RESONANCE PROBLEMS IN PHOTONICS

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1. Introduction and outline

An important area of photonics concerns the propagation and control of light in non-homogeneous or nonlinear media. This is an area of great importance due to the range of physical effects to be exploited in the design of devices for communication and computing, as well as in the study of fundamental phenomena. Progress has involved a rich interplay of experiment, modeling, computation and analysis, which have impact beyond the specific motivating questions. Indeed, the models and analysis that we discuss are closely related those arising in the study of Bose-Einstein condensation in the regime governed by Gross-Pitaevskii type equations.

The propagation of light in a medium is governed by Maxwell's equations with appropriate linear and nonlinear constitutive relations. We consider a setting where there is a separation of scales in the electric field; the carrier wavelength is short compared to a wave envelope width; see Figure 1. An approximate description of the field evolution, in terms of its *envelope* is therefore very natural and effective. In this article we discuss various resonance phenomena leading to and playing a role in the dynamics of such envelope equations.

We begin with the modeling of optical pulses traveling through a nonlinear and periodic medium. At *Bragg resonance*, when the carrier wavelength and medium periodicity satisfy a resonance relation, the wave envelope evolves according to a system of dispersive nonlinear partial differential equations (PDEs), the nonlinear coupled mode equations (NLCME). NLCME has *gap soliton* solutions. These are localized states with frequencies in a photonic band gap. Theory predicts that these can travel at any speed less than the speed of light. It is of fundamental and practical interest to understand whether one can "trap" such soliton pulses. This would have potential application to optical buffers and computing.

We explore strategies for soliton capture via the insertion of suitably designed localized defects. The dynamics are governed by a variant of NLCME, but now with variable coefficient "potentials", analogous to the nonlinear Schrödinger / Gross-Pitaevskii (NLS-GP) equation. Nonlinear scattering interactions of solitons and defect modes, associated with defects of the periodic structure, are explored numerically and a picture of soliton capture, in terms of resonant energy transfer from incoming solitons to "pinned" nonlinear defect states, emerges. Roughly speaking, soliton-defect nonlinear scattering is marked by the approach of an incoming soliton, interaction with defect modes, trapping of some energy by the defect, and a emission of an outgoing soliton and radiation. Finite dimensional models of such soliton capture have been considered, for example, in Refs. 7, 18 and references therein.

In the last part of this article we consider, for the NLS-GP equation, the large time evolution of the *trapped energy*, that part which remains localized around the defect. We prove, that generic small amplitude initial conditions converge, as $t \to \pm\infty$, toward a nonlinear ground state defect mode [35,36]. Experimental verification of this *ground state selection*, described in Theorem 5.2, was recently made in nonlinear optical waveguides by the group of Y. Silberberg at the Weizmann Institute [27].

The structure of this article is as follows:

Section 2: Nonlinear waves in periodic media at Bragg resonance - envelope equations
Section 3: Coherent structures - gap solitons, nonlinear defect modes
Section 4: Controlling nonlinear light pulses - stopping light on a defect
Section 5: Large time distribution of trapped energy of NLS / GP – ground state selection

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2. Nonlinear waves at Bragg resonance - envelope equations

An optical fiber grating is an optical waveguide, whose refractive index varies periodically along its length. Pulse propagation through a optical fiber grating, exhibiting the nonlinear Kerr effect [3], is approximately governed by the nonlinear wave equation, with a refractive index having periodic and nonlinear parts:

$$\partial_t^2 \left[n^2(z,E)E(z,t) \right] = \partial_z^2 E \tag{2.1}$$

$$n^2 = n_0^2 + \varepsilon\kappa \cos(\frac{2\pi}{d} z) + n_2 E^2 \tag{2.2}$$

The coefficient $\varepsilon\kappa$ is the strength of the grating, a measure of the index contrast. The coefficient, n_2, is the Kerr nonlinearity coefficient.

For typical pulse-widths in experiments with a carrier wavelength of 1.5 μm ($1\mu m = 10^{-6}m$), there are $\mathcal{O}(10^5)$ oscillations under the wave envelope. This electromagnetic field is best viewed as a slowly varying envelope, modulating a highly oscillatory carrier wave; see Figure 1. In a linear homogeneous medium

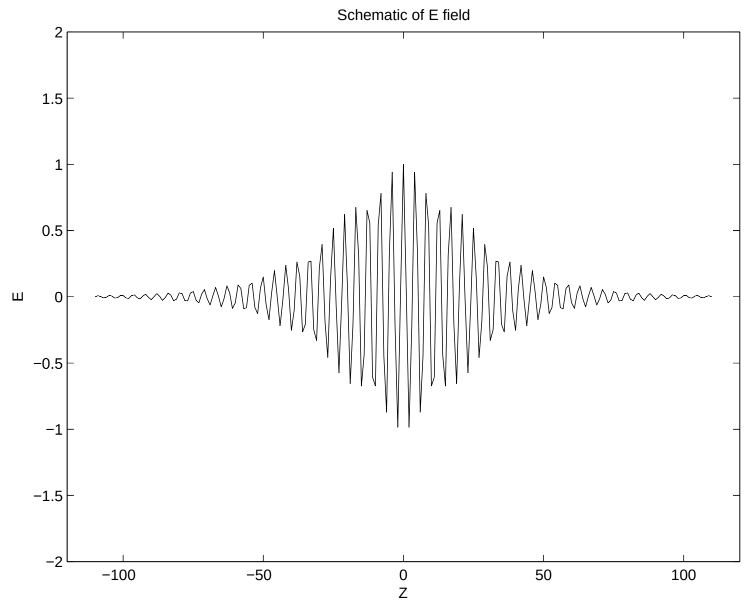

Figure 1. Electric field of an optical pulse consisting of a short wave-length carrier and a slowly varying, spatially localized, envelope

