# EQUILIBRIA IN MULTI-DIMENSIONAL, MULTI-PARTY SPATIAL COMPETITION<sup>1</sup>

James M. Snyder, Jr. Department of Political Science and Department of Economics Massachusetts Institute of Technology

> Michael M. Ting Department of Political Science and SIPA Columbia University

> > January 5, 2007

 $^1 \rm We$  thank Maggie Penn and panel participants at the 2005 Midwest Political Science Association Annual Meeting for comments on an earlier draft, and Alexander Wu for research assistance.

## Abstract

We develop a model of multi-candidate electoral competition in a multi-dimensional policy space with strategic voters. We show that when the distribution of voter ideal points is log-concave, there exist pure-strategy subgame perfect equilibria in weakly undominated strategies in which platforms are located in the minmax set. Analytical and numerical results suggest that the size of the minmax set is relatively "small" under reasonable assumptions.

## 1. Introduction

Almost all democracies have multi-party systems rather than two-party systems, and almost all democracies for which there are good data appear to have a multi-dimensional policy space. Yet, the formal theory literature offers relatively little to help us understand these systems.

In this paper we consider a standard game of multi-dimensional spatial electoral competition with three or more parties. We show that, under rather general conditions, pure-strategy, subgame-perfect Nash equilibria exist. If the minmax set is open, then we can always find equilibria where all parties locate inside the minmax set. We also begin to explore the limits on the set of equilibria. A key idea is that parties that are ranked *last* by too many voters cannot win in equilibrium.

We have not found these results previously in the literature. One reason they may have been overlooked is that most formal work on multi-party electoral politics has adopted the assumption of "sincere," or "naive," voting. And, given sincere voting, pure-strategy Nash equilibria in the multi-party spatial location game rarely exist, even in a one-dimensional setting.<sup>1</sup>

Almost all of the classic papers on multi-party spatial competition assume sincere voting, and most of the more recent papers do as well – e.g., Prescott and Visscher (1977), Palfrey (1984), Greenberg and Weber (1985), Cox (1985, 1987, 1989, 1990a, 1990b), Breyer (1987), Greenberg and Shepsle (1987), Shepsle and Cohen (1990), Weber (1992), Myerson (1993), and Osborne and Slivinski (1996). All of the papers on multi-party competition under probabilistic voting also assume sincere voting.<sup>2</sup> Also, the papers that assume some form of

<sup>&</sup>lt;sup>1</sup>Cox summarizes the situation with the following conjecture: "Multicandidate equilibria are just as rare in one dimension as two-candidate equilibria in many dimensions" (Cox, 1990a, p. 183; original in italics).

<sup>&</sup>lt;sup>2</sup>See, e.g., Wittman (1987), de Palma, Hong, and Thisse (1990), Anderson, Kats, and Thisse (1992), Schofield, Sened and Nixon (1998), Adams (1999a, 1999b), Lin, Enelow, Dorussen, (1999), Schofield (2004), and McKelvey and Patty (2006). This line of work strikes us as quite promising. But many (most?) theorists dislike the idea of relying on a probabilistic voting framework as the starting point for a general theory – even those who think a probabilistic voting framework is more "realistic" than a deterministic one. Also, the Nash equilibria of the these models are complicated, and the characterization of equilibria is typically based on computational work of specific cases. As a result, they convey relatively little intuition.

strategic voting also assume a one-dimensional policy space.<sup>3</sup>

Rather than assume sincere voting, we allow for strategic voting. In fact, we make a very weak assumption about voting strategies, which is that they are weakly undominated. This is what makes the *existence* of an equilibrium a relatively simple problem. There are a variety of different voting equilibria, and parties can have "conservative" beliefs – "we are doing okay where we are, and voters might turn against us if we change our positions" – that are consistent with rational voter behavior. The drawback is that the set of equilibria is large – too large, perhaps, to be very useful. The problem, then, is to *limit* the set of equilibria in some reasonable way. We establish one result on this, but it is only a start.<sup>4</sup>

A potentially promising alternative is to adopt a strategic voting model is in the spirit of Myerson and Weber (1993). This places more restrictions on what voters are allowed to believe and do, and may reduce the set of equilibria.<sup>5</sup>

## 2. Model and Results

#### 2.1 The Game

Assume the policy space is  $X \subset \Re^n$ , where X is convex. Assume there is a continuum

<sup>&</sup>lt;sup>3</sup>See, e.g., Austen-Smith and Banks (1988), Feddersen, Sened and Wright (1990), Feddersen (1992), Cox (1994), and Cox and Shugart (1996). See also Austen-Smith (1996) for a model of multi-district legislative elections with strategic voting. Much of the literature on legislative behavior assumes highly strategic voting, including Miller (1980), Banks (1985), Austen-Smith (1987), and Ordeshook and Palfrey (1988). Also, the most recent literature on voting-as-information-aggregation (Condorcet jury theorem) also assumes highly strategic voting – see, e.g., Austen-Smith and Banks (1996), and Feddersen and Pesendorfer (1996, 1998, 1999). These papers do not model electoral politics, however.

