A theory of time-dependent nonlinear dispersive equations of the Schrödinger or Gross-Pitaevskii and Hartree type is developed. The short, intermediate and large time behavior is found, by deriving nonlinear master equations (NLME), governing the evolution of the mode powers, and by a novel multitime scale analysis of these equations. The scattering theory is developed and coherent resonance phenomena and associated lifetimes are derived. Applications include Bose-Einstein condensate large time dynamics and nonlinear optical systems. The theory reveals a nonlinear transition phenomenon, “selection of the ground state,” and NLME predicts the decay of excited state, with half its energy transferred to the ground state and half to radiation modes. Our results predict the recent experimental observations of Mandelik et al. in nonlinear optical waveguides.
The main results we establish concern the time evolution of solutions to (1) for all data (with small energy). The large time behavior is as follows: For any initial data, with small energy the solution $\psi(t)$ of (1) is asymptotically given by

$$\psi(t) \to e^{-i\theta_0(t)} e^{-iE_1^t} \psi e^{iA_1} \psi_{\pm},$$

as $t \to \pm \infty$, in $L^2$. Generically, $j = 0$, corresponding to the ground state, while there is a finite dimensional set of data for which $j = 1$. Thus, for typical initial conditions we have ground state selection. We analyze the detailed dynamics by reduction of the NLS to a finite dimensional system, which dominates the full dynamics.

The main second result concerns the transient, finite time, behavior of the system, which is very rich: There are three terms (formative stage); once the initial radiative part approaches far enough from the support of $V(x)$, the soliton's position, the second period (embryonic stage) begins. This is marked by a monotonic (exponentially fast) increase in the ground state amplitude, relative to the excited state amplitude. This stage proceeds as long as the excited state part is larger than some fraction of the ground state part. Once the ground state reaches a size comparable with some (fixed) fraction of the excited state, the third and final stage begins (ground state selection); the ground state amplitude increases monotonically and the excited state decreases monotonically, but at a polynomial rate as $t \to \infty$. (This describes the generic case; the asymptotic state is given by an excited state [20] whenever the first stage persists to time equal infinity.) A different approach, based on linearization around the excited state for intermediate times and around the ground state for large times, gives similar results [21].

This phenomenon is quantified by our derivation of the nonlinear master equations (NLME) that govern the dynamics of the coupled ground and excited state modes (and the radiation modes). If we denote by $P_0(t)$ and $P_1(t)$, respectively, the (up to near-identity transformations) squared projection of the system's state onto the ground state and excited states, at time $t$, we have

$$\begin{align*}
\partial_t P_0 &= 2 \Gamma P_0 P_1^2 + \rho_0(t) P_0 P_0^{M_0} + O(t^{-5/2}), \\
\partial_t P_1 &= -4 \Gamma P_0 P_1^2 + \rho_1(t) \sqrt{P_0} P_1^{M_1} + O(t^{-5/2}),
\end{align*}$$

(2)

$$\Gamma = \pi g^2 |\mathcal{F}[dF(\psi_0, \psi_{1*})(\omega)]|^2,$$

$$dF(\psi_0, \psi_{1*})(\omega) \int (G(x - y) \psi_0(y) \psi_{1*}(y) dy) \psi_{1*}(x).$$

(3)

$\rho_{M_0}(t)$ and $\rho_{M_1}(t)$ are small oscillatory functions of time, $M_0, M_1 \geq 3$, and $\mathcal{F}[g](\omega)$ denotes the projection of $g$ onto the generalized eigenfunction of $H_0$ at frequency $\omega$. The crucial number $\Gamma$ is (generically) positive and given explicitly in terms of known eigenstates of $H_0$. $\Gamma \neq 0$ is the nonlinear Fermi golden rule for such systems and it gives the rate of decoherence and relaxation [17,18,22–24]. The behavior of these master equations (2) reflects the three time scales mentioned above, on which the behavior is very different. For large enough time, in the third time domain, the last two terms in (2) can be ignored. In this latter regime, it is easy to see that $P_1(t) \to 0$ as $t \to \infty$ and $\partial_t(2P_0 + P_1) = 0$. Therefore, $2P_0(\infty) = 2P_0(t_0) + P_1(t_0)$. It follows that

