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Transient radiative behavior of Hamiltonian systems in finite domains

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Abstract

We show that Hamiltonian partial differential equations on compact spatial domains can display transient radiative behavior, usually associated with infinite domains. This is done by considering a model of a single oscillator coupled to a wave field, which damps due to the resonant coupling of the oscillator with a discrete frequency with the continuous spectrum of the field. The analysis carried out illustrates that despite the “discretization” of the continuous spectrum due to the finiteness of the domain, a remnant of the resonance mechanism persists. In particular, this explains how numerical computations on bounded domains accurately simulate, on large but finite time scales, phenomena associated with infinite spatial domains. Numerical simulations in the present model show good agreement with our theoretical predictions.

1. Introduction

Hamiltonian partial differential equations (PDEs) defined on unbounded spatial domains have solutions which may exhibit two kinds of behavior:

- *bound state behavior*, characterized by spatially localized and time-periodic or perhaps more complicated temporal behavior, and
- *radiative behavior*, characterized by the escape of energy from any fixed spatially bounded set, or equivalently time-decay of a “local energy”.

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Bound state behavior is associated with discrete spectrum and the radiative behavior with continuous spectrum. There are many mathematical models in which these two types of behavior are coupled by, for example, linear or nonlinear effects, or time-parametric forcing; see, for example, [1–12]. Nonlinearity or time parametric forcing easily gives rise to frequencies which lie in the continuous spectrum. Thus, discrete modes couple to the continuum resulting in energy transfer from bound states to radiation modes; the continuum modes act as a “conservative heat bath”.

A natural question arises: *can systems with discrete spectrum display a radiative character as well?* On the one hand, such energy transfer is based on the coupling of discrete and continuum modes, and where there is no continuous spectrum, one might not expect radiative energy transfer. On the other hand, it is reasonable to expect that for an initial condition of fixed spatial support the initial stages of the evolution of this initial condition can be well-approximated by its evolution on a sufficiently large compact spatial domain, with, say, periodic boundary conditions, for which there is only discrete spectrum. This is precisely what one exploits in numerical simulations; one can solve the initial value problem for PDEs posed on an infinite domain accurately for times of order T on a finite domain of size which is dependent on T and the initial data. Examples of numerical work, in which radiative effects are simulated are the works of [13] on the sine-Gordon equation, of [14] on the ϕ^4 model (see also [15] for a more recent discussion of these examples), and the more recent examples of [16] in the context of the nonlinear Schrödinger equation.

The purpose of this note is to resolve the apparent paradox by considering a concrete example where such calculations can be carried out explicitly. In particular, we consider a particle-wave field model arising in [4]. This system has *metastable states*, bound-state like solutions which slowly decay to zero as time advances. We consider the problem on a finite domain and show how to express the dynamics as those on an infinite domain plus corrections due to finiteness. These corrections tend to zero in the infinite volume limit. Also, for fixed initial data and time intervals, if the domain is sufficiently large these finiteness corrections are negligible.

Our presentation is structured as follows: in Section 2, we present the model and its main features. In Section 3, we analyze the model and compute the radiation rate in the finite system. In Section 4, we present numerical computations and compare them with theoretical predictions. Finally, in Section 5, we briefly summarize our findings and present some interesting directions for future studies.

2. The oscillator–wave field model

Our model consists of a PDE coupled to an ordinary differential equation (ODE); the PDE is a wave equation governing $\eta(x,t)$, a $2L$ periodic function, whereas the ODE is an oscillator equation, governing an amplitude $a(t)$, as follows:

$$\left(\partial_t^2 + B^2\right)\eta(x,t) = \epsilon a^n(t)\phi(x), \tag{1}$$

$$\ddot{a}(t) + \Omega^2 a(t) = n\epsilon a^{n-1}(t)\int_{-L}^{L}\phi(x)\eta(x,t)\,dx, \tag{2}$$

where

$$B^2 = -\partial_x^2 + 1, \tag{3}$$

$\phi(x)$ is a $2L$-periodic L^2 function and $n \geqslant 1$ is an integer. Our analysis extends to a general class of wave equations defined by $B = \omega(p)$, $p = -i\nabla_x$, where $\omega(p)$ is real-valued. For $\epsilon = 0$ the wave field and oscillator are decoupled, while for $\epsilon \neq 0$ they interact.

