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Metastable States in Parametrically Excited Multimode Hamiltonian Systems

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Abstract: Consider a linear autonomous Hamiltonian system with m time periodic bound state solutions. In this paper we study their dynamics under time almost periodic perturbations which are small, localized and Hamiltonian. The analysis proceeds through a reduction of the original infinite dimensional dynamical system to the dynamics of two coupled subsystems: a dominant m-dimensional system of ordinary differential equations (*normal form*), governing the projections onto the bound states and an infinite dimensional dispersive wave equation. The present work generalizes previous work of the authors, where the case of a single bound state is considered. Here, the interaction picture is considerably more complicated and requires deeper analysis, due to a multiplicity of bound states and the very general nature of the perturbation's time dependence. Parametric forcing induces coupling of bound states to continuum radiation modes, of bound states directly to bound states, as well as coupling among bound states, which is mediated by continuum modes. Our analysis elucidates these interactions and we prove the metastability (long life time) and eventual decay of bound states for a large class of systems. The key hypotheses for the analysis are: appropriate local energy decay estimates for the unperturbed evolution operator, restricted to the continuous spectral part of the Hamiltonian, and a matrix Fermi Golden rule condition, which ensures coupling of bound states to continuum modes. Problems of the type considered arise in many areas of application including ionization physics, quantum molecular theory and the propagation of light in optical fibers in the presence of defects.

1. Introduction

1.1. Overview. Consider the autonomous Hamiltonian system:

$$i\partial_t\phi = H_0\,\phi, \tag{1}$$

where H_0 is a self-adjoint operator on a Hilbert space $\mathcal{H}$. Assume that H_0 has eigenvalues $\lambda_1,\ \lambda_2, \ldots,\ \lambda_m$ with a complete set of eigenvectors $\psi_1,\ \psi_2, \ldots,\ \psi_m \in \mathcal{H}$. Hence

$$e^{-i\lambda_j t}\psi_j, \quad j = 1, \ldots, m \tag{2}$$

are time-periodic *bound state* solutions of the dynamical system (1). The general solution of the initial value problem for (1) splits into a noninteracting superposition of states of type (2) and radiation modes. The purpose of this paper is to study the energy exchange among bound states and continuum modes when the system is perturbed by a small time dependent Hamiltonian:

$$i\partial_t\phi \ = \ (\ H_0 \ + \ \varepsilon W(t)\ )\ \phi, \quad |\varepsilon| \text{ small.} \tag{3}$$

Our results concern almost periodic in time perturbations (may have infinite number of frequencies which can be non-commensurate, see [2, 13] or [12, Sect. 9]) of Hamiltonian systems with any finite number of bound states. We prove that the bound states of the unperturbed problem are long lived (metastable) but eventually decay to zero as $t \to \infty$ due to coupling to radiation modes. We give a detailed picture of this process on large intermediate and infinite time scales.

There are many areas in physics, chemistry and engineering in which models like Eq. (3) are used. We mention here ionization phenomena caused general time varying fields [3, 7, 14], quantum theory of molecules (see [8] and references therein) and propagation of light in optical waveguides in the presence of defects [15, 16]. In the latter application, Eq. (3) arises in the *paraxial approximation*. In this approximation, Maxwell's equation for the electromagnetic field reduces to a system of the form (3) where t plays the role of the coordinate along the waveguide, in the direction of propagation.

Due to their wide range of applicability, Eq. (3) has been extensively studied. Previous rigorous results have focused on the cases where (i) the perturbation is time-periodic [8, 16, 24, 27–29] or (ii) the unperturbed equation has a single bound state [12, 20]. In [4, 5] an analysis of certain one-dimensional exactly solvable models has been carried out *without* the requirement that ε be small. In many real physical models the time dependence of the perturbation may consist of discrete (CW) and continuous (time-localized [23]) spectral components. Here, we consider a very general class of time-dependent perturbations with discrete spectral components. Examples include superpositions of electromagnetic fields in the ionization problem, collisions among molecules [8, 26] and the distribution of defects along the length of a waveguide [15]. Moreover, in many applications the unperturbed dynamics supports multiple bound states, *e.g.* heavy atoms, double well potentials in molecules, multimode and/or multicore optical waveguides.

In some particularly interesting cases, the eigenvalues of the unperturbed problem may be nearly degenerate, as in the case of double wells (with large barrier or large separation), or degenerate, as in the case of higher order modes in a waveguide with symmetry. In this case, the dynamics of energy exchange among the modes, and therefore the time-evolution of solutions of the Schrödinger equation, depends on parameter regimes defined in terms of perturbation size and the eigenvalue spacings.

A particular example of this type is related to modeling the localization of symmetric molecules induced by collisions, a phenomenon observed in the ammonium molecule NH_3. See [8, 26] and other references cited therein. Our work applies to the interesting model considered in [8] by Grecchi and Sachetti, where the character of the system is studied on an intermediate time scale, τ_{beat}; see the next subsection. Our analysis extends theirs considerably. For a discussion of our results applied to the model in [8] see Sect. 4. Briefly, our results apply to cases where the eigenvalue splitting is arbitrarily small.

We can therefore treat all scenarios raised in [8]; in particular, the case of localization. Furthermore, our results are more general in that they apply to a large class of almost periodic perturbations on both infinite and intermediate time scales.

The next subsection presents the results for a simplified example and outlines the mathematics behind them.

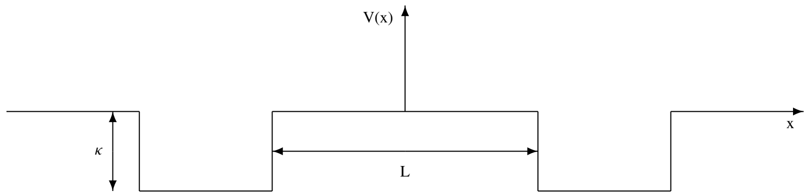
1.2. Outline of the results. To describe our general results and methods consider the Schrödinger operator with a double well potential:

$$H_0 = -\partial_x^2 + V(x) = -\partial_x^2 + V_0(x - L/2) + V_0(x + L/2), \quad x \in \mathbb{R},$$

constructed from the single well potential

$$V_0(x) = -\kappa\chi(\{|x| \le 1\}).$$

Here, $\chi(S)$ denotes the characteristic function of the set S; see the figure below.



The theory of double well potentials (see for example [9]) implies the following. For a fixed κ sufficiently small and for all L sufficiently large:

(1) H_0 is a self adjoint operator on $L^2(\mathbb{R})$ and has exactly two simple eigenvalues, $\lambda_1 < \lambda_2 < 0$, with corresponding orthonormal eigenvectors $\psi_1, \ \psi_2$. The rest of the spectrum consists of the nonnegative real axis and it is absolutely continuous. Let $\mathbf{P_c}$ denote the projection operator associated with the continuous part of the spectrum.

Consequently the unperturbed time dependent Schrödinger equation (1) has two bound state solutions, which are time-periodic and localized in x:

$$e^{-i\lambda_1 t}\psi_1 \quad \text{and} \quad e^{-i\lambda_2 t}\psi_2.$$

Let us point out that the frequency difference between the two bound states, the "eigenvalue splitting" for H_0, decays exponentially with the distance between the wells L, see for example [9]:

$$\delta_* \equiv \delta_*(L) = |\lambda_2 - \lambda_1| \sim e^{-cL} \tag{4}$$

for some positive constant c.

Consider now the perturbed problem (3) where, for simplicity, we choose a perturbation with only one frequency:

(2) Let

$$\varepsilon W(t, x) = \varepsilon \cos(\mu t)\beta(x),$$

where $\beta(x)$ is a real-valued and rapidly decaying function of x as $|x| \to \infty$ and μ is sufficiently large so that $\lambda_1 + \mu > 0$ (and therefore $\lambda_2 + \mu > 0$ as well).

We study the effect of the perturbation by projecting the solution onto the bound states and continuum modes of the unperturbed problem, i.e. we write the solution $\phi(t)$ of (3) in the form:

$$\begin{aligned}\phi(t) &= a_1(t)\psi_1 + a_2(t)\psi_2 + \phi_d(t,x), \quad \phi_d \equiv \mathbf{P_c}\phi \\ &= A_1(t)\, e^{-i\lambda_1 t}\psi_1(x) + A_2(t)\, e^{-i\lambda_2 t}\psi_2(x) + \phi_d(t,x), \\ &\quad \langle \psi_j, \phi_d(t)\rangle = 0, \quad j = 1,2. \end{aligned} \tag{5}$$

Here $A_1(t)$ and $A_2(t)$ are the slowly varying amplitudes of the bound states which in the unperturbed case, $\varepsilon = 0$, are constant. Our theory explains two types behavior of the time-dependent Schrödinger equation (3), which can be classified in terms of whether the eigenvalue splitting of H_0, δ_*, is large (case I) or small (case II) relative to the perturbation size, ε.

Alternatively, we can express these results in terms of naturally entering time scales. To see this, consider the unperturbed equation, $\varepsilon = 0$. Initial data consisting of a general nontrivial superposition (mixed state) of ground and excited states will evolve in this two-dimensional subspace and will exhibit a periodic *beating* on the time scale

$$\tau_{\text{beat}} \;\equiv\; \frac{4\pi}{|\lambda_1 - \lambda_2|} = \frac{4\pi}{\delta_*}. \tag{6}$$

Now consider the perturbed dynamics, $\varepsilon \neq 0$. Focusing on a single bound state ψ_j with energy λ_j, we expect from previous work [20, 12], that if $\mu + \lambda_j > 0$ then on a time scale

$$\tau_{\text{rad damp}} \sim \mathcal{O}\left(\frac{1}{\varepsilon^2}\right) \tag{7}$$

components of the solution in the directions ψ_j will decay due to *radiation damping*. In terms of these time scales, the two types of behavior are characterized according to whether τ_{beat} is much smaller than $\tau_{\text{rad damp}}$ (Case I) or τ_{beat} is comparable or larger than $\tau_{\text{rad damp}}$ (Case II).

For Cases I and II we prove that the bound state amplitude vector in (5):

$$a(t) = \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}$$

satisfies

$$a(t) \;=\; e^{t\left(-i\,\text{diag}[\lambda_1,\lambda_2] + \varepsilon^2[-\Gamma^\# + i\Lambda^\#]\right)} a(0) + R(t),$$

where $\Gamma^\# \geq 0$ and $\Lambda^\#$ are self-adjoint constant matrices, $R(t)$ satisfies an appropriate error bound for times of order ε^{-2} and decay estimate for $|t| \to \infty$. The particular $\Gamma^\#$ and $\Lambda^\#$, as well as estimates on $R(t)$ differ in Cases I and II. The non-negative matrix $\Gamma^\#$ is the analogue of the *Fermi golden rule* [3]. It is generically positive definite and governs the radiation damping on the time scale ε^{-2}.

Case I. Large splitting ($\tau_{\text{rad damp}} \gg \tau_{\text{beat}}$). The first result, a consequence of Theorem 3.1, assumes that $\delta_* \gg \varepsilon^2$ and says that on time scales of order $1/\varepsilon^2$ the coupling between the amplitudes $A_1(t)$ and $A_2(t)$ is negligible. Each amplitude decays exponentially due to its resonant interaction with the continuum modes; see Fig. 1. On time scales much larger than ε^{-2} the amplitudes decay to zero algebraically. The special case of Theorem 3.1 which applies here is the following:

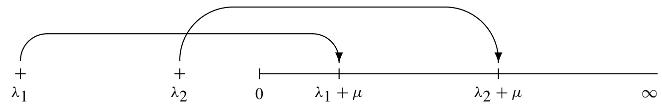


Fig. 1. Interaction picture for large eigenvalue splitting

Theorem 1.1 ($\tau_{\text{rad damp}} \gg \tau_{\text{beat}}$). *Consider the system (3) such that* **(1)** *and* **(2)** *are satisfied. There exists* $\varepsilon_0 > 0$ *such that if*

$$|\varepsilon| + \frac{\varepsilon^2}{\delta_*} \leq \varepsilon_0, \tag{8}$$

then the bound state amplitude vector in (5) satisfies

$$a(t) \;=\; e^{t\left(-i\,\text{diag}[\lambda_1,\lambda_2]+\varepsilon^2[-\Gamma^\# + i\Lambda^\#]\right)}a(0) + R(t), \tag{9}$$

where $\Gamma^\# \geq 0$ *and* $\Lambda^\#$ *are real constant* **diagonal** *matrices given by*[1]

$$\Gamma^\# \equiv \frac{\pi}{4}\begin{pmatrix} \langle \mathbf{P_c}\beta\psi_1, \delta(H_0 - \lambda_1 - \mu)\mathbf{P_c}\beta\psi_1\rangle & 0 \\ 0 & \langle \mathbf{P_c}\beta\psi_2, \delta(H_0 - \lambda_2 - \mu)\mathbf{P_c}\beta\psi_2\rangle \end{pmatrix}, \tag{10}$$

$$\begin{aligned}\Lambda^\# &= \frac{1}{4}\begin{pmatrix} \langle \mathbf{P_c}\beta\psi_1, \text{P.V.}\ (H_0 - \lambda_1 - \mu)^{-1}\mathbf{P_c}\beta\psi_1\rangle & 0 \\ 0 & \langle \mathbf{P_c}\beta\psi_2, \text{P.V.}\ (H_0 - \lambda_2 - \mu)^{-1}\mathbf{P_c}\beta\psi_2\rangle \end{pmatrix} \\ &+ \frac{1}{4}\begin{pmatrix} \langle \mathbf{P_c}\beta\psi_1, (H_0 - \lambda_1 + \mu)^{-1}\mathbf{P_c}\beta\psi_1\rangle & 0 \\ 0 & \langle \mathbf{P_c}\beta\psi_2, (H_0 - \lambda_2 + \mu)^{-1}\mathbf{P_c}\beta\psi_2\rangle \end{pmatrix} \\ &+ \frac{1}{2}\frac{\delta_*}{\delta_*^2 - \mu^2}|\langle \psi_2, \beta\psi_1\rangle|^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{11}$$

Furthermore, for any $T > 0$, $R(t)$ *satisfies*

$$\sup_{0\leq t\leq \frac{T}{\varepsilon^2}} |R(t)| \;=\; C_T\ \left(|\varepsilon| \;+\; \frac{\varepsilon^2}{\delta_*}\right), \tag{12}$$

and the wave part of the solution, $\phi_d(t,x)$*, in (5) can be written as the unperturbed wave plus a small correction:*

$$\begin{aligned} \phi_d(t,x) &= e^{-iH_0 t}\mathbf{P_c}\phi(0,x) + \tilde{\phi}_d(t,x), \\ \|\tilde{\phi}_d(t,\cdot)\|_{L^2_{\text{loc}}} &\leq C\,|\varepsilon| \qquad \text{for some constant } C \text{ and any } t > 0. \end{aligned}$$

With the exception of the non-generic case in which one or both the diagonal terms in $\Gamma^\#$ *are zero, we also have:*

$$\begin{aligned} |R(t)| &= \mathcal{O}(t^{-\frac{3}{2}}) \quad \text{for } t \to \infty, \\ \|\phi(t,\cdot)\|_{L^2_{\text{loc}}} &= \mathcal{O}(t^{-\frac{3}{2}}) \quad \text{for } t \to \infty. \end{aligned} \tag{13}$$

Here $\|\cdot\|_{L^2_{\text{loc}}}$ *denotes a weighted* $L^2(\mathbb{R})$ *norm with the weight function decaying as* $x \to \infty$*; see hypothesis* **(H3)** *in Sect. 2.*

[1] See Sect. 1.5 for comments on the operator $\delta(H_0 - \xi)$. The quantity $\langle \mathbf{P_c}\beta\psi, \delta(H_0 - \xi)\mathbf{P_c}\beta\psi\rangle$ can be viewed as $|\mathcal{F}[\beta\psi](\xi)|^2$, where $\mathcal{F}$ denotes the "Fourier Transform" with respect to the continuous spectral part of H_0.

Note that (9) and (12) imply that the mode amplitudes are very weakly coupled on time scales of order $\mathcal{O}(\varepsilon^{-2})$, and that on this time scale, the mode amplitude $|a_j|$ decays exponentially with approximately the rate $\varepsilon^2\Gamma^{\#}_{jj}$. The frequency of the mode amplitude, $a_j(t)$, is basically the unperturbed one, $-\lambda_j$, plus a small correction given by $\varepsilon^2\Lambda^{\#}_{jj}$.

Case II. Small eigenvalue splitting ($\tau_{\text{rad damp}} \sim \tau_{\text{beat}}$ *or* $\tau_{\text{rad damp}} \ll \tau_{\text{beat}}$). The bounds on the correction R in the previous result break down for $\varepsilon^2\delta_*^{-1}$ large. Our second result, a consequence of Theorem 3.2, is valid for $\delta_* \le \varepsilon$ arbitrarily small. In contrast to *Case I*, the amplitudes $A_1(t)$ and $A_2(t)$ are strongly coupled on the time scale of interest, $\mathcal{O}(\varepsilon^{-2})$. This is due to the fact that the continuum modes with which each bound state resonates have, in this case, approximately equal frequencies; see Fig. 2. As in *Case I*, the amplitudes decay exponentially on the time scale $\mathcal{O}(\varepsilon^{-2})$. However, in this case we have nondiagonal corrections to $\Gamma^{\#}$ (a non-diagonal normal form), which influence the rate of decay. On longer time scales they decay to zero algebraically.

