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A Binomial Lattice Method for Pricing Corporate Debt and Modeling Chapter 11 Proceedings

Mark Broadie and Özgür Kaya*

Abstract

The pricing of corporate debt is still a challenging and active research area in corporate finance. Starting with Merton (1974), many authors proposed a structural approach in which the value of the assets of the firm is modeled by a stochastic process, and all other variables are derived from this basic process. These structural models have become more complex over time in order to capture more realistic aspects of bankruptcy proceedings. The literature in this area emphasizes closed-form solutions that are derived by either partial differential equation methods or analytical pricing techniques. However, it is not always possible to build a comprehensive model with realistic model features and achieve a closed-form solution at the same time. In this paper, we develop a binomial lattice method that can be used to handle complex structural models such as ones that include Chapter 11 proceedings of the U.S. bankruptcy code. Although lattice methods have been widely used in the option pricing literature, they are relatively new in corporate debt pricing. In particular, the limited liability requirement of the equity holders needs to be handled carefully in this context. Our method can be used to solve the Leland (1994) model and its extension to the finite maturity case, the more complex model of Broadie, Chernov, and Sundaresan (2005), and others.

I. Introduction

A model for pricing risky corporate debt is important for both determining optimal capital structure and explaining observed yield spreads. In this paper, we are interested in the numerical evaluation of structural models of corporate debt valuation. In a structural model, the value of the firm's assets or the firm's earnings process is modeled as a primitive variable, and all other variables are derived from this basic variable. There is some freedom as to which aspects of the debt contracts to include in a model because these contracts are not uniform in practice. Contractual agreements and bankruptcy laws may lead to different treatments when the firm fails to make debt payments and declares bankruptcy. For

*Broadie, mnb2@columbia.edu, Columbia University, Graduate School of Business, 3022 Broadway, New York, NY 10027-6902; Kaya, ok94@columbia.edu, Columbia University, Department of Industrial Engineering and Operations Research, 500 West 120th St., New York, NY, 10027-6699. We thank Erwan Morellec (the referee) for many helpful comments and suggestions that improved the paper. This work was supported in part by NSF grant DMS-0410234.

example, bankruptcy may lead to liquidation under Chapter 7 of the bankruptcy code, reorganization under Chapter 11, or the debt may be renegotiated privately between debt holders and equity holders. The literature in this field is still growing as researchers add more complexity to the models in order to make their models more realistic. Our aim in this paper is to introduce a numerical method that can be used to solve complex models when analytical pricing techniques are not available.

Merton (1974) first used a structural model for the valuation of risky zero coupon bonds and risky perpetual coupon bonds. Black and Cox (1976) extended this work by considering the case when asset sales are not allowed and equity dilution is necessary to make coupon payments. Leland (1994) gives closed-form solutions for perpetual coupon bonds in a general setting that includes costly bankruptcy, tax benefits to coupon payments, and cash payouts by the firm. All of these models treat bankruptcy and liquidation as the same event: the firm is taken over by debt holders and liquidated when debt payments are not made in full.

More recent research attempts to model Chapter 11 proceedings by treating bankruptcy and liquidation events separately. In François and Morellec (FM hereafter) (2004), after the firm value hits an endogenous barrier the equity holders start servicing the debt strategically for the duration of a certain grace period. The firm is liquidated only if it stays in bankruptcy more than the granted grace period. The debt service while in bankruptcy is determined by a Nash bargaining game. In other recent work, Broadie, Chernov, and Sundaresan (BCS hereafter) (2005) consider a similar setting, but they assume that instead of strategic debt service while in bankruptcy, the coupon payments are stopped and are recorded in an arrears account. The earnings of the firm are collected in a separate account. The collected earnings are then used to pay the arrears when the firm comes out of bankruptcy. The firm is liquidated if it stays in bankruptcy for more than the granted exclusivity period. Other models that treat bankruptcy and liquidation as separate events include Galai, Raviv, and Wiener (2003), Moraux (2002), and Paseka (2003).

Most of the existing models in the literature attempt to derive analytical valuation formulas for debt and equity values by using simplifications to avoid time and path dependence. To work in a time-independent setting, these models usually price infinite maturity bonds although these bonds are almost never used in practice. It is difficult to obtain analytical solutions in models of bankruptcy proceedings that include automatic stay provisions, arrears payments, and grace periods since these features introduce path dependency. Therefore, the events that happen while the firm is in bankruptcy are simplified by introducing bargaining games at the boundary or by complete debt forgiveness.

Recognizing that it is hard to build a model with realistic features and preserve analytical solutions at the same time, this paper introduces a numerical method that can help extend existing models or build more complex ones. We use lattice methods that are already very common in the option pricing literature but have rarely been used in corporate debt pricing. We model the evolution of the firm's assets on a discrete lattice and then use a backward solution procedure for the valuation of all other securities. We show that the delicate issue of the limited liability of equity holders can easily be handled using our method. Path dependencies are also incorporated into our numerical method by increasing the state space to record the value of path-dependent quantities. Our numerical method can be used to extend models such as Leland (1994) and FM (2004) to finite maturity debt, discrete coupon payments, or to solve new models such as BCS (2005).

Some other authors use numerical techniques in the context of corporate debt pricing. Brennan and Schwartz (1978) solve the partial differential equation (PDE) for a firm that issues a bond paying discrete coupons. Their model is very restrictive with an exogenous bankruptcy boundary, and their valuation method does not give debt and equity values separately. Anderson and Sundaresan (1996) use a binomial lattice method to price finite maturity bonds, but they do not allow equity dilution. They work in a simplified framework in which equity holders and debt holders interact in an extensive form game to determine debt service when firm cash flows are not enough to make the coupon payment. Anderson, Sundaresan, and Tychon (1996) show that it is possible to recast the Anderson-Sundaresan model in continuous time using PDEs. Anderson and Tu (1998) show how to solve the related PDEs using finite difference methods. Fan and Sundaresan (2000) also use PDEs for valuation of finite maturity debt in a very similar setting. However, all these models are models of renegotiation rather than reorganization in the sense of Chapter 11. Galai, Raviv, and Wiener (2003) consider a model in which the firm's excursions in the bankruptcy region can be given different weights based on severity and recentness. They assume there is an exogenous bankruptcy boundary and use Monte Carlo simulation for the valuation of zero coupon bonds. However, it is not straightforward to use simulation methods to price coupon bonds when equity dilution is allowed. The value of equity needs to be known in order to decide if it is possible to make the coupon payment, and this is not available when working forward in time.

As in some of the above-mentioned papers, it may be possible to write the PDEs for equity, debt, and firm values and to solve them using finite difference methods. Although the resulting PDEs are usually simple and easy to solve numerically for plain vanilla models, they become much harder to solve once path-dependent variables such as grace period and arrears are introduced. It is not clear how to treat these variables when discretizing the state space in a finite difference method. Our lattice method is much more intuitive and easier to implement for both simple and complex models. By attaching auxiliary variables to lattice nodes below the bankruptcy boundary and by using interpolation techniques when necessary, these complexities are easily handled.

The rest of the paper is organized as follows. In Section II, we introduce the basic setup. In Sections III–V, we describe the implementation of our method for models with increasing complexity. We give some computational analysis in Section VI. Section VII concludes the paper.

II. Basic Setup

We denote the firm's asset value as V_t and use it as the primitive variable, and hence all other variables can be seen as derivatives with respect to the asset value. We assume that the value of V_t is independent of the capital structure choices and its evolution under the risk-neutral measure $\mathbb{Q}$ is given by

$$\frac{dV_t}{V_t} = (r-q)dt + \sigma dW_t, \tag{1}$$

where W_t is a standard Brownian motion under $\mathbb{Q}$, q is the payout ratio (i.e., cash flow) of the firm, and σ is the volatility of asset returns. We assume that the risk-free rate is constant at r, and that investors may lend and borrow freely at that rate. The process given above for the firm's asset value uses the same representation as in the Black-Scholes model for a stock price that pays dividends at a constant rate q.

The instantaneous cash generated by the firm is denoted by δ_t and is given by

$$\delta_t = qV_t. \tag{2}$$

We assume that the firm issued a bond that promises to pay coupons at constant total rate C, continuously in time, until a default event occurs. The coupon is paid from the firm's generated cash flow δ_t at time t and equity holders receive any surplus $\delta_t - C$ in the form of dividends. There may be cases when $C > \delta_t$, i.e., the cash flows produced by the firm are not enough to make the coupon payment. For modeling purposes, we can treat $(C - \delta_t)^+$ as a negative cash flow for equity holders when $C > \delta_t$. We will demonstrate that this is equivalent to dilution of equity by the firm.

A. Binomial Lattice Method

We take V_t as our primitive variable, and using the representation in (1), build a lattice based on the binomial method of Cox, Ross, and Rubinstein (1979). It is straightforward to adapt our method to other binomial, trinomial, and multinomial lattices. We start with an initial asset value V. Suppose that time horizon is divided into small increments of length Δt. In the next time increment, the value of the assets can increase by a factor of u to become $V_u = uV$, or decrease by a factor of d to become $V_d = dV$. The probability of an up move is p and the probability of a down move is $1-p$. Choosing these parameters to match the mean and variance of the continuous time process and imposing $u = 1/d$, we obtain

$$\begin{aligned} u &= e^{\sigma \Delta t}, \\ d &= e^{-\sigma \Delta t}, \\ p &= \frac{a-d}{u-d}, \end{aligned} \tag{3}$$

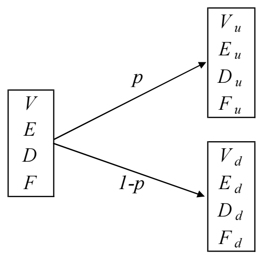
where

$$a = e^{(r-q)\Delta t}. \tag{4}$$

We want to compute the initial values of three quantities on the lattice. The first of these is the claim of the equity holders after the debt is issued, which we

FIGURE 1

Binomial Step



denote by E. The second quantity is the claim of the debt holders, which we denote by D. Finally, we want to compute the total firm value, which is denoted by F. A generic binomial step is shown in Figure 1.

At the current node, the present value of the equity is given by

$$E = e^{-r\Delta t}\left(pE_u + (1-p)E_d\right). \tag{5}$$

The values of D and F can be calculated in a similar way:

$$D = e^{-r\Delta t}\left(pD_u + (1-p)D_d\right), \tag{6}$$
$$F = e^{-r\Delta t}\left(pF_u + (1-p)F_d\right). \tag{7}$$

In computing (5)–(7), we are ignoring any events occuring at the current node. These values will be modified based on events such as coupon payments, liquidation, and distress cost. These model-specific modifications are considered in the sections that follow.

B. Equity Dilution and Limited Liability

In this section, we show how to incorporate the limited liability requirement in our numerical method. Basically, the limited liability requirement says that shareholders should never experience negative cash flows and the most that they can lose is their initial investment.

