

MANAGING CORPORATE LIQUIDITY: STRATEGIES AND PRICING IMPLICATIONS

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Defaults arising from illiquidity can lead to private workouts, formal bankruptcy proceedings or even liquidation. All these outcomes can result in deadweight losses. Corporate illiquidity in the presence of realistic capital market frictions can be managed by (a) equity dilution, (b) carrying positive cash balances, or (c) entering into loan commitments with a syndicate of lenders. An efficient way to manage illiquidity is to rely on mechanisms that transfer cash from “good states” into “bad states” (i.e., financial distress) without wasting liquidity in the process. In this paper, we first investigate the impact of costly equity dilution as a method to deal with illiquidity, and characterize its effects on corporate debt prices and optimal capital structure. We show that equity dilution produces lower firm value in general. Next, we consider two alternative mechanisms: cash balances and loan commitments. Abstracting from future investment opportunities and share re-purchases, which are strong reasons for corporate cash holdings, we show that carrying positive cash balances for managing illiquidity is in general inefficient relative to entering into loan commitments, since cash balances (a) may have agency costs, (b) reduce the riskiness of the firm thereby lowering the option value to default, (c) postpone or reduce dividends in good states, and (d) tend to inject liquidity in both good and bad states. Loan commitments, on the other hand, (a) reduce agency costs, and (b) permit injection of liquidity in bad states as and when needed. Then, we study the trade-offs between these alternative approaches to managing corporate illiquidity. We show that loan commitments can lead to an improvement in overall welfare and reduction in spreads on existing debt for a broad range of parameter values. We derive explicit pricing formulas for debt and equity prices. In addition, we characterize the optimal draw down strategy for loan commitments, and study its impact on optimal capital structure.

Keywords: Loan commitment; cash reserve; liquidity management; security pricing.

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1. Introduction

Theory of optimal capital structure and debt valuation has made rapid strides in recent years. Beginning with the pioneering work of [29], which provided the first structural formulation of debt pricing, a number of recent papers have extended the analysis to consider the valuation of multiple issues of debt, optimal capital structure, debt renegotiation, and the effects of Chapters 7 and 11 provisions on debt valuation and optimal capital structure.¹ These models have tended to abstract from cash holdings by companies or loan commitments held by the companies. Yet these are important ways in which companies deal with illiquidity and avoid potentially costly liquidations or financial distress. In the banking literature, researchers have examined the role of loan commitments and their effects on the welfare of corporate borrowers both in theoretical and empirical work. Papers by [26, 34, 35, 37, 38] have explored the important role played by loan commitments under capital market frictions. The insights of these papers have not yet been brought to bear on the pricing and optimal capital structure implications. This is one of the goals of our paper. A typical company often issues multiple types of debt, with different seniority and covenants. As a result, cash flows to creditors are complicated to model when the firm enters bad states (as in financial distress). In the case of default, the U.S. Bankruptcy Codes (Chapters 11, 13, etc.) allow companies to postpone liquidation by providing the borrowing firm with protection while it undergoes reorganization and debt renegotiation with the creditors. Alternatively, companies can also enter into financial contracts such as loan commitments in good states (when they are solvent), to alleviate their financial distress and thus postponing or avoiding bankruptcy. In this paper, we propose a structural model that incorporates loan commitments as an alternative source of liquidity. We contrast this strategy with two other mechanisms that have been studied in the literature for managing illiquidity: cash balances and costly equity dilution. Our paper does not consider dynamic investment opportunities (future growth options), and this assumption precludes an important incentive for holding cash: the model presented is more relevant for mature firms.

1.1. *Overview of major results*

In our paper we integrate the insights of the banking literature on loan commitments with the insights of debt pricing literature and provide a framework for analyzing loan commitments as a part of optimal capital structure. Our paper also formally explores three distinct ways in which companies deal with illiquidity – equity dilution, cash balances and loan commitments. To our knowledge, this is the first paper to undertake this comprehensive examination of corporate strategies for managing liquidity. In so doing, we extend the theory of optimal capital structure to permit

¹An important subset of this literature includes the papers by [4, 9, 12, 25].

junior and senior debt, where the corporate borrower can optimally select the “draw down” and “repayment” strategy of loan commitments.

We provide an analytical characterization of credit spreads, optimal capital structure and debt capacity when equity dilution is costly, subject to the endogenous default boundary as a solution to a nonlinear equation. Using the base case of zero cost of equity dilution, we characterize the welfare losses due to dilution costs when equity holders decide on the choice of the optimal default boundary.

We show, through an induction argument, that it is never optimal to carry cash balances to guard against default when equity issuance is costless. More importantly, we characterize the threshold levels of managerial frictions associated with holding cash *and* equity dilution costs over which it is never optimal to hold cash balances. Our results show that the dilution costs of equity have to be very significant and the costs of holding cash have to be very low in order for the firm to curtail dividends and hold cash balances.

In the context of previous results, we explore contingent transfer of liquidity through the use of loan commitments where the borrower is able to pay a commitment fee in good states and draw down cash in bad states as needed. We develop a model and computational methodology needed for loan commitments. Using this approach, we characterize the optimal draw down and repayment strategy and the effects of loan commitments on existing debt spreads and the welfare of the borrowing firm. We find that entering into a loan commitment agreement can benefit both equity and debt holders. We also study the impact of a loan commitment on the firm’s optimal capital structure.

Our results should be viewed in the context of our model assumptions. We treat the firm’s investment policy as fixed and exogenous. Availability of future investment and growth options may provide a strong motive for accumulating cash balances when equity dilution costs are significant. We abstract from this important consideration. Our paper does not model strategic debt service considerations. Since strategic debt service (especially when renegotiations are not costly) allows for underperformance in bad states, it acts as a substitute to holding cash balances. This has been noted by [1]. Our analysis, however, can be modified to incorporate strategic debt service. Finally, we do not model the Chapter 11 provision of the bankruptcy code, which also allows illiquid firms to seek bankruptcy protection and coordinate their liquidity problems with lenders and avoid potential liquidation.

1.2. *Review of the literature*

The structural approach to modeling corporate debt begins with the seminal work of [29]. He indicates that risky zero-coupon bonds may be valued using the Black-Scholes-Merton option pricing approach. In this model, default occurs if the asset value of the firm falls below the debt principal at maturity. [9] extend this model for coupon bonds with finite maturity, where they impose that the firm must issue equity to make the coupon payments, and that default is endogenously chosen by

the firm at any time prior to maturity. [25] derives a closed-form solution for prices of perpetual coupon bonds. His model incorporates bankruptcy cost, tax benefit of coupon payments, and dividend payouts. He also derives the expressions for optimal capital structure, debt capacity, and optimal default boundary. [12] consider the presence of Chapters 7 and 11, which leads to reorganization upon bankruptcy prior to liquidation. All of these models restrict the firm from selling its asset.

The work mentioned above implicitly assume that liquidity has no value. This follows from the assumption that the firm can dilute equity freely to finance short-fall of cash from operations to fund the coupon payments. Equity issuance, however, is quite costly in practice, and hence many firms maintain cash reserves or enter into agreements such as a loan commitment to manage liquidity crises.

The literature of cash reserves is related to that of dividend policies. Since the work of [30], who show that dividend policy is irrelevant to the firm value in perfect (frictionless) markets, extensive research has emerged in the study of optimal dividend policies by means of optimization and stochastic optimal control under more realistic market conditions. Examples of stochastic control models include [5, 22, and 33]. These models choose a policy that maximizes the present value of a future dividend stream subject to various cost structures associated with the distribution. [39] provides a more detailed discussion of the literature in this area. In a recent paper, [14] model the cash process as a mean-reverting process, and characterize optimal dividend policy in terms of the amount and timing of dividend distributions. Our model is more similar in spirit to those of [23] and [1], where the focus is on liquidity. [23] develop a three-period model that investigates the trade-offs between holding liquid assets (cash) with lower returns versus facing future uncertainty and costly external funding (equity issuance). They find that the optimal level of cash is increasing with the cost of equity issuance and the volatility of future cash flows. [1] develop a multi-period model that allows the firm to simultaneously keep cash reserves and service debt strategically. They find that the option to carry cash is valuable if the firm also has the option for strategic debt-service. We provide a fully dynamic model of endogenous default and derive implications for cash balances and line of credit.

Existing literature on loan commitments dates back to [19] and [20]. [19] develops a pricing framework for loan commitment contracts under two different models. The first model assumes that the firm's assets are bought or sold when the loan commitment is drawn down or repaid, respectively. The second assumes that the shareholders receive dividends when loan commitment is drawn down, and new shares are issued when it is repaid. [20] focus on the hedging of a loan commitment with bond futures contracts. In their model, they assume that the firm has an option to draw down at a fixed time, and cannot repay it before maturity. [32] discusses these two papers in more detail. [27] formulate an optimization problem for a firm which enters into a loan commitment for future financing of investment opportunities that arrive stochastically. They discuss the optimal size of the loan

commitment, the determinants of the size, and the average usage by the firm. [32] investigates the impact of loan commitment on existing debt. He imposes a covenant that restricts dividend payouts to shareholders when the loan commitment is drawn. In this model, excess cash from revenue after making coupon payments must go back to repay the loan commitment immediately. Our model differs from the above, with the exception of [32], in terms of the purpose of the loan commitment. We use loan commitment solely for managing liquidity, and not for financing investment opportunities. In addition, our model accounts for both the optimal draw down *and* repayment strategies of the loan commitment.

There is also a body of literature that discusses the roles and benefits of loan commitments. [26] argues that loan commitments can increase individual firm's profit in an imperfectly competitive market, when they are used to finance the firm's strategic position as a response to its rival's decisions. [8] develop a two-period model to show that a loan commitment may be an optimal way to finance risky investments. [34] demonstrates that a loan commitment leads to higher optimal debt level and lower cost of debt under similar settings. [36] argues that a loan commitment is a strictly better source of financing than a standard debt, in the presence of the debt overhang problem originally proposed by [31], because of its fixed-fee and low interest rate features. Under our model assumptions, we also find that the loan commitment increases the optimal level of existing debt and may reduce its spread.

On the empirical side, [28] provide empirical analysis on "packages" of loan commitment terms from surveyed data of 132 U.S. non-financial firms. They find that the loan commitment size is positively correlated with the interest mark-up, commitment fee, length of contracts, current ratios, and firm size. [35] provide similar analysis based on data of 2,513 bank loan commitment contracts sold to publicly traded U.S. firms. The data were collected from Securities and Exchange Commission filings by the Loan Pricing Corporation. [2] test the model of [27] empirically against the data for loan commitments made to 712 privately-owned firms, and find the results consistent with the model's predictions. In a recent paper [37] has presented evidence that suggests that banks are unwilling to provide loan commitments to firms that do not have sufficient operating cash flows. In other words, there is a supply side constraint on the ability of borrowing firms to get loan commitment.

Loan commitments often carry Material Adverse Change (MAC) provisions, which may allow the bank to withdraw the lines under specified economic circumstances. This said, MAC is rarely invoked: firms such as Enron were able to draw their lines before they eventually declared bankruptcy². [21] notes that the draw-down of credit lines may help explain the fact that bank commercial and industrial (C&I) lending increased during the fall of 2008 when borrowers and bankers were reporting extremely tight credit market conditions. C&I loans held by U.S. domestic

²See [21], for example.

banks increased from \$1.514 trillion in August 2008 to \$1.582 trillion by year-end, while the Federal Reserves Senior Loan Officer Opinion Survey on Bank Lending Practices found that a record net 83.2% of respondents reported that their institutions had tightened loan standards for large and medium borrowers in the fourth quarter. This pattern of usage of loan commitments is consistent with the theory developed in our paper.

The remainder of the paper is organized as follows. Section 2 describes the basic setup of the model. In Sec. 3, we study the welfare and pricing of the firm's securities under costly equity dilution. We introduce the loan commitment into the model in in Sec. 4. Section 5 concludes and the Appendices contain technical proofs and our computational methodology.

