SIAM J. MATH. ANAL.
Vol. 0, No. 0, pp. 000–000

A REACTION DISPERSION SYSTEM AND RAMAN INTERACTIONS*

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Abstract. We consider the problem of amplification of an optical signal wave with an optical pump wave when both are propagating in the fundamental mode of a single mode optical waveguide. We introduce a system of Ginzburg Landau type and study the radiation loss due to the nonlinear interaction between the signal and the pump waves. The linear dynamics are dispersive, while nonlinearity governs the transfer of energy from the pump wave to the signal wave. The strength of the effect is shown to depend on a dimensionless parameter, which is given by the ratio of the diffraction length and amplification length. If this parameter is small, then the radiation loss is small. This result is established by (i) verifying the absence of resonant terms that can potentially drive the growth of radiative components and (ii) then by estimating the oscillatory (nonresonant) terms by proving the relevant PDE a priori estimates. These estimates require appropriate bounds on the solutions of the PDE, whose only conserved integral is the L^2 norm. However, the special structure of the nonlinear term, dictated by the physics of the Raman effect, implies a weak space-time bound involving the signal and pump intensities. This bound and L^2 conservation are used together with Strichartz (space-time) estimates for the Schrödinger equation to obtain control of stronger classical norms of the signal and pump fields.

Key words. Landau–Ginzburg equations, Raman interaction, nonlinear optics, optical waveguides

AMS subject classifications. 35Q55, 35Q60, 78A60

DOI. 10.1137/S0036141003428172

1. Introduction. In this paper we study a system of nonlinear and dispersive partial differential equations, where nonlinearity is of "reaction" type, i.e., in the absence of dispersion it induces pure energy exchange between the fields. Such systems are reminiscent of reaction-diffusion systems; here the diffusive mechanism is replaced by dispersion. While the latter has been widely studied, the former has received very little attention.

Our reaction-dispersion system arises naturally in mathematical modeling of the stimulated Raman process, but will also arise in other systems (perhaps in somewhat modified form), where two dispersive waves interact nonlinearly, while other nonlinear effects (such as self-phase modulation) and diffusion are negligible. This system can be also considered as a special case of complex Ginzburg–Landau (CGL) system, which has been studied in a different parameter regime [2]. We will often refer to the systems (1.1) and (1.2) as the Raman model, due to their relation to the motivating physical context.

The Raman effect is one where light of one frequency, ω_s ("signal"), is amplified by light of a down shifted frequency, ω_p ("pump"). Taking the energy transfer characteristics of the Raman process into account as well as diffraction leads to the system

*Received by the editors May 21, 2003; accepted for publication (in revised form) August 19, 2004; published electronically DATE.
http://www.siam.org/journals/sima/x-x/42817.html

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of nonlinear evolution equations, discussed in greater detail in Appendix A:

$$\begin{aligned} i\partial_t u_s &= -\Delta u_s + i\epsilon|u_p|^2 u_s, \\ i\partial_t u_p &= -\Delta u_p - i\epsilon|u_s|^2 u_p. \end{aligned} \tag{1.1}$$

More generally, we must include the Kerr effect in (1.1). This would introduce cross-phase and self-phase modulation terms of the types $\alpha_s|u_p|^2 u_s + \beta_s|u_s|^2 u_s$ and $\alpha_p|u_s|^2 u_s + \beta_p|u_p|^2 u_p$ with $\alpha_{s,p}$, $\beta_{s,p}$ real on the right-hand side of (1.1). We remark on the analysis of this more complete model at the end of this section; see Remark 1.1. In a waveguide setting, which is of importance in optical communications, the equations take the form

$$\begin{aligned} i\partial_t u_s &= Hu_s + i\epsilon|u_p|^2 u_s, \\ i\partial_t u_p &= Hu_p - i\epsilon|u_s|^2 u_p, \end{aligned} \tag{1.2}$$

where

$$H = -\Delta + V.$$

Here, $u_s = u_s(x,t)$ and $u_p = u_p(x,t)$ denote, respectively, the signal and pump complex electric field envelopes. Systems (1.1) and (1.2) are valid, assuming the *paraxial approximation.* Δ denotes the Laplace operator with respect to x ($x \in \mathbb{R}^1$ or $x \in \mathbb{R}^2$). The longitudinal coordinate (z), a *time-like* variable with which propagation distance is measured, is denoted by t. In the waveguide setting, the "potential" $V(x)$ is determined by the transverse refractive index profile of the waveguide. The parameter ϵ measures the size of the nonlinear effects relative to the linear effects (e.g., diffraction, dispersion). The particular application to optical communications is discussed in Appendix A.

Our study of systems (1.2) and (1.1) is motivated by a fundamental issue arising in the modeling of the Raman interaction in a waveguide setting. In optical communication applications, the weak signal field whose envelope encodes bits of information is amplified by the strong pump field. This process takes place in an optical fiber waveguide, with one transverse localized state or "guided mode" and radiation modes. Raman amplification of the signal is based on the intended net transfer of energy from the pump to the signal. Physicists have found that good agreement with experiment is achieved by an ODE model, in which one neglects the effect of nonlinear coupling of bound states to radiation modes:

$$\begin{aligned} &\partial_t I_s = \epsilon g_s I_s I_p, \\ &\partial_t I_p = -\epsilon g_p I_s I_p, \\ &I_s \sim |u_s|^2, \ I_p \sim |u_p|^2, \end{aligned} \tag{1.3}$$

where $g_{s,p}$ are coefficients depending on the properties of the fundamental modes, e.g., on frequency and the so-called effective area.

A satisfactory explanation for the above approximation has been lacking; see, for example, [3]. This motivated us to consider the Raman energy transfer problem in the context of the model (1.2). We have found an explanation for the above statement about energy transfer using ideas and methods of resonance and averaging. In particular, in Theorem 3.1 we establish that if the initial field energy, which is not small, is in the guided mode, then this property persists with negligible error on the length scale of physical interest, $\mathcal{O}(\epsilon^{-1})$, and therefore the model (1.3) applies. Note that on the time interval of order ϵ^{-1}, the radiation of size ϵ can grow to become of order $\mathcal{O}(1)$ (since the rate of radiation change is of order $\mathcal{O}(\epsilon)$).

The proof of Theorem 3.1 requires a good understanding of the large time dynamics of the flow defined by (1.2). Thus we consider the question of global existence of the initial value problem for such systems and we have derived results of independent interest. The standard approach to controlling the large time dynamics is to first prove local in time existence of the solutions to the initial value problem in a "natural" Banach space. A "natural" space is often one in which the physically relevant conserved integrals are defined. We formulate initial value problem for system (1.2) as a system of integral equations and prove that there are local solutions using fixed point argument; see sections 2 and 4. Global existence in time then follows from an appropriate a priori bound on the norm of this Banach space. If this norm remains bounded in terms of conserved integrals of the flow, global existence holds.

The ideas we use to prove global existence apply to both systems (1.1) and (1.2). Equations (1.1) and (1.2) have the L^2 (energy) conservation law

$$\mathcal{P}[u_s(t), u_p(t)] \equiv \int_{-\infty}^{+\infty} (|u_s|^2 + |u_p|^2)\, dx = \mathcal{P}[u_s(0), u_p(0)] \equiv \mathcal{P}_0. \tag{1.4}$$

Unfortunately, L^2 is a very weak space in which to control the nonlinear flow. Unlike the nonlinear Schrödinger equation, a Hamiltonian system, (1.2) and (1.1) do not have a second conserved integral (Hamiltonian), which controls $\|\nabla u_{s,p}\|_{L^2}$, and from which sufficient a priori control follows for global existence to hold.

We find that the key to a global existence theory is the following space-time integral a priori bound, which is a consequence of the form of the nonlinear Raman interaction terms:

$$\int_0^T dt \int |u_s|^2 |u_p|^2\, dx \le \frac{1}{2}\mathcal{P}_0. \tag{1.5}$$

Remark 1.1. We believe our theory can be extended to system (1.1) with Kerr effect included. An essential ingredient is the space-time estimate (1.5), which holds for the more general system. However, a more technical analysis is required to obtain closed space-time estimates in the presence of Kerr effect terms. This is work in progress.

Outline of the paper. The paper is structured as follows. We first consider system (1.2) in one space dimension and one time dimension. In section 2 we prove global well-posedness for the solution of the initial value problem. The key to this result are certain a priori estimates, whose point of departure is the L^2 conservation law (1.4) and the space-time a priori bound (1.5). This space-time estimate implies that (1.2) may be viewed as an inhomogeneous system of equations for u_s and u_p, with a source term, which is bounded in a space-time norm. Strichartz estimates [8] are then used to bound u_s and u_p individually in space-time norms and then finally in classical Sobolev norms. In section 3 the energy transfer from the bound mode of a single mode waveguide to radiation modes is studied by estimating nonresonant oscillating terms. Section 4 contains a theory of well-posedness in the case of two transverse dimensions. Finally, there are two appendices: one with a detailed discussion of the motivating application to optical communications and a second in which we prove a normal result, based on the special symmetries of the system, which is of independent interest.

2. Existence theory on $\mathbb{R}^1$. In this section we prove that system (1.2) has a unique global solution in an appropriate (physical) function space. In subsection 2.1 we provide the required operator estimates we shall require. In subsection 2.2 we derive certain a priori bounds which are satisfied by solutions of (1.2). In subsection 2.3 existence in a "weak" space is proved. In section 2.4 it is shown how to extend these results to H^s.

2.1. Estimates for the linear flow. We first introduce the fundamental solution of the Schrödinger equation.

(1) The solution of the initial value problem

$$i\partial_t u = -\partial_x^2 u, \quad u(0,x) = f \tag{2.1}$$

is denoted by $U_0(t)f$ and $U_0(t)$ is called the free propagator.

(2) The solution of the initial value problem

$$i\partial_t u = (-\partial_x^2 + V(x))u = Hu, \quad u(0,x) = f \tag{2.2}$$

is denoted by $U(t)f$.

The operator, H, may have spectrum consisting of continuous and discrete parts, with associated spectral projections P_c and $P_b = I - P_c$. We shall assume that H has finitely many point eigenvalues. Intuitively, on the range of P_c we expect $U(t)$ to behave dispersively in a manner similar to $U_0(t)$. We use dispersive estimates which involve space-time integrals, often referred to as estimates of Strichartz type; see [8] and [9]. The proofs of the space-time estimates for the free Schrödinger equation in the form we use are due to Ginibre and Velo [12] and, in the inhomogeneous case, to Yajima [10] and Cazenave and Weissler [11]. For complete proofs see, for example, Theorems 3.3 and 3.4 of [4].

We now introduce the function spaces and the notation of an *admissible pair* in terms of which the space-time estimates are expressed.

(3) For a real interval I and a Banach space X, we denote by $L^p(I,X)$ the Banach space of functions $u : I \to X$ for which $\int_I \|u(t)\|_X^p \, dt$ is finite.

(4) A pair of real numbers (q,r) is called *admissible* (for dimension $n = 1$) if

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{r}, \quad r \in [2,\infty]. \tag{2.3}$$

We now state Strichartz-type estimates for $U(t)$ for the initial value problem (2.2) and the inhomogeneous problem

$$i\partial_t u = Hu + P_c g. \tag{2.4}$$

THEOREM 2.1. *Assume that V satisfies*

$$\int_{-\infty}^{+\infty} |V(x)|(1+|x|)^{5/2} \, dx < \infty. \tag{2.5}$$

Thus, H has finitely many negative eigenvalues and continuous spectrum extending from 0 to $+\infty$, with associated spectral projections P_b and P_c.[1]

[1]Finiteness of negative discrete spectrum and continuity of spectrum from $[0,\infty)$ follows from sufficient decay of the potential.

Let (q,r) be an admissible pair. For any $f\in L^2$, $U_0(t)f$ and $U(t)P_cf$ are of class $L^q(\mathbb{R},L^r)$ and satisfy the estimates

$$\begin{aligned}\|U_0(\cdot)f\|_{L^q([0,T],L^r)}&\le C\|f\|_{L^2},\\ \|U(\cdot)P_cf\|_{L^q([0,T],L^r)}&\le C\|f\|_{L^2},\end{aligned}\tag{2.6}$$

where C depends only on q.

THEOREM 2.2. *Let V satisfy* (2.5) *and let (γ,ρ) be an admissible pair $(2/\gamma = 1/2-1/\rho)$, $f\in L^{\gamma'}([0,T],L^{\rho'})$, where (γ',ρ') is conjugate to (γ,ρ). Then for any admissible pair (q,r) $(2/q=1/2-1/r)$*

$$\begin{aligned}\left\|\int_0^t U_0(t-\tau)f(\tau)d\tau\right\|_{L^q([0,T],L^r)}&\le C\|f\|_{L^{\gamma'}([0,T],L^{\rho'})},\\ \left\|\int_0^t U(t-\tau)P_cf(\tau)d\tau\right\|_{L^q([0,T],L^r)}&\le C\|f\|_{L^{\gamma'}([0,T],L^{\rho'})},\end{aligned}\tag{2.7}$$

where C depends only on q, γ.

The proofs rely on the $L^p-L^{p'}$ estimates for the free Schrödinger equation

$$\|U_0(t)f\|_{L^p}\le(4\pi|t|)^{-\frac12+\frac1p}\|f\|_{L^{p'}}.\tag{2.8}$$

In the case of a Schrödinger equation with a potential in one space dimension the analogous estimate for $U(t)P_c$ was established by Weder [5]. Adapting the proofs in [4], for the free propagator $U_0(t)$, and using Weder's estimate, one obtains the Strichartz-type estimates for Schrödinger equation with a potential.