($\varepsilon\kappa = 0$, $n_2 = 0$) solutions of the wave equation, 2.1, are superpositions of decoupled forward and backward waves, $E(z,t) = E_f(z,t) + E_b(z,t)$, where $(n_0\partial_t - \partial_z)E_f = 0$ and $(n_0\partial_t + \partial_z)E_b = 0$. Alternatively, the field can viewed as a superposition of plane waves:

$$E(z,t) = E_+ \ e^{i(kz-\omega(k)t)} + E_- \ e^{-i(kz+\omega(k)t)} \ + \ c.c. \quad . \tag{2.3}$$

Here, $\omega(k) = k/n_0$, $E_\pm$ are the forward and backward amplitudes, and *c.c.* denotes the complex conjugate of the preceding expression. Thus propagation of waves in a homogeneous medium is *non-dispersive*; all wavelengths, $\lambda = 2\pi/k$ travel at the same speed, $1/n_0$.

For a periodic structure, $\varepsilon\kappa \neq 0$, there are significant differences. There are certain wavelength intervals, photonic (pass) bands, for which waves are transmitted with little excitation of back-reflection. In other wavelength intervals, the photonic band gaps, significant back-reflection occurs. The simplest case of *Bragg resonance* occurs when the carrier wavelength, λ is twice the medium periodicity, d:

$$\lambda_B = 2d, \quad k_B = \frac{2\pi}{\lambda_B} = \frac{\pi}{d}. \tag{2.4}$$

For small ε the solution can be represented, on non-trivial time-scales, by (2.3), where $E_\pm$ are no longer constants, but rather evolve slowly according to coupled mode equations, a system of PDEs , which tracks the coupled evolution of backward and forward waves. As we shall see below, propagation in a periodic structure is *dispersive*; waves of different wavelengths travel at different speeds.

We now introduce nonlinearity through the refractive index (2.2), $n_2 > 0$, and assume field amplitudes, chosen to balance the linear index modulations. This establishes a balance between dispersive and nonlinear effects. We encode the multiple scale nature of the problem in terms of a small parameter ε, by the relations:

$$\text{Pulse width} \sim \ \varepsilon^{-1}, \quad \text{Pulse amplitude} \sim \ \varepsilon^{1/2}, \ \text{Carrier wavelength} \sim \ 1;$$

see Figure 2. The electric field then has the form

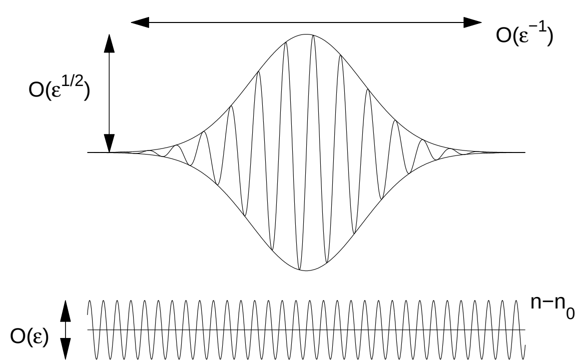

Figure 2. Scalings of pulse width, amplitude and carrier wavelength.

$$\begin{aligned} E^\varepsilon \ &\sim \ \varepsilon^{\frac{1}{2}} \left(E_+(Z,T) e^{i(k_B z - \omega_B t)} + E_-(Z,T) e^{-i(k_B z + \omega_B t)} \right) \quad + \quad c.c. \\ Z &= \varepsilon z, \ T = \varepsilon t, \end{aligned} \tag{2.5}$$

where the amplitudes $E_+(Z,T)$ and $E_-(Z,T)$ satisfy the *Nonlinear Coupled Mode Equations* (NLCME)

$$\begin{aligned} & i(\partial_T + \partial_Z)E_+ + \kappa E_- + g(|E_+|^2 + 2|E_-|^2)E_+ = 0 \\ & i(\partial_T - \partial_Z)E_- + \kappa E_+ + g(|E_-|^2 + 2|E_+|^2)E_- = 0, \quad g > 0. \end{aligned} \tag{2.6}$$

See, for example, Refs. 17 for an overview. A result on the validity of NLCME, as an approximation to solutions of a nonlinear wave system at Bragg resonance, is the following [17]:

Theorem 2.1. *Solutions $E_\pm(Z,T)$ of the nonlinear coupled mode equations (NLCME) yield $H^1(\mathbb{R})$ accurate approximations to solutions of the Anharmonic*

Maxwell-Lorentz model on a time scale of physical interest. Specifically,

$$E^\varepsilon(z,t) \; - \; \Re\, \varepsilon^{\frac{1}{2}} \left(E_+(\varepsilon z, \varepsilon t) e^{i(k_B z - \omega_B t)} + E_-(\varepsilon z, \varepsilon t) e^{-i(k_B z + \omega_B t)} \right)$$

is $\mathcal{O}(\varepsilon)$ in $H^1(\mathbb{R})$ for $t \sim \mathcal{O}(\varepsilon^{-1})$,