$$P_0(\infty) = P_0(t_0) + \frac{1}{2} P_1(t_0);$$

(4)

half of the excited state energy flows to the ground state (the other half goes to radiation), a kind of energy equipartition. (The factor 2 also appears in the linear analysis, and is interpreted as the ratio of relaxation to decoherence time [18,23].) Furthermore, it follows that for any initial state ($P_0(t_0)$, $P_1(t_0)$) with $P_0(t_0) \neq 0$ the system converges to $(P_0(\infty), 0)$ with a rate $\sim (1 + 4 P_0(\infty) P_1(t_0))^{-1}$ (selection of the ground state). Combining the analysis of the NLME (2) for all time scales with the previous statements gives a complete description of the solution for all time scales and, in particular, the asymptotic behavior, relaxation and decoherence rates, the asymptotic profile and energy of the soliton-ground state.

We sketch our method, for the special case of (1) with cubic nonlinearity, $F(|\psi|^2)\psi = |\psi|^2 \phi$. Our approach makes use of ideas from Refs. [13,22,25–27]. We begin with the Ansatz $\psi(t) = e^{-i0(t)} [\psi_0(t) + \psi_1(t) + \phi_2(t)]$ where $\psi_0(t) \equiv \psi_{E_0(t)}$ is a solution of the ground state eigenvalue equation with energy $E_0(t)$, at time $t$. $E_0(t)$ will be determined later by orthogonality conditions [13,22,26]. Similarly $\psi_1(t)$ is an excited state eigenvector with eigenvalue $E_1(t)$, $\theta(t) = \theta_0(t) + \theta_1(t)$; $\theta_0(t) = \int_0^t E_0(s) ds$. $\theta(t)$ will be chosen appropriately; it includes a (logarithmic) divergent phase. Substitution of the above Ansatz for $\psi$ into (1), and complexifying the equations $[\phi_2 \to (\phi_2, \phi_2') \equiv \Phi_2(t), (\psi_j \to (\psi_j, \psi_j')) = \Psi_j(t)$ etc., we derive

$$i \partial_t \Phi_2(t) = \mathcal{H}_0(t) \Phi_2(t) - i \partial_t \Psi_0$$

$$- \{[(E_0 - E_1) + i \partial_\theta] \sigma_3 + i \partial_t \} \Psi_1 + \tilde{E}_NL,$$

(5)

where $\tilde{E}_NL$ is nonlinear in $\Phi_2$, $\Psi_0$, $\Psi_1$, $\theta$ and $\mathcal{H}_0(t)$ is given by the matrix operator

$$\sigma_3 \left[ \begin{array}{c}
H - E_0(t) + 2g |\psi_0(t)|^2 \\
\bar{g} \bar{\psi}_0^2(t)
\end{array} \right] \left[ \begin{array}{c}
\psi_0(t) \\
\bar{\psi}_0^2(t)
\end{array} \right] \left[ \begin{array}{c}
H - E_0(t) + 2g |\psi_0(t)|^2 \\
\bar{g} \bar{\psi}_0^2(t)
\end{array} \right].$$

and $\sigma_3$ is the Pauli matrix diag(1, −1). We consider the spectrum of $\mathcal{H}_0(t)$ for fixed $t$ and $|\psi_0| = |\alpha_0|$ small: (a) The continuous spectrum extends from $-|E_0|$ to $-\infty$, and $|E_0|$ to $\infty$. Let $\mu = E_1 - E_0 + O(|\alpha_0|^2)$. The discrete spectrum is $\{0, \mu, \mu\}$, with $0 < |\mu| < |E_0|$ by assumption. (b) Zero is a generalized eigenvalue of $\mathcal{H}_0$, with generalized eigenspace spanned by $\{\sigma_3 \Psi_0, \partial_\theta \Psi_0\}$.

The discrete spectral subspace has dimension four. Therefore, $\Phi_2$ which lies in the continuous spectral part of $\mathcal{H}_0(t)$, is constrained by four orthogonality conditions.