Suppose at a certain node in the binomial lattice we know the equity values in the next step, and these are given by E_u and E_d. Assume that at the current node, the firm has to make a coupon payment of C, and the cash flow generated by the firm is given by δ. We want to know the value of equity, E, at the current node. The present value of equity ignoring the current coupon payment and the current cash flow is given by

$$\tilde{E} = e^{-r\Delta t}\left(pE_u + (1-p)E_d\right). \tag{8}$$

We denote the difference between the coupon payment and the current cash flow by $\bar{C}$, i.e., we have $\bar{C} = C - \delta$. When the firm cash flow is less than the coupon payment, $\bar{C}$ is positive and it shows the amount that needs to be raised by equity dilution. A negative value of $\bar{C}$ should be interpreted as the excess firm cash flow over the coupon payment that is to be received by equity holders. We show below that the equity value at the current node can be written as

$$E = \begin{cases} 0 & \text{if } \tilde{E} \leq \bar{C} \\ \tilde{E} - \bar{C} & \text{if } \tilde{E} > \bar{C}. \end{cases} \tag{9}$$

This means that we do not need to go through tedious calculations of equity dilution at each step even if the firm's cash flow is not enough to cover the coupon payment. The equity value can be found by checking for the liquidation event and treating the net coupon payment $\bar{C}$ as a negative cash flow to equity holders if the firm is not liquidated. The following proposition shows that this is indeed equivalent to equity dilution.

Proposition 1. If the liquidation event is checked properly as in (9), treating net coupon payments as negative cash flows to equity holders is equivalent to equity dilution, and this does not violate the limited liability requirement.

Proof. If $\bar{C} < 0$, then the current firm cash flow is sufficient to cover the coupon payment, and the excess cash will be received by equity holders. Thus, $\tilde{E} > \bar{C}$ in this case, and $E = \tilde{E} - \bar{C}$ will hold.

If $\bar{C} \geq 0$ and $\tilde{E} \leq \bar{C}$, then the equity holders have a liability that is larger than their current value and, therefore, will choose to liquidate the firm. Thus, $E = 0$ in this case.

On the other hand if $\bar{C} \geq 0$, and $\tilde{E} > \bar{C}$, then equity holders will still have positive holdings after making the net coupon payment. Therefore, they will choose to raise money by the dilution of equity and make the payment. Without loss of generality, we can assume that there is one share outstanding. Equity holders will want to issue x more shares so that $\bar{C}$ is raised from the sale of new shares. After the equity dilution, there will be $(1 + x)$ shares outstanding. The number of new shares to be issued can be found by solving

$$\frac{x}{1+x}\tilde{E} = \bar{C}. \tag{10}$$

The left side of equation (10) is the claim of the new shareholders in total equity value $\tilde{E}$ after making the net coupon payment. Solving this equation gives the value of x as

$$x = \frac{\bar{C}}{\tilde{E} - \bar{C}}. \tag{11}$$

The claim of the original shareholders in the equity value is $1/(1 + x)$. We can plug in the value of x from above to evaluate E:

$$E = \frac{1}{1+x}\tilde{E} = \left(\frac{1}{1 + (\bar{C}/(\tilde{E} - \bar{C}))}\right)\tilde{E} = \left(\frac{\tilde{E} - \bar{C}}{\tilde{E}}\right)\tilde{E} = \tilde{E} - \bar{C}. \tag{12}$$

This can be generalized to all steps and nodes on the lattice by using induction. □

We illustrate this further by a numerical example in Section III.B.

III. Bankruptcy with Immediate Liquidation

In this section, we consider the setting in which equity holders do not have the option of going into default and postponing coupon payments. Either the coupons are paid in full or the firm is liquidated in which case the debt holders receive the proceedings from the liquidation process. Naturally, equity holders will choose to liquidate the firm when the value of equity reaches zero. The limited liability requirement prohibits states of the world in which equity has negative value. If the firm is liquidated, αV_t is incurred as liquidation costs and debt holders receive $(1-\alpha)V_t$.

We use the Leland (1994) model to illustrate our numerical method in this setting. In this model, there is a consol bond with an infinite maturity that pays a constant coupon per unit time. An infinite maturity bond is convenient to work with since this makes all the variables time independent. Leland writes the PDEs satisfied by the equity and debt values and solves them to obtain closed-form solutions. However, his solutions do not extend to the case with finite maturity bonds because of time and path dependency. Using the binomial lattice method, we can easily extend Leland's model to price finite maturity bonds.

A. Finite Maturity Case

Pricing the firm, debt, and equity on a binomial lattice is straightforward once the limited liability requirement is properly included. We assume that the firm has just issued a bond with maturity T that pays a continuous coupon of C per year, and has face value P. We assume that the effective tax rate is τ and all interest payments are tax deductible. Because the tax benefits accrue at rate τC per year, the coupon payments effectively become $(1-\tau)C$ per year from the firm's perspective. We choose a time step Δt and construct the binomial lattice for the unlevered asset value V as described in Section II.A. If the value of the assets at time t is V_t, then the instantaneous cash flow produced by the firm at t is given by $\delta_t = V_t q$. On the binomial lattice, since we are using discrete time steps the total firm cash flow generated at a certain node with asset value V_t is given by $(V_t e^{q\Delta t} - V_t)$. We will slightly abuse notation and use δ_t to denote this value, thus we write

$$\delta_t = V_t e^{q\Delta t} - V_t. \tag{13}$$

This representation accounts for the cash flows accumulated between time steps. This also ensures that the binomial lattice matches the initial asset price V_0 exactly.

At maturity, we know the exact cash flows and the value of the firm's assets, so the debt, equity, and firm values are known. For all nodes at time T, we set

$$\text{(14)}\quad \begin{array}{lrcl} \text{If } V_T+\delta_T \geq (1-\tau)C\Delta t + P: & E & = & V_T+\delta_T-(1-\tau)C\Delta t - P, \\ & D & = & C\Delta t + P, \\ & F & = & V_T+\delta_T+\tau C\Delta t. \\ \text{If } V_T+\delta_T < (1-\tau)C\Delta t + P: & E & = & 0, \\ & D & = & (1-\alpha)(V_T+\delta_T), \\ & F & = & (1-\alpha)(V_T+\delta_T). \end{array}$$

Now we work backward to compute the equity, debt, and firm values at prior times $t < T$. Since liquidation is determined by the value of the equity, we need to compute E first. We compute the present value of equity ignoring anything that happens at the current node, which we denote by $\tilde{E}$,

$$\tilde{E} \quad = \quad e^{-r\Delta t}\left(pE_u + (1-p)E_d\right). \tag{15}$$

If the sum of the firm cash flow, δ_t, and the present value of equity value, $\tilde{E}$, is enough to make the current coupon payment, there is no liquidation. However, if this sum is not enough to make the coupon payment, then liquidation occurs. So, we set:

$$\text{(16)}\quad \begin{array}{lrcl} \text{If } \tilde{E}+\delta_t \geq (1-\tau)C\Delta t: & E & = & \tilde{E}+\delta_t-(1-\tau)C\Delta t, \\ & D & = & C\Delta t + e^{-r\Delta t}\left(pD_u+(1-p)D_d\right), \\ & F & = & \delta_t + e^{-r\Delta t}\left(pF_u+(1-p)F_d\right)+\tau C\Delta t. \\ \text{If } \tilde{E}+\delta_t < (1-\tau)C\Delta t: & E & = & 0, \\ & D & = & (1-\alpha)(V_t+\delta_t), \\ & F & = & (1-\alpha)(V_t+\delta_t). \end{array}$$

Working backward until time zero, we can find the equity, debt, and firm values throughout the lattice. We illustrate this case with an example.

Example 1. We consider a firm with initial asset value $V_0 = 100$, volatility $\sigma = 30\%$, and firm cash flow ratio of $q = 4\%$. The risk-free rate is $r = 6\%$. The firm has just issued a bond with face value of $P = 100$, annual coupon payments of $C = 5$, and with maturity $T = 3$ years. We assume that the liquidation cost and tax rate are zero, i.e., $\alpha = 0\%$, $\tau = 0\%$.

We want to find the equity, debt, and firm values. We use a three-step binomial lattice so that we have $\Delta t = 1$ year. We construct the binomial lattice for the asset value process V_t as described in Section II.A. We also record the firm cash flow δ_t as given in (13) and also the total payment due to debt holders at each step. This lattice is shown in Figure 2.

We then use (14)–(16) and work backward to find the equity, debt, and firm values at each step. Figure 3 shows these values. We find that $E_0 = 21.57$, $D_0 = 78.43$, and $F_0 = 100.00$.

B. Equivalent Equity Dilution Treatment

We return to the limited liability issue mentioned in Section II.B. The equity value E we found in Figure 3 can be decomposed into the cash flows on the paths

FIGURE 2

Binomial Lattice for Example 1

The top number at each node denotes the asset value, the middle number is the firm cash flow, and the bottom number is the payment due to debt holders. Up and down probabilities are $p_u = 0.4587$ and $p_d = 0.5413$, respectively. The discount factor is $e^{-r\Delta t} = 0.9418$. The lattice construction using (3) and writing the firm cash flow using (13) matches the initial asset price exactly. For example, for the first time step we have: $e^{-r\Delta t}(p_u(134.99+5.51)+p_d(74.08+3.02)) = 100.00$. Similar results hold for all time steps and nodes.

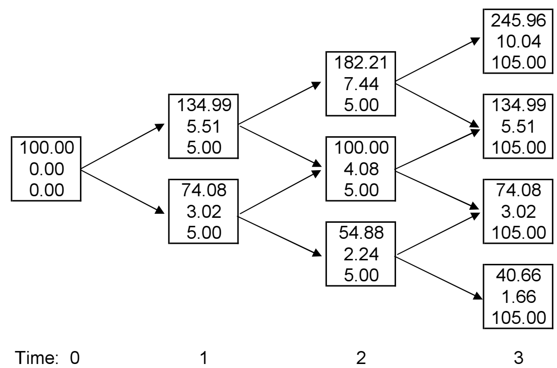
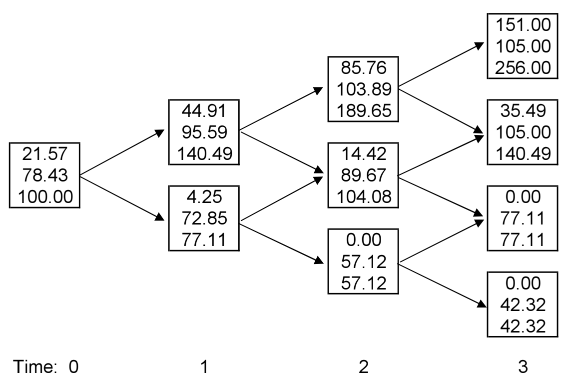


FIGURE 3

Equity, Debt, and Firm Value Computations for Example 1

The top number at each node denotes the equity value, the middle number is the debt value, and the bottom number is the firm value.



of the binomial lattice. If we write explicitly the contribution of each path on the lattice to the value of E, we obtain the values shown in Table 1. The probability of following each path is also shown in the last column of the table. Multiplying the discounted cash flows with the probabilities and summing up, we can obtain the same equity value that we find from the lattice, which is $E = 21.57$. We see from the table that the discounted cash flows on paths 4, 6, 7, and 8 are negative. This seems counterintuitive and in violation of the limited liability requirement. However, we show below that this is just a computational convenience and we obtain the same result when we use explicit dilution of equity.