<sup>&</sup>lt;sup>4</sup>For a more general discussion of multi-candidate, multidimensional electoral competition with strategic voting, see Patty (2006). His work focuses primarily on conditions for equilibrium existence (which turn out to be quite weak), while we are interested primarily in characterizing a plausible set of winning election platforms.

<sup>&</sup>lt;sup>5</sup>A different strategic voting logic, developed in Feddersen, Sened and Wright (1990) and used in Feddersen (1992) and Besley and Coate (1997), makes it less likely that a multi-party equilibrium exists, even under strategic voting. The logic is as follows. Suppose the number of voters is finite. Suppose two or more parties are "winning" – i.e., tied for first place in the election. Then, each voter is pivotal. So, each voter must prefer the lottery over the positions of the set of winning parties to the individual position of each of the winning parties for which the voter did not vote. Otherwise, the voter could change her voting strategy and increase her payoff. This implies, among other things, that each voter must be voting for her top choice among the set of winning parties. Since this logic relies on exact ties, we hesitate to apply it to elections involving thousands or millions of voters. Palfrey (1989) explores another strategic voting logic.

of voters. Each voter has Euclidean preferences over X with an ideal point  $z \in X$ . The distribution of voter ideal points is given by a log-concave density f. Note that many probability distributions are log-concave, including the uniform, normal, exponential, and many elements of the Beta and Gamma families.

There are  $I \geq 3$  political parties. Parties care only about winning votes or offices. Let a voting outcome be a vector  $\mathbf{v} = (v_1, \ldots, v_I)$  such that  $\sum_i v_i = 1$ . Thus,  $v_i$  is the share of votes received by party *i*. Thus, each party *i* receives a payoff of  $u(\mathbf{v})$ , where  $\partial u/\partial v_i \geq 0$  for all  $\mathbf{v}$ , and  $\partial u/\partial v_i > 0$  for some  $\mathbf{v}$ .

The sequence of play is as follows. First, each party *i* takes a position  $x_i \in X$  simultaneously. Let  $\mathbf{x} = (x_1, \ldots, x_I)$  denote a vector of party positions. Second, all voters vote simultaneously for one of the candidates. We denote voter *z*'s strategy by  $e_z : X^I \to I$ , where  $e_z$  is a measurable function of  $\mathbf{x}$ . Finally, the election winner is determined by plurality rule, and the elected politician implements her ideal policy. We assume that election ties are broken fairly.

Voters vote strategically. Since there is a continuum of voters, however, no voter is ever pivotal. Thus, to restrict the possible voting equilibria in a natural way, we assume that all voters use weakly undominated strategies. An *equilibrium* is then a pure-strategy, subgame-perfect Nash equilibrium in weakly undominated strategies.

While many equilibria exist to this game, we focus on equilibria satisfying two criteria. The first is *symmetry*, which requires that all parties receive the same (1/I) proportion of votes.

The second requirement is that some parties choose positions inside the minmax set (Kramer, 1977). This set is defined as follows. For each pair of policies x and y with  $x \neq y$ , let n(x, y) be the fraction of voters who strictly prefer y to x. Let  $\tilde{n}(x) = \sup_{y} n(x, y)$  be the "vulnerability level" of x. And, let  $m^* = \inf_x \tilde{n}(x)$ . This is the minmax number – the minimum vulnerability level. Finally, let  $M^* = \{x | \tilde{n}(x) = m^*\}$ . This is the minmax set – the set of policies with vulnerability level equal to the minmax number.

Note that since the distribution of voter ideal points has a log-concave density f, for any

pair of policies x and y with  $x \neq y$ , the share of voters who are indifferent between x and y is zero. Thus, the fraction of voters who strictly prefer x to y is n(y, x) = 1 - n(x, y).