3. Coherent structures

Note that the linearization of (2.6) about the zero state leads to a linear *dispersive* equation:

$$i(\partial_T + \partial_Z)E_+ + \kappa E_- = 0, \qquad i(\partial_T - \partial_Z)E_- + \kappa E_+ = 0. \tag{3.1}$$

Indeed, (3.1) has plane waves of the form: $\vec{v}e^{-i\Omega T}$ with resulting dispersion relation, $\Omega = \Omega(Q)$ given by

$$\Omega^2 = Q^2 + \kappa^2.$$

Note the gap in the spectrum of admissible plane waves; there are no plane waves with wave-numbers in the interval $[-\kappa, \kappa]$. This corresponds to the fundamental (lowest energy) gap in the spectrum of the Helmholtz wave equation, associated with (2.1) for $n_2 = 0$.

Solitary wave solutions of (2.6) can be sought in the form $(E_-(Z;\omega), E_+(Z;\omega))e^{-i\omega t}$. Explicit solutions were given in Ref. 9, 1, by a generalization of the soliton-form for the massive Thirring (integrable) equation . For each ω in the interval (photonic band gap) $(-\kappa, \kappa)$, there is a solitary wave of finite L^2 norm (optical power). Therefore, these solitary waves are called *gap solitons*. A bifurcation diagram for gap solitons is displayed in Figure 3. In fact, a family of gap solitons can be found, which travel at any speed $0 \leq v < c$.

Experiments by Slusher *et. al.* (1997) have explored nonlinear propagation of light in periodic structures at *Bragg resonance* in fiber gratings. They observe gap soliton propagation with speeds as slow as 50% of the speed of light [14].

Can a gap solitons propagating in a periodic structure be captured by insertion of appropriately engineered defects?

The approach to this question, taken in Ref. 19, is discussed in the following section and leads to some basic questions in the nonlinear scattering of coherent structures.

4. Controlling nonlinear pulses

A simple model of a nonlinear periodic structure with a localized defect is given by the generalization of (2.2)

$$n^2 \; = \; n_0^2(\varepsilon z) \; + \; \varepsilon\kappa(\varepsilon z)\cos\left(\frac{2\pi}{d}z + \Phi(\varepsilon z)\right) \; + \; n_2 E^2 \tag{4.1}$$

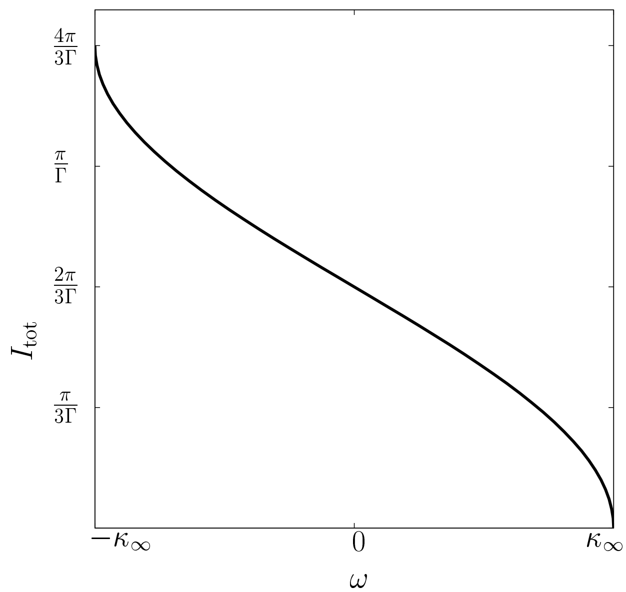

Figure 3. $I = \int |E_+|^2 + |E_-|^2 \, dZ$ vs. ω for $\omega \in (-\kappa, \kappa)$.

Here, $n_0(Z), \kappa(Z)$ and $\Phi(Z)$ are now slowly varying parametric functions, which approach constant limits away from some some compact set. Schematic plots of background linear refractive indices are shown in Figure 4. Now, in a manner

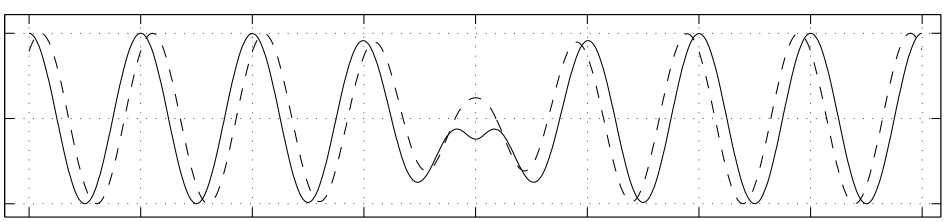

Figure 4. Linear refractive index of a periodic medium with a localized defect for two sets of parametric functions $n_0(\varepsilon z), \kappa(\varepsilon z)$ and $\Phi(\varepsilon z)$

analogous to the derivation of NLCME (2.6), we obtain the following generalization governing propagation in periodic media with defects [19]:

$$
\begin{aligned}
&i(\partial_T + \partial_Z)E_+ + \kappa(Z)E_- + W(Z)E_+ + g(|E_+|^2 + 2|E_-|^2)E_+ = 0 \\
&i(\partial_T - \partial_Z)E_- + \kappa(Z)E_+ + W(Z)E_- + g(|E_-|^2 + 2|E_+|^2)E_- = 0
\end{aligned} \tag{4.2}
$$

The parametric function $W(Z)$ can be computed from the functions $\kappa(\cdot), \Phi(\cdot)$ and $n_0(\cdot)$, which define the linear refractive index [19].