The following corollary, easily derived from the previous estimates by a change of variables, concerns the dependence of space-time estimates on a parameter, which arises when we rescale (1.2).

COROLLARY 2.3. *Consider a one-parameter family of Schrödinger initial value problems*

$$\begin{aligned}i\partial_t u&=\beta Hu,\\ u(x,0)&=f(x),\end{aligned}\tag{2.9}$$

where $\beta\in[\beta_0,\infty)$ with $\beta_0>0$. Assume that the potential, $V(x)$, satisfies (2.5). *Then, the conclusions of Theorems* 2.1 *and* 2.2 *hold with β-dependent constants. In particular, if $U_\beta(t)f=U(\beta t)f$ denotes the solution of the initial value problem* (2.9), *then*

$$\|U_\beta(\cdot)P_cf\|_{L^q([0,T],L^r)}\le C_1(\beta,q)\|f\|_{L^2},\tag{2.10}$$

$$\left\|\int_0^t U_\beta(t-\tau)P_cf(\tau)d\tau\right\|_{L^q([0,T],L^r)}\le C_2(\beta,q,\gamma')\|f\|_{L^{\gamma'}([0,T],L^{\rho'})},\tag{2.11}$$

where

$$C_1(\beta,q)=C_{11}(q)\beta^{-\frac1q},\qquad C_2(\beta,q,\gamma')=C_{22}(q,\gamma')\beta^{-1-\frac1q+\frac1{\gamma'}}.\tag{2.12}$$

In [6], Weder proves continuity of wave operators for (2.2). We use a special case of the main theorem from [6].

PROPOSITION 2.4. *Let V satisfy* (2.5). *Then, there exist wave operators Ω and Ω^* satisfying*

$$\Omega(I-\Delta)\Omega^* = (I+H)P_c. \tag{2.13}$$

These operators are continuous in H^1:

$$\|\Omega f\|_{H^1} \le C\|f\|_{H^1}, \qquad \|\Omega^* f\|_{H^1} \le C\|f\|_{H^1}. \tag{2.14}$$

2.2. A priori space-time estimates. Essential in the proof that system (1.2) defines the solution globally in time and that the solution does not develop singularities are a priori estimates which we now derive. For convenience, we rescale the time $\epsilon t = t_{\rm new}$, so the uniform bound on the interval $t_{\rm new} \in [0,T]$ will correspond to the interval $[0, T/\epsilon]$ in old time. We will continue to use t as the time variable

$$\partial_t u_s + i\beta H u_s = |u_p|^2 u_s, \tag{2.15}$$

$$\partial_t u_p + i\beta H u_p = -|u_s|^2 u_p, \tag{2.16}$$

where $\beta := \epsilon^{-1}$ is a dispersion/diffraction parameter and $\beta \in [\beta_0, \infty]$.

Multiplication of (2.15) by $\overline{u_s}$, taking the real part of the resulting equation and integrating over all gives

$$\frac{d}{dt}\int |u_s|^2\,dx = 2\int |u_s|^2|u_p|^2\,dx. \tag{2.17}$$

Similarly, multiplication of (2.16) by $\overline{u_p}$ yields

$$\frac{d}{dt}\int |u_p|^2\,dx = -2\int |u_s|^2|u_p|^2\,dx. \tag{2.18}$$

Equations (2.17) and (2.18) express the gain of signal energy at the expense of pump energy and the depletion of pump energy at the expense of signal energy; see (1.2).

Addition of (2.17) and (2.18) yields the conservation law

$$\frac{d}{dt}\int |u_s|^2 + |u_p|^2\,dx = 0 \tag{2.19}$$

or

$$\int |u_s|^2 + |u_p|^2\,dx = \int |u_s(0)|^2 + |u_p(0)|^2\,dx \equiv \mathcal{P}_0. \tag{2.20}$$

An important step in our analysis is to use the *energy dissipation identity* (2.18). Integration of (2.18) over time interval $[0,T]$ yields

$$2\int_0^T\int |u_s(x,t)|^2|u_p(x,t)|^2\,dx\,dt = \int |u_p(x,0)|^2\,dx - \int |u_p(x,t)|^2\,dx. \tag{2.21}$$

A simple consequence of (2.21) is the following *space-time bound*.

PROPOSITION 2.5 (a priori space-time estimate). *Let (u_s, u_p) denote a solution of* (2.15)–(2.16) *in the sense of Theorem* 2.9. *Then,*

$$\int_0^T\int |u_s|^2|u_p|^2\,dx\,dt \le \frac{1}{2}\mathcal{P}_0. \tag{2.22}$$

Remark 2.6. Since in Theorem 2.9 we assume that the initial conditions are merely L^2, strictly speaking the above derivation of the bound (2.22) is not valid, since the manipulations require that the solution is a classical solution of the PDE, i.e., pointwise differentiable in space and time. At the end of section 2.3, we sketch a proof of (2.22) for the case of L^2 initial data.

Equations (2.16) and (2.15) can both be viewed as inhomogeneous Schrödinger equations of the form

$$i\partial_t U = \beta H U + g \tag{2.23}$$

with a source term g given by

$$g = |u_p|^2 u_s \quad \text{or} \quad g = -|u_s|^2 u_p. \tag{2.24}$$

We next show that the a priori estimate (2.22) implies bounds on the source terms (2.24) which are suitable for application of the inhomogeneous Strichartz estimate of Theorem 2.2.

PROPOSITION 2.7. *Let g denote either term in* (2.24). *Then, for any $T > 0$ and any $\kappa \in [0, 2]$,*

$$\int_0^T \left| \int |g| \, dx \right|^\kappa dt \le 2^{-\frac{\kappa}{2}} \mathcal{P}_0^\kappa T^{\frac{2-\kappa}{2}}. \tag{2.25}$$

In particular, for $\kappa \in [1, 2]$ and $g \in L^\kappa([0,T], L^1)$,

$$\|g\|_{L^\kappa([0,T],L^1)} \le 2^{-\frac{1}{2}} \mathcal{P}_0 T^{\frac{2-\kappa}{2k}}. \tag{2.26}$$

To prove Proposition 2.7, let $g = -|u_s|^2 u_p$. The proof for $g = |u_p|^2 u_s$ is analogous. By the Cauchy–Schwarz inequality,

$$\left| \int |u_s|^2 |u_p| \, dx \right| \le \left(\int |u_s|^2 \, dx \right)^{\frac{1}{2}} \left(\int |u_s|^2 |u_p|^2 \, dx \right)^{\frac{1}{2}}. \tag{2.27}$$

Squaring this inequality and integrating the result over the time interval $[0, T]$ yields, after using that the L^2 norm of u_s is bounded by $\mathcal{P}_0$, that

$$\int_0^T \left| \int |u_s|^2 |u_p| \, dx \right|^2 dt \le \frac{1}{2} \mathcal{P}_0^2. \tag{2.28}$$

This handles the case $\kappa = 2$. For $\kappa = 0$ the trivial bound of T holds. The result follows by interpolation.

Using the a priori bounds of Proposition 2.7 we can now estimate the solution in $L^q([0,T], L^r)$ spaces, for admissible (q, r).

THEOREM 2.8 (a priori bounds in $L^q([0,T], L^r)$). *Let (q, r) be an admissible pair; see* (2.3). *Then, any solution (u_s, u_p) of system* (2.15)–(2.16) *for $0 \le t \le T$ satisfies the bounds*

$$\begin{aligned} \|u_s\|_{L^q([0,T],L^r)} &\le C \left(\mathcal{P}_0 + \mathcal{P}_0^{\frac{1}{2}} \right) (T+1), \\ \|u_p\|_{L^q([0,T],L^r)} &\le C \left(\mathcal{P}_0 + \mathcal{P}_0^{\frac{1}{2}} \right) (T+1), \end{aligned} \tag{2.29}$$

where C depends only on q, γ, and β_0.

Proof of Theorem 2.8. We estimate $\|u_s\|_{L^q([0,T],L^r)}$. The corresponding estimate for u_p is similar. Equation (2.15) can be rewritten as an equivalent integral equation:

$$u_s(t) = U_\beta(t)u_s(0) + \int_0^t U_\beta(t-\tau)|u_p(\tau)|^2 u_s(\tau)d\tau. \tag{2.30}$$

To estimate the space-time norm of u_s, we apply Corollary 2.3 to the continuous spectral part and estimate the finite-dimensional (bound state) part of u_s separately. For ease of presentation we assume that H has only one spatial localized bound state solution, $\phi(x)$; the proof is the same for any finite number of bound states. Estimation of (2.30) using Corollary 2.3 gives

$$\begin{aligned}
&\|u_s\|_{L^q([0,T],L^r)} \\
&\leq C\left\|U_\beta(t)u_s(0) + \int_0^t U_\beta(t-\tau)|u_p(\tau)|^2 u_s(\tau)d\tau\right\|_{L^q([0,T],L^r)} \\
&\leq C_1(\|u_s(0)\|_{L^2} + \langle u_s(0),\phi\rangle\|\phi\|_{L^q([0,T],L^r)}) + C_2\|u_p^2u_s\|_{L^{\gamma'}([0,T],L^{\rho'})} \\
&\quad + \left\|\int_0^t U_\beta(t-\tau)\langle|u_p(\tau)|^2u_s(\tau),\phi\rangle\phi d\tau\right\|_{L^q([0,T],L^r)} \\
&\leq C_1(1+T^{q^{-1}}\|\phi\|_{L^2}\|\phi\|_{L^r})\|u_s(0)\|_{L^2} + C_2\|u_p^2u_s\|_{L^{\gamma'}([0,T],L^{\rho'})} \\
&\quad + \left\|\int_0^t e^{-i\lambda(t-\tau)}\langle|u_p(\tau)|^2u_s(\tau),\phi\rangle\phi d\tau\right\|_{L^q([0,T],L^r)}.
\end{aligned} \tag{2.31}$$

The last integral is estimated as follows: using that

$$\begin{aligned}
\left|\int_0^t \langle|u_p(\tau)|^2u_s(\tau),\phi\rangle\phi d\tau\right| &\leq \left|\int_0^t |\phi(x)|\int|\phi(x)|\cdot|u_p^2(x,\tau)u_s(x,\tau)|\,dxd\tau\right| \\
&\leq |\phi(x)|\int_0^t \|\phi\|_{L^\rho}\cdot\|u_p^2u_s\|_{L^{\rho'}}d\tau \\
&\leq |\phi(x)|\cdot\|\phi\|_{L^\gamma([0,T],L^\rho)}\cdot\|u_p^2u_s\|_{L^{\gamma'}([0,T],L^{\rho'})}
\end{aligned}$$

we have

$$\begin{aligned}
&\left\|\int_0^t e^{-i\lambda(t-\tau)}\langle|u_p(\tau)|^2u_s(\tau),\phi\rangle\phi d\tau\right\|_{L^q([0,T],L^r)} \\
&\quad\leq \|\phi\|_{L^\gamma([0,T],L^\rho)}\cdot\|\phi\|_{L^q([0,T],L^r)}\cdot\|u_p^2u_s\|_{L^{\gamma'}([0,T],L^{\rho'})} \\
&\quad= \|\phi\|_{L^\rho}\cdot\|\phi\|_{L^r}\cdot\|u_p^2u_s\|_{L^{\gamma'}([0,T],L^{\rho'})}\cdot T^{\gamma^{-1}+q^{-1}}.
\end{aligned}$$

Now Proposition 2.7 implies a bound on $\|u_p^2u_s\|_{L^{\gamma'}([0,T],L^{\rho'})}$, where $\rho' = 1$ and $\gamma' \in [0,2]$. Note that the exponents γ and ρ, dual to γ' and $\rho' = 1$, form an admissible pair provided $\gamma = 4$ and $\gamma' = 4/3$.[2]

Setting $\gamma' = 4/3$ and $\rho' = 1$ in (2.31) and applying Proposition 2.7 with $\kappa = \gamma' = 4/3$ implies

$$\begin{aligned}
\|u_s\|_{L^q([0,T],L^r)} &\leq C_1(1+T^{1/q}\|\phi\|_{L^2}\|\phi\|_{L^r})\|u_s(0)\|_{L^2} \\
&\quad + C_2\|u_p^2u_s\|_{L^{\frac43}([0,T],L^1)} + T^{1/4+1/q}\|\phi\|_{L^\infty}\|\phi\|_{L^r}\|u_p^2u_s\|_{L^{\frac43}([0,T],L^1)} \\
&\leq C_1(1+T^{\frac1q})\mathcal{P}_0^{\frac12} + C_2\mathcal{P}_0T^{\frac14} + C_3\mathcal{P}_0T^{\frac12+\frac1q} \leq C(\mathcal{P}_0+\mathcal{P}_0^{\frac12})(T+1),
\end{aligned} \tag{2.32}$$

[2]Indeed, since $1/\rho' + 1/\rho = 1$, $1/\gamma + 1/\gamma' = 1$, and $2/\gamma = 1/2 - 1/\rho$, we have $\rho = \infty$, $\gamma = 4$, and $\gamma' = 4/3$.

where C_1, C_2, C_3 depend on the corresponding norms of ϕ and we used that $4 \leq q \leq \infty$. This completes the proof of Theorem 2.8. $\square$

2.3. Existence in $L^8(\mathbb{R}_+, L^4) \cap L^\infty(\mathbb{R}_+, L^2)$. In this subsection we prove existence of solutions in a function space $\mathcal{X}(T)$ defined by

$$\mathcal{X}(T) = L^8([0,T], L^4) \cap L^\infty([0,T], L^2) \tag{2.33}$$

with the norm

$$\|u\|_{\mathcal{X}(T)} = \|u\|_{L^8([0,T],L^4)} + \|u\|_{L^\infty([0,T],L^2)} \tag{2.34}$$

$$= \|u\|_{8,4} + \|u\|_{\infty,2}, \tag{2.35}$$

the latter being written when there is no ambiguity. For the two-dimensional field (u_s, u_p), we naturally define the norm

$$\|u_s, u_p\|_{\mathcal{X}(T)} = \|u_s\|_{\mathcal{X}(T)} + \|u_p\|_{\mathcal{X}(T)} = \|u_s\|_{8,4} + \|u_s\|_{\infty,2} + \|u_p\|_{8,4} + \|u_p\|_{\infty,2}. \tag{2.36}$$

Since Theorem 2.8 gives a priori control of solutions in $L^q([0,T], L^r)$ spaces for any admissible pairs (q,r), it is natural to obtain a local existence theorem in a space, where the maximal time of existence depends only on $L^q([0,T], L^r)$ bounds. Then, global existence follows from Theorem 2.8; see the discussion below.