Caplin and Nalebuff (1991a) show that, given a convex set of alternatives X and a logconcave distribution of voter ideal points, the minmax number is bounded from above by  $1-1/e \approx 0.632$ .<sup>6</sup> In the simplest case, where X is unidimensional,  $M^*$  is simply the median voter and  $m^* = 1/2$ . Their result allows us to derive the following propositions.<sup>7</sup>

#### 2.2 The Minmax Set and Winning Platforms

Let  $b_i(\mathbf{x})$  be the number of voters who rank party *i* last given  $\mathbf{x}$ . The first result is that if the minmax set contains at least *I* points, then there exists a symmetric equilibrium in which all platforms are located inside.

**Proposition 1.** Suppose  $|M^*| \ge I$ . Let **x** be any vector of party positions such that  $x_i \in M^*$  for all *i*, and  $x_i \ne x_j$  for all *i* and *j*. Then there exists an equilibrium in which  $v_i = 1/I$  for all *i*.

**Proof.** Evidently, for each party  $i, b_i(\mathbf{x}) \leq \tilde{n}(x_i) = m^* \leq .64$ . This means, in particular, that  $\tilde{n}(x_i) = m^* \leq 2/3$ . Thus, the fraction of voters that do *not* rank party i last is  $1 - b_i(\mathbf{x}) \geq 1/3$ . Thus, we can always construct a voting equilibrium with  $v_i = 1/I$  for all i.

Consider voting strategies satisfying the following: (1) no voter votes for their least favorite party; (2) if the parties adopt the positions specified by  $\mathbf{x}$ , then voters divide their votes so  $v_i = 1/I \leq 1/3$  for all i; (3) if any party i deviates, then all voters who would have voted for it under  $\mathbf{x}$  vote for one of the other parties that they do not rank last, so  $v_i = 0$ .

To show the existence of an allocation of votes satisfying (1) and (2), we construct a voting partition of X iteratively. Pick any partition  $\{X_i\}_{j=1}^I$  of X such that  $e_z = i$  for all  $z \in X_i$  and  $v_i = 1/I$  for each *i*. Consider each  $X_j$  (j = 1, ..., I) in order. For each *j*, let  $\mu_{-j}$ 

<sup>&</sup>lt;sup>6</sup>Their Theorem 1 states that with a  $\rho$ -concave distribution of ideal points, the *mean* point of the density f has a vulnerability level of  $1 - \left[\frac{n+1/\rho}{n+1+1/\rho}\right]^{n+1/\rho}$ , where n is the dimensionality of X. This expression is monotonically increasing in n, less than .64, and has a limit as  $\rho \to 0$  (i.e., log-concavity) of 1-1/e. However, the mean point is not necessarily in the minmax set.

<sup>&</sup>lt;sup>7</sup>For an overview of other applications of these developments, see Caplin and Nalebuff (1991b).

be the measure of voters in  $X_j$  who rank party j last. If  $\mu_{-j} = 0$ , then iteration j is finished. Otherwise, the minmax set and log-concavity imply the existence of a measure of at least  $\overline{\mu}_{-j} = (I-1)/I - .64 + \mu_{-j}$  voters not voting for party j who do not rank party j last. Since  $I \ge 3$ ,  $\overline{\mu}_{-j} > \mu_{-j}$ . Now for the set of voters in  $X_j$  ranking party j last, we can find a set of measure  $\mu_{-j}$  of voters not voting for party j who do not rank party j last. Finally, exchange the votes of these two sets. Thus no voters in  $X_j$  choose their lowest-ranked candidate and no voters who switch votes to party j rank it last, while the voting outcome remains  $v_i = 1/I$ for each i.

Thus each voter is choosing a weakly undominated strategy, and no party wants to deviate, so we have a subgame perfect equilibrium in weakly undominated strategies. Q.E.D.

The intuition of Proposition 1 is that since  $m^* < 2/3$ , then at least 1/3 > 1/I voters do not rank each candidate last. This makes it possible for 1/I voters not to choose a weakly dominated strategy in voting for each candidate.<sup>8</sup>

There are subgame perfect equilibria in which some parties adopt the same positions in  $M^*$ , but not all of these parties can win in such equilibria. Thus, equilibria featuring policy convergence are not necessarily symmetric. As an example, suppose that I = 3 and  $x_1^* = x_2^*$ . Then all voters who prefer  $x_3^*$  to  $x_2^*$  must vote for party 3. This is true for at least 36% of voters, so party 3 wins with certainty if parties 1 and 2 split the remaining 64% evenly. Note, however, that this argument does not hold when there are at least three distinct platforms.