The path cash flows can be decomposed in a different way when we look at the situation from the equity dilution point of view. We will need to calculate

TABLE 1

Equity Cash Flow Decomposition

The letter u denotes an up move in the lattice and d denotes a down move. Corresponding probabilities are $p_u = 0.4587$ and $p_d = 0.5413$. The discount factor is $e^{-r\Delta t} = 0.9418$.

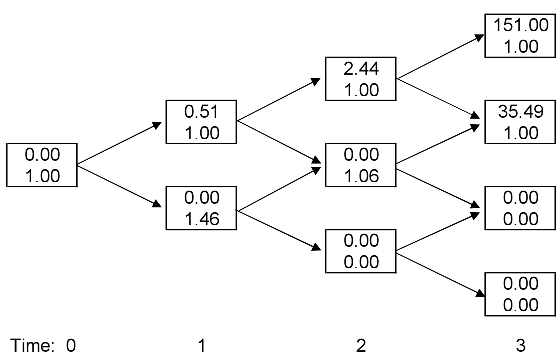
Path No.	Path Direction	Cash Flows 1	2	3	Discounted Payoff	Path Prob.
1	*u–u–u*	0.51	2.44	151.00	128.76	0.0965
2	*u–u–d*	0.51	2.44	35.49	32.29	0.1139
3	*u–d–u*	0.51	−0.92	35.49	29.31	0.1139
4	*u–d–d*	0.51	−0.92	0.00	−0.34	0.1344
5	*d–u–u*	−1.98	−0.92	35.49	26.97	0.1139
6	*d–u–d*	−1.98	−0.92	0.00	−2.68	0.1344
7	*d–d–u*	−1.98	0.00	0.00	−1.86	0.1344
8	*d–d–d*	−1.98	0.00	0.00	−1.86	0.1586

the dilution proportion $(1 + x)$ where x is as given in (11) for each node, and use this to calculate the total number of shares outstanding. Once we do this, we can assume that the equity holders are exposed only to positive cash flows since the negative cash flows are handled by equity dilution. Therefore, we compute the cash flows $(\delta_t - C)^+$ for the intermediate nodes and $(V + \delta_t - C - P)^+$ for the terminal nodes. The lattice in Figure 4 shows these calculations. The top number is the net cash flow to be shared by total equity holders, and the bottom number is the equity dilution proportion. The dilution proportion at a node is set to zero if the firm has been liquidated.

FIGURE 4

Equity Dilution Calculations for Example 1

The top number at each node shows positive cash flows to equity holders, and the bottom number shows the equity dilution ratios.



We again write each path separately and look at the contribution of each path. The shares are diluted along the way to make the coupon payments. Therefore, if there are N shares outstanding at a particular node, the original shareholders will have a claim of $1/N$ in the cash flow occurring at that node. N is found by multiplying the dilution proportions of the nodes along each path. Table 2 shows these values explicitly. We obtain the net discounted contribution from each path by discounting all the cash flows back to time zero and adding them up.

Multiplying the contribution of each path with path probability and summing up, we find the equity value as $E = 21.57$.

TABLE 2

Equity Dilution Path Decomposition

The numbers below cash flows show the total number of outstanding shares. If this number is zero, it means the firm has been liquidated.

Path No.	Path Direction	Cash Flows 1	Cash Flows 2	Cash Flows 3	Discounted Payoff	Path Prob.
1	*u–u–u*	0.51 (1.00)	2.44 (1.00)	151.00 (1.00)	128.76	0.0965
2	*u–u–d*	0.51 (1.00)	2.44 (1.00)	35.49 (1.00)	32.29	0.1139
3	*u–d–u*	0.51 (1.00)	0.00 (1.06)	35.49 (1.06)	28.35	0.1139
4	*u–d–d*	0.51 (1.00)	0.00 (1.06)	0.00 (0.00)	0.48	0.1344
5	*d–u–u*	0.00 (1.46)	0.00 (1.56)	35.49 (1.56)	19.02	0.1139
6	*d–u–d*	0.00 (1.46)	0.00 (1.56)	0.00 (0.00)	0.00	0.1344
7	*d–d–u*	0.00 (1.46)	0.00 (0.00)	0.00 (0.00)	0.00	0.1344
8	*d–d–d*	0.00 (1.46)	0.00 (0.00)	0.00 (0.00)	0.00	0.1586

The equity dilution approach demonstrated in Table 2 and the cash flow approach demonstrated in Table 1 give exactly the same results. This is a numerical illustration of Proposition 1. This result is very useful because the cash flow approach is computationally much easier to apply on a binomial lattice.

C. Infinite Maturity Case

The procedure described above can also be used to price a consol bond with infinite maturity. We need to use a very long time horizon, such as $T = 200$ years, and change the terminal condition in (14). The terminal condition is not very important since the time horizon is very long and the effect of the terminal nodes on initial prices is very small. If the bond were riskless, its price would be C/r. So, we treat the bond as if it has face value C/r and maturity T. At the final nodes, we set:

$$\begin{aligned}
\text{If } V_T > \frac{C}{r}: \quad E &= V_T - \frac{C}{r}, \\
D &= \frac{C}{r}, \\
F &= V_T. \\
\text{If } V_T < \frac{C}{r}: \quad E &= 0, \\
D &= (1-\alpha)V_T, \\
F &= (1-\alpha)V_T.
\end{aligned} \tag{17}$$

We then work backward and use (16) to update the values at the nodes other than the terminal node.

Leland (1994) solves for the liquidation boundary, that is, the value of V that gives zero equity value using PDE methods and invoking the smooth pasting condition. The above procedure achieves this boundary endogenously. In the consol bond setting, the liquidation boundary is constant and time independent. As we work backward in the lattice, we observe zero equity value below a certain level of nodes. The recursive procedure given in (16) computes the equity value by comparing continuation and stopping values as seen from the perspective of equity holders. Thus, the usual smooth pasting condition will be obtained in the limit as Δt goes to zero. See Dixit and Pindick ((1994), pp. 130–132) for a discussion of the optimal stopping problem and the smooth pasting condition.

D. Convergence of the Method

Leland (1994) gives closed-form solutions for the infinite maturity bond in the setting described in the previous sections. We include these formulas in the Appendix for reference. In this section, we analyze the convergence rate and behavior of our numerical method by comparing our numerical results with analytical results.

The pricing of equity is similar to the pricing of a call option. Because of the limited liability principle, when equity holders' net worth is less than the coupon payment due, they will default on the debt obligation, and hand the firm over to the debt holders. Thus, there will be an implicit default boundary on the lattice below which the equity has value zero. The pricing of debt is similar to the pricing of a barrier option. When equity holders default on their debt, the firm is liquidated and debt holders receive what is left of the firm. So the default boundary acts as a knockout barrier on which debt value achieves its liquidation value.

Figure 5 shows the convergence of debt and equity pricing errors as the number of time steps is increased. Here the pricing error is defined as the difference between the formula value and the value from the lattice. As the number of time steps is increased, the spacing of the nodes on the lattice becomes finer. We use $T = 200$ as the maturity of the bond on the lattice to approximate an infinite maturity bond. We find that increasing this maturity further does not have a significant effect on the prices produced by the lattice. Figure 6 shows a magnified version of the same convergence graph.

We see that the convergence of the equity error is smooth while debt error convergence exhibits an oscillation similar to the ones observed in the pricing of a barrier option (see, for example, Boyle and Lau (1994)). There are two sources of error on the lattice. One is the discretization error caused by approximating the continuous processes by their discretized versions on the lattice. The other is the barrier error caused by approximating the values of the quantities on the default boundary. The value of equity is already close to zero near the default boundary, therefore, its value is not affected significantly by the barrier error. However, debt is sensitive to both the value and the location of the boundary since its value is affected by the liquidation event. The barrier error can further be decomposed into two components. One source of error comes from the uncertainty of the exact

FIGURE 5

Convergence of Debt and Equity Errors for the Leland (1994) Model

Model parameters are $V_0 = 100$, $\sigma = 20\%$, $C = 3.0$, $r = 5\%$, $q = 4\%$, $\alpha = 50\%$, and $\tau = 0\%$. The true value of debt is 49.8527, and the true value of equity is 46.0567.

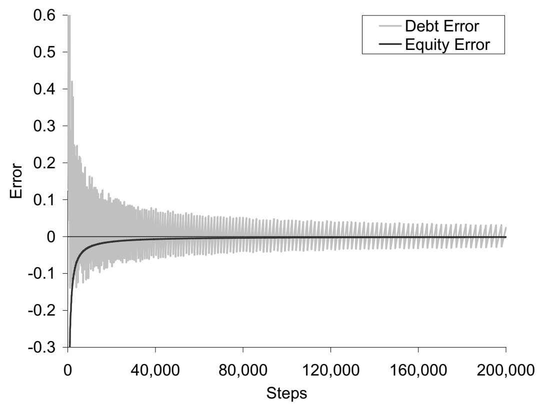
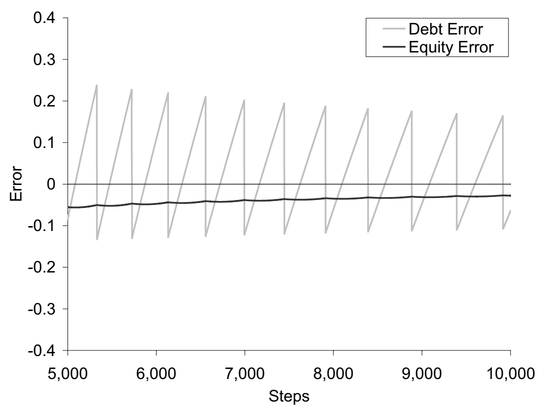


FIGURE 6

A Magnified View of the Convergence Graph in Figure 5

Figure 6 shows the oscillating pattern in the convergence of the debt value.



default boundary. The default boundary is also approximated on the lattice by looking at the first point on which equity fails to make the coupon payment. This may differ slightly from the true bankruptcy point because of the discretization error. The second component of barrier error comes from the relative positioning of the barrier between the nodes. On a lattice, even if the barrier is between the lattice nodes, it will effectively be moved to coincide with a level of nodes since the calculations are done on the nodes of the lattice only. These two effects contribute to the debt error and cause the oscillatory behavior of the debt values.