Define the mapping

$$(u_s, u_p) \mapsto A_\beta(u_s, u_p) \equiv \left(A_\beta^{(s)}(u_s, u_p), A_\beta^{(p)}(u_s, u_p) \right), \tag{2.37}$$

where

$$A_\beta^{(s)}(u_s, u_p) = U_\beta(t) u_s(0) + \int_0^t U_\beta(t-\tau) |u_p(\tau)|^2 u_s(\tau) d\tau, \tag{2.38}$$

$$A_\beta^{(p)}(u_s, u_p) = U_\beta(t) u_p(0) - \int_0^t U_\beta(t-\tau) |u_s(\tau)|^2 u_p(\tau) d\tau. \tag{2.39}$$

Then, the above evolution equation has the equivalent formulation as a fixed point problem.

For initial data $(u_s(0), u_p(0)) \in L^2$, find $(u_s, u_p) \in \mathcal{X}(T)$ for some $T > 0$ such that

$$(u_s, u_p) = A_\beta(u_s, u_p). \tag{2.40}$$

Our local existence theorem is the following.

THEOREM 2.9 (local existence).

(1) *Given initial data* $(u_s(0), u_p(0)) \in L^2$, *there exist a* $T > 0$ *and a unique solution* $(u_s, u_p) \in \mathcal{X}(T)$ *of* (2.40). *This local solution satisfies the a priori estimate* (2.22).

(2) *Let* $T_{\max} > 0$ *denote the maximal time of existence. Either* $T_{\max} = \infty$ *(global existence in time) or*

$$T_{\max} < \infty \quad \text{and} \quad \limsup_{T \to T_{\max}} \|(u_s, u_p)\|_{\mathcal{X}(T)} = \infty. \tag{2.41}$$

Using the local existence theory of Theorem 2.9 and the a priori bounds of Theorem 2.8, we have $T_{\max} = \infty$. Therefore, the following result holds.

THEOREM 2.10 (global existence). *For any initial data* $(u_s(0), u_p(0))$ *in* L^2, (2.40) *has a unique global solution of class* $L^8(\mathbb{R}_+, L^4) \cap L^\infty(\mathbb{R}_+, L^2)$.

We need only prove the local existence Theorem 2.9. The proof follows from the next two propositions in which we establish that for L^2 initial conditions and T sufficiently small,

(i) the transformation A_β maps a specified ball $\mathcal{B}(T)$ in $\mathcal{X}(T)$ into itself and that

(ii) A_β is a contraction mapping on $\mathcal{B}(T)$.

PROPOSITION 2.11. *Let* $(u_s(0), u_p(0))$ *be in* L^2. *Define the ball in* $\mathcal{X}(T)$

$$\mathcal{B}(T) = \{(u_s, u_p) \in \mathcal{X}(T) : \|(u_s, u_p)\|_{\mathcal{X}(T)} \le 2C(\|u_s(0)\|_2 + \|u_p(0)\|_2)\}, \tag{2.42}$$

where C *is found in the proof below. There exists* $T_0 > 0$ *such that for any* $T < T_0$, *the ball is mapped into itself, i.e.,* $A_\beta(\mathcal{B}(T)) \subset \mathcal{B}(T)$ *for any* $\beta \in [\beta_0, \infty]$, *with* T_0 *depending on* β_0.

Proof of Proposition 2.11. We estimate the action of $A_\beta^{(s)}$. The estimation for $A_\beta^{(p)}$ is similar.

Following the proof of Theorem 2.8, we obtain a similar inequality

$$\begin{aligned}\|A_\beta^{(s)}(u_s, u_p)\|_{q,r} &\le \|U_\beta(t)u_s(0)\|_{q,r} + \left\|\int_0^t U_\beta(t-\tau)|u_p|^2 u_s\right\|_{q,r} \\ &\le C_1(1+T)\|u_s(0)\|_2 + C_2(1+T)\|u_p^2 u_s\|_{\gamma',\rho'},\end{aligned}$$

where C_1, C_2 depend on ϕ.

Estimation of the cubic term proceeds as follows. By the Cauchy–Schwarz inequality,

$$\begin{aligned}\|u_p^2 u_s\|_{\gamma',\rho'} &= \left[\int_0^T \left(\int |u_p^2 u_s|^{\rho'}\,dx\right)^{\gamma'/\rho'} dt\right]^{1/\gamma'} \\ &\le \left[\int_0^T \left(\int |u_p|^{4\rho'}\,dx\right)^{\gamma'/\rho'} dt\right]^{1/2\gamma'} \left[\int_0^T \left(\int |u_s|^{2\rho'}\,dx\right)^{\gamma'/\rho'} dt\right]^{1/2\gamma'}.\end{aligned}$$

Set $\rho' = 1$ and therefore $\gamma' = 4/3$. Then the last expression becomes

$$= \left[\int_0^T \left(\int |u_p|^4\,dx\right)^{4/3} dt\right]^{3/8} \left[\int_0^T \left(\int |u_s|^2\,dx\right)^{4/3} dt\right]^{3/8} \tag{2.43}$$

and by Hölder's inequality, applied to each factor, we have the bound

$$\begin{aligned}&\le \left[\left[\int_0^T \left(\int |u_p|^4\,dx\right)^2 dt\right]^{(2/3)\cdot(3/8)} \left[\int_0^T 1^3\,dt\right]^{(1/3)}\right]^{3/8} \left[\sup_t \|u_s(t)\|_2^{2\cdot(4/3)} T\right]^{3/8} \\ &\quad\le \|u_p\|_{8,4}^2 \|u_s\|_{\infty,2} T^{1/2}.\end{aligned}$$

Adding up all the terms, we obtain

$$\begin{aligned}\left\|A_\beta^{(s)}(u_s, u_p)\right\|_{\mathcal{X}(T)} &\le C(\|u_s(0)\|_2 + \|u_p\|_{8,4}^2 \|u_s\|_{\infty,2} T^{\frac{1}{2}})(1+T), \\ \left\|A_\beta^{(p)}(u_s, u_p)\right\|_{\mathcal{X}(T)} &\le C(\|u_p(0)\|_2 + \|u_s\|_{8,4}^2 \|u_p\|_{\infty,2} T^{\frac{1}{2}})(1+T),\end{aligned}$$

where $C = \max\{C_1, C_2\}$. Finally, combining the last two terms, we have

$$\|A_\beta(u_s,u_p)\|_{\mathcal{X}(T)} \le C(\|u_s(0)\|_2 + \|u_p(0)\|_2 + \|(u_s,u_p)\|^3_{\mathcal{X}(T)} T^{\frac{1}{2}})(1+T). \tag{2.44}$$

Assume now that $\|(u_s,u_p)\|_{\mathcal{X}(T)} \le 2C(\|u_s(0)\|_2 + \|u_p(0)\|_2)$ and that T is sufficiently small; then A_β maps $\mathcal{B}(T)$ into itself. This completes the proof of Proposition 2.11. ∎

Proposition 2.12. *For $T < T_1 \le T_0$ sufficiently small, the transformation, A_β, is a contraction on $\mathcal{B}(T)$. That is,*

$$\|A_\beta(u_s,u_p) - A_\beta(v_s,v_p)\|_{\mathcal{X}(T)} \le q\|(u_s - v_s, u_p - v_p)\|_{\mathcal{X}(T)}, \tag{2.45}$$

where $0 < q < 1$.

Proof of Proposition 2.12. Consider the first component of the map. By Corollary 2.3,

$$\begin{aligned}\|A^{(s)}_\beta(u_s,u_p) - A^{(s)}_\beta(v_s,v_p)\|_{q,r} &\le C_2\||u_p|^2 u_s - |v_p|^2 v_s\|_{\frac{4}{3},1}\\ &\le \|u_p^2(u_s - v_s)\|_{\frac{4}{3},1} + \|u_p v_s(u_p - v_p)\|_{\frac{4}{3},1}\\ &\quad + \|v_p v_s(u_p - v_p)\|_{\frac{4}{3},1}.\end{aligned}$$

These terms are all estimated in a similar manner. We focus on the second term. First, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\|u_p v_s(u_p - v_p)\|_{\frac{4}{3},1} &\le \left[\int_0^T \left(\int |u_p|^2|v_s|^2\,dx\right)^{4/3} dt\right]^{3/8} \left[\int_0^T \left(\int |u_p - v_p|^2 dx\right)^{4/3} dt\right]^{3/8}\\ &\le \left[\int_0^T \left(\int |u_p|^2|v_s|^2\,dx\right)^{4/3} dt\right]^{3/8} T^{3/8}\|u_p - v_p\|_{\infty,2}.\end{aligned}$$

Another application of the Cauchy–Schwarz inequality to the spatial integral in the first factor in the previous expression and then Hölder's inequality to the time integral gives

$$\begin{aligned}&\left[\int_0^T \left(\int |u_p|^4\,dx\right)^{4/3} dt\right]^{3/16} \left[\int_0^T \left(\int |v_s|^4\,dx\right)^{4/3} dt\right]^{3/16} T^{3/8}\|u_p - v_p\|_{\infty,2}\\ &\quad\le \|u_p\|_{8,4}\|v_s\|_{8,4} T^{\frac{2}{16}} T^{\frac{3}{8}}\|u_p - v_p\|_{\infty,2} = \|u_p\|_{8,4}\|v_s\|_{8,4} T^{1/2}\|u_p - v_p\|_{\infty,2}\\ &\quad\le \|u_p\|_{8,4}\|v_s\|_{8,4} T^{1/2}\|(u_s - v_s, u_p - v_p)\|_{\mathcal{X}(T)}.\end{aligned}$$

Adding the estimates for

$$\left\|A^{(s)}_\beta(u_s,u_p) - A^{(s)}_\beta(v_s,v_p)\right\|_{\mathcal{X}(T)} \quad \text{and} \quad \left\|A^{(p)}_\beta(u_s,u_p) - A^{(p)}_\beta(v_s,v_p)\right\|_{\mathcal{X}(T)} \tag{2.46}$$

and choosing, if necessary, $T_1 < T_0$, we obtain the contraction estimate. This completes the proof. ∎

Remark 2.13. Finally, we give a proof of the space-time bound for solutions with data in L^2.

(1) *Existence of solutions for very regular data.* Using that H^s is an algebra for $s > \frac{1}{2}$, it is standard to prove, by a contraction mapping argument, that for

data in H^s with $s \geq s_0 \geq 2$ there is a unique classical solution. However, this argument requires differentiation of the original system and as a consequence $V(x)$. This requires imposing unnecessary smoothness assumptions on V. To avoid such restrictions on V, we observe that the norms $\|\cdot\|_{H^2} = \|(I+H)P_c\cdot\|_{L^2}$ and $\|(I-\Delta)\cdot\|_{L^2}$ are equivalent, by Proposition 2.4; see also Proposition 2.15. Therefore, applying $(I+H)P_c$, which commutes with $U_\beta(\cdot)$, to the integral equation for (u_s, u_p), we can use standard estimates to obtain a classical solution. An argument of this type is implemented in section 2.4. Therefore, by the computation of section 2.2, this classical solution satisfies (2.22).

(2) *Continuity of solutions with respect to variations in the initial data.* Let (u_s, u_p) denote the solution corresponding to data $(u_s(0), u_p(0))$ and (v_s, v_p) denote the solution corresponding to data $(v_s(0), v_p(0))$. Both of these are fixed points of the operator A_β (see (2.37)) with the corresponding data. By the same estimate as in the proof of Proposition 2.12 we have (2.45) plus an additional data term on the right-hand side: $\|(u_s(0) - v_s(0), u_p(0) - v_p(0))\|_{L^2}$, where the Strichartz estimate for the free propagator is applied to the difference of initial conditions. In other words,

$$\|(u_s, u_p) - (v_s, v_p)\|_{q,r} \leq C\|(u_s(0) - v_s(0), u_p(0) - v_p(0))\|_{L^2}. \tag{2.47}$$

(3) *Convergence.* Finally, take a sequence of initial data in $H^s, s \geq s_0 \geq 2$, which converges in L^2 to a limit. For each member of this sequence, the solution satisfies the space-time bound (2.22). The right-hand side of (2.22) converges by convergence in L^2 of the data and the left-hand side of (2.22) converges by (2.47). Therefore, (2.22) holds on the interval of existence for any solution with L^2 data.

2.4. Existence in $H^1(\mathbb{R}^1_+)$. We consider the existence theory in H^1. In this section we prove the following theorem.

THEOREM 2.14. *Let* $(u_s(0), u_p(0))$ *be in* H^1 *and let the potential* V *satisfy* (2.5). *Then there exists a unique global solution for system* (1.2) *in* $L^\infty(\mathbb{R}^1_+, H^1)$.