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The system (1) is a Hamiltonian one with a conserved energy functional

$$H(t) = \frac{1}{2}\int \eta_t^2(x,t) + \eta_x^2(x,t) + \eta^2(x,t)\,dx + \frac{1}{2}\big(\dot{a}^2(t) + \Omega^2 a^2(t)\big) - \epsilon a^n \int \phi(x)\eta(x,t)\,dx$$
$$= H_{\text{wave}}(t) + H_{\text{oscillator}}(t) + H_{\text{coupling}}(t). \tag{4}$$

For the infinite domain, $L = \infty$, this model arises naturally as a reduction of a class of nonlinear Klein–Gordon equations with metastable states, studied in [4]. The variable $a(t)$ describes the amplitude of the time-periodic and spatially localized component of the full solution and $\eta(x,t)$ its dispersive part. It is shown that for $t \to \infty$, $a(t)$ decays to zero slowly. For $n = 3$ and $3\Omega > 1$, the very long time decay of $a(t)$ is governed essentially by a closed equation, in which the effect of the dispersive wave field is modeled by an effective dissipation [4]:

$$\ddot{a} + \big(\Omega^2 + \epsilon^2 \mathcal{O}(|a|^2)\big)a \sim -\Gamma a^4 \dot{a}, \quad \Gamma = \mathcal{O}(\epsilon^2) \geqslant 0. \tag{5}$$

Γ is the nonlinear radiation damping coefficient, which is generically strictly positive. In terms of a slowly varying complex envelope function (see Eq. (14)), Eq. (5) is equivalent to

$$\dot{A}(t) \sim (i\Lambda - \Gamma)|A|^4 A. \tag{6}$$

The mechanism for this internal damping is that the unperturbed ($\epsilon = 0$) oscillation of frequency Ω leads, via the cubic nonlinearity, to frequencies $\pm 3\Omega$ and higher harmonics. Although the frequencies of the unperturbed problem, $\pm\Omega$ lie in the *gap* of the continuous spectrum of $\pm B$, $\{E\colon |E| \geqslant 1\}$, the nonlinearity induced third harmonic, $\pm 3\Omega$, does fall in the continuous spectrum. This coupling of discrete to continuum modes is responsible for the energy transfer from localized to radiation modes.

In a similar fashion, this model is relevant to the dynamics of any dispersive nonlinear PDE or differential difference equation (DDE) that has internal modes [17,18] due to the (nonlinearity induced) coupling of various harmonics of the point spectrum eigenmodes with the continuous spectrum; furthermore, its dynamics are relevant to the time evolution of such systems (and even conjectured to be relevant to issues of the integrability of the PDE/DDE [19,20]).

Our goal is to study the system on the *finite* interval $[-L, L]$ and to show how *transient* radiative behavior emerges for this problem, which at small amplitude, is described in terms of a basis of states corresponding to *discrete* spectrum. We shall derive the analogue of Eq. (6) (see also, (5)) for finite L.

3. Analysis of the model

The analysis proceeds by Fourier series, where we use the following conventions:

$$f(x) = \sum_{m=-\infty}^{\infty} f_m e^{\frac{i\pi m x}{L}}, \quad f_m = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-\frac{i\pi m x}{L}}\,dx. \tag{7}$$

We consider the system (1)–(2) with initial conditions, in which the oscillator is excited but not the wave field:

$$\eta(x,0) = 0, \qquad \eta_t(x,0) = 0, \tag{8}$$
$$a(0) = a_0, \qquad \dot{a}(0) = a_1. \tag{9}$$

Expansion of $\eta(x,t)$ in a Fourier series, we obtain the following system of equations for the $\eta_m(t)$:

$$\eta_{m,tt} + \omega_m^2 \eta_m = \epsilon a^n \phi_m, \quad m = 0, \pm 1, \pm 2, \ldots. \tag{10}$$

Here,

$$\omega_m = \sqrt{k_m^2 + 1}, \quad k_m = \frac{\pi m}{L}. \tag{11}$$

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3.1. Reduction to a closed oscillator equation

The solution of (10) with the initial data (8) is given by

$$\eta_m(t) = \epsilon \frac{\phi_m}{\omega_m} \int_0^t a^n(s) \sin\bigl(\omega_m(t-s)\bigr)\, ds. \tag{12}$$

Substitution of (12) into (2) yields the following integro-differential equation for $a(t)$:

$$\ddot{a} + \Omega^2 a = 2n\epsilon^2 a^{n-1} L \sum_{m=-\infty}^{\infty} \frac{|\phi_m|^2}{\omega_m} \int_0^t a^n(s) \sin\bigl(\omega_m(t-s)\bigr)\, ds. \tag{13}$$

We used the fact that ϕ is real valued and so $\phi_{-m}\phi_m = |\phi_m|^2$.

