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Exact Simulation of Stochastic Volatility and Other Affine Jump Diffusion Processes

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The stochastic differential equations for affine jump diffusion models do not yield exact solutions that can be directly simulated. Discretization methods can be used for simulating security prices under these models. However, discretization introduces bias into the simulation results, and a large number of time steps may be needed to reduce the discretization bias to an acceptable level. This paper suggests a method for the exact simulation of the stock price and variance under Heston's stochastic volatility model and other affine jump diffusion processes. The sample stock price and variance from the exact distribution can then be used to generate an unbiased estimator of the price of a derivative security. We compare our method with the more conventional Euler discretization method and demonstrate the faster convergence rate of the error in our method. Specifically, our method achieves an $O(s^{-1/2})$ convergence rate, where s is the total computational budget. The convergence rate for the Euler discretization method is $O(s^{-1/3})$ or slower, depending on the model coefficients and option payoff function.

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1. Introduction

Financial models usually specify the dynamics of the state variables, e.g., stock price, volatility, and interest rate, as stochastic differential equations (SDE). If these SDEs yield closed-form solutions, then Monte Carlo simulation can be used to generate an unbiased estimator of the price of a derivative security. When using Monte Carlo simulation, many sample paths of the state variables are generated and the payoff of the derivative is evaluated for each path. Discounting and averaging over all paths gives an estimator of the derivative price. The error in the Monte Carlo estimator can be calculated using the central limit theorem and converges to zero as the number of sample paths used increases. If generating each sample path requires a roughly equal amount of time, then the convergence rate for such an unbiased Monte Carlo estimator is $O(s^{-1/2})$, where s is the total computational budget.

If the SDEs that define the dynamics of the state variables do not yield closed-form solutions, it is still possible to use Monte Carlo simulation by discretizing the time interval and simulating the state process dynamics on this discrete-time grid. However, the approximation of continuous-time processes by discrete-time processes introduces bias into the simulation estimator. This bias causes several important problems when estimating the prices or Greeks of derivative securities. First, because the magnitude of the bias is unknown, it is difficult to obtain valid confidence intervals. Second, many time steps may be necessary to reduce the bias to an acceptable level, and even more computational effort is needed to verify that the bias is small enough. Finally, both the number of time steps and number of sample paths need to be increased together to decrease the total error of the simulation estimator, but the optimal choice of these parameters is difficult to specify in advance. Duffie and Glynn (1995) study optimal rules for allocating the total computational budget between the number of time steps and the number of simulation trials. They show that, under some regularity conditions, the error for a first-order method such as Euler discretization has $O(s^{-1/3})$ convergence.

In this paper, we propose a method that recovers the $O(s^{-1/2})$ convergence rate of an unbiased Monte Carlo estimator for simulating derivative prices under some affine jump diffusion models. We first consider the stochastic volatility (SV) model of Heston (1993), which models the variance as a square-root process that is correlated with the stock price. Scott (1996) uses Fourier inversion methods to sample from the integral of a square-root process in an interest rate setting. Willard (1997) observes that the stock price is lognormally distributed conditional on a path of the Brownian motion driving the variance process. A similar "mixing result" was independently discovered in Romano and Touzi (1997). We build on the ideas from these papers to generate an exact sample from the distribution of the terminal stock price. We first generate a sample from the final value of the variance. Then, using Fourier inversion methods and conditioning, we get a sample from the integral of the variance. Finally, conditional on the values for the variance process, we are able to generate a sample from the stock price. We also show how to extend the method for models involving jumps in addition to the stochastic volatility. Strong empirical evidence of stochastic volatility and jumps in financial markets is documented in many recent papers, including Bates (1996), Bakshi et al. (1997), Broadie et al. (2006), Chernov et al. (2003), Eraker et al. (2003), and others.

Some other authors used Monte Carlo simulation for pricing derivatives under the stochastic volatility models. When the stock price and volatility are instantaneously uncorrelated, Hull and White (1987) show that the stock price has a lognormal distribution conditional on the integral of the variance process. They write the price of a call option as a Black-Scholes price given the integral of the variance. They simulate paths of the variance process and find the unconditional price of the option by taking the average of the conditional Black-Scholes prices. Willard (1997) considers the case when the stock price and volatility are instantaneously correlated. He observes that the stock price is lognormally distributed conditional on the entire path of the Brownian motion driving the variance process, and the call option price can be written as a Black-Scholes price conditional on this path. He simulates the variance process on a discrete-time grid and uses a conditional Monte Carlo approach to get the option price. His method still suffers from discretization bias because of the discretization of the variance process. Heath and Platen (2002) propose a variance reduction technique based on Itô calculus and show how their method can be applied for simulation under Heston's stochastic volatility model. Their approach uses discretization methods to simulate the state processes, and therefore does not eliminate the discretization bias.

The rest of this paper is organized as follows: §2 introduces the SV model dynamics and Euler discretization method. Section 3 is a step-by-step introduction to our method for exact simulation. In §4, we give some numerical results and compare our method to Euler discretization. In §5, we show that the efficiency of our simulation method can be further improved using conditional Monte Carlo techniques. Section 6 shows how to extend the method to models that involve jumps. In §7, we give an application of the exact simulation method to the pricing of forward start options. Section 8 contains concluding remarks. The derivation of the characteristic function of the integral of the variance is given in the appendix.

2. SV Model and Euler Discretization

2.1. Specification of the SV Model

Heston's model (1993) is based on the following equations, which represent the dynamics of the stock price and the variance processes under the risk-neutral measure:

$$dS_t = rS_t dt + \sqrt{V_t} S_t \big[\rho\, dW_t^{(1)} + \sqrt{1-\rho^2}\, dW_t^{(2)}\big], \tag{1}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t}\, dW_t^{(1)}. \tag{2}$$

The first equation gives the dynamics of the stock price: S_t denotes the stock price at time t, r is the risk-neutral drift, and $\sqrt{V_t}$ is the volatility. The second equation gives the evolution of the variance, which follows the square-root process: θ is the long-run mean variance, κ represents the speed of mean reversion, and σ_v is a parameter that determines the volatility of the variance process. $W_t^{(1)}$ and $W_t^{(2)}$ are two independent Brownian-motion processes, and ρ represents the instantaneous correlation between the return process and the volatility process.

2.2. Euler Discretization

Euler discretization can be used to approximate the paths of the stock price and variance processes on a discrete-time grid. Let $[0 = t_0 < t_1 < \cdots < t_M = T]$ be a partition of a time interval into M equal segments of length Δt, i.e., $t_i = iT/M$ for each $i = 0, 1, \ldots, M$. The discretization for the stock price process is

$$S_{t_i} = S_{t_{i-1}} + rS_{t_{i-1}}\Delta t + \sqrt{V_{t_{i-1}}}\, S_{t_{i-1}}\big[\rho \Delta W_{t_i}^{(1)} + \sqrt{1-\rho^2}\,\Delta W_{t_i}^{(2)}\big], \tag{3}$$

where $\Delta W_{t_i}^{(j)} = W_{t_i}^{(j)} - W_{t_{i-1}}^{(j)}$, $j = 1, 2$. The discretization for variance process is

$$V_{t_i} = V_{t_{i-1}} + \kappa(\theta - V_{t_{i-1}})\Delta t + \sqrt{V_{t_{i-1}}}\,\sigma_v \Delta W_{t_i}^{(1)}. \tag{4}$$

To simulate the Brownian increments, the fact that each increment $W_{t_i}^{(j)} - W_{t_{i-1}}^{(j)}$ is independent of others is used. Each such increment is normally distributed with mean 0 and standard deviation $\sqrt{\Delta t}$. In particular, we can simulate these increments by first generating a uniform random variate U and then replacing $\Delta W_{t_i}^{(j)}$ by $\sqrt{\Delta t} F^{-1}(U)$, where $F(x)$ is the cumulative distribution function of a standard normal random variable. There are efficient algorithms for accurately approximating the inverse function of a normal distribution function. We use the algorithm by Moro (1995). For each increment, we use an independent uniform random variate, so we need a total of $2M$ uniform random variates to simulate a path of stock price and variance processes. When using Euler discretization with a large number of simulation trials and a large number of time steps, the number of uniform random variates needed may be huge. We use the random number generator by Matsumoto and Nishimura (1998), which has a very long period. To avoid the negative values for variance and stock price, we set these to zero if we encounter negative values during the simulation.

By repeating this procedure, many paths can be generated. Because no-arbitrage derivative prices are given by their expected discounted payoffs under the risk-neutral measure, the price of a derivative, written as $E[f(S_T)]$, can be estimated by Monte Carlo simulation using

$$\frac{1}{N}\sum_{k=1}^{N} f(\hat{S}_T^M), \tag{5}$$

where N is the number of sample paths used in simulation and $\hat{S}_T^M$ denotes the simulated value of S_T over each sample path using M time steps. This Monte Carlo estimator converges to the correct price $E[f(S_T)]$ as the number of time steps M and the number of samples N become large.

