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The Effect of Jumps and Discrete Sampling on Volatility and Variance Swaps*

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Abstract

We investigate the effect of discrete sampling and asset price jumps on fair variance and volatility swap strikes. Fair discrete volatility strikes and fair discrete variance strikes are derived in different models of the underlying evolution of the asset price: the Black-Scholes model, the Heston stochastic volatility model, the Merton jump-diffusion model and the Bates and Scott stochastic volatility and jump model. We determine fair discrete and continuous variance strikes analytically and fair discrete and continuous volatility strikes using simulation and variance reduction techniques and numerical integration techniques in all models. Numerical results show that the well-known convexity correction formula may not provide a good approximation of fair volatility strikes in models with jumps in the underlying asset. For realistic contract specifications and model parameters, we find that the effect of discrete sampling is typically small while the effect of jumps can be significant.

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1 Introduction

Volatility and variance swaps are forward contracts in which one counterparty agrees to pay the other a notional amount times the difference between a fixed level and a realized level of variance and volatility, respectively. The fixed level is called the variance strike for variance swaps and the volatility strike for volatility swaps. This is typically set initially so that the net present value of the payoff is zero. Realized variance, or the floating leg of the swap, is determined by the average variance of the asset over the life of the swap.

The variance swap payoff is defined as

$$(V_d(0,n,T) - K_{var}(n)) \times N$$

where $V_d(0,n,T)$ is the realized stock variance (as defined below) over the life of the contract, $[0,T]$, n is the number of sampling dates, $K_{var}(n)$ is the variance strike, and N is the notional amount of the swap in dollars. The holder of a variance swap at expiration receives N dollars for every unit by which the stock's realized variance $V_d(0,n,T)$ exceeds the variance strike $K_{var}(n)$. The variance strike is quoted as volatility squared, e.g., $(20\%)^2$.

The volatility swap payoff is defined as

$$(\sqrt{V_d(0,n,T)} - K_{vol}(n)) \times N$$

where $\sqrt{V_d(0,n,T)}$ is the realized stock volatility (quoted in annual terms as defined below) over the life of the contract, n is the number of sampling dates, $K_{vol}(n)$ is the volatility strike, and N is the notional amount of the swap in dollars. The volatility strike $K_{vol}(n)$ is typically quoted as volatility, e.g., 20%. The procedure for calculating realized volatility and variance is specified in the contract and includes details about the source and observation frequency of the price of the underlying asset, the annualization factor to be used in moving to an annualized volatility and the method of calculating the variance.

Let $0 = t_0 < t_1 < ... < t_n = T$ be a partition of the time interval $[0,T]$ into n equal segments of length $\triangle t$, i.e., $t_i = iT/n$ for each $i = 0,1,...,n$. Most traded contracts define the realized variance to be

$$V_d(0,n,T) = \frac{AF}{m}\sum_{i=0}^{n-1}\left(\ln\left(\frac{S_{i+1}}{S_i}\right)\right)^2 \tag{1}$$

for a swap covering n return observations. Typically contracts m is set to $n-1$, though n is sometimes used. Here S_i is the price of the asset at the i^{th} observation time t_i and AF is the annualization factor, e.g., 252 ($= n/T$) if the maturity of the swap, T, is one year with daily sampling. This definition of realized variance differs from the usual sample variance because the sample average is not subtracted from each observation. Since the sample average is approximately zero, the realized variance is close to the sample variance.

Demeterfi, Derman, Kamal and Zou (1999) examined properties of variance and volatility swaps. They derived an analytical formula for the variance strike in the presence of a volatility skew. Brockhaus and Long (2000) provided an analytical approximation for the pricing of volatility swaps. Javaheri, Wilmott and Haug (2002) discussed the valuation of volatility swaps in the GARCH(1,1) stochastic volatility model. They used a partial differential equation approach to determine the first two moments of the realized variance and then used a convexity approximation formula to price the volatility swaps. Little and Pant (2001) developed a finite difference method for the valuation of variance swaps in the case of discrete sampling in an extended Black-Scholes framework. Detemple and Osakwe (2000) priced European and American options on spot volatility when volatility follows a diffusion process. Carr, Geman, Madan and Yor (2005) priced options on realized variance by directly modeling the quadratic variation of the underlying asset using a Lévy process. Carr and Lee (2005) priced arbitrary payoffs of realized variance under a zero correlation assumption between the stock price process and variance process. Lipton (2000) priced volatility swaps using a partial differential equation approach. Sepp (2008) priced options on realized variance in the Heston stochastic volatility model by solving a partial differential equation. Buehler (2006) proposed a general approach to model a term structure of variance swaps in an HJM-type framework. Meddahi (2002) presented quantitative measures of the realized volatility in the eigenfunction stochastic volatility model for different sampling sizes.

The analysis in most of these papers is based on an idealized contract where realized variance and volatility are defined with continuous sampling, e.g., a continuously sampled realized variance, $V_c(0,T)$, defined by:

$$V_c(0,T) \equiv \lim_{n\to\infty} V_d(0,n,T) \tag{2}$$

In this paper we analyze the differences between actual contracts based on discrete sampling and idealized contracts based on continuous sampling. Another objective of this paper is to analyze the effect of ignoring jumps in the underlying on fair variance swap strikes.

Financial models typically specify the dynamics of the stock price and variance using stochastic differential equations (SDE) and discrete and continuous realized variance depend on the modeling assumptions. The Black-Scholes model proposed in the early 1970s assumes that a stock price follows geometric Brownian motion with a constant volatility term. This constant volatility assumption is not typically satisfied by options trading in the market and subsequently many different models have been proposed. Merton (1973) extended the constant volatility assumption in Black-Scholes model to a term structure of volatility, i.e., $\sigma = \sigma(t)$. Derman and Kani (1994) and Derman, Kani and Zou (1996) extended this to local volatility models where volatility is a function of two parameters, time and the current level of the underlying, i.e., $\sigma = \sigma(t, S(t))$. Several models have been developed where volatility is modeled as a stochastic process often including mean reversion. Hull and White (1987) proposed a lognormal model for the variance process with independence between the driving Brownian motions of the stock price and variance processes. Heston (1993) proposed a mean reverting model for variance that allows for correlation between volatility and the asset level. Stein and Stein (1991) and Schobel and Zhu (1999) proposed a stochastic volatility model in which volatility of underlying asset follows Ornstein-Uhlenbeck process. Bates (1996) and Scott (1997) proposed a stochastic volatility with jumps model by adding log-normal jumps in stock price process in the Heston stochastic volatility model.