2. The Model

In a perfect capital market where there are no frictions and where distress can be resolved costlessly, the question of corporate liquidity is not relevant. Empirical evidence suggests, however, that there are at least two important sources of frictions which render corporate liquidity a very important question. First, there are costs associated with security issuance. Such costs reflect not only the costs associated with underwriting and distribution, but also with asymmetric information that characterize the relationship between the suppliers of capital and corporations. A survey by [24] finds that direct equity issuance cost (underwriter spreads and administrative fees) on average is over 13% for \$2–10 million and over 8% for \$10–20 million of seasoned equity offerings. The average cost for initial public offerings is even higher, 17% for \$2–10 million and over 11% for \$10–20 million. In addition, financial distress can lead to private workouts, bankruptcy proceedings and even liquidation, all of which are known to be costly. [10] reports total bankruptcy-related costs to firm and claimholders to be between 13% and 21% of pre-distressed value. [11] provide a detailed analysis of bankruptcy costs under Chapters 7 and 11. They find that the average change in asset value maybe as high as 84% before and after the filing of Chapter 7. In this paper, we construct a simple trade-off model of capital structure, where corporate liquidity plays a pivotal role. Our structural model is an extension of [25] with net cash payouts by the firm (Sec. VI. B). We introduce two costs explicitly. First, we model the cost of maintaining liquidity by introducing an exogenous cost on cash balances. In addition, we also recognize that equity dilution is costly. A corporate borrower chooses a level of debt by trading off tax benefits and the possibility of costly liquidation in the future. We ask how this decision is informed by managing the liquidity of the firm through three possible tools: (a) equity dilution, (b) carrying cash balances, and (c) using loan commitment, which are fairly priced. In order to focus squarely on liquidity strategies of the corporation in a trade-off model, we abstract from a detailed modeling of the bankruptcy code, which in itself is an institution that is designed, in part, to address the question of resolving illiquidity. It will be of interest to extend our analysis to

examine how the liquidity strategies of a firm will be influenced by the availability of a well designed bankruptcy code.³

Since the investment policy is fixed, we restrict the sale and purchase of the firm's assets, and hence the firm can only use its revenue or proceeds from equity dilution to make the coupon payments in absence of cash balances or loan commitments. We consider costs associated with holding cash explicitly. As an alternative to cash or equity dilution, the firm may also enter into a loan commitment to transfer cash from the solvent state to the state when the firm is not generating enough revenue to cover the coupon payment. We assume that the firm has to pay a one-time fee at the time the loan commitment contract is signed. Since the firm's asset cannot be purchased or sold, the sole purpose of cash and a loan commitment is to provide liquidity when the firm is in distress, and not for investment purposes.

2.1. Basic setup

We assume that the firm's asset value, denoted by V_t , is independent of its capital structure, and follows a diffusion process whose evolution under the risk neutral measure \mathbb{Q} is given by:

$$dV_t = (r - q)V_t dt + \sigma V_t dW_t$$

where r is the risk-free rate, q is the instantaneous rate of the return on asset, σ is the volatility of asset value, and W_t is a standard Brownian motion under \mathbb{Q} . The instantaneous revenue generated by the firm is given by:

$$\delta_t = qV_t.$$

The parameter q determines the internal liquidity of the firm from its cash flow. Holding the current asset value fixed, a higher value of q implies that the firm produces more cash flow per unit time than a lower value of q . In this regard, lower q is consistent with a higher growth rate, holding other factors constant.

The firm issues debt with principal P , a continuous coupon rate c , and a maturity T . The firm uses its revenue δ_t to make the coupon payment, and uses any surplus to pay dividends to the shareholders. In absence of cash balances and loan commitment, when revenue is not enough to cover the payment, the firm can either dilute equity to raise money to make the payment or declare bankruptcy. In the latter case, the firm liquidates its assets and the proceeds go to the debt holders. There is an equity dilution cost of γ and a liquidation cost of α , which will be explained in the next section. We next describe the three strategies to manage corporate illiquidity.

³Examples of papers that examine alternative bankruptcy procedures include [3, 6, 7, 17]. [18] summarizes the general consensus on the goals and the characteristics of efficient bankruptcy procedures.

2.2. Equity dilution

When faced with a liquidity crisis, the firm may raise capital by issuing additional equity. We assume that the cost of equity dilution is proportional to the proceeds from the issuance. Let γ be the proportion of the proceeds lost due to deadweight costs of issuance. In other words, if x is the percentage of outstanding shares to be issued, then the proceeds from equity dilution are $\frac{x}{1+x}E$. However, the firm only receives $(1-\gamma)\frac{x}{1+x}E$. The claim value of the original shareholders reduces to $\frac{1}{1+x}E$.

Suppose the firm needs to raise \bar{C} dollars from equity dilution, then the percentage of shares it needs to issue is given by:

$$x = \frac{\bar{C}}{(1-\gamma)E - \bar{C}}$$

and the claim value of the original shareholders reduces to:

$$\frac{1}{1+x}E = \left(\frac{1}{1 + (\bar{C}/((1-\gamma)E - \bar{C}))} \right) E = E - \frac{\bar{C}}{1-\gamma}$$

which is equivalent to the shareholders receiving a *negative dividend* of $\frac{\bar{C}}{1-\gamma}$. In our model, equity issuance would not be undertaken if it does not lead to an increase in the value of equity.

2.3. Cash balance

An alternative to issuing equity is to build up a cash balance. The firm may choose a dividend payout rate d strictly less than the revenue rate δ in the good states when the revenues exceed coupon payments, and thus accumulate a cash balance, x , over time. On this cash balance, we assume that the firm earns an interest at rate $r_x \leq r$, reflecting the possible costs associated with leaving cash inside the firm. In our model, the firm chooses the payout rate d that maximizes its equity value.

2.4. Loan commitment

Finally, the firm may enter into a loan commitment with another lender. A loan commitment is a contractual agreement between lender (the “bank”) and the firm, that gives the right, but not an obligation, to the firm to borrow in the future at the pre-specified terms. The terms may include, but are not limited to, the size of a loan commitment P^l , a rate of interest (coupon rate) c^l , and an expiration date. Generally, there are *drawn* and *undrawn* fees associated with the loan commitment.

In our model, the bank imposes a covenant that forbids the firm from using the proceeds of the loan to pay shareholder dividends. The firm does not keep a cash balance, so as a result, it only considers drawing down the loan when revenue is not sufficient to cover the coupon payment. Furthermore, the firm can at most draw down the amount of the shortfall. The bank also allows the firm to repay any amount of the loan at any time prior to maturity. We assume that the firm makes all its decisions to maximize the equity value.

We describe in detail how the firm makes the decisions, and how the fees are determined in Sec. 5.

3. Welfare and Pricing Under Costly Equity Dilution

In this section, we show how equity, debt and total firm value can be determined when equity dilution is costly. We derive closed-form solutions for these values for the infinite maturity case. We show how equity dilution cost affects credit spread, debt capacity and optimal capital structure.

3.1. Exogenous bankruptcy boundary

First, we derive closed-form solutions for equity, debt and total firm values when the bankruptcy boundary is determined exogenously. We assume that there is an asset value boundary V_B , below which the firm will declare bankruptcy. Upon bankruptcy, the firm liquidates its assets and incurs a liquidation cost of α , i.e., the proceeds from liquidation are $(1 - \alpha)(V + \delta)$.

It is well known (e.g. [15]) that the value, $f(V_t, t)$, of any security whose payoff depends on the asset value, V_t , must satisfy the following PDE:

$$\frac{1}{2}\sigma^2 V_t^2 f_{VV} + (r - q)V_t f_V - rf + f_t + g(V_t, t) = 0 \tag{3.1}$$

where $g(V_t, t)$ is the payout received by holders of security f .

In the perpetuity case, the security becomes time-independent, and hence the above PDE reduces to the following ODE:

$$\frac{1}{2}\sigma^2 V^2 f_{VV} + (r - q)V f_V - rf + g(V) = 0 \tag{3.2}$$

Equation (3.2) has the general solution:

$$f(V) = A_0 + A_1 V + A_2 V^{-Y} + A_3 V^{-X} \tag{3.3}$$

where

$$X = \frac{\left(r - q - \frac{\sigma^2}{2}\right) + \sqrt{\left(r - q - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}}{\sigma^2} \tag{3.4}$$

$$Y = \frac{\left(r - q - \frac{\sigma^2}{2}\right) - \sqrt{\left(r - q - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}}{\sigma^2} \tag{3.5}$$

3.1.1. Equity value

Consider a firm that issues perpetual debt which pays a continuous coupon at the rate C . When the firm's revenue rate exceeds the coupon rate, $qV \geq C$, equity holders receive dividends at rate $(qV - C)$, otherwise the firm dilutes equity to make the coupon payment. As explained above, the latter is equivalent to a negative dividend of $\beta(qV - C)$, where $\beta = 1/(1 - \gamma)$.

Therefore, the equity value, E , must satisfy:

$$\frac{1}{2}\sigma^2V^2E_{VV} + (r - q)VE_V - rE + \beta(qV - C) = 0 \quad \text{for } V_B \leq V < C/q \quad (3.6)$$

$$\frac{1}{2}\sigma^2V^2E_{VV} + (r - q)VE_V - rE + qV - C = 0 \quad \text{for } V \geq C/q \quad (3.7)$$

The general solution (3.3) becomes:

$$E(V) = \begin{cases} A_0 + A_1V + A_2V^{-Y} + A_3V^{-X} & \text{for } V_B \leq V < C/q \\ \tilde{A}_0 + \tilde{A}_1V + \tilde{A}_2V^{-Y} + \tilde{A}_3V^{-X} & \text{for } V \geq C/q \end{cases} \quad (3.8)$$

To solve for the coefficients in (3.8), we use the fact that $E(V)$ must also satisfy the following boundary, continuity and smooth pasting conditions:

$$\begin{aligned} \text{(BC I):} & \quad E(V_B) = 0 \\ \text{(BC II):} & \quad \lim_{V \rightarrow \infty} E(V) = V - \frac{C}{r} \\ \text{(CC):} & \quad E((C/q)^-) = E(C/q) \\ \text{(SP):} & \quad E_V((C/q)^-) = E_V(C/q) \end{aligned} \quad (3.9)$$

(BC I) follows from our assumption that the shareholders get nothing when the firm declares bankruptcy. (BC II) is the value of equity when the firm becomes riskless as the asset value approaches infinity. (CC) and (SP) ensures a smooth and continuous transition in and out of equity dilution region. This follows from the fact that the equity value fully anticipates the future events.

By solving the system of equations above, the equity value is given by:

$$E(V) = \begin{cases} \beta \left(V - \frac{C}{r} \right) + A_2V^{-Y} + A_3V^{-X} & \text{for } V_B \leq V < C/q \\ \left(V - \frac{C}{r} \right) + \tilde{A}_3V^{-X} & \text{for } V \geq C/q \end{cases} \quad (3.10)$$

where

$$\begin{aligned} A_2 &= \frac{(\beta - 1) \left(\frac{C}{q} \right)^Y \left(X \frac{C}{r} - X \frac{C}{q} - \frac{C}{q} \right)}{X - Y} \\ A_3 &= \left[\beta \left(\frac{C}{r} - V_B \right) - A_2V_B^{-Y} \right] V_B^X \\ \tilde{A}_3 &= \left[(\beta - 1) \left(\frac{C}{q} - \frac{C}{r} \right) V_B^{-X} + \beta \left(\frac{C}{q} \right)^{-X} \left(\frac{C}{r} - V_B \right) \right] V_B^{X+Y} \\ &\quad + \frac{(\beta - 1) \left(X \frac{C}{r} - X \frac{C}{q} - \frac{C}{q} \right) \left(V_B^{-X} - V_B^{-Y} \left(\frac{C}{q} \right)^{Y-X} \right)}{X - Y} V_B^{X+Y} \end{aligned}$$

The details of this derivation can be found in Appendix A.

3.1.2. *Debt value*

Debt holders receive a constant payout rate of C as long as the firm remains solvent, i.e., when $V \geq V_B$. Therefore the debt value, D , must satisfy:

$$\frac{1}{2}\sigma^2V^2D_{VV} + (r - q)VD_V - rE + C = 0 \quad \text{for } V \geq V_B \quad (3.11)$$

Equation (3.11) has a general solution:

$$D(V) = B_0 + B_1V + B_2V^{-X} \quad (3.12)$$

where X is given by (3.4).

Similar to solving for the equity value, we use the fact that $D(V)$ must satisfy the following boundary conditions:

$$\begin{aligned} \text{(BC I):} \quad & D(V_B) = (1 - \alpha)V_B \\ \text{(BC II):} \quad & \lim_{V \rightarrow \infty} D(V) = \frac{C}{r} \end{aligned} \quad (3.13)$$

(BC I) follows from our assumption on bankruptcy cost, and (BC II) is the value of a perpetual debt as it becomes riskless.

Then we can show that the debt value is given by:

$$D(V) = \frac{C}{r} + \left((1 - \alpha)V_B - \frac{C}{r} \right) \left(\frac{V}{V_B} \right)^{-X} \quad \text{for } V \geq V_B \quad (3.14)$$

The details of this derivation can be found in Appendix A.