We can estimate the convergence rate of the debt and equity errors by positing a functional form in which $|\text{error}| \approx M^{\beta}$, i.e., absolute value of the error is proportional to M, the number of time steps on the lattice. We can estimate β by using the error values from the convergence graph in Figure 6. For equity error, we obtain a β value of -1.00, which indicates that the equity error converges linearly as the number of time steps is increased. For debt error, we use the values on the tips of the zigzag patterns for estimation and obtain a β value of -0.58. Gobet (1999) shows that the convergence rate for a barrier option is $o(M^{-\frac{1}{2}})$. According to our numerical results, the convergence rate of debt error is indeed close to square root convergence similar to a barrier option.

Interpolation methods such as the one given in Derman, Kani, Ergener, and Bardhan (1995) may help in reducing the size of the oscillations in the debt value. Even without any numerical improvements, the highest debt error when using around 5,000 time steps is less than 0.5% of the debt value. This level of accuracy is sufficient for corporate debt pricing where one is interested in qualitative comparisons with different parameter values.

Table 3 shows further convergence results for various parameter combinations. The linear convergence of the equity error is easily recognized from the values in the table for all cases. The debt error convergence is not as smooth because of the oscillations mentioned above. The highest relative debt error is 0.79% when 2,000 steps are used in the lattice, and 0.20% when 20,000 steps are used. When convergence is smooth, as is the case for equity error, Richardson extrapolation can be used to improve the convergence rate. For a general reference on extrapolation methods, see, e.g., Brezinski and Zaglia (1991). See Broadie and Detemple (1996) for an application of Richardson extrapolation to option pricing in lattice methods.

TABLE 3

Convergence of Binomial Lattice Method for Leland's (1994) Model Using Various Parameter Combinations

Fixed parameters are $V_0 = 100$, $\sigma = 20\%$, $r = 5\%$, $q = 3\%$, and $\alpha = 30\%$. The error columns report the formula value minus the lattice estimate.

C	τ	Steps	Equity Estimate	Equity Error	Debt Estimate	Debt Error
2.0	15%	200	67.9332	−0.8308	36.9565	0.9161
		2,000	67.1860	−0.0836	37.7997	0.0728
		20,000	67.1106	−0.0082	37.8436	0.0290
4.0	15%	200	40.2075	−1.6106	64.1990	3.0701
		2,000	38.7672	−0.1703	67.1840	0.0851
		20,000	38.6138	−0.0169	67.3027	−0.0336
2.0	35%	200	75.1869	−0.6353	37.3689	1.0512
		2,000	74.6158	−0.0642	38.3156	0.1045
		20,000	74.5580	−0.0064	38.4156	0.0045
4.0	35%	200	52.5637	−1.2629	69.1531	1.3927
		2,000	51.4299	−0.1291	70.2787	0.2671
		20,000	51.3136	−0.0128	70.4801	0.0658

IV. Bankruptcy with Grace Period and Bargaining

In the previous section, we assumed that bankruptcy and liquidation are equivalent events. Coupon payments are always needed to be made in full by the firm. If the sum of equity value and firm cash flow is not enough to make a coupon payment, then the firm is liquidated and the debt holders acquire what is left of the firm's assets after the liquidation costs are deducted. In reality, however, the equity holders can either liquidate the firm under Chapter 7 of the U.S. Bankruptcy Code or renegotiate debt payments under Chapter 11. When a firm declares bankruptcy under Chapter 11, the bankruptcy court grants the firm a certain observation period during which the company is allowed to restructure its debt. Chapter 11 also implies the automatic stay of assets while in bankruptcy, which prevents the debt holders from liquidating the firm's assets. Therefore, a firm in financial distress may declare bankruptcy under Chapter 11, spend some time as a bankrupt firm without making the full coupon payments, and then recover to continue as a healthy firm. On the other hand, if the firm spends too much time in bankruptcy and exceeds the grace period granted by the court, it will be liquidated and debt holders will acquire the firm's assets less liquidation costs.

There are different ways of modeling how the debt is serviced once the firm is in bankruptcy. In this section, we consider the approach of FM (2004). We assume that, at a certain level of the firm asset value V_B, equity holders decide to declare bankruptcy. A grace period of G is granted by the bankruptcy court. If the firm does not come out of bankruptcy at the end of this period, the firm is liquidated. There is a distress cost ω that reduces the net firm cash flow when the firm is in bankruptcy, i.e., the firm cash flow rate is reduced from q to $q - \omega$. While the firm asset value is under the default boundary V_B, the debt is serviced strategically. The exact amount of the debt service is the result of a bargaining game between debt holders and equity holders at the time bankruptcy is declared. We follow Fan and Sundaresan (2000) to determine the debt service using a Nash bargaining game. We assume that the bargaining power of the equity holders is η and the bargaining power of the debt holders is $1 - \eta$. As in the previous section, α denotes the proportional liquidation cost.

The bargaining process between equity holders and debt holders works as follows. If the firm is liquidated at the bankruptcy point, then debt holders receive $(1-\alpha)V_B$ and equity holders receive nothing. However, if the firm is not liquidated, its value will be F_B and this amount will be shared between equity holders and debt holders. If we denote the sharing rule at the bankruptcy point as θ, then the incremental value gained by equity holders is θF_B and the incremental value gained by debt holders is $(1-\theta)F_B - (1-\alpha)V_B$. Therefore, the optimal sharing rule satisfies

$$\theta^* = \arg\max \left\{ [\theta F_B]^\eta [(1-\theta)F_B - (1-\alpha)V_B]^{1-\eta} \right\}, \tag{18}$$

and its solution is

$$\theta^* = \eta \left(1 - \frac{(1-\alpha)V_B}{F_B} \right). \tag{19}$$

As a result, at the bankruptcy point, the value of the claim of the equity holders is

$$\theta^* F_B = \eta \left(F_B - (1-\alpha)V_B \right), \tag{20}$$

and the value of the claim of the debt holders is

$$(1-\theta^*)F_B = (1-\eta)\left(F_B - (1-\alpha)V_B \right) + (1-\alpha)V_B. \tag{21}$$

The bargaining game conveniently determines the value of equity and debt at the bankruptcy point through equations (20) and (21). Thus, we do not need to know explicitly how the debt payments are made once the firm is in bankruptcy—knowing the total firm value, F_B, is enough.

A. Binomial Lattice Computations

We first set up the binomial lattice as described in Section II.A. We assume that the bankruptcy boundary V_B is known for each time step in the lattice. If the bond that we are pricing is a consol bond with infinite maturity, this boundary will be constant and time independent. However, if the bond has finite maturity, the bankruptcy boundary will typically be time dependent; for example, it may be given exogenously as a function of the present value of the face value of the bond. We start with infinite maturity debt since it is easier to illustrate the numerical method, and consider finite maturity debt in Section IV.D.

We assume that the default boundary V_B coincides with a level of nodes on the lattice. If this is not the case, we can approximate V_B with the first node level that is higher than V_B. In order to do the calculations, we need to distinguish among three types of nodes as follows:

• *Nodes with* $V > V_B$. The firm is in a healthy state in these nodes, the coupons are paid using the firm cash flow and from equity dilution if necessary. Equity, debt, and firm values are updated in the following way:

$$
\begin{aligned}
(22)\quad &\text{If } \tilde{E} + \delta_t \geq (1-\tau)C\Delta t: & E &= \tilde{E} + \delta_t - (1-\tau)C\Delta t, \\
& & D &= C\Delta t + e^{-r\Delta t}\left(pD_u + (1-p)D_d\right), \\
& & F &= \delta_t + e^{-r\Delta t}\left(pF_u + (1-p)F_d\right) + \tau C\Delta t. \\
&\text{If } \tilde{E} + \delta_t < (1-\tau)C\Delta t: & E &= 0, \\
& & D &= (1-\alpha)(V_t + \delta_t), \\
& & F &= (1-\alpha)(V_t + \delta_t),
\end{aligned}
$$

where $\tilde{E}$ is as given in (15) and δ_t is as given in (13).

• *Nodes with* $V < V_B$. The firm is in bankruptcy. The debt is served strategically based on the outcome of the bargaining game. We do not know explicitly how the firm cash flow is shared between debt holders and equity holders. However, since equations (20) and (21) determine the value of debt and equity at the bankruptcy point, we only need to keep track of the firm value F when the firm is in bankruptcy. There are no tax benefits for payments to debt holders while the firm is in bankruptcy. Since the payments to debt holders are not in the form of pre-determined coupon payments anymore, there are no tax benefits for those payments.

The total time spent in bankruptcy needs to be recorded so that it can be checked against the allowed grace period G. Let g record the length of time the firm spends in bankruptcy. Since we are working with discrete time steps on the binomial lattice, g can only take discrete values. Therefore, we will represent g in terms of the number of time steps rather than absolute terms. Let $\overline{g}$ denote the maximum number of time steps that the firm can spend in bankruptcy. We have $\overline{g} = G/\Delta t$, where G is the grace period and Δt is the time step. Assume for simplicity that $\overline{g}$ is an integer. Then, g will take values in $[0, 1, \ldots, \overline{g} - 1, \overline{g}]$. For a given node and a given g, there are three possibilities in the next time step. First, the firm can come out of bankruptcy, i.e., the asset value may move to the state where $V = V_B$. Second, if $g = \overline{g} - 1$ in the current node, and $V < V_B$ in the next node, then the grace period will expire and the firm will be liquidated. Finally, the firm can still be in bankruptcy without an expired grace period, in which case the current node will connect to a state that has value of g one higher than the current one in the next node. For each node, we need to keep track of the firm value in every possible state of g. Thus, $F[i]$ will denote the firm value at the current node when $g = i$. We can update the firm value in the following way:

$$(23) \quad F[i] \ = \ \begin{cases} \overline{\delta}_t + e^{-r\Delta t}\left(pF_u[i+1] + (1-p)F_d[i+1]\right) & \text{for } i = 1, \ldots, \overline{g} - 1 \\ (1-\alpha)(V + \overline{\delta}_t) & \text{for } i = \overline{g} \end{cases},$$

where

$$(24) \qquad \overline{\delta_t} \ = \ V_t e^{(q-\omega)\Delta t} - V_t.$$

Here, $\overline{\delta_t}$ represents the distress cost adjusted (i.e., net) cash flow of the firm.

