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Optimal Replication of Contingent Claims under Portfolio Constraints

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We determine the minimum cost of super-replicating a nonnegative contingent claim when there are convex constraints on portfolio weights. We show that the optimal cost with constraints is equal to the price of a related claim without constraints. The related claim is a dominating claim, that is, a claim whose payoffs are increased in an appropriate way relative to the original claim. The results hold for a variety of options, including some path-dependent options. Constraints on the gamma of the replicating portfolio, constraints on portfolio amounts, and constraints on the number of shares are also considered.

Since the pioneering option pricing work of Black and Scholes (1973) and Merton (1973), much research has

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focused on relaxing the assumptions of a perfect market. The types of market imperfections that have been studied include transactions costs, differential borrowing and lending rates, trading restrictions, and leverage constraints. Although there is an extensive transactions cost literature,[1] option hedging under portfolio (or leverage) constraints has received much less attention. Naik and Uppal (1994) first studied the effects of leverage constraints on the pricing and hedging of stock and bond options in discrete time.[2]

In this article we extend the work of Naik and Uppal (1994) to the continuous-time framework. In particular, we solve for the minimum cost portfolio which super-replicates the payoff of a contingent claim when there are convex constraints on the portfolio weights. Super-replication allows the hedging strategy to generate portfolio values that strictly exceed the contingent claim payoff in some states. Edirisinghe, Naik, and Uppal (1993) noted that a super-replication may be much cheaper than exact replication. Our solution is fairly simple and intuitively appealing. To price an option with portfolio constraints we first create a *dominating claim*, that is, one whose payoffs are increased in an appropriate way relative to the original claim. We show that the price of the original claim with constraints is the price of the dominating claim without constraints. The latter can be priced using standard risk-neutral valuation procedures. This solution provides an intuitive view of the increased cost due to constraints, namely the additional hedging cost arises from pricing a claim with a higher payoff. Our solution applies to a wide variety of contingent claims, including American options, options on multiple assets, and some path-dependent options (e.g., lookback options). The solution offers numerical advantages as well.

The dominating claim solution ties together two strands of literature on portfolio constraints. It joins the finance literature initiated in Naik and Uppal (1994) with the mathematical finance literature in Cvitanić and Karatzas (1993), Bardhan (1995), El Karoui and Quenez (1995), and Karatzas and Kou (1995, 1996). Naik and Uppal (1994) derive an explicit recursive solution to a linear programming formulation of the minimum cost hedging problem with leverage constraints. The latter articles derive an abstract stochastic control representation for the same problem in continuous time and provide some bounds

[1] The effect of transaction costs on option pricing and hedging has been studied in discrete time in Leland (1985), Boyle and Vorst (1992), Edirisinghe, Naik, and Uppal (1993), Boyle and Tan (1994), and Rutkowski (1996). Continuous-time transaction costs are treated in Flesaker and Hughston (1993), Wilmott, Dewynne, and Howison (1993), Soner, Shreve, Cvitanić (1995), and Cvitanić and Karatzas (1996), and Barles and Soner (1998).

[2] In somewhat different contexts, leverage constraints are also studied in Grossman and Vila (1992) and Marín and Olivier (1996).

and complex approximation schemes for calculating them. In Karatzas and Kou (1996, 1998) it is shown that the minimum cost of super-replication for the buyer, and the corresponding cost for the seller of a claim, form an interval collapsing to the Black–Scholes price if there are no constraints. There is no arbitrage in the constrained market if and only if the price of the claim is contained in the interval. The dominating claim approach provides an explicit solution to the continuous-time stochastic control formulation and represents an extension of the discrete-time linear programming solution given in Naik and Uppal (1994).[3]

The model and the main results of the article are given in the next section. Examples and illustrations are given in Section 2. Constraints on the gamma of the portfolio are treated in Section 3. Extensions to path-dependent options and alternative types of constraints are given in Section 4. Proofs are deferred to the Appendix.

1. The Model and Main Results

We consider a Black–Scholes–Merton financial market consisting of a riskless bond and d risky assets which are traded continuously on the finite time span $[0, T]$. The bond price and the d-dimensional vector of asset prices evolve according to the stochastic differential equations

$$dB_t = rB_t dt \tag{1}$$

$$dS_t^i = S_t^i \left[\mu_i dt + \sum_{j=1}^{d} \sigma_{ij} dW_t^j \right], \tag{2}$$

with initial conditions $B_0 = 1$ and $S_0^i > 0$ given.[4] In Equation (2), $W = (W^1, \ldots, W^d)$ is a d-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ which is the $\mathbf{P}$-augmentation of the natural filtration $\mathcal{F}_t^W \triangleq \sigma\{W_s : 0 \leq s \leq t\}$ generated by W. The parameters of the market are the interest rate $r > 0$, the vector of return rates $\mu \triangleq (\mu_1, \ldots, \mu_d)'$, and the nonsingular volatility matrix $\sigma \triangleq \{\sigma_{ij}\}_{1 \leq i,j \leq d}$. Let S_t denote the vector of asset prices $(S_t^1, \ldots, S_t^d)$.

A *portfolio process* is represented by the vector $\pi_t = (\pi_t^1, \ldots, \pi_t^d)'$, where π_t^i is the proportion of wealth held in asset i at time t. Also let $\pi_t^0 = 1 - \sum_{i=1}^{d} \pi_t^i$ represent the proportion of wealth held in the

[3] The linear programming approach is also discussed in Musiela and Rutkowski (1997).

[4] Throughout the article we use superscripts to indicate a dimension, not to indicate a power. The results of this article also carry through with a constant dividend yield δ^i for asset i, $i = 1, \ldots, d$.

riskless bond. We model portfolio constraints by requiring $\pi_t \in C$, where C is a given closed convex set in $\mathbb{R}^d$. Throughout the article we assume that the portfolio value process is nonnegative. In this case, the no-borrowing constraint, $\pi^0 \geq 0$, is equivalent to setting $C = \{\pi \in \mathbb{R}^d : \sum_{i=1}^d \pi^i \leq 1\}$. Short selling of assets can be restricted by setting $C = \{\pi \in \mathbb{R}^d : \pi^i \geq -l_i\}$. Taking $l_i = 0$ for all i prohibits short sales in all assets. The ratio of debt to equity can be restricted in a similar fashion.

Let $b : \mathbb{R}^d_+ \mapsto \mathbb{R}_+$ be a given lower semicontinuous payoff function.[5] First we focus on European contingent claims whose payoff at time T is $b(S_T)$. Later we analyze American claims whose payoff is $b(S_t)$ when exercised at time t. Consider a seller of the European claim b who wants to hedge his short position as cheaply as possible by trading in the underlying assets and riskless bond while satisfying restrictions on the composition of the hedging portfolio. In particular, the seller wishes to super-replicate the claim in the least expensive way while keeping the super-replicating portfolio process π_t in C for all $t \in [0, T]$.[6] By super-replication we mean that the wealth process almost surely dominates the value of the claim $b(S_T)$ at time $t = T$. We consider super-replication since exact replication is generally not possible when there are portfolio constraints, and even when exact replication is possible super-replication may be cheaper.