3.2. *Endogenous bankruptcy boundary*

In the previous section, we assume that the bankruptcy boundary is determined exogenously. In practice, the management of the firm acts in the best interest of its shareholders. Thus in the absence of a covenant imposed by the debt holders, the firm will choose a bankruptcy boundary that maximizes its equity value. From the expression of equity value given in (3.10), we can see that the equity value only depends on V_B through A_3 (assuming that optimal V_B is in the equity dilution region, $V < \frac{C}{q}$). Thus, the optimal V_B can be found by solving $\frac{\partial E}{\partial V_B} = 0$, which is equivalent to solving $\frac{\partial A_3}{\partial V_B} = 0$. Consequently, the optimal bankruptcy boundary, V_B^* , is given by the following implicit expression:

$$X\beta\frac{C}{r}(V_B^*)^{-1} - (1 + X)\beta - (X - Y)A_2(V_B^*)^{-Y-1} = 0 \quad (3.15)$$

The equity and debt values with endogenous bankruptcy boundary is therefore given by (3.8) and (3.14), evaluated at V_B^* .

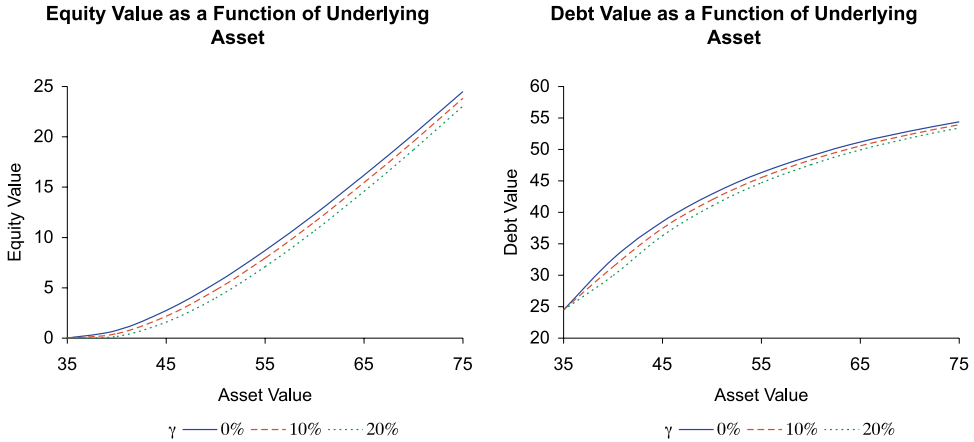


Fig. 1. Plots of equity, $E(V)$, and debt values, $D(V)$, as a function asset value at different levels equity dilution cost, γ . Both $E(V)$ and $D(V)$ are increasing in V , decreasing in γ , and converge to the Leland results as $V \rightarrow \infty$ for all values of γ . ($V = 100, C = 3.36, \sigma = 0.2, r = 0.05, q = 0.03, \alpha = 0.3, \tau = 0.15$).

3.3. Impacts of costly equity dilution

3.3.1. Security values

Figure 1 plots equity and debt values as a function of the underlying asset value as given by (3.8) and (3.14). We observe that the optimal bankruptcy boundary, V_B , increases as γ increases. This makes sense because as asset value declines, the equity value declines at a faster rate with costly equity dilution, and hence bankruptcy is declared at a higher asset value. Equity value is also decreasing in γ . The impact of dilution cost converges to zero as $V \rightarrow \infty$.

We observe that debt value is decreasing in γ . Since $D(V)$ is only a function of γ through V_B and V_B is increasing in γ , from (3.14), we can deduce that $D(V)$ is decreasing in γ if and only if $(1 - \alpha)V_B \leq \frac{C}{r} \frac{X}{1+X}$. Similar to the equity value, the effect of dilution cost converges to zero as $V \rightarrow \infty$.

3.3.2. Debt capacity and optimal capital structure

The debt capacity of a firm is defined as the maximum value of debt that can be sustained by the firm. This is determined by the value of coupon, C^{\max} , which maximizes the debt value, denoted by D^{\max} . In this section, we illustrate numerically how the cost of equity dilution, γ , affects D^{\max} . The left panel of Fig. 2 shows that C^{\max} , as well as D^{\max} , decreases as equity dilution becomes more costly. In this example, a dilution cost of 10% reduces the debt capacity from \$86 to \$84.

The optimal capital structure of the firm is defined as the coupon value, C^* , that maximizes the total firm value, denoted by v^* . The right panel of Fig. 2 illustrates how the optimal coupon, C^* , as well as the firm value, v^* , decreases as equity

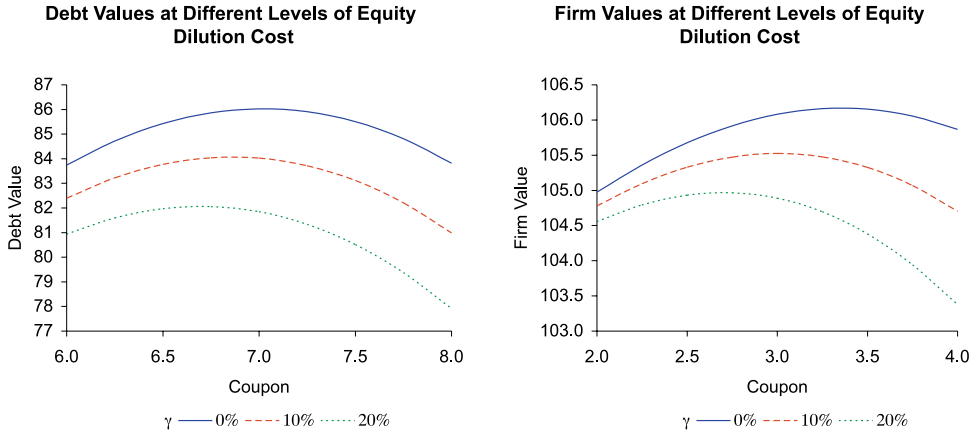


Fig. 2. The left panel plots debt value vs. coupon rate. This illustrates how the debt capacity is affected by the equity dilution cost. The debt capacity decreases as equity dilution becomes more costly. The right panel plots total firm value vs. coupon rate. This illustrates how the optimal capital structure is affected by the dilution cost. The firm issues lower coupon when dilution cost is higher under the optimal structure. ($V = 100, \sigma = 0.2, r = 0.05, q = 0.03, \alpha = 0.3, \tau = 0.15$).

dilution becomes more costly. Therefore, neglecting the costliness of equity dilution can result in over-leveraging the firm. In this example, a dilution cost of 10% reduces the optimal coupon from \$3.3 to \$3 (9%). This translates to a 0.6% change in the firm value.

4. Cash Balances to Manage Illiquidity

In this section, we explore the optimality of building cash balances in good states in order to confront illiquidity in bad states of the world. This strategy may make sense when equity dilution costs are high and the costs of holding cash balances are low. We model the costs of holding cash by assigning a rate of return r_x on cash balances, where $r_x \leq r$, and r is the risk-free rate. We refer to the difference, $r_x - r$, as the cost associated with holding cash inside the firm. We continue to model equity dilution costs as in the previous section. Note that the strategy of building cash balances has several disadvantages. First, building cash balances requires that the equity holders forego some dividends in good states. Second, the firm wastes liquidity by carrying cash in good states when it is not needed. Third, the accumulation of cash balances makes the existing debt less risky and thereby may induce a transfer of wealth from the equity holders to the existing creditors. We prove the following general result:

Proposition 4.1. *If there is no equity dilution cost and the interest on cash reserve is not greater than the risk-free rate, then paying out the entire cash reserve as dividend is an optimal strategy when the firm wishes to maximize its equity value.*

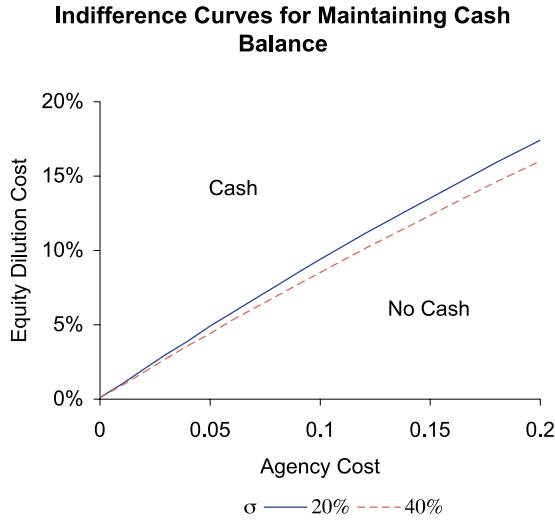


Fig. 3. This figure plots an indifference region for maintaining a positive cash balance for different levels of equity dilution and costs of holding cash balances. The upper-left region is where it is optimal to carry a positive cash balance, while the lower-right region is where it is optimal to always pay everything out as dividend. The plot shows that the region for keeping cash is larger for firms with higher volatility. ($V = 100, \sigma = 0.2, r = 0.05, q = 0.03, \alpha = 0.3, \tau = 0.15$).

The proof of this proposition is in Appendix B.⁴

This base case result establishes a benchmark for evaluating the following question: how high must the equity dilution costs be in order for the firm to hold cash? Note here that there are no other motives such as growth options or investments for carrying cash in our setting. The results presented in the remainder of this section are obtained using the binomial approach described in Appendix C. We use the infinite horizon implementation with $T = 200, N = 2, 400$ and $M = 40$.

Figure 3 shows the magnitude of equity dilution costs for various costs of carrying cash under which it is optimal for the firm to carry cash balances. The indifference curve shows that for high costs of equity dilution and low costs of holding cash inside the firm, cash balances will be carried by the firm to avoid illiquidity induced liquidation. It also shows that the region for holding cash is larger for the more volatile firm.

4.1. Optimal level of cash

In this section, we discuss the optimal level of cash for a firm that maximizes equity value. Note that cash has the following pros and cons: carrying cash exposes the firm

⁴This result is similar to the Proposition 4 of [16] where it is shown that it is optimal to pay all extra cash flows in good states as dividends. But their result assumes that the cash flows are reinvested in the technology of the firm and no cash balances were modeled. Similar result has been shown by [23] in a static setting.

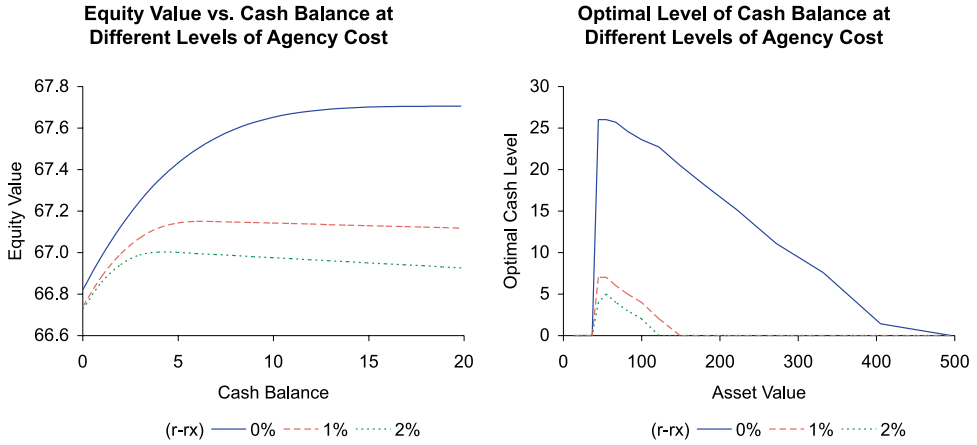


Fig. 4. The left panel plots equity value, E , vs. cash balance, x , for different levels of costs. With no cost, E is increasing in x and levels off beyond a certain value of x . With positive cost, E is increasing in x up to a critical value of x when it starts to decrease. This results in a unique optimal level of x that maximizes E . The right panel plots the optimal level of cash balance, x^* , vs. asset value, V . $x^* = 0$ for values of V below a certain critical level, and it is decreasing in V beyond that level. This critical level is close to the bankruptcy boundary. ($V = 67, \sigma = 0.2, r = 0.05, q = 0.03, C = 3.36, \alpha = 0.3, \tau = 0.15, \gamma = 0.15$).

to costs and forces equity holders to forgo dividends in good states. This makes the firm relatively safe, benefiting the debt, ex-post. But, this latter effect also results in lower spreads, ex-ante, which is good for the equity holders. The left panel of Fig. 4 plots the equity values as a function of the cash balance for different levels of costs of holding cash balances. It shows that when there is no cost to holding cash balances, equity value is maximized as long as the cash balance is above a certain level.⁵ However, with strictly positive agency cost, there is a unique optimal level of cash. The right panel of Fig. 4 plots the optimal level of cash as a function of asset values for different levels of costs of holding cash balances. As expected, the optimal level is decreasing with the asset value. We also observe that below a certain level of asset value, the optimal cash level is zero. This is because the firm is anticipating bankruptcy and is passing the cash to shareholders so that debt holders will not receive it upon liquidation.