Let

$$\underline{\lambda} = (\lambda_1 + \lambda_2)/2. \tag{14}$$

The special case of Theorem 3.2 which applies here is the following:

Theorem 1.2 ($\tau_{\text{rad damp}} < \tau_{\text{beat}}$). *Consider the system (3) and assume* **(1)** *and* **(2)** *are satisfied and*

$$\delta_* \le C\varepsilon$$

for some constant C. Then there exists $\varepsilon_0 > 0$ such that for

$$\varepsilon \le \varepsilon_0$$

the bound state amplitude vector, $a(t)$, is given by

$$a(t) = e^{t\left(-i\,\text{diag}[\lambda_1,\lambda_2]+\varepsilon^2[-\Gamma^{\#}+i\Lambda^{\#}]\right)}a(0) + R(t), \tag{15}$$

where $\Gamma^{\#} \ge 0$ and $\Lambda^{\#}$ are now **non-diagonal** *self adjoint constant matrices given by*

$$\Gamma^{\#} \equiv \frac{\pi}{4}\begin{pmatrix} \langle \mathbf{P_c}\beta\psi_1, \delta(H_0-\underline{\lambda}-\mu)\mathbf{P_c}\beta\psi_1\rangle & \langle \mathbf{P_c}\beta\psi_1, \delta(H_0-\underline{\lambda}-\mu)\mathbf{P_c}\beta\psi_2\rangle \\ \langle \mathbf{P_c}\beta\psi_2, \delta(H_0-\underline{\lambda}-\mu)\mathbf{P_c}\beta\psi_1\rangle & \langle \mathbf{P_c}\beta\psi_2, \delta(H_0-\underline{\lambda}-\mu)\mathbf{P_c}\beta\psi_2\rangle \end{pmatrix}, \tag{16}$$

$$\begin{aligned}\Lambda^{\#} \equiv\ & \frac{1}{4}\begin{pmatrix} \langle \mathbf{P_c}\beta\psi_1, \text{P.V. } (H_0-\underline{\lambda}-\mu)^{-1}\mathbf{P_c}\beta\psi_1\rangle & \langle \mathbf{P_c}\beta\psi_1, \text{P.V. } (H_0-\underline{\lambda}-\mu)^{-1}\mathbf{P_c}\beta\psi_2\rangle \\ \langle \mathbf{P_c}\beta\psi_2, \text{P.V. } (H_0-\underline{\lambda}-\mu)^{-1}\mathbf{P_c}\beta\psi_1\rangle & \langle \mathbf{P_c}\beta\psi_2, \text{P.V. } (H_0-\underline{\lambda}-\mu)^{-1}\mathbf{P_c}\beta\psi_2\rangle \end{pmatrix} \\ &+ \frac{1}{4}\begin{pmatrix} \langle \mathbf{P_c}\beta\psi_1, (H_0-\underline{\lambda}+\mu)^{-1}\mathbf{P_c}\beta\psi_1\rangle & \langle \mathbf{P_c}\beta\psi_1, (H_0-\underline{\lambda}+\mu)^{-1}\mathbf{P_c}\beta\psi_2\rangle \\ \langle \mathbf{P_c}\beta\psi_2, (H_0-\underline{\lambda}+\mu)^{-1}\mathbf{P_c}\beta\psi_1\rangle & \langle \mathbf{P_c}\beta\psi_2, (H_0-\underline{\lambda}+\mu)^{-1}\mathbf{P_c}\beta\psi_2\rangle \end{pmatrix},\\ \underline{\lambda} \equiv\ & \frac{\lambda_1+\lambda_2}{2}. \end{aligned} \tag{17}$$

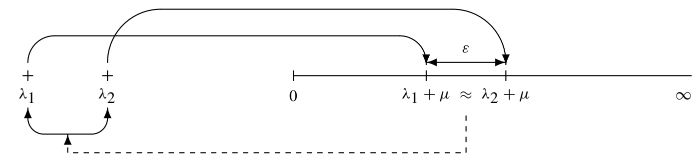


Fig. 2. Interaction picture for small eigenvalue splitting

Furthermore, for any $T > 0$, $R(t)$ satisfies the estimate (independent of δ_*):

$$\sup_{0 \le t \le \frac{T}{\varepsilon^2}} |R(t)| = C_T |\varepsilon|, \tag{18}$$

and the wave part of the solution, $\phi_d(t,x)$, can be written as the unperturbed wave plus a small correction:

$$\phi_d(t,x) = e^{-iH_0 t}\mathbf{P_c}\phi(0,x) + \tilde{\phi}_d(t,x),$$
$$\|\tilde{\phi}_d(t,\cdot)\|_{L^2_{\text{loc}}} \le C\,|\varepsilon| \qquad \text{for some constant } C \text{ and any } t > 0.$$

If in addition $\Gamma^\# \ge \theta_0 > 0$, where θ_0 is a constant which is independent of ε and δ_*, then

$$\begin{aligned} |R(t)| &= \mathcal{O}(t^{-\frac{3}{2}}) \quad \text{for } t \to \infty, \\ \|\phi(t,\cdot)\|_{L^2_{\text{loc}}} &= \mathcal{O}(t^{-\frac{3}{2}}) \quad \text{for } t \to \infty. \end{aligned} \tag{19}$$

Here $\|\cdot\|_{L^2_{\text{loc}}}$ is as in Theorem 1.1.

Remark 1.1. In contrast to Theorem 1.1 the additional coupling in this regime is manifested in the off diagonal terms of $\Gamma^\#$ and $\Lambda^\#$. The term corresponding to the last term in (11) is omitted, since it is of higher order. It is now part of $R(t)$.

The off diagonal terms show that the bound states $\psi_1,\ \psi_2$ of the unperturbed problem no longer form the right basis for describing the evolution. Instead one should use a pair of linear combinations of $\psi_1,\ \psi_2$ to diagonalize or at least obtain an upper triangular matrix as an exponent in (15). In Sect. 4 we show that for a particular example the right basis is formed by $\psi_1 + \psi_2$, respectively $\psi_1 - \psi_2$ which are localized in the left, respectively in the right well. Moreover, for the example considered, the perturbation is localized in the left well and we find that the amplitude of $\psi_1 + \psi_2$ decays on a much shorter time scale compared to the amplitude of $\psi_1 - \psi_2$. Hence the system "localizes" in the right well.

1.3. Outline of the analysis. In this subsection we outline the mathematics behind these theorems. Consider the solution, $\phi(t)$, of (3) in the form (5). A careful expansion and analysis to second order in the perturbation $\varepsilon W(t)$, which is presented in Appendix 6 and constituting an extension of the one in [12, 20], reveals the following system for $A(t) = (A_1(t), A_2(t))$, and $\phi_d(t)$:

$$\partial_t A(t) = (-\varepsilon^2\Gamma + i\varepsilon\eta(t) + i\varepsilon^2\Lambda + \varepsilon^2\rho(t)\ )\ A(t) + E(t;A(t),\phi_d(t)), \tag{20}$$
$$i\partial_t\phi_d(t,x) = H_0\phi_d(t,x) + \mathbf{P_c}\ F(t,x;A(t),\phi_d(t)). \tag{21}$$

Here

$$\mathbf{P_c} f \equiv f - \langle\psi_1, f\rangle\psi_1 - \langle\psi_2, f\rangle\psi_2$$

defines the projection onto the continuous spectral part of H_0, the term $E(t)$ can be neglected on times up to order $1/\varepsilon^2$ while on larger time scales both $E(t;A,\phi_d)$ and $F(t,x;A,\phi_d)$ tend to zero provided A and the "local energy" of ϕ_d do so. The matrices Γ, respectively Λ are diagonal with real constant coefficients given by (10), respectively (11) without the last term. $\eta(t)$ and $\rho(t)$ have quasiperiodic coefficients (almost periodic in general) with frequencies $\mu,\ \lambda_2 - \lambda_1,\ \lambda_2 - \lambda_1 \pm \mu$ and $\lambda_2 - \lambda_1 \pm 2\mu$.

An important step in our analysis is assessing the effect of the oscillatory terms $\eta(t)$ and $\rho(t)$ in (20). We construct a near-identity change of variables:

$$a(t) \mapsto [I + M_\varepsilon(t)]b(t), \tag{22}$$

where $M_\varepsilon(t)$ is almost periodic (thus uniformly bounded in t) with $M_\varepsilon(t) = \mathcal{O}(\varepsilon)$ which maps the bound state amplitude system to one of the form

$$\frac{db}{dt} = \left[-i \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} - \varepsilon^2 \Gamma^\# + i\varepsilon^2 \Lambda^\# \right] b + \tilde{R}_\varepsilon(t). \tag{23}$$

Here $M_\varepsilon(t)$ is bounded and almost periodic in t. The details of $\Gamma^\# = \Gamma + \ldots$, $\Lambda^\# = \Lambda + \ldots$, $\tilde{R}_\varepsilon$ and $M_\varepsilon(t)$ depend on whether one considers *Case I* or *Case II*. In *Case I*, $\Gamma^\#$ and $\Lambda^\#$ are diagonal and self-adjoint. In Case II, $\Gamma^\#$ and $\Lambda^\#$ are both non-diagonal and self-adjoint, with the non-diagonal part potentially having large effect on the *lifetime* of localized states.

1.4. Outline of the paper. The paper is structured as follows. In Sect. 2 we give a general formulation of the problem. The hypotheses on the unperturbed Hamiltonian H_0 and the perturbation $W(t)$ are introduced and discussed. A general result for the case of large eigenvalue splitting $\delta_* \sim 1$, Theorem 2.1, is stated. In Sect. 3 we study the case of small eigenvalue splitting. In order to focus on how the bound states are affected by the interaction with the wave part we rule out any bound state – bound state direct resonance, see hypothesis **(H6)'**. We show that the bound states slowly decay, but, in contrast to the case of large eigenvalue splitting, the rates of decay are now influenced by the relative sizes of the perturbation and eigenvalue gap. This also determines if the nearly degenerate bound states will evolve uncoupled or coupled. The results are stated in Theorems 3.1 and 3.2 which are generalizations of *Cases I* and *II* presented in Sect. 1.2. In Sect. 4 we discuss two examples related to double well potentials. The first one has been previously considered in [8]. Based on our results, in particular Theorem 3.2, we solve their "localization" conjecture. The second example is a rather general double well problem: $H_0 = -\Delta + V(x_1 - L/2, x_2, \ldots, x_n) + V(x_1 + L/2, x_2, \ldots, x_n)$. For large L, H_0 has the eigenvalue splitting roughly $\exp(-cL)$, $c > 0$. We discuss our results in this context. Section 5 contains the essentials of the proofs. Much of the work lies in constructing near-identity transformations which map the system (20) to a normal form, appropriate for the regimes of *Cases I* and *II*. We require, in particular, an extension of the normal form approach developed in [12, 20, 21]. For completeness we present it in Appendix 6. It is here where the interaction among bound states and the continuum modes is made explicit. The three subsections in 5 show how to further simplify the normal form by using a (non)stationary phase type computation. Their outcome is a system of ODE's having the general form (68). The long time behavior of the solution of the latter is then analyzed in Proposition 5.2. This last and rather technical result is proven in Appendix 7.

1.5. Some notation and terminology. For $z \in \mathbb{C}$, $\Re z$, $\Im z$,denote, respectively, its real and imaginary parts.

For $a = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m$, $|a| = \sqrt{\sum_{i=1}^m |a_i|^2}$ denotes its Euclidean norm.

Generic constants will be denoted by C, D, etc.

$\langle x\rangle = \left(1+|x|^2\right)^{\frac{1}{2}}$.
$\delta_{jk} = 1$, if $j = k$ and 0 if $j \neq k$.
$\mathcal{L}(\mathcal{A}, \mathcal{B}) =$ the space of bounded linear operators from $\mathcal{A}$ to $\mathcal{B}$; $\mathcal{L}(\mathcal{A}, \mathcal{A}) \equiv \mathcal{L}(\mathcal{A})$.
$M^* \in \mathcal{L}(\mathcal{B}^*, \mathcal{A}^*)$ denotes the adjoint operator of $M \in \mathcal{L}(\mathcal{A}, \mathcal{B})$.
$[M_1, M_2] = M_1M_2 - M_2M_1$ denotes the commutator of the operators $M_1, M_2 \in \mathcal{L}(\mathcal{A})$.
For f, g in a Hilbert space $\mathcal{H}$, their inner product is denoted by $\langle f, g\rangle$, and the norm of f is denoted by $\|f\|$.
Functions of self-adjoint operators are defined via the spectral theorem; see for example [18]. The operators containing boundary value of resolvents or singular distributions applied to self-adjoint operators are defined in [12, Sect. 8]. In particular we will frequently use the operators $\delta(H-\lambda)$ and P.V.$(H-\lambda)^{-1}$.

2. General Formulation and Results for Large Eigenvalue Splitting

Consider the Schrödinger equation for the function of time, $\phi(t)$, with values in a complex Hilbert space $\mathcal{H}$:

$$\begin{aligned} i\partial_t\phi(t) &= (H_0 + W(t))\,\phi(t), \\ \phi|_{t=0} &= \phi(0). \end{aligned} \tag{24}$$

Note. The perturbation of H_0 is written as $W(t)$ instead of $\varepsilon W(t)$, used in the introduction. We have done this to make the notation less cumbersome. The smallness condition $\varepsilon \le \varepsilon_0$ will be expressed in terms of an appropriate norm of W, $|||W|||$; see (30).

We now introduce some general hypotheses on the dynamical system (1).
(H1) H_0 is self-adjoint on the Hilbert space $\mathcal{H}$.
(H2) The spectrum of H_0 is assumed to consist of an absolutely continuous part, $\sigma_{\mathrm{cont}}(H_0)$, with associated spectral projection $\mathbf{P_c}$ and isolated eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ (counting multiplicity) with an orthonormalized set of eigenvectors $\psi_1, \psi_2, \ldots, \psi_m$, i.e. for $k,\ j = 1, \ldots, m$,

$$H_0\psi_k = \lambda_k\psi_k,\ \langle\psi_k, \psi_j\rangle = \delta_{kj}, \tag{25}$$

where δ_{kj} is the Kronecker-delta symbol.
(H3) Local decay estimates on e^{-iH_0t}: There exist self-adjoint "weights", w_-, w_+, number $r_1 > 1$ and a constant $\mathcal{C}$ such that

(i) w_+ is defined on a dense subspace of $\mathcal{H}$ and on which $w_+ \ge cI,\ c > 0$;
(ii) w_- is bounded, i.e. $w_- \in \mathcal{L}(\mathcal{H})$, and Range$(w_-) \subseteq$ Domain(w_+);
(iii) $w_+\, w_-\, \mathbf{P_c} = \mathbf{P_c}$ on $\mathcal{H}$ and $\mathbf{P_c} = \mathbf{P_c}\, w_-\, w_+$ on the domain of w_+;

and for all $f \in \mathcal{H}$ satisfying $w_+f \in \mathcal{H}$ we have

$$\textbf{(a)}\ \ \|w_-e^{-iH_0t}\mathbf{P_c}f\| \le \mathcal{C}\,\langle t\rangle^{-r_1}\|w_+f\|,\ \text{ for } t \in \mathbb{R}; \tag{26}$$

$$\textbf{(b)}\ \ \|w_-e^{-iH_0t}(H_0-\lambda_k-\mu_j-i0)^{-1}\,\mathbf{P_c}f\| \le\ \mathcal{C}\,\langle t\rangle^{-r_1}\ \|w_+f\|,\ \text{ for } t \ge 0, \tag{27}$$

where $k = 1, 2, \ldots, m$ while $\mu_j \in \mathbb{Z}$ are the Fourier exponents of the perturbation (see below). For $t < 0$ estimate (27) is assumed to hold with $-i0$ replaced by $+i0$.

Remark 2.1. In the case $H_0 = -\Delta + V(x),\ x \in \mathbb{R}^n$ condition **(a)** is satisfied for generic potentials $V(x)$, with sufficiently rapid decay at infinity; see [10, 17, 25]. As for condition **(b)** we showed in [12, Sect. 3] how to obtain a constant $\mathcal{C}$ uniform in frequencies μ_j for generic, localized $V(x)$ and $\lambda_k + \mu_j$ bounded away from zero. In particular, both (a) and (b) are satisfied for our example in the introduction provided the zero energy resonance is excluded, i.e. $\kappa \notin \{(k\pi/2)^2,\ k \in \mathbb{N}\}$, see [25].

(H4) Hypotheses on the perturbation $W(t)$: We consider time-dependent symmetric perturbations of the form [2]

$$W(t) = \frac{1}{2}\beta_0 + \sum_{j\in\mathbb{N}} \cos(\mu_j t)\ \beta_j,\ \text{ with } \beta_j^* = \beta_j \text{ and } \sum_{j\in\mathbb{N}_0} \|\beta_j\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{28}$$

Equivalently,

$$W(t) = \frac{1}{2} \sum_{j\in\mathbb{Z}} \exp(-i\mu_j t)\beta_j, \tag{29}$$

where, $\mu_0 = 0$ and for $j < 0$, $\mu_j = -\mu_{-j}$, $\beta_j = \beta_{-j}$. Thus, $W(t)$ is an almost periodic function with values in the Banach space $\mathcal{L}(\mathcal{H})$ with the Fourier exponents $\{\mu_j\}_{j\in\mathbb{Z}}$ and corresponding Fourier coefficients $\{\beta_j\}_{j\in\mathbb{Z}}$; see, for example, [13].