Definition 1. *Define the seller's cost of the claim b to be the minimum initial amount of money (possibly infinite) which is needed to super-replicate $b(S_T)$ with a self-financing portfolio strategy π_t which satisfies $\pi_t \in C$ for all $t \in [0, T]$. Denote by $P(t, S_t)$ the corresponding minimum super-replicating value process for the seller at time t.*

For exact definitions of self-financing strategies and precise mathematical descriptions of the above definition we refer the interested reader to Karatzas and Kou (1996). Moreover, it is shown there that the defined processes and values exist, unless they are infinite.

To state our results in this general framework, we need to intro-

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[5] Recall that $b : \mathbb{R}^d_+ \mapsto \mathbb{R}_+$ is lower semicontinuous means

$$b(x) \leq \liminf_{\substack{y \to x \\ y \in \mathbb{R}^d_+}} b(y) \text{ for all } x \in \mathbb{R}^d_+.$$

[6] Similarly, we could consider the problem from the perspective of the buyer of the claim. The buyer wishes to borrow the maximum amount of money against the claim and hedge his long position by trading in the underlying assets and riskless bond while satisfying portfolio restrictions. Thus the buyer wishes to borrow the maximum amount in order to super-replicate $-b(S_T)$ at time T while satisfying π_t in C for all $t \in [0, T]$. All of the results of this article carry over to the buyer's case with the appropriate modifications.

duce the support function $\delta(\nu) \triangleq \sup_{\pi \in C}(-\pi'\nu)$ of $-C$, defined on its effective domain $\tilde{C} \triangleq \{\nu \in \mathbb{R}^d : \delta(\nu) < \infty\}$, which is a closed convex cone (see, e.g., Rockafellar 1970). The function δ is positively homogeneous and convex. We assume that it is continuous and that $0 \in C$, so that δ is nonnegative. Denote by $\mathcal{D}$ the set of all progressively measurable processes ν_t taking values in $\tilde{C}$. Also introduce for a given process ν_t in $\mathbb{R}^d$, the auxiliary shadow economy vector of asset prices $S_t(\nu)$ by

$$dS_t^i(\nu) = S_t^i(\nu)\left[(r - \nu_t^i)dt + \sum_{j=1}^{d} \sigma_{ij} dW_t^j\right]. \tag{3}$$

With these definitions and assumptions, we restate Theorem 6.4 from Cvitanić and Karatzas (1993), as follows:

Theorem 1. *The value process for the seller is given by*

$$P(t, S_t) = \sup_{\nu \in \mathcal{D}} E\left[b(S_T(\nu))e^{-\int_t^T (r+\delta(\nu_s))ds} \middle| S_t(\nu) = S_t\right]. \tag{4}$$

We will show that this complex looking stochastic control problem has a simple solution. Given a claim b and a closed convex set of portfolio constraints C, we define the dominating claim $\hat{b}$ by

$$\hat{b}(S) = \sup_{\nu \in \tilde{C}} b(Se^{-\nu})e^{-\delta(\nu)}, \tag{5}$$

where $Se^{-\nu} \triangleq (S^1 e^{-\nu^1}, \ldots, S^d e^{-\nu^d})'$. (We use the same notation for the componentwise product of two vectors throughout the article.) Since $0 \in \tilde{C}$, it follows from Equation (5) that $\hat{b}(S) \geq b(S)$ for all $S \in \mathbb{R}_+^d$, which justifies the term dominating. In what follows we use the term Feynman–Kac assumptions to refer to those assumptions under which the relevant expected values satisfy the corresponding PDEs. A set of such assumptions is given in Duffie (1996).

Here is our main result:

Theorem 2. *The seller's cost $P(t, S_t)$ of super-replicating the claim $b(S_T)$ with the closed convex set of constraints C is the Black–Scholes cost function for the dominating claim $\hat{b}(S_T)$ without constraints. In particular, if $\hat{b}$ satisfies the Feynman–Kac assumptions, denoting $a = \sigma\sigma'$, $P(t, S)$ is the solution to the PDE*

$$P_t + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij} S^i S^j P_{S^i S^j} + r\left(\sum_{i=1}^{d} S^i P_{S^i} - P\right) = 0, \tag{6}$$

with the terminal condition

$$P(T, S) = \hat{b}(S), \quad S \in \mathbb{R}_+^d. \tag{7}$$

Moreover, the corresponding self-financing portfolio strategy satisfies the constraints $\pi_t \in C$ *for all* $t \in [0, T]$*. In particular, under the Feynman–Kac assumptions, it is given by*

$$\pi_t^i = S_t^i \frac{P_{S^i}(t, S_t)}{P(t, S_t)}, \quad i = 1, \ldots, d. \tag{8}$$

The proof of Theorem 2 is given in the Appendix. It should be remarked that from the results in Karatzas and Kou (1996), it follows that if $P(t, S_t) = 0$ in Equation (8), then we can take π_t to be equal to any vector in C. Using a result from Karatzas and Kou (1998), stating that for American options one gets the same type of representation as in Equation (4) by taking an additional supremum over all stopping times, we get

Corollary 1. *The seller's cost* $P(t, S_t)$ *for super-replicating an* American *claim* $b(S_t)$ *with closed convex constraints* C *is the cost function of the unconstrained American dominating claim* $\hat{b}(S_t)$.

2. Examples

We first consider the case of a single asset, that is, $d = 1$, and constraints of the type

$$C = [-l, u] \tag{9}$$

with $0 \leq l, u \leq +\infty$, and with the understanding that the interval C is open to the right (left) if $u = +\infty$ (respectively, if $l = +\infty$). Here l represents the limit imposed on short selling and u the limit imposed on borrowing. The support function $\delta(\nu)$ can be written compactly as

$$\delta(\nu) = l\nu^+ + u\nu^-. \tag{10}$$

The effective domain $\tilde{C}$ is given by

$$\tilde{C} = \begin{cases} \mathbb{R} & \text{if } l, u < \infty, \\ [0, \infty) & \text{if } u = +\infty, \\ (-\infty, 0] & \text{if } l = +\infty. \end{cases} \tag{11}$$

Next we proceed to solve for the dominating claim for standard call options, put options, and digital options. Later we consider options on multiple assets and lookback options. Examples are given for American and European options.