If the parameter ϵ is small, we anticipate that $a(t)$ will consist of

(i) "fast oscillations" coming from the natural frequency Ω and its nonlinearly generated harmonics, as well as
(ii) slow variations on top of the fast modes.

Thus, we extract from $a(t)$ the dominant "fast oscillations" of frequency Ω

$$a(t) = A(t)e^{i\Omega t} + A^*(t)e^{-i\Omega t}. \tag{14}$$

Imposing the constraint

$$\dot{A}(t)e^{i\Omega t} + \dot{A}^*(t)e^{-i\Omega t} = 0, \tag{15}$$

we find that Eq. (2) for $a(t)$ is equivalent to the following equation for the complex amplitude $A(t)$:

$$\dot{A}(t) = \frac{ne^{-i\Omega t}}{i\Omega} \epsilon^2 L \bigl(Ae^{i\Omega t} + A^* e^{-i\Omega t}\bigr)^{n-1} \cdot \sum_{m=-\infty}^{\infty} \frac{|\phi_m|^2}{\omega_m} \int_0^t \sin\bigl(\omega_m(t-s)\bigr)\bigl(Ae^{i\Omega s} + A^* e^{-i\Omega s}\bigr)^n\, ds. \tag{16}$$

3.2. Dominant resonant behavior

Eq. (16) is an exact reduction of the oscillator dynamics to a closed-form equation for the complex amplitude $A(t)$. In this section, we make various approximations intended to single out the terms which are relevant to energy transfer from the oscillator to the wave field. We assume that only $n\Omega$ lies in the continuous spectrum of the operator, B, while all lower harmonics lie in its spectral gap[1]:

$$|n\Omega| > 1. \tag{17}$$

Keeping only these relevant terms, we find that Eq. (16) simplifies to

$$\dot{A}(t) \sim \frac{-n\epsilon^2 L}{2\Omega} A^{*(n-1)}(t) \sum_{m=-\infty}^{\infty} \frac{|\phi_m|^2}{\omega_m} e^{-i(n\Omega-\omega_m)t} \int_0^t A^n e^{-i(\omega_m - n\Omega)s}\, ds. \tag{18}$$

[1] Terms giving rise to frequencies which lie in the spectral gap and do not resonate with the continuum can be systematically removed by near-identity changes of variables; see, for example, [4,22].

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Inserting the explicit expression for the Fourier coefficients, ϕ_m, gives

$$\dot{A}(t) = \frac{-n\epsilon^2}{8\pi\Omega} A^{*(n-1)}(t) \sum_{m=-\infty}^{\infty} \frac{\pi}{L}\frac{1}{\omega_m}\left|\int_{-L}^{L}\phi(x)e^{\frac{-i\pi mx}{L}}\,dx\right|^2 e^{-i(n\Omega-\omega_m)t}\int_0^t A^n e^{-i(\omega_m-n\Omega)s}\,ds. \tag{19}$$

Note that the sum in (18) can be written in the form of a Riemann sum $\sum_{m=-\infty}^{\infty} f(mh)h$, where

$$f(y) = \frac{e^{-i(n\Omega-\omega(y))t}}{\omega(y)}\left|\mathcal{F}_L\phi(y)\right|^2\int_0^t A^n e^{-i(\omega(y)-n\Omega)s}\,ds, \tag{20}$$

where $y = mh$, $h = \pi/L$, $\omega(y) = \sqrt{1+y^2}$ and

$$\mathcal{F}_L\phi(y) = \int_{-L}^{L}\phi(x)e^{-ixy}\,dx. \tag{21}$$

The key to relating the finite L to infinite L dynamics is to re-express this sum using the Poisson summation formula [21]:

$$\sum_{m=-\infty}^{\infty} f(mh)h = \sum_{m=-\infty}^{\infty}\mathcal{F}[f](m/h), \tag{22}$$