There can also be subgame perfect equilibria where some parties locate "far away" from the minmax set, but such parties cannot win in a plurality-rule contest. In particular, if party *i* is ranked last by at least (I - 1)/I of the voters, then it cannot come in first.

One issue with Proposition 1 is that the number of parties may exceed the size of the minmax set. This will be true when n = 1 (in which case the minmax set is simply the median voter), and in some cases for higher dimensions as well.<sup>9</sup> Caplin and Nalebuff (1988)

 $<sup>^{8}\</sup>mathrm{The}$  result can be also be stated more generally in terms of the minmax number, rather than the minmax set.

<sup>&</sup>lt;sup>9</sup>Kramer (1977) argues that the size of the minmax set is increasing in the level of social agreement, and so more uniform distributions of voter ideal points will tend to have smaller minmax sets.

show that if f is uniform and X is a triangle, then the minmax set is the triangle's center of gravity. For such cases, as well as others in which the minmax set is not open, it is still possible to show the existence of a symmetric equilibrium in which some platforms are in the minmax set.

**Proposition 2.** There exists an equilibrium in which  $x_i \in M^*$  for some  $i, x_i \neq x_j$  for  $i \neq j$ , and  $v_i = 1/I$  for all i.

**Proof.** Without loss of generality, suppose that  $x_1^* \in M^*$ . We construct a vector of I platforms **x** such that  $b_i < 1 - 1/I$  for each platform  $x_i$ . Since  $m^* < .64$ , this holds trivially for  $x_1$ . Observe that for any platform z, there exists another platform y such that n(z, y) > 0.36. Let  $z = x_1$  and choose any such y as  $x_2$ . Repeat this operation for i > 2 by substituting  $x_{i-1}$  for z and setting  $x_i = y$ , where  $x_i \neq x_j$  for all j < i. To ensure that distinct platforms can be chosen, note that because f is continuous and preferences are Euclidean, for any y' and z and  $\lambda \in (0, 1)$ ,  $n(z, \lambda z + (1 - \lambda)y')$  is strictly increasing in  $\lambda$ . Thus there are an infinite number of points  $(\{y'\})$  satisfying the desired criteria for y.

As constructed, at least .36 of citizens prefer each  $x_i$  to some  $x_j$   $(j \neq i)$ . Thus,  $b_i(\mathbf{x}) < 1 - 1/I$  for  $I \geq 3$ . We may therefore apply the argument from Proposition 1, establishing the result. Q.E.D.

This result uses the continuity of f, which ensures that for any platform z there exists a nearby platform y that many voters will prefer to z. Since  $m^* \ge 1/2$  for all n, it is always possible to find y such that n(z, y) is near 1/2. As a result, y will not be the least-preferred choice of enough voters, which makes the construction of a symmetric equilibrium possible.

#### 2.3 A Partial Characterization of the Minmax Set

For many distributions, the minmax set is difficult to characterize analytically. However, we may exploit a relationship between the minmax set and Pareto sets to derive an analytical bound of the minmax set. To see this, denote by  $B_z(p)$  the ball centered around z containing proportion p of voter ideal points. The next result shows that the minmax set must be contained within the intersection of all  $B_z(m^*)$ .

## **Proposition 3.** $M^* \subseteq \bigcap_{z \in X} B_z(m^*).$

**Proof.** Let  $P(m^*)$  be any Pareto set that contains at least proportion  $m^*$  of voter ideal points. Since preferences are Euclidean and f is continuous and log-concave, for any  $x \notin P(m^*)$ , there exists some  $x' \in P(m^*)$  such that  $n(x, x') > m^* + \epsilon$  for some  $\epsilon > 0$ . Thus,  $M^* \subseteq P(m^*)$ . Since any  $B_z(m^*)$  is a Pareto set,  $M^*$  must belong to all such balls. Q.E.D.

With a log-concave distribution of ideal points,  $M^*$  is then a subset of all balls B(2/3), since  $M^*$  lies in the intersection of all Pareto sets with more than  $m^*$  voters in them, and  $m^* < 2/3$ . Thus, for equilibria of the type identified in Propositions 1 and 2, one can derive an "outer bound" on platform locations simply by constructing the intersection of balls with probability mass 2/3.

## **3.** Examples

#### 3.1 Numerical Examples

The following numerical example illustrates the preceding construction. The program, written in R, first draws a random sample of N points from a multivariate normal distribution. The distribution is centered at **0** and has uniform variance and no covariance.<sup>10</sup> It then estimates the boundary of the intersection of the two-thirds balls, and categorizes points as inside or outside this boundary.