Furthermore, we observe that optimal level of cash decreases drastically when the cost of holding cash balances increases from 0% to 1%. Later, we will see that the impact of cash quickly diminishes as the cost increases for this reason.

4.2. Impacts of cash balance

In this section we discuss the effect of cash balances on debt and total firm value. The left panel of Fig. 5 shows that optimally carrying a cash balance increases debt

⁵The equity value converges to a constant as cash balance increases.

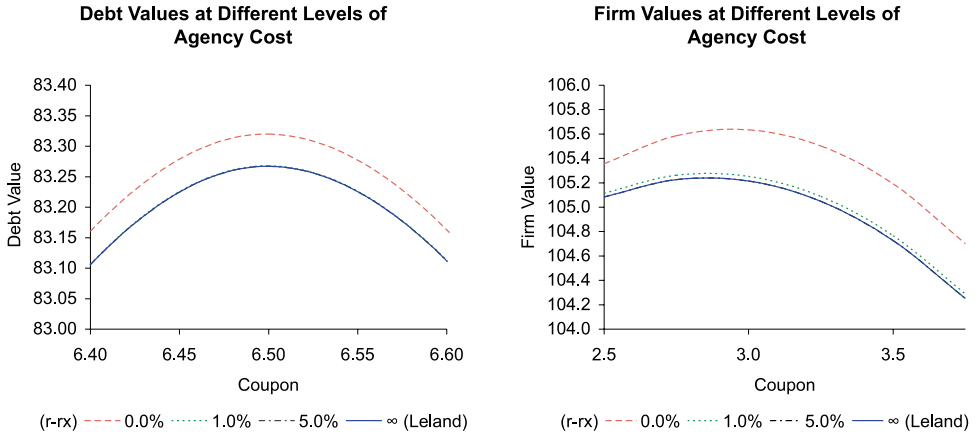


Fig. 5. The left panel plots debt value vs. coupon rate. This illustrates how debt capacity is affected by cash balance. Cash balance virtually has no effect on debt value when there is a positive cost of holding cash. At zero cost, it increases the debt value. The right panel plots total firm value vs. coupon rate. This illustrates how the optimal capital structure is affected by the cash balance. Cash balance increases the value of the firm and also increases the optimal coupon rate. ($V = 100, \sigma = 0.2, r = 0.05, q = 0.03, \alpha = 0.3, \tau = 0.15, \gamma = 0.15$).

value as well as equity values. However, the impact becomes negligible when the cost is strictly positive.

When the cash balance is carried optimally, it can alter the optimal capital structure of the firm as well. The right panel of Fig. 5 illustrates this effect. The optimal capital structure increases relative to the Leland benchmark (i.e., no cash balance or infinite cost of holding cash). The availability of cash enables the firm to increase the proportion of debt it holds in the optimal capital structure. Note that when the cost increases from zero to 1% there is a dramatic shift in the firm value as well as in the optimal level of coupon. For subsequent increases in costs, there is only a marginal variation in the firm values and optimal coupon. This is because the optimal level of cash decreases rapidly when cost increases, as illustrated in the right panel of Fig. 4.

We characterize the impact of optimal cash balances on spreads in Fig. 6. We have seen earlier that as costs of holding cash increase, the debt value decreases and the optimal capital structure carries a lower proportion of debt. Note that the spreads decline as the costs decline. This is due to the fact that higher cash balances can be optimally carried at low costs. This makes the firm less risky and provides it with cushions from premature liquidation, and the firm’s debt consequently becomes more attractive to investors.

Although the cash balance helps the firm avoid equity dilution, it does not significantly postpone the expected time to default, or its distribution, when carried optimally. This result is somewhat surprising. But as we mentioned before, as the firm approaches bankruptcy, it depletes the cash balance by paying shareholders

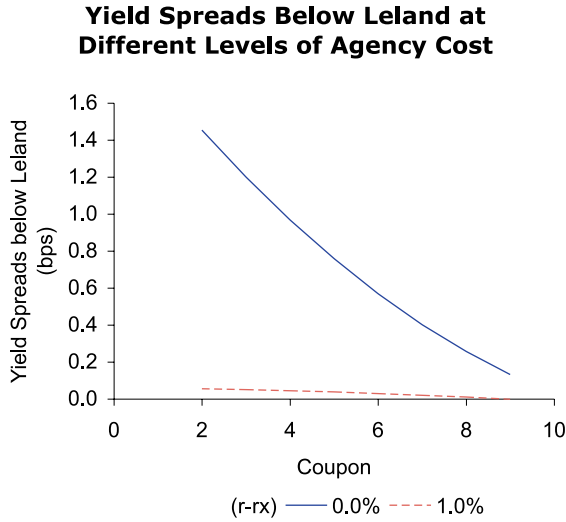


Fig. 6. This figure shows how much cash balance reduces the spread of the existing debt. ($V = 100, \sigma = 0.2, r = 0.05, q = 0.03, \alpha = 0.3, \tau = 0.15, \gamma = 0.15$).

dividend. The result is different when the firm is maximizing total firm value instead of equity value. Appendix F discusses this difference in more detail. Also in Sec. 5.2, we will see that a loan commitment significantly impacts the distribution of default times even in the equity maximizing case.

We have characterized in this section how cash balances may be optimally used to manage illiquidity in the presence of costs of holding cash balance and equity dilution costs. We now turn to the use of a loan commitment as an alternative to managing illiquidity.

5. Loan Commitment

In this section, we show how the firm can use a loan commitment to manage its liquidity problem and relieve it from expensive equity dilution costs and costs associated with carrying cash.

We first define a stylized loan commitment contract. The firm pays a one-time fee (*undrawn fee*), F , at the time it enters into the loan commitment. Let p be the amount of loan commitment outstanding, and $C^l(p)$ be the associated coupon obligation (*drawn fee*). Then when the firm faces a cash flow shortage, i.e., when its revenue is not sufficient to cover the coupon payments, the firm has the option to draw down the loan commitment as long as $p < P^l$. We assume that the firm chooses between drawing down the loan and diluting equity to maximize its equity value. The decision that the firm faces is how much to draw down or repay the loan commitment at each time. We denote this decision variable by λ , where $\lambda > 0$ corresponds to drawing down the loan and $\lambda < 0$ corresponds to repaying the loan.

At any given time, if the firm is still solvent, then one of the following events can happen:

First, in the case where there is no cash flow shortage, i.e., $\delta \geq C + C^l(p)$, the firm can decide how much of the outstanding loan commitment to repay, and the remaining cash surplus goes to the shareholders as dividends. Note that since the firm can neither use the proceeds from the loan for dividends, nor keep a cash balance, it cannot draw down the loan commitment in this case. Hence λ must be in the interval $[-p, 0]$. If $|\lambda| > \delta - C + C^l(p)$, then the firm is also diluting equity to repay the loan commitment.

Second, in the case where there is a cash flow shortage, i.e., $\delta < C + C^l(p)$, the firm has to decide how to manage this shortfall. It needs to raise an additional $C + C^l(p) - \delta$ to meet its coupon obligations. The firm can decide between diluting equity or drawing down the loan commitment, or both. Again, since we restrict the firm from using the proceeds from the loan for dividends, it can draw down at most the shortfall amount, as long as it does not exceed the credit line. We also restrict the firm from repaying the loan commitment in this case. Thus λ must be in the interval $[0, \min\{C + C^l(p) - \delta, P^l - p\}]$.

If the firm declares bankruptcy, then we apply an absolute priority rule to the distribution of proceeds from liquidation. The bankruptcy cost is shared between the bank and the debt holders. Let $L(V, p)$ be the value of the loan commitment. Then the value of the debt and the loan commitment before liquidation costs are:

$$\begin{aligned} \hat{L}(V, p) &= \min\{V + \delta, p\} \\ \hat{D}(V, p) &= V + \delta - L(V, p) \end{aligned}$$

And then we apply the bankruptcy cost pro rata. Thus the debt and loan commitment values ex-liquidation costs are:

$$\begin{aligned} L(V, p) &= \hat{L}(V, p) - \frac{\hat{L}(V, p)}{\hat{L}(V, p) + \hat{D}(V, p)}\alpha(V + \delta) \\ D(V, p) &= \hat{D}(V, p) - \frac{\hat{D}(V, p)}{\hat{L}(V, p) + \hat{D}(V, p)}\alpha(V + \delta) \end{aligned}$$

The one-time fee, F , is chosen such that the loan commitment has zero value to the bank at the time of signing. Therefore, it is the difference between the present value of the loan commitment and the present value of all expected future draws on the loan commitment:

$$F = \int_0^T e^{-rt} \lambda_t^+ dt - L(V_0, 0) \tag{5.1}$$

The coupon rate of the loan commitment, c^l , is determined as follows. Given the current capital structure of the firm, the bank first determines the level of firm's asset value at which it will start drawing down the loan ($qV \leq C$). Then for a loan

commitment of size P^l , the bank decides what the cost of borrowing for the firm should be in that state. In particular, it chooses c^l to solve

$$P^l = D(C/q, c^l P^l),$$

where $D(V, C)$ is the value of a perpetual debt that pays continuous coupon of rate C dollars issued by the firm with asset value V .

This choice of c^l is reasonable because when the firm faces a liquidity crisis at $V = C/q$, its alternative to a loan commitment is to issue additional perpetual debt. If the bank charges any amount greater than c^l , the firm might as well issue additional perpetual debt instead of using a loan commitment.

The size of the loan commitment P^l is usually chosen by the firm. Larger P^l provides more liquidity to the firm, but it also requires larger coupon C^l and larger fee F . Thus the firm chooses P^l that maximizes the benefit to the shareholders. The left panel of Fig. 7 illustrate this result for different levels of α , at the respective level of optimal coupon in absence of the loan commitment. There is also a maximum value of P^l that the firm can support. This is similar to the concept of debt capacity. The right panel of Fig. 7 shows how the loan commitment capacity changes with liquidation cost, α .

The results in this section are obtained by using the binomial approach described in Appendix D. We use the infinite horizon implementation with $T = 200$, $N = 2,400$ and $M = 40$.

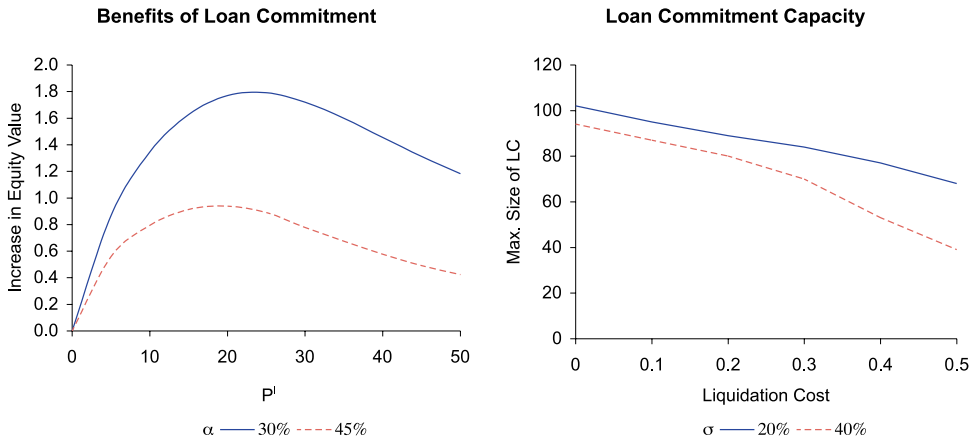


Fig. 7. The left panel plots impact of loan commitment on equity value vs. size of the loan commitment, P^l . The figure shows that there is an optimal value, P^{l*} , where shareholders benefit the most. Furthermore, P^{l*} as well as the impact itself decreases as liquidation cost, α , increases. The right panel plots firm’s capacity for loan commitment vs. liquidation cost. The firm is able to take on smaller size of loan commitment as α gets larger. ($V = 100, C = 3.17, \sigma = 0.2, r = 0.05, q = 0.03, \tau = 0.15, \gamma = 0.05$).