There are two types of error associated with the Monte Carlo estimator given in (5). First, there is a statistical error that decreases at the rate $O(1/\sqrt{N})$. This is due to the central limit theorem: The standard error of the simulation estimator is $\sigma_f/\sqrt{N}$, where σ_f is the standard deviation of $f(\hat{S}_T^M)$. Second, there is a discretization error or discretization bias defined as $|E[f(S_T)] - E[f(\hat{S}_T^M)]|$. This error is caused by simulation of continuous-time processes using discrete-time approximations. Kloeden and Platen (1992) show that, under the conditions stated in their Theorem 14.5.2, this error decreases at the rate $O(1/M)$ for Euler discretization. Talay and Tubaro (1990) and Bally and Talay (1996) give similar results and show that the error can be expanded in powers of $1/M$ under some regularity conditions on the coefficients of the SDEs and the function f. The conditions for first-order convergence do not hold for the SDEs in (1) and (2), so the actual convergence may be slower in this case.

Duffie and Glynn (1995) study the optimal allocation of a computational budget between the number of time steps and the number of simulation trials. Their result implies that it is optimal to choose the number of time steps proportional to the square root of the number of simulation trials for first-order methods, although the optimal constant of proportionality is left undetermined. They show that with their allocation rule the convergence rate of the error for the Euler discretization is $O(s^{-1/3})$, where s is the total computational budget. More generally, if p is the convergence order of the bias, then an optimal allocation has error convergence of $O(s^{-p/(2p+1)})$. However, because the conditions for first-order convergence do not hold, $O(s^{-1/3})$ convergence rate is not guaranteed in this case. Indeed, the numerical examples we consider in §4 demonstrate that Euler discretization may achieve a lower convergence rate depending on the convergence rate of the bias.

There are higher-order discretization schemes that can be used instead of Euler for discretization of stock price and variance processes. For example, the Milstein scheme is a discretization scheme that achieves second-order convergence for the bias under the conditions given in Kloeden and Platen (1992). In theory, such a second-order method would improve the convergence of the total simulation error to $O(s^{-2/5})$, but again, because the conditions for convergence are not satisfied by the dynamics of the state variables, this method is not guaranteed to achieve a second-order convergence rate. Also, the convergence is not very smooth. For an example of this for the SV model, see Glasserman (2003). Another disadvantage of the Milstein scheme is that it is harder to implement and takes more time per replication than the Euler discretization method.

3. Exact Simulation of the SV Model

In this section, we give the details of the exact simulation method. In particular, we show how to generate an exact sample from the distribution of S_t by conditioning on the values generated by the variance process.

The stock price at time t, given the values of S_u and V_u for $u < t$, can be written as

$$S_t = S_u \exp\left[r(t-u) - \frac{1}{2}\int_u^t V_s\,ds + \rho\int_u^t \sqrt{V_s}\,dW_s^{(1)} + \sqrt{1-\rho^2}\int_u^t \sqrt{V_s}\,dW_s^{(2)}\right] \tag{6}$$

and the variance at time t is given by

$$V_t = V_u + \kappa\theta(t-u) - \kappa\int_u^t V_s\,ds + \sigma_v\int_u^t \sqrt{V_s}\,dW_s^{(1)}. \tag{7}$$

We go through the following steps to sample from the distribution of (S_t, V_t).

Exact Simulation Algorithm for the SV Model

Step 1. Generate a sample from the distribution of V_t given V_u.

Step 2. Generate a sample from the distribution of $\int_u^t V_s\,ds$ given V_t and V_u.

Step 3. Recover $\int_u^t \sqrt{V_s}\,dW_s^{(1)}$ from (7) given V_t, V_u, and $\int_u^t V_s\,ds$.

Step 4. Generate a sample from the distribution of S_t given $\int_u^t \sqrt{V_s}\,dW_s^{(1)}$ and $\int_u^t V_s\,ds$.

Next, we specify the details of each of these steps.

3.1. Sampling from V_t Given V_u

The distribution of V_t given V_u for some $u < t$ is, up to a scale factor, a noncentral chi-squared distribution as noted by Cox et al. (1985). The transition law of V_t can be expressed as

$$V_t = \frac{\sigma_v^2(1-e^{-\kappa(t-u)})}{4\kappa}\chi_d'^2\left(\frac{4\kappa e^{-\kappa(t-u)}}{\sigma_v^2(1-e^{-\kappa(t-u)})}V_u\right), \quad t > u, \tag{8}$$

where $\chi_d'^2(\lambda)$ denotes the noncentral chi-squared random variable with d degrees of freedom, and noncentrality parameter λ, and

$$d = \frac{4\theta\kappa}{\sigma_v^2}. \tag{9}$$

This says that, given V_u, V_t is distributed as $\sigma_v^2(1-e^{\kappa(t-u)})/(4\kappa)$ times a noncentral chi-squared random variable with d degrees of freedom and noncentrality parameter

$$\lambda = \frac{4\kappa e^{-\kappa(t-u)}}{\sigma_v^2(1-e^{-\kappa(t-u)})} V_u. \tag{10}$$

Thus, we can sample from the distribution of V_t exactly, provided that we can sample from the noncentral chi-squared distribution.

Let χ_d^2 denote a chi-squared random variable with d degrees of freedom. Johnson et al. (1994) show that for $d > 1$, the following representation is valid:

$$\chi_d'^2(\lambda) = \chi_1'^2(\lambda) + \chi_{d-1}^2. \tag{11}$$

To generate $\chi_d'^2(\lambda)$, $d > 1$, we can generate χ_{d-1}^2 and an independent standard normal random variable Z and set

$$\chi_d'^2(\lambda) = (Z + \sqrt{\lambda})^2 + \chi_{d-1}^2. \tag{12}$$

Thus, sampling from a noncentral chi-squared distribution is reduced to sampling from an ordinary chi-squared and an independent normal when $d > 1$.

For any $d > 0$, a noncentral chi-squared random variable can be represented as an ordinary chi-squared random variable with a random degrees of freedom parameter. In particular, if N is a Poisson random variable with mean $(1/2)\lambda$, then χ_{d+2N}^2 has the same distribution as $\chi_d'^2(\lambda)$. Therefore, to sample from $\chi_d'^2(\lambda)$, we can first generate a Poisson random variable N and then, conditional on N, we can sample a chi-squared random variable with $d+2N$ degrees of freedom.

In the simulation of V_t, we use the representation given in (12) when $d > 1$, and we use the Poisson method described above otherwise. To sample from the chi-squared distribution, the methods to sample from gamma distribution can be used because chi-squared is a special case of this distribution. We use algorithms GS* and GKM3 in Fishman (1996) to sample from the gamma distribution. The first of these is used when the degrees of freedom parameter for chi-squared distribution is less than one, and the second one is used otherwise.

3.2. Sampling from $\int_u^t V_s\,ds$ Given V_t and V_u

Once we have a sample for V_t, we want to sample from the distribution of $\int_u^t V_s\,ds$ given V_t and V_u. Scott (1996) uses Fourier inversion techniques to invert the characteristic function of this distribution to sample from the same distribution in an interest rate setting. We follow a similar approach to generate a sample for the integral. The Laplace transform

$$E\left[\exp\left(-a^*\int_u^t V_s\,ds\right) \,\Big|\, V_u, V_t\right]$$

can be derived using the results in Pitman and Yor (1982). Details of the derivation are given in the appendix. The characteristic function follows by setting $a^* = -ia$:

$$\begin{aligned}\Phi(a) &= E\left[\exp\left(ia\int_u^t V_s\,ds\right) \,\Big|\, V_u, V_t\right]\\ &= \frac{\gamma(a)e^{(-1/2)(\gamma(a)-\kappa)(t-u)}(1-e^{-\kappa(t-u)})}{\kappa(1-e^{-\gamma(a)(t-u)})}\\ &\quad\times \exp\left\{\frac{V_u+V_t}{\sigma^2}\left[\frac{\kappa(1+e^{-\kappa(t-u)})}{1-e^{-\kappa(t-u)}} - \frac{\gamma(a)(1+e^{-\gamma(a)(t-u)})}{1-e^{-\gamma(a)(t-u)}}\right]\right\}\\ &\quad\times \frac{I_{0.5d-1}\left[\sqrt{V_uV_t}\frac{4\gamma(a)e^{-0.5\gamma(a)(t-u)}}{\sigma^2(1-e^{-\gamma(a)(t-u)})}\right]}{I_{0.5d-1}\left[\sqrt{V_uV_t}\frac{4\kappa e^{-0.5\kappa(t-u)}}{\sigma^2(1-e^{-\kappa(t-u)})}\right]},\end{aligned} \tag{13}$$

where $\gamma(a) = \sqrt{\kappa^2 - 2\sigma^2 ia}$, d is as given in (9), and $I_\nu(x)$ is the modified Bessel function of the first kind.