Continuous realized variance depends on the model assumed for the underlying asset price. Depending on the model, discrete realized variance and continuous realized variance can be different. The fair strike of a variance swap (with discrete or continuous sampling) is defined to be the strike which makes the net present value of the swap equal to zero. We call it the fair variance strike. The fair discrete volatility strike and fair discrete variance strikes are defined similarly. In this paper we analyze discrete variance swaps and continuous variance swaps and the effect of the number of sampling dates on fair variance strikes and fair volatility strikes. Various authors have proposed to replicate a variance swap using a static portfolio of out-of-the-money call and put options. This ignores the effect of jumps in the underlying. The fair variance swap strike will differ from the static replicating portfolio of options if the underlying has jumps. In this paper we investigate the following questions:

- What is the effect of ignoring jumps in the underlying on fair variance swap strikes?
- What is the relationship between fair variance strikes and fair volatility strikes?
- How do fair variance strikes and fair volatility strikes vary in different models?

- What is the convergence rate of expected discrete realized variance to expected continuous realized variance with the number of sampling dates? Are fair discrete variance strikes and fair discrete volatility strikes with daily, weekly or monthly sampling significantly different than fair continuous variance strikes and fair continuous volatility strikes, respectively?

- How well does the convexity correction formula approximate fair volatility strikes?

In this paper, we analyze all these issues under four different models of underlying evolution of asset price: the Black Scholes model (BS), the Heston stochastic volatility model (SV), the Merton jump-diffusion model (J) and stochastic volatility with jumps model (SVJ).

The rest of the paper is organized as follows. We begin briefly by introducing volatility derivatives in section 2 and provide formulas to price these derivatives. Convergence rates of discrete variance strikes to continuous variance strikes in the Black-Scholes model are derived. In sections 3, 4 and 5, we present analysis for the models J, SV, and SVJ, respectively. In section 6 we present numerical results and concluding remarks are given in section 7.

2 Volatility derivatives

2.1 Variance swaps

In this section, we provide definitions of discretely sampled realized variance and continuously sampled realized variance and review how to replicate variance swaps when the stock price process is continuous. We assume the risk neutral dynamics of the underlying asset S_t are given by:

$$\frac{dS_t}{S_t} = rdt + \sigma_t dW_t^Q \tag{3}$$

where r is the risk free rate, W is a standard Brownian motion under the risk-neutral measure Q. We assume throughout in this paper that there exists a unique risk-neutral measure Q. The parameter σ_t represents the level of volatility. The standard Black-Scholes model assumes that this parameter is constant, while in stochastic volatility models σ_t is specified by another diffusion process, as in the Heston stochastic volatility model (SV) in section 4.

A variance swap is a forward contract on the realized variance of underlying security. The floating leg of variance swap is the realized variance and is calculated using the second moment of log returns of the underlying asset:

$$R_i = \ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right), i = 1, 2, ..., n$$

where $0 = t_0 < t_1 < ... < t_n = T$ is a partition of the time interval $[0, T]$ into n equal segments of length $\triangle t$, i.e., $t_i = iT/n$ for each $i = 0, 1, ..., n$. The discrete realized variance, $V_d(0, n, T)$, from equation (1) can be written as:

$$V_d(0, n, T) = \frac{1}{(n-1)\Delta t}\sum_{i=1}^{n} R_i^2 = \frac{\sum_{i=0}^{n-1}(\ln(\frac{S_{i+1}}{S_i}))^2}{(n-1)\Delta t} \tag{4}$$

The floating leg of the variance swap, or the discretely sampled realized variance, in the limit approaches the continuously sampled realized variance, $V_c(0, T)$, that is:

$$V_c(0, T) \equiv \lim_{n\to\infty} V_d(0, n, T) = \lim_{n\to\infty} \frac{n}{(n-1)T}\sum_{i=1}^{n} R_i^2 \tag{5}$$

Jacod and Protter (1998) provide necessary and sufficient conditions for the rate of convergence of the Euler scheme approximation of the solution to a stochastic differential equation to be $1/\sqrt{n}$. The discrete realized variance, $V_d(0, n, T)$, is the Euler scheme approximation of the stochastic differential equation (3) followed by underlying asset S_t when sampling size is n. Thus, the rate of convergence of discrete realized variance, $V_d(0, n, T)$, to continuous realized variance, $V_c(0, T)$, is $1/\sqrt{n}$.

In the case of the Black-Scholes model and the Heston stochastic volatility model, continuous realized variance is given by:[1]

$$V_c(0, T) = \frac{1}{T}\int_0^T \sigma_t^2 dt \tag{6}$$

Continuous realized variance can be replicated by a static position in a log contract (Demeterfi et al. 1999) and a dynamic trading strategy in the underlying asset. Applying Itô's lemma to equation (3) we get

$$\ln\left(\frac{S_T}{S_0}\right) = \int_0^T \left(r - \frac{1}{2}\sigma_t^2\right)dt + \int_0^T \sigma_t dW_t^Q \tag{7}$$

Subtracting equation (7) from equation (3) and rearranging we get

$$\frac{2}{T}\left(\int_0^T \frac{dS_t}{S_t} - \ln\frac{S_T}{S_0}\right) = \frac{1}{T}\int_0^T \sigma_t^2 dt = V_c(0, T) \tag{8}$$

[1]Equation (6) holds for asset price models following the dynamics in (3). When jumps are introduced the definition of $V_c(0, T)$ will be different.

Equation (8) shows that continuous realized variance can be replicated by a short position in the log contract and payoffs from a dynamic trading strategy which holds $1/S_t$ shares of the underlying stock at each instant of time t. In particular, equation (8) holds in the Black-Scholes model and the Heston stochastic volatility model.

Next, we give definitions of realized variance and accumulated variance with discrete sampling and continuous sampling. Continuous realized variance between time t and T is given by

$$V_c(t,T) = \frac{1}{T}\int_t^T \sigma_s^2 ds \tag{9}$$

Continuous accumulated variance from the start of the contract (time 0) until time t is defined by

$$I_c(t,T) = \frac{1}{T}\int_0^t \sigma_s^2 ds \tag{10}$$

From equations (9) and (10) we can write

$$V_c(0,T) = I_c(t,T) + V_c(t,T)$$

We define $P_c(t,T,K,I_c)$ to be the expected present value at time t of the payoff of a continuous variance swap with variance strike, K, i.e.,

$$P_c(t,T,K,I_c) = E_t^Q\Big(e^{-r(T-t)}\Big(I_c + V_c(t,T) - K\Big)\Big) \tag{11}$$

where the superscript Q denotes the risk-neutral measure and the subscript t denotes expectation at time t and $I_c = I_c(t,T)$. Throughout this paper expectation is always in the risk-neutral measure so we will drop the superscript. The fair continuous variance strike, K^*_{var}, is defined to be the strike such that the net present value of the swap at time $t = 0$ is zero, i.e.,

$$P_c(0,T,K^*_{var},I_c) = E_0\Big(e^{-rT}\Big(V_c(0,T) - K^*_{var}\Big)\Big) = 0 \tag{12}$$

where $I_c = I_c(0,T) = 0$. Solving (12) for K^*_{var} gives

$$K^*_{var} = E_0[V_c(0,T)] \;=\; E_0\left[\frac{1}{T}\int_0^T \sigma_s^2 ds\right] \tag{13}$$