We first observe that our proof of local existence, via the contraction mapping principle, extends to the space

$$\mathcal{X}_1(T) \equiv C([0,T], H^1) \cap \mathcal{X}(T). \tag{2.48}$$

In particular, one needs only to prove that A_β maps a ball to a ball in this smaller space and it is a contraction mapping there, for $T < T_2$, where $T_2 \leq T_1 \leq T_0$. This can be proven by applying $(H+I)^{\frac{1}{2}}P_c$ to the equations, using equivalence of norms (in the appropriate spaces): $\|(H+I)^{\frac{1}{2}}P_c \cdot\|_{L^2}$ and $\|\cdot\|_{H^1}$ and carrying out the standard energy estimates. We use $(H+I)P_c$ rather than $I-\Delta$ because functions of H commute with H and thus we avoid differentiation of the potential $V(x)$. Otherwise, we would require bounds on norms of $\partial_x V$.

If $T^*_{\max}$ denotes the maximal time of existence for the solution in $\mathcal{X}_1(T)$, then in view of the a priori estimates in $\mathcal{X}(T)$, global existence ($T^*_{\max} = \infty$) will follow from a priori bounds on (u_s, u_p) in H^1.

Let

$$\mathcal{A}_c \equiv (I+H)P_c. \tag{2.49}$$

Applying to system (2.15)–(2.16) operator $\mathcal{A}_c^{1/2}$, we obtain the inequality

$$\frac{\partial}{\partial t}\int (|\mathcal{A}_c^{1/2}u_s|^2 + |\mathcal{A}_c^{1/2}u_s|^2)dx \tag{2.50}$$

$$\leq 2\left|\int \left(\overline{\mathcal{A}_c^{1/2}u_s}\mathcal{A}_c^{1/2}(|u_p|^2u_s) - \overline{\mathcal{A}_c^{1/2}u_p}\mathcal{A}_c^{1/2}(|u_s|^2u_p)\right)\, dx\right|. \tag{2.51}$$

Proposition 2.15. *Assume that V satisfies condition* (2.5). *Then the operator $\mathcal{A}_c^{1/2}(I-\Delta)^{-1/2}$ and its inverse are bounded in L^2; that is, for any $f \in L^2$ both $\mathcal{A}_c^{1/2}(I-\Delta)^{-1/2}f$ and $(I-\Delta)^{1/2}\mathcal{A}_c^{-1/2}f$ are bounded in L^2 and*

$$\|\mathcal{A}_c^{1/2}(I-\Delta)^{-1/2}f\|_{L^2} \le C\|f\|_{L^2}, \tag{2.52}$$

$$\|(I-\Delta)^{1/2}\mathcal{A}_c^{-1/2}f\|_{L^2} \le C\|f\|_{L^2}. \tag{2.53}$$

Proof. This proposition states that the $\|(I-\Delta)^{\frac{1}{2}}\|_{L^2}$ norm and $\|\mathcal{A}_c^{1/2}\cdot\|_{L^2}$ are equivalent. Then our strategy will be similar to the proof in the potential-free case (as if $\mathcal{A}_c^{1/2}$ were ∂_x).

We prove the proposition using Weder's result on the continuity of wave operators [6]. Under the conditions stated above, Weder proves that there exists wave operator Ω such that

$$\Omega(I-\Delta)\Omega^* = \mathcal{A}_c,$$

where Ω is a bounded continuous operator on H^1. Then, taking the square root, we obtain

$$\Omega(I-\Delta)^{-1/2}\Omega^* = \mathcal{A}_c^{-1/2}. \tag{2.54}$$

The square root exists since $\mathcal{A}_c = (I+H)P_c$ is a positive operator on the subspace corresponding to the continuous spectrum.

Now it is easy to verify (2.52):

$$\begin{aligned}\|\mathcal{A}_c^{1/2}(I-\Delta)^{-1/2}f\|_{L^2} &= \|\Omega(I-\Delta)^{1/2}\Omega^*(I-\Delta)^{-1/2}f\|_{L^2}\\ &= \|(I-\Delta)^{1/2}\Omega^*(I-\Delta)^{-1/2}f\|_{L^2} \le C\|\Omega^*(I-\Delta)^{-1/2}f\|_{H^1}\\ &\le C\|(I-\Delta)^{-1/2}f\|_{H^1} \le C\|f\|_{L^2},\end{aligned}$$

where we have used that Ω, Ω^* are isometries in L^2 and continuous in H^1. The other inequality (2.53) can be proved similarly. ◻

Corollary 2.16. *Let $f \in H^1 \cap \mathrm{Range}(P_c)$. Then*

$$\|\mathcal{A}_c^{1/2}f\|_2 \le C\|(I-\Delta)^{1/2}f\|_2, \tag{2.55}$$

$$\|(I-\Delta)^{1/2}f\|_2 \le \|\mathcal{A}_c^{1/2}f\|_2. \tag{2.56}$$

Proof. Let $f = (I-\Delta)^{1/2}g$ in (2.52), with $g \in H^1 \cap \mathrm{Range}(P_c)$. Then we have

$$\|\mathcal{A}_c^{1/2}g\|_{L^2} \le \|(I-\Delta)^{1/2}g\|_{L^2}.$$

To prove the other inequality (2.56), we write

$$\|(I-\Delta)^{1/2}f\|_{L^2} = \|(I-\Delta)^{1/2}\mathcal{A}_c^{-1/2}\mathcal{A}_c^{1/2}f\|_{L^2} \le \|\mathcal{A}_c^{1/2}f\|_{L^2}. \qquad ◻$$

Proof of Theorem 2.14. First, using the above proposition, we estimate

$$\begin{aligned}&\left|\int\left(\overline{\mathcal{A}_c^{1/2}u_s}\mathcal{A}_c^{1/2}(|u_p|^2u_s)\right)\,dx\right| \le \|\mathcal{A}_c^{1/2}u_s\|_2\cdot\|\mathcal{A}_c^{1/2}(|u_p|^2u_s)\|_2\\ &\quad= \|\mathcal{A}_c^{1/2}u_s\|_2\cdot\|\mathcal{A}_c^{1/2}(I-\Delta)^{-1/2}(I-\Delta)^{1/2}(|u_p|^2u_s)\|_2\\ &\quad\le \|\mathcal{A}_c^{1/2}u_s\|_2\cdot\|\mathcal{A}_c^{1/2}(I-\Delta)^{-1/2}\|_{\mathcal{B}(L^2,L^2)}\cdot\|(I-\Delta)^{1/2}(|u_p|^2u_s)\|_2.\end{aligned}$$

Using Leibnitz formula for the fraction, see [7],

$$\|(I-\Delta)^{1/2}(fg)\|_2 \le \|f\|_\infty \cdot \|(I-\Delta)^{1/2}g\|_2 + \|(I-\Delta)^{1/2}f\|_2 \cdot \|g\|_\infty,$$

we obtain

$$\|(I-\Delta)^{1/2}(|u_p|^2 u_s)\|_2 \le \|u_p\|_\infty^2 \|(I-\Delta)^{1/2}u_s\|_2 + 2\|u_p\|_\infty \|u_s\|_\infty \|(I-\Delta)^{1/2}u_p\|_2.$$

Combining the last two estimates, we obtain

$$\begin{aligned}
&\left| \int \left(\overline{\mathcal{A}_c^{1/2} u_s} \mathcal{A}_c^{1/2}(|u_p|^2 u_s) \right) dx \right| \\
&\quad \le C\|\mathcal{A}_c^{1/2}u_s\|_2 \cdot (\|u_p\|_\infty^2 + \|u_s\|_\infty^2)(\|(I-\Delta)^{1/2}u_p\|_2 + \|(I-\Delta)^{1/2}u_s\|_2) \\
&\quad \le C\|\mathcal{A}_c^{1/2}u_s\|_2 \cdot (\|u_p\|_\infty^2 + \|u_s\|_\infty^2)(\|(I-\Delta)^{1/2}P_c u_p\|_2 + \|(I-\Delta)^{1/2}\langle u_p, \phi\rangle \phi\|_2 \\
&\qquad\qquad + \|(I-\Delta)^{1/2}P_c u_s\|_2 + \|(I-\Delta)^{1/2}\langle u_s, \phi\rangle \phi\|_2) \\
&\quad \le C(\|\mathcal{A}_c^{1/2}u_s\|_2^2 + \|\mathcal{A}_c^{1/2}u_p\|_2^2 + 1) \cdot (\|u_p\|_\infty^2 + \|u_s\|_\infty^2).
\end{aligned}$$

Finally, adding the s and p components of the differential inequalities, we obtain

$$\partial_t(\|\mathcal{A}_c^{1/2}u_s\|_2^2 + \|\mathcal{A}_c^{1/2}u_p\|_2^2) \le C(\|u_p\|_\infty^2 + \|u_s\|_\infty^2) \cdot (\|\mathcal{A}_c^{1/2}u_s\|_2^2 + \|\mathcal{A}_c^{1/2}u_p\|_2^2 + 1)$$

which implies that

$$\begin{aligned}
&\|\mathcal{A}_c^{1/2}u_s\|_2^2 + \|\mathcal{A}_c^{1/2}u_p\|_2^2 \\
&\qquad \le \exp\left(C \int_0^T [\|u_s\|_{L^\infty}^2 + \|u_p\|_{L^\infty}^2] \right) (\|\mathcal{A}_c^{1/2}u_s(0)\|_2^2 + \|\mathcal{A}_c^{1/2}u_p(0)\|_2^2 + 1).
\end{aligned}$$

Applying Hölder's inequality to the time integral in the exponent we have

$$\begin{aligned}
\int_0^T [\|u_s\|_{L^\infty}^2 + \|u_p\|_{L^\infty}^2]\, dt &\le C\|u_s\|_{L^4([0,T],L^\infty)}^2 T^{\frac{1}{2}} + C\|u_p\|_{L^4([0,T],L^\infty)}^2 T^{\frac{1}{2}} \\
&= C(\|u_s\|_{L^4([0,T],L^\infty)}^2 + \|u_p\|_{L^4([0,T],L^\infty)}^2) T^{\frac{1}{2}} \\
&\le C(\mathcal{P}_0 + \mathcal{P}_0^{\frac{1}{2}})(T^{\frac{1}{4}} + T^{\frac{3}{4}}) T^{\frac{1}{2}}.
\end{aligned}$$

The last inequality follows from the a priori space-time estimate of Theorem 2.8 and the fact that $(4, \infty)$ is an admissible pair. We, thus, establish the boundedness of (u_s, u_p) in $\|\mathcal{A}_c(\cdot)\|_2$ norm:

$$\|\mathcal{A}_c^{1/2}u_s\|_2^2 + \|\mathcal{A}_c^{1/2}u_p\|_2^2 \le K_1 e^{K_2(\mathcal{P}_0+1)(T^2+1)} (\|\mathcal{A}_c^{1/2}u_s(0)\|_2^2 + \|\mathcal{A}_c^{1/2}u_p(0)\|_2^2 + 1).$$

Therefore, using the equivalence of norms, see Corollary 2.16, we obtain

$$\|u_s(t)\|_{H^1} + \|u_p(t)\|_{H^1} \le \tilde{K}_1 e^{K_2(\mathcal{P}_0+1)(T^2+1)} (\|u_s(0)\|_{H^1} + \|u_p(0)\|_{H^1} + 1) \tag{2.57}$$

for some $K_1, K_2 > 0$. This completes the proof of global existence in H^1. □

3. Energy transfer from the guided mode to radiation modes. In this section, we prove that, over time scales of interest ($t \leq \mathcal{O}(\epsilon^{-1})$), radiation terms remain small during the amplification process and the finite-dimensional model (1.3) is a valid approximation. In the discussion of this section we return to the time-scale, where nonlinear terms are of order ϵ:

$$i\partial_t u_s - Hu_s = i\epsilon|u_p|^2 u_s, \tag{3.1}$$

$$i\partial_t u_p - Hu_p = -i\epsilon|u_s|^2 u_p. \tag{3.2}$$

For this system, we are going to show that radiation is indeed bounded by $C\epsilon$ on a time scale of order $1/\epsilon$.

To proceed, we first orthogonally decompose a solution of (3.1)–(3.2) into its bound state and continuous spectral (radiative) parts:

$$u_s(x,t) = a_s(t)\phi(x) + U_s(x,t), \tag{3.3}$$

$$u_p(x,t) = a_p(t)\phi(x) + U_p(x,t). \tag{3.4}$$

We prove the following theorem.

THEOREM 3.1. *Let $(u_s(0), u_p(0)) \in H^1$ and $P_c u_s(0) = P_c u_p(0) = 0$. Assume that $0 < \epsilon < \epsilon_0 < \infty$ and that V satisfies* (2.5)*. Then, for any $T > 0$ there exists $C(T, \epsilon_0)$ so that*

$$\max\{\|U_s(t)\|_{H^1}, \|U_p(t)\|_{H^1}\} \leq C(T,\epsilon_0)\epsilon \tag{3.5}$$

on the interval $t \in [0, T/\epsilon]$.

We begin the proof with the following proposition, which follows from the ϵ-independent bounds $\|u_{s,p}\|_{H^1} \leq C(T, \epsilon_0)$, (2.57).

PROPOSITION 3.2. *Let $0 < \epsilon < \epsilon_0 < \infty$. Then for any $T > 0$ there exists $C(T, \epsilon_0)$ such that*

$$\|U_s\|_{H^1}, \|U_p\|_{H^1}, |a_s|, |a_p| \leq C \tag{3.6}$$

on the interval $t \in [0, T/\epsilon]$.