4.1. *Nonlinear scattering*

We have numerically investigated the interaction of incoming gap solitons, of different amplitudes and phases, with a variety of defects. Some representative results [19] of such simulations appear in Figures 5 and 6; see also Ref. 18. Simulations show

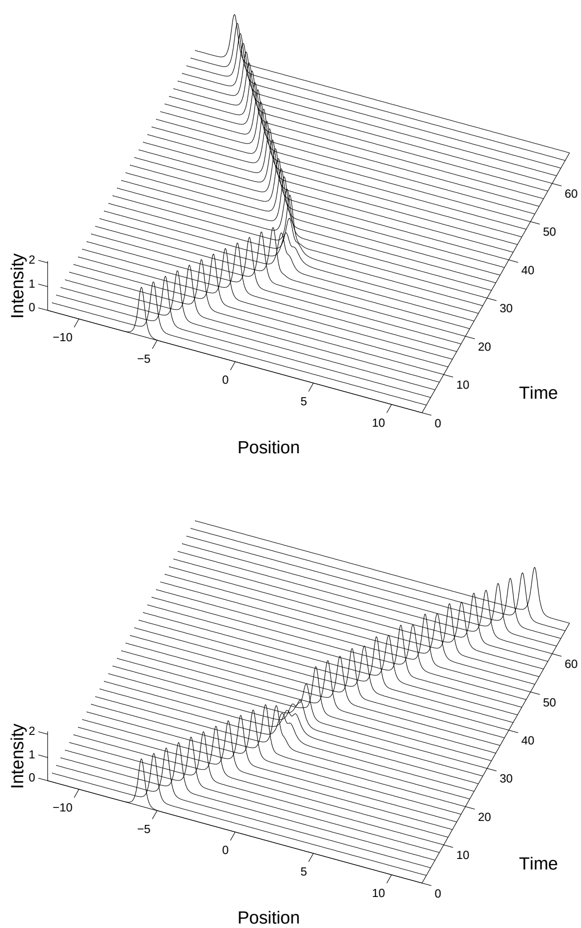

Figure 5. Reflection and transmission of gap soliton

a soliton incident on the defect and the resulting redistibution of its energy into (a) a reflected soliton, (b) localized states within the defect region, (c) a transmitted soliton and (d) outgoing radiation.

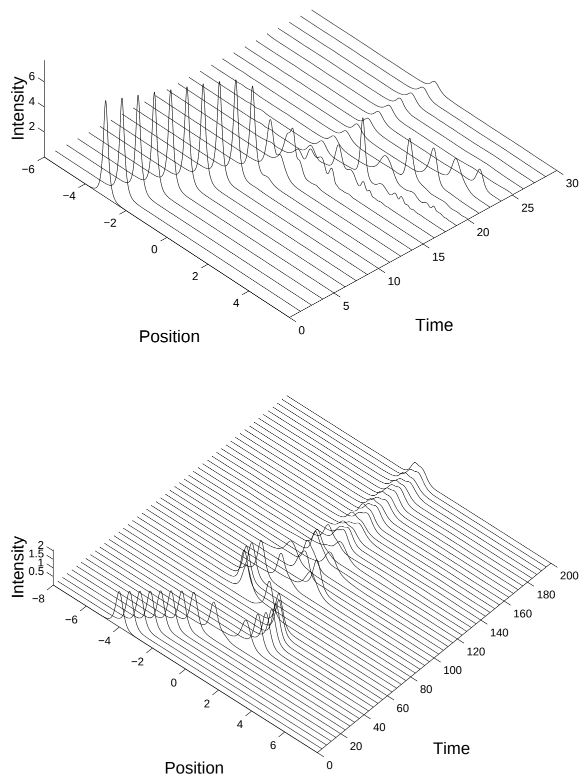

Figure 6. Partial capture and capture of gap soliton

We next identify these localized (pinned) states within the defect region as being *nonlinear defect modes* related to the background linear refractive index.

4.2. *Nonlinear defect modes*

Nonlinear defect modes are standing wave solutions of (4.2), which are spatially localized ("pinned") at the defect. Setting $(E_+(Z,T), E_-(Z,T) = (\mathcal{E}_+(Z), \mathcal{E}_Z(Z))e^{-i\omega T}$, we have

$$\begin{aligned}(\omega + i\partial_Z)\mathcal{E}_+ + \kappa(Z)E_- + W(Z)\mathcal{E}_+ + g(|\mathcal{E}_+|^2 + 2|\mathcal{E}_-|^2)\mathcal{E}_+ = 0\\ (\omega - i\partial_Z)\mathcal{E}_- + \kappa(Z)E_+ + W(Z)\mathcal{E}_- + g(|\mathcal{E}_-|^2 + 2|\mathcal{E}_+|^2))\mathcal{E}_- = 0\end{aligned} \tag{4.3}$$

Consider first the linearization of (4.2) about the zero state. We have

$$\begin{aligned}i(\partial_T + \partial_Z)E_+ + \kappa(Z)E_- + W(Z)E_+ = 0\\ i(\partial_T - \partial_Z)E_- + \kappa(Z)E_+ + W(Z)E_- = 0\end{aligned} \tag{4.4}$$