The probability function can be computed using Fourier inversion methods. If we let $V(u,t)$ denote the random variable that has the same distribution as the integral $\int_u^t V_s\,ds$ conditional on V_u and V_t, then we can write (see Feller 1971)

$$\begin{aligned}F(x) \equiv \Pr\{V(u,t) \leqslant x\} &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ux}{u}\Phi(u)\,du\\ &= \frac{2}{\pi}\int_0^{\infty} \frac{\sin ux}{u}\mathrm{Re}[\Phi(u)]\,du.\end{aligned} \tag{14}$$

We use the trapezoidal rule to compute the probability distribution numerically:

$$\Pr\{V(u,t) \leqslant x\} = \frac{hx}{\pi} + \frac{2}{\pi}\sum_{j=1}^{\infty} \frac{\sin hjx}{j}\mathrm{Re}[\Phi(hj)] - e_d(h), \tag{15}$$

where h is a grid size to be set to achieve any desired level of accuracy, and $e_d(h)$ is the resulting discretization error. The trapezoidal rule works well for periodic and other oscillating integrands, because the errors tend to cancel. The discretization error $e_d(h)$ can be bounded above and below by using a Poisson summation formula as shown in Abate and Whitt (1992):

$$\begin{aligned}0 \leqslant e_d(h) &= \sum_{k=1}^{\infty}\left[F\left[\frac{2k\pi}{h}+x\right] - F\left[\frac{2k\pi}{h}-x\right]\right]\\ &\leqslant F^c\left[\frac{2\pi}{h}-x\right],\end{aligned} \tag{16}$$

where $F^c(x) = 1 - F(x)$.

If we want to achieve a discretization error ϵ, then the step size should be

$$h = \frac{2\pi}{x+u_\epsilon} \geqslant \frac{\pi}{u_\epsilon}, \quad \text{where } F^c(u_\epsilon) = \epsilon \text{ and } 0 \leqslant x \leqslant u_\epsilon. \tag{17}$$

Finding the correct u_ϵ is not straightforward, but the moments of the distribution can easily be found using the characteristic function in (13). We use the mean, plus five or more standard deviations, to get a value with a small-tail probability.

To be able to calculate $\Pr\{V(u,t) \leqslant x\}$ using (15), we need to determine a point at which the summation can be terminated. Let N represent the last term to be calculated so that the approximation becomes

$$\Pr\{V(u,t) \leqslant x\} = \frac{hx}{\pi} + \frac{2}{\pi}\sum_{j=1}^{N}\frac{\sin hjx}{j}\operatorname{Re}[\Phi(hj)] - e_d(h) - e_T(N), \quad (18)$$

where $e_T(N)$ denotes the truncation error caused by using N terms in the finite sum. The magnitude of the characteristic function, $|\Phi(u)|$, has value one at $u=0$ and it decreases as u increases. Because $|\sin(ux)| \leqslant 1$, the integrand in (14) is bounded by

$$\frac{2|\operatorname{Re}[\Phi(u)]|}{\pi u} \leqslant \frac{2|[\Phi(u)]|}{\pi u} = \beta(u). \quad (19)$$

Because the integrand is oscillating, the bound for the last term gives a good estimate for the truncation error, i.e., we set $e_T(N) = h\beta(Nh)$. The summation is terminated at $j = N$ when

$$\frac{|\Phi(hN)|}{N} < \frac{\pi\epsilon}{2}, \quad (20)$$

where ϵ is the desired truncation error.

It should be noted that very often the error bounds stated above for determination of the grid size h and the truncation point N are conservative. By trial and error one can get h and N that achieve the desired accuracy with a smaller number of terms.

The hardest and numerically most time-consuming part of (15) is the evaluation of the characteristic function. The characteristic function given in (13) involves two modified Bessel functions of the first kind, and the one in the numerator has a complex argument. The modified Bessel function of the first kind is characterized by the following power series:

$$I_\nu(z) = \left(\frac{1}{2}z\right)^{\nu}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^j}{j!\Gamma(\nu+j+1)}, \quad (21)$$

where $\Gamma(x)$ is the gamma function and z is a complex number. We used a routine written by Amos (1986) to calculate these functions. The routine uses the power series for small $|z|$, the asymptotic expansion for large $|z|$, the Miller Algorithm normalized by the Wronskian and a Neumann Series for intermediate magnitudes, and the uniform asymptotic expansion for large orders. The routine can also calculate a scaled version of the Bessel function, $e^{-|\operatorname{Re}(z)|}I_\nu(z)$, which makes it possible to compute the function value for very large orders.

The representation of $I_\nu(z)$ given in (21) is not complete because the function z^ν, which is a factor on the right-hand side, is a multivalued function and needs precise specification. It is defined to be $\exp(\nu \log z)$, where the argument of z is given its principal value so that

$$-\pi < \arg(z) \leqslant \pi.$$

This is the definition most software packages and programming library routines use to calculate the function value. However, the function $I_\nu(z)$ defined in this way is discontinuous at each point along the negative x-axis, as are all functions defined on a specific branch of $\arg(z)$. Therefore, to get around this problem and accomplish the continuity of the characteristic function, we need to keep track of $\arg(z)$ and change the branch when necessary. We use the following continuation formula from Abramowitz and Stegun (1972) for this purpose, which makes it possible to calculate the value of $I_\nu(z)$ for $\arg(z)$ different than its principal value:

$$I_\nu(ze^{m\pi i}) = e^{m\nu\pi i}I_\nu(z) \quad \{m \text{ integer}\}. \quad (22)$$

As this formula shows, to calculate the function on a different branch, we need to multiply the value on the principal branch by a factor $e^{m\pi\nu i}$.

To simulate the value of the integral, we use the inverse transform method. We generate a uniform random variable U and then find the value of x for which $\Pr\{V(u,t) \leqslant x\}$ is equal to U. We use a second-order Newton's method to find the solution for x. This method is used to solve equations of the form $f(x) = 0$ using the iteration

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}\left(1 - \sqrt{1 - \frac{2f(x_n)f''(x_n)}{f'(x_n)^2}}\right).$$

The conditional distribution of the integral is approximately normal and the initial guess for x is calculated using the inverse normal distribution function with the mean and standard deviation of the correct distribution. The moments of the correct distribution are found using (13). We set the starting value to a small value, such as 0.01 times the mean, if the normal value is negative. The first and second derivatives of the distribution function are calculated numerically, using the derivatives of the approximation (15). The iterative procedure achieves five-decimal-place accuracy in three to four iterations. We use a bisection search method when the abovementioned method fails to converge to true value, which typically occurs when U is very close to 0 or 1.

3.3. Generating a Sample for S_t

After having generated samples from V_t and $\int_u^t V_s\,ds$, we use (7) to get

$$\int_u^t \sqrt{V_s}\,dW_s^{(1)} = \left(\frac{1}{\sigma_v}\right)\left(V_t - V_u - \kappa\theta(t-u) + \kappa\int_u^t V_s\,ds\right). \quad (23)$$

Furthermore, because the process for V_t is independent of the Brownian motion $W_t^{(2)}$, the distribution of $\int_u^t \sqrt{V_s}\,dW_s^{(2)}$, given the path generated by V_t, is normal, with mean 0 and variance $\int_u^t V_s\,ds$.

Using these two results, we get that the conditional distribution of $\log S_t$ is normal with mean

$$m(u,t)=\log S_u+\left[r(t-u)-\frac{1}{2}\int_u^t V_s\,ds+\rho\int_u^t \sqrt{V_s}\,dW_s^{(1)}\right]$$

and variance

$$\sigma^2(u,t)=(1-\rho^2)\int_u^t V_s\,ds.$$

To generate a sample from S_t, we generate a standard normal random variable Z and then set

$$S_t=e^{m(u,t)+\sigma(u,t)Z}.$$

We can use the value of S_t generated in this way to find the price of any derivative that depends on the value of S_t. Repeating over many sample paths, taking the average and discounting gives us an unbiased estimator of the derivative price.

4. Numerical Results

In this section, we present some numerical comparisons of the Euler and the exact simulation method described above. We use a European call option for this purpose. Heston (1993) gives a closed-form solution for the price of this option, so we are able to calculate and compare the two methods through their root-mean-squared (RMS) errors. If $\hat{\alpha}$ is the simulation estimator used for the derivative price and α is the true price, then the bias of the estimator is given by $(E[\hat{\alpha}]-\alpha)$ and the variance is given by $E[(\hat{\alpha}-E[\hat{\alpha}])^2]$. RMS error is then defined as $(\text{bias}^2+\text{variance})^{1/2}$. The bias for Euler discretization with a specific number of time steps can be estimated using a very large number of simulation trials to estimate $E[\hat{\alpha}]$, and then taking the difference with the true price. The variance for each simulation experiment is estimated by the sample variance of the simulation output.