To measure the size of the perturbation W, we introduce the norm

$$|||W||| \ \equiv\ \frac{1}{2}\sum_{j\in\mathbb{Z}} \|w_+\ \beta_j\|_{\mathcal{L}(\mathcal{H})}\ +\ \frac{1}{2}\sum_{j\in\mathbb{Z}} \|w_+\ \beta_j w_+\|_{\mathcal{L}(\mathcal{H})}, \tag{30}$$

which is assumed to be finite. We shall require in addition that $|||W|||$ is small.

(H5) Resonance condition – Fermi golden rule: Consider the self-adjoint complex matrix Γ with elements

$$\Gamma_{ij}\ \equiv\ \frac{\pi}{4} \sum_{\substack{k,n\in\mathbb{Z}\\ \lambda_i+\mu_k=\lambda_j+\mu_n\in\sigma_{\text{cont}}(H_0)}} \langle \mathbf{P_c}\beta_k\psi_i,\ \delta(H_0-\lambda_j-\mu_n)\mathbf{P_c}\beta_n\psi_j\rangle, \tag{31}$$

where $i, j \in 1,\ldots m$. In Appendix 6 Γ is defined and shown to be nonnegative definite. Let γ denote the smallest eigenvalue of Γ. We assume that there exists a constant $\theta_0 > 0$ such that

$$\gamma\ \geq\ \theta_0\ |||W|||^2. \tag{32}$$

Remark 2.2. The assumption (32) is used only to obtain infinite time scale asymptotics. Our results for times of up to order $|||W|||^{-2}$ (i.e. ε^{-2} in the notation for the introduction) are valid even if Γ has zero eigenvalues.

(H6) Control of small denominators (large spectral gap): Define the first and second order resonance sets:

$$\begin{aligned} I^1 &= \{1,\ldots,m\}\times\{1,\ldots,m\}\times\mathbb{Z},\\ I^1_{res} &= \{(i,j,k)\in I:\ \lambda_i+\mu_k=\lambda_j\},\\ I^2 &= \{1,\ldots,m\}\times\{1,\ldots,m\}\times\mathbb{Z}\times\mathbb{Z},\\ I^2_{res} &= \{(i,j,k,n)\in I:\ \lambda_i+\mu_k=\lambda_j+\mu_n\}. \end{aligned}$$

We assume

$$\delta_* \equiv \min\left(\inf_{I^1\setminus I^1_{res}} |\lambda_i+\mu_k-\lambda_j|,\ \inf_{I^2\setminus I^2_{res}} |\lambda_i+\mu_k-\lambda_j-\mu_n|\right) > 0. \tag{33}$$

[2] $W(t) = \frac{1}{2}\beta_0 + \sum_{j\in\mathbb{N}} \cos(\mu_j t + \delta_j)\ \beta_j$ can be handled as well.

Remark 2.3. Note that for the example in the introduction $\delta_* = |\lambda_2 - \lambda_1|$ provided μ is fixed and L is sufficiently large such that $\mu > 2(\lambda_2 - \lambda_1) \approx 2\exp(-cL)$. Also note that (H6) is satisfied if H_0 is fixed and $W(t)$ is periodic or a trigonometric polynomial in "t".

We now state the main result of this section:

Theorem 2.1 (*Large spectral gap*)**.** *Assume the hypotheses* **(H1)–(H6)** *hold. Then there exist ε_0 and the constants C_T, C such that if*

$$|||W||| + \frac{|||W|||}{\delta_*} + \frac{|||W|||}{\delta_*^2} < \varepsilon_0, \tag{34}$$

then any solution of (24) with $w_+\phi(0) \in \mathcal{H}$, satisfies

$$\begin{aligned} \phi(t) &= \sum_{j=1}^{m} e^{-i\lambda_j t} A_j(t)\psi_j + \phi_d(t), \quad A = (A_1, A_2, \dots, A_m)^T, \\ A(t) &= e^{(-\Gamma^{\#} + i\Lambda^{\#})t} A(0) + R(t), \\ \phi_d(t) &\equiv \mathbf{P_c}\phi(t) = e^{-iH_0 t}\mathbf{P_c}\phi(0) + \tilde{\phi}_d(t). \end{aligned} \tag{35}$$

Here $\Gamma^{\#} = \Gamma \geq 0$ is the self adjoint matrix given in **(H5)**, *$\Lambda^{\#} = \Lambda + \eta_1 + \eta_3$, is constant and self adjoint and displayed in (120), (74), (82). Finally, for any fixed $T > 0$,*

$$|R(t)| \leq C_T \left(|||W||| + \frac{|||W|||}{\delta_*} + \frac{|||W|||}{\delta_*^2} \right), \ 0 \leq t \leq T|||W|||^{-2}, \tag{36}$$

$$\begin{aligned} |R(t)| &= \mathcal{O}(\langle t \rangle^{-r_1}), \quad \text{for } t \to \infty, \\ \|w_-\tilde{\phi}_d(t)\| &\leq C\ |||W|||, \quad \text{for } t > 0, \\ \|w_-\phi(t)\| &= \mathcal{O}(\langle t \rangle^{-r_1}), \quad \text{for } t \to \infty. \end{aligned} \tag{37}$$

The theorem explicitly computes the dominant evolution for the amplitudes of the eigenvectors on times of order $|||W|||^{-2}$, (35). From an examination of the formulae for Γ in **(H5)** and (120), (74), (82), we infer that coupling between the modes ψ_i and ψ_j, $(i \neq j)$, occurs on time scales of order $|||W|||^{-2}$ only if the perturbation W has frequencies μ_k and μ_n such that:

$$\lambda_i + \mu_k = \lambda_j + \mu_n. \tag{38}$$

If none of these resonance relations hold, the matrix Γ is diagonal and $|A_j(t)| \sim e^{-\Gamma_{jj}t}$. Theorem 2.1 is an extension to the multibound state case of the result in [12] for the case where the unperturbed Hamiltonian, H_0, has one bound state.

Theorem 2.1 does not apply if $\delta_*^2 \sim |||W|||$ or $\delta_*^2 \ll |||W|||$. An important example is the double-well problem, in which two single wells are separated by a distance L; see the introduction. In this case, the eigenvalues of H_0 occur in pairs which are exponentially close for L large and hence δ_* is not bounded below uniformly in L. In particular,

$$\begin{aligned} &\delta_* \leq \inf\{\, |\lambda_i - \lambda_j| : \lambda_i \neq \lambda_j,\ \lambda_i, \lambda_j \in \sigma_{\text{discrete}}(H_0)\,\} = \mathcal{O}(e^{-cL}) \to 0 \\ &\quad \text{as } L \to \infty. \end{aligned}$$

The results of the next section provide a description of the dynamics in problems with nearly degenerate eigenvalues.

3. Small Eigenvalue Splitting

In this section we study the case when the unperturbed Hamiltonian has very small eigenvalue spacings. Motivated by the case of double well potentials, whose eigenvalues come in nearly degenerate pairs in the large separation or large barrier limit (see Sect. 4), we assume in addition to **(H1)**:

(H2)' The spectrum of H_0 consists of an absolutely continuous part, $\sigma_{\rm cont}(H_0)$, with associated spectral projection $\mathbf{P_c}$ and isolated eigenvalues which group in nearly degenerate pairs: $\lambda_1, \lambda_2; \lambda_3, \lambda_4; \dots; \lambda_{2N-1}, \lambda_{2N}$. Corresponding to these eigenvalues is an orthonormal set of eigenvectors $\psi_1, \psi_2, \dots, \psi_{2N}$.

We assume that the distances

$$\delta_*^j = |\lambda_{2j-1} - \lambda_{2j}|,$$

where $j = 1, \dots, N$ are small compared with the size of the perturbation frequencies and the distances among the pairs of eigenvalues in a manner which is made precise in hypothesis **(H6)'** below. Let

$$\underline{\lambda}^j = \frac{\lambda_{2j-1} + \lambda_{2j}}{2}, \qquad j = 1, \dots, N.$$

We found it to be much simpler to state the theorems for the case of pairs of close eigenvalues. However, these results have straightforward generalizations to clusters of more than two eigenvalues and we will refer to them after each theorem.

In order to prove our results we assume that **(H3), (H4), (H5)** are satisfied and replace **(H6)** with the following condition which ensures that there are no significant couplings among modes corresponding to different pairs of nearly degenerate eigenvalues:

(H6)' Assume that:

$$\underline{\lambda}^i \pm \mu_k = \underline{\lambda}^j \text{ if and only if } i = j \text{ and } \mu_k = 0,$$
$$\underline{\lambda}^i \pm \mu_k = \underline{\lambda}^j \pm \mu_n \text{ if and only if } i = j \text{ and } \mu_k = \mu_n.$$

In all the other cases we assume that there exists a constant $D > 0$ such that:

$$|\underline{\lambda}^i \pm \mu_k - \underline{\lambda}^j| \geq D > 0,$$
$$|\underline{\lambda}^i \pm \mu_k - \underline{\lambda}^j \pm \mu_n| \geq D > 0.$$

Furthermore assume that

$$\delta_*^j \leq \frac{D}{4}, \qquad j = 1, \dots, N.$$

Denote:

$$\delta_* = \min_{j=1,\dots,N} \delta_*^j, \tag{39}$$

$$\delta^* = \max_{j=1,\dots,N} \delta_*^j. \tag{40}$$

Recall that the formulas in Theorem 2.1 contain correction terms of size $|||W|||/\delta_*$ which may be large in the settings of this section (one can easily see that δ_* in this section coincides with the one in the previous section). However the next result shows that under the the modified hypothesis **(H2)'** and **(H6)'** and for mean zero perturbations (i.e. $\beta_0 \equiv 0$), the correction terms are much smaller allowing us to infer an uncoupled evolution of all the bound states provided $\delta_* \gg |||W|||^2$:

Theorem 3.1. *Suppose that the Hamiltonian H_0 satisfies* **(H1), (H2)'** *and* **(H3)**. *Assume that the perturbation, W is chosen such that* **(H4), (H5)** *and* **(H6)'** *hold and in addition that it has mean value zero, i.e. $\beta_0 = 0$. Then there exist $\varepsilon_0 > 0$ such that if*

$$|||W||| + \frac{|||W|||^2}{\delta_*} \le \varepsilon_0,$$

then any solution of (53) with $w_+\phi(0) \in \mathcal{H}$, satisfies

$$\phi(t) = \sum_{j=1}^{m} e^{-i\lambda_j t} A_j(t)\psi_j + \phi_d(t), \quad A = (A_1, A_2, \ldots, A_{2N})^T,$$

$$A(t) = e^{(-\Gamma^\# + i\Lambda^\#)t} A(0) + R(t),$$

$$\phi_d(t) \equiv \mathbf{P_c}\phi(t) = e^{-iH_0 t}\mathbf{P_c}\phi(0) + \tilde{\phi}_d(t).$$

Here $\Gamma^\# \ge 0$ and $\Lambda^\#$ are constant, real-valued **diagonal** *matrices given by:*

$$\Gamma^\#_{jj} = \frac{\pi}{4} \sum_{\substack{n\in\mathbb{Z} \\ \lambda_j+\mu_n\in\sigma_{\text{cont}}(H_0)}} \langle \mathbf{P_c}\beta_n\psi_j,\ \delta(H_0 - \lambda_j - \mu_n)\mathbf{P_c}\beta_n\psi_j\rangle,$$

$$\begin{aligned}\Lambda^\#_{jj} = &\frac{1}{4} \sum_{\substack{n\in\mathbb{Z} \\ \lambda_j+\mu_n\in\sigma_{\text{cont}}(H_0)}} \langle \mathbf{P_c}\beta_n\psi_j, \text{P.V.}(H_0 - \lambda_j - \mu_n)^{-1}\mathbf{P_c}\beta_n\psi_j\rangle \\ &+\frac{1}{4} \sum_{\substack{n\in\mathbb{Z} \\ \lambda_j+\mu_n\notin\sigma_{\text{cont}}(H_0)}} \langle \mathbf{P_c}\beta_n\psi_j, (H_0 - \lambda_j - \mu_n)^{-1}\mathbf{P_c}\beta_n\psi_j\rangle, \\ &+\frac{1}{2} \sum_{\substack{n\in\mathbb{N} \\ 1\le p\le 2N}} \frac{\lambda_p - \lambda_j}{(\lambda_j - \lambda_p)^2 - \mu_n^2} \left|\langle \psi_p, \beta_n\psi_j\rangle\right|^2,\end{aligned}$$

where $j = 1,\ 2, \ldots, 2N$.

Finally, for any fixed $T > 0$,

$$|R(t)| \ \le\ C_T\left(|||W||| + \frac{|||W|||^2}{\delta_*}\right),\ 0 \le t \le T|||W|||^{-2},$$

and for any $t > 0$

$$\|w_-\tilde{\phi}_d(t)\| \ \le\ C\ |||W|||.$$

If, in addition, for all $j = 1, 2, \ldots, 2N$ we have $\Gamma^\#_{jj} \ge \theta_0|||W|||^2$ for some constant θ_0, which is independent of $|||W|||$ and δ_*, then

$$|R(t)| = \mathcal{O}(\langle t\rangle^{-r_1}), \qquad \text{for } t \to \infty,$$

$$\|w_-\phi(t)\| = \mathcal{O}(\langle t\rangle^{-r_1}), \qquad \text{for } t \to \infty.$$

Note that if one deals with clusters of more than two nearly degenerate eigenvalues the above result carries out without modifications.

The next result is valid as $\delta_* \to 0$. The tradeoff is that it contains correction terms of size δ^*, consequently it is aimed at Hamiltonians with nearly degenerate eigenvalue pairs and complements Theorem 3.1:

Theorem 3.2 (*Arbitrarily small spectral splitting*)**.** *Consider the initial value problem (53). Suppose* **(H1)**, **(H2)'** *and* **(H3)** *hold, the perturbation satisfies* **(H4)** *and* **(H6)'**. *Then there exists* ε_0 *such that if*

$$\delta^* + |||W||| \le \varepsilon_0,$$

then any solution of (53) with $w_+\phi(0) \in \mathcal{H}$, *satisfies*

$$\phi(t) = \sum_{j=1}^{m} a_j(t)\psi_j + \phi_d(t), \quad a = (a_1, a_2, \ldots, a_{2N})^T,$$
$$a(t) = e^{(-i\,\mathrm{diag}[\lambda_1,\lambda_2,\ldots,\lambda_{2N}]-\Gamma^{\#}+i\Lambda^{\#})t}a(0) + R(t),$$
$$\phi_d(t) \equiv \mathbf{P_c}\phi(t) = e^{-iH_0t}\mathbf{P_c}\phi(0) + \tilde{\phi}_d(t).$$

Here

$$\Gamma^{\#} = \mathrm{diag}\left[\Gamma_1^{\#}, \Gamma_2^{\#}, \ldots, \Gamma_N^{\#}\right] \ge 0,$$
$$\Lambda^{\#} = \mathrm{diag}\left[\Lambda_1^{\#}, \Lambda_2^{\#}, \ldots, \Lambda_N^{\#}\right]$$

are constant, self adjoint, block-diagonal matrices with each block of size 2×2. $\Gamma^{\#} \ge 0$ *and its blocks are given by*

$$\Gamma_j^{\#} = \frac{\pi}{4} \sum_{\substack{n\in\mathbb{Z} \\ \underline{\lambda}_j+\mu_n\in\sigma_{\mathrm{cont}}(H_0)}} \Gamma_j^n, \tag{41}$$

where

$$\Gamma_j^n = \begin{pmatrix} \langle \mathbf{P_c}\beta_n\psi_{2j-1}, \delta(H_0-\underline{\lambda}_j-\mu_n)\mathbf{P_c}\beta_n\psi_{2j-1}\rangle & \langle \mathbf{P_c}\beta_n\psi_{2j-1}, \delta(H_0-\underline{\lambda}_j-\mu_n)\mathbf{P_c}\beta_n\psi_{2j}\rangle \\ \langle \mathbf{P_c}\beta_n\psi_{2j}, \delta(H_0-\underline{\lambda}_j-\mu_n)\mathbf{P_c}\beta_n\psi_{2j-1}\rangle & \langle \mathbf{P_c}\beta_n\psi_{2j}, \delta(H_0-\underline{\lambda}_j-\mu_n)\mathbf{P_c}\beta_n\psi_{2j}\rangle \end{pmatrix}. \tag{42}$$