Substitution of (3.3)–(3.4) into (3.1)–(3.2) and projection onto ϕ and the range of P_c gives

$$\begin{aligned} i\partial_t a_s - \lambda a_s = i\epsilon[&|a_p|^2 a_s\langle\phi^3|\phi\rangle + \overline{a_p}a_s\langle\phi^3|U_p\rangle + a_p a_s\langle\phi^3|\overline{U_p}\rangle \\ &+ |a_p|^2\langle\phi^3|U_s\rangle + \cdots + \langle|U_p|^2 U_s|\phi\rangle], \end{aligned} \tag{3.7}$$

$$\begin{aligned} i\partial_t a_p - \lambda a_p = -i\epsilon[&|a_s|^2 a_p\langle\phi^3|\phi\rangle + \overline{a_s}a_p\langle\phi^3|U_s\rangle + a_p a_s\langle\phi^3|\overline{U_s}\rangle \\ &+ |a_s|^2\langle\phi^3|U_p\rangle + \cdots + \langle|U_s|^2 U_p|\phi\rangle], \end{aligned} \tag{3.8}$$

$$\begin{aligned} i\partial_t U_s - HU_s = i\epsilon[&a_s|a_p|^2 P_c\phi^3 + \overline{a_p}a_s P_c\phi^2 U_p + a_p a_s P_c\phi^2\overline{U_p} \\ &+ |a_p|^2 P_c\phi^2 U_s + \cdots + P_c|U_p|^2 U_s], \end{aligned} \tag{3.9}$$

$$\begin{aligned} i\partial_t U_p - HU_p = -i\epsilon[&a_p|a_s|^2 P_c\phi^3 + \overline{a_s}a_p P_c\phi^2 U_s + a_p a_s P_c\phi^2\overline{U_s} \\ &+ |a_s|^2 P_c\phi^2 U_p + \cdots + P_c|U_s|^2 U_p], \end{aligned} \tag{3.10}$$

where $H = -\partial_x^2 + V(x)$ and $H\phi = \lambda\phi$.

COROLLARY 3.3. *The fundamental modes are slowly varying with the rate ϵ*

$$|\partial_t|a_s||, |\partial_t|a_p|| \leq C(T, \epsilon_0)\epsilon.$$

Proof. This follows from Proposition 3.2 and (3.7)-(3.8). Indeed,

$$|\partial_t|a_s\| = |\partial_t|ia_s e^{it\lambda}\| \leq |\partial_t i(a_s e^{it\lambda})| = |(i\partial_t - \lambda)a_s| \leq C(T, \epsilon_0)\epsilon.$$

The same argument leads to a similar estimate for $|\partial_t|a_p\|$. This ends the proof of the corollary. □

The following estimates are used in the proof of the theorem and can be easily verified.

LEMMA 3.4.

$$\begin{aligned}
&\|P_c f\|_2 \leq \|f\|_2,\\
&\|P_c f\|_\infty \leq \|f\|_\infty(1 + \|\phi\|_1 \cdot \|\phi\|_\infty),\\
&\|(H-\lambda)^{-1} e^{i(H-\lambda)t} P_c f\|_2 \leq C\|f\|_2,\\
&\|(H-\lambda)^{-1} P_c\|_{H^1} \leq \frac{C}{\operatorname{dist}(H_c, \lambda)} \leq \frac{C}{|\lambda|}.
\end{aligned} \tag{3.11}$$

Proof of Theorem 3.1. We now make transformations $a_s = e^{-i\lambda t} A_s$ and $U_s = \epsilon e^{-iHt} W_s$ to remove rapid oscillations and explicitly show the smallness of radiation. By hypothesis of Theorem 3.1, we have $\|W_{s,p}(0)\|_{H^1} \leq C$. Note that by the bounds of Proposition 3.2 we have $\|W_{s,p}\|_{H^1} \leq C(T, \epsilon_0)/\epsilon$ and $|A_{s,p}| \leq C(T, \epsilon_0)$.

The slowly varying amplitudes A_s, A_p satisfy

$$\begin{aligned}
\partial_t A_s = {}& \epsilon |A_p|^2 A_s \langle \phi^3 | \phi \rangle + \epsilon^2 \overline{A_p} A_s \langle \phi^3 | e^{-i(H-\lambda)t} W_p \rangle + \epsilon^2 A_p A_s \langle \phi^3 | e^{i(H-\lambda)t} \overline{W_p} \rangle\\
&+ \epsilon^2 |A_p|^2 \langle \phi^3 | e^{-i(H-\lambda)t} W_s | \phi \rangle + \cdots + \epsilon^4 \langle | e^{-iHt} W_p|^2 e^{-i(H-\lambda)t} W_s \rangle,
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\partial_t A_p = {}& -\epsilon |A_s|^2 A_p \langle \phi^3 | \phi \rangle - \epsilon^2 \overline{A_s} A_p \langle \phi^3 | e^{-i(H-\lambda)t} W_s \rangle - \epsilon^2 A_p A_s \langle \phi^3 | e^{i(H-\lambda)t} \overline{W_s} \rangle\\
&- \epsilon^2 |A_s|^2 \langle \phi^3 | e^{-i(H-\lambda)t} W_p \rangle - \cdots - \epsilon^4 \langle | e^{-iHt} W_s|^2 e^{-i(H-\lambda)t} W_p | \phi \rangle.
\end{aligned} \tag{3.13}$$

Further, $W_{s,p}$ satisfy

$$\begin{aligned}
\partial_t W_s = {}& e^{i(H-\lambda)t} A_s |A_p|^2 P_c \phi^3\\
&+ e^{iHt} [\epsilon \overline{A_p} A_s P_c \phi^2 e^{-iHt} W_p + \epsilon A_p A_s P_c \phi^2 e^{iHt} \overline{W_p}\\
&\quad + \epsilon |A_p|^2 P_c \phi^2 e^{-iHt} W_s + \cdots + \epsilon^3 P_c | e^{-iHt} W_p|^2 e^{-iHt} W_s],
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\partial_t W_p = {}& e^{i(H-\lambda)t} A_p |A_s|^2 P_c \phi^3\\
&+ e^{iHt} [\epsilon \overline{A_s} A_p P_c \phi^2 e^{-iHt} W_s + \epsilon A_p A_s P_c \phi^2 e^{iHt} \overline{W_s}\\
&\quad + \epsilon |A_s|^2 P_c \phi^2 e^{-iHt} W_p + \cdots + \epsilon^3 P_c | e^{-iHt} W_s|^2 e^{-iHt} W_p].
\end{aligned} \tag{3.15}$$

The goal is now to show that given initial data where W is $\mathcal{O}(1)$ (which corresponds to radiation $\mathcal{O}(\epsilon)$) during the evolution W will remain $\mathcal{O}(1)$ on time interval $\mathcal{O}(1/\epsilon)$.

In order to do this we integrate the above equations:

$$W_s(t) = W_s(0) + \int_0^t e^{i(H-\lambda)s} A_s |A_p|^2 P_c \phi^3 ds + \epsilon \int_0^t R_s ds, \tag{3.16}$$

where ϵR_s is the ϵ-order part in (3.14), i.e., the second and the third lines. Integrating by parts

$$\begin{aligned}
W_s(t) = {}& W_s(0) + \frac{e^{i(H-\lambda)t} - 1}{i(H-\lambda)} A_s |A_p|^2 P_c \phi^3\\
&- \int_0^t \frac{e^{i(H-\lambda)s}}{i(H-\lambda)} \partial_s (A_s |A_p|^2) P_c \phi^3 ds + \epsilon \int_0^t R_s ds
\end{aligned} \tag{3.17}$$

and applying $\|\mathcal{A}_c^{\frac{1}{2}}\cdot\|_{L^2}$, we obtain

$$\|\mathcal{A}_c^{\frac{1}{2}}W_s(t)\|_{L^2} \le \|\mathcal{A}_c^{\frac{1}{2}}W_s(0)\|_{L^2} + \left\|\frac{e^{i(H-\lambda)t}-1}{i(H-\lambda)}A_s|A_p|^2\mathcal{A}_c^{\frac{1}{2}}P_c\phi^3\right\|_{L^2} \tag{3.18}$$

$$+C(T,\epsilon_0)\epsilon\int_0^t\left\|\frac{e^{i(H-\lambda)t}-1}{i(H-\lambda)}\mathcal{A}_c^{\frac{1}{2}}P_c\phi^3\right\|_{L^2} \tag{3.19}$$

$$+\epsilon\int_0^t\|\mathcal{A}_c^{\frac{1}{2}}R_s\|_{L^2}ds.$$

Therefore, we have

$$\|\mathcal{A}_c^{\frac{1}{2}}W_s(t)\|_{L^2} \le \|\mathcal{A}_c^{\frac{1}{2}}W_s(0)\|_{L^2} + C\|(H-\lambda)^{-1}\mathcal{A}_c^{\frac{1}{2}}P_c\phi^3\|_{L^2} \tag{3.20}$$

$$+\epsilon\int_0^t\Big(\|\mathcal{A}_c^{\frac{1}{2}}P_c\phi^2W_p\|_{L^2} \tag{3.21}$$

$$+\cdots+\epsilon^2\|\mathcal{A}_c^{\frac{1}{2}}P_c|e^{-iHt}W_p|^2e^{-iHt}W_s\|_{L^2}\Big)ds.$$

The terms on the right-hand side in the first line are bounded by a constant. To estimate the other terms we use the above properties of $\mathcal{A}_c$, $(H-\lambda)^{-1}$, etc. We illustrate how one proceeds with the estimates using the last term:

$$\epsilon^2\|\mathcal{A}_c^{\frac{1}{2}}P_c|e^{-iHt}W_p|^2e^{-iHt}W_s\|_{L^2} \tag{3.22}$$

$$\le \epsilon^2\|(I-\Delta)^{\frac{1}{2}}|e^{-iHt}W_p|^2e^{-iHt}W_s\|_{L^2}$$

$$\le \epsilon^2\|(I-\Delta)^{\frac{1}{2}}e^{-iHt}W_p\|_{L^2}^2\cdot\|(I-\Delta)^{\frac{1}{2}}e^{-iHt}W_s\|_{L^2} \tag{3.23}$$

$$\le \epsilon^2\|\mathcal{A}_c^{\frac{1}{2}}e^{-iHt}W_p\|_{L^2}^2\cdot\|\mathcal{A}_c^{\frac{1}{2}}e^{-iHt}W_s\|_{L^2}$$

$$\le \epsilon^2\|\mathcal{A}_c^{\frac{1}{2}}W_p\|_{L^2}^2\cdot\|\mathcal{A}_c^{\frac{1}{2}}W_s\|_{L^2} \tag{3.24}$$

$$\le \epsilon^2\|W_p\|_{H^1}^2\cdot\|\mathcal{A}_c^{\frac{1}{2}}W_s\|_{L^2} \le C\|\mathcal{A}_c^{\frac{1}{2}}W_s\|_{L^2},$$

where we used equivalence of norms, Leibnitz rule, and the uniform bound $\|W_{s,p}\|_{H^1} \le C/\epsilon$. Thus, the inequality takes the form

$$\|\mathcal{A}_c^{\frac{1}{2}}W_s(t)\|_{L^2} \le B + \epsilon K\int_0^t(\|\mathcal{A}_c^{\frac{1}{2}}W_p\|_{L^2}+\cdots+\|\mathcal{A}_c^{\frac{1}{2}}W_s\|_{L^2})ds, \tag{3.25}$$

where B and K do not depend on $\epsilon < \epsilon_0$. Adding the last inequality with the similar one for the p-component, and then using the notation $z(t) = \|\mathcal{A}_c^{\frac{1}{2}}W_s(t)\|_{L^2} + \|\mathcal{A}_c^{\frac{1}{2}}W_p(t)\|_{L^2}$, we obtain the inequality with modified B and K (but still independent of ϵ):

$$z(t) \le B + \epsilon K\int_0^t z(s)ds. \tag{3.26}$$

Using the standard Gronwall's result, we find

$$z(t) \le Be^{\epsilon Kt} \Rightarrow z(t) \le Be^{KT}, \tag{3.27}$$

which proves the bound $\|W_s\|_{H^1} \le C(T,\epsilon_0)$. □

Remark 3.5. Using dispersive properties of e^{itH}, it is possible to establish smallness of radiation in weaker spaces, namely, in $\|\cdot\|_{L^\infty}$ norm. Taking (3.9)–(3.10) for $U_{s,p}$ and rewriting them in the integral form, we are led to estimate the terms

$$\epsilon \int_0^t e^{itH} a_s |a_p|^2 P_c \phi^3 ds \tag{3.28}$$

and

$$\epsilon \int_0^t e^{itH} R_s ds. \tag{3.29}$$

After some changes of variables with the aid of standard L^∞ decay estimates for the Schrödinger evolution, one obtains that

$$\|U_{s,p}\|_{L^\infty} \leq C\sqrt{\epsilon}. \tag{3.30}$$

This argument also extends to the two-dimensional case with even better decay in ϵ (see the end of section 4.4).

4. Two-dimensional problem. We now consider the Raman system in the case of two transverse spatial dimensions

$$i\partial_t u_s - \beta H u_s = i|u_p|^2 u_s, \tag{4.1}$$
$$i\partial_t u_p - \beta H u_p = -i|u_s|^2 u_p, \tag{4.2}$$

where $H = -\Delta + V(x, y)$ and we prove analogous existence results and energy transfer estimates. Our strategy in the two-dimensional case is similar to the one-dimensional case; therefore, we omit some calculations which can be found in the previous sections.

In the two-dimensional case we require stronger conditions on the potential.

Assumption 4.1. The potential $V(x, y)$ is twice differentiable and

$$|D^\alpha V| \leq C_\alpha (1 + x^2 + y^2)^{-a},$$

where $a > 6$ and $|\alpha| \leq 2$.

Assumption 4.2. We assume that potential $V(x, y)$ has no zero energy eigenvalues or resonances.[3]

These assumptions are required to obtain space-time estimates in the next section.