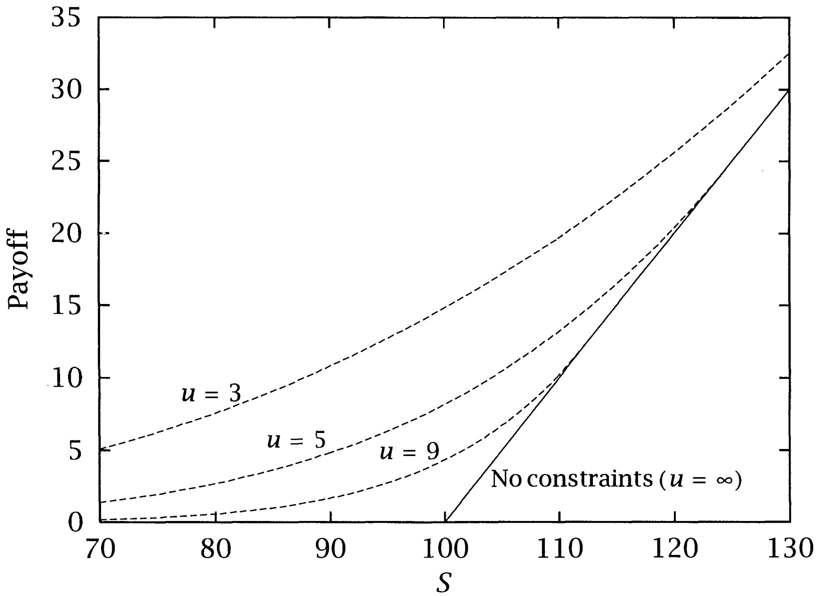

Figure 1
Standard call option
Illustration of $b(S) = (S - K)^+$ (with $K = 100$) and $\hat{b}(S)$ for several values of u.

2.1 Standard call option

The payoff function of a call option is $b(S) = (S - K)^+$. In this case short-selling constraints do not matter, that is, $\hat{b}$ does not depend on l. For $u < 1$, $\hat{b}(S) = \infty$, for $u = 1$, $\hat{b}(S) = S$, and for $u = \infty$, $\hat{b}(S) = b(S)$. For $1 < u < \infty$, ordinary calculus gives

$$\hat{b}(S) = \begin{cases} S - K & \text{if } S \geq \frac{Ku}{u-1}, \\ \frac{K}{u-1}\left(\frac{(u-1)S}{Ku}\right)^u & \text{if } S < \frac{Ku}{u-1}. \end{cases} \tag{12}$$

Figure 1 illustrates $\hat{b}$ for several values of u. As the borrowing constraint is tightened, that is, as u decreases, $\hat{b}(S)$ increases, and so does the seller's cost. For fixed u, the effect of the constraint decreases as the option moves in-the-money, that is, as S increases beyond K. This is reasonable, since replicating an in-the-money option requires less leverage than an at-the-money option.

Figure 2 illustrates how the delta of the portfolio strategy varies for several values of u. Recall that the delta is the number of units of the asset in the super-replicating portfolio. From Equation (8), the delta is $\Delta(S_t) \triangleq \pi_t P(t, S_t)/S_t = P_S(t, S_t)$. For deep in-the-money options, the

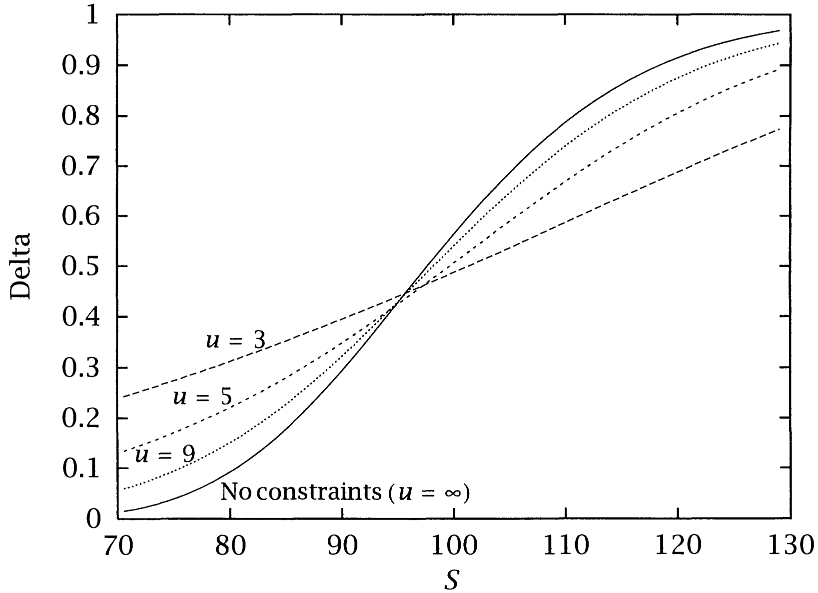

Figure 2
Standard call option
Illustration of $\Delta(S)$ for several values of u. The option parameters are $K = 100$, $r = 0.05$, $\sigma = 0.3$, and $T = 0.25$.

portfolio delta decreases as the borrowing constraint is tightened, that is, as u decreases. The reverse happens for deep out-of-the-money options. The results in Figure 2 are consistent with the observations in Naik and Uppal (1994). The gamma of the portfolio is defined by $\Gamma(S_t) \triangleq \partial\Delta(S_t)/\partial S_t$. Figure 3 shows how the gamma of the portfolio varies with S and u. For a large range of asset prices S around the strike K, the gamma decreases as the borrowing constraint is tightened. However, the pattern reverses for deep out-of-the-money and in-the-money options. We will return to the gamma of the portfolio in the next section.

In Naik and Uppal (1994), for the limited borrowing European call case in a discrete-time framework, the authors find a "critical stock price" boundary, that is, a curve below which the constraint is binding and above which it is not. Here, in continuous time, the constraint is never binding for $t < T$. Indeed, using Equation (12) one can check that $S\frac{\partial \hat{b}}{\partial S}(S)/\hat{b}(S) \leq u$, for all $S > 0$, with strict inequality for some S. Then using an argument similar to the one after the proof of Theorem 2 in the Appendix, the strong maximum principle implies that portfolio process π_t of Equation (8) satisfies $\pi_t < u$ for $t < T$.