• *Nodes with* $V = V_B$. This is the last healthy state before the firm goes into bankruptcy or the first healthy state after the firm comes out of bankruptcy. The equity and debt values at this level are found through equations (20) and (21) after the firm value is computed. We update firm, equity, and debt values as follows:

$$(25) \qquad \begin{aligned} F[0] &= \delta_t + e^{-r\Delta t}\left(pF_u + (1-p)F_d[1]\right), \\ F[i] &= \overline{\delta}_t + e^{-r\Delta t}\left(pF_u + (1-p)F_d[1]\right) \text{ for } i = 1, \ldots, \overline{g}, \\ E &= \eta\left(F[0] - (1-\alpha)V_B\right), \\ D &= (1-\eta)\left(F[0] - (1-\alpha)V_B\right) + (1-\alpha)V_B. \end{aligned}$$

Here, $F[0]$ represents the value of the firm at the bankruptcy boundary V_B without the firm having been in bankruptcy, and this value will be used as input for the nodes reaching V_B from above. The $F[i]$ values represent the values of the firm just coming out of bankruptcy, and these will be used as inputs for the nodes reaching V_B from below. Therefore, the calculation of $F[i]$ takes into account the distress cost, while the calculation of $F[0]$ does not.

Note that in the above calculations, for nodes with $V > V_B$, we do not have to keep track of the grace period and, therefore, we write them using the format F rather than $F[.]$. Finally, at contract termination, bankruptcy will occur if the full debt payment is not made, so this case can be handled using the equations given in (14).

B. Optimal Bankruptcy Boundary

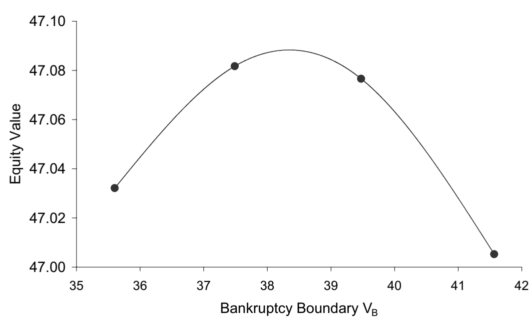
The above procedure gives us the debt, equity, and firm values for a chosen level of the bankruptcy boundary. Often we want to be able to choose this boundary endogenously. Since default is usually the equity holders' decision, the default boundary will be chosen to maximize the equity value. In the case of infinite maturity debt, we can do a numerical optimization to find the default boundary that maximizes the equity value as well as the debt, equity, and firm values on that boundary.

We first choose an arbitrary bankruptcy boundary that is likely to be lower than the optimal boundary. One natural choice is the Leland (1994) liquidation boundary given in equation (40) on which equity has value zero. Any boundary lower than that will not be effective and will degenerate to the liquidation boundary. We then start increasing the bankruptcy boundary on the lattice and reprice. The equity value first increases and then starts to decrease after it achieves its maximum value as we move the bankruptcy boundary up on the lattice. Therefore, we stop moving the boundary when the equity value starts to decrease. Thus, we obtain equity value as a function of the bankruptcy boundary at discrete observation points. We can fit a cubic spline to approximate the exact functional form and use this spline to find the maximum value of equity and the maximizing boundary. After that, we can fit cubic splines to the debt and firm values and obtain the values that correspond to the equity maximizing boundary. Figure 7 illustrates the cubic spline interpolation to find the equity maximizing boundary for the FM (2004) model. We use the cubic spline routines given in Section 3.3 of Press, Teukolsky, Vetterling, and Flannery ((1992), pp. 113–116) in our numerical experiments. Using the procedure described above, we are effectively solving the first-order condition $dE/dV_B = 0$ numerically. The second-order condition, i.e., concavity, is also verified numerically as shown in Figure 7.

FIGURE 7

Illustration of Cubic Spline Interpolation for Equity

The four circles mark the four data points used in the interpolation. These points plot the equity value as the bankruptcy boundary is moved up on the nodes of the lattice. The model parameters are the same as in Figure 8, and 3,000 time steps are used in the lattice. The maximum equity value and the corresponding bankruptcy boundary are $E^* = 47.0883$ and $V_B^* = 38.3460$.



C. Convergence of the Method

FM (2004) give analytical formulas for debt and equity values for an infinite maturity bond in the setting described above. We include these formulas in the Appendix for reference. We analyze the convergence of our numerical method by comparing the results from the binomial lattice method with their closed-form formulas.

Figure 8 shows the convergence of debt and equity errors as the number of time steps is increased. In a bankruptcy model with a grace period, both equity and debt values on the bankruptcy boundary are significant. Since we do an interpolation on the bankruptcy boundary, both of these values are affected by the relative positioning of the nodes and the boundary. This change in the positioning of the nodes as the number of steps increases causes the oscillating behavior of the errors. The size of the oscillations is relatively small. Even the largest error value is less than 0.2% of the true value when we use around 5,000 time steps.

FIGURE 8

Convergence of Debt and Equity Errors for the FM (2004) Model

Model parameters are $V_0 = 100$, $\sigma = 20\%$, $C = 3.0$, $r = 5\%$, $q = 4\%$, $\alpha = 50\%$, $\tau = 0\%$, $\omega = 2\%$, $\eta = 50\%$, and grace period $G = 1$ year. The true value of debt is 49.8761, and the true value of equity is 46.9983.

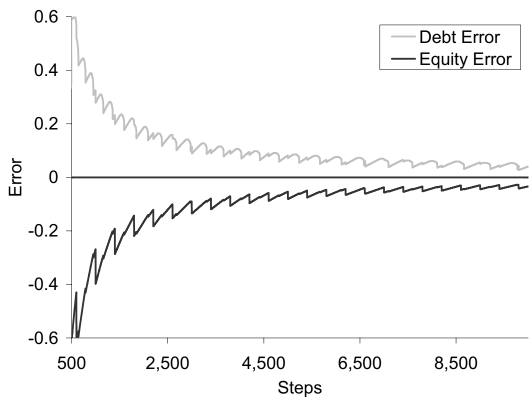


Figure 9 shows a similar convergence graph in log scale as the number of time steps is increased. Although not very smooth because of the oscillations, the slopes of the lines are very close to -1. Thus, the errors seem to converge linearly. This shows that the interpolation method helps recover the usual linear convergence similar to pricing a plain vanilla option on a binomial lattice instead of the slower barrier option convergence.

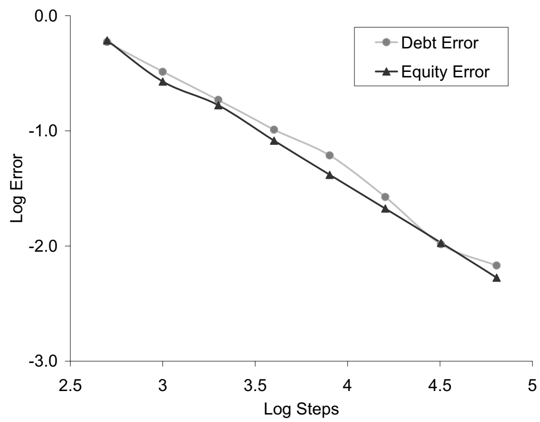
D. Pricing Finite Maturity Debt

We can use the procedure described in Section IV.A for pricing a finite maturity bond with coupon C, face value P, and maturity T. We need to specify the values at the terminal nodes to reflect the payment of the face value. At maturity, the full face value of the bond plus any accumulated coupons have to be paid, otherwise the firm will be liquidated. Also, if the firm value is still under the bankruptcy boundary V_B when the bond matures, the firm will be liquidated. So, the terminal values will be calculated in the following way:

FIGURE 9

Convergence of Debt and Equity Errors for the FM (2004) Model in Log Scale

Model parameters are the same as in Figure 8.



• *Nodes with* $V > V_B$*:*

$$
\begin{array}{llll}
(26) & \text{If } V_T + \delta_T \geq (1-\tau)C\Delta t + P: & E &= V_T + \delta_T - (1-\tau)C\Delta t - P, \\
& & D &= C\Delta t + P, \\
& & F &= V_T + \delta_T + \tau C\Delta t. \\
& \text{If } V_T + \delta_T < (1-\tau)C\Delta t + P: & E &= 0, \\
& & D &= (1-\alpha)(V_T + \delta_T), \\
& & F &= (1-\alpha)(V_T + \delta_T),
\end{array}
$$

where δ_T is as given in (13).

• *Nodes with* $V < V_B$*:*

$$
(27) \qquad F[i] = (1-\alpha)(V_T + \overline{\delta}_T) \text{ for } i = 1, \ldots, \overline{g},
$$

where $\overline{\delta}_T$ is as given in (24).

• *Nodes with* $V = V_B$*:*

$$
\begin{array}{llll}
(28) & \text{If } V_T + \delta_T \geq (1-\tau)C\Delta t + P: & E &= V_T + \delta_T - (1-\tau)C\Delta t - P, \\
& & D &= C\Delta t + P, \\
& & F[0] &= V_T + \delta_T + \tau C\Delta t, \\
& & F[i] &= V_T + \overline{\delta}_T + \tau C\Delta t \text{ for } i = 1, \ldots, \overline{g}. \\
& \text{If } V_T + \delta_T < (1-\tau)C\Delta t + P: & E &= 0, \\
& & D &= (1-\alpha)(V_T + \delta_T), \\
& & F[0] &= (1-\alpha)(V_T + \delta_T), \\
& & F[i] &= (1-\alpha)(V_T + \overline{\delta}_T) \text{ for } i = 1, \ldots, \overline{g}.
\end{array}
$$

The valuation for the other nodes in the lattice will be done as in Section IV.A. It is thus straightforward to compute the bond price for a given V_B. Here we assume that V_B is a vector that contains the bankruptcy boundary for each time step on the lattice. If the bankruptcy boundary is given exogenously by a covenant, for example, then we can use the above procedure by using the appropriate value of V_B for each time step.

In Section IV.B, we show how to determine the V_B that maximizes the equity value for an infinite maturity bond. In general, the optimal V_B in the finite maturity setting will not be constant, but rather will be time dependent since the remaining value of the bond is changing over time. Determining a V_B that will maximize the equity value is significantly harder in this case, even using a numerical method. However, we may assume a functional form for the bankruptcy boundary and let the equity holders choose a parameter of that function to maximize the equity value. One alternative in this case is taking the bankruptcy boundary as a linear function of the riskless bond price. So, we may have

$$V_B^t = \beta P_t, \tag{29}$$

where V_B^t is the bankruptcy boundary at an intermediate time t, P_t is the riskless bond price at time t, and β is a positive number that is time independent. This construction makes intuitive sense since equity holders will want to declare bankruptcy at a lower level for a cheaper bond, and at a higher level for a more expensive bond. Given this form of the bankruptcy boundary, equity holders will choose β to maximize the equity value. Since an infinite maturity bond has the same riskless price for all t, this construction would imply a constant V_B for each time step, and thus is consistent with what we do in Section IV.B for an infinite maturity bond. The optimization can be done as explained in Section IV.B, but β will be the variable parameter to change in the search algorithm.

In Table 4, we give some numerical examples for the pricing of finite maturity bonds using the method described. Note that for a discount bond, the default boundary will be an increasing function of time t; for a par bond, it will be constant; and for a premium bond, it will be a decreasing function of t.