The simulation experiments in this and the other sections of the paper were performed on a desktop PC with an AMD Athlon 1.66 GhZ processor and 624 MB of RAM, running Windows XP Professional. The codes were written in the C programming language, and the compiler used was Microsoft Visual C++ 6.0.

Table 1 gives simulation results for a European call option. The set of parameters for the SV model are taken from Duffie et al. (2000). These were found by minimizing the mean squared errors for market option prices for S&P 500 on November 2, 1993. The bias column is estimated using 40 million simulation trials. The number of time steps for the Euler discretization is set equal to the square root of the number of simulation trials.

Table 1. Simulation results under the SV model for a European call option.

(a) Simulation with the exact method

No. of simulation trials	RMS error	Computing time (sec.)
10,000	0.0750	3.8
40,000	0.0373	15.2
160,000	0.0186	60.0
640,000	0.0093	239.4
2,560,000	0.0046	955.7
10,240,000	0.0023	3,822.6

(b) Simulation with the Euler discretization

No. of simulation trials	No. of time steps	Bias	Standard error	RMS error	Computing time (sec.)
10,000	100	0.1543	0.0772	0.1725	0.2
40,000	200	0.1003	0.0381	0.1073	1.9
160,000	400	0.0662	0.0189	0.0689	15.2
640,000	800	0.0395	0.0094	0.0406	121.3
2,560,000	1,600	0.0267	0.0047	0.0272	970.0
10,240,000	3,200	0.0161	0.0023	0.0163	7,758.6

Note. Option parameters: $S=100$, $K=100$, $V_0=0.010201$, $\kappa=6.21$, $\theta=0.019$, $\sigma_v=0.61$, $\rho=-0.70$, $r=3.19\%$, $T=1.0$ year, true option price $=6.8061$.

The time needed for the exact method is more than the Euler discretization for a small number of simulation trials. However, the exact method requires less computation time as the desired accuracy is increased.

There are some cases for which the bias of the Euler discretization is very large even if a large number of time steps is used. This is especially true when $(2\kappa\theta/\sigma_v^2)\leqslant 1$ and σ_v is large relative to θ. In this case, the variance process for Euler discretization often hits to 0, causing a large discretization bias. Table 2 demonstrates this case. The bias column is estimated using 40 million simulation trials.

As seen from the simulation results in Table 2, significant bias remains in the Euler discretization even when 3,200 time steps are used with 10,240,000 simulation trials. The RMS error for our exact method with the same number of trials is about 30 times smaller with much less computation time. Figures 1 and 2 show the convergence of the RMS graphically for both methods. The faster convergence rate of the exact method is demonstrated by the steeper slope in the graphs.

One apparent problem when using Euler discretization is the choice of the number of time steps to be used in the simulation. Our choice of setting the number of time steps equal to the square root of the number of simulation trials is somewhat arbitrary. Duffie and Glynn (1995) show that, asymptotically, it is optimal to increase the number of time steps proportional to the square root of the number of simulation trials for first-order methods such as Euler discretization. However, the optimal constant of proportionality is not easy to determine. Ideally, for a specific number

Table 2. Simulation results under the SV model for a European call option.

(a) Simulation with the exact method

No. of simulation trials	RMS error	Computing time (sec.)
10,000	0.6125	3.8
40,000	0.2904	15.3
160,000	0.1464	61.3
640,000	0.0726	244.5
2,560,000	0.0362	978.6
10,240,000	0.0181	3,916.5

(b) Simulation with the Euler discretization

No. of simulation trials	No. of time steps	Bias	Standard error	RMS error	Computing time (sec.)
10,000	100	2.1962	0.6568	2.2923	0.2
40,000	200	1.6413	0.3247	1.6731	1.9
160,000	400	1.2151	0.1544	1.2248	15.4
640,000	800	0.9265	0.0761	0.9296	122.6
2,560,000	1,600	0.7093	0.0379	0.7103	980.4
10,240,000	3,200	0.5367	0.0188	0.5370	7,838.4

Note. Option parameters: $S = 100$, $K = 100$, $V_0 = 0.09$, $\kappa = 2.00$, $\theta = 0.09$, $\sigma_v = 1.00$, $\rho = -0.30$, $r = 5.00\%$, $T = 5.0$ years, true option price $= 34.9998$.

of simulation trials, one should choose the number of time steps such that the magnitude of the discretization bias and the standard error will be close. Then, as more accuracy is needed, the number of time steps should be increased proportional to the square root of the number of simulation trials. This way, the simulation estimate converges to the true value with neither of the errors dominating the other. Of course, in practice, we do not usually know the true value of the quantity we want to estimate, and therefore it is not possible to know what the bias is. Even for the cases where we do know the true value, we might need to do a convergence study with many simulation trials to estimate the bias. Thus, the choice of the number of time steps is another shortcoming of the Euler discretization method.

Figure 1. Convergence of the RMS errors of the two methods for the option in Table 1 (SV model).

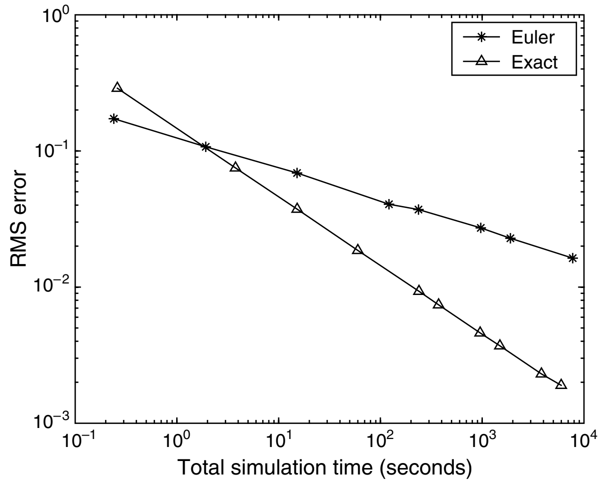


Figure 2. Convergence of the RMS errors of the two methods for the option in Table 2 (SV model).

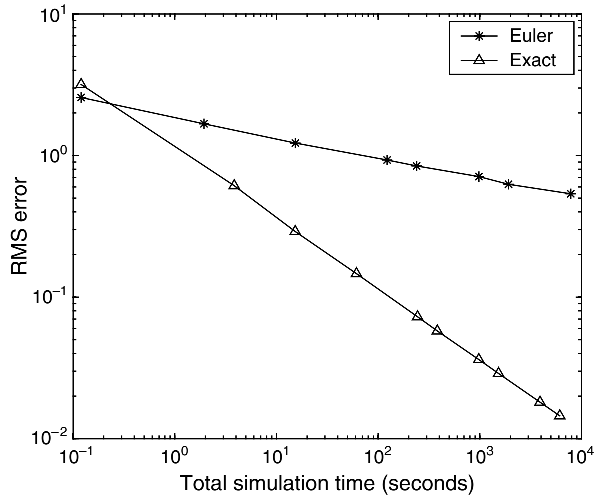


Table 3 shows the bias values for various time steps for the two options considered before. The bias values are estimated using 40 million simulation trials. For both options, the bias dominates the standard error when number of time steps, M, is set to the square root of the number of simulation trials, N. For the first option, the number of time steps that makes bias and standard error approximately equal is between $\sqrt{N}$ and $10\sqrt{N}$, while for the second option this value is greater than $10\sqrt{N}$.

Careful examination of the bias values in Table 3 reveals another problem with the convergence of the Euler discretization method. The bias values do not decrease to half of their values when the number of time steps is doubled. We mentioned in §2.2 that the conditions for first-order convergence do not hold for the SDEs in (1) and (2). These examples demonstrate that the convergence for discretization bias may indeed be significantly slower. As shown in Duffie and Glynn (1995), if p is the convergence order of the bias, then an optimal allocation has error convergence of $O(s^{-p/(2p+1)})$. The convergence rate of the bias for these examples is close to $O(1/\sqrt{M})$. Assuming that this rate is the real convergence rate, the convergence rate of the total error for an optimal allocation would be $O(s^{-1/4})$. In practical cases, a nonoptimal allocation may be used because the real convergence rate of the bias is unknown, leading to even slower convergence rates. These arguments provide another motivation for using an exact simulation method instead of biased methods.

Table 3. Effect of the number of time steps in Euler discretization.