The categorization algorithm works as follows. Along each dimension in a given sample, points are sorted from lowest to highest. The distances between **0** and the points at which two-thirds of the points have lower and higher values provide two estimates of the bounds of  $\bigcap_{z \in X} B_z(m^*)$ . This is based on the fact that a ball centered sufficiently far from **0** along this dimension has an arc approximated by an orthogonal hyperplane intersecting either of the aforementioned points. Each point with a rank order higher than one third of the points along

<sup>&</sup>lt;sup>10</sup>Code available upon request.

this dimension is therefore within the ball  $B_z(2/3)$ , where the element of z in this dimension is  $\infty$ ; likewise, each point with a rank lower than two thirds of the points along this dimension is within  $B_z(2/3)$ , where the element of z in this dimension is  $-\infty$ . Note that these twothirds boundaries provide the smallest estimate of the theoretical radius of  $\bigcap_{z \in X} B_z(2/3)$ along this dimension, as they are the limiting case for all circles. Due to radial symmetry, finding the value in one dimension provides an estimate of the radius in all directions, but estimating in more than one direction provides greater accuracy. The estimated radius is calculated by averaging across both estimates from all dimensions. Distances from **0** can be easily calculated for each point, and all points further than the radius are classified as outside  $\bigcap_{z \in X} B_z(2/3)$ .

Table 1 summarizes the results from 150 trials. The program was run 10 times for each combination of dimensions  $(1, \ldots, 5)$  and variance (1, 2, 3). Each trial drew N = 10,000 points, except for the five-dimension trials, which drew N = 100,000 for greater accuracy. As intuition suggests, the proportion of points within all  $B_z(2/3)$  is about one third for the unidimensional case. Note that for each dimension, the proportion within the ball is invariant with respect to variance, and that for each variance level, the proportion within the ball is invariant with respect to dimension. The table clearly shows that the proportion of points in the intersection falls dramatically as the number of dimensions increases. Even with three dimensions, only about 2% of the points are located within our estimated upper bound of the minmax set.

#### [Table 1 here]

The following figure plots the results of one of the trials for the two dimensional case.

#### [Figure 1 here]

One possible objection to this example is that the symmetry of the normal distribution implies the satisfaction of the Plott (1967) conditions, which are sufficient for the existence of a majority rule core. We therefore modified the simulation slightly to accommodate multivariate Type I extreme value distributions. As the next figure illustrates, the results remain similar to those of the normal case.

## [Figure 2 here]

#### 3.2 An Analytical Example

In some cases, the bounds on the minmax set suggested by Proposition 3 can be derived analytically. Suppose that  $X \equiv \left\{ \mathbf{x} \mid -1 \leq x_1 \leq 1, 0 \leq x_2 \leq \sqrt{1-x_1^2} \right\}$  is a unit half-circle, and  $f(z) = 2/\pi$  is uniform over z. Using the same argument as in the proof of Proposition 3, it is straightforward to derive some Pareto sets which bound the minmax set.

Consider the following four Pareto sets;  $P_1 \equiv \{\mathbf{x} \in X \mid x_1 \leq k_1\}, P_2 \equiv \{\mathbf{x} \in X \mid x_1 \geq k_2\}, P_3 \equiv \{\mathbf{x} \in X \mid x_2 \leq k_3\}$ , and  $P_4 \equiv \{\mathbf{x} \in X \mid x_2 \geq k_4\}$ . We are therefore interested in finding  $k_i$  such that each  $P_i$  has probability mass 2/3. The integral of the boundary of X is  $[\sqrt{1-x^2} + \sin^{-1}x]/2 + c$ . From this, it is easily seen that  $k_1 \approx 0.265$  and  $k_3 \approx 0.553$ . By symmetry,  $k_1 = -k_2 = k_4$ . Taking the intersection of these sets yields a rectangular superset of the minmax set with an area of 0.153, or about 9.7% of the probability mass of the ideal points under f.

## 4. An Extension: The Centrality of Platforms

Our results suggest that the minmax set deserves some attention as a "solution set" for multi-dimensional electoral competition. The main intuition behind our results has been the importance of not being the lowest-ranked candidate by at least 1/I of the voters. Here we push that logic a step further, and derive a more general result on the centrality of equilibrium platform locations.