TABLE 4

Numerical Examples for the FM (2004) Model Using Finite Maturity Bonds

Fixed parameters are $V_0 = 100$, $\sigma = 20\%$, $r = 5\%$, $q = 3\%$, $G = 1$ year, $\omega = 1\%$, $\eta = 0.5$, $\tau = 25\%$, and $\alpha = 50\%$. The face value of the bond is $P = 60$. The bond maturity T is given in years, and the time increment used in the lattice is $\Delta t = 0.005$.

T	C	Riskless Price	Equity Estimate	Debt Estimate	Firm Estimate	Yield Spread (bps)
10	2.0	52.13	54.13	46.79	100.92	130
	2.5	56.07	51.37	50.33	101.70	134
	3.0	60.00	48.68	53.77	102.45	141
	3.5	63.93	46.04	57.08	103.11	150
	4.0	67.87	43.46	60.29	103.75	160
15	2.0	49.45	58.56	43.92	102.49	105
	2.5	54.72	54.89	48.60	103.49	110
	3.0	60.00	51.31	53.10	104.41	118
	3.5	65.28	47.84	57.43	105.26	127
	4.0	70.55	44.48	61.45	105.93	142

V. Bankruptcy with Grace Period, Automatic Stay, and Arrears Account

We mentioned in the previous section that even in the existence of Chapter 11 bankruptcy there can be different ways of modeling the events that happen once the firm declares bankruptcy. In this section, we consider the approach of BCS (2005) and show how to solve this model using our binomial lattice method.

Different from the models considered in the previous sections, this model uses as its primitive variable the earnings before interest and taxes (EBIT). The treatment of taxes is slightly different in this case, but the basics are the same. The earning process corresponds to a firm asset value process as given in equation (1). Therefore, we can still build our binomial lattice using the firm's asset value as the primitive variable. Equity value needs to be multiplied by $(1 - \tau)$ after the computations are done to account for the tax payments. Also, the firm value at time zero is found using the sum of debt and tax deducted equity values, i.e., $F_0 = (1 - \tau)E_0 + D_0$.

The bankruptcy event is triggered when the firm's asset value reaches a certain value V_B. We consider a consol bond with infinite maturity, and thus this bankruptcy level is time independent. When the firm asset value satisfies $V > V_B$, the firm is in a healthy state and, as in the previous models considered, coupon payments are made from the firm cash flow and by dilution of equity if necessary. When the firm asset value reaches V_B, equity holders declare bankruptcy under Chapter 11. The bankruptcy court grants a grace period of length G, and the firm is liquidated if it does not recover to a healthy state before that grace period expires. While the firm is in bankruptcy, all the coupon payments are stopped, and unpaid coupons are recorded in an arrears account A_t. If the firm returns to a healthy state at some future point T, the firm will pay θA_T to debt holders, $0 \leq \theta \leq 1$. The parameter θ controls the debt forgiveness when the firm is in bankruptcy.

When the firm is in default, the entire firm cash flow or EBIT is accumulated in a separate account, S_t. If the firm reaches a healthy state at a future time T, the amount S_T is used to pay the arrears θA_T. If there is any leftover, this is distributed to shareholders. If S_T is not enough to cover θA_T, the rest of arrears is paid by equity dilution. If the firm spends too much time in default or if the equity value reaches zero, the firm is liquidated with a proportional liquidation cost of α. In the event of liquidation, the value of S_T is added to the value of the firm's asset. Finally, when in bankruptcy, the firm is exposed to a continuously accruing proportional distress cost ω. This distress cost is reflected as a direct cost to equity holders.

A. Binomial Lattice Computations

In this case, we will first set up our binomial lattice and do the computations for a given default boundary. We can then do a numerical optimization to maximize equity value by changing the default boundary, and observe how equity, debt, and firm values change with the changing default boundary. Here, we assume that we are pricing an infinite maturity bond. However, the method described below can be used for pricing finite maturity bonds after making slight modifications as explained in Section IV.D for the FM model.

We first set up the binomial lattice as described in Section II.A. We assume that the default boundary V_B coincides with a level of nodes on the lattice. If this is not the case, we can approximate V_B with the first node level that is higher than V_B. In order to do the calculations, we need to distinguish among three types of nodes:

• *Nodes with* $V > V_B$. The firm is in the healthy state in these nodes, and the coupons are paid using the firm cash flow and from equity dilution if necessary.

• *Nodes with* $V < V_B$. The firm is in bankruptcy. The coupons are recorded in an arrears account to be paid when the firm goes out of bankruptcy, and the firm cash flows are accumulated in a separate account. The total time spent in bankruptcy needs to be recorded to be checked against the allowed grace period.

• *Nodes with* $V = V_B$. This is the first healthy state after the firm goes through bankruptcy. All arrears have to be cleared and the firm's automatic stay payoffs have to be distributed accordingly when the firm comes out of bankruptcy.

We next explain how to update the variable values for the three different types of nodes in the lattice.

1. Nodes with $V_t > V_B$

This is similar to the Leland (1994) case in Section III. Note that even though we say the firm is in a healthy state, there may still be cases when the firm's cash flow is less than the coupon payment, and we may need equity dilution to make the coupon payment. This in turn may result in the liquidation of the firm if the equity value reaches zero. The notion of healthiness here means that the firm did not declare bankruptcy and all coupon payments have to be made in full.

For these nodes, we update the equity and debt values in the following way:

$$
\begin{array}{lrcl}
\text{If } \tilde{E} + \delta_t \geq C\Delta t: & E & = & \tilde{E} + \delta_t - C\Delta t, \\
 & D & = & C\Delta t + e^{-r\Delta t}\left(pD_u + (1-p)D_d\right). \\
\text{If } \tilde{E} + \delta_t < C\Delta t: & E & = & 0, \\
 & D & = & (1-\alpha)(V_t + \delta_t),
\end{array}
\tag{30}
$$

where $\tilde{E}$ is as given in (15) and δ_t is as given in (13). Note that we do not reduce the coupon payments for equity by multiplying with $(1-\tau)$ because of the EBIT modeling of the cash flows. The cash flows of the firm represent the EBIT, which implies that there is no tax deduction for interest payments.

2. Nodes with $V_t < V_B$

In this region, the firm is in bankruptcy. The coupon payments are accumulated in an arrears account A and the firm cash flows are accumulated in an automatic stay payoff account S. Also, there is a grace period G, which is the maximum amount of time that the firm can spend in bankruptcy. If the firm is not out of bankruptcy after G years, it is liquidated. Therefore, there are three variables to keep track of once the firm's asset value falls below the bankruptcy boundary: accumulated coupons A, accumulated payoffs S, and the time spent in bankruptcy g. However, since the coupons are constant, keeping track of g is enough, and accumulated coupons can be deduced from the value of g. We will add two state variables to each node to represent the values of g and S for each node.

We have explained how to keep track of g on the lattice in Section IV.A. To keep track of the automatic stay payoffs S, we will use a discretized grid for its values and then use linear interpolation. An upper bound on the value of S is given by $\overline{S} = V_B(e^{q\Delta t} - 1)G$. Assume we want to use M values in the discretized grid. Then S is represented by the values on the grid $S_j = j\overline{S}/M$, where j takes values in the set $[0, 1, \ldots, M-1, M]$. We use $M = 20$ points for the numerical results reported in this paper. Our numerical experiments show that the results are not very sensitive to the choice of M, and using a higher value does not change the results significantly.

For a given node, the variables equity and debt will have values for each state of g and S. Therefore, we will use an index representation; for example, $E[i, j]$ denotes the equity value in the current node when $g = i$ and $S = S_j$. We assume we know the equity values $E_u[i, j]$ in the up state in the next time step and $E_d[i, j]$ in the down state in the next time step for all i and j. Clearly, $E[i, j]$ will connect to a state with $g = i + 1$ in the next time step and the move is either up or down. We need to determine what will happen to S. If $S = S_j$ in the current node, then it will be

$$S_u = S_j e^{r\Delta t} + \delta_u, \tag{31}$$
$$S_d = S_j e^{r\Delta t} + \delta_d \tag{32}$$

for the up and down moves, respectively. Here δ_u and δ_d denote the firm cash flows in the next time step. We find the equity values that connect to the current node in the next time step by linearly interpolating on the value of S:

$$\tilde{E}_u[i+1, j] = E_u[i+1, j] + \frac{S_u - S_j}{S_{j+1} - S_j}\left(E_u[i+1, j+1] - E_u[i+1, j]\right), \tag{33}$$
$$\tilde{E}_d[i+1, j] = E_d[i+1, j] + \frac{S_d - S_j}{S_{j+1} - S_j}\left(E_d[i+1, j+1] - E_d[i+1, j]\right). \tag{34}$$

More elaborate interpolation schemes can be used at the expense of increased computation time. We can compute the interpolated values of debt, $\tilde{D}_u[i+1, j]$ and $\tilde{D}_d[i+1, j]$, in the same way. Next, we compute the present value of equity ignoring anything that happens at the current node, which we denote by $\tilde{E}[i, j]$:

$$\tilde{E}[i, j] = e^{-r\Delta t}\left(p\tilde{E}_u[i+1, j] + (1-p)\tilde{E}_d[i+1, j]\right). \tag{35}$$

Before we finalize the values, we need to check for liquidation. Liquidation in this case may occur because of the distress cost. This cost is proportional to the asset value and is given by $\omega V_t \Delta t$, where V_t is the current asset value. The values are updated by checking for liquidation:

$$\begin{array}{lrcl} \text{If } \tilde{E}[i,j] \geq \omega V_t \Delta t: & E[i,j] & = & \tilde{E}[i,j] - \omega V_t \Delta t, \\ & D[i,j] & = & e^{-r\Delta t}\left(p\tilde{D}_u[i+1,j] + (1-p)\tilde{D}_d[i+1,j]\right). \\ \text{If } \tilde{E}[i,j] < \omega V_t \Delta t: & E[i,j] & = & 0, \\ & D[i,j] & = & (1-\alpha)(V_t + S_j). \end{array}$$

This procedure is repeated for all states of the current node, and then all nodes of the current time step that are below V_B.

3. Nodes with $V_t = V_B$

We consider this level of nodes separately since this is the first level when the firm reaches a healthy state after being in bankruptcy. We will need to account for the arrears A, and the automatic stay payoffs S once the firm reaches this level after coming out of bankruptcy.

We first find the present value of equity ignoring anything that happens in the current node. If the next move is up, the firm will be in a healthy state, but if the next move is down, then the firm will go into bankruptcy. Therefore, present value of equity is given by

$$\tilde{E} = e^{-r\Delta t}\left(pE_u + (1-p)\tilde{E}_d[1,0]\right),$$

where $\tilde{E}_d[1,0]$ is the interpolated value of equity in the down state as in (34).