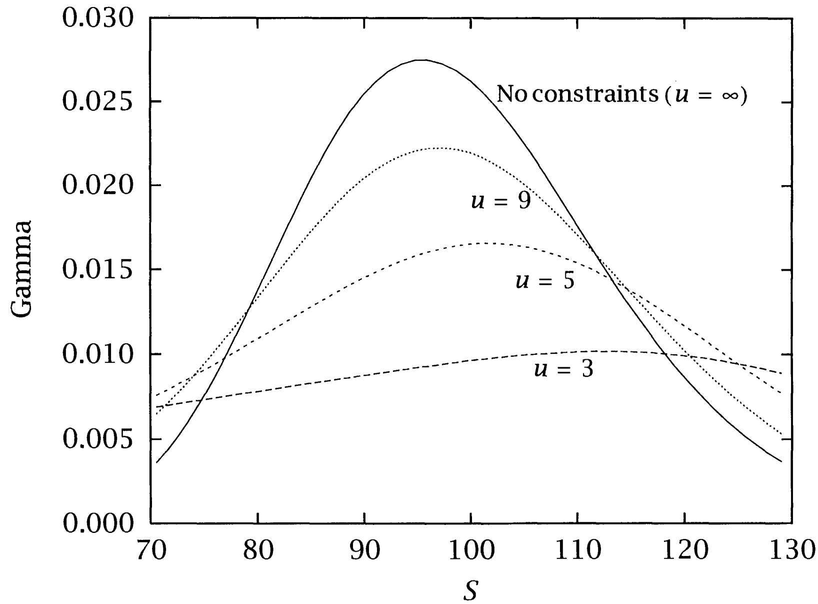

Figure 3
Standard call option
Illustration of $\Gamma(S)$ for several values of u. The option parameters are $K = 100$, $r = 0.05$, $\sigma = 0.3$, and $T = 0.25$.

Numerical results for European call options are given in Table 1. To be specific, consider the case with $K = 100$ and $\sigma = 0.3$. In the unconstrained case, $u = \infty$, the option is initially worth 9.635, the delta is 0.589, and $\pi_0^1 = 6.11$. If π_t^1 is restricted to 20 or less, the minimum super-replication cost rises to 9.863, restricted to 10 or less the cost rises to 10.509. In the case $\pi^1 \leq 1$, the seller must hold the stock alone and the minimum cost for super-replication is $S_0 = 100$. Table 1 shows that constraints have a greater relative and absolute impact for lower volatilities than higher volatilities. This observation is consistent with Figure 1, which shows that $\hat{b}$ is significantly different from b for S near K, while $\hat{b}$ is equal or approximately equal to b for $S \gg K$ and $S \ll K$. When S_0 is near K, the probability that S_T will be close to K is larger under low volatility than high volatility. Hence low volatility leads to terminal asset prices where $\hat{b}$ differs from b, that is, the impact of constraints tends to be greater when volatility is low.

2.2 Standard put option

The payoff function of a put option is $b(S) = (K - S)^+$. In this case borrowing constraints do not matter, that is, $\hat{b}$ does not depend on u. For $l = \infty$, $\hat{b} = b$ and for $l = 0$, $\hat{b} = K$. For $0 < l < \infty$, the

Table 1
Minimum super-replication cost for European options

K	u	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
	∞	13.499 (0.840)	15.486 (0.764)	17.763 (0.727)
	20	13.681 (0.827)	15.656 (0.759)	17.907 (0.724)
90	10	14.155 (0.796)	16.119 (0.745)	18.319 (0.717)
	5	15.963 (0.707)	17.812 (0.698)	19.859 (0.691)
	2	29.056 (0.581)	29.790 (0.595)	30.817 (0.613)
	1	100.000 (1.000)	100.000 (1.000)	100.000 (1.000)
	∞	6.889 (0.598)	9.635 (0.589)	12.385 (0.591)
	20	7.211 (0.591)	9.863 (0.587)	12.560 (0.591)
100	10	8.058 (0.571)	10.509 (0.579)	13.076 (0.588)
	5	10.891 (0.522)	12.821 (0.553)	15.021 (0.574)
	2	26.151 (0.523)	26.812 (0.536)	27.756 (0.554)
	1	100.000 (1.000)	100.000 (1.000)	100.000 (1.000)
	∞	2.906 (0.335)	5.587 (0.411)	8.370 (0.457)
	20	3.255 (0.343)	5.831 (0.414)	8.555 (0.459)
110	10	4.247 (0.353)	6.560 (0.418)	9.124 (0.462)
	5	7.526 (0.372)	9.218 (0.424)	11.322 (0.465)
	2	23.773 (0.475)	24.375 (0.487)	25.239 (0.504)
	1	100.000 (1.000)	100.000 (1.000)	100.000 (1.000)

Option parameters: $S_0 = 100$, $r = 0.05$, and $T = 0.5$. Δ given in parentheses.

dominating claim is given by

$$\hat{b}(S) = \begin{cases} K - S & \text{if } S \leq \frac{Kl}{l+1}, \\ \frac{K}{l+1}\left(\frac{Kl}{(l+1)S}\right)^l & \text{if } S > \frac{Kl}{l+1}. \end{cases} \tag{13}$$

2.3 Digital call option

A digital call option pays \$$D$ at maturity if $S_T > K$ and zero otherwise. Its payoff function can be written compactly as $b(S) = D\mathbf{1}_{\{S>K\}}$. For all $0 \leq l \leq \infty$ and $0 \leq u < \infty$, the dominating claim is given by

$$\hat{b}(S) = \begin{cases} D & \text{if } S > K, \\ D\left(\frac{S}{K}\right)^u & \text{if } S \leq K. \end{cases} \tag{14}$$

Numerical results for four different types of options are given in Table 2. For American calls to have value in excess of their European counterparts, a constant dividend rate of $\delta = 10\%$ is used. For standard European calls, setting $u = 40$ increases the super-replicating cost relative to the unconstrained case by a few cents. But for European digital calls, the effect on the price is much larger, for example, over 50 cents when $\sigma = 20\%$. Even mild constraints on the replicating portfolio can have significant price implications for digital options. For European digital options in the extreme case of $u = 0$, the optimal strategy is to invest De^{-rT} in the riskless bond. For American digital

Table 2
Minimum replication cost for American calls, European and American digital calls