Linear defect modes are solutions of (4.4) of the form $E_\pm = e^{-i\Omega T}F_\pm(Z;\Omega)$ with $F \in L^2$, or equivalently are L^2 eigenstates of the self-adjoint linear operator,

$$i\sigma_3\partial_Z \ + \ \kappa(Z)\sigma_2 \ + \ W(Z),$$

where σ_2 and σ_3 are standard Pauli matrices. Thus,

$$\begin{aligned}(\Omega + i\partial_Z)F_+ + \kappa(Z)F_- + W(Z)F_+ = 0\\ (\Omega - i\partial_Z)F_- + \kappa(Z)F_+ + W(Z)F_- = 0\end{aligned} \tag{4.5}$$

Remark 4.1. Illustrated in Figure 4 are two different linear periodic structures with defects, (4.1) with $n_2 = 0$, for which the (envelope) spectral problem (4.5) is identical.

In analogy with a result for the nonlinear Schrödinger / Gross-Pitaevskii equation (see section 5) [28], we have [19]:

Theorem 4.1. *Nonlinear defect modes of (4.2) bifurcate from the zero state at the linear discrete eigenvalues of the eigenvalue problem (4.5). These are standing wave states which are localized or "pinned" at the defect site.*

Figure 7 shows two branches of states. The dark curve corresponds to gap solitons for translation invariant equation (2.6); see also Figure 3. These bifurcate from the zero state at the edge of the continuous spectrum. The lighter curve corresponds to a family of nonlinear defect states bifurcating from the zero state at the eigenvalue, $\omega = -1$, of the linear eigenvalue problem (4.5). Such states have recently been considered as well for a two-dimensional variant of (4.2); see Ref. 13. Note that if the linear problem (4.5) has multiple defect states, then there are multiple branches bifurcating from the linear eigenvalues [19,20]. In section 5 we consider, for the NLS-GP equation, the implications for the detailed nonlinear dynamics of multiple branches of defect states.

The problem of capturing a gap soliton by a defect can be seen in terms of the transfer of energy of an incoming soliton to nonlinear defect states. For low velocity incoming solitons, we find that an incoming soliton transfers its energy to a nonlinear defect state (is trapped) if there is an energetically accessible (lower L^2 norm) defect state of approximately the same frequency with which it can resonate [19,18]; see Figure 7. Figures 8 and 9 show the location of the soliton center of mass, z_{cm}, as a function of time in the non-resonant and resonant cases. In the non-resonant case, the soliton has been reflected, while in the resonant case, much of its energy has been trapped. The lower plots are of mode projections and indicate the degree of energy exchange between soliton and defect mode.

A rigorous theory of soliton capture is an open problem. In the recent papers, Refs. 24, 25, fast soliton scattering from a delta-function defect is studied analytically and numerically.

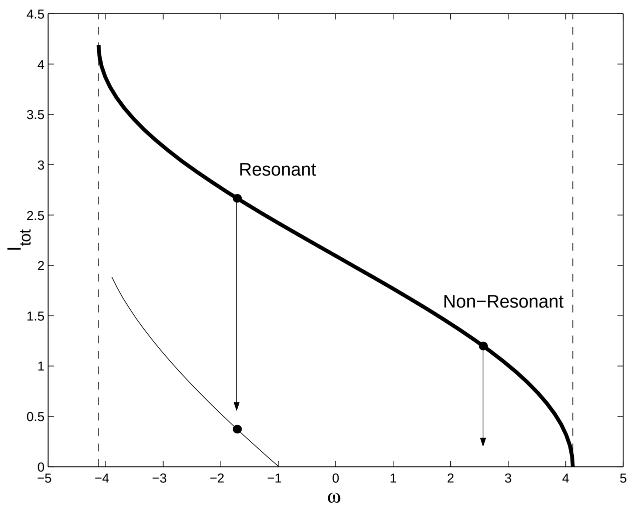

Figure 7. Bifurcation curves and resonant energy transfer

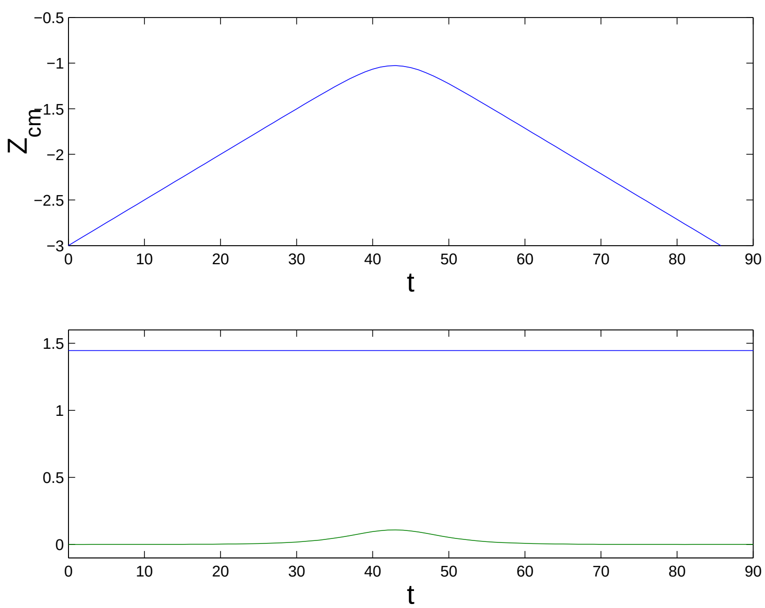

Figure 8. Nonresonance - minimal energy transfer to defect mode

5. Large time evolution of a captured soliton - ground state selection for NLS / G-P

Suppose a soliton is trapped by a defect, say as in Figure 6. The defect may have multiple linear bound states and therefore multiple bifurcating nonlinear bound