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Furthermore, $\partial_t\tilde{\alpha}$ is chosen to remove divergent logarithmic phase contributions. In the weakly nonlinear (perturbative) regime, bound states have expansions $\psi_E = \alpha_j(\psi_j(x) + g|\alpha_j|^2\psi_j^{(1)}(x) + O(g^2|\alpha_j|^4))$ and $E_g = E_j + O(|\alpha_j|^2)$. The system for $\Phi_2$ and $\tilde{\alpha} = (\tilde{\alpha}_0, \tilde{\alpha}_1)$ can be written in the form: $i\partial_t\tilde{\alpha} = \mathcal{H}(t)\tilde{\alpha} + F_a$, $i\partial_t\Phi_2 = \mathcal{H}(t)\Phi_2 + F_q$, where $\tilde{\alpha}_j = (\alpha_j, \alpha_j^*)$.

To study the energy exchange between $\alpha_0$ and $\alpha_1$, it is important to express $\alpha_1(t)$ as a slow amplitude modulation of a rapidly varying phase. With this goal in mind, we study the equation for $\alpha_1$, expressible as $i\partial_t\tilde{\alpha}_1 = A(t)\tilde{\alpha}_1 + \tilde{F}$. Freezing $t$ at some arbitrary large time $T$ in $A(t)$, we solve $i\partial_t\tilde{\alpha}_1 = A(T)\tilde{\alpha}_1$ in terms of the Floquet solution matrix $X(t)$. We use this to eliminate the (fast) oscillation in $\tilde{\alpha}_1 \equiv X(t)\tilde{\beta}_1$. The difference $A(t) - A(T)$ and other similar differences are higher order corrections, uniformly in $t$, as $T \to \infty$, and therefore can be neglected.

To proceed further we decompose $\Phi_2$ into its continuous spectral (dispersive) part, $\eta$, relative to $\mathcal{H}_0(t)$, and its components along the discrete modes. The latter are higher order and controllable. Thus NLS at low energy is equivalent to a system of the form:

$$i\partial_t\eta = \mathcal{H}_0(t)\eta + \mathcal{F}_a(t;\alpha_0, \beta_1, \eta),$$

$$i\partial_t\beta_1 = 2g(\psi_{00}, \psi_{10}^2)|\beta_1|^2\alpha_0 e^{i(\mathcal{E}_1 - \mathcal{E}_0)t} + 2g(\psi_{00}, \psi_{10}^2(\Phi_2)\beta_1|\alpha_0^2 e^{i(\mathcal{E}_1 - \mathcal{E}_0)t} + \mathcal{R}_0,$$

$$i\partial_t\alpha_0 = g(\psi_{00}, \psi_{10}^2)|\beta_1|^2 e^{-2(\mathcal{E}_1 - \mathcal{E}_0)t} + \mathcal{R}_1,$$

where $\mathcal{R}_j$ denotes corrections of a similar form and higher order.

The above system can be viewed as an infinite dimensional Hamiltonian system consisting of two subsystems: a finite dimensional subsystem governing "oscillators," $(\alpha_0, \beta_1)$, and an infinite dimensional subsystem governing the field, $\eta$. Although this system has time-rapidly varying coefficients and no evident direction of energy flow, we claim we have now made explicit the key aspects, which give rise to resonant energy exchange. This is achieved via a detailed analysis of nonresonant and resonant terms. Nonresonant oscillatory contributions can be transformed by successive near-identity changes of variables, $(\alpha_0, \beta_1) \mapsto (\tilde{\alpha}_0, \tilde{\beta}_1)_r$, to higher order in the energy of the data (assumed small) and perturbatively controlled. Resonant oscillatory terms cannot be transformed to higher order and contribute to the finite dimensional reduction.