The discrete realized variance between times $t_i = iT/n$ and T when there are n sampling dates between the start of contract at $t = 0$ and its maturity at $t = T$ is given by

$$V_d(i,n,T) = \frac{\sum_{j=i}^{n-1}(\ln(\frac{S_{j+1}}{S_j}))^2}{(n-1)\Delta t} \tag{14}$$

The discrete accumulated variance from the start of the contract, $t = 0$, until time t_i is

$$I_d(i, n, T) = \frac{\sum_{j=0}^{i-1} (\ln(\frac{S_{j+1}}{S_j}))^2}{(n-1)\Delta t} \tag{15}$$

From equations (14) and (15) we can write

$$V_d(0, n, T) = I_d(i, n, T) + V_d(i, n, T)$$

We define $P_d(i, n, T, K, I_d(i, n, T))$ to be the expected present value at time $t_i = iT/n$ of the payoff of a discrete variance swap with strike K and it is given by

$$P_d(i, n, T, K, I_d) = E_{t_i}\left(e^{-r(T-t_i)}\left(I_d + V_d(i, n, T) - K\right)\right) \tag{16}$$

where $I_d = I_d(i, n, T)$. The fair discrete variance strike, $K^*_{var}(n)$, is defined to be the strike such that the expected net present value of the swap at time $t = 0$ is zero. i.e.,

$$P_d(0, n, T, K^*_{var}(n), I_d) = E_0\left(e^{-rT}\left(V_d(0, n, T) - K^*_{var}(n)\right)\right) = 0 \tag{17}$$

where $I_d = I_d(0, n, T) = 0$.At time $t = 0$, $P_d(0, n, T, K, I_d)$ can be written as

$$\begin{aligned} P_d(0, n, T, K, I_d) &= P_c(0, T, K, I_c) + E_0\left(e^{-rT}\left(V_d(0, n, T) - V_c(0, T)\right)\right) \\ &= P_c(0, T, K, I_c) + e^{-rT}\left(K^*_{var}(n) - K^*_{var}\right) \end{aligned} \tag{18}$$

In above equation both I_d and I_c equals zero. We will use these definitions to show the linear convergence rate of $P_d(0, n, T, K, I_d)$ to $P_c(0, T, K, I_c)$.

2.2 Volatility swaps

The floating leg of a volatility swap on an asset S is the realized volatility of that asset's price. This volatility is commonly calculated using the square root of the realized variance defined in equation (4). The fair strike K^*_{vol} of a continuous volatility swap is set at the initiation of the contract so that the contract's net present value is equal to zero, i.e.,

$$E_0\left[e^{-rT}(\sqrt{V_c(0, T)} - K^*_{vol})\right] = 0 \tag{19}$$

Solving (19) for the fair continuous volatility strike, K^*_{vol}, we get

$$K^*_{vol} = E[\sqrt{V_c(0, T)}] = E_0\left[\sqrt{\frac{1}{T}\int_0^T \sigma_s^2 ds}\right]$$

Similarly, the fair discrete volatility strike is given by:

$$K^*_{vol}(n) = E_0\left[\sqrt{V_d(0, n, T)}\right] \tag{20}$$

2.3 Convexity correction formula

In this section we present the convexity correction formula (Brockhaus and Long 2000) to approximate fair volatility strikes. Then we present an argument to show why in some cases it may not provide an accurate approximation.

Jensen's inequality shows that the fair volatility strike is bounded above by the square root of the fair variance strike:[2]

$$K_{vol}^* = E_0[\sqrt{V_c(0,T)}] \le \sqrt{E_0[V_c(0,T)]} = \sqrt{K_{var}^*} \tag{21}$$

A similar result holds in the discrete case:

$$K_{vol}^*(n) = E_0[\sqrt{V_d(0,n,T)}] \le \sqrt{E_0[V_d(0,n,T)]} = \sqrt{K_{var}^*(n)} \tag{22}$$

Brockhaus and Long (2000) provide a convexity correction formula for calculating the fair volatility strike using a Taylor's expansion of the square root function. A second order Taylor's expansion of $f(x) = \sqrt{x}$ around x_0 gives

$$\sqrt{x} = \sqrt{x_0} + \frac{(x-x_0)}{2\sqrt{x_0}} - \frac{(x-x_0)^2}{8{x_0}^{\frac{3}{2}}} + f^{(3)}(\varepsilon)\frac{(x-x_0)^3}{3!} \tag{23}$$

where $f^{(3)}$ is the 3^{rd} derivative of function $f(x)$ for some ε in (x_0, x). The first three terms on the right hand side provide a good approximation of $\sqrt{x}$ for all values of x in the neighborhood of x_0 for which Taylor's series converges. For Taylor's series to converge, $x - x_0$ should lie in the radius of convergence, which for the square root function is

$$|x - x_0| \le x_0 \tag{24}$$

When this condition holds, the last term in equation (23) is bounded and the first three terms provide a good estimate to compute the value of function at a point, in this case $\sqrt{x}$. Now, substitute $x = V_c(0,T)$ and $x_0 = E[V_c(0,T)]$ in equation (23) to get:

$$\sqrt{V_c(0,T)} \approx \sqrt{E[V_c(0,T)]} + \frac{(V_c(0,T) - E[V_c(0,T)])}{2\sqrt{E[V_c(0,T)]}} - \frac{(V_c(0,T) - E[V_c(0,T)])^2}{8E[V_c(0,T)]^{\frac{3}{2}}} \tag{25}$$

[2]For the concave square root function Jensen's inequality is:

$$E(\sqrt{x}) \le \sqrt{E(x)}$$

The terms on the right hand side in equation (25) provide a good estimate of the square root of the realized variance $\sqrt{V_c(0,T)}$ on a single stock price path if the realized variance $V_c(0,T)$ satisfies the condition:

$$|V_c(0,T) - E(V_c(0,T))| \leq E(V_c(0,T)) \tag{26}$$

which can also be rewritten as

$$0 \leq V_c(0,T) \leq 2E(V_c(0,T)) \tag{27}$$

If condition (27) holds on all stock price paths under the risk-neutral measure then the right hand side of equation (25) provides a good estimate of square root of realized variance $\sqrt{V_c(0,T)}$ on all stock price paths. Taking expectations under the risk-neutral measure on both sides of (25) gives:

$$K^*_{vol} \approx \sqrt{K^*_{var}} - \frac{\mathrm{Var}(V_c(0,T))}{8E[V_c(0,T)]^{\frac{3}{2}}} \tag{28}$$

The convexity correction formula (28) can be used to approximate fair volatility strikes. The 2^{nd} order term in (28) is the convexity correction term. This approximation should work well if condition (27) holds on all sample paths. Noting that the excess probability

$$p \equiv P(V_c(0,T) \geq 2E(V_c(0,T))), \tag{29}$$

condition (27) is equivalent to the excess probability, p, being equal to zero. A discrete version of (28) can be used to approximate fair discrete volatility strikes and this approximation should work well if (27) is satisfied by the discrete realized variance.