where

$$\mathcal{F}[f](\xi) = \int_{\mathbb{R}} e^{-i\xi y} f(y)\,dy \tag{23}$$

is the Fourier transform of f. For (22) to hold, it suffices that f and $\mathcal{F}[f]$ have sufficient decay at infinity. From the properties of $\phi(y)$ and $\omega(y)$, we can verify the bounds

$$\left|f(y)\right| \leqslant C\bigl(1+|y|\bigr)^{-2}\bigl(\|\phi\|_\infty + \|\phi'\|_1\bigr), \tag{24}$$

$$\left|\mathcal{F}[f](\xi)\right| \leqslant C_1\bigl(1+|\xi|\bigr)^{-2}\bigl(1+|t|\bigr)^3, \tag{25}$$

where C_1 depends on the energy of the initial data, $H(0)$. The latter estimate uses $|\partial_y^r\omega(y)| \leqslant C(1+|y|)^{-r+1}$, $r = 0, 1$, $|\partial_y^r\omega(y)| \leqslant C(1+|y|)^{-r-1}$, $r \geqslant 2$, and $|\mathcal{F}_L\phi(y)| \leqslant \frac{C}{1+|y|}(\|x^r\phi(x)\|_\infty + \|\frac{d}{dx}(x^r\phi(x))\|_1)$, for $r \geqslant 0$.

The Poisson summation formula yields

$$\dot{A}(t) = \frac{-n\epsilon^2}{8\pi\Omega}A^{*(n-1)}\Bigl(\mathcal{F}[f](0) + \sum_{m\neq 0}\mathcal{F}[f](mL/\pi)\Bigr). \tag{26}$$

We will show that first term in brackets, $\mathcal{F}[f](0)$, yields the infinite volume ($L = \infty$) result, while the remaining sum over $m \neq 0$ represents corrections due to finite L. By the bound (25),

$$\left|\sum_{m\neq 0}\mathcal{F}[f](mL/\pi)\right| \leqslant C_1\bigl(1+|t|\bigr)^3\sum_{m\neq 0}\left(1+\left|\frac{mL}{\pi}\right|\right)^{-2} \leqslant C_1\bigl(1+|t|\bigr)^3 L^{-2}. \tag{27}$$

We now focus on the leading term, which we will now see corresponds to the $L = \infty$ limit,

$$\mathcal{F}[f](0) = \int_{-\infty}^{\infty}\left|\mathcal{F}_L\phi(y)\right|^2\frac{e^{-i(n\Omega-\omega(y))t}}{\omega(y)}\int_0^t A^n(s)e^{i(n\Omega-\omega(y))s}\,ds\,dy. \tag{28}$$

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Now, we consider the inner ds integral in (28). We have

$$\int_0^t A^n(s)e^{i(n\Omega-\omega(y))s}\,ds$$
$$=\lim_{\zeta\to 0}\int_0^t A^n(s)e^{i(n\Omega-\omega(y)-i\zeta)s}\,ds$$
$$=\lim_{\zeta\to 0}\left(\frac{A^n(t)(e^{i(n\Omega-\omega(y)-i\zeta)t}-1)}{i(n\Omega-\omega(y)-i\zeta)}-n\int_0^t A^{n-1}(s)\dot{A}(s)\frac{(e^{i(n\Omega-\omega(y)-i\zeta)s}-1)}{i(n\Omega-\omega(y)-i\zeta)}ds\right). \tag{29}$$

By (26), the integral on the right-hand side of (29) is $\mathcal{O}(\epsilon^2)$ and therefore contributes only at order $\mathcal{O}(\epsilon^4)$. Therefore, the dominant contribution to (29) is

$$\int_0^t A^n(s)e^{i(n\Omega-\omega(y))s}\,ds\sim\lim_{\zeta\to 0}\frac{A^n(t)(e^{i(n\Omega-\omega(y)-i\zeta)t}-1)}{i(n\Omega-\omega(y)-i\zeta)}. \tag{30}$$

Substitution of (30) into (28) gives

$$\mathcal{F}[f](0)\sim A^n(t)\left(\int_{-\infty}^{\infty}\left|\mathcal{F}_L\phi(y)\right|^2\frac{1}{i\omega(y)(n\Omega-\omega(y)-i0)}dy\right.$$
$$\left.-\int_{-\infty}^{\infty}\left|\mathcal{F}_L\phi(y)\right|^2\frac{e^{-i(n\Omega-\omega(y))t}}{i\omega(y)(n\Omega-\omega(y)-i0)}dy\right). \tag{31}$$