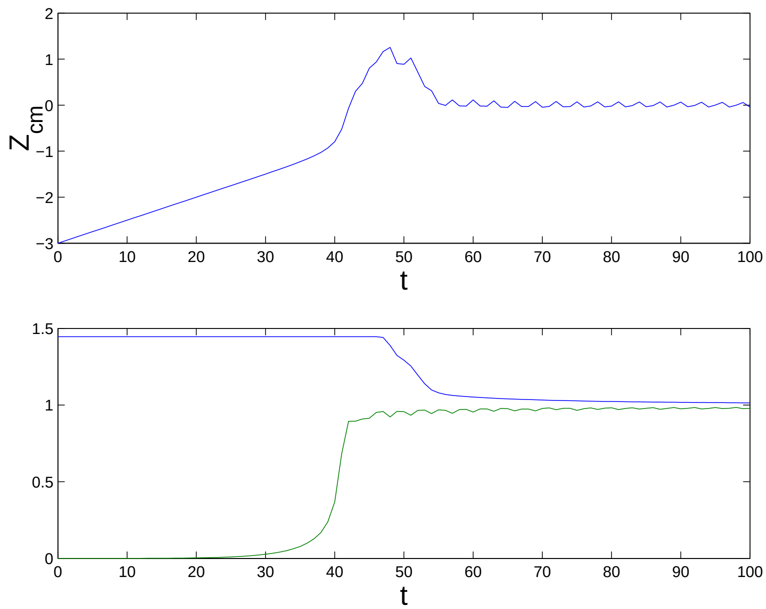

Figure 9. Resonance: Strong energy transfer to defect mode

state families. These compete for the soliton's energy. In this section we consider the question of how the energy distributes among the available modes in the large time limit.

This is a question of general interest and we consider it in the context of the nonlinear Schrödinger / Gross-Pitaevskii (NLS-GP) equation:

$$\begin{aligned} i\partial_t\Phi \ &= \ (\ -\Delta \ + \ V(x)\ )\,\Phi \ + \ g|\Phi|^2\Phi \\ &= \ H\Phi \ + \ g|\Phi|^2\Phi \qquad (5.1) \\ \Phi(x,0) \ &= \ \Phi_0(x),\ \ x\in \mathbb{R}^3, \ \ \text{small norm and localized.} \end{aligned}$$

Here, $V(x)$ is a potential which decays rapidly as $|x| \to \infty$.

NLS-GP can be derived in the context of nonlinear optics [3,37] and in the study of the macroscopic behavior of a large ensemble of quantum particles (Bosons), *e.g.* Bose-Einstein condensates (BEC); see, for example, Refs. 15, 2. In (5.1), g is a constant, corresponding to, in nonlinear optics, the negative of the Kerr nonlinear coefficient and, for Bose-Einstein condensation, the scattering length of the inter-particle quantum potential .

Assume that H has two bound states:

$$H\psi_{j*} \ = \ E_{j*}\psi_{j*}, \ \ \|\psi_{j*}\|_2 = 1, \ \ j=0,1$$

First consider the linear case, where $g=0$. If Φ_0 is sufficiently localized, then the solution decomposes into a spatially localized, time quasi-periodic part and a part which decays to zero (in L^∞) [26,42] due to dispersion, as $t \to \pm\infty$:

$$\Phi(x,t) \;=\; \underbrace{c_0 e^{-iE_{0*}t}\psi_{0*}(x) \;+\; c_1 e^{-iE_{1*}t}\psi_{1*}(x)}_{\text{quasi}-\text{periodic}} \;+\; \mathcal{O}(t^{-\frac{3}{2}}).$$

How does the solution of NLS/G-P ($g \neq 0$) resolve as $t \to \infty$?

To study this question, we begin by introducing the *nonlinear bound states* or *defect modes* of the NLS / Gross Pitaevskii Equation. These are solutions of (5.1) of the form $\Psi(x; E)e^{-iEt}$, where

$$H\Psi \;+\; g|\Psi|^2\Psi \;=\; E\Psi$$

The following result, analogous to Theorem 4.1, shows that there are families of nonlinear bound states, which bifurcate from the discrete eigenstates of H [28].

Theorem 5.1. *There exist nonlinear bound states bifurcating from the zero state at the linear eigenvalues, E_{j*}.*

$$\begin{aligned} \Psi_{\alpha_j} \;&=\; \alpha_j \left(\psi_{j*} \;+\; \mathcal{O}(|\alpha_{j*}|^2)\right),\; \alpha_j \in \mathbb{C} \\ E_j \;&=\; E_{j*} \;+\; \mathcal{O}(|\alpha_{j*}|^2), \quad |\alpha_j| \to 0 \end{aligned} \tag{5.2}$$

How do the nonlinear bound states Ψ_{α_0} and Ψ_{α_1} participate in the dynamics on small, intermediate and infinite time scales?