Of course, the firm may have reached the nodes $V_t = V_B$ either from above, being in a healthy state, or from below, being in bankruptcy. To accommodate the former case, we define a special state $[0,0]$ and define the associated values as

$$\begin{array}{lrcl} \text{If } \tilde{E} + \delta_t \geq C\,\Delta t: & E[0,0] & = & \tilde{E} + \delta_t - C\Delta t, \\ & D[0,0] & = & C\Delta t + e^{-r\Delta t}\left(pD_u + (1-p)\tilde{D}_d[1,0]\right). \\ \text{If } \tilde{E} + \delta_t < C\Delta t: & E[0,0] & = & 0, \\ & D[0,0] & = & (1-\alpha)(V_t + \delta_t). \end{array} \tag{36}$$

For the case when the firm comes out of bankruptcy, we interpret $E[i,j]$ as the value of equity after the firm has been in bankruptcy for $g = i$ time steps and accumulated a payoff of S_j. This will act as a boundary condition and will be used to find the values under the bankruptcy boundary. We denote the accumulated arrears by A_i, the coupon amount accumulated by staying in bankruptcy for i time steps.

For the firm to continue in a healthy state, it needs to clear the arrears. Therefore, the sum of automatic stay payoffs S_j and the present value of equity should be larger than A_i, otherwise the firm will be liquidated. This does not mean that there are cases when the firm comes out of bankruptcy and is immediately liquidated in a healthy state. It is actually a convenient boundary condition so that the values under V_B can be correctly calculated—the actual liquidation will occur under V_B before reaching the healthy state. Equity holders will choose to liquidate the firm if they accumulate arrears to the point that it is not possible to clear them when V_B is reached. The variable values for all remaining states $[i,j]$ are updated according to

$$\text{(37)} \quad \begin{array}{rrcl} \text{If } \tilde{E}+S_j \geq A_i: & E[i,j] & = & \tilde{E}+S_j-A_i, \\ & D[i,j] & = & A_i+e^{-r\Delta t}\left(pD_u+(1-p)\tilde{D}_d[1,0]\right). \\ \text{If } \tilde{E}+S_j < A_i: & E[i,j] & = & 0, \\ & D[i,j] & = & (1-\alpha)(V_t+S_j). \end{array}$$

B. Optimal Bankruptcy Boundary

For a given bankruptcy boundary V_B, we can find the equity, debt, and firm values by using the above procedure. If V_B is chosen endogenously to maximize one of these values, we need to do a numerical optimization similar to the one described in Section IV.B, and solve the first-order condition numerically.

For example, let us assume that we want to find the value V_B that maximizes the equity value, as well as the debt and firm values for this equity maximizing V_B. We perform a numerical optimization as follows. We start with an initial value of V_B that is equal to Leland's (1994) bankruptcy boundary as explained in Section III. We then obtain discrete observation points that give the equity, debt, and firm values at various values of V_B. We fit a cubic spline to these discrete observation points and find the V_B^* that maximizes equity value on that spline. We can also find the corresponding debt and firm values by fitting cubic splines for these and reading off the values for V_B^*. This gives a way of obtaining a good approximation to an optimal bankruptcy boundary even though our ability to choose a specific bankruptcy boundary on the lattice is limited.

VI. Computational Analysis and Comparisons

A. Pricing Finite Maturity Coupon Bonds

In this section, we study the prices and yield spreads of coupon bonds with finite maturity and compare them with infinite maturity bonds. We consider the setting described in Section III in which bankruptcy leads to immediate liquidation. The length of the granted grace period in a Chapter 11 setting may be different for bonds with different maturities. Therefore, we ignore the alternative of going into Chapter 11 bankruptcy with a grace period in order to make uniform comparisons and observe the effects of maturity on prices. Leland and Toft (1996) consider the pricing of finite maturity coupon bonds using a stationary debt structure setting in which the firm always replaces retired debt with the same amount of new debt. This simplification makes the default boundary a time-independent constant and they are able to obtain closed-form solutions. However, when the firm does not roll over its debt in this way, the debt structure will not be stationary and the default boundary will be time dependent. No closed-form solutions exist for this general case, so we use our numerical method introduced in Section III.A. We use bonds that pay continuous coupons to be consistent with the rest of the literature, however, pricing coupon bonds that pay discrete coupons is also straightforward with our method.

We consider a bond that pays a continuous coupon of C per year with maturity T years and face value of P. For an infinite maturity bond that pays C per year, the riskless price would be C/r. We choose the face value of the finite maturity bond such that $P = C/r$. Thus, at maturity, the bondholders will have to be paid the riskless value of an infinite maturity bond. Of course, they may not be able to obtain the full amount if equity holders default on their debt. In this setting, as the maturity date increases, the price of the finite maturity bond converges to the price of an infinite maturity bond. By looking at the effects of maturity on variable values, we can see how fast this convergence is, and when infinite maturity bonds become good approximations for finite maturity bonds.

Figure 10 shows the graphs of equity, debt, and spread values as the maturity is increased. The yield spread is calculated by first solving the equation,

$$\frac{C}{y}(1 - e^{-yT}) + Pe^{-yT} \quad = \quad D, \tag{38}$$

for y. Here, y is the yield of the bond, and D is the debt value from the lattice. Thus, y is the rate that produces a bond's market price when used to discount the bond's cash flows. The yield spread is then defined as $s = y - r$, where r is the riskless interest rate. As expected, when maturity increases, the equity value increases and the debt value decreases. The curves seem to flatten at around a maturity of 30 years. The equity value curve moves higher and the debt value curve moves lower with increasing volatility. The behavior of yield spreads is broadly consistent with empirical results. Sarig and Warga (1989) study the empirical term structure of yield spreads for zero coupon bonds. They find that the yield spread curve is upward sloping for high rated bonds, humped for medium rated bonds, and downward sloping for low rated bonds. Although they do not study coupon bonds extensively because of a lack of data, their sample results show that the same may hold for coupon bonds. Usually, low rated firms are associated with higher volatilities and high rated firms with lower volatilities. Therefore, the yield spreads in Figure 10 show close resemblance to their empirical findings.

Table 5 shows yield spreads for a variety of values for σ, τ, and C. The leverage of the firm increases as the coupon value increases. Merton (1974) defines the quasi debt firm value ratio as the ratio of the riskless value of debt to the firm value. We denote this ratio by Q, and define it for coupon bonds as

$$Q \quad = \quad \frac{(C/r)(1 - e^{-rT}) + Pe^{-rT}}{V_0}. \tag{39}$$

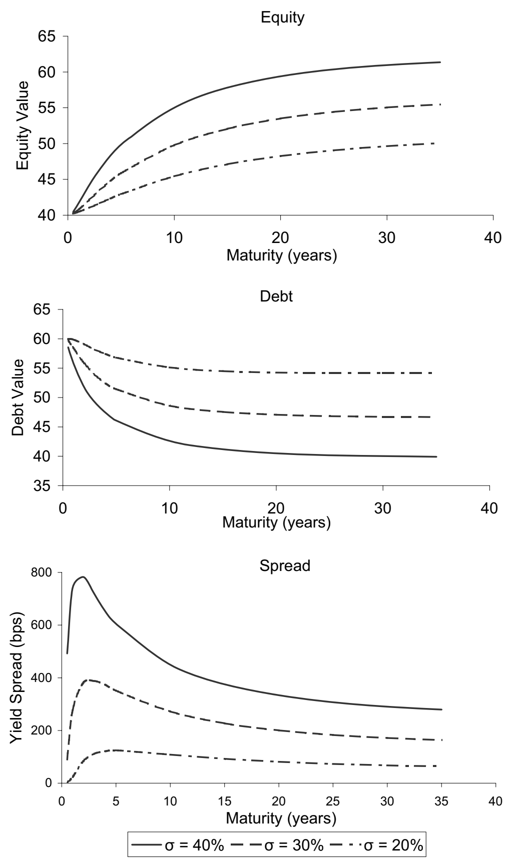
So the coupon values of $C = 3$, 4, and 5 correspond to $Q = 60\%$, 80%, and 100%, respectively.

We see that even for a relatively long maturity of 20 years, the finite maturity bond may have a yield spread that is about 130 basis points higher than the infinite maturity bond. Yield spreads increase with increasing firm volatility and leverage, and decrease with an increasing tax rate. The leverage has a more pronounced effect on shorter than longer maturities. The reverse is true for the effect of the tax rate. Since the weight of interest payments in the total debt value increases with maturity, taxes become more effective for longer maturities. Merton (1974) shows

FIGURE 10

Effect of Maturity on Equity, Debt, and Spread for a Coupon Bond for Various Firm Volatility Values

The model parameters are $V_0 = 100$, $C = 3.0$, $P = C/r = 60.0$, $r = 5\%$, $q = 2\%$, $\alpha = 50\%$, and $\tau = 15\%$. The time increment used in the lattice is $\Delta t = 0.0003$ years.



that, for zero coupon bonds, yield spread is a decreasing function of maturity when the quasi debt firm value ratio satisfies $Q \geq 1$. The results in Table 5 show that coupon bonds show similar behavior: as coupon value increases, the yield spreads become very high for maturities close to zero. This is because when leverage is high, the firm becomes insolvent as maturity T approaches zero and the spreads approach infinity.

B. Comparison of Alternative Bankruptcy Procedures

In this section, we analyze the effects of different bankruptcy procedures on equity, firm, and spread values. We consider three alternatives correspond-

TABLE 5

Yield Spreads as σ, C, and τ Change

Fixed parameters are $V_0 = 100$, $r = 5\%$, $q = 2\%$, $\alpha = 50\%$, and $P = C/r$.

		Yield Spread (in basis points)				
σ	C	$T = 2$	$T = 5$	$T = 10$	$T = 20$	$T = \infty$
$\tau = 15\%$						
0.2	3.0	83	125	108	81	52
0.3	3.0	381	351	272	200	143
0.4	3.0	783	605	449	334	254
0.2	4.0	586	369	237	153	95
0.3	4.0	1099	645	421	286	202
0.4	4.0	1542	902	603	424	324
0.2	5.0	1730	752	417	253	159
0.3	5.0	2093	981	583	381	272
0.4	5.0	2446	1212	753	516	398
$\tau = 35\%$						
0.2	3.0	82	120	100	69	33
0.3	3.0	377	342	258	179	113
0.4	3.0	777	592	430	307	218
0.2	4.0	573	349	213	124	59
0.3	4.0	1083	622	393	251	158
0.4	4.0	1523	877	572	386	275
0.2	5.0	1666	694	361	196	95
0.3	5.0	2047	936	535	328	207
0.4	5.0	2408	1171	708	464	333

ing to the three models described in detail in the previous sections. The first of these is the Leland (1994) model, which treats bankruptcy and liquidation as the same events. The second is the FM (2004) model in which equity holders can declare bankruptcy and start servicing the debt strategically for an allowed grace period. Through a bargaining process, equity holders and debt holders share the surplus generated by the prevention of immature liquidation. The last model is BCS (2005) in which, again, the equity holders can declare bankruptcy and stop servicing the debt for an allowed grace period. Debt payments are recorded in an arrears account, and there is no explicit bargaining but some of the debt may be forgiven.