The second term contributes a part that is decaying as $t\to\infty$ [4] so the first term dominates and we have:

$$\mathcal{F}[f](0)\sim A^n(t)\int_{-\infty}^{\infty}\left|\mathcal{F}_L\phi(y)\right|^2\frac{1}{i\omega(y)(n\Omega-\omega(y)-i0)}dy. \tag{32}$$

The expression (32) can be evaluated using the classical Plemelj distributional formula:

$$\lim_{\zeta\downarrow 0}\frac{1}{x-i\zeta}=\mathrm{P.V.}\frac{1}{x}+i\pi\delta(x). \tag{33}$$

We obtain

$$\mathcal{F}[f](0)\sim\lim_{\zeta\to 0}A^n(t)\int_{-\infty}^{\infty}\left|\mathcal{F}_L\phi(y)\right|^2\frac{e^{\zeta t}}{i\omega(y)(n\Omega-\omega(y)-i\zeta)}dy$$
$$=-iA^n(t)\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{|\mathcal{F}_L\phi(y)|^2}{\omega(y)(n\Omega-\omega(y))}dy+\pi A^n(t)\int_{-\infty}^{\infty}\frac{|\mathcal{F}_L\phi(y)|^2}{\omega(y)}\delta\big(n\Omega-\omega(y)\big)\,dy. \tag{34}$$

Evaluation of the second term in (34) gives

$$\mathcal{F}[f](0)\sim\left[-i\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{|\mathcal{F}_L\phi(y)|^2}{\omega(y)(n\Omega-\omega(y))}dy+\pi\left(\sum_{\{y\in\mathbb{R}:\,\omega(y)=n\Omega\}}1\right)\frac{|\mathcal{F}_L\phi(\omega^{-1}(n\Omega))|^2}{\omega^{-1}(n\Omega)}\right]A^n(t). \tag{35}$$

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Finally, if we use (35) in (26) we obtain

$$\dot{A}(t) \sim (i\Lambda - \Gamma)|A|^{2(n-1)}A, \tag{36}$$

where $\Lambda = \mathcal{O}(\epsilon^2)$ is real and, for $f \in C^\tau$,

$$\Gamma_L \equiv \Gamma_\infty + \mathcal{O}(L^{-\tau}) = \frac{n\epsilon^2}{8\Omega}\Bigg(\sum_{\{y\in\mathbb{R}:\ \omega(y)=n\Omega\}} 1\Bigg)\frac{|\mathcal{F}\phi(\omega^{-1}(n\Omega))|^2}{\omega^{-1}(n\Omega)} + \mathcal{O}(L^{-\tau}). \tag{37}$$

In the special case of the Klein–Gordon equation, where $\omega_{\mathrm{KG}}(y) = \sqrt{1+y^2}$, there are two distinct solutions of $\omega_{\mathrm{KG}}(y) = n\Omega$ and we obtain

$$\Gamma_{\mathrm{KG},\infty} = \frac{n\epsilon^2}{4\Omega\sqrt{(n\Omega)^2-1}}\big|\mathcal{F}\phi(\omega^{-1}(n\Omega))\big|^2 + \mathcal{O}(L^{-\tau}). \tag{38}$$

Summary. *Let $T > 0$ be arbitrary and let ϵ be sufficiently small. Consider the dynamics (1)–(2) on $[-L, L]$ with order one initial data (8)–(9). If L is chosen so that T^3/L^2 is sufficiently small, then the initial value problem starting with $\mathcal{O}(1)$ energy in the bound state, i.e. and zero (or negligible) energy in the dispersive part, is governed by (36)–(37) on the time interval $0 \leqslant t \leqslant T/\epsilon^2$.*

4. Verification via numerical simulations

In this section we present typical numerical simulations, which validate the leading order behavior obtained analytically. We consider the system of Eqs. (1), (2), with $B^2 = I - \partial_x^2$, $n = 1$ with initial data $a(0) = 0$, $\dot{a}(0) = 1$, $\eta(x,0) = \eta_t(x,0) = 0$. Thus, $\omega(y) = (1+y^2)^{1/2}$. $\phi(x) = \exp(-x^2/2)$ is selected as a typical Gaussian kernel of interaction. For the particular example displayed in Fig. 1, we choose $L = 500$ and $\epsilon = 0.1$ and $\Omega = 1.2$. Since $\Omega = 1.2 > 1$, the unperturbed frequencies $\pm\Omega$ themselves, are embedded in the continuous spectrum and hence the decay predicted by Eq. (36) should be (a modulated) exponential in good agreement with the findings of Fig. 1. The upper plot in Fig. 1 demonstrates the transfer of energy from the oscillator to the wave field and resulting transient exponential decay of the oscillator energy. The lower plot in Fig. 1, a plot of the evolving logarithm of the oscillator energy (solid curve) and its best linear approximation (dashed line), illustrates the transient exponential behavior more directly.