When the excess probability (29) is not equal to zero then the higher order terms in the Taylor's expansion are not negligible compared to the first three terms in the expansion. If we include the 3^{rd} and 4^{th} order expansion terms in (28) we get

$$\begin{aligned} E[\sqrt{V_c(0,T)}] \approx \sqrt{E[V_c(0,T)]} &+ \frac{(V_c(0,T) - E[V_c(0,T)])}{2\sqrt{E[V_c(0,T)]}} - \frac{(V_c(0,T) - E[V_c(0,T)])^2}{8E[V_c(0,T)]^{\frac{3}{2}}} \\ &+ \frac{(V_c(0,T) - E[V_c(0,T)])^3}{16E[V_c(0,T)]^{\frac{5}{2}}} - \frac{5(V_c(0,T) - E[V_c(0,T)])^4}{128E[V_c(0,T)]^{\frac{7}{2}}} \end{aligned} \tag{30}$$

When p is not equal to zero then the higher moments of $V_c(0,T) - E(V_c(0,T))$ are not negligible.

In the Black-Scholes model, the excess probability is equal to zero in the continuous case and the higher moments of continuous realized variance are zero since volatility is constant. Hence, the convexity correction formula holds with equality in (28) in the Black-Scholes model and can be used to compute the fair continuous volatility strike.

In the discrete case, i.e., for a finite number of sampling dates n, the excess probability is not equal to zero and the higher moments of the discrete realized variance in the Black-Scholes model are not zero. The magnitude of the 3^{rd} and 4^{th} order terms are comparable to first two terms and the excess probability p is not zero with discrete sampling in the Black-Scholes model. Hence, the convexity correction formula (28) will not provide a good approximation of the fair volatility strike in the Black-Scholes model when the number of sampling dates n is small.

In the Heston stochastic volatility model, the excess probability p is not equal to zero, the 3^{rd} and 4^{th} order terms in equation (30) are not small and hence the convexity correction approximation will not provide a good estimate of the fair volatility strike. This is true in the Merton jump-diffusion model as well. Section 6 provides numerical results illustrating the computation of volatility strikes from the convexity correction formula, the 3^{rd} and 4^{th} order terms in equation (30) and the excess probability in all three models.

3 Merton jump-diffusion model

In this section, we present an analysis of the convergence of discrete variance strikes to continuous variance strikes with number of sampling dates in the Merton jump-diffusion (J) model. The risk-neutral dynamics of the jump-diffusion model are given by:

$$\frac{dS_t}{S_t^-} = (r - \lambda m)dt + \sigma dW_t^Q + dJ_t \tag{31}$$

where $J_t = \sum_{i=1}^{N_t}(Y_j - 1)$ and N_t is a Poisson process with rate λ and Y_j is the relative jump size in the stock price. When jump occurs at time τ_j, then $S(\tau_j^+) = S(\tau_j^-)Y_j$, where the distribution of Y_j is LN$[a, b^2]$ and m is the mean proportional size of jump $E(Y_j - 1) = m$. The parameters a and m are related to each other by the equation: $e^{a+\frac{1}{2}b^2} = m+1$ and only one of them needs to be specified. Results for the Black-Scholes model are given by setting the jump parameter, λ, to zero.

In the case of continuous sampling, realized variance consists of two components. The first is the accumulated variance contributed from the diffusive component of the underlying asset price process and second is the contribution from jumps. If there are $N(T)$ price jumps in $[0, T]$, the contribution to the realized variance from jumps is

$$\frac{1}{T}\left(\sum_{i=1}^{N(T)}(\ln(Y_i))^2\right)$$

Thus, the continuous realized variance in the Merton jump-diffusion model can be expressed as

$$V_c(0,T) = \frac{1}{T}\int_0^T \sigma^2 dt + \frac{1}{T}\left(\sum_{i=1}^{N(T)}(\ln(Y_i))^2\right) = \sigma^2 + \frac{1}{T}\left(\sum_{i=1}^{N(T)}(\ln(Y_i))^2\right) \tag{32}$$

The fair continuous variance strike is obtained by taking the expectation of the continuous realized variance:

$$K^*_{var} = E_0[V_c(0,T)] = \sigma^2 + \lambda(a^2 + b^2) \tag{33}$$

In the jump-diffusion model, the fair continuous variance strike depends on the continuous volatility parameter σ and the volatility of the stock from the jumps during the life of contract. Depending on the relative size of the jump parameters, realized variance can be significantly different than in other models.

3.1 Jump-Diffusion Model: Continuous Volatility Strike

In this section we derive a formula for the fair continuous volatility strike in the Merton jump-diffusion model. The continuous realized variance in the Merton jump-diffusion model is given by equation (32). The square root function can be expressed (Schürger 2002) as:

$$\sqrt{x} = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1 - e^{-sx}}{s^{\frac{3}{2}}}ds \tag{34}$$

Taking expectations on both sides of (34) and interchanging the expectation and integral using Fubini's theorem we get

$$E(\sqrt{x}) = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1 - E(e^{-sx})}{s^{\frac{3}{2}}}ds \tag{35}$$

The fair continuous volatility strike can be evaluated using (35), numerical integration, and the expression for expectation in the integral given in the next proposition.

Proposition 1 *In the Merton jump-diffusion model, the Laplace transform of the continuous realized variance* $E(\exp(-sV_c(0,T)))$ *is given by:*

$$E(\exp(-sV_c(0,T))) = \exp\Bigg(- s\sigma^2 + \lambda T\Bigg(\frac{\exp(\frac{-sa^2}{T+2sb^2})}{\sqrt{(1+\frac{2sb^2}{T})}} - 1\Bigg)\Bigg) \tag{36}$$

All proofs are given in Appendix A.

3.2 Merton jump-diffusion model: discrete variance strike

Proposition 2 *In the Merton jump-diffusion model*

$$\begin{aligned} E_0\Bigg(V_d(0,n,T)\Bigg) &= E_0\Bigg(V_c(0,T)\Bigg) + f(r,a,b,\sigma,\lambda,m,T,n) \\ &= \sigma^2 + (a^2+b^2)\lambda + f(r,a,b,\sigma,\lambda,m,T,n) \end{aligned} \tag{37}$$

where the function $f(r,a,b,\sigma,\lambda,m,T,n)$ *converges to zero linearly with number of sampling dates* n *and*

$$f(r,a,b,\sigma,\lambda,m,T,n) = \frac{\sigma^2 + (a^2+b^2)\lambda + (r-\lambda m - \frac{1}{2}\sigma^2)^2T + a^2\lambda^2T + 2(r-\lambda m-\frac{1}{2}\sigma^2)a\lambda T}{n-1}$$

The expectation of discrete realized variance converges to the expected continuous realized variance linearly with the sampling size ($n = T/\Delta t$)*. Consequently*

$$K^*_{var}(n) = K^*_{var} + f(r,a,b,\sigma,\lambda,m,T,n) \tag{38}$$

and the fair discrete variance strike converges to the fair continuous variance strike linearly with number of sampling dates ($n = T/\Delta t$)*.*

Hence, from equation (18) the initial value of a discrete variance swap, $P_d(0,n,T,K,I)$, converges linearly to the initial value of a continuous variance swap, $P_c(0,T,K,I)$, with the number of sampling dates.