The blocks which form $\Lambda^{\#}$ *are given by*

$$\Lambda_j^{\#} = -\frac{1}{2}\begin{pmatrix} \langle\psi_{2j-1}, \beta_0\psi_{2j-1}\rangle & \langle\psi_{2j-1}, \beta_0\psi_{2j}\rangle \\ \langle\psi_{2j}, \beta_0\psi_{2j-1}\rangle & \langle\psi_{2j}, \beta_0\psi_{2j}\rangle \end{pmatrix}$$
$$+\frac{1}{4}\sum_{\substack{n\in\mathbb{Z} \\ \underline{\lambda}_j+\mu_n\in\sigma_{\mathrm{cont}}(H_0)}} \Lambda_{j1}^n + \frac{1}{4}\sum_{\substack{n\in\mathbb{Z} \\ \underline{\lambda}_j+\mu_n\notin\sigma_{\mathrm{cont}}(H_0)}} \Lambda_{j2}^n$$
$$+\frac{1}{2}\sum_{\substack{n>0 \\ 1\le p\le N}} \Lambda_j^{np} + \frac{1}{4}\sum_{\substack{p\ne j \\ 1\le p\le N}} \Lambda_j^{0p}, \tag{43}$$

where

$$\Lambda_{j1}^n =$$
$$\begin{pmatrix} \langle \mathbf{P_c}\beta_n\psi_{2j-1}, \mathrm{P.V.}\,(H_0-\underline{\lambda}_j-\mu_n)^{-1}\mathbf{P_c}\beta_n\psi_{2j-1}\rangle & \langle \mathbf{P_c}\beta_n\psi_{2j-1}, \mathrm{P.V.}\,(H_0-\underline{\lambda}_j-\mu_n)^{-1}\mathbf{P_c}\beta_n\psi_{2j}\rangle \\ \langle \mathbf{P_c}\beta_n\psi_2, \mathrm{P.V.}\,(H_0-\underline{\lambda}_j-\mu_n)^{-1}\mathbf{P_c}\beta\psi_{2j-1}\rangle & \langle \mathbf{P_c}\beta_n\psi_{2j}, \mathrm{P.V.}\,(H_0-\underline{\lambda}_j-\mu_n)^{-1}\mathbf{P_c}\beta_n\psi_{2j}\rangle \end{pmatrix},$$

$$\Lambda_{j2}^n =$$
$$\begin{pmatrix} \langle \mathbf{P_c}\beta_n\psi_{2j-1},\ (H_0-\underline{\lambda}_j-\mu_n)^{-1}\mathbf{P_c}\beta_n\psi_{2j-1}\rangle & \langle \mathbf{P_c}\beta_n\psi_{2j-1},\ (H_0-\underline{\lambda}_j-\mu_n)^{-1}\mathbf{P_c}\beta_n\psi_{2j}\rangle \\ \langle \mathbf{P_c}\beta_n\psi_2,\ (H_0-\underline{\lambda}_j-\mu_n)^{-1}\mathbf{P_c}\beta\psi_{2j-1}\rangle & \langle \mathbf{P_c}\beta_n\psi_{2j},\ (H_0-\underline{\lambda}_j-\mu_n)^{-1}\mathbf{P_c}\beta_n\psi_{2j}\rangle \end{pmatrix},$$

$$\Lambda_j^{np} = \frac{\underline{\lambda}_p - \underline{\lambda}_j}{(\underline{\lambda}_j - \underline{\lambda}_p)^2 - \mu_n^2} \times \begin{pmatrix} \left|\langle\psi_{j*}, \beta_n\psi_{p*}\rangle\right|^2 + \left|\langle\psi_{j*}, \beta_n\psi_{2p}\rangle\right|^2 & \langle\psi_{j*}, \beta_n\psi_{p*}\rangle\langle\psi_{p*}, \beta_n\psi_{2j}\rangle + \langle\psi_{j*}, \beta_n\psi_{2p}\rangle\langle\psi_{2p}, \beta_n\psi_{2j}\rangle \\ \langle\psi_{2j}, \beta_n\psi_{p*}\rangle\langle\psi_{p*}, \beta_n\psi_{j*}\rangle + \langle\psi_{2j}, \beta_n\psi_{2p}\rangle\langle\psi_{2p}, \beta_n\psi_{j*}\rangle & \left|\langle\psi_{2j}, \beta_n\psi_{p*}\rangle\right|^2 + \left|\langle\psi_{2j}, \beta_n\psi_{2p}\rangle\right|^2 \end{pmatrix},$$

$$\Lambda_j^{0p} = \frac{1}{\underline{\lambda}_p - \underline{\lambda}_j} \times \begin{pmatrix} \left|\langle\psi_{j*}, \beta_0\psi_{p*}\rangle\right|^2 + \left|\langle\psi_{j*}, \beta_0\psi_{2p}\rangle\right|^2 & \langle\psi_{j*}, \beta_0\psi_{p*}\rangle\langle\psi_{p*}, \beta_0\psi_{2j}\rangle + \langle\psi_{j*}, \beta_0\psi_{2p}\rangle\langle\psi_{2p}, \beta_0\psi_{2j}\rangle \\ \langle\psi_{2j}, \beta_0\psi_{p*}\rangle\langle\psi_{p*}, \beta_0\psi_{j*}\rangle + \langle\psi_{2j}, \beta_0\psi_{2p}\rangle\langle\psi_{2p}, \beta_0\psi_{j*}\rangle & \left|\langle\psi_{2j}, \beta_0\psi_{p*}\rangle\right|^2 + \left|\langle\psi_{2j}, \beta_0\psi_{2p}\rangle\right|^2 \end{pmatrix},$$

here $j*$, *respectively* $p*$, *denote* $2j-1$, *respectively* $2p-1$.

Finally, for any fixed $T > 0$,

$$|R(t)| \le C_T \left(|||W||| + \delta^*\right), \; 0 \le t \le T|||W|||^{-2},$$

and for any $t > 0$

$$\|w_- \tilde{\phi}_d(t)\| \le C \; |||W|||.$$

If, in addition, the smallest eigenvalue of $\Gamma^\#$, γ, *satisfies* $\gamma \ge \theta_0|||W|||^2$ *for some constant* θ_0, *which is independent of* $|||W|||$ *and* δ_* *then*

$$|R(t)| = \mathcal{O}(\langle t\rangle^{-r_1}), \qquad \text{for } t \to \infty,$$
$$\|w_-\phi(t)\| = \mathcal{O}(\langle t\rangle^{-r_1}), \qquad \text{for } t \to \infty.$$

In the case of a cluster of $n > 2$ eigenvalues the corresponding 2×2 block will be replaced by a $n \times n$ block on the diagonal and with elements which are straightforward generalizations of the $n = 2$ case.

4. Examples

In this section we illustrate our results with two examples.

4.1. Double well with large barrier. An interesting example studied in [8] by Grecchi and Sachetti is a one-dimensional model of a double well potential with a barrier. The mathematical formulation is:

$$i\partial_t\phi(t,x) = \left(-\partial_x^2 + V(x)\right)\phi(t,x) + \varepsilon W(t,x)\phi(t,x),$$
$$\phi(t=0,x) = \phi_0(x). \tag{44}$$

The potential, $V(x)$, is given by

$$V(x) = -b\delta(x+a) - b\delta(x-a) + \rho\delta(x), \tag{45}$$

where $\delta(x-\xi)$ denotes the Dirac distribution centered at $x = \xi$. a, b and ρ are real positive parameters.

Consequently, the unperturbed Hamiltonian $H_0 \equiv -\partial_x^2 + V(x)$ is defined on the $H^2(\mathbb{R})$ Sobolev space and is self adjoint on $L^2(\mathbb{R})$. If $ab > 1$ and $\rho \geq \rho_0 > 0$ it has exactly two eigenvalues:

$$\begin{aligned} \lambda_1 < \lambda_2 < 0, \\ \delta_* \equiv \lambda_2 - \lambda_1 = \mathcal{O}(\rho^{-1}). \end{aligned} \tag{46}$$

The ground state eigenfunction, $\psi_1(x)$ is symmetric and concentrated in a neighborhood of the interval $x \in [-a,\ a]$. The excited state, $\psi_2(x)$ is anti-symmetric and concentrated in a neighborhood of the interval $x \in [-a,\ a]$. It is approximately equal to $\psi_1(x)$ in a neighborhood of $x \in [-a,\ 0]$ and $-\psi_1(x)$ in a neighborhood of $x \in [0,\ a]$.

Moreover the rest of the spectrum is absolutely continuous and equal to the positive real line. The spectral theory is worked out in [1]. Concerning hypothesis **(H3)**, Grecchi and Sachetti verify the local decay estimate (26) with $r_1 = 3/2$, $w_\pm \equiv (1 + x^2)^{\pm\frac{\sigma}{2}}$, $\sigma > 7/2$, and a constant which is uniform in ρ. By the analysis in [12, Sect. 3] we can obtain the singular decay estimate (27) from (26) provided

$$|\lambda_{1,2} \pm \mu_j| \geq D > 0, \tag{47}$$

where μ_j are the frequencies of the perturbation $W(t, x)$. In this case the estimate will be uniform in μ's and ρ.

The perturbation $W(t, x)$ in [8] is periodic in time, has a finite number of frequencies:

$$W(t, x) = \beta_0(x) + \sum_{n=1}^{N} \cos(\mu_n t)\beta_n(x), \quad \mu_n = n\mu_1, \tag{48}$$

and $\beta_n(x)$ decay sufficiently fast as $|x| \to \infty$ such that

$$\int_{-\infty}^{+\infty} (1 + x^2)^{2\sigma} |\beta_n(x)|^2 dx < \infty. \tag{49}$$

We can treat more general perturbations. For example a general trigonometric polynomial in "t" with non-commensurate frequencies, i.e. $\mu_n \neq n\mu_1$, or a time periodic perturbation having an infinite number of harmonics, i.e. $N = \infty$, and sufficiently smooth in time so that (30) holds. In both cases all our hypothesis are satisfied.

If the "barrier height parameter", ρ, is not very large we can apply Theorem 2.1. The correction terms are of size $\mathcal{O}(\rho\varepsilon)$. If in addition $\beta_0 \equiv 0$, the correction terms are of size $\mathcal{O}(\rho\varepsilon^2)$ due to Theorem 3.1. If ρ is so large that the above corrections are significant then one can apply Theorem 3.2 and obtain correction terms of size $\mathcal{O}(\varepsilon + \rho^{-1})$ which **decrease** as ρ **increases**. Consequently, this latter result is uniform in $\rho \geq \rho_0$ for some large ρ_0. The analysis in [8] holds for $\rho \leq C\varepsilon^{-2}$, $C > 0$.

We now show how our results apply to prove the "localization" phenomenon conjectured in [8]. There it is claimed that for

$$\delta_* \sim \rho^{-1} \ll \varepsilon^2 \tag{50}$$

and for a perturbation localized in one well the system will tend to stay in the other well. Indeed consider

$$W(t, x) = \cos(\mu t)\beta(x) \tag{51}$$

with $\beta(x)$ being the characteristic function of the interval $(-a, 0)$, i.e. localized in the left well. μ is chosen such that $\lambda_1 + \mu > 0$. The particular case of Theorem 3.2 which applies has already been explicitly stated in Theorem 1.2. By the properties of β, ψ_1 and ψ_2 we have

$$\mathbf{P_c}\beta\psi_1 = \mathbf{P_c}\beta\psi_2 + \mathcal{O}(\delta_*). \tag{52}$$

For the formulae for $\Gamma^\#$ and $\Lambda^\#$ in Theorem 1.2, the order δ_* term above can be omitted by adding it to the correction term $R(t)$. The resulting increase in the size of the error, $\mathcal{O}(\delta_*)$ is much smaller than the actual size of $R(t) \sim \varepsilon$, see (12) and (50). Now one can proceed to find the eigenvalues and eigenvectors of the constant matrix in the exponent of (15). It is a matter of tedious but elementary calculations to show that the eigenvectors are $(1, 1)'$ and $(1, -1)'$. They correspond to the following states:

$$\frac{\psi_1 + \psi_2}{\sqrt{2}}, \qquad \frac{\psi_1 - \psi_2}{\sqrt{2}}.$$

The first state is localized in the left well while the second one is in the right well. The real part of the eigenvalue corresponding to the first state is

$$\frac{-\varepsilon^2}{2}\langle \mathbf{P_c}\beta\psi_1, \delta(H_0 - \underline{\lambda} - \mu - i0)\mathbf{P_c}\beta\psi_1\rangle + \varepsilon^2\mathcal{O}((\frac{\delta_*}{\varepsilon^2})^2) \sim \varepsilon^2$$

which insures a significant decay of this state on ε^{-2} time scales. The real part of the eigenvalue corresponding to the second state is of order $\varepsilon^2\mathcal{O}\left((\frac{\delta_*}{\varepsilon^2})^2\right) \ll \varepsilon^2$ which leaves the size of the second state practically unchanged over ε^{-2} time scales. As the second state is localized in the right well we conclude that the system localizes in this well.

4.2. Double wells with large separation. In this example the equation is

$$\begin{aligned} &i\partial_t\phi(t) = (H_0 + \varepsilon W(t))\,\phi(t),\\ &\phi|_{t=0} = \phi(0), \end{aligned} \tag{53}$$

where

$$\begin{aligned} H_0 &\equiv -\Delta + V(x),\\ V(x) &\equiv V_0(x_1 - L/2, x_2, \dots, x_n) + V_0(x_1 + L/2, x_2, \dots, x_n). \end{aligned}$$

Here Δ is the Laplacian with respect to the variables $(x_1, \dots, x_n)$, $L > 0$ is a parameter measuring the distance between the wells, and $V_0(x_1, \dots, x_n)$ is a real valued potential defined on $\mathbb{R}^n$, with sufficiently rapid decay as $|x| \to \infty$, i.e.

$$|V_0(x)| \le C\,(1 + |x|)^{-\sigma} \tag{54}$$

for a sufficiently large and positive σ, see [10, 17, 25].

We assume that the single well Hamiltonian $-\Delta + V_0(x)$ has simple eigenvalues, $\underline{\lambda}^1, \underline{\lambda}^2, \dots, \underline{\lambda}^N < 0$ and that the rest of its spectrum, $[0, \infty)$, is absolutely continuous. It follows from [10, 17, 25] that for sufficiently large L, the unperturbed, double well Hamiltonian H_0 has a spectrum consisting of the absolutely continuous part, $[0, \infty)$, with associated continuous spectral projection $\mathbf{P_c} = \mathbf{P_c}(L)$. Moreover it has pairs of simple eigenvalues $\lambda_{2j-1}(L), \lambda_{2j}(L)$, $j = 1, \dots, N$ such that for any positive and small ϵ, as L tends to infinity,

$$\lambda_{2j-1}(L), \lambda_{2j}(L) \to \underline{\lambda}^j, \tag{55}$$

$$\delta_*^j \equiv |\lambda_{2j-1}(L) - \lambda_{2j}(L)| = \mathcal{O}(L^{1-n}e^{-2L\sqrt{-\underline{\lambda}^j-\epsilon}}); \tag{56}$$

see [9, relation (5.13)].

Following the technique in [12, Sect. 3] and relying on the decaying properties of V_0, one can verify that both the local and singular local decay estimates in **(H3)** hold for a fixed L. We choose $W(t)$ such that **(H4)** and **(H5)** hold. For small L ($\delta_* \sim 1$) we assume **(H6)** and for large L (δ_* small) we assume **(H6)'**. Hence Theorems 2.1, 3.1 and 3.2 hold.

We note that due to hypotheses on decay of $V(x)$, see [10, 17, 25], the constant in the local decay estimates (26) and (27) may grow with L at some polynomial rate, $L^{\tilde{\sigma}}$, $\tilde{\sigma} > 0$. Thus, our proof gives an ε_0 in Theorem 3.2 which decreases with increasing L, e.g. $\varepsilon_0 = \varepsilon_1 L^{-\tilde{\sigma}}$. Therefore, for large L the condition for the validity of Theorem 3.2 is

$$\varepsilon + \delta^* \leq \varepsilon_1 L^{-\tilde{\sigma}}.$$

Since $\delta^* = \max_j \delta_*^j$ is exponentially small in L, for large enough L we have Theorem 3.2 if

$$\varepsilon \leq \varepsilon_1' L^{-\tilde{\sigma}}.$$

A result which is uniform in L for all $L \geq L_0$ would hold if local decay estimates of type (26) and (27) in **(H3)** were known for $V(x)$ in, say, appropriate non-weighted $L^p(\mathbb{R}^n)$ spaces. This appears to be an open problem.

5. Decomposition and Normal Form

The goal of this section is to rewrite the perturbed Schrödinger equation (24) in an equivalent form in which the dominant flow of energy among bound states and radiation modes is made explicit. Initially we follow the path used in [12]. We then refine and extend the approach to obtain asymptotics which are valid uniformly in the eigenvalue splitting, δ_*. We start with a review and extension of the technique in [12]. Details are provided in the Appendix of Sect. 6.