We performed numerical simulations for various values of L and calculated the slope, $S_{\mathrm{numerical}}$, of the best linear fit of the logarithm of the oscillator energy, is given by

$$H_{\mathrm{oscillator}}(t) = \frac{1}{2}\big(\dot{a}^2 + \Omega^2 a^2\big) = 2\Omega^2|A|^2. \tag{39}$$

In this example, the reduced oscillator equation is of the form $\dot{A} \sim (i\Lambda_{\mathrm{KG}} - \Gamma_{\mathrm{KG}})A$, which implies $\partial_t|A|^2 \sim -2\Gamma_{\mathrm{KG}}|A|^2$. Therefore, we must have

$$S_{\mathrm{numerical}} \sim \frac{d}{dt}\log H_{\mathrm{oscillator}} \sim -2\Gamma_{\mathrm{KG}}. \tag{40}$$

We find fairly good agreement between the analytical result presented in the previous section and our numerical computations; in particular, for large L we observe: $S_{\mathrm{numerical}} \sim 0.0237$, while $2\Gamma_{\mathrm{KG}} \sim 0.0252$, where Γ_{KG} is given by the theoretical expression (38). The relative error is less than 6%.

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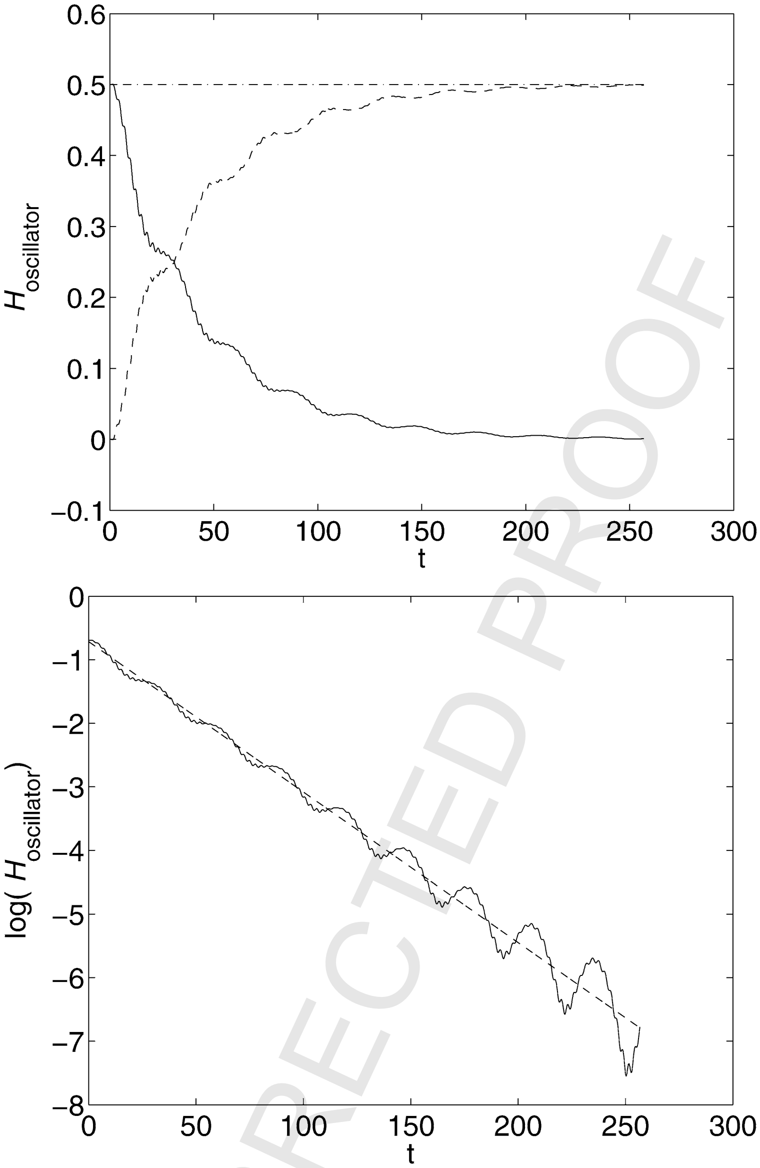