3.3 Merton jump-diffusion model: discrete volatility strike

In this section we compute the fair discrete volatility strike in the Merton jump-diffusion model. We can compute the fair discrete volatility strike by using formula (35).

Proposition 3 *In the Merton jump-diffusion model, the Laplace transform of the discrete realized variance $E(\exp(-sV_d(0,n,T)))$ is given by:*

$$E\bigg(\exp(-sV_d(0,n,T))\bigg) = \bigg(\sum_{n_i=0}^{\infty}\frac{\exp(-\lambda\Delta t)(\lambda\Delta t)^{n_i}}{n_i!}\Big(\frac{\exp\Big(\frac{-s((r-\lambda m-\frac{1}{2}\sigma^2)\Delta t+an_i)^2}{(n-1)\Delta t+2s(\sigma^2\Delta t+b^2 n_i)}\Big)}{\sqrt{(1+\frac{2s(\sigma^2\Delta t+b^2 n_i)}{(n-1)\Delta t})}}\Big)\bigg)^n \quad (39)$$

The expectation in (39) can be computed numerically since the sum converges very fast. We use the Laplace transform and formula (35) to compute the fair discrete volatility strike in the Merton jump-diffusion model.

4 Heston stochastic volatility model

In this section, we present an analysis of the convergence of discrete variance strikes to continuous variance strikes with number of sampling dates in the Heston stochastic volatility (SV) model. The Heston (1993) model is given by:

$$dS_t = rS_t dt + \sqrt{v_t} S_t(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2) \quad (40)$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma_v \sqrt{v_t} dW_t^1 \quad (41)$$

Equation (40) gives the dynamics of the stock price: S_t denotes the stock price at time t, r is the risk-neutral drift, and $\sqrt{v_t}$ is the volatility. Equation (41) gives the evolution of the variance which follows a square root process: θ is the long run mean variance, κ represents the speed of mean reversion, and σ_v is a parameter which determines the volatility of the variance process. The processes W_t^1 and W_t^2 are independent standard Brownian motions under risk-neutral measure Q, and ρ represents the instantaneous correlation between the return process and the volatility process. First we derive the continuous variance strike.

4.1 SV model: continuous variance strike

Proposition 4 *In the Heston stochastic volatility model, the fair continuous variance strike $K_{var}^* = E[V_c(0,T)]$ is given by:*

$$E\bigg(\frac{1}{T}\int_0^T v_s ds\bigg) = \theta + \frac{v_0-\theta}{\kappa T}(1-e^{-\kappa T}) \quad (42)$$

The fair continuous variance strike in the Heston stochastic volatility model is independent of the volatility of variance σ_v. Similarly, the variance of the continuous realized variance, $\mathrm{Var}(V_c(0,T))$, can be derived by calculating the second moment of the Laplace transform:

$$\mathrm{Var}\left(\frac{1}{T}\int_t^T v_s ds\right) = \frac{\sigma_v^2 e^{-2\kappa(T-t)}}{2\kappa^3 T^2}\Big(2(e^{2\kappa(T-t)} - 2e^{\kappa(T-t)}\kappa(T-t) - 1)(v_t - \theta) + (4e^{\kappa(T-t)} - 3e^{2\kappa(T-t)} + 2e^{2\kappa(T-t)}\kappa(T-t) - 1)\theta\Big) \quad (43)$$

The variance of the continuous realized variance (43) depends on the volatility of variance. Since the variance of the continuous realized variance is not equal to zero, there will be a convexity correction (28) in the volatility strike and the fair volatility strike will not be equal to the square root of the fair variance strike. However, in the Heston stochastic volatility model, the realized variance on a sample path doesn't satisfy condition (27), and the convexity correction formula (28) may not provide a good estimate of the fair volatility strike. Numerical results are given in section 6.

We compute the fair continuous volatility strike in the stochastic volatility model by using the formula (34) and the Laplace transform of the realized variance from equation (42). Broadie and Jain (2008) present an alternative partial differential equation approach to compute the same quantities, as well as to price variance options. Next, we compute the fair discrete variance strike in the Heston stochastic volatility model and show that the expected discrete realized variance converges linearly to the expected continuous realized variance with the number of sampling dates.

4.2 SV model: discrete variance strike

Proposition 5 *In the Heston stochastic volatility model:*

$$E_0\Big(V_d(0,n,T)\Big) = E_0\Big(V_c(0,T)\Big) + g(r,\rho,\sigma_v,\kappa,\theta,n) \quad (44)$$

The function $g(\cdot)$ is given explicitly in appendix A. It converges to zero linearly with the number of sampling dates:

$$g(r,\rho,\sigma_v,\kappa,\theta,n) = O\left(\frac{1}{n}\right)$$

and the expectation of discrete realized variance converges to the expected continuous realized variance linearly with the sampling size ($n = T/\Delta t$). Hence,

$$K^*_{var}(n) = K^*_{var} + g(r,\rho,\sigma_v,\kappa,\theta,n) \quad (45)$$

and the discrete variance strike converges to the continuous variance strike linearly with the number of sampling dates ($\Delta t = T/n$).

A proof is given in appendix A. Hence, from equation (18) the initial value of a discrete variance swap, $P_d(0, n, T, K, I)$, converges linearly to the initial value of a continuous variance swap, $P_c(0, T, K, I)$, with the number of sampling dates.

5 Stochastic volatility with jumps model

In this section, we present an analysis of the convergence of discrete variance strikes to continuous variance strikes with number of sampling dates in the stochastic volatility (SVJ) model with jumps. The Bates (1996) and Scott (1997) stochastic volatility with jumps model (SVJ) is an extension of the SV model to include jumps in the stock price process. The risk-neutral dynamics are:

$$\begin{aligned}
\frac{dS_t}{S_t^-} &= (r - \lambda m)dt + \sqrt{v_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2) + dJ_t \\
dv_t &= \kappa(\theta - v_t)dt + \sigma_v \sqrt{v_t} dW_t^1
\end{aligned} \tag{46}$$

The specifications of different parameters are same as in the Heston stochastic volatility model specified in equations (40) and (41) and the Merton jump-diffusion (J) model specified by equation (31). The jump process, N_t, and the Brownian motion are independent.

From equation (32) the continuous realized variance in the SVJ model is

$$V_c(0,T) = \frac{1}{T}\int_0^T v_t dt + \frac{1}{T}\left(\sum_{i=1}^{N(T)} (\ln(Y_i))^2\right) \tag{47}$$

The fair continuous variance strike in the SVJ model is obtained by taking the expectation of the continuous realized variance and using equations (42) and (33) we get:

$$K_{var}^* = E_0[V_c(0,T)] = \theta + \frac{v_0 - \theta}{\kappa T}(1 - e^{-\kappa T}) + \lambda(a^2 + b^2) \tag{48}$$

5.1 SVJ model: continuous volatility strike

In this section we derive the fair continuous volatility strike in SVJ model. The continuous realized variance in the SVJ model is given by equation (47).