The following theorem addresses the above questions [35,36].

Theorem 5.2. *For generic small initial data, the solution of the nonlinear Schrödinger / Gross-Pitaevskii equation (NLS-GP) (5.1) evolves toward a nonlinear ground state as $t \to \pm\infty$.*

More precisely, assume

- *(H1) H has 2 bound states ($\Longrightarrow$ 2 families of nonlinear defect modes)*
- *(H2) Φ_0 is smooth and spatially localized and small (weakly nonlinear regime)*
- *(H3) $\Gamma \;\equiv\; \pi\left|\mathcal{F}_{H_c}[\psi_{0*}\psi_{1*}^2](\omega_{res})\right|^2 \;>\; 0$, where $\omega_{res} = 2E_{1*} - E_{0*}$; (nonlinear variant of Fermi Golden Rule). Here, $\mathcal{F}_{H_c}[f](\omega_{res})$ denotes the projection of f onto the generalized plane-wave eigenfunction of H (generalized Fourier transform) at frequency ω_{res}.*

Then,

- *(1) As $t \to \pm\infty$,*

$$\Phi(t) \;=\; e^{i\omega_j^\pm(t)}\Psi_{\alpha_j^\pm}(x) \;+\; \mathcal{O}(t^{-\frac{1}{2}})$$

where either $j = 0$ (nonlinear ground state) or $j = 1$ (nonlinear excited state).

- *(2) Generically, $j = 0$. $\omega_0^\pm(t) = E_0^\pm t + \mathcal{O}(\log t)$.*

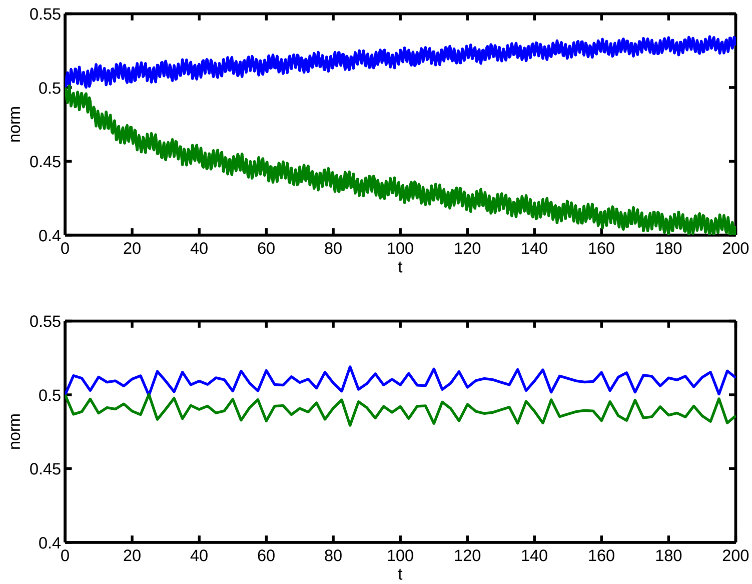

Figure 10. Trapped state's projections on ground and excited states. Top plot is for the case where $\omega_{res} = 2E_{1*} - E_{0*} > 0$, $\Gamma > 0$. Bottom plot is for the case where $\omega_{res} < 0$, $\Gamma = 0$

Remark 5.1. The detailed analysis indicates that if one considers initial data, which is a superposition of a nonlinear ground state and a nonlinear excited state, then half the excited state energy is radiated and half goes into to forming a new asymptotic ground state [35,36,40]: $|\alpha_0^{\pm}|^2 \sim |\alpha_0(0)|^2 + \frac{1}{2}|\alpha_1(0)|^2$; see (5.9).

Remark 5.2. This *ground state selection* has been observed in experiments in optical waveguides[27].

Remark 5.3. For related work on asymptotic behavior for NLS type equations see, for example, Refs. 5, 34, 6, 10, 11, 38, 39, 16, 40.

Remark 5.4. The "emission" of energy from the excited state into the ground state and dispersive radiation channels is a nonlinear variant of phenomena such as spontaneous emission, associated with the embedded eigenvalues in the continuous spectrum; see, for example, Refs. 29, 33, 12 and references cited therein.

Sketch of the analysis:
We view the full infinite dimensional Hamiltonian system (PDE) as being comprised of two weakly coupled subsystems:

- a finite dimensional (nonlinear oscillators - ODEs) governing the interacting nonlinear bound states (particles), and

- an infinite dimensional (wave equation - PDE) governing dispersive radiation.

We obtain this equivalent formulation beginning with the following Ansatz:

$$\Phi(x,t) \ = \ \Psi_{\alpha_0(t)}e^{-i\Theta_0(t)} \ + \ \Psi_{\alpha_1(t)}e^{-i\Theta_1(t)} \ + \ \eta_{\rm rad}(t) \tag{5.3}$$

The functions $\alpha_j(t)$ and $\Theta_j(t),\ j=0,1$ are "collective coordinates" on nonlinear bound state manifolds of equilibria and $\eta_{\rm rad}(t)$ denotes dispersive radiation; see, for example, Refs. 41, 30, 31, 32, 4, 22.