Fig. 1. The top panel shows the time evolution of the oscillator energy, $H_{\text{oscillator}}(t)$ (solid line), the wave field energy, $H_{\text{wave}}(t)$ (dashed line) and the total energy, $H(t) = H(0) = 0.5$ (dash-dotted, which is indistinguishable from the line $E = 0.5$). The bottom panel shows the time evolution of the logarithm of the oscillator energy, illustrating its practically exponential dependence on time.

5. Summary and discussion

In this Letter we considered a coupled oscillator–wave field model of interactions between discrete (bound state) modes and continuum (radiation) modes. Resonant interaction of discrete and continuum modes is known to result in energy transfer from bound states to radiation. Although, for a bounded spatial domain of size L the spectrum is all discrete, characteristics of the infinite L limit, for which there is continuous spectrum, emerge in the large but finite time transient dynamics. In particular, we derive *transient* radiative energy transfer by obtaining a description

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of the bound state dynamics in terms of the infinite domain limit ($L = \infty$) description plus $\mathcal{O}(L^{-\tau})$ corrections. A key tool in the derivation is the Poisson summation formula. Our analytical predictions were tested against direct numerical simulations of the full oscillator–wave field model and were found to be in good agreement. It would be particularly interesting to examine such effects systematically in the context of more complex models such as ones where the internal and radiation modes exist in the linearization around a coherent structure, and the traveling or collision dynamics of the coherent structure(s) is monitored (together with the projection of the dynamics to the internal mode eigenspace). It would also be of interest to understand the resonance picture in settings where multiple internal modes exist and various combinations of their frequencies may give rise to resonances. Such studies are currently in progress.

References

[1] V.S. Buslaev, G.S. Perel'man, Amer. Math. Soc. Transl. Ser. 2 164 (1995) 75.
[2] A. Soffer, M.I. Weinstein, Geom. Funct. Anal. 8 (1998) 1086.
[3] A. Soffer, M.I. Weinstein, J. Stat. Phys. 93 (1998) 359.
[4] A. Soffer, M.I. Weinstein, Invent. Math. 136 (1999) 9.
[5] A. Komech, M. Kunze, H. Spohn, Commun. Math. Phys. 203 (1999) 1.
[6] A. Soffer, M.I. Weinstein, Lett. Math. Phys. 48 (1999) 339.
[7] P.D. Miller, A. Soffer, M.I. Weinstein, Nonlinearity 13 (2000) 507.
[8] E.W. Kirr, M.I. Weinstein, SIAM J. Math. Anal. 33 (2001) 16.
[9] M. Kunze, A. Rendall, Class. Quantum Grav. 18 (2001) 3573.
[10] T.-P. Tsai, H.-T. Yau, Internat. Math. Res. Notices 31 (2002) 2363.
[11] E.W. Kirr, M.I. Weinstein, Commun. Math. Phys. 236 (2003) 335.
[12] A. Soffer, M.I. Weinstein, Rev. Math. Phys. 16 (2004) 977.
[13] M. Peyrard, M.D. Kruskal, Physica D 14 (1984) 88.
[14] D.K. Campbell, J.F. Schonfeld, C.A. Wingate, Physica D 9 (1983) 1.
[15] P.G. Kevrekidis, M.I. Weinstein, Physica D 142 (2000) 113.
[16] M. Johansson, S. Aubry, Phys. Rev. E 61 (2000) 5864.
[17] P.G. Kevrekidis, C.K.R.T. Jones, Phys. Rev. E 61 (2000) 3114.
[18] P.G. Kevrekidis, M.I. Weinstein, Math. Comput. Simulation 62 (2003) 65.
[19] P.G. Kevrekidis, Phys. Lett. A 285 (2001) 383.
[20] N.R. Quintero, P.G. Kevrekidis, Physica D 170 (2002) 31.
[21] G.B. Folland, Real Analysis—Modern Techniques and their Applications, Wiley, New York, 1984.
[22] J. Guckenheimer, P.J. Holmes, Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, Springer-Verlag, New York, 1983.