Proposition 6 *In the SVJ model, the fair continuous volatility strike is given by:*

$$K_{vol}^{*} = E\sqrt{V_c(0,T)} = \frac{1}{2\sqrt{\pi}} \int_0^{\infty} \frac{1 - E(e^{-sV_c(0,T)})}{s^{\frac{3}{2}}} ds \tag{49}$$

where the Laplace transform of the continuous realized variance $E(\exp(-sV_c(0,T)))$ *is:*

$$E(\exp(-sV_c(0,T))) = \exp\left(A(T,s) - B(T,s)v_0 + \lambda T\left(\frac{\exp(\frac{-sa^2}{T+2sb^2})}{\sqrt{(1+\frac{2sb^2}{T})}} - 1\right)\right) \tag{50}$$

$A(T,s)$ *and* $B(T,s)$ *are given by equation (A-12).*

Proof: Equation (49) follows from (34) and equation (50) follows from Propositions 1 and 5. □

5.2 SVJ model: discrete variance strike

Proposition 7 *In the SVJ model*

$$E_0\left(V_d(0,n,T)\right) = E_0\left(V_c(0,T)\right) + h(r,\rho,\sigma_v,\kappa,\theta,\lambda,m,b,n) \tag{51}$$

The function $h(\cdot)$ *is given explicitly in appendix A. It converges to zero linearly with the number of sampling dates:*

$$h(r,\rho,\sigma_v,\kappa,\theta,\lambda,m,b,n) = O\left(\frac{1}{n}\right)$$

and the expectation of discrete realized variance converges to the expected continuous realized variance linearly with the sampling size ($n = T/\Delta t$)*. Hence*

$$K_{var}^{*}(n) = K_{var}^{*} + h(r,\rho,\sigma_v,\kappa,\theta,\lambda,m,b,n) \tag{52}$$

and the discrete variance strike converges to the continuous variance strike linearly with the number of sampling dates ($\Delta t = T/n$)*.*

A proof is given in appendix A. Hence, from equation (18) the initial value of a discrete variance swap, $P_d(0,n,T,K,I)$, converges linearly to the initial value of a continuous variance swap, $P_c(0,T,K,I)$, with the number of sampling dates.

6 Numerical Results

In this section we answer the questions posed in section 1. In section 6.1 we analyze the effect of ignoring jumps in the underlying on the fair variance strike. Demeterfi et al. (1999) show that when the price process for the underlying asset is continuous, a variance swap can be replicated by a static portfolio of out-of-the-money put and call options. We analyze the impact of jumps in the SVJ model by computing the fair variance strike and value of the portfolio of options and show the difference between them. In section 6.2 we analyze the effect of discrete sampling on variance and volatility strikes and present different numerical and analytical approaches to compute these strikes. We also present the computation of volatility strikes using the convexity correction formula and show that in some cases it does not provide a good approximation to the fair volatility strike. We also illustrate how variance swap strikes change with respect to the swap maturity. In practice many variance swaps are actively traded and price quotes are readily available in the market. Fair volatility strikes must be priced consistently with market prices of variance swaps. We investigate how fair volatility strikes depend on model assumptions by choosing model parameters so that the fair variance strike is the same in all models under comparison.

6.1 Effect of jumps on fair variance strikes

In this section we analyze the effect of ignoring jumps in the computation of the fair variance strike. When there are no jumps in the underlying asset price (e.g., in the BS and SV models), realized variance can be replicated (Demeterfi et al. 1999) by a short position in a log contract and payoffs from a dynamic trading strategy which holds $1/S_t$ shares of the underlying stock at each instant of time t. Using equation (8) Demeterfi et al. (1999) further showed that the continuous realized variance (or floating leg of a variance swap contract) can be replicated using a static portfolio of out-of-the-money call and put options. The discretized version of (8) is used to compute the value of the VIX index. The VIX index provides the one-month volatility computed from a portfolio of out-of-the-money S&P 500 (SPX) index put and call options of one-month maturity. Carr and Wu (2006) show that the square of the VIX index is an approximation of the one-month variance swap rate up to discretization error (from using a finite number of options in the VIX definition) under the assumption that the SPX index does not jump. They also showed that the effect of jumps is third order. Broadie and Jain (2007) analyze the discretization error, and use $\widetilde{\text{VIX}}$ to denote the theoretical value of the VIX without any discretization error. If the underlying has no jumps, $\widetilde{\text{VIX}}$ and the fair variance strike are the same and a static portfolio of out-of-the-money call and put options replicates the one-month continuous realized variance. When there are jumps in the underlying asset price, $\widetilde{\text{VIX}}$ (i.e., the value of the portfolio of options) and the square-root of the fair variance strike are not equal. We investigate the magnitude of this difference, i.e., we quantify the effect of ignoring jumps and incorrectly computing the fair variance strike from the portfolio of options. Broadie and Jain (2007) show that $\widetilde{\text{VIX}}$ in the SVJ model is given by

$$\widetilde{\text{VIX}} = \sqrt{\theta + \frac{1 - e^{-\kappa\tau}}{\kappa\tau}(v_0 - \theta) + 2\lambda(m - a)} \tag{53}$$

where $\tau = 30/365$. The fair variance strike in the SVJ model is given by (48). Hence the effect of ignoring jumps is:

$$\sqrt{K^*_{var}} - \widetilde{\text{VIX}} = \sqrt{\theta + \frac{1 - e^{-\kappa\tau}}{\kappa\tau}(v_t - \theta) + \lambda(a^2 + b^2))} - \sqrt{\theta + \frac{1 - e^{-\kappa\tau}}{\kappa\tau}(v_t - \theta) + 2\lambda(m - a)} \tag{54}$$

Equation (54) shows the difference between the fair variance strike value and the $\widetilde{\text{VIX}}$ value which is computed from the market prices of a weighted portfolio of S&P 500 (SPX) index options. Recall λ represents the arrival rate of jumps and the jump size follows a lognormal distribution, $\text{LN}[a, b^2]$, where m is the mean proportional jump size. The parameters a and m are related to each other by $e^{a+\frac{1}{2}b^2} = m + 1$. Equation (54) shows that when there are no jumps, i.e., $\lambda = 0$, the fair variance strike is the same as the $\widetilde{\text{VIX}}$ index value (which can replicated from a portfolio of options). In the case of jumps the two values will be different. We expand terms on the right hand side of equation (54) to understand the effect of jumps on the fair variance swap strike. On squaring the variance swap strike and $\widetilde{\text{VIX}}$ and then expanding the difference we get

$$K^*_{var} - \widetilde{\text{VIX}}^2 = -\lambda\left(ab^2 + \frac{1}{4}b^4 + \frac{1}{3}(a + \frac{1}{2}b^2)^3 + O((a + \frac{1}{2}b^2)^3)\right) \tag{55}$$

We compute the variance strike and $\widetilde{\text{VIX}}$ value using the parameters of the SVJ model in Table 2 and obtain:

$$\sqrt{K^*_{var}} = 12.29\% \quad \text{and} \quad \widetilde{\text{VIX}} = 12.11\% \tag{56}$$

Thus for the parameters in Table 2 (negative jumps) the $\widetilde{\text{VIX}}$ value under-approximates the square root of the variance strike, in other words the weighted portfolio of options under-approximates the variance strike value.