Under the hypotheses **(H1)** and **(H2)** the solution of (24) can be written as:

$$\phi(t) = \sum_{l=1}^{m} a_l(t)\psi_l + \mathbf{P_c}\phi(t), \tag{57}$$

where

$$a_l(t) = \langle \psi_l, \phi(t) \rangle, \quad l = 1, \ldots, m. \tag{58}$$

$\mathbf{P_c}\phi(t)$ denotes the projection onto the continuous spectrum associated with H_0. Denote

$$\phi_d(t) = \mathbf{P_c}\phi(t), \tag{59}$$

$$A_l(t) = e^{i\lambda_l t} a_l(t), \quad l = 1, \ldots, m, \tag{60}$$

and $A(t)$ the column vector with components $A_l(t)$. In Sect. 6 we prove the following

Proposition 5.1. *The initial value problem for Eq. (24) is equivalent to the system*

$$\partial_t A(t) = [-\Gamma + i(\Lambda + \eta(t)) + \rho(t)]\, A(t) + E(t), \tag{61}$$

$$\partial_t \phi_d(t) = -iH_0\phi_d(t) - i\mathbf{P_c}W(t)\phi_d(t) - i\sum_{l=1}^{m} e^{-i\lambda_l t} A_l(t)\mathbf{P_c}W(t)\psi_l, \tag{62}$$

where Γ *is given in* **(H5)** *and* $\Lambda,\ \eta(t),\ \rho(t)$ *and* $E(t)$ *are explicitly displayed in (120–123). Consequently*

$$\phi_d(t) = e^{-iH_0 t}\phi_d(0) + \tilde{\phi}(t),$$

where, for any $t > 0$ *and* $0 \le r \le r_1$*,*

$$\|w_-\tilde{\phi}(t)\| \le C|||W|||\,, \tag{63}$$

$$\langle t\rangle^r \|w_-\phi_d(t)\| \le C + D|||W||| \sup_{0\le s\le t} \langle s\rangle^r |A(s)|\,. \tag{64}$$

Moreover, the coefficient matrices have the following important properties:

(a) Γ *and* Λ *are self-adjoint constant coefficient complex matrices of order* $\mathcal{O}(|||W|||^2)$.
(b) Γ *is nonnegative definite.*
(c) $\eta(t)$ *is of order* $\mathcal{O}(|||W|||)$ *and self-adjoint.*
(d) $\rho(t)$ *is an almost periodic matrix with mean* 0 *and order* $\mathcal{O}(|||W|||^2)$.
(e) $E(t) = E(t, A(t), \phi_d(t))$ *is such that for any* $0 \le r \le r_1$ *and sufficiently small perturbation* W*, i.e.* $|||W|||$ *is sufficiently small, there exists the constants* C *and* D *such that for all* $t > 0$*:*

$$|E(t)| \le \frac{C}{\langle t\rangle^{r_1}}|||W||| + \frac{D}{\langle t\rangle^{r}}|||W|||^3 \sup_{0\le s\le t} \langle s\rangle^r |A(s)|. \tag{65}$$

Note that relation (63) says that the wave part is within $|||W|||$ from the unperturbed wave. This is part of the conclusion in all our Theorems 2.1, 3.1 and 3.2. To show that the full solution of the unperturbed problem decays polynomially in time it is now sufficient to prove that $\langle t\rangle^{r_1}|A(t)|$ is bounded for $t > 0$, since

$$\begin{aligned}\langle t\rangle^{r_1}\|w_-\phi(t)\| &\le \langle t\rangle^{r_1}|A(t)| + \langle t\rangle^{r_1}\|w_-\phi_d(t)\| \\ &\le C + (1 + D\,|||W|||) \sup_{0\le s\le t}\langle s\rangle^{r_1}|A(s)|,\end{aligned} \tag{66}$$

where we used (64).

In order to show that $\langle t\rangle^{r_1}|A(t)|$ is bounded for $t > 0$ and to obtain a more precise dynamics for $A(t)$ on $|||W|||^{-2}$ time scales we need to further refine (61). We will use a near identity change of variables, $A \mapsto B$, to reduce (61) to a system of the form:

$$B(t) = (I - M(t))A(t), \tag{67}$$

$$\partial_t B = (-\Gamma^{\#} + i\Lambda^{\#})B + \alpha B + F. \tag{68}$$

Here,

(p1) $M(0) = 0$ and $M(t)$ is a time dependent matrix of order $\mathcal{O}(|||W||| + \epsilon)$ uniformly in t, i.e. there exists a constant C such that for all $t \in \mathbb{R}$ we have

$$|M(t)| \le C\,(|||W||| + \epsilon)\,.$$

(p2) $\Gamma^{\#}$ is a constant, self adjoint, nonnegative definite matrix of order $\mathcal{O}(|||W|||^2)$.
(p3) $\Lambda^{\#}$ is a constant, self adjoint matrix of order $\mathcal{O}(|||W|||)$.
(p4) $\alpha(t)$ is a time dependent matrix of higher order, i.e. there exists a constant C such that for all $t \in \mathbb{R}$ we have

$$|\alpha(t)| \le C_\alpha\,|||W|||^2\,(|||W||| + \epsilon).$$

(p5) $F(t) = F(B, \phi_d; t)$ satisfies an estimate in terms of the norm of W and B of the form (65), i.e. there exists the constants C and D such that for all $t > 0$ and $0 \le r \le r_1$:

$$|F(t)| \le C\frac{1+\epsilon}{\langle t\rangle^{r_1}}|||W||| + D\frac{1+\epsilon}{\langle t\rangle^{r}}\,|||W|||^3 \sup_{0\le s\le t}\langle s\rangle^r|B(s)|. \tag{69}$$

The different settings of Theorems 2.1, 3.1 and 3.2 lead to different M in (67) and, consequently, different $\Gamma^{\#}$, $\Lambda^{\#}$, α, F and ϵ in (68) and (p1)–(p5). They arise due to the matrix character of (61). In [12] it was relatively simple to infer the behavior of the scalar $A(t)$ from (61); the fundamental solution associated with its homogeneous part can be factored as

$$e^{-\Gamma(t-s)+\int_s^t i(\Lambda+\eta(\tau))+\rho(\tau)d\tau} = e^{-\Gamma(t-s)}e^{i\int_s^t \Lambda+\eta(\tau)d\tau}e^{\int_s^t \rho(\tau)d\tau}, \tag{70}$$

and we could analyze the norm of each operator in the right-hand side.

However, in the multiple bound state case, the above splitting is valid only if the $m \times m$ complex matrices Γ, Λ, $\eta(t)$, $\int_s^t \eta(\tau)d\tau$, $\rho(t)$ and $\int_s^t \rho(\tau)d\tau$ commute for all t, $s \in \mathbb{R}$, $t \ge s$. Since this is typically false for $m \ge 2$, a detailed analysis of the fundamental solution of the homogeneous part of (61) is required. In carrying this out, we exploit the fact that $\partial_t A$ is of order $\mathcal{O}(|||W|||)$ and we carefully integrate by parts appropriate terms on the right-hand side of (61) to obtain a system of the form (67–68). The proofs are then finished by applying the following proposition:

Proposition 5.2. *Suppose $B(t)$ is a solution of (68) and* (p2)–(p5) *are satisfied. Then, there exists $\varepsilon_0 > 0$ such that whenever $|||W||| + \epsilon \le \varepsilon_0$, we have*

$$B(t) = e^{(-\Gamma^{\#}+i\Lambda^{\#})t}B(0) + \mathcal{O}(|||W||| + \epsilon) \quad \text{for } |t| = \mathcal{O}(|||W|||^{-2}).$$

If, in addition, the smallest eigenvalue of $\Gamma^{\#}$, γ, satisfies $\gamma \ge \theta_0|||W|||^2$ for some constant θ_0, which is independent of $|||W|||$ and ϵ then

$$B(t) = O(\langle t\rangle^{-r_1}) \quad \text{as } |t| \to \infty.$$

The proof of this general proposition is rather technical and is presented in the Appendix of Sect. 7.

We now observe that to prove Theorems 2.1, 3.1 and 3.2 it suffices to verify that in each case (61–62) can be reduced to a system of (68) with (p1)–(p5) satisfied. This is the purpose of the next three subsections.

To verify this assertion we note that if the solution, $A(t)$, of (61) is related to those of (68) through the change of variable (67), then

$$A(t) = (I-M(t))^{-1}B(t) = e^{(-\Gamma^{\#}+i\Lambda^{\#})t}A(0)+\mathcal{O}(|||W|||+\epsilon) \quad \text{for } |t| = \mathcal{O}(|||W|||^{-2}), \tag{71}$$

provided (p1)–(p5) hold.

If in addition $\gamma \ge \theta_0|||W|||^2$ for some constant θ_0 then clearly

$$A(t) = (I - M(t))^{-1}B(t) = \mathcal{O}(\langle t\rangle^{-r_1}) \quad \text{as } |t| \to \infty. \tag{72}$$

Using now the estimate (66) the theorems are completely proven.

5.1. Expansion and normal form for Theorem 2.1. We begin by splitting the coefficient, $\eta(t)$, in (61) and displayed in (121) into its time independent average and time dependent oscillating part:

$$\eta(t) \equiv \eta_1 - i\eta_2(t), \tag{73}$$

$$(\eta_1)_{lj} = -\frac{1}{2} \sum_{\substack{n\in\mathbb{Z} \\ \lambda_l = \lambda_j + \mu_n}} \langle \psi_l, \beta_n \psi_j \rangle, \tag{74}$$

$$(\eta_2(t))_{lj} = -\frac{i}{2} \sum_{\substack{n\in\mathbb{Z} \\ \lambda_l \neq \lambda_j + \mu_n}} e^{i(\lambda_l - \lambda_j - \mu_n)t} \langle \psi_l, \beta_n \psi_j \rangle. \tag{75}$$

Then, (61) becomes:

$$\partial_t A(t) = [-\Gamma + i\Lambda + i\eta_1]\, A(t) + [\eta_2(t) + \rho(t)]\, A(t) + E(t). \tag{76}$$

We next integrate (76) from 0 to t:

$$A(t) = A(0) + \int_0^t [-\Gamma + i\Lambda + i\eta_1]\, A(s)ds + \int_0^t E(s)ds + \int_0^t [\eta_2(s) + \rho(s)]\, A(s)\, ds.$$

Let

$$M_1(t) = \int_0^t \eta_2(s) + \rho(s) ds. \tag{77}$$

Integration by parts of the last integral yields:

$$\begin{aligned} [I - M_1(t)]\, A(t) = [I - M_1(0)]\, A(0) + \int_0^t [-\Gamma + i\Lambda + i\eta_1]\, A(s)ds \\ + \int_0^t E(s)ds - \int_0^t M_1(s)\partial_s A(s)ds, \end{aligned} \tag{78}$$

where

$$\begin{aligned} &(M_1(t))_{lj} \\ &\quad = -\frac{1}{2} \sum_{\substack{n\in\mathbb{Z} \\ \lambda_l \neq \lambda_j + \mu_n}} \frac{e^{i(\lambda_l - \lambda_j - \mu_n)t}}{\lambda_l - \lambda_j - \mu_n} \langle \psi_l, \beta_n \psi_j \rangle \\ &\qquad + \frac{1}{4} \sum_{\substack{n,k\in\mathbb{Z} \\ \lambda_l + \mu_k \neq \lambda_j + \mu_n}} \frac{e^{i(\lambda_l + \mu_k - \lambda_j - \mu_n)t}}{\lambda_l + \mu_k - \lambda_j - \mu_n} \langle \mathbf{P_c} \beta_k \psi_l, (H_0 - \lambda_j - \mu_n - i0)^{-1} \mathbf{P_c} \beta_n \psi_j \rangle. \end{aligned} \tag{79}$$

Note that due to **(H4)** and **(H6)**,

$$M_1(t) = \mathcal{O}\left(\frac{|||W|||}{\delta_*} \right). \tag{80}$$

The next step is to replace $\partial_s A(s)$ in (78) using (76):

$$[I - M_1(t)]\, A(t) = [I - M_1(0)]\, A(0) + \int_0^t [I - M_1(s)]\, [-\Gamma + i\Lambda + i\eta_1]\, A(s) ds$$
$$- \int_0^t M_1(s)\eta_2(s) A(s) ds - \int_0^t M_1(s)\rho(s) A(s) ds$$
$$+ \int_0^t [I - M_1(s)]\, E(s) ds.$$

Commuting $[I - M_1(s)]$ and $[-\Gamma + i\Lambda + i\eta_1]$ in the first integral, we obtain

$$[I - M_1(t)]\, A(t) = [I - M_1(0)]\, A(0) + \int_0^t [-\Gamma + i\Lambda + i\eta_1]\, [I - M_1(s)]\, A(s) ds$$
$$-i \int_0^t [M_1(s), \eta_1]\, A(s) ds - \int_0^t M_1(s)\eta_2(s) A(s) ds$$
$$- \int_0^t [M_1(s), (-\Gamma + i\Lambda)]\, A(s) ds - \int_0^t M_1(s)\rho(s) A(s) ds$$
$$+ \int_0^t [I - M_1(s)]\, E(s)\, ds. \tag{81}$$

A direct calculation gives:

$$\begin{aligned}
&(i\,[M_1(s), \eta_1] \;+\; M_1(s)\eta_2(s))_{lj} \\
&= \frac{i}{4} \sum_{k,n\in\mathbb{Z}} \sum_{\substack{1\le p\le m \\ \lambda_p \neq \lambda_l+\mu_k}} \frac{e^{i(\lambda_l+\mu_k-\lambda_j-\mu_n)s}}{\lambda_l + \mu_k - \lambda_p} \langle \beta_k\psi_l, \psi_p\rangle \langle \psi_p, \beta_n\psi_j\rangle \\
&\quad - \frac{i}{4} \sum_{\substack{k,n\in\mathbb{Z} \\ \lambda_l+\mu_k\neq\lambda_j+\mu_n}} \frac{e^{i(\lambda_l+\mu_k-\lambda_j-\mu_n)s}}{\lambda_l + \mu_k - \lambda_j - \mu_n} \sum_{\substack{1\le p\le m \\ \lambda_p=\lambda_l+\mu_k}} \langle \beta_k\psi_l, \psi_p\rangle \langle \psi_p, \beta_n\psi_j\rangle \\
&\quad + \mathcal{O}\left(\frac{|||W|||^3}{\delta_*}\right) \\
&\equiv -i\,(\eta_3)_{lj} - (\eta_4(s))_{lj} + \mathcal{O}\left(\frac{|||W|||^3}{\delta_*}\right).
\end{aligned}$$

Here, $-i\eta_3$ is the average and $\eta_4(t)$ the oscillating part. Specifically,

$$(\eta_3)_{lj} = -\frac{1}{4} \sum_{\substack{k,n\in\mathbb{Z} \\ \lambda_l+\mu_k=\lambda_j+\mu_n}} \sum_{\substack{1\le p\le m \\ \lambda_p\neq\lambda_l+\mu_k}} \frac{1}{\lambda_l + \mu_k - \lambda_p} \langle \beta_k\psi_l, \psi_p\rangle \langle \psi_p, \beta_n\psi_j\rangle, \tag{82}$$

$$\begin{aligned}
(\eta_4(t))_{lj} &= -\frac{i}{4} \sum_{\substack{k,n\in\mathbb{Z} \\ \lambda_l+\mu_k\neq\lambda_j+\mu_n}} \sum_{\substack{1\le p\le m \\ \lambda_p\neq\lambda_l+\mu_k}} \frac{e^{i(\lambda_l+\mu_k-\lambda_j-\mu_n)t}}{\lambda_l + \mu_k - \lambda_p} \langle \beta_k\psi_l, \psi_p\rangle \langle \psi_p, \beta_n\psi_j\rangle \\
&\quad + \frac{i}{4} \sum_{\substack{k,n\in\mathbb{Z} \\ \lambda_l+\mu_k\neq\lambda_j+\mu_n}} \frac{e^{i(\lambda_l+\mu_k-\lambda_j-\mu_n)t}}{\lambda_l + \mu_k - \lambda_j - \mu_n} \sum_{\substack{1\le p\le m \\ \lambda_p=\lambda_l+\mu_k}} \langle \beta_k\psi_l, \psi_p\rangle \langle \psi_p, \beta_n\psi_j\rangle.
\end{aligned} \tag{83}$$

Note that **(H4)** and **(H6)** imply

$$\eta_3 = \mathcal{O}\left(\frac{|||W|||^2}{\delta_*}\right). \tag{84}$$

Hence (81) can be written as:

$$\begin{aligned}
[I - M_1(t)]\, A(t) = [I - M_1(0)]\, A(0) \\
&+ \int_0^t [-\Gamma + i(\Lambda + \eta_1 + \eta_3)]\, [I - M_1(s)]\, A(s)ds \\
&+ \int_0^t \eta_4(s) A(s)ds + \int_0^t \mathcal{O}\left(\frac{|||W|||^3}{\delta_*} + \frac{|||W|||^3}{\delta_*^2}\right) A(s)ds \\
&+ \int_0^t [I - M_1(s)]\, E(s)\, ds.
\end{aligned} \tag{85}$$

The last step is to integrate by parts $\int_0^t \eta_4(s)A(s)ds$. Let $M_2(t)$ be the antiderivative of $\eta_4(t)$ given by

$$\begin{aligned}
(M_2(t))_{lj} = -\frac{1}{4} \sum_{\substack{k,n\in\mathbb{Z} \\ \lambda_l+\mu_k\neq\lambda_j+\mu_n}} \sum_{\substack{1\leq p\leq m \\ \lambda_p\neq\lambda_l+\mu_k}} \frac{e^{i(\lambda_l+\mu_k-\lambda_j-\mu_n)t}}{(\lambda_l+\mu_k-\lambda_p)(\lambda_l+\mu_k-\lambda_j-\mu_n)} \\
\times \langle \beta_k\psi_l, \psi_p\rangle\langle\psi_p, \beta_n\psi_j\rangle + \frac{1}{4} \sum_{\substack{k,n\in\mathbb{Z} \\ \lambda_l+\mu_k\neq\lambda_j+\mu_n}} \frac{e^{i(\lambda_l+\mu_k-\lambda_j-\mu_n)t}}{(\lambda_l+\mu_k-\lambda_j-\mu_n)^2} \\
\times \sum_{\substack{1\leq p\leq m \\ \lambda_p=\lambda_l+\mu_k}} \langle \beta_k\psi_l, \psi_p\rangle\langle\psi_p, \beta_n\psi_j\rangle.
\end{aligned} \tag{86}$$

By **(H4)** and **(H6)**,

$$M_2(t) = \mathcal{O}\left(\frac{|||W|||^2}{\delta_*^2}\right). \tag{87}$$

Hence, (85) becomes:

$$\begin{aligned}
&[I - M_1(t) - M_2(t)]\, A(t) \\
&\quad = [I - M_1(0) - M_2(0)]\, A(0) \\
&\quad + \int_0^t [-\Gamma + i(\Lambda + \eta_1 + \eta_3)]\, [I - M_1(s) - M_2(s)]\, A(s)ds \\
&\quad + \int_0^t \mathcal{O}\left(\frac{|||W|||^3}{\delta_*} + \frac{|||W|||^3}{\delta_*^2}\right) A(s)ds \\
&\quad + \int_0^t [I - M_1(s) - M_2(s)]\, E(s)ds.
\end{aligned} \tag{88}$$

Now introduce into (88) the near identity change of variable:

$$B(t) = [I - M_1(t) - M_2(t)]\, A(t). \tag{89}$$

Differentiation with respect to t yields:

$$\partial_t B(t) = [-\Gamma + i(\Lambda + \eta_1 + \eta_3)]\, B(t) \\ +\mathcal{O}\left(\frac{|||W|||^3}{\delta_*} + \frac{|||W|||^3}{\delta_*^2}\right) B(t) + [I - M_1(t) - M_2(t)]\, E(t). \quad (90)$$

Comparing (89–90) with (67–68) gives the identities:

$$M(t) = M_1(t) + M_2(t) = \mathcal{O}\left(\frac{|||W|||}{\delta_*}\right) + \mathcal{O}\left(\frac{|||W|||^2}{\delta_*^2}\right),$$
$$J = \Lambda + \eta_1 + \eta_3,$$
$$\alpha(t) \le C_\alpha |||W|||^2 \left(\frac{|||W|||}{\delta_*} + \frac{|||W|||}{\delta_*^2}\right),$$
$$F(t) = [I - M_1(t) - M_2(t)]\, E(t).$$

Now, hypotheses (p1)–(p5) can be readily verified. Thus Theorem 2.1 is completely proven.