4.1. Space-time estimates for the propagator. The definition of admissible pair is modified: (q, r) is admissible (in dimension $n = 2$) if

$$\frac{1}{q} = \frac{1}{2} - \frac{1}{r}, \quad r \in [2, \infty].$$

THEOREM 4.3. *Assume that the potential V satisfies both assumptions and let (q, r) be an admissible pair. Then for any $f \in L^2$ we have that $U_0(t)f$ and $U(t)P_c f$ are in $L^q(\mathbb{R}, L^r)$ and*

$$\begin{aligned} \|U_0(\cdot)f\|_{L^q([0,T],L^r)} &\leq C\|f\|_{L^2}, \\ \|U(\cdot)P_c f\|_{L^q([0,T],L^r)} &\leq C\|f\|_{L^2}, \end{aligned} \tag{4.3}$$

[3]Zero eigenvalues and resonances are obstructions to the optimal time-decay estimates for e^{-iHt}. Their absence holds generically; see, for example, [10].

where C depends only on q.

THEOREM 4.4. *Assume that the potential V satisfies both assumptions, let (γ, ρ) be an admissible pair, and let $f \in L^{\gamma'}([0,T], L^{\rho'})$, where (γ', ρ') is conjugate to (γ, ρ). Then for any admissible pair (q, r)*

$$\left\| \int_0^t U_0(t-\tau) f(\tau) d\tau \right\|_{L^q([0,T],L^r)} \le C \|f\|_{L^{\gamma'}([0,T],L^{\rho'})},$$

$$\left\| \int_0^t U(t-\tau) P_c f(\tau) d\tau \right\|_{L^q([0,T],L^r)} \le C \|f\|_{L^{\gamma'}([0,T],L^{\rho'})}, \tag{4.4}$$

where C depends only on q, γ.

Both Theorems 4.3 and 4.4 are proven by using Yajima's results [10] on $W^{k,p}$ continuity of wave operators. The argument follows the proof of Proposition 2.15 and is omitted here. Finally, Corollary 2.3 is valid in the current setting without any changes as the scaling is independent of the space dimension.

Remark 4.5. The application of Yajima's results on $W^{k,p}$ continuity of wave operators is the origin of the more restrictive smoothness assumptions on the potential $V(x)$. In one space dimension smoothness of $V(x)$ is not required [6].

4.2. A priori space-time estimates. The same argument as in the one-dimensional case applies here and we obtain the a priori bound of Proposition 2.5, as well as the bound (2.28) on nonlinear terms.

THEOREM 4.6 (a priori bounds in $L^q([0,T], L^r)$). *Let (q, r) be an admissible pair. Then any solution (u_s, u_p) satisfies the bounds*

$$\|u_s\|_{L^q([0,T],L^r)} \le C(\mathcal{P}_0 + 1)(T+1), \tag{4.5}$$

$$\|u_p\|_{L^q([0,T],L^r)} \le C(\mathcal{P}_0 + 1)(T+1). \tag{4.6}$$

Proof. As in the one-dimensional case, this estimate is proved by a straightforward application of space-time estimate for Schrödinger equation with a potential and using a priori estimate on nonlinear terms. Following the proof of Theorem 2.8, we obtain the same bounds with different ρ, ρ', γ, γ'. We must impose $\rho' = 1$ with $\rho = \infty$, but $\gamma = 2$ (since in $1/\gamma = 1/2 - 1/\rho$) with $\gamma' = 2$. This results in

$$\begin{aligned} \|u_s\|_{L^q([0,T],L^r)} &\le C_1 \|u_s(0)\|_{L^2} + C_2 \|u_p^2 u_s\|_{L^2([0,T],L^1)} \\ &\quad + \|\phi\|_{L^\infty} \|\phi\|_{L^r} T^{1/2+1/q} \|u_p^2 u_s\|_{L^2([0,T],L^1)} \\ &\le C_1 \mathcal{P}_0^{\frac{1}{2}} + C_2 \mathcal{P}_0 + C_3 \mathcal{P}_0 (T + T^{\frac{1}{2}}), \end{aligned}$$

since $q \ge 2$. ∎

4.3. Local existence in $H^2(\mathbb{R}^2)$. We now prove local existence of solutions in H^2 and will extend it to a global solution using our space-time estimates.

THEOREM 4.7 (local existence).

(1) *Given initial data $(u_s(0), u_p(0)) \in H^2(\mathbb{R}^2)$, there exist $T > 0$ and a unique solution $(u_s, u_p) \in L^\infty([0,T], H^2(\mathbb{R}^2))$.*

(2) *Let $T_{\max} > 0$ denote the maximal time of existence. Then, either $T_{\max} = \infty$ (global existence in time) or*

$$T_{\max} < \infty \quad \text{and} \quad \limsup_{t \to T_{\max}} \|(u_s, u_p)\|_{L^\infty([0,T],H^2(\mathbb{R}^2))} = \infty.$$

To prove the local existence theorem we have to show that a ball in H^2 is mapped into itself and that the mapping is a contraction.

Consider the same mapping as in the one-dimensional case (2.38)–(2.39). We will first show that it maps a ball into a ball:

$$\|A_\beta^{(s)}(u_s,u_p)\|_{H^2} \le \|U_\beta(t)u_s(0)\|_{H^2} + \left\|\int_0^t U_\beta(t-\tau)|u_p|^2 u_s\right\|_{H^2}. \tag{4.7}$$

The first term is estimated as follows:

$$\begin{aligned}\|U_\beta(t)u_s(0)\|_{H^2} &= \|(I-\Delta)U_\beta(t)u_s(0)\|_{L^2} = \|(I-\Delta)(H+i)(H+i)^{-1}U_\beta(t)u_s(0)\|_{L^2}\\ &\le \|(I-\Delta)(H+i)^{-1}\|_{\mathcal{B}(L^2,L^2)}\|(H+i)U_\beta(t)u_s(0)\|_{L^2}.\end{aligned}$$

Note that the operator $(I-\Delta)(H+i)^{-1}$ is bounded in L^2-operator norm. This follows from the identity

$$(I-\Delta)(H+i)^{-1} = -I + (I+i-V)(H+i)^{-1}$$

and the boundedness of V in L^∞ and of $(H+i)^{-1}$ in L^2.

Next, we have to establish the bound for

$$\begin{aligned}\|(H+i)U_\beta(t)u_s(0)\|_{L^2} &= \|(H+i)u_s(0)\|_{L^2} = \|(H+i)(I-\Delta)^{-1}(I-\Delta)u_s(0)\|_{L^2}\\ &\le \|(H+i)(I-\Delta)^{-1}\|_{\mathcal{B}(L^2,L^2)}\|(I-\Delta)u_s(0)\|_{L^2} \le C\|u_s(0)\|_{H^2},\end{aligned}$$

where the operator $(H+i)(I-\Delta)^{-1}$ is bounded by similar calculations as for $(I-\Delta)(H+i)^{-1}$.

Now, we estimate the second term in (4.7):

$$\left\|\int_0^t U_\beta(t-\tau)|u_p|^2u_s d\tau\right\|_{H^2} \le C\left\|\int_0^t (H+i)U_\beta(t-\tau)|u_p|^2u_s d\tau\right\|_{L^2}.$$

To bound this term we write it in the form

$$\int dx\left\{\int_0^t\int_0^t (H+i)U_\beta(t-\tau_1)|u_p(\tau_1)|^2u_s(\tau_1)(H-i)\overline{U_\beta}(t-\tau_2)|u_p(\tau_2)|^2\overline{u_s}(\tau_2)d\tau_1 d\tau_2\right\}$$

(using Hölder inequality and isometry of U_β in L^2)

$$\begin{aligned}&\le \int_0^t\int_0^t d\tau_1 d\tau_2 \left|\int |(H+i)|u_p(\tau_1)|^2u_s(\tau_1)|^2\,dx\right|^{\frac12}\left|\int |(H+i)|u_p(\tau_2)|^2u_s(\tau_2)|^2\,dx\right|^{\frac12}\\ &\quad\le t^2\sup_{\tau\in[0,t]}\|(H+i)u_p^2(\tau)u_s(\tau)\|_2 \le Ct^2\sup_{\tau\in[0,t]}\|u_p^2(\tau)u_s(\tau)\|_{H^2}^2\\ &\quad\le Ct^2\sup_{\tau\in[0,t]}\|u_p(\tau)\|_{H^2}^4\|u_s(\tau)\|_{H^2}^2.\end{aligned}$$

Therefore, we finally obtain the bound on the s-part of the map

$$\|A_\beta^{(s)}(u_s,u_p)\|_{H_2} \le C\|u_s(0)\|_{H^2} + Ct\sup_{\tau\in[0,t]}\|u_p(\tau)\|_{H^2}^2\|u_s(\tau)\|_{H^2}$$

and the full map

$$\begin{aligned}\|A_\beta(u_s,u_p)\|_{L^\infty([0,t],H^2)} &\le C\left(\|u_s(0)\|_{H^2} + \|u_p(0)\|_{H^2}\right)\\ &\quad + Ct\big(\|u_p\|^2_{L^\infty([0,t],H^2)}\|u_s\|_{L^\infty([0,t],H^2)}\\ &\qquad + \|u_p\|_{L^\infty([0,t],H^2)}\|u_s\|^2_{L^\infty([0,t],H^2)}\big).\end{aligned}$$

With the obtained inequality it is easy to establish the following proposition.

PROPOSITION 4.8. *Let $(u_s(0), u_p(0))$ be in H^2. Define the ball in $L^\infty([0,T], H^2)$ as*

$$\mathcal{B}(T) = \big\{(u_s, u_p) \in L^\infty([0,T], H^2) : \|(u_s, u_p)\|_{L^\infty([0,T],H^2)} \\ \le 2C(\|u_s(0)\|_{H^2} + \|u_p(0)\|_{H^2})\big\}.$$

Then there exists $T_0 > 0$ such that for any $T < T_0$, the ball is mapped into itself, i.e., $A_\beta(\mathcal{B}(T)) \subset \mathcal{B}(T)$ with T_0 independent of β.

Next, we have to show that the mapping is a contraction in $L^\infty([0,T], H^2)$. Consider the first component of the map applied to two different pairs (u_s, u_p) and (v_s, v_p) with the same initial data $(u_s(0), u_p(0)) = (v_s(0), v_p(0))$:

$$\begin{aligned} &\|A_\beta^{(s)}(u_s, u_p) - A_\beta^{(s)}(v_s, v_p)\|_{L^\infty([0,t],H^2)} \\ &\quad \le CT \sup_{t\in[0,T]} \||u_p(t)|^2 u_s(t) - |v_p(t)|^2 v_s(t)\|_{H^2} \\ &\quad \le CT \sup_{t\in[0,T]} (\|u_p^2(u_s - v_s)\|_{H^2} + \|u_p v_s(u_p - v_p)\|_{H^2} + \|v_p v_s(u_p - v_p)\|_{H^2}) \\ &\quad \le C(\|u_p, u_s, v_p, v_s\|_{L^\infty([0,T],H^2)}) T \|(u_p - v_p, u_s - v_s)\|_{L^\infty([0,T],H^2)}. \end{aligned}$$

Adding both s and p components of the map we obtain

$$\begin{aligned} &\|A_\beta(u_p, u_s) - A_\beta(v_p, v_s)\|_{L^\infty([0,T],H^2)} \\ &\quad \le TC(\|u_p, u_s, v_p, v_s\|_{L^\infty([0,T],H^2)}) \|(u_p - v_p, u_s - v_s)\|_{L^\infty([0,T],H^2)}. \end{aligned}$$

This inequality implies the following proposition.

PROPOSITION 4.9. *There exists $T_1 : 0 < T_1 < T_0$ sufficiently small such that the map A_β is a contraction in the ball $\mathcal{B}(T)$ for any $T : 0 < T < T_1$:*

$$\|A_\beta(u_p, u_s) - A_\beta(v_p, v_s)\|_{L^\infty([0,T],H^2)} \le q \|(u_p - v_p, u_s - v_s)\|_{L^\infty([0,T],H^2)},$$

where $q < 1$.

Remark 4.10. Whenever there exists a local solution on $t \in [0,T]$, it is bounded in $L^q([0,T], L^r(\mathbb{R}^2))$. Indeed, a solution in $L^\infty([0,T], H^2)$ is also in $L^q([0,T], L^r(\mathbb{R}^2))$ and the earlier obtained a priori bounds apply.

4.4. Global existence in $H^2(\mathbb{R}^2)$. In this section we will show that the local solution obtained via the contraction mapping principle in the previous section can be extended to a global solution using space-time estimates. We start with establishing uniform bound in H^1 space.

A priori estimates in H^1. Proceeding as in the one-dimensional problem (2.4), we obtain H^1 bound. Since $(\infty, 2)$ is an admissible pair, we obtain that the solutions are even uniformly bounded in time on the interval of existence of a local solution:

$$\|(u_s, u_p)\|_{L^\infty([0,T],H^1)} \le C(\mathcal{P}_0, T) \|(u_s(0), u_p(0))\|_{H^1}. \tag{4.8}$$

Continuation to a global solution using Theorem 4.7 requires an H^2 estimate, which we now derive.

Global existence in H^2. Having established uniform bound in H^1, we are ready to obtain a bound in H^2, which will rule out the first alternative in Theorem 4.7 (on local existence) and, thus, leave the global existence as the only possibility.