No. of simulation trials (N)	Standard error	Bias values for various time steps (M)					
		$M = 0.1\sqrt{N}$	Bias	$M = \sqrt{N}$	Bias	$M = 10\sqrt{N}$	Bias
(a) Bias values for the option in Table 1							
10,000	0.0772	10	0.5348	100	0.1543	1,000	0.0364
40,000	0.0381	20	0.3897	200	0.1003	2,000	0.0225
160,000	0.0189	40	0.2667	400	0.0662	4,000	0.0163
640,000	0.0094	80	0.1745	800	0.0395	8,000	0.0099
2,560,000	0.0047	160	0.1156	1,600	0.0267	16,000	0.0067
10,240,000	0.0023	320	0.0739	3,200	0.0161	32,000	0.0029
(b) Bias values for the option in Table 2							
10,000	0.6568	10	6.0489	100	2.1962	1,000	0.8408
40,000	0.3247	20	4.5423	200	1.6413	2,000	0.6257
160,000	0.1544	40	3.2997	400	1.2151	4,000	0.5142
640,000	0.0761	80	2.4358	800	0.9265	8,000	0.3731
2,560,000	0.0379	160	1.7914	1,600	0.7093	16,000	0.2935
10,240,000	0.0188	320	1.3501	3,200	0.5367	32,000	0.2273

5. Conditional Monte Carlo for the SV Model

Willard (1997) proposed a conditional Monte Carlo method to improve the efficiency of the simulation estimators under stochastic volatility models. His method is applicable to path-independent derivatives that have closed-form solutions under the Black-Scholes model. In this section, we apply the conditional Monte Carlo method to improve the efficiency of the exact simulation method we described in §3. We also compare the improved results with the results of Euler discretization in the same setting.

We follow the approach of Willard and use a European call option to demonstrate the conditional Monte Carlo method. We assume that the dynamics of the stock price and the variance processes are as given in (1) and (2). Let $BS(S_0, \sigma)$ denote the Black-Scholes formula for a European call option with maturity T, strike K, and written on a stock with initial price S_0 and constant volatility σ. Let $\tilde{\sigma}$ be the average volatility of the stock over the time horizon:

$$\tilde{\sigma} \equiv \sqrt{\frac{1}{T}\int_0^T V_s\,ds}. \tag{24}$$

Willard observes that conditional on a path of the Brownian motion driving the variance process, the price of the call option can be written as

$$BS\big(S_0\xi, \tilde{\sigma}\sqrt{1-\rho^2}\big), \tag{25}$$

where

$$\xi = \exp\left(-\frac{\rho^2}{2}\int_0^T V_s\,ds + \rho\int_0^T \sqrt{V_s}\,dW_s^{(1)}\right). \tag{26}$$

Then, using the law of iterated expectations, the unconditional price of the option can be written as

$$E\big[BS\big(S_0\xi, \tilde{\sigma}\sqrt{1-\rho^2}\,\big|\, W_s^{(1)}\big)\big]. \tag{27}$$

The notation used emphasizes that the Black-Scholes price inside the expectation is conditional on a path of the Brownian motion $W_s^{(1)}$. Actually, it is not necessary to condition on the entire path of the Brownian motion. We can just write

$$E\left[BS\left(S_0\xi, \tilde{\sigma}\sqrt{1-\rho^2}\,\middle|\, \int_0^T V_s\,ds, \int_0^T \sqrt{V_s}\,dW_s^{(1)}\right)\right] \tag{28}$$

because we only need $\int_0^T V_s\,ds$ and $\int_0^T \sqrt{V_s}\,dW_s^{(1)}$ to compute the Black-Scholes price. We can now use the above representation to estimate the option price using conditional Monte Carlo.

The above approach can also be used to generate unbiased estimators of the Greeks under the SV model. For example, to calculate the delta of a European call option, we can differentiate the expression in (28) and get

$$E\left[\Delta\left(S_0\xi, \tilde{\sigma}\sqrt{1-\rho^2}\,\middle|\, \int_0^T V_s\,ds, \int_0^T \sqrt{V_s}\,dW_s^{(1)}\right)\xi\right], \tag{29}$$

where $\Delta(S, v)$ denotes the Black-Scholes delta. See Broadie and Glasserman (1996) for more details about the pathwise method to estimate option Greeks in a Monte Carlo simulation. Extensions of the exact simulation method in this paper to the development of unbiased simulation estimates of Greeks for various exotic options are given in Broadie and Kaya (2004).

In §3, we described the steps to sample from the final variance value V_T, and the integral of the variance value $\int_0^T V_s\,dW_s^{(1)}$, and also showed how to recover $\int_0^T \sqrt{V_s}\,dW_s^{(1)}$ given these two. When using the exact simulation method with conditional Monte Carlo, we can complete these steps and then use (25) to get the conditional option price. By repeating over many paths and taking the average, we can get an unbiased estimator of the option price.

When using Euler discretization with conditional Monte Carlo, only the variance process needs to be simulated using the discretization in (4). Simulation of the stock price is not needed, because after simulating a path of the variance process, one can use (25) to find the conditional option price. Using conditional Monte Carlo cuts the simulation time for Euler discretization roughly in half because only one process is simulated instead of two. However, it does not eliminate the discretization bias because the variance process is still being simulated on a discrete-time grid.

Tables 4 and 5 show the simulation results for the two options considered in the previous section, this time using conditional Monte Carlo with the exact simulation and Euler discretization methods. Although the conditional Monte Carlo method decreases the simulation time and the standard error for the Euler discretization, the bias is almost unchanged. Figures 3 and 4 show the convergence of the RMS errors.

Comparing the RMS error for the exact method using conditional Monte Carlo and unconditional Monte Carlo, we see that conditional Monte Carlo can lead to significant variance reduction. For the second option considered, more than 50-fold variance reduction is achieved using conditional Monte Carlo.

In addition to the conditional Monte Carlo method described above, Willard (1997) uses quasi-Monte Carlo (low-discrepancy) methods and shows that it results in significant efficiency improvements. Quasi Monte Carlo helps in improving the convergence of the simulation error, but it does not decrease the discretization bias. Therefore, qualitative results similar to the ones above will hold in comparing the exact method and the Euler discretization method when quasi Monte Carlo is used for both.

Table 4. Simulation results for a European call option using conditional Monte Carlo.

(a) Simulation with the exact method

No. of simulation trials	RMS error	Computing time (sec.)
10,000	0.0395	3.6
40,000	0.0199	14.5
160,000	0.0099	57.7
640,000	0.0050	230.4
2,560,000	0.0025	921.6
10,240,000	0.0012	3,687.4

(b) Simulation with the Euler discretization

No. of simulation trials	No. of time steps	Bias	Standard error	RMS error	Computing time (sec.)
10,000	100	0.1574	0.0421	0.1630	0.1
40,000	200	0.1000	0.0207	0.1022	1.0
160,000	400	0.0649	0.0102	0.0657	8.1
640,000	800	0.0419	0.0050	0.0422	64.1
2,560,000	1,600	0.0267	0.0025	0.0268	511.1
10,240,000	3,200	0.0170	0.0012	0.0170	4,082.8

Note. Option parameters: $S = 100$, $K = 100$, $V_0 = 0.010201$, $\kappa = 6.21$, $\theta = 0.019$, $\sigma_v = 0.61$, $\rho = -0.70$, $r = 3.19\%$, $T = 1.0$ year, true option price $= 6.8061$.

Table 5. Simulation results for a European call option using conditional Monte Carlo.

(a) Simulation with the exact method

No. of simulation trials	RMS error	Computing time (sec.)
10,000	0.0803	3.8
40,000	0.0398	15.1
160,000	0.0199	60.1
640,000	0.0100	239.7
2,560,000	0.0050	958.9
10,240,000	0.0025	3,835.7

(b) Simulation with the Euler discretization

No. of simulation trials	No. of time steps	Bias	Standard error	RMS error	Computing time (sec.)
10,000	100	2.1046	0.1002	2.1070	0.1
40,000	200	1.5870	0.0469	1.5877	1.0
160,000	400	1.2025	0.0227	1.2027	8.4
640,000	800	0.9154	0.0110	0.9155	64.2
2,560,000	1,600	0.7022	0.0054	0.7022	512.1
10,240,000	3,200	0.5391	0.0026	0.5391	4,086.5

Note. Option parameters: $S = 100$, $K = 100$, $V_0 = 0.09$, $\kappa = 2.00$, $\theta = 0.09$, $\sigma_v = 1.00$, $\rho = -0.30$, $r = 5.00\%$, $T = 5.0$ years, true option price $= 34.9998$.

6. Extensions to Other Models

6.1. Simulation Under the SVJ Model

The stochastic volatility with jumps (SVJ) model is an extension of the SV model to include jumps in the stock

Figure 3. Convergence of the RMS errors of the two methods for the option in Table 4 (SV model).