5.2. Expansion and normal form for Theorem 3.1. We start as in the previous section and after one integration by parts we have:

$$[I - M_1(t)]\, A(t) = [I - M_1(0)]\, A(0) + \int_0^t [-\Gamma + i\Lambda]\, [I - M_1(s)]\, A(s)ds \\ - \int_0^t M_1(s)\eta_2(s)A(s)ds \\ - \int_0^t [M_1(s), (-\Gamma + i\Lambda)]\, A(s)ds - \int_0^t M_1(s)\rho(s)A(s)ds \\ + \int_0^t [I - M_1(s)]\, E(s)\, ds, \quad (91)$$

where $M_1(t)$ and $\eta_2(t)$ are given as before by (79) respectively (75). Note though that due to (h6) and $\beta_0 \equiv 0$ we have $\eta_1 \equiv 0$ and

$$M_1(t) = \mathcal{O}\left(\frac{|||W|||}{D} + \frac{|||W|||^2}{\delta_*}\right). \quad (92)$$

Terms which need to further be integrated by parts lie in the kernel

$$(M_1(s)\eta_2(s))_{lj} = \frac{i}{4} \sum_{k,n\in\mathbb{Z}} \sum_{\substack{1\le p\le 2N \\ \lambda_p \ne \lambda_l + \mu_k}} \frac{e^{i(\lambda_l+\mu_k-\lambda_j-\mu_n)s}}{\lambda_l + \mu_k - \lambda_p} \langle \beta_k\psi_l, \psi_p\rangle\langle\psi_p, \beta_n\psi_j\rangle \\ -\frac{i}{4}\sum_{k,n\in\mathbb{Z}} \frac{e^{i(\lambda_l-\lambda_j-\mu_n)s}}{\lambda_l - \lambda_{l*}} \langle \mathbf{P_c}\beta_k\psi_l, (H_0 - \lambda_{l*} - i0){-1}\mathbf{P_c}\psi_{l*}\rangle\langle\psi_{l*}, \beta_n\psi_j\rangle \\ +\mathcal{O}\left(\frac{|||W|||^3}{D}\right) \\ \equiv -i\eta_3 - (\eta_4(s))_{lj} + \mathcal{O}\left(\frac{|||W|||^3}{D}\right).$$

Here $l* = l+1$ respectively $l* = l-1$ if l is odd respectively if l is even. By integrating by parts $\int_0^t \eta_4(s)A(s)ds$ in (91) we obtain

$$\begin{aligned}[I - M_1(t) - M_2(t)]\,A(t) &= [I - M_1(0) - M_2(0)]\,A(0)\\ &\quad + \int_0^t [-\Gamma + i\Lambda + i\eta_3]\,[I - M_1(s) - M_2(s)]\,A(s)ds\\ &\quad + \int_0^t M_2(s)\eta_2(s)A(s)ds\\ &\quad + \int_0^t \mathcal{O}\left(\frac{|||W|||^3}{D} + \frac{|||W|||^4}{\delta_*}\right)A(s)ds\\ &\quad + \int_0^t [I - M_1(s) - M_2(s)]\,E(s)ds. \end{aligned} \tag{93}$$

Here

$$M_2(t) = \int_0^t \eta_4(s)ds = \mathcal{O}\left(\frac{|||W|||^2}{\delta_* D}\right).$$

We still have to do one integration by parts as the following expansion suggests:

$$\begin{aligned}(M_2(s)\eta_2(s))_{lj} &= \frac{1}{4}\sum_{k,n\in\mathbb{Z}}\sum_{\substack{1\le p\le 2N\\ \lambda_p\neq\lambda_l+\mu_k}} \frac{e^{i(\lambda_l-\lambda_j-\mu_n)s}}{(\lambda_l+\mu_k-\lambda_p)(\lambda_l-\lambda_{l*})}\\ &\qquad \times \langle\beta_k\psi_l,\psi_p\rangle\langle\psi_p,\beta_k\psi_{l*}\rangle\langle\psi_{l*},\beta_n\psi_j\rangle + \mathcal{O}\left(\frac{|||W|||^3}{D^2} + \frac{|||W|||^4}{\delta_* D}\right)\\ &\equiv -(\eta_5(t))_{lj} + \mathcal{O}\left(\frac{|||W|||^3}{D^2} + \frac{|||W|||^4}{\delta_* D}\right). \end{aligned} \tag{94}$$

Now let

$$M_3(t) = \int_0^t \eta_5(s)ds = \mathcal{O}\left(\frac{|||W|||^3}{\delta_* D^2}\right).$$

Finally after integrating by parts $\int_0^t \eta_5(s)A(s)ds$ in (93) we get

$$\begin{aligned}&[I - M_1(t) - M_2(t) - M_3(t)]\,A(t)\\ &\quad = [I - M_1(0) - M_2(0) - M_3(0)]\,A(0)\\ &\qquad + \int_0^t [-\Gamma + i\Lambda + i\eta_3]\,[I - M_1(s) - M_2(s) - M_3(s)]\,A(s)ds\\ &\qquad + \int_0^t \mathcal{O}\left(\frac{|||W|||^3}{D} + \frac{|||W|||^4}{\delta_*} + \frac{|||W|||^4}{\delta_* D^2}\right)A(s)ds\\ &\qquad + \int_0^t [I - M_1(s) - M_2(s) - M_3(s)]\,E(s)ds. \end{aligned} \tag{95}$$

The bottom line is that we are going to use a change of variables:

$$B(t) = [I - M_1(t) - M_2(t) - M_3(t)]\,A(t) \tag{96}$$

and diferentiating (95) we obtain

$$\begin{aligned}\partial_t B(t) &= [-\Gamma + i\Lambda + i\eta_3]\, B(t)\\ &\quad + \mathcal{O}\left(\frac{|||W|||^3}{D} + \frac{|||W|||^4}{\delta_*} + \frac{|||W|||^4}{\delta_* D^2}\right) B(t)\\ &\quad + [I - M_1(t) - M_2(t) - M_3(t)]\, E(t). \qquad (97)\end{aligned}$$

Equations (96–97) can be identified with (67–68) where:

$$\begin{aligned}M(t) &= M_1(t) + M_2(t) + M_3(t)\\ &= \mathcal{O}\left(\frac{|||W|||}{D} + \frac{|||W|||^2}{\delta_*}\right) + \mathcal{O}\left(\frac{|||W|||^2}{\delta_* D}\right) + \mathcal{O}\left(\frac{|||W|||^3}{\delta_* D^2}\right),\\ J &= \Lambda + \eta_3,\\ \alpha(t) &\le C_\alpha |||W|||^2 \left(\frac{|||W|||}{D} + \frac{|||W|||^2}{\delta_*} + \frac{|||W|||^2}{\delta_* D^2}\right),\\ F(t) &= [I - M_1(t) - M_2(t) - M_3(t)]\, E(t).\end{aligned}$$

Hypothesis (p1)–(p5) hold under the assumptions that $D > 0$ is a fixed constant. This concludes the proof of Theorem 3.1.

5.3. Expansion and normal form for Theorem 3.2. For the case of Theorem 3.2 the procedure is even more natural. We proceed in a manner similar to that in Subsect. 5.1. However, note that integrals with integrands containing the factors $\exp(\pm i(\lambda_{2i-1} - \lambda_{2i})t),\ i = 1, \ldots, N$ are *not* integrated by parts. Thus "small" denominators are avoided.

In conclusion we obtain a change of variables:

$$B(t) = [I - M_1(t) - M_2(t)]\, A(t) \qquad (98)$$

such that:

$$\partial_t B(t) = U(t)B(t) + \mathcal{O}(|||W|||^3)B(t) + [I - M_1(t) - M_2(t)]\, E(t). \qquad (99)$$

Here M_1, respectively M_2 are as in (79), respectively (86), but *without* the terms having denominators of the form $\pm(\lambda_{2l-1} - \lambda_{2l})$. Thus M_1, respectively M_2, are of order $\mathcal{O}(|||W|||)$, respectively $\mathcal{O}(|||W|||^2)$ uniformly in $\delta_* \searrow 0$ and $t \in \mathbb{R}$.

$U(t)$ is a tridiagonal matrix formed by 2×2 blocks on the diagonal:

$$U(t) = \text{diag}\left[U^1(t), U^2(t), \ldots, U^N(t)\right].$$

Each 2×2 block, $U^l(t),\ l = 1, 2, \ldots N$ corresponds to the pair of close eigenvalues $(2l - 1, 2l),\ l = 1, \ldots, N$ and has the form:

$$U^l = \begin{pmatrix} u^l_{11} & e^{i(\lambda_{2l-1} - \lambda_{2l})t} u^l_{12} \\ e^{i(\lambda_{2l} - \lambda_{2l-1})t} u^l_{21} & u^l_{22} \end{pmatrix}, \qquad (100)$$

where $u^l_{jk},\ j, k = 1, 2$ are *constant* complex numbers. Before we explicitly write them let us briefly explain why $U(t)$ has this form. Given that we follow the procedure in Sect. 5.1, $U(t)$ contains the dominant constant coefficient matrix we found in that section, namely $-\Gamma + i\Lambda + i\eta_1 + i\eta_3$. Due to **(H6)'** this matrix is diagonal, see (31), (120),

(74) and (82). In addition $U(t)$ contains all the terms we avoid integrating by parts, i.e. the ones having as factors $e^{\pm i(\lambda_{2l-1}-\lambda_{2l})t}$, $l = 1, 2, \ldots, N$. It is a matter of actually doing the procedure to see that such terms only occur in the positions $(2l-1, 2l)$ and $(2l, 2l-1)$, $l = 1, 2, \ldots, N$ of the matrix $U(t)$. If one removes the restriction **(H6)'** the matrix $U(t)$ may have slowly varying almost periodic in time terms outside the 2×2 blocks. As the theory for such systems is not yet well developed, they will prevent us from analyzing the dynamics.

Below are the formulas for the constants u^l_{jk}, $j, k = 1, 2$ in (100):

$$
\begin{aligned}
u^l_{11} &= i\lambda_{2l-1} - \frac{i}{2}\langle\psi_{2l-1}, \beta_0\psi_{2l-1}\rangle \\
&\quad + \frac{i}{4}\sum_{n\in\mathbb{Z}}\langle \mathbf{P_c}\beta_n\psi_{2l-1}, (H_0 - \lambda_{2l-1} - \mu_n - i0)^{-1}\mathbf{P_c}\beta_n\psi_{2l-1}\rangle \\
&\quad - \frac{i}{4}\sum_{\substack{1\le p\le 2N \\ p\ne 2l-1,\ p\ne 2l}} \frac{1}{\lambda_{2l-1}-\lambda_p}\left|\langle\psi_{2l-1},\ \beta_0\psi_p\rangle\right|^2 \\
&\quad - \frac{i}{4}\sum_{1\le p\le 2N}\sum_{n>0}\left|\langle\psi_{2l-1}, \beta_n\psi_p\rangle\right|^2\left(\frac{1}{\lambda_{2l-1}-\lambda_p-\mu_n} + \frac{1}{\lambda_{2l-1}-\lambda_p-\mu_{-n}}\right) \\
&= i\lambda_{2l-1} - \frac{i}{2}\langle\psi_{2l-1}, \beta_0\psi_{2l-1}\rangle \\
&\quad + \frac{i}{4}\sum_{n\in\mathbb{Z}}\langle \mathbf{P_c}\beta_n\psi_{2l-1}, (H_0 - \underline{\lambda}_l - \mu_n - i0)^{-1}\mathbf{P_c}\beta_n\psi_{2l-1}\rangle \\
&\quad + \frac{i}{4}\sum_{\substack{1\le p\le N \\ p\ne l}} \frac{1}{\underline{\lambda}_p-\underline{\lambda}_l}\left(\left|\langle\psi_{2l-1},\ \beta_0\psi_{2p-1}\rangle\right|^2 + \left|\langle\psi_{2l-1},\ \beta_0\psi_{2p}\rangle\right|^2\right) \\
&\quad + \frac{i}{2}\sum_{1\le p\le N}\sum_{n>0}\frac{\underline{\lambda}_p-\underline{\lambda}_l}{(\underline{\lambda}_l-\underline{\lambda}_p)^2-\mu_n^2}\left(\left|\langle\psi_{2l-1}, \beta_n\psi_{2p-1}\rangle\right|^2 + \left|\langle\psi_{2l-1}, \beta_n\psi_{2p}\rangle\right|^2\right) \\
&\quad + \mathcal{O}(\delta^*|||W|||^2), \\
u^l_{12} &= -\frac{1}{2}\langle\psi_{2l-1}, \beta_0\psi_{2l}\rangle + \frac{1}{4}\sum_{n\in\mathbb{Z}}\langle \mathbf{P_c}\beta_n\psi_{2l-1}, (H_0 - \lambda_{2l} - \mu_n - i0)^{-1}\mathbf{P_c}\beta_n\psi_{2l}\rangle \\
&\quad - \frac{1}{4}\sum_{\substack{1\le p\le 2N \\ p\ne 2l-1,\ p\ne 2l}} \frac{1}{\lambda_{2l-1}-\lambda_p}\langle\psi_{2l-1},\ \beta_0\psi_p\rangle\ \langle\psi_p,\ \beta_0\psi_{2l}\rangle \\
&\quad - \frac{1}{4}\sum_{1\le p\le 2N}\sum_{n>0}\langle\psi_{2l-1}, \beta_n\psi_p\rangle\langle\psi_p, \beta_n\psi_{2l}\rangle \\
&\quad \times\left(\frac{1}{\lambda_{2l-1}-\lambda_p-\mu_n} + \frac{1}{\lambda_{2l-1}-\lambda_p-\mu_{-n}}\right) \\
&= -\frac{1}{2}\langle\psi_{2l-1}, \beta_0\psi_{2l}\rangle + \frac{1}{4}\sum_{n\in\mathbb{Z}}\langle \mathbf{P_c}\beta_n\psi_{2l-1}, (H_0 - \underline{\lambda}_l - \mu_n - i0)^{-1}\mathbf{P_c}\beta_n\psi_{2l}\rangle \\
&\quad + \frac{1}{4}\sum_{\substack{1\le p\le N \\ p\ne l}} \frac{1}{\underline{\lambda}_p-\underline{\lambda}_l}\big(\langle\psi_{2l-1},\ \beta_0\psi_{2p-1}\rangle\ \langle\psi_{2p-1},\ \beta_0\psi_{2l}\rangle
\end{aligned}
$$

$$+\langle\psi_{2l-1},\ \beta_0\psi_{2p}\rangle\ \langle\psi_{2p},\ \beta_0\psi_{2l}\rangle\big)$$
$$+\frac{1}{2}\sum_{1\le p\le N}\sum_{n>0}\frac{\underline{\lambda}_p-\underline{\lambda}_l}{(\underline{\lambda}_l-\underline{\lambda}_p)^2-\mu_n^2}\big(\langle\psi_{2l-1},\beta_n\psi_{2p-1}\rangle\langle\psi_{2p-1},\beta_n\psi_{2l}\rangle$$
$$+\langle\psi_{2l-1},\beta_n\psi_{2p}\rangle\langle\psi_{2p},\beta_n\psi_{2l}\rangle\big)+\mathcal{O}(\delta^*|||W|||^2),$$

where we used $\mu_{-n}=-\mu_n$ and the resolvent identity

$$\Big((H_0-\lambda_{2l-1}-\mu_n-i0)^{-1}-(H_0-\underline{\lambda}_l-\mu_n-i0)^{-1}\Big)\mathbf{P_c}$$
$$=(\lambda_{2l-1}-\underline{\lambda}_l)(H_0-\lambda_{2l-1}-\mu_n-i0)^{-1}(H_0-\lambda_{2l-1}-\mu_n-i0)^{-1}\mathbf{P_c}=\mathcal{O}(\delta^*) \tag{101}$$

in weighted norms, see [12, p. 36]. u^l_{22}, respectively u^l_{21}, can be obtained from u^l_{11}, respectively u^l_{12}, by interchanging $2l-1$ with $2l$. The terms $\mathcal{O}(\delta^*|||W|||^2)$ can be added to $\alpha(t)$ in (68). Thus $\epsilon=\delta^*$ in (p4).