To arrive at the reduction, we solve the $\eta$ equation, making explicit all terms through second order in $g$, using the Green’s function $G(t,t') = e^{-i\mathcal{H}(t)(t-t')}$, we focus on the key terms coming from the sources in $\mathcal{F}_a$ or the type $\alpha_i^0\alpha_i^j$, $0 \leq i, j \leq 2$ and having oscillatory phases $e^{i\mathcal{E}_j(t)}. Their contribution to $\eta$ is of the form

$$\sim \int_0^t e^{-i\mathcal{H}(t)(t-t')} |\chi| e^{i\mathcal{E}_j(t')} \alpha_i^0(t') \alpha_i^j(t') dt'$$

where $\alpha_0, \alpha_1$ is a component of either $\tilde{\alpha}_0$ or $\tilde{\alpha}_1$ and where $|\chi|$ is an (exponentially localized) function of position, expressible in terms of $\psi_{00}$ and $\psi_{10}$. We insert this solution into the $\alpha_0, \alpha_1$ equations, in place of $\Phi_2$. We obtain integro-differential equations for $\alpha_0, \alpha_1, (\beta_1).$ Terms of the form (6) are solutions to a forced linear system and among the forcing terms made explicit in (6) are oscillatory terms with the frequency $\omega_\ast$, which are resonant with the continuous spectrum. Internal dissipation resulting in non-linear resonant energy transfer from the excited state to the ground state and to dispersive radiation is derived from these resonant terms; see also the derivation of internal dissipation in both linear and nonlinear resonance theories, recently developed by us [17,22,24,25]. The dissipation coefficient is $\Gamma$, the rate of decoherence and relaxation. The above described scheme gives $\alpha_0(t) = (-\lambda (\mathcal{E}_1 - \mathcal{E}_0) + R_0(t), i\partial_t\beta_1 = (\lambda - i\Gamma)|\alpha_0|^2|\beta_1|^2 + R_1(t)$.

Introducing the squared projections of the system’s state onto the ground state and excited states, $P_0 = |\alpha_0|^2, P_1 = |\beta_1|^2$ we obtain NLME, (2). The system (2) is analyzed in terms of renormalized powers, $Q_0$ and $Q_1$. It is shown that there exist transition times $t_0$ and $t_1$, such that: $Q_0(t) \text{ decays rapidly on } [0, t_0], Q_0(t)/Q_1(t) \text{ grows rapidly on } [t_0, t_1]$, and then finally on $[t_1, \infty)$ the following system governs: $Q_0(t) = 2Q_0Q_1^2, Q_1(t) = -4Q_0Q_1^2.$ This gives $Q_0(t) \uparrow Q_0(\infty)$ and $Q_1(t) \downarrow 0$ at rates discussed above.

The decay of the nonlinear excited state can also be understood as a linear instability. Let $\mathcal{H}_1$ denote the linearization about the nonlinear excited state, the operator in (5) with $\psi_{10}$ replaced by the excited state $\psi_{10}$. Since $\omega_\ast > 0$, in the limit of vanishing nonlinear terms, $\mathcal{H}_1$ has an embedded discrete eigenvalue in the continuous spectrum. Perturbation theory of embedded eigenvalues (see, for example, the time-dependent approach in Ref. [25,21]) can be applied to show that this embedded eigenvalue perturbs to a linear exponential instability with exponential rate $-\Gamma$ associated with the linear propagator $\exp(-i\mathcal{H}_1(t)).$
to optical devices is the use of periodic photonic micro-
structures with appropriately designed defects to trap co-
herent light pulses. For example, it has been shown theo-
rettically that (Kerr) nonlinear and periodic structures
support gap solitons [28] and trap pulses [29] traveling at
any speed less than the speed of light. Experiments have
demonstrated soliton propagation at about 50% of c [30] and
theoretical studies [31] show that gap solitons can be
trapped with appropriately designed defects. This has po-
tential applications to optical buffering, high-density stor-
gage and optical gates. The light stopping mechanism,
is based on the transfer of energy from incoming solitons to
the pinned nonlinear modes of the defect. The ground state
based on the transfer of energy from incoming solitons to
the ground state is related to trap co-
ping among two families of nonlinear bound states (soliton-
type) and dispersive radiation. We have derived nonlinear
master equations, which govern the dynamics. The pheno-
nomenon of ground state selection was demonstrated, the
rate of decay or relaxation was computed, and the energy
distribution of the asymptotic state of the system was
derived. These phenomena were recently observed experi-
mentally in Ref. [4]. Finally, we have shown the applica-
ibility of our theoretical results to nonlinear optical devices
and to BEC dynamics and decoherence phenomena.

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