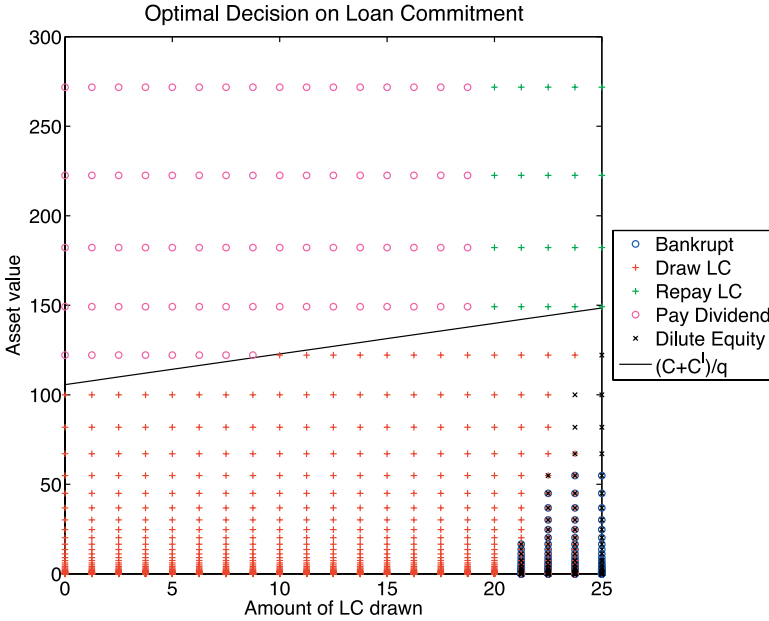


Fig. 8. This figure shows the optimal decisions on the loan commitment. The black solid line is the boundary below which the firm does not generate enough revenue to cover coupon payments. The firm draws down the loan commitment below this line, and only dilutes equity or declare bankruptcy when the credit limit is reached. Above this line, there is a critical level of amount the loan commitment drawn, below which the firm uses excess revenue to pay dividend and above which the firm uses it to repay loan commitment. ($V = 100, \sigma = 0.2, r = 0.05, q = 0.03, \alpha = 0.3, \tau = 0, \gamma = 0.05, C = 3.17, P^l = 25, C^l = 1.285$ (5.14%)).

5.1. Optimal draw down and repayment decisions

When a loan commitment is available to the firm, we find that it will gradually draw it down until the credit limit is reached before diluting equity. The firm will not declare bankruptcy as long as the loan commitment is still available. While the firm is above the cash flow shortage line $(C + C^l(p_t))/q$, there is a level of drawn amount, p_R , above which the it will use excess revenue to repay the principle of the loan commitment, and below which it will use it to pay dividend. Figure 8 illustrates these observations.

5.2. Impacts of loan commitment

In this section, we first discuss the impact of the loan commitment on existing debt and firm values. The left panel of Fig. 9 shows how equity value changes for various sizes of the loan commitment. We find that the presence of the loan commitment increases the equity value. This result is intuitive since the loan commitment helps the firm avoid costly equity dilution. However, we observe that the size of the

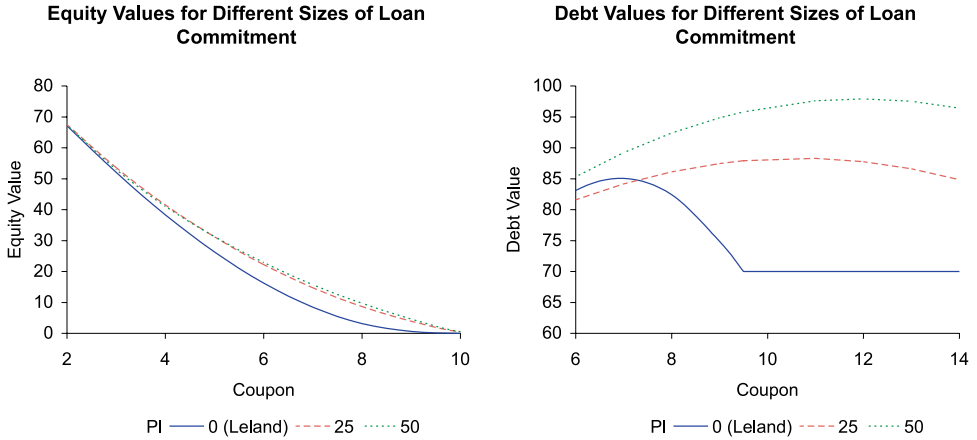


Fig. 9. The left panel plots equity value vs. coupon rate for different sizes of loan commitment. Loan commitment always increases the equity value but its size has little impact on the difference. The right panel plots debt value vs. coupon rate. Debt holders do not always benefit from the loan commitment. High leveraged firms benefit more than low leveraged firms. The debt capacity of the firm increases significantly in presence of loan commitment, and is increasing in its size, P^l . ($V = 100, \sigma = 0.2, r = 0.05, q = 0.03, \alpha = 0.3, \tau = 0.15, \gamma = 0.05$).

loan commitment has very small impact on the change in equity value, just as we previously observed in Fig. 7.

We find more interesting results for debt values. While the shareholders benefit from the loan commitment regardless of the size of the existing debt, we find that the existing debt holders only benefit if its size, as characterized by the coupon C , is larger than a certain level. When C is small, the debt holders are better off without the loan commitment. The right panel of Fig. 9 shows that with a loan commitment of size \$25, the debt holders are better off if the coupon is greater than \$7, whereas for a loan commitment of size \$50, the debt holders are always better off. The loan commitment has both positive and negative impacts on the debt holders. First, since it is more senior than the existing debt, in the case of bankruptcy, the debt holders receive less money when the firm is liquidated. This, and the cost of the loan commitment itself, reduces the existing debt value. Second, it can help the firm avoid or postpone bankruptcy and hence increase the existing debt value. Therefore, it is not surprising that the debt holders are worse off when C is small. This is because the firm is not likely to declare bankruptcy in the future and the costs of having the loan commitment outweigh the benefits. However, at higher leverage, the probability of declaring bankruptcy is more significant and the benefits of the loan commitment dominates the costs.

Notice also in the absence of loan commitments, the debt value is constant when the coupon is above a certain level (\$9 in our example). At this level of coupon, the optimal bankruptcy boundary is above the current asset value, and thus the firm declares bankruptcy immediately, leaving the debt holder with only

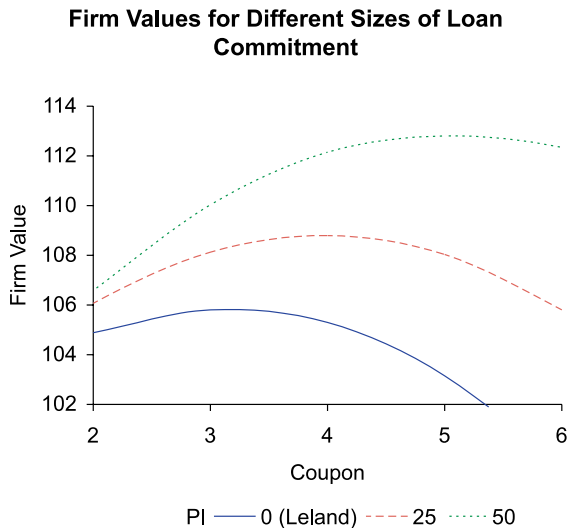


Fig. 10. This figure plots total firm value vs. coupon rate for different sizes of loan commitment. The firm value increases with the size of the loan commitment. This figure also illustrates how loan commitment impacts the optimal capital structure of the firm. The firm with loan commitment will issue debt with much higher coupon rate at optimal capital structure. ($V = 100, \sigma = 0.2, r = 0.05, q = 0.03, \alpha = 0.3, \tau = 0.15, \gamma = 0.05$).

the liquidated asset value less bankruptcy cost, $(1-\alpha)V$. As we discussed previously, it is never optimal to declare bankruptcy when the firm can still draw down the loan commitment. For this reason, we observe that the debt value does not behave the same way when a loan commitment is present. Furthermore, the benefit of a loan commitment increases with its size, because it can prolong the life of the firm further, and the debt holder gets to collect more coupon payments. In addition, the debt capacity of the firm significantly increases in the presence of a loan commitment for the same reason.

Figure 10 shows how the loan commitment increases the total firm value. What is more interesting here is how it impacts the optimal capital structure. We observe that the optimal leverage of the firm increases with the size of the loan commitment. This can be explained by the fact that the loan commitment allows the firm take on more debt to utilize the tax benefits, while providing it with protection from bankruptcy.

The results from this section leads to the following important observation.

Even in the absence of equity dilution and liquidation costs, it is generally beneficial for the firm to enter into a loan commitment contract.

Note that the firm is able derive benefits by exercising the loan commitment in bad states to meet any liquidity needs. The critical point is that the firm can optimally stagger the draw on a loan commitment and only pay (higher) interest on the drawn portion of it. It is this property that makes the loan commitment

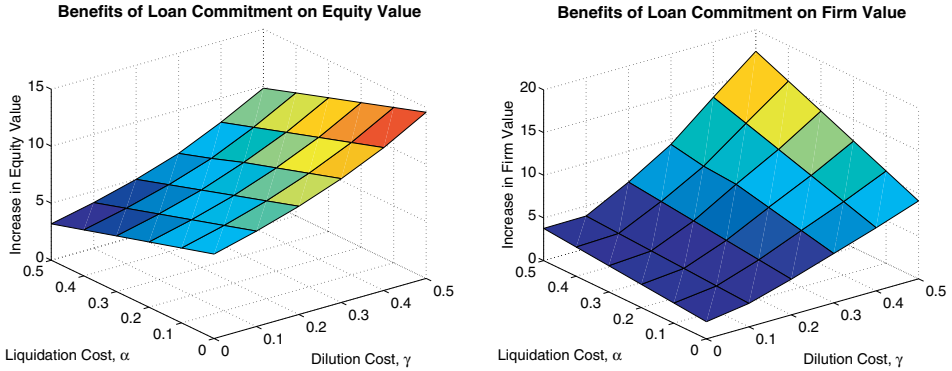


Fig. 11. The left panel plots the increase in equity values from loan commitment for different level of dilution and liquidation costs. The increase in equity value is increasing in γ , but decreasing in α . The right panel plots the increase in total firm value. The increase in the firm value is increasing in γ and α . In both plots, we fix the coupon rate of original debt to \$5.27, which is the optimal level for $\alpha = 0$ and $\gamma = 0$. ($V = 100, \sigma = 0.2, r = 0.05, q = 0.03, C = 5.27, \alpha = 0.3, \tau = 0.15, \gamma = 0.05$).

attractive. Figure 11 plots the increase in equity and total firm value from the loan commitment. The left panel shows that the increase in equity value is increasing in γ , but decreasing in α . This is because equity holders benefit mostly from avoiding costly dilution. Hence as liquidation cost increases, the increase in cost of loan commitment outweighs the increase in benefits. On the other hand, the right panel shows that the increase total firm value is increasing in both γ and α . This is

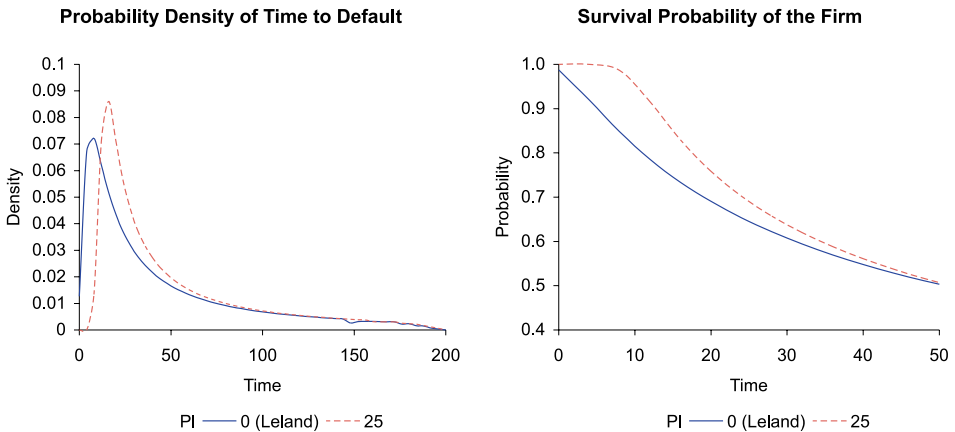


Fig. 12. The left panel plots the probability density function of the default time with and without the loan commitment. The right panel plots the firm’s survival probability, or the probability that the firm will not default before a certain time. The expect default time conditioned on ever defaulting increases from 47 to 50 years. Moreover, the survival probability over the *near* future increases significantly in the presence of loan commitment. ($V = 100, \sigma = 0.2, r = 0.05, q = 0.03, \alpha = 0.3, \tau = 0.15, \gamma = 0.05$).

intuitive because the loan commitment helps to avoid both equity dilution and liquidation costs.