Option	u	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
	∞	1.567 (0.207)	3.800 (0.307)	6.307 (0.369)
	40	1.636 (0.211)	3.854 (0.308)	6.349 (0.370)
Standard	20	1.860 (0.220)	4.023 (0.312)	6.482 (0.372)
European	10	2.763 (0.245)	4.715 (0.322)	7.029 (0.378)
call	5	5.874 (0.292)	7.304 (0.344)	9.187 (0.390)
	2	21.511 (0.430)	22.056 (0.441)	22.839 (0.457)
	1	95.123 (0.951)	95.123 (0.951)	95.123 (0.951)
	∞	1.637 (0.219)	3.923 (0.320)	6.471 (0.382)
	40	1.699 (0.222)	3.971 (0.321)	6.509 (0.383)
Standard	20	1.912 (0.230)	4.129 (0.324)	6.632 (0.384)
American	10	2.792 (0.250)	4.794 (0.331)	7.155 (0.388)
call	5	5.878 (0.293)	7.336 (0.348)	9.259 (0.397)
	2	22.727 (0.455)	22.727 (0.455)	22.841 (0.457)
	1	100.000 (1.000)	100.000 (1.000)	100.000 (1.000)
	∞	1.740 (0.018)	2.442 (0.015)	2.784 (0.012)
	40	2.256 (0.021)	2.834 (0.016)	3.091 (0.012)
European	20	2.851 (0.022)	3.255 (0.016)	3.413 (0.013)
digital	10	3.981 (0.022)	4.080 (0.016)	4.055 (0.013)
call	5	5.558 (0.019)	5.369 (0.015)	5.143 (0.012)
	2	7.516 (0.012)	7.241 (0.010)	6.940 (0.008)
	1	8.505 (0.007)	8.302 (0.006)	8.074 (0.005)
	0	9.753 (0.000)	9.753 (0.000)	9.753 (0.000)
	∞	4.111 (0.045)	5.796 (0.037)	6.712 (0.031)
	40	4.222 (0.044)	5.836 (0.037)	6.730 (0.030)
American	20	4.506 (0.043)	5.947 (0.036)	6.782 (0.030)
digital	10	5.226 (0.039)	6.283 (0.033)	6.958 (0.028)
call	5	6.399 (0.031)	6.970 (0.028)	7.384 (0.025)
	2	8.264 (0.017)	8.264 (0.017)	8.279 (0.016)
	1	9.091 (0.009)	9.091 (0.009)	9.091 (0.009)
	0	10.000 (0.000)	10.000 (0.000)	10.000 (0.000)

Option parameters: $S_0 = 100$, $K = 110$, $r = 0.05$, $\delta = 0.10$, and $T = 0.5$. Δ given in parentheses. For the digital options $D = 10$.

calls with $u = 0$, an amount equal to the initial asset price S_0 is invested in the riskless bond.

2.4 Options on multiple assets

With multiple assets there are many economically reasonable sets of constraints. We focus on two types of constraints in the case $d = 2$. First, suppose that borrowing is restricted. This can be modeled by taking $C_1 = \{\pi \in \mathbb{R}^2 : \pi^1 + \pi^2 \leq u\}$. Second, we consider bounds on the first asset only, that is, $C_2 = \{\pi \in \mathbb{R}^2 : \pi^1 \leq u\}$. The payoff of a call option on the maximum of two asset prices is $b(S^1, S^2) = (\max(S^1, S^2) - K)^+$. With constraints C_1 and $u > 1$, it can be shown

Table 3
Minimum replication cost for European max options on two assets

Constraints	u	$\sigma = 0.2$	$\sigma = 0.4$
	∞	11.65 (0.44, 0.44)	21.56 (0.46, 0.46)
	20	11.90 (0.43, 0.43)	21.72 (0.46, 0.46)
C_1	10	12.55 (0.41, 0.41)	22.18 (0.45, 0.45)
	5	14.96 (0.35, 0.35)	23.86 (0.43, 0.43)
	2	30.29 (0.30, 0.30)	36.37 (0.36, 0.36)
	1	107.97 (0.54, 0.54)	115.85 (0.58, 0.58)
	∞	11.65 (0.44, 0.44)	21.56 (0.46, 0.46)
	20	11.80 (0.44, 0.43)	21.65 (0.46, 0.46)
C_2	10	12.25 (0.43, 0.41)	21.93 (0.46, 0.45)
	5	14.02 (0.41, 0.34)	23.09 (0.45, 0.43)
	2	26.78 (0.49, 0.09)	32.36 (0.47, 0.28)
	1	100.00 (1.00, 0.00)	100.23 (0.99, 0.02)

Option parameters: $S_0^1 = S_0^2 = 100$, $K = 100$, $r = 0.05$, $\sigma_1 = \sigma_2 = \sigma$, $\rho = 0.0$, and $T = 0.5$. (Δ^1, Δ^2) given in parentheses.

that the dominating claim is given by

$$\hat{b}(S^1, S^2) = \begin{cases} \max(S^1, S^2) - K & \text{if } \max(S^1, S^2) \geq \frac{Ku}{u-1}, \\ \frac{K}{u-1}\left(\frac{(u-1)\max(S^1,S^2)}{Ku}\right)^u & \text{if } \max(S^1, S^2) < \frac{Ku}{u-1}. \end{cases} \quad (15)$$

With constraints C_2 and $u > 1$, the dominating claim is given by

$$\hat{b}(S^1, S^2) = \begin{cases} S^1 - K & \text{if } S^1 \geq \frac{Ku}{u-1},\ S^1 \geq S^2, \\ S^2 - K & \text{if } S^1 \geq \frac{Ku}{u-1},\ S^2 \geq S^1, \\ \frac{K}{u-1}\left(\frac{(u-1)S^1}{Ku}\right)^u & \text{if } S^1 < \frac{Ku}{u-1},\ S^1 \geq S^2, \\ \max\left(S^2 - K, \frac{K}{u-1}\left(\frac{(u-1)S^1}{Ku}\right)^u\right) & \text{if } S^1 < \frac{Ku}{u-1},\ S^2 \geq S^1. \end{cases} \quad (16)$$

Numerical results for European max options on $d = 2$ assets are given in Table 3. Since $C_1 \subset C_2$, the minimum replication cost is higher with constraints C_1 compared to C_2. As with the other examples, the absolute and relative impact of the constraints is greater at lower volatilities than at high volatilities. With constraints C_1 and $u = 1$, the option cost equals the common value of the initial asset prices plus the value of an exchange option [see Margrabe (1978)].

The examples in this section illustrate both the generality and ease of applicability of Theorem 2 and Corollary 1. As we saw in Figure 3, the gamma of the replicating portfolio typically decreases as the portfolio constraints are tightened. In the next section we consider the problem of directly constraining the gamma of the super-replicating portfolio.

3. Gamma Bounds

In this section we set $d = 1$ for simplicity and consider only European claims of the form $b(S(T))$. It is often of practical interest to have some bounds on the "gamma" of the hedging portfolio, namely $P_{SS}(\cdot)$. This is because if gamma is too large, so is the trading volume. A conservative way of approaching the problem is to notice that function $(t, S) \mapsto S^2 P_{SS}(t, S) - gS^2 e^{-t(\sigma^2+r)}$ also solves the Black–Scholes equation (here, g is a positive constant). Therefore, if the terminal condition $\hat{b}(S)$ has a second derivative in S bounded above by $\tilde{g} = ge^{-T(\sigma^2+r)}$, the corresponding gamma will satisfy $P_{SS}(t, S) \leq ge^{-t(\sigma^2+r)}$, for any $0 \leq t \leq T$ and $S > 0$ by the maximum principle. In particular, gamma will satisfy $P_{SS}(t, S) \leq g$. However, this is not necessarily the least expensive way of bounding gamma from above. In order to find the least expensive way, one would have to find the function that always satisfies Black–Scholes PDE as inequality, with equality if $P_{SS}(t, S) < g$. In other words, this is related to an American option problem, with the condition $P_{SS} \leq g$. Finding an analytical solution to this problem seems quite difficult. Instead we illustrate the conservative approach described above. More precisely, if the payoff is given by $b(S)$, we look for the minimal function $\hat{b}$ dominating b and having a prescribed bound on the second derivative, if such exists. We show that this indeed will be the case, if we only prescribe an upper bound on gamma. Roughly speaking, this is because the second derivative of the minimum of two functions is smaller than the minimum of their second derivatives. This is not the case for a lower bound (although in many cases convexity will be preserved, and the lower bound will be zero).