To get the dominant dynamics described in Theorem 3.2, we switch to the fast oscillating amplitudes:

$$\begin{aligned} b_{2l-1}&=e^{-i\lambda_{2l-1}t}B_{2l-1},\\ b_{2l}&=e^{-i\lambda_{2l}t}B_{2l}\end{aligned} \tag{102}$$

for $l=1,2,\ldots,N$. We then have:

$$\partial_t b(t)=\Big(-i\,\mathrm{diag}\,[\lambda_1,\lambda_2,\ldots,\lambda_{2N}]+\tilde{U}\Big)\,b(t)+\mathcal{O}(|||W|||^3+|||W|||^2\delta^*)b(t)$$
$$+\Big[I-\tilde{M}_1(t)-\tilde{M}_2(t)\Big]\,E(t), \tag{103}$$

where $\tilde{U}$ has the same components as $U(t)$ except that the factors $\exp(\pm i(\lambda_{2i-1}-\lambda_{2i})t),\ i=1,\ldots,N$ have disappeared. Thus $\tilde{U}$ is a constant matrix. We further expand it using the formulas for u^l_{jk} above and

$$(H_0-\lambda-\mu-i0)^{-1}=\mathrm{P.V.}(H_0-\lambda-\mu)^{-1}+i\pi\delta(H_0-\lambda-\mu)$$

for $\lambda+\mu\in\sigma_{\mathrm{cont}}(H_0)$. Upon grouping the results in a self adjoint matrix and an anti-self adjoint one we get

$$\tilde{U}=-\Gamma^{\#}+i\Lambda^{\#},$$

where $\Gamma^{\#}$ and $\Lambda^{\#}$ are the ones given in Theorem 3.2.

We only have to show that $\Gamma^{\#}\ge 0$ in order to apply Proposition 5.2. Since $\Gamma^{\#}$ is formed by the 2×2 blocks, $\Gamma^{\#}_l$, $l=1,2,\ldots,N$ the problem reduces to showing that each of this block is nonnegative definite. But this is straightforward from the fact that $\delta(H_0-\lambda-\mu),\ \lambda+\mu\in\sigma_{\mathrm{cont}}(H_0)$ induces a scalar product in weighted Hilbert spaces, see [12, Appendix].

The theorem is now completely proven.

6. Appendix – Proof of Proposition 5.1

In this appendix we prove Proposition 5.1, *i.e.* we show that (24) is indeed equivalent to (61–62) under hypothesis (H1–H4). The latter system is the starting point of Sect. 5 which leads to the normal form for the amplitude equations. The computation is an extension to multiple bound states of the one implemented in [12, 19, 21, 20] for the one bound state case.

We recall that (24) is:

$$i\partial_t \phi(t) = (H_0 + W(t))\,\phi(t), \tag{104}$$

and under hypothesis (H1–H2) its general solution can be written in the form, see also (57–59)

$$\phi(t) = \sum_{l=1}^{m} a_l(t)\psi_l + \phi_d(t), \tag{105}$$

where

$$a_l(t) = \langle \psi_i, \phi(t)\rangle,\ \ l = 1, \ldots, m, \tag{106}$$

and

$$\phi_d(t) = \mathbf{P_c}\phi(t) \tag{107}$$

is the projection of the solution onto the continuous spectrum associated with H_0.

We proceed by first inserting (105) into (104), which yields the equation:

$$\begin{aligned} i\sum_{j=1}^{m} \partial_t a_j(t)\psi_j \ +\ i\partial_t\phi_d(t) \ =\ & \sum_{j=1}^{m} \lambda_j a_j(t)\psi_j \ + H_0\phi_d(t) \\ & + \sum_{j=1}^{m} a_j(t)W(t)\psi_j \ +\ W(t)\phi_d(t). \end{aligned} \tag{108}$$

Taking the inner product of (108) with each of the eigenvectors, $\psi_l,\ l = 1, \ldots, m$, we get the following system of equations for the amplitudes $a_l(t),\ l = 1, \ldots, m$:

$$i\partial_t a_l(t) \ =\ \lambda_l a_l(t) \ +\ \sum_{j=1}^{m}\langle \psi_l, W(t)\psi_j\rangle a_j(t) \ +\ \langle \psi_l, W(t)\phi_d\rangle. \tag{109}$$

In deriving (109) we have used that $\psi_l,\ l = 1, \ldots, m$ are orthonormal and satisfy the following orthogonality relations:

$$\langle \psi_l, \phi_d(t)\rangle = 0, \quad l = 1, \ldots, m. \tag{110}$$

Applying $\mathbf{P_c}$ to (108), we obtain an equation for ϕ_d:

$$i\partial_t\phi_d(t) \ =\ H_0\phi_d(t) \ +\ \mathbf{P_c}W(t)\phi_d(t) \ +\ \sum_{j=1}^{m} a_j(t)\mathbf{P_c}W(t)\psi_j. \tag{111}$$

Since we are after a slow resonant decay phenomenon, it will prove advantageous to extract the fast oscillatory behavior of $a_l(t)$. We therefore define:

$$A_l(t) \equiv e^{i\lambda_l t} a_i(t), \quad l = 1, \ldots, m. \tag{112}$$

Then, for $l = 1, \dots, m$, (109) reads

$$\partial_t A_l \;=\; -i \sum_{j=1}^{m} e^{-i(\lambda_j - \lambda_l)t} \langle \psi_l, W(t)\psi_j \rangle A_j \;-\; i e^{i\lambda_l t} \, \langle \psi_l, W(t)\phi_d(t) \rangle. \tag{113}$$

Note that since W is small, $A(t)$ is slowly varying. We write (111) in an integral form using Duhamel's principle:

$$\begin{aligned}\phi_d(t) = e^{-iH_0 t}\phi_d(0) \;-\; i \sum_{j=1}^{m} \int_0^t & e^{-iH_0(t-s)} \mathbf{P_c} W(s) a_j(s)\psi_j \, ds \\ -i \int_0^t & e^{-iH_0(t-s)} \mathbf{P_c} W(s)\phi_d(s)\, ds \\ \equiv \phi_0(t) \;+\; \phi_1(t) \;+\; \phi_2(t). & \end{aligned} \tag{114}$$

By standard methods, the system (113)–(114) for $A(t) = (A_1(t), A_2(t), \dots, A_m(t))$ and $\phi_d(t)$ has a global solution in t with

$$A \in C^1(\mathbb{R}, \mathbb{R}^m), \;\; \|\phi_d(t)\| \in C^0(\mathbb{R}), \;\; \|w_- \phi_d(t)\| \in C^0(\mathbb{R}).$$

Our analysis of the $|t| \to \infty$ behaviour is based on a study of this system.

Let us prove first the wave estimates (63–64). By comparing (114) with (63) we have

$$\tilde{\phi}(t) \;=\; \phi_1(t) \;+\; \phi_2(t).$$

The decay properties of the unperturbed wave operator $e^{-iH_0 t}$ combined with the conservation of energy

$$|a(t)|^2 + \|\phi_d(t)\|^2 = \|\phi(0)\|^2$$

which in particular gives $|a(t)|, \;\|\phi_d(t)\| \le \|\phi(0)\|$, readily implies (63). As for (64) let us multiply (114) by w_- and apply the norm:

$$\begin{aligned}\|w_-\phi_d(t)\| \le \|w_- e^{-iH_0 t}\phi_d(0)\| &+ \int_0^t \|w_- e^{-iH_0(t-s)} \mathbf{P_c} W(s)\| \, |A(s)| \, ds. \\ &+ \int_0^t \|w_- e^{-iH_0(t-s)} \mathbf{P_c} W(s) w_+\| \, \|w_-\phi_d(s)\| ds.\end{aligned}$$

Using now the local decay estimates in **(H3)** we have

$$\begin{aligned}\langle t\rangle^r \|w_-\phi_d(t)\| \le \mathcal{C}\|w_+\phi_d(0)\| &+ \mathcal{C} \sup_{0\le s\le t} \langle s\rangle^r |A(s)| \langle t\rangle^r \int_0^t \langle t-s\rangle^{-r_1} \|w_+ W(s)\| \langle s\rangle^{-r} ds \\ &+ \mathcal{C} \sup_{0\le s\le t} \langle s\rangle^r \|w_-\phi_d(s)\| \langle t\rangle^r \int_0^t \langle t-s\rangle^{-r} \|w_+ W(s) w_+\| ds.\end{aligned}$$

Finally by $\|w_+ W(s)\| \le |||W|||$ and $\|w_+ W(s) w_+\| \le |||W|||$, see **(H4)** and the standard inequality

$$\langle t\rangle^r \int_0^t \langle t-s\rangle^{-r_1} \langle s\rangle^{-r} ds \le D$$

valid for some constant D independent of t we get

$$(1 - C|||W|||) \sup_{0\le s\le t} \langle s\rangle^r \|w_-\phi_d(s)\| \le \mathcal{C}\|w_+\phi_d(0)\| + C \sup_{0\le s\le t} \langle s\rangle^r |A(s)|.$$

Hence, for sufficiently small $|||W|||$ there exists the constants C and D such that for any $t > 0$,

$$\sup_{0\le s\le t} \langle s\rangle^r \|w_-\phi_d(s)\| \le C + D \sup_{0\le s\le t} \langle s\rangle^r |A(s)|. \tag{115}$$

In particular this implies (64).

Now, in order to obtain (61) we need to insert (114) into (113). Before doing so let us expand the $\phi_1(t)$ term. We first replace $a_j(t) = e^{-i\lambda_j t} A_j(t)$ and then make explicit the frequency content of $W(t)$ by using the expression (29) for $W(t)$:

$$W(t) = \frac{1}{2} \sum_{k\in\mathbb{Z}} \exp(-i\mu_k t)\beta_k,$$

which by **(H4)**, is a convergent series uniformly in $t \in \mathbb{R}$. We get:

$$\begin{aligned}\phi_1(t) &= -\frac{i}{2} \sum_{j=1}^{m} \sum_{k\in\mathbb{Z}} \int_0^t e^{-iH_0(t-s)} e^{-i(\lambda_j+\mu_k)s} A_j(s)\mathbf{P_c}\beta_k\psi_j\, ds \\ &= -\frac{i}{2} \sum_{j=1}^{m} \sum_{k\in\mathbb{Z}} \int_0^t e^{-iH_0 t} e^{i[H_0-(\lambda_j+\mu_k)]s} A_j(s)\mathbf{P_c}\beta_k\psi_j\, ds. \end{aligned} \tag{116}$$

We wish to obtain the dominant contributions from ϕ_1. These will come from resonances, terms where $\lambda_j + \mu_k \in \sigma_{cont}(H_0)$. These contributions are calculated by careful integration by parts. To carry this out we first regularize ϕ_1 by defining:

$$\phi_1^\eta(t) = -\frac{i}{2} \sum_{j=1}^{m} \sum_{k\in\mathbb{Z}} \int_0^t e^{-iH_0(t-s)} e^{-i(\lambda_j+\mu_k+i\eta)s} A_j(s)\mathbf{P_c}\beta_k\psi_j\, ds \tag{117}$$

for η positive and arbitrary and $t > 0$. Note that $\phi_1(t) = \lim_{\eta\searrow 0} \phi_1^\eta(t)$ uniformly with respect to t on compact intervals.

Now, integration by parts for each integral in expression (117) and letting η tend to zero from above gives the following expansion of $\langle\psi_l, W(t)\phi_1(t)\rangle$, $l = 1, \dots, m$:

$$\begin{aligned}&\langle\psi_l, W(t)\phi_1(t)\rangle \\ &= \langle W(t)\psi_l,\ -\frac{1}{2}\sum_{j=1}^{m}\sum_{k\in\mathbb{Z}} e^{-i(\lambda_j+\mu_k)t} A_j(t)(H_0 - \lambda_j - \mu_k - i0)^{-1}\mathbf{P_c}\beta_k\psi_j\rangle \\ &\quad + \left\langle W(t)\psi_l,\ \frac{1}{2}\sum_{j=1}^{m}\sum_{k\in\mathbb{Z}} A_j(0)e^{-iH_0 t}(H_0 - \lambda_j - \mu_k - i0)^{-1}\mathbf{P_c}\beta_k\psi_j\right\rangle \\ &\quad + \left\langle W(t)\psi_l,\ \frac{1}{2}\sum_{j=1}^{m}\sum_{k\in\mathbb{Z}} \int_0^t e^{-iH_0(t-s)}(H_0 - \lambda_j - \mu_k - i0)^{-1}\mathbf{P_c} e^{-i(\lambda_j+\mu_k)s}\partial_s A(s)\beta_k\psi_j ds\right\rangle. \end{aligned} \tag{118}$$

For a detailed discussion of the singular operators in the above computation and a justification of the calculation using hypothesis **(H3)**, see [12, Sect. 8]. The choice of regularization, $+i\eta$, in (117) ensures that the latter two terms in the expansion of ϕ_1, (118), decay dispersively as $t \to +\infty$; see hypothesis **(H3)**. For $t < 0$, we replace $+i\eta$ with $-i\eta$ in (117).

To further expand the first series in (118) we use the distributional identities for the singular terms:

$$\begin{aligned}\langle \mathbf{P_c} f, (H_0 - \lambda - \mu \mp i0)^{-1} \mathbf{P_c} g\rangle = \ &\langle \mathbf{P_c} f, \text{P.V.}(H_0 - \lambda - \mu)^{-1} \mathbf{P_c} g\rangle \\ &\pm i\pi \langle \mathbf{P_c} f, \delta(H_0 - \lambda - \mu) \mathbf{P_c} g\rangle,\end{aligned}$$

which according to [12, Sect. 8] holds whenever f, g satisfy $w_+ f,\ w_+ g \in \mathcal{H}$.

Finally, using the Fourier expansion for $W(t)$ in (118) and substitution into (113) we find

$$\partial_t A(t) = [-\Gamma + i(\Lambda + \eta(t)) + \rho(t)]\, A(t) + E(t), \tag{119}$$

where Γ is given in (31), $\Lambda,\ \eta(t)$ and $\rho(t)$ are $m \times m$ matrices with components:

$$\begin{aligned}(\Lambda)_{lj} = \ &\frac{1}{4} \sum_{\substack{n,k\in\mathbb{Z} \\ \lambda_l+\mu_k=\lambda_j+\mu_n\in\sigma_{\text{cont}}(H_0)}} \langle \mathbf{P_c}\beta_k\psi_l, \text{P.V.}(H_0 - \lambda_j - \mu_n)^{-1}\mathbf{P_c}\beta_n\psi_j\rangle \\ &+\frac{1}{4} \sum_{\substack{n,k\in\mathbb{Z} \\ \lambda_l+\mu_k=\lambda_j+\mu_n\notin\sigma_{\text{cont}}(H_0)}} \langle \mathbf{P_c}\beta_k\psi_l, (H_0 - \lambda_j - \mu_n)^{-1}\mathbf{P_c}\beta_n\psi_j\rangle, \end{aligned}\tag{120}$$

$$\eta_{lj}(t) = -e^{i(\lambda_l-\lambda_j)t}\langle \psi_l, W(t)\psi_j\rangle, \tag{121}$$

$$\rho_{lj}(t) = \frac{i}{4} \sum_{\substack{n,k\in\mathbb{Z} \\ \lambda_l+\mu_k\neq\lambda_j+\mu_n}} e^{i(\lambda_l+\mu_k-\lambda_j-\mu_n)t}\langle \mathbf{P_c}\beta_k\psi_l, (H_0 - \lambda_j - \mu_n - i0)^{-1}\mathbf{P_c}\beta_n\psi_j\rangle. \tag{122}$$

$E(t)$ is a column vector with components:

$$\begin{aligned}E_l(t) = &-\frac{i}{4}\sum_{j=1}^{m}\sum_{n,k\in\mathbb{Z}} e^{i(\lambda_l+\mu_k)t}\langle \mathbf{P_c}\beta_k\psi_l, e^{-iH_0t}(H_0-\lambda_j-\mu_n - i0)^{-1}\mathbf{P_c}\beta_n\psi_j\rangle A_j(0) \\ &-\frac{i}{4}\sum_{j=1}^{m}\sum_{n,k\in\mathbb{Z}} \int_0^t e^{i(\lambda_l+\mu_k)t-i(\lambda_j+\mu_n)s} \\ &\times\langle \mathbf{P_c}\beta_k\psi_l, e^{-iH_0(t-s)}(H_0 - \lambda_j - \mu_n - i0)^{-1}\mathbf{P_c}\beta_n\psi_j\rangle\partial_s A_j(s)ds \\ &-ie^{i\lambda_l t}\langle W(t)\psi_l, e^{-iH_0t}\phi_d(0)\rangle - e^{i\lambda_l t}\langle W(t)\psi_l, \int_0^t e^{-iH_0(t-s)}\mathbf{P_c}W(s)\phi_d(s)ds\rangle.\end{aligned}\tag{123}$$

Note that (119) respectively (111) are exactly (61) respectively (62). They form a closed system equivalent to (104) and hence to (24).