As stated above, entering into a loan commitment is generally beneficial for the firm since it postpones costly equity dilutions and liquidations. Figure 12 shows the probability density of the time of default with and without the loan commitment. In this example, the expected time to default increases from 47 to 50 years with a \$25 loan commitment. More importantly, it significantly increases the survival probability in the near future. In this example, the survival probability is one up to almost ten years, compared with 80% at ten years without the loan commitment. The probability remains higher for about 50 years. The methodology for determining the default time distribution is described in Appendix E. The results were obtained by using $T = 200$, $N = 2,400$ and $K = 50$.

In contrast to cash balance, a loan commitment postpones the default time because the firm does not need to worry about debt holders taking cash upon liquidation. Moreover, the firm must first accumulate the cash balance before it can be of any use, while the loan commitment is available immediately.

6. Conclusion

Our paper characterizes liquidity management strategies and their impact on pricing. We provide an analytical solution to pricing of the firm's securities when equity dilution is costly, and show that costly dilution reduces the firm's debt capacity and optimal level of debt. We provide a methodology to investigate the effectiveness of cash balances and loan commitments as means of managing illiquidity. We find that carrying cash is generally not efficient even in absence of costs associated with holding cash. This is because (a) it is not immediately available since the firm must accumulate cash first, and (b) in doing so, equity holders forego dividends in good states. On the contrary, we show that loan commitments provide an efficient way to manage illiquidity and can be a better strategy when equity dilution and liquidation costs are significant. The staggered draw feature of the loan commitment allows the firm to inject liquidity when needed without foregoing dividends in the good states. Furthermore, we find that the loan commitment increases both the debt capacity and the optimal level of debt, reduces default probability, and increases the expected time to default of the firm.

An interesting extension might be to incorporate strategic debt service in our framework. This is likely to provide a lower incentive to hold cash balances as strategic debt service is another way to deal with illiquidity which imposes greater costs ex-ante to the borrower. Another extension would be to model the Chapter 11 provisions explicitly in our framework as Chapter 11 is yet another institution to deal with liquidity problems of companies. We have not modeled the possibility of carrying cash balances to take advantage of future investment opportunities. In addition, we have not addressed the fact that companies issue debt to raise cash for "general corporate purposes." These are fruitful avenues for further research.

Appendix A

Derivation of equity value from Section 3.1.1

Substituting the general solution (3.8) into (3.6), we get:

$$\begin{aligned} & \frac{1}{2}\sigma^2 A_2 Y(Y+1)V^{-Y} + \frac{1}{2}\sigma^2 A_3 X(X+1)V^{-X} - (r-q)A_1 V \\ & - (r-q)A_2 YV^{-Y} - (r-q)A_3 XV^{-X} \\ & - r(A_0 + A_1 V + A_2 V^{-Y} + A_3 V^{-X}) + \beta(qV - C) = 0 \end{aligned} \quad (\text{A.1})$$

We solve for X and Y by collecting the coefficients of V^{-X} and V^{-Y} in (A.1) and setting it to zero:

$$\begin{aligned} & \frac{1}{2}\sigma^2 A_2 X(X+1) - (r-q)A_2 X - rA_2 = 0 \\ & \Rightarrow \frac{1}{2}\sigma^2 X^2 + \left(\frac{1}{2}\sigma^2 - (r-q)\right) X - r = 0 \\ & \Rightarrow X = \frac{\left(r-q - \frac{\sigma^2}{2}\right) + \sqrt{\left(r-q - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}}{\sigma^2} \\ & \text{and } Y = \frac{\left(r-q - \frac{\sigma^2}{2}\right) - \sqrt{\left(r-q - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}}{\sigma^2} \end{aligned}$$

A_0 and A_1 can be solved by collecting the constant and the coefficient of the linear term in (A.1) as follows:

$$\begin{aligned} -rA_0 - \beta C &= 0 \Rightarrow A_0 = -\frac{\beta C}{r} \\ (r-q)A_1 - rA_1 + \beta q &= 0 \Rightarrow A_1 = \beta \end{aligned}$$

Similarly, substituting (3.8) into (3.7), and solve for \tilde{A}_0 and \tilde{A}_1 we get:

$$\begin{aligned} -r\tilde{A}_0 - C &= 0 \Rightarrow \tilde{A}_0 = -\frac{C}{r} \\ (r-q)\tilde{A}_1 - r\tilde{A}_1 + q &= 0 \Rightarrow \tilde{A}_1 = 1 \end{aligned}$$

Since $Y \leq -1$, the condition (BC II) in (3.9) implies that $\tilde{A}_2 = 0$. The remaining coefficients are determined by solving the system of linear equations imposed by the other three conditions in (3.9).

$$\text{(BC I):} \quad \beta \left(-\frac{C}{r} + V_B\right) + A_2 V_B^{-Y} + A_3 V_B^{-X} = 0$$

$$\text{(CC):} \quad \beta \left(-\frac{C}{r} + \frac{C}{q}\right) + A_2 \left(\frac{C}{q}\right)^{-Y} + A_3 \left(\frac{C}{q}\right)^{-X} = -\frac{C}{r} + \frac{C}{q} + \tilde{A}_3 \left(\frac{C}{q}\right)$$

$$\text{(SP):} \quad \beta - A_2 Y \left(\frac{C}{q}\right)^{-Y-1} - A_3 X \left(\frac{C}{q}\right)^{-X-1} = 1 - \tilde{A}_3 X \left(\frac{C}{q}\right)^{-X-1}$$

And we get the expression for A_2 , A_3 and \tilde{A}_3 as given in (3.10).

Derivation of debt value from Section 3.1.2

Substituting the general solution (3.12) into (3.11), we get:

$$\begin{aligned} \frac{1}{2}\sigma^2 B_2 X(X+1)V^{-X} + (r-q)B_1V - (r-q)B_2XV^{-X} \\ - r(B_0 + B_1V + B_2V^{-X}) + C = 0 \end{aligned} \tag{A.2}$$

B_0 and B_1 can be solved by collecting the constant and the coefficient of the linear term in (A.2) as follows:

$$\begin{aligned} -rB_0 + C = 0 \Rightarrow B_0 = \frac{C}{r} \\ (r-q)B_1 - rB_1 = 0 \Rightarrow B_1 = 0 \end{aligned}$$

The condition (BC I) in (3.13) implies that:

$$\frac{C}{r} + B_2V_B^{-X} = (1-\alpha)V_B \Rightarrow B_2 = \left((1-\alpha)V_B - \frac{C}{r} \right) \frac{1}{V_B^{-X}}$$

Hence the debt value is as given in (3.14).

Appendix B

Proof of Proposition 4.1

Define the notation:

- x_t = cash balance
- δ_t = revenue generated during time interval $(t-h, t]$
- ζ_t = coupon payment during time interval $(t-h, t]$
- d = dividend payout
- $E_t(x_t, d)$ = Equity value at time t , with cash balance x , and dividend payout d
- r_x = interest earned on cash balance
- p = probability of an up move

Proof. We will establish by induction that $E_t(x, d)$ is increasing in d for all t and x , and hence it is optimal to pay as much dividend as possible, leaving no cash inside the firm. For convenience, we set $\tau = 0$ without loss of generality.

Let x^+ be the value of cash balance at the next time period, i.e.

$$x^+ = e^{r_x h}(\delta_t + x - \zeta - d) \tag{B.1}$$

At maturity,

$$E_T(x) = (x + \delta_T + V_T - \zeta - P)^+ \tag{B.2}$$

Let $\mathbb{E}_t[E_{t+h}(x)] = pE_{t+h}^u(x) + (1-p)E_{t+h}^d(x)$, where $E_{t+h}^u(x)$ and $E_{t+h}^d(x)$ are the equity values at the up and down node at time $t+h$ respectively.

Then, for non-bankruptcy nodes at time $T - h$,

$$\begin{aligned} E_{T-h}(x, d) &= e^{-rh} \mathbb{E}_{T-h}[E_T(x^+)] + d \\ &= e^{-rh} \mathbb{E}_{T-h}[(x^+ + V_T + \delta_T - \zeta - P)^+] + d \end{aligned}$$

Case I: For a range of d such that both $E_T^u(x^+)$ and $E_T^d(x^+)$ are not bankrupt,

$$\begin{aligned} E_{T-h}(x, d) &= e^{-rh} \mathbb{E}_{T-h}[x^+ + V_T + \delta_T - \zeta - P] + d \\ &= e^{-rh}(x^+ + \mathbb{E}_{T-h}[V_T + \delta_T] - \zeta - P) + d \\ \Rightarrow \frac{\partial E_{T-h}}{\partial d} &= e^{-rh} \frac{\partial x^+}{\partial d} + 1 \\ &= 1 - e^{-(r-r_x)h} \geq 0 \quad \text{for } r_x \leq r \end{aligned} \tag{B.3}$$

Case II: $E_T^d(x^+)$ is bankrupt, but $E_T^u(x^+)$ is not,

$$\begin{aligned} E_{T-h}(x, d) &= e^{-rh} p(x^+ + V_T^u + \delta_T^u - \zeta - P) + d \\ \Rightarrow \frac{\partial E_{T-h}}{\partial d} &= 1 - pe^{-(r-r_x)h} \geq 0 \quad \text{for } r_x \leq r \end{aligned}$$

Case III: Both $E_T^u(x^+)$ and $E_T^d(x^+)$ are bankrupt

$$E_{T-h}(x, d) = d \Rightarrow \frac{\partial E_{T-h}}{\partial d} = 1$$

Since, E_{T-h} is increasing in d for all cases, the optimal dividend at time $T - h$, d_{T-h}^* is given by

$$d_{T-h}^* = (x + \delta_{T-h} - \zeta)^+ \tag{B.4}$$

When the firm does not have enough revenue and cash to cover the coupon payments, it has to dilution equity, which is equivalent to paying a negative dividend. So to cover the case where dilution occurs, we simply write the optimal dividend as

$$d_{T-h}^* = x + \delta_{T-h} - \zeta \tag{B.5}$$

Let $E_{T-h}^*(x) = E_{T-h}(x, d_{T-h}^*)$. Then for the three cases above, we have

$$\begin{aligned} E_{T-h}^*(x) &= \begin{cases} e^{-rh}(\mathbb{E}_{T-h}[V_T + \delta_T] - \zeta - P) + x + \delta_{T-h} - \zeta \\ e^{-rh}p(V_T^u + \delta_T^u - \zeta - P) + x + \delta_{T-h} - \zeta \\ x + \delta_{T-h} - \zeta \end{cases} \\ \Rightarrow \frac{\partial E_{T-h}^*}{\partial x} &= 1 \end{aligned}$$

Assume that for non-bankruptcy nodes at time $t + h$, $\frac{\partial E_{t+h}^*}{\partial x} = 1$. Then at time t ,

Case I: For a range of d such that both $E_{t+h}^u(x^+)$ and $E_{t+h}^d(x^+)$ are not bankrupt,

$$\begin{aligned} E_t(x, d) &= e^{-rh} \mathbb{E}_t[E_{t+h}^*(x^+)] + d \\ \Rightarrow \frac{\partial E_t^*}{\partial d} &= e^{-rh} \left(\frac{\partial E_{t+h}^*}{\partial x^+} \frac{\partial x^+}{\partial d} \right) + 1 \\ &= 1 - e^{-(r-r_x)h} \geq 0 \quad \text{for } r_x \leq r \end{aligned}$$

Case II: $E_{t+h}^d(x^+)$ is bankrupt, but $E_{t+h}^u(x^+)$ is not,

$$\begin{aligned} E_t(x, d) &= e^{-rh} p E_{t+h}^{u*}(x^+) + d \\ \Rightarrow \frac{\partial E_t}{\partial d} &= p e^{-rh} \left(\frac{\partial E_{t+h}^*}{\partial x^+} \frac{\partial x^+}{\partial d} \right) + 1 \\ &= 1 - p e^{-(r-r_x)h} \geq 0 \quad \text{for } r_x \leq r \end{aligned}$$

Case III: Both nodes $E_{t+h}^u(x^+)$ and $E_{t+h}^d(x^+)$ are bankrupt

$$\begin{aligned} E_t(x, d) &= d \\ \Rightarrow \frac{\partial E_t}{\partial d} &= 1 \end{aligned}$$

Hence $E_t(x, d)$ is increasing in d , and $d_t^* = x + \delta_t - \zeta$ is the optimal dividend payout. We can write the optimal equity value for the three cases as,

$$\begin{aligned} E_t^*(x) &= \begin{cases} e^{-rh} \mathbb{E}_t[E_{t+h}^*(0)] + x + \delta_t - \zeta \\ e^{-r} p E_{t+h}^{u*}(0) + x + \delta_t - \zeta \\ x + \delta_t - \zeta \end{cases} \\ \Rightarrow \frac{\partial E_t^*}{\partial x} &= 1 \end{aligned}$$

which completes the proof. □

Appendix C

Binomial lattice for cash balance

We extend the binomial lattice method of [13] to price corporate debt in the presence of cash balances. Following Cox, Ross and Rubinstein (1979), we divide time into N increments of length $h = T/N$, then we write the evolution of the firm asset value V_t as follows:

$$V_{t+h} = \begin{cases} uV_t & \text{with risk-neutral probability } p \\ dV_t & \text{with risk-neutral probability } 1 - p \end{cases} \quad (\text{C.1})$$

Let $V_t^u = uV_t$ and $V_t^d = dV_t$. By imposing that $u = 1/d$, we have:

$$u = e^{\sigma\sqrt{h}} \tag{C.2}$$

$$d = e^{-\sigma\sqrt{h}} \tag{C.3}$$

$$a = e^{(r-q)h} \tag{C.4}$$

$$p = \frac{a - d}{u - d} \tag{C.5}$$

In this setting, we approximate revenue during time interval h by:

$$\int_t^{t+h} qV_t dt \approx V_t(e^{qh} - 1) =: \delta_t$$

Similarly, we approximate the coupon payment by:

$$\int_t^{t+h} cP dt \approx P(e^{ch} - 1) =: \zeta \tag{C.6}$$

Incorporating the tax benefit of rate τ , the effective coupon payment is $(1 - \tau)\zeta$.