What follows is a rough description of how to construct such $\hat{b}$. All the statements can be proved, under reasonable conditions, using standard arguments of optimal stopping [see, e.g., Oksendal (1992)]. Consider the following optimal stopping problem

$$\hat{b}(s) = \sup_{\tau} E^s[b(S(\tau))]$$

where τ is a stopping time on an infinite horizon, and $dS = SdW$, $S(0) = s$. Then $\hat{b}(s)$ is the smallest superharmonic function (i.e., satisfying $\hat{b}_{SS} \leq 0$), such that $\hat{b} \geq b$. Similarly, the smallest function $\hat{b} \geq b$ for which $\hat{b}_{SS} \leq \tilde{g}$ is the value function (under some conditions) of the optimal stopping problem

$$\hat{b}(s) = \sup_{\tau} E^s \left[b(S(\tau)) - \frac{\tilde{g}}{2} \int_0^{\tau} S^2(t)dt \right]$$

To solve this problem, one looks for the function for which $\hat{b}_{SS} = \tilde{g}$ in the continuation region, with inequality outside, and which dominates b. Typically the solution is of the quadratic form inside the continuation region, and equal to $b(s)$ on the boundary and outside, and one uses the smooth fit conditions to get the coefficients (matching the first derivatives).

Let us consider the case of the European call $b(S) = (S - K)^+$. Assume first that $K - 1/(2\tilde{g}) > 0$. We look for a pair (s_0, s_1), $s_0 \le K$, $s_1 \ge K$, and a function $\hat{b}(S)$, such that $\hat{b}(S) = aS^2 + bS + c$ on $[s_0, s_1]$, $\hat{b}(S) = b(S)$ for $S \le s_0$ and $S \ge s_1$, $\hat{b}'(s_i) = b'(s_i)$, $i = 0, 1$, and such that $\hat{b}''(S) = \tilde{g}$ on (s_0, s_1). It is straightforward to see that the unique function satisfying those conditions is described by

$$\hat{b}(S) = \frac{\tilde{g}}{2}\left(S - \left(K - \frac{1}{2\tilde{g}}\right)\right)^2, \quad \text{for } S \in \left[K - \frac{1}{2\tilde{g}},\ K + \frac{1}{2\tilde{g}}\right] \tag{17}$$

and is otherwise equal to $b(S)$. In the case $K - 1/(2\tilde{g}) < 0$, we don't have to worry about smooth fit at zero, so we draw a parabola going through origin and smoothly hitting $S - K$. The solution is

$$\hat{b}(S) = \frac{\tilde{g}}{2}S^2 + \left(1 - \sqrt{2K\tilde{g}}\right)S, \quad \text{for } S \le \sqrt{2K/\tilde{g}}, \tag{18}$$

and $\hat{b}(S) = S - K$ otherwise.

To recap, the function $\hat{b}$ given by Equation (17) or (18) defines a payoff which dominates $b(S)$ and whose Black–Scholes hedging strategy will have its gamma no greater than $\tilde{g}e^{(T-t)(\sigma^2+r)}$ at time t. In particular, the gamma of the hedging portfolio will not exceed $\tilde{g}e^{T(\sigma^2+r)} = g$. Similarly, if one prices an American option with the payoff (at time t) $\hat{b}(S)e^{(T-t)(\sigma^2+r)}$ such that $\hat{b}_{SS}(S) \le g$, the corresponding gamma will not exceed $ge^{(T-t)(\sigma^2+r)}$ at time t.

3.1 Example

Consider a European call option to be super-replicated by a portfolio whose gamma should not exceed $g = 0.01$. Suppose for illustration that $K = 100$, $r = 5\%$, $\sigma = 30\%$, and $T = 0.5$. Set $\tilde{g} = ge^{-T(\sigma^2+r)} = 0.00932$. Since $K - 1/(2\tilde{g}) > 0$, set $\hat{b}$ according to Equation (17). At the maturity of the option it is guaranteed that the portfolio gamma is bounded above by $\tilde{g} < g$. For $0 \le t \le T$ it is guaranteed that the gamma is bounded above by g. In fact, at $t = 0$ the maximum value of gamma is 0.00995, that is, slightly less than the desired bound of 0.01. See Figure 4 for an illustration.

This example illustrates several difficult features of the problem. Bounding the terminal gamma by g does not lead to a bound of g on the portfolio gamma for $t < T$. The conservative approach illustrated

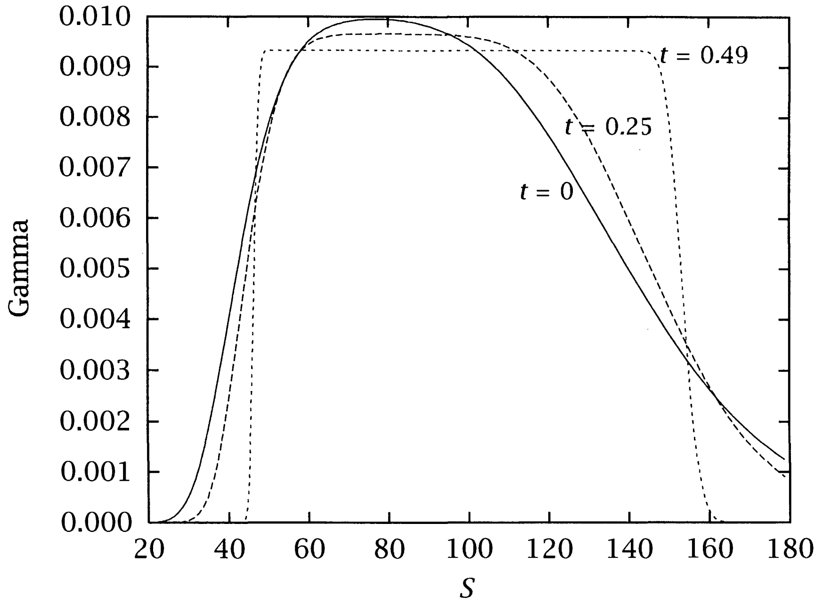

Figure 4
Illustration of gamma for the $\hat{b}$ option for several values of t
The European call option parameters are $K = 100$, $r = 0.05$, $\sigma = 0.3$, and $T = 0.5$.

here is clearly not optimal, since the portfolio gamma is bounded above by a constant that is strictly smaller than g. Finally, notice in Figure 4 that the maximum value of gamma depends on S, so an exact procedure would need to account for this dependence. One could discretize the problem and formulate a linear program to solve the discrete problem as in Naik and Uppal (1994).