Substitution into (5.1) and projecting with respect to an appropriate biorthogonal basis of the adjoint problem yields an equivalent system in terms of $\alpha_0(t),\alpha_1(t)$ and $\eta_{\rm res}(t)$ having the form of

Oscillators interacting with a field:

$$\begin{aligned}
i\partial_t\alpha_0 \ &= \ \mathcal{C}_{\alpha_0}(\alpha_0,\alpha_1,\eta_{\rm rad})\\
i\partial_t\alpha_1 \ &= \ \mathcal{C}_{\alpha_1}(\alpha_0,\alpha_1,\eta_{\rm rad})\\
i\partial_t\eta_{\rm rad} \ &= \ H\eta_{\rm rad} \ + \ \mathcal{P}_c(H)\ \mathcal{R}[\alpha_0,\alpha_1,\eta_{\rm rad}]
\end{aligned} \tag{5.4}$$

Approximate finite dimensional reduction: In a manner analogous to centre-manifold reduction of dissipative systems [8,23], we next attempt to find a closed system by approximately solving for the radiation components, $\eta_{\rm rad}$, as functional of oscillator variables, α_j. *In particular, we find the contributions responsible for resonant energy exchange between oscillator and field degrees of freedom.* These involve spectral components in a neighborhood of frequency $\omega_{res} \ = \ 2E_{1*}-E_{0*} \ >0$:

$$\eta_{\rm rad} \ \sim \ \eta_{\rm rad}^{\rm res}[\alpha_0,\alpha_1].$$

We obtain a finite dimensional system, a set of ODEs in **normal form** [21], which captures, up to controllable corrections, the energy loss from the oscillators due to *radiation damping.* This normal form is weakly coupled to a dispersive wave equation, whose effect decreases with advancing time. For large time t, this effect can be estimated in the spirit of low energy scattering phenomena in the absence of coherent structures.

For concreteness, we illustrate the steps of the argument, beginning with a *model oscillator - field system*, closely related to our analysis:

$$\begin{aligned}
i\partial_t A_0 \ &= \ \langle\chi,\overline{\eta(\cdot,t)}\rangle\ A_1^2 e^{-i\omega_{res}t} \ + \ \dots\\
i\partial_t A_1 \ &= \ 2\langle\chi,\eta(\cdot,t)\rangle\ \overline{A_1}A_0 e^{i\omega_{res}t} + \dots\\
i\partial_t\eta \ &= \ -\Delta\eta \ + \ \chi\overline{A_0}A_1^2 e^{-i\omega_{res}t} + \dots\\
\text{where } +\dots \ &= \ \text{dispersive PDE corrections,}
\end{aligned} \tag{5.5}$$

and χ denotes a spatially localized function.

The key contribution to the radiation field, due to resonance (because $0<\omega_{res}\in$ spec$(-\Delta)=[0,\infty)$) is

$$\eta_{\rm res} \sim -i\pi\ e^{-i\omega_{res}t}\ \overline{A_0}A_1^2\ \Im\left(-\Delta-\omega_{res}-i0\right)^{-1}[\chi\ \cdot\ ] \ + \ \dots \tag{5.6}$$

Using (5.6) to approximately close the system for A_0 and A_1 yields the *dispersive normal form*

$$\begin{aligned} i\partial_t A_0 &= (\Lambda_0 + i\Gamma)\,|A_1|^4 A_0 + \ldots \\ i\partial_t A_1 &= (\Lambda_1 - 2i\Gamma)\,|A_0|^2|A_1|^2 A_1 + \ldots, \quad \Gamma > 0 \end{aligned}$$

The precise character of the dynamics is made transparent if we introduce (renormalized) ground state and excited state energies:

$$P_0(t) \sim |A_0(t)|^2, \qquad P_1(t) \sim |A_1(t)|^2. \tag{5.7}$$

Here $\sim$ refers to equality up to near-identity change of variables. For sufficiently large times, $t_1(\Phi_0) \leq t$ we have the *nonlinear master equations:*

$$\frac{dP_0}{dt} \sim \Gamma P_1^2\ P_0 + \ldots, \qquad \frac{dP_1}{dt} \sim -2\Gamma P_1^2\ P_0 + \ldots, \quad \Gamma > 0 \tag{5.8}$$

From this we can show, for generic initial conditions, that as $t \to \pm\infty$ the system

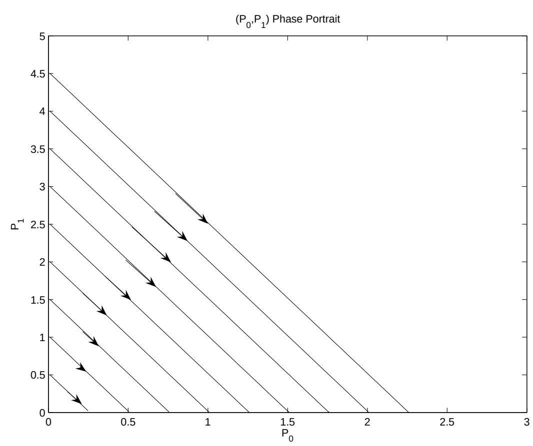

Figure 11. Phase portrait, corresponding to leading terms in (5.8), governing large t ground state selection

crystallizes on the ground state; see Figure 11. Furthermore, it follows from (5.8) that $2P_0(t) + P_1(t) \sim 2P_0(0) + P_1(0)$. Taking the limit as $t \to \infty$ and using the generic decay of $P_1(t)$ gives:

$$P_0(\infty) \;=\; P_0(0) \;+\; \frac{1}{2}P_1(0), \tag{5.9}$$

see Remark 5.1.

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