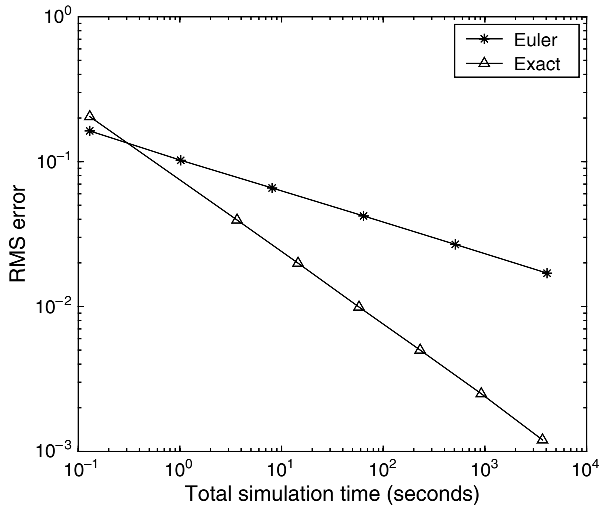
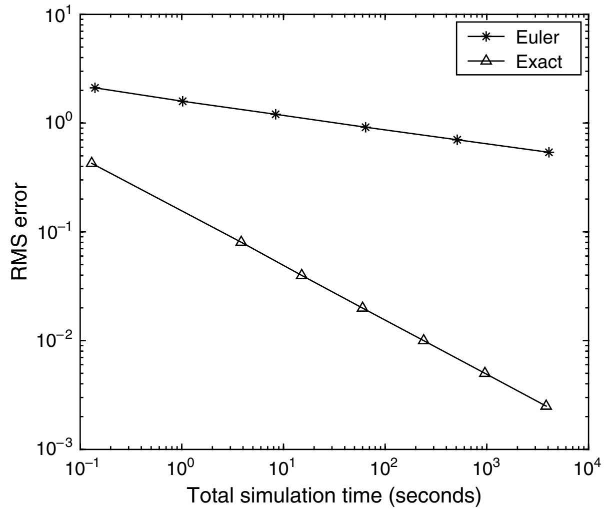


Figure 4. Convergence of the RMS errors of the two methods for the option in Table 5 (SV model).



price process. The risk-neutral dynamics are

$$dS_t = (r-\lambda\bar{\mu})S_t dt + \sqrt{V_t}S_t\left[\rho dW_t^{(1)} + \sqrt{1-\rho^2}dW_t^{(2)}\right] + (\xi^s - 1)dN_t, \tag{30}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v\sqrt{V_t}dW_t^{(1)}, \tag{31}$$

where N_t is a Poisson process with constant intensity λ, and ξ^s is the relative jump size in the stock price. In particular, when a jump occurs at time t, then $S_{t+} = S_{t-}\xi^s$. The distribution of ξ^s is lognormal with mean μ_s and variance σ_s^2. The parameters μ_s and $\bar{\mu}$ are related to each other by the equation

$$\mu_s = \log(1+\bar{\mu}) - \tfrac{1}{2}\sigma_s^2$$

and only one of them needs to be specified.

To sample from the final stock price, the jump part and the diffusion part can be simulated separately. In other words, we can first simulate the diffusion part of the stock price, and then multiply by the realized jump sizes. This makes it possible to extend the exact simulation method to simulate under the SVJ model.

To generate a sample for the derivative payoff that has maturity T, we use the following procedure.

Exact Simulation Algorithm for the SVJ Model

Step 1. Disregard the jump part, and simulate the stock price S_T as in the SV model using the exact simulation method described in §3.

Step 2. Generate a Poisson random variable with mean (λT) to determine the number of jumps that occurred in the time horizon, and call this number J.

Step 3. Generate independent jump sizes ξ_i^s for $i = 1, \ldots, J$, from the lognormal distribution with mean μ_s and variance σ_s^2.

Step 4. Find the adjusted final stock price $\tilde{S}_T$ by multiplying the price from Step 1 with the jump sizes, i.e., set $\tilde{S}_T = S_T \prod_{i=1}^{i=J} \xi_i^s$.

Step 5. Compute the derivative payoff using $\tilde{S}_T$.

Table 6 gives numerical results for a European call option using exact simulation and Euler discretization methods. The model parameters are taken from Duffie et al. (2000); these are fitted parameters for S&P 500 on a particular day. The bias of the Euler discretization method is relatively small for the set of parameters used. Therefore, we choose the number of time steps, M, to be equal to $0.1\sqrt{N}$, where N is the number of simulation trials. The bias column is estimated using 500 million simulation trials. Figure 5 shows the convergence of the RMS errors for the two simulation methods used. The exact method has a better RMS error than the Euler discretization method when more than one second of computation time is used.

6.2. Simulation Under the SVCJ Model

The SVCJ model (stochastic volatility with contemporaneous jumps in the stock price and variance) introduced in Duffie et al. (2000) is similar to the SVJ model, but it also includes jumps in the variance process. The governing equations are

$$dS_t = (r-\lambda\bar{\mu})S_t dt + \sqrt{V_t}S_t\left[\rho dW_t^{(1)} + \sqrt{1-\rho^2}dW_t^{(2)}\right]$$

Table 6. Simulation results under the SVJ model for a European call option.

(a) Simulation with the exact method

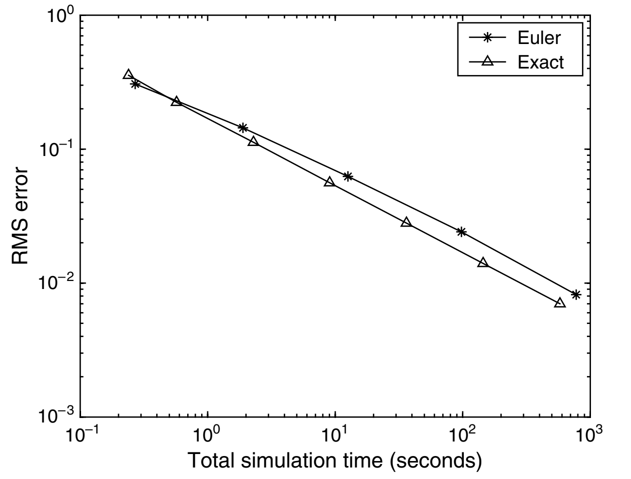
No. of simulation trials	RMS error	Computing time (sec.)
10,000	0.2232	0.6
40,000	0.1124	2.3
160,000	0.0560	9.1
640,000	0.0280	36.2
2,560,000	0.0140	144.7
10,240,000	0.0070	579.4

(b) Simulation with the Euler discretization

No. of simulation trials	No. of time steps	Bias	Standard error	RMS error	Computing time (sec.)
10,000	10	0.5902	0.2480	0.6402	0.1
40,000	20	0.2834	0.1169	0.3065	0.3
160,000	40	0.1322	0.0575	0.1442	1.9
640,000	80	0.0557	0.0283	0.0625	12.6
2,560,000	160	0.0196	0.0141	0.0241	97.7
10,240,000	320	0.0043	0.0070	0.0082	776.1

Note. Option parameters: $S = 100$, $K = 100$, $V_0 = 0.008836$, $\kappa = 3.99$, $\theta = 0.014$, $\sigma_v = 0.27$, $\rho = -0.79$, $\lambda = 0.11$, $\bar{\mu} = -0.12$, $\sigma_s = 0.15$, $r = 3.19\%$, $T = 5.0$ years, true option price $= 20.1642$.

Figure 5. Convergence of the RMS errors of the two methods for the option in Table 6 (SVJ model).



$$+(\xi^s-1)dN_t, \tag{32}$$

$$dV_t=\kappa(\theta-V_t)dt+\sigma_v\sqrt{V_t}dW_t^{(1)}+\xi^v dN_t, \tag{33}$$

where N_t is again a Poisson process with constant intensity λ, ξ^s is the relative jump size of the stock price, and ξ^v is the jump size of the variance. The jumps in stock price and the variance occur concurrently, and their magnitudes have a correlation determined by the parameter ρ_J. The distribution of ξ^v is exponential with mean μ_v. Given ξ^v, ξ^s is lognormally distributed with mean $(\mu_s+\rho_J\xi^v)$ and variance σ_s^2. The parameters μ_s and $\bar{\mu}$ are related to each other by the equation

$$\mu_s=\log[(1+\bar{\mu})(1-\rho_J\mu_v)]-\tfrac{1}{2}\sigma_s^2$$

and only one of them needs to be specified.

The jumps in variance prevent us from using the same method as in a SVJ model, but it is still possible to extend the method to generate exact simulation estimators under the SVCJ model. We need to divide the simulation horizon according to the jump occurrences and simulate the values of variance and stock price at each jump time.

To generate a sample for the derivative payoff that has maturity T, we use the procedure described below. We denote the initial stock price by S_0, the initial variance value by V_0, and the current time by t_0.