In the previous example, the unconstrained Black–Scholes option value is \$9.635 and the option gamma is 0.0183 at time $t = 0$. Constraining the super-replicating portfolio gamma to 0.01 and applying the conservative approach outlined above leads to a replication cost of \$16.506. The effect on the option value is large because the terminal value of gamma with $S = K$ is infinite. Table 4 shows how the super-replication cost varies with the gamma constraint for a range of option parameters.

4. Extensions

In this section we briefly consider extensions of the results to path-dependent options. We also consider constraints on portfolio amounts and constraints on the number of units (e.g., shares) of each asset.

Table 4
Super-replication cost for European options with a constraint on gamma

K	g	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
	∞	13.499 (0.840)	15.486 (0.764)	17.763 (0.727)
	0.20	13.523 (0.838)	15.507 (0.764)	17.782 (0.726)
90	0.10	13.595 (0.834)	15.572 (0.762)	17.837 (0.726)
	0.05	13.879 (0.818)	15.824 (0.756)	18.059 (0.723)
	0.02	15.680 (0.740)	17.453 (0.721)	19.525 (0.706)
	0.01	20.582 (0.639)	22.028 (0.651)	23.865 (0.661)
	∞	6.889 (0.598)	9.635 (0.589)	12.385 (0.591)
	0.20	6.920 (0.597)	9.657 (0.589)	12.403 (0.591)
100	0.10	7.013 (0.597)	9.723 (0.589)	12.456 (0.591)
	0.05	7.375 (0.594)	9.983 (0.588)	12.666 (0.591)
	0.02	9.576 (0.575)	11.682 (0.584)	14.083 (0.592)
	0.01	15.006 (0.544)	16.506 (0.563)	18.451 (0.581)
	∞	2.906 (0.335)	5.587 (0.411)	8.370 (0.457)
	0.20	2.931 (0.336)	5.605 (0.411)	8.385 (0.458)
110	0.10	3.003 (0.339)	5.660 (0.412)	8.429 (0.458)
	0.05	3.291 (0.349)	5.876 (0.417)	8.607 (0.461)
	0.02	5.179 (0.399)	7.345 (0.440)	9.835 (0.474)
	0.01	10.360 (0.448)	11.872 (0.472)	13.864 (0.498)

Option parameters: $S_0 = 100$, $r = 0.05$, $\delta = 0.0$, and $T = 0.5$. Δ given in parentheses.

4.1 Path-dependent options

We first consider lookback options with payoffs that depend on the terminal asset price as well as the maximum or minimum price over a given time period. For example, the payoff of a lookback call option is $b(S_T, S_-) = S_T - S_-$, where $S_- \triangleq \min_{0 \leq t \leq T} S_t$. Theorem 2 is not directly applicable in this case. For these and other path-dependent options, there is an abstract formula corresponding to Equation (4); see Cvitanić and Karatzas (1993) or Karatzas and Kou (1996). It is easier in this case to use the PDE arguments indicated in the Appendix, if one knows the PDE of the option in the unconstrained case. In the lookback call case, it can be checked that

$$\hat{b}(S, y) = \sup_{\nu \leq 0} \left\{ [Se^{-\nu} - \min(y, Se^{-\nu})] e^{-\delta(\nu)} \right\}, \tag{19}$$

where in fact $\min(y, Se^{-\nu}) = y$ on the domain $y \leq S$ and for $\nu \leq 0$. That this is the lower bound follows from an argument such as the one of the proof of Theorem 2, part (ii). To see that the value of this bound can be replicated, one can use the maximum principle for PDEs again, and the fact that the corresponding value function here is a function of (S, y) defined on $S \geq y$, and the PDE is again the Black–Scholes PDE in the S variable for each y (with Neumann boundary conditions), and the replicating portfolio is given by Equation (8) [see Wilmott, Dewynne, and Howison (1993)].

Table 5
Minimum replication cost for European lookback call options

u	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
∞	11.95 (0.12)	16.91 (0.17)	21.65 (0.22)
20	12.10 (0.12)	17.01 (0.17)	21.72 (0.22)
10	12.57 (0.13)	17.33 (0.17)	21.96 (0.22)
5	14.66 (0.15)	18.78 (0.19)	23.05 (0.23)
2	28.64 (0.29)	30.67 (0.31)	32.99 (0.33)
1	100.00 (1.00)	100.00 (1.00)	100.00 (1.00)

Option parameters: $S_0 = 100$, $r = 0.05$, and $T = 0.5$. Δ given in parentheses.

If $u = \infty$, Equation (19) gives $\hat{b} = b$. If $1 < u < \infty$, $\hat{b}$ is of the form as in Equation (12), with K replaced by y. Numerical results are given in Table 5.

Unfortunately, although we were "lucky" in the case of lookback options, it seems unlikely that there is a general result for path-dependent options, and so the analysis has to be done on a case-by-case basis. The reason is partly because the PDEs for the pricing function differ from one path-dependent option to another. To obtain results on the seller's cost with constraints, one would have to find a corresponding PDE for the price without constraints and then check whether there is a (minimal) way of modifying boundary conditions so that the constraints become satisfied. It has recently been shown in Wystup (1997) that this is possible for barrier options.

4.2 Constraints on portfolio amounts

Suppose instead of constraints on portfolio weights, that there are constraints on portfolio amounts, described by set C. In this case it is possible to show, using methods of Cvitanić and Karatzas (1993), that the (dominating) value process for the seller is given by

$$P(t, S_t) = \sup_{\nu \in \mathcal{D}} E\left[b(S_T(\nu))e^{-r(T-t)} - \int_t^T e^{-rs}\delta(\nu_s)ds \,\middle|\, S_t(\nu) = S_t \right]. \tag{20}$$

This corresponds to the terminal payoff

$$\hat{b}(S) = \sup_{\nu \in \tilde{C}}\{b(Se^{-\nu}) - \delta(\nu)\}.$$

Again, by similar arguments as in the case with constrained portfolio weights (provided in the Appendix), the Black–Scholes price of this payoff gives the minimal seller's cost of super-replication.