As before, let  $B_z(p)$  be a ball centered at z containing proportion p of the ideal points, and let r(p) be its radius. Let  $B_z^3(p)$  be the ball centered at z with radius 3r(p). The next result uses weak dominance (but not the minmax set) to establish that in a symmetric equilibrium, if all platforms but one are centrally located in the sense of being located within some  $B_z(p)$ , then none can be outside of  $B_z^3(p)$ . **Proposition 4.** Let p > (I-1)/I. For any symmetric equilibrium and  $B_z(p)$ , if  $|\{x_i \mid x_i \in B_z(p)\}| = (I-1)/I$ , then  $x_i \in B_z^3(p)$  for all *i*.

**Proof.** Let the equilibrium platform be **x**. Suppose to the contrary (and without loss of generality) that  $x_1, \ldots, x_{I-1} \in B_z(p)$  and  $x_I \notin B_z^3(p)$ . Clearly, all voters in  $B_z(p)$  rank party I last. It follows that  $b_I(\mathbf{x}) > (I-1)/I$ , or equivalently that less than 1/I do not rank party I last. But for party I to win under weakly undominated strategies, at least 1/I of the voters must not rank it last: contradiction. Q.E.D.

To see the intuition for Proposition 4, let I = 3 and suppose that parties 1 and 2 both locate inside  $B_z(2/3)$  and party 3 locates outside  $B_z^3(2/3)$ . Then all voters in  $B_z(2/3)$  must rank party 3 lowest, which implies that less than one third of voters do not rank party 3 lowest. Thus, party 3 cannot win without some voters using weakly dominated strategies, and no symmetric equilibrium exists.

This result applies to all values of p and any ball center z, and therefore restricts the extent to which platforms may be dispersed across X. Taking p = 2/3 allows us to connect the result with those of the minmax set. If I - 1 platforms are located in the minmax set, then by Proposition 3 they are also located within  $\bigcap_{z \in X} B_z(2/3)$ . In a symmetric equilibrium, the remaining platform is therefore located within  $\bigcap_{z \in X} B_z^3(2/3)$ .

Note finally that since Proposition 4 bounds the locations of possible election winners, it also has implications for games of electoral competition with endogenous and costly entry. Consider a citizen-candidate model with strategic voters. Given the presence of at least two "centrist" citizen-candidates, no sufficiently extreme citizen can enter and expect to win with positive probability. Thus the result provides some modest limits the set of candidates that can be expected in citizen-candidate competition.

## 5. Discussion

We have developed a model of multi-party elections in a multi-dimensional policy space. While this scenario describes electoral competition in many countries, there are relatively few results that characterize the equilibria of such games. In our game, voters are strategic and play subgame-perfect Nash equilibrium strategies. We show that the minmax set can provide considerable guidance in predicting the location of party platforms. Additionally, these locations can be shown to be relatively small subsets of the policy space under reasonable conditions.

A useful direction for future work will be to impose restrictions on voting behavior that may further restrict the set of plausible platform locations. One natural alternative is to examine the role of voter beliefs. A simpler one may be to explore different voter coordination strategies. For example, voters may coordinate on a candidate if she is preferred by a majority over all other candidates. In this case, there must be one platform in every  $B_z^3(1/2)$  ball, as a party could "defect" to a location within  $B_z(1/2)$  and attract majority support. Under either approach, it may be possible to explore the intuition that in multi-party competition, losing (extremist) parties can play a role in constraining winning (centrist) parties.