Each node in the lattice is a vector of dimension M , where each element in the vector corresponds to a different level of cash balance, x . The values of x are linearly spaced from 0 to x^{\max} .⁶

We distinguish the cash before and cash after revenue and dividend by x and x^+ , respectively.

Let $E_t(V_t, x_t)$ be the equity value at time t when the firm's asset value is V_t , and its cash balance level is x_t . Similarly, $D_t(V_t, x_t)$ is the corresponding debt value. We also let $\mathbb{E}_t[f_{t+h}(V_{t+h}, x_{t+h})]$ be the expected value of the security $f(\cdot)$ at the next time period $t + h$, given the information at time t , where $f(\cdot)$ can represent the equity value, $E(\cdot)$, or the debt value, $D(\cdot)$, namely,

$$\mathbb{E}_t[f_{t+h}(V_{t+h}, x_{t+h})] = pf_{t+h}(V_{t+h}^u, x_{t+h}) + (1 - p)f_{t+h}(V_{t+h}^d, x_{t+h}).$$

Note that x_{t+h} is deterministic because it only depends on the decision at time t . Since we only know the values of $f_{t+h}(V, x)$ for a discrete set of values of x , we use linear interpolation to estimate $f_{t+h}(V_{t+h}^u, x_{t+h})$ and $f_{t+h}(V_{t+h}^d, x_{t+h})$.

Figure C.1 illustrates what each node in the lattice represents.

At maturity, the firm returns everything to shareholders after meeting its debt obligations.

If $V_T + x_T + \delta_T \geq (1 - \tau)\zeta + P$:

$$E_T(V_T, x_T) = V_T + x_T + \delta_T - (1 - \tau)\zeta - P$$

$$D_T(V_T, x_T) = \zeta + P$$

If (V_T, x_T) is such that the firm does not have enough value to meet its debt obligation, it declares bankruptcy and liquidates its assets.

⁶In our routine, we pick x^{\max} arbitrarily and check if the optimal cash balance, x^* , is attained at x^{\max} . If it does, then we increase x^{\max} and repeat the routine.

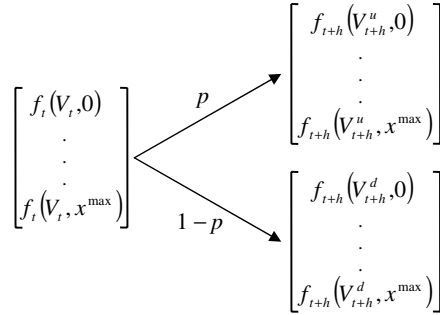


Fig. C.1. This figure illustrates the vector in each node of the lattice. $f_t(\cdot)$ represents the security value at each element of the vector, where f_t can represent the equity value, E_t , or the debt value, D_t .

If $V_T + x_T + \delta_T < (1 - \tau)\zeta + P$:

$$E_T(V_T, x_T) = 0$$

$$D_T(V_T, x_T) = (1 - \alpha)(V_T + x_T + \delta_T)$$

At other times t , if (V_t, x_t) is such that the firm has enough cash from revenue and cash balance to meet its debt obligations, then it has an option to retain some excess revenue in the cash balance, or pay it out to shareholders as dividend.

(1) If $x_t + \delta_t \geq (1 - \tau)\zeta$:

Choose $x_t^+ \in [0, x_t + \delta_t - (1 - \tau)\zeta]$

$$x_{t+h} = e^{r_h} x_t^+$$

$$E_t(V_t, x_t) = e^{-r_h} \mathbb{E}_t[E_{t+h}(V_{t+h}, x_{t+h})] + x_t + \delta_t - (1 - \tau)\zeta - x_t^+$$

$$D_t(V_t, x_t) = e^{-r_h} \mathbb{E}_t[D_{t+h}(V_{t+h}, x_{t+h})] + \zeta$$

x_t^+ is chosen such that the objective value (equity or firm value) is maximized at time t . The optimization is done by performing a search over K discrete points of feasible values of x_t^+ as given above.

If (V_t, x_t) is such that the firm does not have enough from revenue and cash balance, then it has to raise money by diluting equity.

(2) If $x_t + \delta_t < (1 - \tau)\zeta$:

If (V_t, x_t) is such that the firm has enough equity to cover the shortfall:

2A. If $(1 - \gamma)e^{-r_h} \mathbb{E}_t[E_{t+h}(V_{t+h}, 0)] \geq (1 - \tau)\zeta - x_t - \delta_t$:

$$E_t(V_t, x_t) = e^{-r_h} \mathbb{E}_t[E_{t+h}(V_{t+h}, 0)] + (x_t + \delta_t - (1 - \tau)\zeta)/(1 - \gamma)$$

$$D_t(V_t, x_t) = e^{-r_h} \mathbb{E}_t[D_{t+h}(V_{t+h}, 0)] + \zeta$$

If (V_t, x_t) is such that the firm does not have enough equity value to cover the shortfall, then it declares bankruptcy:

$$2B. \text{ If } (1 - \tau)\zeta - x_t - \delta_t > (1 - \gamma)e^{-rh}\mathbb{E}_t[E_{t+h}(V_{t+h}, 0)]$$

$$E_t(V_t, x_t) = 0$$

$$D_t(V_t, x_t) = (1 - \alpha)(V_t + x_t + \delta_t)$$

In the infinite maturity case, we use an arbitrarily large maturity T and modify the terminal values of states (V_T, x_T) such that the firm is not bankrupt as follows:

If $V_T + x_T + \delta_T \geq (1 - \tau)\zeta + P$:

$$E_T(V_T, x_T) = V_T + x_T - (1 - \tau)\zeta / (1 - r)$$

$$D_T(V_T, x_T) = \zeta / (1 - r)$$

We use the infinite horizon implementation with $T = 200$, $N = 2,400$ and $M = 40$ for the results in Figs. 3–6.

Appendix D

Binomial lattice for loan commitment

The basic setup of the lattice is as described in Appendix C. Each node in the lattice is a vector of dimension M , where each element in the vector corresponds to the different level of outstanding loan commitment, p . The values of p are linearly spaced from 0 to P^l .

Let λ_t be the amount the loan commitment drawn at time t . In addition to $E_t(p_t)$ and $D_t(p_t)$, we let $L_t(p_t)$ be the loan commitment value to the bank at time t , with the drawn amount of p_t , and $\Lambda_t(p_t)$ be its value to the firm (i.e., future cash flow from the loan commitment to the firm). Again, we let $\mathbb{E}_t[f_{t+h}(V_{t+h}, x_{t+h})]$ be the expected value of the security $f(\cdot)$ at the next time period $t + h$, given the information at time t , where $f(\cdot)$ can represent $L(\cdot)$ or $\Lambda(\cdot)$, in addition to $E(\cdot)$ and $D(\cdot)$.

At maturity, the firm returns everything to shareholders after meeting all of its debt obligations.

If $V_T + \delta_T \geq (1 - \tau)(\zeta + \zeta^l(p_T)) + P + p_T$:

$$E_T(V_T, p_T) = V_T + \delta_T - (1 - \tau)(\zeta + \zeta^l(p_T)) - (P + p_T)$$

$$L_T(V_T, p_T) = \zeta^l(p_T) + p_T$$

$$D_T(V_T, p_T) = \zeta + P$$

$$\Lambda_T(V_T, p_T) = 0$$

If (V_T, p_T) is such that the firm does not have enough value to meet its debt obligation, then it declares bankruptcy and the liquidation cost is shared among the debt holders as described in Sec. 5.

If $V_T + \delta_T < (1 - \tau)(\zeta + \zeta^l(p_T)) + P + p_T$:

$$E_T(V_T, p_T) = 0$$

$$\hat{L}_T(V_T, p_T) = \min \{V_T + \delta_T, \zeta^l(p_T) + p_T\}$$

$$\begin{aligned}\hat{D}_T(V_T, p_T) &= V_T + \delta_T - \hat{L}_T(V_T, p_T) \\ L_T(V_T, p_T) &= \hat{L}_T(V_T, p_T) - \frac{\hat{L}_T(V_T, p_T)}{\hat{L}_T(V_T, p_T) + \hat{D}_T(V_T, p_T)}\alpha(V_T + \delta_T) \\ D_T(V_T, p_T) &= \hat{D}_T(V_T, p_T) - \frac{\hat{D}_T(V_T, p_T)}{\hat{L}_T(V_T, p_T) + \hat{D}_T(V_T, p_T)}\alpha(V_T + \delta_T) \\ \Lambda_T(V_T, p_T) &= 0\end{aligned}$$

At other times t , if (V_T, p_T) is such that the firm generates enough revenue to meet its debt obligations, then it has an option to repay some of the loan commitment by using excess cash from revenue or equity dilution, or both. Any remaining cash goes to shareholders as dividend.

(1) If $\delta_t \geq (1 - \tau)(\zeta + \zeta^l(p_t))$:

Choose $\lambda_t \in [-p_t, 0]$

$$p_{t+h} = p_t + \lambda_t$$

$$\Lambda_t(V_t, p_t) = e^{-rh} \mathbb{E}_t[\Lambda_{t+h}(V_{t+h}, p_{t+h})]$$

If (V_t, p_t) is such that the firm has enough equity value to repay the chosen amount of loan commitment, λ_t :

(1A) If $(1 - \gamma)\mathbb{E}_t[E_{t+h}(V_{t+h}, p_{t+h})] \geq (1 - \tau)(\zeta + \zeta^l(p_t)) - \delta_t - \lambda_t$:

$$E_t(V_t, p_t) = e^{-rh} \mathbb{E}_t[E_{t+h}(V_{t+h}, p_{t+h})] + \min\{\delta_t - (1 - \tau)(\zeta + \zeta^l(p_t)) + \lambda_t, (\delta_t - (1 - \tau)(\zeta + \zeta^l(p_t)) + \lambda_t) / (1 - \gamma)\}$$

$$L_t(V_t, p_t) = e^{-rh} \mathbb{E}_t[L_{t+h}(V_{t+h}, p_{t+h})] + \zeta^l(p_t) - \lambda_t$$

$$D_t(V_t, p_t) = e^{-rh} \mathbb{E}_t[D_{t+h}(V_{t+h}, p_{t+h})] + \zeta$$

If (V_t, p_t) is such that the firm does not have enough equity value, it declares bankruptcy:

(1B) If $(1 - \tau)(\zeta h + \zeta^l(p_t)) - \delta_t - \lambda_t > (1 - \gamma)e^{-rh} \mathbb{E}_t[E_{t+h}(V_{t+h}, p_{t+h})]$:

$$E_t(V_t, p_t) = 0$$

$$\hat{L}_t(V_t, p_t) = \min\{V_t + \delta_t, \zeta^l(p_t) + p_t\}$$

$$\hat{D}_t(V_t, p_t) = V_t + \delta_t - \hat{L}_t(V_t, p_t)$$

$$L_t(V_t, p_t) = \hat{L}_t(V_t, p_t) - \frac{\hat{L}_t(V_t, p_t)}{\hat{L}_t(V_t, p_t) + \hat{D}_t(V_t, p_t)}\alpha(V_t + \delta_t)$$

$$D_t(V_t, p_t) = \hat{D}_t(V_t, p_t) - \frac{\hat{D}_t(V_t, p_t)}{\hat{L}_t(V_t, p_t) + \hat{D}_t(V_t, p_t)}\alpha(V_t + \delta_t)$$

If (V_t, p_t) is such that the firm does not generate enough revenue, then it has to raise money by either drawing down the loan commitment or diluting equity. The maximum it can draw from the loan commitment is limited by

the credit limit P^l , and the maximum it can raise from equity dilution is limited by its equity value.