In case $d = 1$, with constraints $C = [-l, u]$ and $u < \infty$, for those payoffs for which $sb_s(s) \to \infty$ as $s \to \infty$, we have $\hat{b}(S) \equiv \infty$, that is, super-replication is impossible. On the other hand, for example, for the put option $b(S) = (K - S)^+$, we get $\hat{b}(S) = b(S)$ if $K \leq l$ and, if $l < K$,

$$\hat{b}(S) = \begin{cases} K - S & \text{if } S \leq l, \\ K - l + l\log(l/S) & \text{if } l < S < le^{K/l-1}, \\ 0 & \text{if } S > le^{K/l-1}. \end{cases}$$

4.3 Constraints on the number of shares

Assume for simplicity that $d = 1$. Requiring that the number of shares of the stock in the hedging portfolio takes values in C at time t is equivalent to requiring that the portfolio amounts take values in $S_t \cdot C$. Since the results of Cvitanić and Karatzas (1993) extend to random constraint sets, denoting by $\delta(\nu, S_t)$ the (random) support function of the set $-S_t \cdot C$, we get, from Equation (20)

$$P(t, S_t) = \sup_{\nu \in \mathcal{D}} E\left[b(S_T(\nu))e^{-r(T-t)} - \int_t^T e^{-rs}\delta(\nu_s, S_s(\nu))ds \,\middle|\, S_t(\nu) = S_t \right].$$

In case $d = 1$ we have $\delta(\nu, S) = \delta(\nu)S$ and, as in the Appendix, by taking limits as $t \to T$, it is seen that the terminal payoff has to be

$$\hat{b}(S) = \sup_{\nu \in \tilde{C}} \left\{ b(Se^{-\nu}) - \delta(\nu)S\frac{1 - e^{-\nu}}{\nu} \right\}.$$

Again, the seller's price is given by the Black–Scholes price of $\hat{b}(S)$.

The example of European call, $b(S) = (S - K)^+$, with constraints $C = [-l, u]$, is not very interesting. If $u < 1$ then super-replication is impossible, and if $u \geq 1$ the constraint is redundant, that is, $\hat{b}(S) = b(S)$. In the case of the put, $b(S) = (K - S)^+$, the constraint is redundant if $l \geq 1$, and if $l < 1$, the dominating payoff is another put-like payoff, $\hat{b}(S) = (K - lS)^+$. Formally speaking, we are fitting the smallest piecewise linear function $\hat{b}(S)$ dominating $b(S)$, and with the slope between $-l$ and u.

Appendix

Proof of Theorem 2

(i) We first show, using results from Cvitanić and Karatzas (1993), that portfolio π that replicates $\hat{b}(S_T)$, also satisfies the constraints. Let $\nu \in \mathcal{D}$ and observe that, from the properties of the support function and the cone property of $\tilde{C}$,

$$\hat{\hat{b}} = \hat{b}$$

$$\int_t^T \delta(\nu_s)ds \geq \delta\left(\int_t^T \nu_s ds\right),$$

$$\int_t^T \nu_s ds \text{ is an element of } \tilde{C},$$

where $\int_t^T \nu_s ds \triangleq (\int_t^T \nu_s^1 ds, \ldots, \int_t^T \nu_s^d ds)'$. Moreover, we have

$$S_T^i(\nu) = S_T^i(0)e^{-\int_t^T \nu_s^i ds},$$

because the processes on the left-hand side and the right-hand side satisfy the same linear SDE. Then, for every $\nu \in \mathcal{D}$, we have (setting $t = 0$ without loss of generality)

$$\begin{aligned} E[\hat{b}(S_T(\nu))e^{-\int_0^T (r+\delta(\nu_s))ds}] &\leq E[\hat{b}(S_T(0)e^{-\int_0^T \nu_s ds})e^{-\delta(\int_0^T \nu_s ds)}e^{-rT}] \\ &\leq E[\sup_{\nu\in\tilde{C}} \hat{b}(S_T(0)e^{-\nu})e^{-\delta(\nu)}e^{-rT}] \\ &= E[\hat{\hat{b}}(S_T(0))e^{-rT}] \\ &= E[\hat{b}(S_T(0))e^{-rT}]. \end{aligned} \tag{21}$$

Therefore the supremum (over $\mathcal{D}$) of the initial expression is obtained for $\nu = 0$. Similarly for conditional expectations of Equation (4). Now it follows from Theorems 6.6 and 6.7 in Cvitanić and Karatzas (1993) that $\hat{b}(S_T)$ can be replicated by a portfolio that satisfies the constraints. Moreover, under Feynman–Kac assumptions, its value function is the solution to Equations (6) and (7), and the portfolio is given by Equation (8).

(ii) To conclude we have to show that to hedge $b(S_T)$ we have to hedge at least $\hat{b}(S_T)$. Denote by $P(t, S_t; b)$ the value function corresponding to claim $b(S_T)$, that is, $P(t, S_t; b) = P(t, S_t)$ of Equation (4).

It remains to prove that the left limit of $P(t, S; b)$ at $t = T$ is larger than $\hat{b}(S_T)$. For this, let $\{\nu^k\}$ be the maximizing sequence in the cone $\tilde{C}$ attaining $\hat{b}(S)$, that is, such that $b(Se^{-\nu^k})e^{-\delta(\nu^k)}$ converges to $\hat{b}(S)$ as k goes to infinity. Then, using (for fixed $t < T$) constant deterministic controls $\nu^k/(T-t)$ in Equation (4), we get

$$P(t, S; b) \geq E\left[b(S_T(0)e^{-\nu^k})e^{-\delta(\nu^k)}e^{-r(T-t)} \middle| \, S_t(0) = S\right],$$

and hence by lower semicontinuity of b

$$\liminf_{t \to T} P(t, S; b) \geq b(Se^{-\nu^k})e^{-\delta(\nu^k)}.$$

Now letting k increase to infinity finishes the proof. ■

Here is a sketch of a PDE proof for part (i) in the proof above: Let P be the solution to Equations (6) and (7). For a given $\nu \in \tilde{C}$, consider the function $W_\nu = (SP_S)'\nu + \delta(\nu)P = \sum_{i=1}^d S^i P_{S^i} \nu^i + \delta(\nu)P$, where P_S is the vector of partial derivatives of P with respect to S^i, $i = 1, \ldots, d$. By Theorem 13.1 in Rockafellar (1970), to prove that portfolio π of Equation (8) takes values in C, it is sufficient (and necessary) to prove that W_ν is nonnegative for all $\nu \in \tilde{C}$. It is not difficult to see (assuming enough smoothness) that W_ν solves the PDE [Equation (6)], too. Moreover, it is also possible to check that the vector $S\hat{b}_S(S)$ belongs to the set $\hat{b}(S) \times C \triangleq \{x \in \mathbb{R}^d : x = \hat{b}(S)c; c \in C\}$, which implies $W_\nu(S, T) \geq 0$. So, by the maximum principle, $W_\nu \geq 0$ everywhere.

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