X_i = X_j$ whenever i, j are connected by a path in the directed graph $G(B)$. But the latter is strongly connected because B is irreducible. It follows that all components of X are equal hence all the 0-eigenvectors are parallel to r_0 . This together with B symmetric implies that 0 is simple. □

Lemma 5.2. *If B is infinite dimensional, then B is a bounded linear operator on ℓ^1 with $\|B\|_1 \leq 2$. In addition, for $|\varepsilon| \leq 1$, the operator $T_\varepsilon = \mathbb{I} - \varepsilon^2 B$ transforms positive vectors (i.e. vectors with all components positive) into positive vectors and conserves their ℓ^1 norm.*

Proof. We need to show:

$$\|B\|_1 = \sup_j \sum_{i=1}^{\infty} |b_{ij}| \leq 2. \quad (140)$$

Fix an arbitrary $j \in \{1, 2, \dots\}$ and consider the j^{th} vector in the standard basis of ℓ^1 :

$$X = (X_1, X_2, \dots)', \quad X_i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}. \quad (141)$$

Clearly $X \in \ell^2$, $\|X\|_2 = 1$. Define the contractive operator in ℓ^2 :

$$A = (a_{ij})_{1 \leq i, j < \infty}, \\ a_{ij} = \alpha_{ij} \hat{g}_0(-\Delta_{ij}) = \langle \psi_i, \beta \psi_j \rangle \int_{-\infty}^{\infty} g_0(t) e^{i(\lambda_i - \lambda_j)t} dt. \quad (142)$$

A is contractive because for any $Y \in \ell^2$, $\|Y\|_2 = 1$:

$$\begin{aligned} |Y^* A Y| &= \left| \sum_{j,k=1}^{\infty} a_{jk} \bar{Y}_j Y_k \right| = \left| \int_{-\infty}^{\infty} g_0(t) \langle \sum_{j=1}^{\infty} e^{i\lambda_j t} Y_j \psi_j, \sum_{k=1}^{\infty} e^{i\lambda_k t} Y_k \beta \psi_k \rangle dt \right| \\ &= \left| \int_{-\infty}^{\infty} g_0(t) \langle Y(t), \beta Y(t) \rangle dt \right| \leq \int_{-\infty}^{\infty} |g_0(t)| \cdot |\langle Y(t), \beta Y(t) \rangle| dt \\ &\leq \int_{-\infty}^{\infty} |g_0(t)| \cdot \|\beta\|_{\mathcal{H}} \|Y(t)\|^2 dt = \|\beta\|_{\mathcal{H}} \int_{-\infty}^{\infty} |g_0(t)| dt = \|\beta\|_{\mathcal{H}} \|g_0\|_{L^1} = 1, \end{aligned}$$

where, $Y(t) = \sum_{j=1}^{\infty} e^{i\lambda_j t} X_j \psi_j$, $Y(t) \in \mathcal{H}$, $\|Y(t)\| \equiv 1$, and, at the very end, we used **(H2)** and **(H3)**.

By a direct calculation we have

$$\begin{aligned} \sum_{i=1}^{\infty} |b_{ij}| &= \sum_{i=1}^{\infty} |\bar{X}_i (A \cdot AX)_i - (\overline{AX})_i (AX)_i| \\ &\leq \|X\|_2 \cdot \|A \cdot AX\|_2 \leq \|A\|_2^2 \|X\|_2 + \|AX\|_2^2 \\ &\leq 2\|A\|_2^2 \|X\|_2^2 \leq 2. \end{aligned} \quad (143)$$

Inequality (140) is now proven. In addition, because $\sum_i |b_{ij}| = 2b_{ii}$, see (138), we get

$$0 \leq b_{ii} \leq 1. \quad (144)$$

Consider,

$$T_\varepsilon = \mathbb{I} - \varepsilon^2 B, \quad T_\varepsilon = (t_{ij})_{1 \leq i, j < \infty}.$$

From (138), (144) and $|\varepsilon| \leq 1$ we deduce that T_ε has nonnegative coefficients and

$$\sum_{i=1}^{\infty} t_{ij} = \sum_{i=1}^{\infty} t_{ji} = 1. \quad (145)$$

Now let

$$X = (X_1, X_2, \dots)' \in \ell^1, \quad X_j > 0 \forall j = 1, 2, \dots$$

Then

$$(T_\varepsilon X)_i = \sum_{j=1}^{\infty} t_{ij} X_j > 0,$$

since all terms in the sum are nonnegative with at least one being strictly positive. Moreover

$$\|T_\varepsilon X\|_1 = \sum_{i=1}^{\infty} |(T_\varepsilon X)_i| = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} t_{ij} X_j = \sum_{j=1}^{\infty} X_j \sum_{i=1}^{\infty} t_{ij} = \sum_{j=1}^{\infty} X_j = \|X\|_1,$$

where we exchanged the order of summation because we are dealing with convergent series with nonnegative terms and we also used (145). \square

Lemma 5.3. *If B is infinite dimensional, then B is a bounded, linear, self adjoint, non-negative operator on ℓ^2 with spectral radius less than or equal to 2. Moreover, 0 is not an eigenvalue for B .*

Proof. Consider the 2-form induced by B on ℓ^2 :

$$\begin{aligned} X^* B X &= \sum_{i,j=1}^{\infty} b_{ij} \bar{X}_i X_j \leq \sum_{i=1}^{\infty} |b_{ii}| |X_i|^2 + 1/2 \sum_{i=1}^{\infty} \sum_{j,j \neq i}^{\infty} |b_{ij}| |X_i|^2 + |X_j|^2 \\ &\leq \left(\sup_i \sum_j |b_{ij}| + \sup_j \sum_i |b_{ij}| \right) \|X\|_2^2. \end{aligned} \quad (146)$$

Because B is symmetric the two supremums above are equal to $\|B\|_1 \leq 2$, see Lemma 5.2. Therefore $|X^* B X| \leq 2 \|X\|_2^2$ and the 2-form induced by B together with its ℓ^2 norm and spectral radius are all bounded by 2. Since now B is both a symmetric and bounded operator on ℓ^2 it is self adjoint.

The argument at the end of Lemma 5.1 can be easily generalized to show that any eigenvector corresponding to the zero eigenvalue for the irreducible operator B should have all components equal. However such a vector is not in ℓ^2 unless it is trivial. Therefore 0 is not an eigenvalue for B . \square

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