Exact Simulation Algorithm for the SVCJ Model

Step 1. Simulate a Poisson process with arrival rate λ and determine the time of the next jump, and denote this time by t_j. If $t_j>T$, then set $t_j=T$.

Step 2. Disregard the jump part, and simulate the variance value V_{t_j} and stock price S_{t_j} using the exact simulation method described in §3 using a time interval of $\Delta t=t_j-t_0$.

Step 3. If $t_j=T$, then set $S_T=S_{t_j}$ and go to Step 6. Otherwise, generate ξ^v by sampling from an exponential distribution with mean μ_v. Update the variance value by setting $\widetilde{V}_{t_j}=V_{t_j}+\xi^v$.

Step 4. Generate ξ^s by sampling from a lognormal distribution with mean $(\mu_s+\rho_J\xi^v)$ and variance σ_s^2. Update the stock price by setting $\widetilde{S}_{t_j}=S_{t_j}\xi^s$.

Step 5. Set $S_0=\widetilde{S}_{t_j}$, $V_0=\widetilde{V}_{t_j}$, $t_0=t_j$ and go to Step 1.

Step 6. Compute the derivative payoff using S_T.

Dividing the simulation horizon and using more time steps affects the computation speed; therefore, the simulation of the SVCJ model is slower than the SV or SVJ models. If, for example, the arrival rate of jumps is one per year, then the simulation of a five-year option will be about five times slower. Some of the advantage of simulating the stock price over a long horizon is lost, but still, the method is practical and provides unbiased simulation estimators of derivative prices.

The Fourier inversion step in the exact simulation method may run into some problems when the simulation interval is too small. Such cases may occur in the simulation of the SVCJ model when two jumps are very close to each other or when a jump is too close to maturity. If the simulation interval is less than 0.01 year, instead of using the exact simulation method, we use Euler discretization with 100 time steps. This introduces negligible bias into the

Table 7. Simulation results under the SVCJ model for a European call option.

(a) Simulation with the exact method

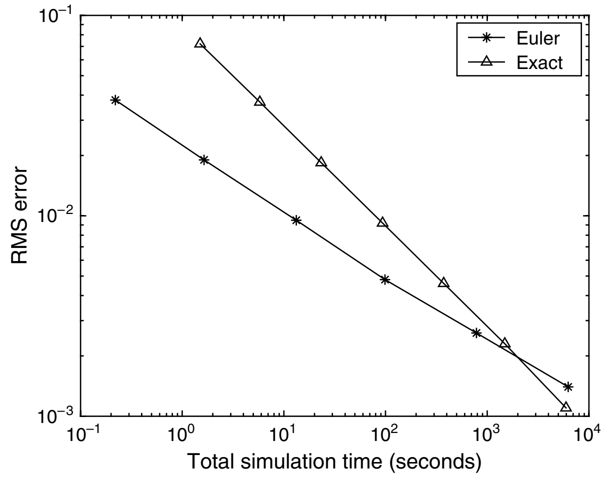
No. of simulation trials	RMS error	Computing time (sec.)
10,000	0.0720	1.5
40,000	0.0369	5.8
160,000	0.0184	23.3
640,000	0.0092	93.5
2,560,000	0.0046	373.0
10,240,000	0.0023	1,491.3
40,960,000	0.0011	5,967.5

(b) Simulation with the Euler discretization

No. of simulation trials	No. of time steps	Bias	Standard error	RMS error	Computing time (sec.)
10,000	10	0.0148	0.0729	0.0744	0.1
40,000	20	0.0090	0.0367	0.0378	0.2
160,000	40	0.0050	0.0183	0.0190	1.6
640,000	80	0.0024	0.0092	0.0095	13.3
2,560,000	160	0.0014	0.0046	0.0048	98.8
10,240,000	320	0.0012	0.0023	0.0026	784.4
40,960,000	640	0.0008	0.0011	0.0014	6,254.7

Note. Option parameters: $S=100$, $K=100$, $V_0=0.007569$, $\kappa=3.46$, $\theta=0.008$, $\sigma_v=0.14$, $\rho=-0.82$, $\lambda=0.47$, $\bar{\mu}=-0.1$, $\sigma_s=0.0001$, $\mu_v=0.05$, $\rho_J=-0.38$, $r=3.19\%$, $T=1.0$ year, true option price $=6.8619$.

Figure 6. Convergence of the RMS errors of the two methods for the option in Table 7 (SVCJ model).



simulation estimate and solves the complexities for very small time intervals.

Table 7 gives some numerical results for simulation under the SVCJ model with the exact and Euler discretization methods. The model parameters are again the fitted parameters for S&P 500 and taken from Duffie et al. (2000). For the Euler discretization, we choose the number of time steps, M, to be equal to $0.1\sqrt{N}$, where N is the number of simulation trials. The bias column is estimated using 500 million simulation trials. Figure 6 shows the convergence of the RMS errors for both methods. In this case, the point where the exact method starts dominating Euler discretization is closer to the right of the graph. This is because the bias in the Euler discretization is relatively small for this set of parameters, and because the jumps increase the computation time of the exact simulation method.

7. An Application: Pricing of Forward Start Options

The exact simulation method of this paper can be applied to the pricing of many exotic options. To illustrate this point, in this section we consider the pricing of forward start options. These are options whose strike is set at a future date. In particular, if T_1 is the time when strike is set, T_2 is the option expiration, S_i is the stock price at time T_i, and k is the constant that determines the strike, then the forward start option payoff at time T_2 is given by $(S_2 - kS_1)^+$. For example, if $k = 1$, then at time T_1, the option becomes an at-the-money option with expiration T_2. Kruse and Nögel (2005) recently developed an expression for this option under the SV model by integrating the pricing formula with the conditional density of the variance value at time T_1. However, numerical evaluation of their expression is quite complicated because it includes another level of integration to already complex integrals in the Heston formula. Therefore, simulation may be considered as a more practical alternative for finding prices and sensitivities of this option. Also, for the SVJ and SVCJ models, no similar expressions are available, and so other numerical methods are necessary for pricing forward start options under these models.

It is straightforward to price a forward start option using simulation: For each path, simulate the stock price values at T_1 and T_2, and then evaluate the payoff function using these two values. However, we can use an alternative method for simulating a forward start option that will allow us to get more efficient estimators. Note that at time T_1 we know the stock price, strike, and the expiration of the option. Therefore, the option price at time T_1 can be written using closed-form formulas. Let $C_E(S, K, T, V)$ denote the price of a European call option with initial stock price S, strike K, time to expiration T, and initial variance V. With this notation, at time T_1 the holder of the forward start option has a call option with value $C_E(S_1, kS_1, T_2 - T_1, V_1)$, where V_1 is the variance value at T_1. Note also that the option price is linearly homogenous with respect to the stock price and the strike, i.e., we can write $C_E(S, kS, T, V) = SC_E(1, k, T, V)$. Using this expression, and following the above arguments, the price of a forward start option can be written as

$$C_{FW} = E[e^{-rT_1} S_1 C_E(1, k, T_2 - T_1, V_1)]. \tag{34}$$

Thus, we can price forward start options by simulating the stock price and the variance values at time T_1, and then using the European option pricing formula to evaluate the option payoff. Using the analytical formula in expression (34) eliminates the variance that occurs due to the uncertainty after T_1. Therefore, this estimator is expected to give a better estimate than plain Monte Carlo simulation.

In Table 8, we give simulation results for a forward start option under each model. The model parameters are taken from Duffie et al. (2000), and are the same as given in

Table 8. Simulation estimates for a forward start option.

	Model		
	SV	SVJ	SVCJ
Plain simulation price	7.0213	7.0352	7.1376
Standard error	0.0778	0.0777	0.0798
No. of simulation trials	10,000	10,000	10,000
Computing time (sec.)	6.1	1.7	2.9
Formula simulation price	6.9708	6.8978	7.0593
Standard error	0.0088	0.0149	0.0136
No. of simulation trials	8,600	2,800	3,300
Computing time (sec.)	6.1	1.7	2.9

Note. Option parameters: $S_0 = 100$, $k = 1$, $T_1 = 1$ year, $T_2 = 2$ years. SV parameters are the same as in Table 1; SVJ parameters are the same as in Table 6; and SVCJ parameters are the same as in Table 7.

the previous sections. The price estimator given in (34) is denoted as the "formula simulation price," and the plain Monte Carlo estimate that simulates stock prices at T_1 and T_2 is denoted as the "plain simulation price." The computation time per simulation path is different for the two methods. We adjust the number of simulation trials for the formula simulation method such that it takes roughly the same amount of time as the plain simulation method. Both simulation estimators are unbiased because we use the exact simulation algorithm, but combining the algorithm with the analytical European call formula results in significant variance reduction.