## References

- Adams, James. 1999a. "Multiparty Spatial Competition with Probabilistic Voting." *Public Choice* 99: 259-274.
- Adams, James. 1999b. "Policy Divergence in Multicandidate Probabilistic Spatial Voting." Public Choice 100: 103-122.
- Anderson, Simon P., Amoz Kats, and Jacques-Francois Thisse. 1994. "Probabilistic Voting and Platform Selection in Multi-Party Elections." Social Choice and Welfare 11: 305-322.
- Austen-Smith, David. 1987. "Sophisticated Sincerity: Voting Over Endogenous Agendas." American Political Science Review 81: 1323-1330.
- Austen-Smith, David. 1996. "Electing Legislatures." In Collective Decision-making: Social Choice and Political Economy, edited by Norman Schofield. Boston: Kluwer Academic Publishers.
- Austen-Smith, David and Jeffrey S. Banks. 1988. "Elections, Coalitions and Legislative Outcomes." American Political Science Review 82: 405-22.
- Austen-Smith, David and Jeffrey S. Banks. 1996. "Information Aggregation, Rationality, and the Condorcet Jury Theorem." *American Political Science Review* 90: 34-45.
- Banks, Jeffrey S. 1985. "Sophisticated Voting Outcomes and Agenda Control." Social Choice and Welfare 1: 295-306.
- Besley, Timothy and Stephen Coate. 1997. "An Economic Model of Representative Democracy." *Quarterly Journal of Economics* 112: 85-114.
- Breyer, Friedrich. 1987. "On the Existence of Political Equilibrium in a Three-Party System with Plurality Voting." In *The Logic of Multiparty Systems*, edited by Manfred J. Holler. Dordrecht: Kluwer Academic Publishers.
- Caplin, Andrew, and Barry Nalebuff. 1988. "On 64% Majority Rule." *Econometrica* 56: 787-814.
- Caplin, Andrew, and Barry Nalebuff. 1991a. "Aggregation and Social Choice: A Mean Voter Theorem." *Econometrica* 59: 1-23.
- Caplin, Andrew, and Barry Nalebuff. 1991b. "Aggregation and Imperfect Competition: On the Existence of Equilibrium ." *Econometrica* 59: 25-59.
- Cox, Gary W. 1985. "Electoral Equilibrium under Approval Voting." American Journal of Political Science 29: 112-118.
- Cox, Gary W. 1987. "Electoral Equilibrium under Alternative Voting Institution." American Journal of Political Science 31: 82-108.

- Cox, Gary W. 1989. "Undominated Candidate Strategies under Alternative Voting Rules." Mathematical Computer Modeling 12: 451-459.
- Cox, Gary W. 1990a. "Multicandidate Spatial Competition." In Advances in the Spatial Theory of Voting, edited by James M. Enelow and Melvin J. Hinich. Cambridge: Cambridge University Press.
- Cox, Gary W. 1990b. "Centripetal and Centrifugal Incentives in Electoral Systems." American Journal of Political Science 34: 903-935.
- Cox, Gary W. 1994. "Strategic Voting Equilibria Under the Single Nontransferable Vote." American Political Science Review 88: 608-621.
- Cox, Gary W. and Matthew Soberg Shugart. 1996. "Strategic Voting under Proportional Representation." Journal of Law, Economics, and Organization 12: 299-324.
- de Palma, Andre, Gap-Seon Hong, and Jacques-Francois Thisse. 1990. "Equilibria in Multi-Party Competition under Uncertainty." Social Choice and Welfare 7: 247-259.
- Feddersen, Timothy J. 1992. "A Voting Model Implying Duverger's Law and Positive Turnout." American Journal of Political Science 36: 938-962.
- Feddersen, Timothy J. and Wolfgang Pesendorfer. 1996. "The Swing Voter's Curse." American Economic Review 86: 408-424.
- Feddersen, Timothy J. and Wolfgang Pesendorfer. 1998. "Convicting the Innocent: The Inferiority of Unanimous Jury Verdicts under Strategic Voting." American Political Science Review 92: 23-35.
- Feddersen, Timothy J. and Wolfgang Pesendorfer. 1999. "Abstention in Elections with Asymmetric Information and Diverse Preferences." American Political Science Review 93: 381-398.
- Feddersen, Timothy J., Itai Sened, and Stephen G. Wright. 1990. "Sophisticated Voting and Candidate Entry Under Plurality Rule." American Journal of Political Science 34: 1005-1016.
- Greenberg, Joseph, and Shlomo Weber. 1985. "Multiparty Equilibria under Proportional Representation." *American Political Science Review* 79: 693-703.
- Greenberg, Joseph, and Kenneth Shepsle. 1987 "The Effect of Electoral Rewards in Multiparty Competition with Entry." *American Political Science Review* 81: 525-537.
- Kramer, Gerald H. 1977. "A Dynamic Model of Political Equilibrium." Journal of Economic Theory 16: 310-334.
- Lin, Tse-min, James M. Enelow, and Han Dorussen. 1999. "Equilibrium in Multicandidate Probabilistic Spatial Voting." *Public Choice* 98: 59-82.