Equation (55) shows that the difference between $\widetilde{\text{VIX}}$ index and the fair variance strike depends on the jump model parameters a, λ and b. The parameters λ and b are always positive. When $a \geq 0$, the value of the right hand side in equation (55) is always negative and hence $\widetilde{\text{VIX}}$ index is more than the fair variance strike. In the case $a < 0$, the relative magnitude of the parameters a and b determine the value of the right hand side in equation (55). For large negative values of a, the right hand side in equation (55) is positive and hence $\widetilde{\text{VIX}}$ index is less than the fair variance strike. To study the effect of different parameters on relative value between variance strike and $\widetilde{\text{VIX}}$ value we compute both quantities by varying the jump parameters (a, b and λ) in Table 1. We keep the other parameters the same as in Table 2. These jump parameters are representative of the SVJ model parameters obtained by different authors using historical data from different time periods as reported in Gatheral (2006).

Table 1: Effect of jumps in the fair variance strike

	a								
$\lambda = 0.3$	-0.2	0	0.2	-0.2	0	0.2	-0.2	0	0.2
b	$\sqrt{K_{var}^*}$ (%)			$\widetilde{\text{VIX}}$ (%)			Diff. (%)		
0.0	15.00	10.24	15.00	14.74	10.24	15.27	0.26	0.00	-0.28
0.1	15.96	11.61	15.96	15.55	11.62	16.43	0.41	0.00	-0.47
0.2	18.57	15.00	18.57	17.79	15.04	19.53	0.78	-0.04	-0.96
0.3	22.25	19.36	22.25	21.06	19.52	23.89	1.19	-0.16	-1.64
0.4	26.55	24.18	26.55	25.03	24.59	29.05	1.52	-0.40	-2.50

This table shows the fair variance strike and the $\widetilde{\text{VIX}}$ value and their difference with different jump parameters. The first column shows the jump size volatility b and the first row shows the jump size mean, a. The other parameters are taken from the SVJ model parameters of Table 2.

The first three columns in Table 1 show the fair variance strikes with different jump size mean a and jump size volatility b. The next three columns show the $\widetilde{\text{VIX}}$ value (i.e., the portfolio of options value) and the last three columns show the difference between the fair variance strike and $\widetilde{\text{VIX}}$ index value. When there are positive jumps ($a = 0.2$), the strike from the portfolio of options, or the $\widetilde{\text{VIX}}$ index value, over approximates the fair variance strike. When the jump size is equal to zero ($a = 0$), the $\widetilde{\text{VIX}}$ index value is still larger than the variance strike value. As jump size becomes negative, the $\widetilde{\text{VIX}}$ index value is smaller than the variance strike value. For small negative jump sizes the difference between $\widetilde{\text{VIX}}$ index value and the fair variance strike depends on the magnitude of volatility parameter (b). Hence the jump size mean strongly influences whether the portfolio of options (or the $\widetilde{\text{VIX}}$ index) under- or over-approximates the variance strike value. We know from theoretical results that the fair discrete variance strike is more than the fair continuous variance strike. Hence when the underlying asset has negative jumps (which is typically the case in equity markets), the effect of discrete sampling and jumps can cause the fair discrete variance strike to be significantly different from the continuous fair variance strike value.

6.2 Computation of fair strikes in the SVJ model

In this section we present the numerical techniques used to compute the fair discrete and continuous variance and volatility strikes in the SVJ model. We price variance swaps and volatility swaps of one-year maturity with monthly, weekly and daily sampling, i.e., with $n = 12, 52, 252$ respectively. For each sampling size n we compute variance strikes and volatility strikes using analytical formulas and simulation. Using Monte Carlo simulation, we calculate the realized variance, the realized volatility, the convexity correction term and the 3^{rd} and 4^{th} order correction terms in equation (30). We use model parameters similar to those in Duffie, Pan and Singleton (2000) which were found by minimizing the mean squared errors for market S&P500 option prices on November 2, 1993. To analyze the effect of different models on the fair volatility strike, we adjust the parameters slightly so that the fair continuous variance strike, $(13.261\%)^2$, is the same in all models. We used the stochastic volatility jump model parameters and equation (42) to calculate the volatility in the Merton jump-diffusion (J) model. Table 2 gives these parameters.

In the stochastic volatility with jumps model (SVJ) we compute fair discrete variance strikes from equation (52) and fair continuous variance strikes from equation (48). We compute fair continuous volatility strikes by numerical integration using (49). Equation (35) can be written:

$$E\left(\sqrt{V_c(0,T)}\right) = \frac{1}{2\sqrt{\pi}}\int_0^c \frac{1-E(e^{-sV_c(0,T)})}{s^{\frac{3}{2}}}ds + \frac{1}{2\sqrt{\pi}}\int_c^\infty \frac{1-E(e^{-sV_c(0,T)})}{s^{\frac{3}{2}}}ds \tag{57}$$

We can bound the second integral as follows:

$$\frac{1}{2\sqrt{\pi}}\int_c^\infty \frac{1-E(e^{-sV_c(0,T)})}{s^{\frac{3}{2}}}ds \leq \frac{1}{2\sqrt{\pi}}\int_c^\infty \frac{1}{s^{\frac{3}{2}}}ds = \frac{1}{\sqrt{\pi c}} \tag{58}$$

There are two types of errors in computing the fair volatility strike numerically using the integration formula. The first one is the discretization error in evaluating the first integral in equation

Table 2: Model parameters used in numerical experiments

Parameters	BS model	SV model	J model	SVJ model
risk free rate r	3.19%	3.19%	3.19%	3.19%
initial volatility $\sqrt{V_0}$	13.26%	10.11%	11.39%	9.94%
correlation ρ	n/a	-0.70	n/a	-0.79
long run mean variance θ	n/a	0.019	n/a	0.014
speed of mean reversion κ	n/a	6.21	n/a	3.99
volatility of variance σ_v	n/a	0.31	n/a	0.27
jump arrival rate λ	n/a	n/a	0.11	0.11
jump size mean a	n/a	n/a	-0.14	-0.12
jump size volatility b	n/a	n/a	0.15	0.15

(57) and second is the tail sum error in the second integral in equation (57). We compute discrete volatility strikes so that both errors are less than 10^{-8}. Thus, we choose the parameter $c = 10^{16}$ and evaluate first integral between 0 and $c = 10^{16}$ such that the discretization error in the first integral is less than 10^{-8}.