To complete Proposition 5.1 it remains to prove that the properties (a)–(e) hold. These concern the symmetries and norms of the matrices Γ, Λ, η and ρ and are a direct consequence of the symmetries and norms of the operators $\delta(H_0 - \lambda)$, $\text{P.V.}(H_0 - \lambda)^{-1},\ \lambda \in$

$\sigma_{\rm cont}(H_0)$ and W. In the proofs we focus primarily on Γ; the proofs for Λ, η and ρ are similar.

Lemma 6.1. Γ, Λ *and* η *are self-adjoint matrices.*

Proof of Lemma 6.1. We prove self-adjointness of Γ.

Recall from **H5** (31) that

$$(\Gamma)_{lj} = \frac{\pi}{4} \sum_{\substack{n,k\in\mathbb{Z}\\ \lambda_l+\mu_k=\lambda_j+\mu_n}} \langle \mathbf{P_c}\beta_k\psi_l, \delta(H_0-\lambda_j-\mu_n)\mathbf{P_c}\beta_n\psi_j\rangle. \tag{124}$$

Then

$$\begin{aligned}(\Gamma)^*_{jl} &= \left[\frac{\pi}{4} \sum_{\substack{n,k\in\mathbb{Z}\\ \lambda_j+\mu_k=\lambda_l+\mu_n}} \langle \mathbf{P_c}\beta_k\psi_j, \delta(H_0-\lambda_l-\mu_n)\mathbf{P_c}\beta_n\psi_l\rangle\right]^*\\ &= \frac{\pi}{4} \sum_{\substack{n,k\in\mathbb{Z}\\ \lambda_j+\mu_k=\lambda_l+\mu_n}} \langle \delta(H_0-\lambda_l-\mu_n)\mathbf{P_c}\beta_n\psi_l, \mathbf{P_c}\beta_k\psi_j\rangle.\end{aligned}$$

Replacing now $\lambda_l+\mu_n$ with $\lambda_j+\mu_k=\lambda_l+\mu_n$ in the argument of δ and then switching the summing indices k, n we get

$$\begin{aligned}(\Gamma)^*_{jl} &= \frac{\pi}{4} \sum_{\substack{n,k\in\mathbb{Z}\\ \lambda_l+\mu_k=\lambda_j+\mu_n}} \langle \delta(H_0-\lambda_j-\mu_n)\mathbf{P_c}\beta_k\psi_l, \mathbf{P_c}\beta_n\psi_j\rangle\\ &= (\Gamma)_{lj}\,,\end{aligned}$$

where the last equality comes from (124) and the fact that $\delta(H_0-\lambda_j-\mu_n)$, $j\in\{1,2,\dots,N\}$, $n\in\mathbb{Z}$ are self adjoint operators, see [12, Sect. 8]. This proves that Γ is self-adjoint. $\square$

Lemma 6.2. Γ, Λ *have matrix norms which are* $\mathcal{O}(|||W|||^2)$. $\eta(t)$*, respectively* $\rho(t)$*, are almost periodic matrices of norms* $\mathcal{O}(|||W|||)$*, respectively* $\mathcal{O}(|||W|||^2)$*, independent of "*t*".*

Proof of Lemma 6.2. Let "$a=(a_1,a_2,\dots,a_m)^T$" be an arbitrary $\mathbb{C}^m$-column vector. If

$$b=\Gamma a, \qquad b=(b_1,b_2,\dots,b_m),$$

then by left multiplying it with b^* and expanding the left-hand side using (124) we have successively

$$\begin{aligned}|b|^2 &= \sum_{l,j=1}^{m} \frac{\pi}{4} \sum_{\substack{n,k\in\mathbb{Z}\\ \lambda_l+\mu_k=\lambda_j+\mu_n}} \langle \mathbf{P_c}\beta_k b_l\psi_l, \delta(H_0-\lambda_j-\mu_n)\mathbf{P_c}\beta_n a_j\psi_j\rangle\\ &= \frac{\pi}{4}\sum_{n,k\in\mathbb{Z}} \sum_{\substack{l,j\in\{1,2,\dots,m\}\\ \lambda_l+\mu_k=\lambda_j+\mu_n}} \langle \mathbf{P_c}\beta_k b_l\psi_l, \delta(H_0-\lambda_j-\mu_n)\mathbf{P_c}\beta_n a_j\psi_j\rangle\end{aligned}$$

$$\leq \frac{C_0}{4} \sum_{n,k\in\mathbb{Z}} \|w_+\beta_k\| \, \|w_+\beta_n\| \sum_{\substack{l,j\in\{1,2,\ldots,m\} \\ \lambda_l+\mu_k=\lambda_j+\mu_n}} |b_l| \, |a_j|$$

$$\leq \frac{C_0}{4} \, m \, |b| \, |a| \, |||W|||^2,$$

where we used the estimate [12, relation (8.8)], the Cauchy-Buniakowski-Schwartz inequality:

$$\sum_{j=1}^{m} |c_j| \leq \sqrt{m} \sqrt{\sum_{j=1}^{m} |c_j|^2}$$

applied to both sums over l and j and the definition of $|||W|||$ given in (30).

In conclusion, for any $\mathbb{C}^m$-vector a we have

$$|\Gamma a| \leq m \frac{\mathbb{C}_0}{4} |||W|||^2 \, |a|.$$

Hence

$$|\Gamma| \leq m \frac{\mathbb{C}_0}{4} |||W|||^2 = \mathcal{O}(|||W|||^2). \tag{125}$$

This completes the proof of Lemma 6.2. □

Note however that the size of Γ seems to depend on "m", the number of bound states. Actually there is a method to prove the bound (125) with a constant which is independent of "m". Here is how. First remark that:

$$(\tilde{\rho})_{lj} \equiv (-\Gamma + i(\Lambda + \rho(t)))$$
$$= -\frac{1}{4} \sum_{n,k\in\mathbb{Z}} e^{i(\lambda_l+\mu_k-\lambda_j-\mu_n)t} \langle \mathbf{P_c}\beta_k\psi_l, (H_0 - \lambda_j - \mu_n - i0)^{-1} \mathbf{P_c}\beta_n\psi_j \rangle.$$

We are going to show that the right-hand side is an almost periodic 2-form on $\mathbb{C}^m \times \mathbb{C}^m$ bounded by $C|||W|||^2$ for some constant C independent of t or dimension "m". As a consequence its mean $-\Gamma + i\Lambda$, hence both Γ and Λ, and its mean zero part $\rho(t)$ will be dominated roughly by the same bound, see [12, Theorem 9.5] for estimating the mean value and use $\|f - M(f)\| \leq \|f\| + \|M(f)\|$ to estimate the zero mean part.

Let us fix "t" and apply $\tilde{\rho}$ to the arbitrary vectors $a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots) \in \mathbb{C}^m$. We have:

$$b^*\tilde{\rho}a = -\frac{1}{4} \sum_{l,j=1}^{m} b_l^* a_j \sum_{n,k\in\mathbb{Z}} e^{i(\lambda_l+\mu_k-\lambda_j-\mu_n)t} \langle \mathbf{P_c}\beta_k\psi_l, (H_0-\lambda_j-\mu_n - i0)^{-1} \mathbf{P_c}\beta_n\psi_j \rangle$$
$$= \left\langle w_+ W(t)\psi_b(t), \lim_{\xi\searrow 0} w_- \int_0^\infty e^{-i(H_0-i\xi)s} \mathbf{P_c} w_- w_+ W(t-s)\psi_a(t-s) ds \right\rangle, \tag{126}$$

where $\psi_c(t) = \sum_j c_j e^{-i\lambda_j t}\psi_j$ for any vector $c = (c_j)$. All in all the 2-form ρ is a composition of almost periodic functions $c, t \mapsto \psi_c(t)$ and $t \mapsto W(t)$ with the continuous function

$$\left\langle w_+ \cdot, \lim_{\xi\searrow 0} w_- \int_0^\infty e^{-i(H_0-i\xi)s} \mathbf{P_c} w_- w_+ \cdot ds \right\rangle.$$

Therefore, the 2-form is almost periodic by [12, Theorem 9.3]. Using now the the relation (26) in Hypothesis **(H3)** and the fact that

$$\|\psi_c(t)\| \equiv |c|,$$

for any $t \geq 0$, we get

$$|b^* \tilde{\rho} a| \leq C|||W|||^2 \, |b| \, |a|,$$

which implies

$$\sup_{t\in\mathbb{R}} |\tilde{\rho}(t)| \leq C|||W|||^2,$$

where now $\tilde{\rho}(t)$ is viewed as an operator on $\mathbb{C}^m$.

In conclusion Γ, Λ and $\tilde{\rho}(t)$ have the required sizes, independent of the dimension, and the latter is almost periodic. Note that in the above argument it was essential to sum over the components of the vectors in $\mathbb{C}^m$ before we applied norms.

As for the symmetry and norm of $\eta(t)$ they are an immediate consequence of the symmetry and norm of W (again one should sum over the vector components as before and then apply norms to avoid the dependence of the estimate on the dimensionality).

Lemma 6.3. Γ *is a nonnegative semidefinite matrix.*

Proof of Lemma 6.3. For any vector $b = (b_1, b_2, \ldots) \in \mathbb{C}^m$ we have

$$b^*\Gamma b = \frac{\pi}{4} \sum_{\alpha\in\sigma_{\text{cont}}(H_0)} \langle \psi_\alpha^b, \delta(H_0 - \alpha)\psi_\alpha^b \rangle \geq 0.$$

Here $\alpha \in \sigma_{\text{cont}}(H_0)$ is such that there exist $l \in \{1, 2, \ldots\}$ and $k \in \mathbb{Z}$ with the property $\lambda_l + \mu_k = \alpha$, while ψ_α^b is given by

$$\psi_\alpha^b = \sum_{l=1}^{m} \sum_{\substack{k \in \mathbb{Z} \\ \lambda_l + \mu_k = \alpha}} b_l \mathbf{P_c} \beta_k \psi_l.$$

This completes the proof of Lemma 6.3. □

It remains to prove part (e) of Proposition 5.1. Such estimates have already been obtained in [12, Proposition 6.2] for the case of one bound state, i.e. $E(t)$ a scalar. There are no new ideas in generalizing it for "m" bound states except the tricks used above to make the estimates independent on "m".

7. Appendix – Proof of Proposition 5.2

By Duhamel's principle, the solutions of the system (68) can be written as:

$$B(t) = e^{(-\Gamma^\# + i\Lambda^\#)t} B(0) + \int_0^t e^{(-\Gamma^\# + i\Lambda^\#)(t-s)} \alpha(s) B(s) ds + \int_0^t e^{(-\Gamma^\# + i\Lambda^\#)(t-s)} F(s) ds. \tag{127}$$

Hence, for any $0 \le r \le r_1$,

$$\begin{aligned} |B(t)| &\le e^{-\gamma t}B(0) + C_\alpha |||W|||^2(|||W||| + \epsilon)\int_0^t e^{-\gamma(t-s)}|B(s)|ds \\ &+C(1+\epsilon)|||W|||\int_0^t e^{-\gamma(t-s)}\langle s\rangle^{-r_1}ds \\ &+D(1+\epsilon)|||W|||^3 \sup_{0\le s\le t}\langle s\rangle^r|B(s)|\int_0^t e^{-\gamma(t-s)}\langle s\rangle^{-r}ds, \end{aligned} \tag{128}$$

where we used the estimates for $\alpha(s)$ and $F(s)$ in (p4) respectively (p5) and $|e^{(-\Gamma^\# + i\Lambda^\#)t}| \le e^{-\gamma t}$ due to the symmetry of both $\Gamma^\#$ and $\Lambda^\#$.

Let us focus first on the result on time scales of order $1/|||W|||^2$. Fix $T \le c|||W|||^{-2}$, where c is an arbitrary constant. First we show that there exists a constant C such that

$$\sup_{0\le t\le T}|B(t)| \le C. \tag{129}$$

To this end we use $r = 0$ in (p5) and $e^{-\gamma t} \le 1$, for $t \ge 0$, since by **(H5)** $\gamma \ge 0$. We plug them in (128) and obtain for all $0 \le t \le T$:

$$|B(t)| \le B(0) + \ t\ |||W|||^2(|||W||| + \epsilon)\sup_{0\le s\le T}|B(s)|(C_\alpha + D) + C(1+\epsilon)|||W|||.$$

Then since $0 \le t \le T \le c|||W|||^{-2}$ we get

$$(1 - \tilde{C}(|||W||| + \epsilon))\sup_{0\le t\le T}|B(t)| \le B(0) + C(1+\epsilon)|||W|||$$

which for sufficiently small $|||W||| + \epsilon$ implies (129).

Now, by comparing (127) with the conclusion of Proposition 5.2 we find that we must show $R(t) = \mathcal{O}(|||W||| + \epsilon)$, where

$$R(t) \equiv \int_0^t e^{(-\Gamma^\# + i\Lambda^\#)(t-s)}\alpha(s)B(s)ds + \int_0^t e^{(-\Gamma^\# + i\Lambda^\#)(t-s)}F(s)ds. \tag{130}$$

Hence for any $0 \le t \le T$,

$$|R(t)| \le (C_\alpha + D)\ t\ |||W|||^2(|||W||| + \epsilon)\sup_{0\le s\le T}|B(s)| + C(1+\epsilon)|||W|||,$$

and since $0 \le t \le c|||W|||^{-2}$ and $B(s)$ is bounded we have

$$R(t) \le \tilde{C}(|||W||| + \epsilon).$$

This completes the $|||W|||^{-2}$ time scale estimates.

For infinite time behavior we again proceed in two steps. First we show that $B(t)$ is bounded on the positive half line. We start from (128) and use $\int_0^\infty e^{-\gamma s}ds \le \gamma^{-1}$ together with $r = 0$ in (p5), to obtain for all $t \ge 0$:

$$|B(t)| \le B(0) + \frac{|||W|||^2}{\gamma}(|||W||| + \epsilon)(C + D)\sup_{s\ge 0}|B(s)| + C(1+\epsilon)|||W|||.$$

Then since $\gamma \geq \theta_0 |||W|||^2$ we get

$$(1 - \tilde{C}_1(|||W||| + \epsilon)) \sup_{s\geq 0} |B(s)| \leq B(0) + C(1+\epsilon)|||W|||$$

($\tilde{C}_1 = (C+D)/\theta_0$) which for sufficiently small $|||W|||+\epsilon$ implies that $B(t)$ is uniformly bounded for $t > 0$.

Next we multiply (128) by $\langle t\rangle^{r_1}$ and split the integrals into $\int_0^{t/2} + \int_{t/2}^t$. We use (p5) with $r = 0$ for integrals up to $t/2$ and $r = r_1$ for integrals from $t/2$. We obtain for all $t \geq 0$,

$$\begin{aligned}
&\langle t\rangle^{r_1} |B(t)| \\
&\quad \leq \langle t\rangle^{r_1} e^{-\gamma t} B(0) + \langle t\rangle^{r_1} e^{-\gamma t/2} |||W|||^2 (|||W||| + \epsilon) \\
&\qquad \times (C+D) \sup_{s\geq 0} |B(s)| \int_0^{t/2} e^{-\gamma(t/2-s)} ds \\
&\qquad + (C+D)|||W|||^2 (|||W||| + \epsilon) \langle t\rangle^{r_1} \sup_{s\geq 0} \langle s\rangle^{r_1} |B(s)| \int_{t/2}^t e^{-\gamma(t-s)} \langle s\rangle^{-r_1} ds \\
&\qquad + C(1+\epsilon)|||W||| \langle t\rangle^{r_1} \left(e^{-\gamma t/2} \int_0^{t/2} e^{-\gamma(t/2-s)\langle s\rangle^{-r_1}} ds + \int_{t/2}^t e^{-\gamma(t-s)} \langle s\rangle^{-r_1} ds \right).
\end{aligned}$$

For the terms in the first row we take into account that $B(t)$ and the product of a power function with positive exponent and a decaying exponential is always bounded and that $\int_0^{t/2} e^{-\gamma(t/2-s)} ds \leq \gamma^{-1}$. For the terms in the second row we note that for $s \geq t/2$ we have $\langle s\rangle^{-r_1} \leq \langle t/2\rangle^{-r_1}$ which helps annihilate the $\langle t\rangle^{r_1}$ factor in front of the integrals. All in all we get

$$\begin{aligned}
\langle t\rangle^{r_1} |B(t)| \leq C_1 B(0) + C_2 \frac{|||W|||^2}{\gamma} (|||W||| + \epsilon) \\
+ C_3 \frac{|||W|||^2}{\gamma} (|||W||| + \epsilon) \sup_{s\geq 0} \langle s\rangle^{r_1} |B(s)| + C_4 |||W|||,
\end{aligned}$$

which after maximizing over all $t \geq 0$ in the left-hand side and using $\gamma > \theta_0 |||W|||^2$ leads to

$$\left(1 - \frac{C_3}{\theta_0}(|||W||| + \epsilon)\right) \sup_{t\geq 0} \langle t\rangle^{r_1} |B(t)| \leq C_1 B(0) + \frac{C_2}{\theta_0}(|||W||| + \epsilon) + C_4 |||W|||,$$

which for sufficiently small $|||W||| + \epsilon$ implies $\sup_{t\geq 0} \langle t\rangle^{r_1} |B(t)|$ bounded. □

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