- McKelvey, Richard D., and John W. Patty. 2006. "A Theory of Voting in Large Elections." *Games and Economic Behavior*, forthcoming.
- Miller, Nicholas. R. 1980. "A New Solution Set for Tournaments and Majority Voting." American Journal of Political Science 24: 68-96.
- Myerson, Roger B. 1993. "Incentives to Cultivate Favored Minorities Under Alternative Electoral Systems." *American Political Science Review* 87: 856-869.
- Myerson, Roger B., and Robert J. Weber. 1993. "A Theory of Voting Equilibria." American Political Science Review 87: 102-114.
- Osborne, Martin J. and Al Slivinski. 1996. "A Model of Political Competition with Citizen-Candidates." *Quarterly Journal of Economics* 111: 65-96.
- Ordeshook, Peter C., and Thomas R. Palfrey. 1988. "Agendas, Strategic Voting, and Signaling with Incomplete Information." American Journal of Political Science 32: 441-466.
- Palfrey, Thomas R. 1984. "Spatial Equilibrium with Entry." *Review of Economic Studies* 51: 139-156.
- Palfrey, Thomas R. 1989. "A Mathematical Proof of Duverger's Law." In Models of Strategic Choice in Politics, edited by Peter C. Ordeshook. Cambridge: Cambridge University Press.
- Patty, John W. 2006. "Two's Company, Three's and Equilibrium: Strategic Voting and Electoral Equilibrium in Multicandidate Elections." Unpublished manuscript, Harvard University.
- Plott, Charles R. 1967. "A Notion of Equilibrium and its Possibility Under Majority Rule." American Economic Review 57(4): 787-806.
- Prescott, E. and M. Visscher. 1977. "Sequential Location Among Firms with Foresight." Bell Journal of Economics 8: 378-393.
- Schofield, Norman. 2004. "Equilibrium in the Spatial 'Valence' Model of Politics." Journal of Theoretical Politics 16: 447-481.
- Schofield, Norman, Itai Sened and David Nixon. 1998. "Nash Equilibria in Multiparty Competition with Stochastic Voters." Annals of Operations Research 84: 3-27.
- Shepsle, Kenneth A. and Ronald N. Cohen. 1990 "Multiparty Competition, Entry, and Entry Deterrence in Spatial Models of Elections." In Advances in the Spatial Theory of Voting, edited by James M. Enelow and Melvin J. Hinich. Cambridge: Cambridge University Press.
- Weber, Sholomo. 1992. "On Hierarchical Spatial Competition." Review of Economic Studies 59: 407-425.

Wittman, Donald A. 1987. "Elections with N voters, M Candidates and K Issues." In *The Logic of Multiparty Systems*, edited by Manfred J. Holler. Dordrecht: Kluwer Academic Publishers.

| Table 1   Approximate Bounds of Minmax Set |         |   |   |   |
|--|---------|---|---|---|
| Normally distributed ideal points          |         |   |   |   |
| Dimensions                                 | N       | Variance                                | $\begin{array}{c} \text{Mean Radius} \\ \text{of } \cap_{z \in X} B_z(2/3) \end{array}$ | Mean Proportion<br>in $\bigcap_{z \in X} B_z(2/3)$        |
| 1  | 10,000  | 1<br>2<br>3                             | $0.4367 \\ 0.6078 \\ 0.7478$  | $\begin{array}{c} 0.3334 \\ 0.3334 \\ 0.3329 \end{array}$ |
| 2  | 10,000  | $\begin{array}{c}1\\2\\3\end{array}$    | $\begin{array}{c} 0.4310 \\ 0.6077 \\ 0.7463 \end{array}$                               | $\begin{array}{c} 0.0886 \\ 0.0887 \\ 0.0888 \end{array}$ |
| 3  | 10,000  | $\begin{array}{c} 1\\ 2\\ 3\end{array}$ | $0.4300 \\ 0.6087 \\ 0.7451$  | $\begin{array}{c} 0.0202 \\ 0.0208 \\ 0.0206 \end{array}$ |
| 4  | 10,000  | 1<br>2<br>3                             | $0.4288 \\ 0.6123 \\ 0.7477$  | $\begin{array}{c} 0.0038 \\ 0.0044 \\ 0.0040 \end{array}$ |
| 5  | 100,000 | 1<br>2<br>3                             | $0.4308 \\ 0.6094 \\ 0.7461$  | 0.0007<br>0.0008<br>0.0007                                |

\* averaged over ten trials per dimension/variance set.



Figure 1: Approximate upper bound on the minmax set. Bivariate Normal distribution with variance 2 and zero covariance; N = 10,000. The central white area is the approximate intersection of all balls with mass of 2/3. It has a radius of 0.61 and contains 9% of the ideal points.



Figure 2: Approximate upper bound on the minmax set. Bivariate Type I extreme value distribution with  $\alpha = 0.6$ ; N = 10,000. The central white area is the approximate intersection of all balls with mass of 2/3. It has a radius of 0.49 and contains 10.9% of the ideal points.