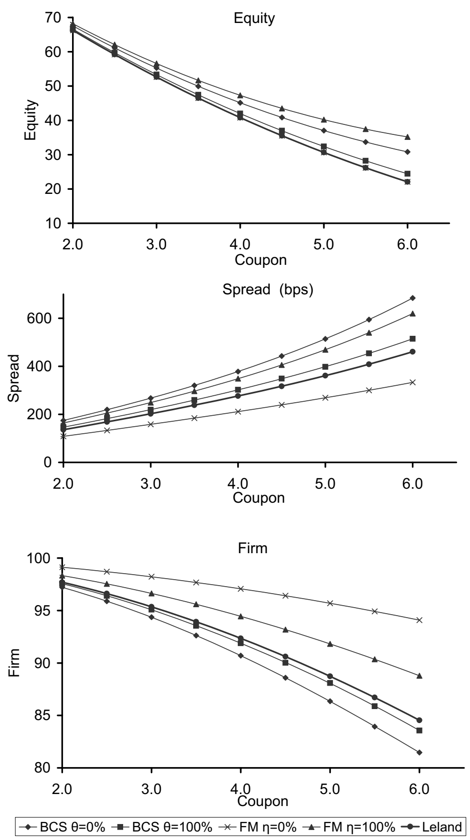
We compare these models using infinite maturity bonds since this is the original setting in which the models are presented. We investigate how the Chapter 11 alternative in the FM and the BCS models changes the spreads and other variables as compared to immediate liquidation in the Leland model. In the FM model, the bargaining power of the equity holders, denoted by η, determines how the surplus generated by the bankruptcy proceedings is shared between equity holders and debt holders. There is no bargaining in the BCS model, but a similar parameter θ, expressed in percentage, shows how much of the arrears has to be cleared when the firm comes out of bankruptcy. Although there is not a one-to-one correspondence between η in the FM model and θ in the BCS model, we try to gain some understanding of the effects of these parameters by looking at the extreme values of 0% and 100% for both parameters.

Figure 11 shows the firm, equity, and spread values for the three models as the coupon value increases. We ignore any tax benefits in order to observe only the differences caused by bankruptcy alternatives. We assume that equity holders choose the bankruptcy boundary, V_B, in both the FM and BCS models. One immediate observation is that both the FM and BCS models generate equity values that are at least as high as the Leland model. This is expected since the equity holders are assumed to be in control of the firm and they choose the bankruptcy boundary to maximize their own wealth. They can always choose the Leland boundary in the worst case scenario, so equity holders are expected to benefit from Chapter 11 proceedings. How much they benefit depends on the model parameters: equity holders' bargaining power in the FM model, which is represented by η, and the fraction of arrears to pay in the BCS model, which is represented by θ. When $\eta = 0\%$ in the FM model, the equity holders have no bargaining power and they obtain the same equity value as in the Leland model.

FIGURE 11

Comparison of the Leland (1994), FM (2004), and BCS (2005) Models for Infinite Maturity Bonds

Model parameters are $V_0 = 100$, $\sigma = 30\%$, $r = 5\%$, $q = 4\%$, $\alpha = 50\%$, and $\tau = 0\%$. For the FM and the BCS models, distress cost $\omega = 0\%$ and the grace period while in default is two years. It is assumed that equity holders choose V_B in both the FM and BCS models. The time increment used in the lattice for the BCS model is $\Delta t = 0.02$, $M = 20$ points are used in the automatic stay payoffs grid, and $T = 200$ years is used to approximate the infinite maturity bond. The Leland and FM models are solved using analytical formulas.



We observe that in the BCS model, spreads are higher than the Leland model, while in the FM model they can be higher or lower depending on the value of η. If η is low, debt holders will get most of the surplus from the bankruptcy proceedings and they will be better off; if it is high, then the debt holders may be worse off and the spreads increase.

Total firm values seem to contrast for the FM and the BCS models. The FM model generates higher firm values than the Leland model, while the BCS model generates lower firm values. Together with the results for equity and spread values this shows that in the BCS model, equity holders use the bankruptcy procedures to transfer wealth from debt holders and while doing this they increase the total liquidation costs incurred. By introducing a bargaining process at the bankruptcy boundary, the FM model prevents this wasteful liquidation. This is merely because the bargaining is done on the excess wealth generated by the bankruptcy proceedings. The results for the BCS model show that if equity holders are allowed to go into bankruptcy without any motivation to generate excess wealth, they can actually cause more wasteful liquidation to maximize their own wealth. BCS (2005) show how this can be prevented by shifting some of the power to debt holders. They show that it is possible to generate excess wealth when debt holders are given some control, but this need not necessarily be through a bargaining process on the wealth. They consider a setting in which equity holders choose the bankruptcy level and debt holders choose the granted grace period. They show that this kind of power shift may help in generating higher firm values than the Leland model.

VII. Conclusion

In this paper, we present a lattice method for pricing risky corporate debt using structural models. Our method takes the asset value of the firm as the primitive variable and prices other quantities as derivatives of this basic variable. We show that our method generates results that are consistent with the limited liability of equity principle. Since a backward valuation method is used, the continuation value of equity is known at each time step, which in turn is used to make bankruptcy decisions.

We show that our method is easily extendable to the case when the firm has a Chapter 11 bankruptcy alternative. By adding a bankruptcy boundary and increasing the state space on the lattice nodes if necessary, models of different complexity can be solved. We described the details of the implementation for three different models: Leland (1994), François and Morellec (2004), and Broadie, Chernov and Sundaresan (2005). We also illustrate the convergence of our method by comparing numerical results from the lattice method with analytical results when they exist.

Our method can be beneficial to corporate debt pricing in several ways. Many existing models use infinite maturity bonds to obtain closed-form solutions. Our method can be used to solve these models for finite maturity bonds. It can also be used for pricing bonds with discrete coupon payments, as well as pricing multiple bonds with different coupon payments and different maturities. Our method is intuitive and easily extendable compared to the alternative of using finite difference methods on PDEs. Especially in a Chapter 11 setting when there are different variables such as grace period and arrears in consideration, our approach provides a convenient solution method.

We analyze the term structure of yield spreads for finite maturity coupon bonds when there is no Chapter 11 bankruptcy option. The yield spread behavior is broadly consistent with empirical findings. We also analyze some alternative bankruptcy procedures. The assumptions about what happens once the firm is in bankruptcy and the balance of power between equity holders and debt holders have a significant effect on the variable values. Therefore, further research in this area may be directed to experiment with the complexities of bankruptcy procedures.

Appendix. Formulas for the Leland and François and Morellec Models

In this appendix, we give the closed-form formulas from Leland (1994) and François and Morellec (2004) for their respective models. We assume that the initial firm value V, the firm asset volatility σ, the firm payout ratio q, and the risk-free interest rate r are given. The firm has just issued a perpetual bond that pays a continuous coupon of C per year. The effective tax rate is τ, and the proportional liquidation cost is α. The variables of interest are the equity maximizing bankruptcy boundary V_B and the corresponding values for debt D, equity E, and firm F.

Leland (1994) derives the following closed-form formulas:

$$V_B = \left[\frac{(1-\tau)C}{r}\right]\left[\frac{X}{1+X}\right], \tag{40}$$

$$D = \frac{C}{r} + \left[(1-\alpha)V_B - \frac{C}{r}\right]\left[\frac{V}{V_B}\right]^{-X}, \tag{41}$$

$$E = V - (1-\tau)\frac{C}{r} + \left[(1-\tau)\frac{C}{r} - V_B\right]\left[\frac{V}{V_B}\right]^{-X}, \tag{42}$$

$$F = V + \frac{\tau C}{r}\left[1 - \left(\frac{V}{V_B}\right)^{-X}\right] - \alpha V_B\left(\frac{V}{V_B}\right)^{-X}, \tag{43}$$

where

$$X = \left[(r - q - 0.5\sigma^2) + \left((r - q - 0.5\sigma^2)^2 + 2\sigma^2 r\right)^{1/2}\right]/\sigma^2.$$

For the François and Morellec model, we have the following additional parameters: G is the grace period while in default, φ is the distress cost, and η is the bargaining power of equity holders. François and Morellec (2004) derive the following closed-form formulas:

$$V_B = \frac{\xi}{\xi+1}\frac{C[1-\tau+\eta\tau(1-d)]}{r - r\eta[\alpha(1-c) - \frac{\varphi}{q}(qa-c)]}, \tag{44}$$

$$D = \frac{C}{r}\left[1 - \left(\frac{V}{V_B}\right)^{-\xi}\right] + (1-\alpha)V_B\left(\frac{V}{V_B}\right)^{-\xi} + (1-\eta)R\left(\frac{V}{V_B}\right)^{-\xi}, \tag{45}$$

$$E = V - V_B\left(\frac{V}{V_B}\right)^{-\xi} - \frac{(1-\tau)C}{r}\left[1 - \left(\frac{V}{V_B}\right)^{-\xi}\right] + \eta R\left(\frac{V}{V_B}\right)^{-\xi}, \tag{46}$$

$$F = V + \frac{\tau C}{r} - \left[\frac{\varphi}{q}(qa-c)V_B + \alpha c V_B + \frac{\tau C}{r}d\right]\left(\frac{V}{V_B}\right)^{-\xi}, \tag{47}$$

where $b = (1/\sigma)(r - q - 0.5\sigma^2)$, $\lambda = \sqrt{2r + b^2}$, and $\xi = (1/\sigma)(b + \lambda)$. In the equations above, we have

$$
\begin{aligned}
a &= \frac{1}{\lambda}\left[\frac{1}{\lambda + b + \sigma} + \frac{1}{\lambda - b - \sigma}\frac{\Phi(-\lambda\sqrt{G})}{\Phi(\lambda\sqrt{G})}\right], \\
c &= \frac{\Phi\left[-(\sigma + b)\sqrt{G}\right]}{\Phi(\lambda\sqrt{G})}, \\
d &= \frac{\lambda - b}{2\lambda} + \frac{\lambda + b}{2\lambda}\frac{\Phi(-\lambda\sqrt{G})}{\Phi(\lambda\sqrt{G})}, \\
R &= \alpha V_B(1 - c) - \frac{\varphi}{q}(qa - c)V_B + \frac{\tau C}{r}(1 - d), \\
\Phi(x) &= 1 + x\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)\mathcal{N}(x)
\end{aligned}
$$

with $\mathcal{N}(\cdot)$ the cumulative standard normal distribution function.

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