(2) If $\delta_t < (1 - \tau)(\zeta + \zeta^l(p_t))$:

Choose $\lambda_t \in [0, \min\{(1 - \tau)(\zeta + \zeta^l(p_t)) - \delta_t, P_{\max}^l - p_t\}]$

$$p_{t+h} = p_t + \lambda_t$$

$$\Lambda_t(V_t, p_t) = e^{-rh} \mathbb{E}_t[\Lambda_{t+h}(V_{t+h}, p_{t+h})] + \lambda_t$$

(2A) If (V_t, p_t) is such that λ_t alone is enough to cover the shortfall, i.e., $\lambda_t = (1 - \tau)(\zeta + \zeta^l(p_t)) - \delta_t$, then:

$$E_t(p_t) = e^{-rh} \mathbb{E}_t[E_{t+h}(p_{t+h})]$$

$$L_t(p_t) = e^{-rh} \mathbb{E}_t[L_{t+h}(p_{t+h})] + \zeta^l(p_t)$$

$$D_t(p_t) = e^{-rh} \mathbb{E}_t[D_{t+h}(p_{t+h})] + \zeta$$

If (V_t, p_t) is such that λ_t is not enough to cover the shortfall, i.e., $\lambda_t < (1 - \tau)(\zeta + \zeta^l(p_t)) - \delta_t$, and the firm has enough equity to cover the difference:

(2B) If $(1 - \gamma)e^{-rh} \mathbb{E}_t[E_{t+h}(V_{t+h}, p_{t+h})] \geq (1 - \tau)(\zeta + \zeta^l(p_t)) - \delta_t - \lambda_t$:

$$E_t(V_t, p_t) = e^{-rh} \mathbb{E}_t[E_{t+h}(V_{t+h}, p_{t+h})] + (\delta_t - (1 - \tau)(\zeta + \zeta^l(p_t)) + \lambda_t)/(1 - \gamma)$$

$$L_t(V_t, p_t) = e^{-rh} \mathbb{E}_t[L_{t+h}(V_{t+h}, p_{t+h})] + \zeta^l(p_t)$$

$$D_t(V_t, p_t) = e^{-rh} \mathbb{E}_t[D_{t+h}(V_{t+h}, p_{t+h})] + \zeta$$

If at the chosen level of λ_t , (V_t, p_t) is such that the firm does not have enough equity value to cover the shortfall, then it declares bankruptcy:

(2C) If $(1 - \tau)(\zeta + \zeta^l(p_t)) - \delta_t - \lambda_t > (1 - \gamma)e^{-rh} \mathbb{E}_t[E_{t+h}(V_{t+h}, p_{t+h})]$

$$E_t(V_t, p_t) = 0$$

$$\hat{L}_t(V_t, p_t) = \min \{V_t + \delta_t, \zeta^l(p_t) + p_t\}$$

$$\hat{D}_t(V_t, p_t) = V_t + \delta_t - \hat{L}_t(p_t)$$

$$L_t(V_t, p_t) = \hat{L}_t(V_t, p_t) - \frac{\hat{L}_t(V_t, p_t)}{\hat{L}_t(V_t, p_t) + \hat{D}_t(V_t, p_t)} \alpha(V_t + \delta_t)$$

$$D_t(V_t, p_t) = \hat{D}_t(V_t, p_t) - \frac{\hat{D}_t(V_t, p_t)}{\hat{L}_t(V_t, p_t) + \hat{D}_t(V_t, p_t)} \alpha(V_t + \delta_t)$$

In the infinite maturity case, we use an arbitrarily large maturity T and modify the terminal values of states (V_T, x_T) such that the firm is not bankrupt as follows:

If $V_T + \delta_T \geq (1 - \tau)(\zeta + \zeta^l(p_T)) + P + p_T$:

$$E_T(V_T, p_T) = V_T + \delta_T - (1 - \tau)(\zeta + \zeta^l(p_T))/(1 - r)$$

$$L_T(V_T, p_T) = \zeta^l(p_T)/(1 - r)$$

$$D_T(V_T, p_T) = \zeta/(1 - r)$$

We use the infinite horizon implementation with $T = 200$, $N = 2,400$ and $M = 40$ for the results in Figs. 7–12.

Appendix E

Estimating default distribution in binomial lattice

In this section, we describe a method for estimating the distribution of default time in the binomial lattice. For simplicity, we assume that there is no cash balance or loan commitment, but the method can be easily applied to both cases.

We discretize the probability density of default time into K buckets of time intervals of length $\Delta = T/K$. The k th bucket corresponds to the time interval $(t_k, t_{k+1}]$, for $k = 0, 1, \dots, K - 1$, where $t_k = k\Delta$. K can be any positive integer no greater than the number of discrete time steps, N , in the lattice. Let $\Pi_t(V_t, k)$ be the time t probability of defaulting during the k th interval when the firm's asset value is V_t .

At maturity, if V_T is such that the firm is not bankrupt, then the default probability is zero for all intervals k .

$$\Pi_T(V_T, k) = 0 \quad \text{for all } k$$

Otherwise, if V_T is such that the firm is bankrupt, then the probability of defaulting at $t = T$ is one, and is zero everywhere else:

$$\Pi_T(V_T, k) = \begin{cases} 0 & \text{for } k \neq K - 1 \\ 1 & \text{for } k = K - 1 \end{cases}$$

because T is in the $(K - 1)$ st interval (the last interval).

At other times t , if V_t is such that the firm is not bankrupt, then the time t distribution of default time is the expected value of time $t + h$ distribution.

$$\Pi_t(V_t, k) = \mathbb{E}_t[\Pi_{t+h}(V_{t+h}, k)] \quad \text{for all } k$$

Otherwise, if V_t is such that the firm is bankrupt, then probability of defaulting during the interval where t belongs in is one, and is zero everywhere else.

$$\Pi_t(V_t, k) = \begin{cases} 0 & \text{for } k \text{ such that } t \notin (t_k, t_{k+1}] \\ 1 & \text{for } k \text{ such that } t \in (t_k, t_{k+1}] \end{cases}$$

Figure E.1 provides an example of the default time distribution in a binomial lattice with $K = N = 4$ and $p = 0.5$.

We use $T = 200$, $N = 2,400$ and $K = 50$ for the results in Figs. 12 and F.2.

Appendix F

Effects of cash balance on default time

As discussed in Sec. 4.2, cash balance does not have a significant impact on default time when the firm is maximizing the equity value. This is because the firm

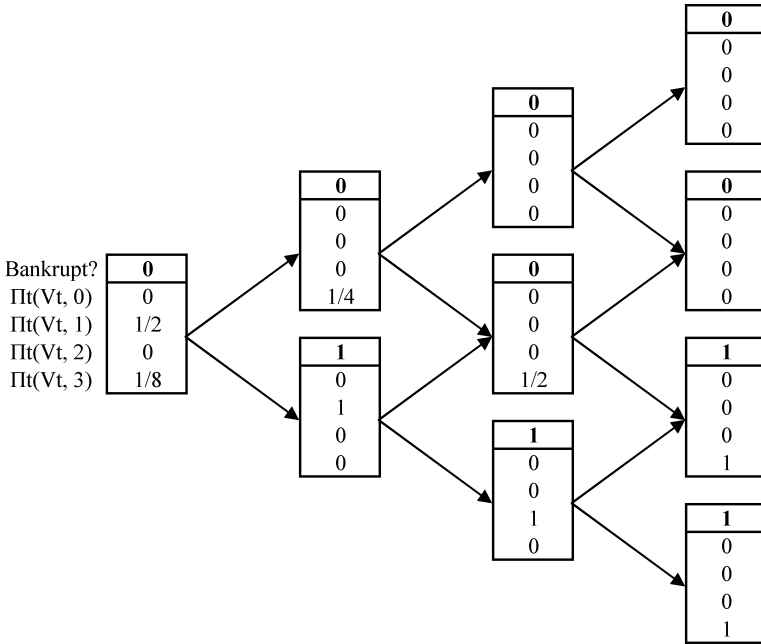


Fig. E.1. This figure shows the default time distribution at each node of the binomial lattice for $K = N = 4$ and $p = 0.5$. Each box represents a node in the lattice. The first (bold) number is 1 if the firm is bankrupt at that node, or is 0 otherwise. The remaining four elements are the probabilities of defaulting at time 0, 1, ..., 4, respectively.

Optimal Level of Cash Balance at Different Levels of Agency Cost (Firm Value Maximization)

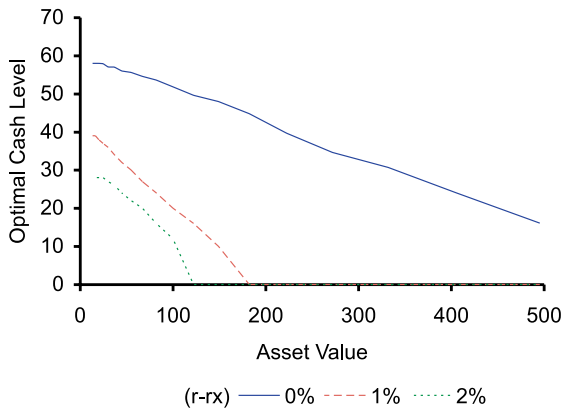


Fig. F.1. This figure plots the optimal level of cash balance, x^* , vs. asset value, V , under firm value maximization. Unlike equity value maximization, x^* is decreasing in V for all values of V . ($V = 100, C = 3.36, \sigma = 0.2, r = 0.05, q = 0.03, \alpha = 0.3, \tau = 0.15$).

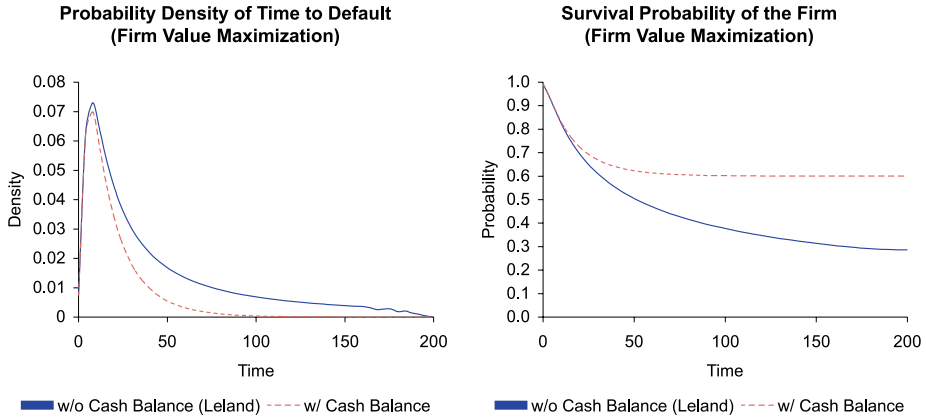


Fig. F.2. The left panel plots the probability density function of the default time with and without cash balance under firm value maximization. The distribution is significantly different because the firm keeps cash inside even as it approaches bankruptcy. The right panel plots the survival probability. The survival probability is almost the same for $t \leq 10$, but starts to deviate (upward) significantly for larger values of t . ($V = 100, C = 3.36, \sigma = 0.2, r = 0.05, q = 0.03, \alpha = 0.3, \tau = 0.15$).

depletes the cash balance so that shareholders can receive more dividends prior to bankruptcy. The impact of cash balance on default time is significantly different under total firm value maximization. Figure F.1 plots the optimal level of cash balance as a function of asset value. Unlike the equity maximization case, the optimal cash level increases as the asset value decreases, regardless of how close it is to bankruptcy. Consequently, as the firm heads towards bankruptcy, it will not empty out the cash balance and will be able to remain solvent longer. This has a big impact on the probability of default. In our example, the probability of default reduces from 71.3% to 40.0%. However, since there is a huge reduction in probability of default, the expected default time, conditioned on ever defaulting, reduces from 44.6 to 19.6. Figure F.2 shows the distribution of default time and the survival probability. We find that the probability of defaulting within ten years is practically the same in both cases. The probabilities start to deviate at later times. The reason for this is because in the next few years, the firm is not likely to be able to accumulate significant amount of cash to make a difference. If the firm survives the initial period, then the survival probability is significantly different.

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