Applying $H+I$ to the s component of (4.2) and using energy estimates, we obtain

$$\begin{aligned} \partial_t \int |(H+I)u_s|^2 &\leq \int |(H+I)u_s| \cdot |(H+I)(|u_p|^2 u_s)| \\ &\leq \|(H+I)u_s\|_2 \|(H+I)(|u_p|^2 u_s)\|_2. \end{aligned} \tag{4.9}$$

To estimate the last term, we write

$$\begin{aligned} \int |\Delta(|u_p|^2 u_s)|^2 &= \int |u_p|^4 |\Delta u_s|^2 + 2\int |\nabla u_p|^2 |\nabla u_s|^2 |u_p|^2 + \int |\nabla u_p|^4 |u_s|^2 \\ &\leq C(\|u_p\|_\infty^4 \cdot \|u_s\|_{H^2}^2 + \|u_s\|_\infty^2 \cdot \|u_p\|_{H^2}^2 \cdot \|u_p\|_{H^1}^2 \\ &\quad + \|u_p\|_\infty^2 \cdot \|u_s\|_{H^2}^2 \cdot \|u_s\|_{H^1}^2 + \|u_p\|_\infty^2 \cdot \|u_p\|_{H^2}^2 \cdot \|u_p\|_{H^1}^2) \end{aligned}$$

and therefore we have

$$\int |\Delta(|u_p|^2 u_s)|^2 \leq C\left(1 + \|u_p\|_\infty^4 + \|u_s\|_\infty^4\right) \|(u_s, u_p)\|_{H^1}^2 \cdot \|(u_s, u_p)\|_{H^2}^2.$$

Now, adding the s and p components, we obtain

$$\begin{aligned} &\partial_t \int (|(H+I)u_s|^2 + |(H+I)u_s|^2) \\ &\quad \leq C(1 + \|u_p\|_\infty^2 + \|u_s\|_\infty^2)(\|(H+I)u_s\|_{H^2}^2 + \|(H+I)u_p\|_{H^2}^2), \end{aligned}$$

where we have used that $\|(u_s, u_p)\|_{H^1}$ is bounded (see previous paragraph on H^1 estimates).

Since $(2, \infty)$ is an admissible pair, both $\int \|u_p\|_\infty^2 \, dt$ and $\int \|u_s\|_\infty^2 \, dt$ are bounded, and we obtain the required bound:

$$\|(H+I)u_s\|_2^2 + \|(H+I)u_p\|_2^2 \leq C(\mathcal{P}_0, T)e^{kT}(\|(H+I)u_s(0)\|_2^2 + \|(H+I)u_p(0)\|_2^2)$$

on the interval $t \in [0, T]$. Because of the norm equivalence, we also obtain

$$\|u_s(t)\|_{H^2}^2 + \|u_p(t)\|_{H^2}^2 \leq C(\mathcal{P}_0, T)e^{kT}(\|u_s(0)\|_{H^2}^2 + \|u_p(0)\|_{H^2}^2).$$

This bound implies the global existence of solutions in H^2.

Radiation losses in two-dimensional amplification model. The analogue of Theorem 3.1 on the boundedness of radiation in H^2 holds with the proof carrying over from the one-dimensional case. One can also obtain boundedness in L^∞ as described in Remark 3.5 with even faster decay: $\epsilon \log \epsilon$ rather than $\sqrt{\epsilon}$.

Appendix A. Application to optical communications. In this section we provide the details on the Raman model in optical communication systems as well as the derivation of the reaction-dispersion system. In modern long-haul optical communication systems the signal propagates in the fundamental mode of a single mode fiber. In an "ideal" lossless optical fiber waveguide the transverse shape of the wave envelope does not change and there is no transfer of energy from the fundamental mode to radiation modes. However, in the real systems Rayleigh scattering causes attenuation of signal power, thus requiring periodic amplification [1]. A current strategy for amplification of a signal is based on the stimulated Raman effect. Here, light of a second *pump* wavelength is co- or counter-propagated in the medium. The stimulated Raman effect is a parametric process in which light of the pump frequency is transferred to that of the signal frequency. This amplification process is inherently nonlinear and therefore is expected to cause deformation of the transverse mode shape. There are other linear and nonlinear effects which may need to be taken into account such as group velocity dispersion, self-phase modulation, and four wave mixing. Also, refractive index depends on the frequency shift between the pump and signal frequencies.

Regarding linear effects, like group velocity dispersion, in practice they are weaker compared to the Raman effect assuming that pulses are not too short. However, in our case, we consider a model problem with both pump and signal being continuous (constant amplitude) waves. Then dispersion just vanishes.

The refractive index depends on the light frequency. As a result fundamental modes would be slightly different for pump and signal waves. Here, we assume that the modes are the same as it simplifies the exposition. All our results can be obtained for the frequency-dependent dispersion/diffraction coefficients with minimal modifications.

Approximate equations for the Raman interaction of signal and pump in the waveguide have been derived in [3]. These authors derived a pair of coupled ODEs for the signal and pump intensities, based on the assumption that all energy is contained in the fundamental modes. This model compares well with experiment [3] (see also [1] and the references therein). The authors [3] also discussed why the approximation was so accurate. They suggested that radiative losses (energy transfer from bound to radiation modes) is negligible due to the fact that "the wave-guiding action of the fiber reforms the pump and Stokes waves so that they always have intensities distributions which are close approximations to those which would exist in the absence of Raman interaction." However, this explanation has some limitations as the energy transfer could occur adiabatically (e.g., like the ionization of an atom). In other words, a weak process may lead to non-negligible changes after sufficiently long time. In particular, one might expect that the modes would undergo continuous deformation while also shedding radiation, so that after the full energy exchange a non-negligible amount of energy would accumulate in radiative modes and would constitute a significant loss.

To understand these effects, equations which take into account the effects of diffraction, wave-guiding and amplification should be studied. Naturally, the model will contain a small parameter: the ratio of diffraction and amplification lengths.

Raman stimulated emission describes the amplification of signal photons (with frequency ω_s) with Stokes down shifted pump photons (with frequency ω_p) and is governed by [3]

$$\frac{\partial n_s}{\partial z} = g(\omega_p - \omega_s) n_s n_p, \tag{A.1}$$

$$\frac{\partial n_p}{\partial z} = -g(\omega_p - \omega_s) n_s n_p, \tag{A.2}$$

where n_s, n_p are the number densities of signal and pump photons, respectively, and the total number of photons per unit volume is conserved $n_s + n_p = N$. Since we wish to focus on the effects due to the resonant coupling of the two wave fields (pump and signal), we ignore other effects, such as amplified spontaneous emission [1] which is always present, though a small effect.

Introducing the intensities

$$I_s = h\omega_s n_s, \qquad I_p = h\omega_p n_p, \tag{A.3}$$

where h denotes Planck's constant, we obtain the corresponding equations

$$\frac{\partial I_s}{\partial z} = \frac{g(\omega_p - \omega_s)}{h\omega_p} I_s I_p, \tag{A.4}$$

$$\frac{\partial I_p}{\partial z} = -\frac{g(\omega_p - \omega_s)}{h\omega_s} I_s I_p. \tag{A.5}$$

These equations satisfy the photon number conservation relation

$$\frac{I_s}{\omega_s} + \frac{I_p}{\omega_p} = \text{constant}. \tag{A.6}$$

In the case of radiative loss, this conservation law would be violated, since some photons would be lost from the bound waveguide mode to radiation modes. Equations (A.4)–(A.5) describe the plane wave Raman interaction.

Consider now the propagation of light in a dielectric cylinder waveguide with longitudinal coordinate, z. Maxwell's equations [1] imply

$$\Delta \mathbf{E} - \frac{1}{c^2}\mathbf{E}_{tt} - \nabla(\nabla \cdot \mathbf{E}) = \frac{1}{c^2}[\chi^{(1)}(\mathbf{r}, t) * \mathbf{E}_{tt}]_{tt} = 0,$$

where $\mathbf{E} \in \mathbb{R}^3$ is the electric field and $\chi^{(1)}(\mathbf{r}, t)$ is the linear susceptibility. Neglecting vector effects (see, e.g., [1]), we find that the time Fourier transform of $\mathbf{E}$, $\hat{\mathbf{E}}$, satisfies

$$\Delta \hat{\mathbf{E}} + \frac{\omega^2 n^2(\mathbf{x}_\perp, \omega)}{c^2} \hat{\mathbf{E}} = 0.$$

Each component of $\mathbf{E}$ satisfies

$$E_{zz} + \Delta_\perp E + \frac{\omega^2 n^2(\mathbf{x}_\perp, \omega)}{c^2} E = 0. \tag{A.7}$$

Next, we introduce the paraxial approximation. Let δ be a small parameter, and assume the following structure for the refraction index dependence on $\mathbf{x}_\perp$:

$$n^2(\mathbf{x}_\perp, \omega) = n_0^2(\omega) + \delta^2 n_1^2(\delta \mathbf{x}_\perp/\lambda_0, \omega). \tag{A.8}$$

We also seek E, in the form

$$E = A((\delta/\lambda_0)\mathbf{x}_\perp, (\delta^2/\lambda_0)z)e^{ikz}, \tag{A.9}$$

where λ_0 is the light wavelength and

$$\frac{2\pi}{\lambda_0} = k = \omega n_0(\omega)/c.$$

Thus, E varies more rapidly in the transverse than longitudinal directions.

Substituting (A.8), (A.9) into (A.7) and multiplying by $\lambda_0^2 \delta^{-4}$ we obtain

$$A_{ZZ} + 2ik\lambda_0 \delta^{-2} A_Z + \delta^{-2}\Delta_\perp A + \delta^{-2}\lambda_0^2(\omega^2/c^2)n_1^2(\mathbf{X}_\perp) = 0,$$

where $Z = (\delta^2/\lambda_0)z$, $\mathbf{X}_\perp = (\delta/\lambda_0)\mathbf{x}_\perp$, and $(*)_\perp$ denotes differentiation with respect to $\mathbf{X}_\perp$. For δ small, we keep the dominant terms, those of order δ^{-2} and obtain, after using the relation between k and λ_0,

$$i4\pi A_Z + \Delta_\perp A + \frac{4\pi^2 n_1(\mathbf{X}_\perp, \omega)}{n_0^2} A = 0. \tag{A.10}$$

Equation (A.10) governs the linear propagation of any light field (signal or pump) in the paraxial approximation. To obtain a model governing the Raman interaction of signal and pump fields, u_s and u_p, we argue as follows. The signal field envelope propagates through a medium with refractive index (A.8) corrected by an imaginary term proportional to $i|u_p|^2$ corresponding to the Raman amplification by pump. The pump field envelope, u_p, propagates through a medium with refractive index (A.8) corrected by an imaginary term proportional to $-i|u_s|^2$ corresponding to pump depletion by the signal. The coupled signal and pump envelopes are then taken to satisfy the system

$$i\partial_z u_s + \Delta_\perp u_s - V(\omega_s, \mathbf{x}_\perp)u_s = i\epsilon_s|u_p|^2 u_s, \tag{A.11}$$

$$i\partial_z u_p + \Delta_\perp u_p - V(\omega_p, \mathbf{x}_\perp)u_p = -i\epsilon_p|u_s|^2 u_p, \tag{A.12}$$

where $\epsilon_{p,s}$ is the parameter which measures the ratio of the diffraction and nonlinear lengths. Usually, $\epsilon_{p,s}$ is very small [1]. We further assume[4] $\epsilon = \epsilon_s = \epsilon_p$ and neglect the dependence of the refractive index on frequency, i.e., $V(x_\perp, \omega) = V(x_\perp)$.

System (A.12) models the Raman energy exchange between the two continuous waves. We have not included the effects of losses due to the Rayleigh scattering in order not to burden the exposition. In reality, the Raman amplification length might be comparable to the effective (loss) length (20 km).

Thus, we have

$$\begin{aligned} i\partial_t u_s - Hu_s &= i\epsilon|u_p|^2 u_s, \\ i\partial_t u_p - Hu_p &= -i\epsilon|u_s|^2 u_p, \end{aligned} \tag{A.13}$$

where we use "t" to denote the "time-like" direction, z, $x_\perp = x$, and

$$H = -\Delta + V(x). \tag{A.14}$$

We study system (A.13) in the case where H has spectrum consisting of one point eigenvalue, $\lambda < 0$, with corresponding eigenfunction ϕ, $\|\phi\|_{L^2} = 1$. The components of u_s and u_p, which are orthogonal to ϕ, are called radiative components. Our goal is to prove if for $t = 0$ the order-one energy is concentrated in ϕ alone, then on time scales of order ϵ^{-1} the energy in radiative components is at most of order ϵ.

Appendix B. Normal form theorem. In this section we state a normal form result on the absence of the terms driving the radiation to any order of the perturbation theory. While in the main part of the paper we have used only the fact that these terms can be removed at the first order, we find this result important and potentially useful in the search for sharper estimates of radiation growth. The results in this section are formal in the sense that we do not verify the validity of transformations and of obtained systems.

[4]Nonlinear coefficient ϵ is then the same for both fields. Therefore, in this system the energy is conserved rather than photon number. This is done to simplify the presentation. All results hold for the "true" model as well.