8. Conclusion

In this paper, we propose a method for the exact simulation of the stock price and variance under Heston's stochastic volatility and some other affine jump diffusion models. Our method is based on Fourier inversion techniques and conditioning arguments. The proposed simulation method recovers the error convergence rate of $O(s^{-1/2})$ for unbiased Monte Carlo simulation, where s is the total computation time. Using the more conventional Euler discretization method leads to $O(s^{-1/3})$ or slower convergence, depending on the convergence rate of the bias.

The proposed simulation method may be used to generate unbiased price estimators for path-independent derivatives, and also for some path-dependent derivatives with mild path dependence. By mild path dependence, we mean that the derivative payoff should depend on the stock price and variance values at few points during the life of the derivative. Bermudan options and American call options with discrete dividends are some examples. In the case of path-independent derivatives, any type of payoff that depends on the final stock price and variance can be handled. Also, under the SV model, it is possible to generate unbiased estimators of the Greeks of path-independent derivatives using the conditional Monte Carlo method.

The exact simulation method has several advantages over the discretization methods. The bias for the discretization methods may be large, and many time steps may be needed to reduce the bias to an acceptable level, which increases the computation time. Also, because the bias is unknown, the standard error may be a poor estimate of the actual error, and valid confidence intervals are not available. Finally, it is difficult to determine the optimal tradeoff between the time steps and simulation trials because the convergence rate of the bias is unknown.

We verified the faster convergence rate of our exact simulation methods through numerical examples. We demonstrated that they achieve a smaller RMS error than the Euler discretization under the SV and SVJ models, except when a very small amount of computation time is used. The exact simulation method is relatively slower for simulation under the SVCJ model, but it is still more efficient than the Euler discretization approach when high accuracy is needed. We have also shown that the conditional Monte Carlo methods of Willard (1997) can be used to improve the efficiency of the exact method for simulation of certain path-independent options under the SV model.

Appendix. Derivation of the Laplace Transform

In this appendix, we derive the Laplace transform for the integral of $\int_u^t V(s)\,ds$ given $V(u)$ and $V(t)$. We follow the approach of Chesney et al. (1993) to get a square-root process with volatility parameter $\sigma = 2$, and then use the results of Pitman and Yor (1982) to write the Laplace transform.

$V(t)$ follows the following square-root process:

$$V(t) = V(0) + \int_0^t \kappa[\theta - V(s)]\,ds + \int_0^t \sigma\sqrt{V(s)}\,dW_s. \tag{35}$$

Now,

$$\begin{aligned}\int_0^t \sigma\sqrt{V(s)}\,dW_s &= 2\int_0^t \sqrt{V(s)}\frac{\sigma}{2}\,dW_s\\ &\cong 2\int_0^t \sqrt{V(s)}\,dW_{\sigma^2 s/4} \quad \text{(in law)}\\ &= 2\int_0^t \sqrt{V\left(\frac{4}{\sigma^2}\frac{\sigma^2}{4}s\right)}\,dW_{\sigma^2 s/4}.\end{aligned} \tag{36}$$

The second equality in (36) follows from the scaling property of Brownian motion. Define $u = \sigma^2 s/4$, so we have $du = (\sigma^2/4)ds$. We can write (35) as

$$\begin{aligned}V\left(\frac{4}{\sigma^2}\frac{\sigma^2}{4}t\right) &= V(0) + \frac{4}{\sigma^2}\int_0^{\sigma^2 t/4} \kappa\left[\theta - V\left(\frac{4}{\sigma^2}u\right)\right]du\\ &\quad + 2\int_0^{\sigma^2 t/4} \sqrt{V\left(\frac{4}{\sigma^2}u\right)}\,dW_u.\end{aligned}$$

Defining the process $\rho(u) = V(4u/\sigma^2)$, we obtain

$$\begin{aligned}\rho(\sigma^2 t/4) &= \rho(0) + \frac{4}{\sigma^2}\int_0^{\sigma^2 t/4} \kappa[\theta - \rho(u)]\,du\\ &\quad + 2\int_0^{\sigma^2 t/4} \sqrt{\rho(u)}\,dW_u.\end{aligned} \tag{37}$$

Setting $n = 4\kappa\theta/\sigma^2$ and $j = -2\kappa/\sigma^2$, we get

$$\rho(t) = \rho(0) + \int_0^t [2j\rho(u) + n]\,du + 2\int_0^t \sqrt{\rho(u)}\,dW_u. \tag{38}$$

The infinitesimal generator of this process is $2xD^2 + (2jx + n)D$, where $D = d/dx$. Pitman and Yor (1982) give the following formula for the squared Bessel process $X(s)$, which has generator $2xD^2 + nD$:

$$\begin{aligned}&\tilde{E}\left[\exp\left\{-\frac{b^2}{2}\int_0^t X(s)\,ds\right\} \,\middle|\, X(0) = x,\ X(t) = y\right]\\ &\quad = \left(\frac{bt}{\sinh bt}\right)\exp\left\{\frac{x+y}{2t}(1 - bt\coth bt)\right\}\\ &\qquad \times I_\nu\left(\frac{\sqrt{xy}\,b}{\sinh bt}\right)\Big/ I_\nu\left(\frac{\sqrt{xy}}{t}\right),\end{aligned} \tag{39}$$

where $\nu = n/2 - 1$, $I_\nu(.)$ is the modified Bessel function of the first kind, and $\tilde{E}$ denotes the expectation taken with respect to the law of the squared Bessel process. To be able to use this formula for the process in (38), we need to apply a change of law formula to eliminate the random component of the drift term and get a process with $j = 0$.

We now write the Laplace transform we want to derive:

$$\begin{aligned}
&E\left[\exp\left\{-a\int_u^t V(s)\,ds\right\}\Bigg|\, V(u),\, V(t)\right]\\
&\quad = E\left[\exp\left\{-a\int_u^t \rho\left(\frac{\sigma^2 s}{4}\right)ds\right\}\Bigg|\,\rho\left(\frac{\sigma^2 u}{4}\right),\,\rho\left(\frac{\sigma^2 t}{4}\right)\right]\\
&\quad = E\left[\exp\left\{-\frac{4a}{\sigma^2}\int_{\sigma^2 u/4}^{\sigma^2 t/4}\rho(s)\,ds\right\}\Bigg|\,\rho\left(\frac{\sigma^2 u}{4}\right),\,\rho\left(\frac{\sigma^2 t}{4}\right)\right]\\
&\quad = \tilde{E}\left[\exp\left\{-\left(\frac{4a}{\sigma^2}+\frac{j^2}{2}\right)\right.\right.\\
&\qquad\qquad \left.\left.\cdot\int_{\sigma^2 u/4}^{\sigma^2 t/4}\rho(s)\,ds\right\}\Bigg|\,\rho\left(\frac{\sigma^2 u}{4}\right),\,\rho\left(\frac{\sigma^2 t}{4}\right)\right]\\
&\qquad \Bigg/\tilde{E}\left[\exp\left\{-\frac{j^2}{2}\int_{\sigma^2 u/4}^{\sigma^2 t/4}\rho(s)\,ds\right\}\Bigg|\,\rho\left(\frac{\sigma^2 u}{4}\right),\rho\left(\frac{\sigma^2 t}{4}\right)\right].
\end{aligned}$$

The last equality follows from the change of law formula $(6.d)$ of Pitman and Yor (1982). The expectations in the numerator and the denominator are calculated using formula (39).

After arranging the terms and changing back to the $V(.)$ domain, we get the Laplace transform as

$$\begin{aligned}
&E\left[\exp\left\{-a\int_u^t V(s)\,ds\right\}\Bigg|\, V(u),\, V(t)\right]\\
&\quad = \frac{\gamma(a)e^{(-1/2)(\gamma(a)-\kappa)(t-u)}(1-e^{-\kappa(t-u)})}{\kappa(1-e^{-\gamma(a)(t-u)})}\\
&\qquad \times\exp\left\{\frac{V(u)+V(t)}{\sigma^2}\left[\frac{\kappa(1+e^{-\kappa(t-u)})}{1-e^{-\kappa(t-u)}}\right.\right.\\
&\qquad\qquad \left.\left.-\frac{\gamma(a)(1+e^{-\gamma(a)(t-u)})}{1-e^{-\gamma(a)(t-u)}}\right]\right\}\\
&\qquad \times\frac{I_{0.5n-1}\left[\frac{4\gamma(a)\sqrt{V(u)V(t)}}{\sigma^2}\frac{e^{-0.5\gamma(a)(t-u)}}{(1-e^{-\gamma(a)(t-u)})}\right]}{I_{0.5n-1}\left[\frac{4\kappa\sqrt{V(u)V(t)}}{\sigma^2}\frac{e^{-0.5\kappa(t-u)}}{(1-e^{-\kappa(t-u)})}\right]},
\end{aligned}\tag{40}$$

where $\gamma(a) = \sqrt{\kappa^2 + 2\sigma^2 a}$.

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