For the Raman energy exchange system

$$i\partial_t u_s - Hu_s = i\epsilon|u_p|^2 u_s, \tag{B.1}$$

$$i\partial_t u_p - Hu_p = -i\epsilon|u_s|^2 u_p, \tag{B.2}$$

we now use representation

$$u_s = u_0^s\phi_0^s(x) + \int_0^\infty u_\lambda^s\phi_\lambda^s(x)d\lambda,$$
$$u_p = u_0^p\phi_0^p(x) + \int_0^\infty u_\lambda^p\phi_\lambda^p(x)d\lambda,$$

where H becomes diagonal

$$H\phi_0 = \lambda_0\phi_0,$$
$$H\phi_\lambda = \lambda\phi_\lambda, \qquad \langle u_\lambda, \phi_\lambda\rangle = u_\lambda,$$

where $\lambda_0 < 0$ corresponds to the fundamental mode and the remaining part of the spectrum ($\lambda > 0$) corresponds to the continuous spectrum. In this representation the equations take the form

$$\begin{aligned}
i\partial_t u_0^s - \lambda_0 u_0^s &= i\epsilon C_{00}^{00}u_0^p\overline{u}_0^p u_0^s + i\epsilon\int_0^\infty C_{\lambda_1 0}^{00}u_0^p\overline{u}_0^p u_{\lambda_1}^s d\lambda_1 \\
&\quad + \cdots + i\epsilon\int_0^\infty\int_0^\infty\int_0^\infty C_{\lambda_1 0}^{\mu_1\mu_2}u_{\mu_1}^p\overline{u}_{\mu_2}^p u_{\lambda_1}^s d\mu_1 d\mu_2 d\lambda_1, \\
i\partial_t u_0^p - \mu_0 u_0^p &= i\epsilon C_{00}^{00}u_0^s\overline{u}_0^s u_0^p + i\epsilon\int_0^\infty C_{00}^{\mu_1 0}u_0^s\overline{u}_0^s u_{\mu_1}^p d\mu_1 \\
&\quad + \cdots + i\epsilon\int_0^\infty\int_0^\infty\int_0^\infty C_{\lambda_1\lambda_2}^{\mu_1 0}u_{\lambda_1}^s\overline{u}_{\lambda_2}^s u_{\mu_1}^p d\lambda_1 d\lambda_2 d\mu_1, \\
i\partial_t u_\lambda^s - \lambda u_\lambda^s &= i\epsilon C_{0\lambda}^{00}u_0^p\overline{u}_0^p u_0^s + i\epsilon\int_0^\infty C_{\lambda_1\lambda}^{00}u_0^p\overline{u}_0^p u_{\lambda_1}^s d\lambda_1 \\
&\quad + \cdots + i\epsilon\int_0^\infty\int_0^\infty\int_0^\infty C_{\lambda_1\lambda}^{\mu_1\mu_2}u_{\mu_1}^p\overline{u}_{\mu_2}^p u_{\lambda_1}^s d\mu_1 d\mu_2 d\lambda_1, \\
i\partial_t u_\mu^p - \mu u_\mu^p &= i\epsilon C_{00}^{0\mu}u_0^s\overline{u}_0^s u_0^p + i\epsilon\int_0^\infty C_{00}^{\mu_1\mu}u_0^s\overline{u}_0^s u_{\mu_1}^p d\mu_1 \\
&\quad + \cdots + i\epsilon\int_0^\infty\int_0^\infty\int_0^\infty C_{\lambda_1\lambda_2}^{\mu_1\mu}u_{\lambda_1}^s\overline{u}_{\lambda_2}^s u_{\mu_1}^p d\lambda_1 d\lambda_2 d\mu_1,
\end{aligned}$$

where

$$C_{\lambda_1\lambda_2}^{\mu_1\mu_2} = \int_{-\infty}^{+\infty}\phi_{\mu_1}^p\overline{\phi}_{\mu_2}^p\phi_{\lambda_1}^s\overline{\phi}_{\lambda_2}^s\,dx.$$

The natural consequence of the stimulated emission process is the invariance with respect to the phase shifts of both the signal and the pump modes:

$$u_{\lambda_i}^s, u_{\mu_j}^p \to u_{\lambda_i}^s e^{i\psi_s}, u_{\mu_j}^p e^{i\psi_p}.$$

This torus action will be called G_{sp}-action below.

Consider the class of polynomial vector fields which stay invariant under this group action. We are going to consider near-identity transformations, which commute with the torus action. Therefore these transformations map a G_{sp}-invariant vector field to another G_{sp}-invariant vector field. We will now invoke these transformations to remove nonresonant terms from the equations. We first observe that a polynomial vector field that is invariant with respect to the G_{sp}-action is generated by the monomials of this form

$$\mathbf{e}^{\mathbf{s}}_{\lambda} u^p_{\mu_1}\overline{u}^p_{\mu_2}\cdots u^p_{\mu_{m-1}}\overline{u}^p_{\mu_m} u^s_{\lambda_1}\overline{u}^s_{\lambda_2}\cdots u^s_{\lambda_{k-1}}\overline{u}^s_{\lambda_k} u^s_{\lambda_{k+1}}, \tag{B.3}$$

$$\mathbf{e}^{\mathbf{p}}_{\mu} u^s_{\lambda_1}\overline{u}^s_{\lambda_2}\cdots u^s_{\lambda_{m-1}}\overline{u}^s_{\lambda_m} u^p_{\mu_1}\overline{u}^p_{\mu_2}\cdots u^p_{\mu_{k-1}}\overline{u}^p_{\mu_k} u^p_{\mu_{k+1}}, \tag{B.4}$$

where m and k are even numbers.

Definition. *The monomial of type* (B.3) *is called resonant if*

$$\mu_1-\mu_2+\cdots+\mu_{m-1}-\mu_m+\lambda_1-\lambda_2+\cdots+\lambda_{k-1}-\lambda_k+\lambda_{k+1}-\lambda=0$$

and the monomial of type (B.4) *is called resonant if*

$$\lambda_1-\lambda_2+\cdots+\lambda_{m-1}-\lambda_m+\mu_1-\mu_2+\cdots+\mu_{k-1}-\mu_k+\mu_{k+1}-\mu=0.$$

If in a G_{sp}-invariant vector field initially all the energy is concentrated in fundamental modes, then the radiative modes can be excited only through the terms

$$\mathbf{e}^{\mathbf{s}}_{\lambda}|u^p_0|^k|u^s_0|^l u^s_0 \quad \text{and} \quad \mathbf{e}^{\mathbf{p}}_{\mu}|u^s_0|^k|u^p_0|^l u^p_0,$$

while no other radiation driving terms can appear after application of a G_{sp}-invariant transformation. These terms are nonresonant, since the corresponding arithmetic combinations are $\lambda-\lambda_0$ and $\mu-\lambda_0$, where $\lambda,\mu>0$ and $\lambda_0<0$. Therefore, $|\lambda-\lambda_0|>|\lambda_0|>0$ and $|\mu-\lambda_0|>|\lambda_0|>0$.

According to the standard normal form procedure, these terms can be removed by employing the transformations of the form[5]

$$u^s_{\lambda}=U^s_{\lambda}+\epsilon^{\frac{k+l}{2}}C^s_{\lambda}|U^p_0|^k|U^s_0|^lU^s_0, \tag{B.5}$$

$$u^p_{\mu}=U^p_{\mu}+\epsilon^{\frac{k+l}{2}}C^p_{\mu}|U^s_0|^k|U^p_0|^lU^p_0. \tag{B.6}$$

Now we formulate the normal form theorem.

Theorem B.1. *For any $N\geq 1$ there exists a sequence of transformations of the form* (B.5)–(B.6)*, which bring the system to the form*

$$\begin{aligned}
i\partial_t U^s_0-\lambda_0U^s_0 &= i\left[\epsilon C^{00}_{00}U^p_0\overline{U}^p_0U^s_0+\sum_{n_1,n_2>0}^{n_1+n_2<N+1}\epsilon^{\frac{n_1+n_2}{2}}C^s_{n1,n_2}|U^p_0|^{n_1}|U^s_0|^{n_2}U^s_0\right]\\
&\quad+\epsilon R^s_0(U,\epsilon)+O(\epsilon^{N+1}),\\
i\partial_t U^p_0-\lambda_0U^p_0 &= -i\left[\epsilon C^{00}_{00}U^s_0\overline{U}^s_0U^p_0+\sum_{n_1,n_2>0}^{n_1+n_2<N+1}\epsilon^{\frac{n_1+n_2}{2}}C^p_{n1,n_2}|U^p_0|^{n_1}|U^s_0|^{n_2}U^p_0\right]\\
&\quad+\epsilon R^p_0(U,\epsilon)+O(\epsilon^{N+1}),\\
i\partial_t U^s_{\lambda}-\lambda U^s_{\lambda} &= \epsilon R^s_{\lambda}(U,\epsilon)+O(\epsilon^{N+1}),\\
i\partial_t U^p_{\mu}-\mu U^p_{\mu} &= \epsilon R^p_{\mu}(U,\epsilon)+O(\epsilon^{N+1}),
\end{aligned}$$

[5]Capitals denote new variables while small ones denote old variables.

where $R_*^{s,p}(U,\epsilon)$ *are the terms which vanish if there is no energy in radiative modes* ($U_\lambda^s = U_\mu^p = 0$ *for all* $\lambda, \mu > 0 \Rightarrow R = 0$)*.*

Proof. We will prove the theorem by applying a series of nearly identical transformations. We start with the transformation

$$u_\lambda^s = U_\lambda^s + \epsilon C_\lambda^s U_0^p \overline{U_0^p} U_0^s.$$

Straightforward calculations show that in order to remove the corresponding radiation driving terms, the coefficient C_λ^s must be of the form

$$C_\lambda^s = \frac{C_{0\lambda}^{00}}{i(\lambda_0 - \lambda)}$$

and similarly to remove pump radiation driving terms, we apply

$$u_\mu^p = U_\mu^p + \epsilon C_\mu^p U_0^s \overline{U_0^s} U_0^p$$

with

$$C_\mu^p = \frac{C_{0\mu}^{00}}{i(\mu_0 - \mu)}.$$

Our results from the previous sections indicate that the transformations are valid and the new system is well defined. Indeed, with the first-order radiation driving terms removed, the obtained system is equivalent to (3.14)–(3.15).

Next, we formally remove quadratic radiation driving terms. We observe that all transformations are near-identity ones differing only by fundamental mode amplitudes. No small denominators arise due to the gap in the spectrum (between the eigenvalue and the continuum spectrum). Continuing these transformations, we remove higher-order radiation driving terms to order N. ∎

Appendix C. Numerical simulations. We verify some of the results obtained in this paper by carrying out numerical simulations. We simulate system (1.2) in one dimension, where the potential is chosen to be $V = \operatorname{sech}^2(x)$, so that the fundamental mode can be explicitly calculated. We use the initial data with all the power (L^2 norm) contained in fundamental modes.

The numerical simulation uses a Fourier split-step scheme, where evolutions due to dispersive, potential, and nonlinear interactions are calculated separately. Nonlinear interaction is solved exactly using the standard solution of the corresponding ODE [1]. Time step is chosen to be $\Delta t = 0.01$ and there are 2^{12} Fourier modes. In Figure 1, the L^2 norm of total radiation is calculated after sufficiently long evolution, so that power exchange between the fields is almost complete. This is done for $\epsilon \in [0.005, 0.05]$. One can see that the time interval is sufficiently long from Figure 2, where for the smallest $\epsilon = 0.005$, the power of both fields contained in the fundamental modes is computed. Even in this case with the smallest ϵ (so that the energy exchange takes longer) there is enough time for the pump field to transfer almost all the power to the signal field. It appears from Figure 1 that losses due to radiation (L^2 norm) scale linearly with nonlinearity strength ϵ, as predicted by our analysis.

Acknowledgment. Part of this research was carried out while V. Zharnitsky was visiting the Program in Applied and Computational Mathematics at Princeton University. He would like to thank Ingrid Daubechies for her hospitality and for providing stimulating research environment.

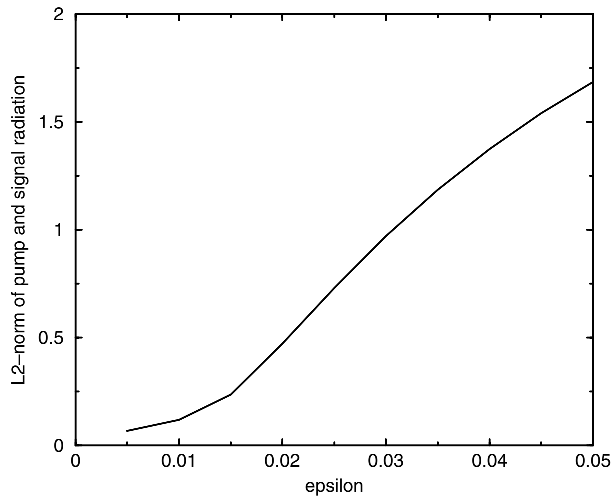

FIG. 1. *Dependence of radiation power on time. The radiation is "measured" after the evolution for* 75 *units of dimensionless time.*

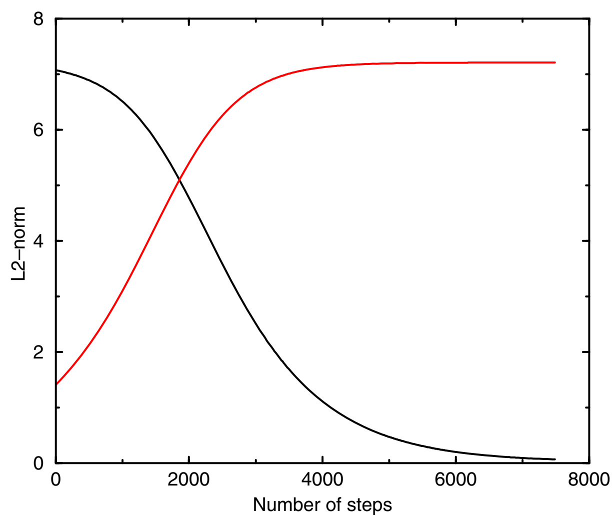

FIG. 2. *Power exchange between fundamental modes for the smallest* $\epsilon = 0.005$ *after the evolution for the same time* $T = 75$ *(it corresponds to* 7500 *steps). Note that the pump field is almost completely "depleted." This indicates that time interval* $T = 75$ *is of sufficient length for the power exchange to take place.*

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