We also compute fair discrete variance strikes and fair discrete volatility strikes using Monte Carlo simulation. To simulate the stochastic volatility component of the SVJ model, we used the Euler discretization with modified drift to simulate the paths of the stock price and the variance process on a discrete time grid with continuous approximation to the drift part. Let $0 = t_0 < t_1 < ... < t_n = T$ be a partition of $[0, T]$ into n equal segments of length $\Delta t = T/n$, i.e. $t_i = iT/n$ for each $i = 0, 1, ..., n$. The discretization of the variance process is:

$$v_{t_i} = \theta(1 - e^{-\kappa\Delta t}) + v_{t_{i-1}} e^{-\kappa\Delta t} + \sqrt{v_{t_{i-1}}}\sigma_v \Delta W^1_{t_i}$$

Here, we used the exact solution of the drift part of the variance process. In our simulation we set the variance process to zero if variance goes negative. The discretization of the stock price process is:

$$S^*_{t_i} = S_{t_{i-1}} + rS_{t_{i-1}}\Delta t + \sqrt{v_{t_{i-1}}} S_{t_{i-1}}(\rho\Delta W^1_{t_i} + \sqrt{1-\rho^2}\Delta W^2_{t_i})$$

where $\Delta W^j_{t_i} = W^j_{t_i} - W^j_{t_{i-1}}, j = 1, 2$. Then we add the jumps in the stock price at each date:

$$S_{t_i} = S^*_{t_i} \prod_{j=N(t_i)}^{N(t_{i+1})} Y_j \tag{59}$$

where $N(t_i)$ refers to total jumps in time $[0, t_i]$. We used the parameters in Table 2 for simulation. We simulated $N = 1{,}000{,}000$ paths of stock prices to compute the fair strikes and the convexity approximation terms for different number of sampling dates.

Table 3: Fair variance strikes and fair volatility strikes with different numbers of sampling dates in the SVJ model

	Simulation		Analytical	Simulation		
n	$K^*_{var}(n)(\%^2)$	$SE(K^*_{var}(n))$	$K^*_{var}(n)(\%^2)$	ρ_n	$K^*_{vol}(n)(\%)$	$SE(K^*_{vol}(n))$
Monthly	13.854	0.796	14.061	0.937	12.756	0.017
Weekly	13.398	0.749	13.440	0.958	12.313	0.017
Daily	13.311	0.757	13.299	0.968	12.236	0.017
Cont.			13.263		12.222	

The first column shows the sampling size in computing the realized variance of one-year maturity swap in the SVJ model. The second column shows the fair variance strike for the respective number of sampling dates computed using simulation. The third column shows the standard error in the estimate of the fair variance strike computed using simulation. The fourth column shows the fair variance strike values computed using analytical formula in proposition 7. The fifth column shows the correlation coefficient ρ_n. The sixth column shows fair volatility strikes obtained using simulation with a control variate. The seventh column shows the standard error in the estimate of the fair volatility strike and last column shows the fair volatility strike computed using numerical integration. The last row shows the fair continuous variance strike and the fair continuous volatility strike.

Table 3 shows fair discrete and continuous variance and volatility strikes in the SVJ model. Since we know the exact value of the fair discrete variance strike, we used the control variate method to obtain more reliable estimates of fair discrete volatility strikes. We used the following equation to compute the fair discrete volatility strike:

$$K^*_{vol}(n) = E(\sqrt{V_d(0,n,T)}) - b_n\Big(E((V_d(0,n,T)) - K^*_{var}(n)\Big) \tag{60}$$

where $E(\sqrt{(V_d(0,n,T)})$ is the simulation estimate of the fair discrete volatility strike with n sampling dates, $E((V_d(0,n,T))$ is the simulation estimate of the fair discrete variance strike with the same sampling size computed using same simulation paths, $K^*_{var}(n)$ is the fair discrete variance strike computed using equation (52) and b_n is the optimal coefficient which minimizes

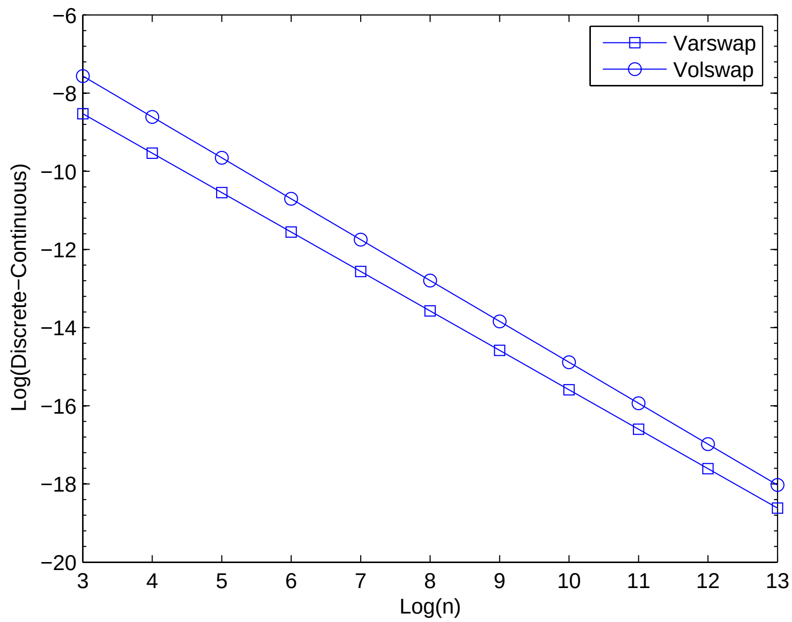

Figure 1: Convergence of fair strikes with sampling size in SVJ model. This figure plots on log-log scale difference in the fair discrete strike and the fair continuous strike versus the number of sampling dates.

the variance and is given by:

$$b_n = \frac{\text{Cov}((V_d(0,n,T), \sqrt{(V_d(0,n,T)})}{\text{Var}((V_d(0,n,T))}$$

The ratio of the variance of the optimally controlled estimator $K^*_{vol}(n)$ to that of the uncontrolled estimator is $1-\rho_n^2$ where ρ_n is the correlation coefficient between $(V_d(0,n,T)$ and $\sqrt{(V_d(0,n,T)}$. We also report the correlation coefficient ρ_n for all sampling sizes n in Table 3.

In the SVJ model, the fair discrete variance strike in the case of monthly sampling ($n = 12$) is $(14.061\%)^2$ versus the fair continuous variance strike $(13.26\%)^2$. This corresponds to a relative difference of 12.4%. The fair discrete volatility strike 12.756% in the monthly sampling ($n = 12$) differs from the fair continuous volatility strike 12.222%. This corresponds to a relative difference of 4.43%.

From the analytical results we plot $\log(K^*_{var}(n) - K^*_{var})$ versus $\log(n)$. We plot the same for the volatility strikes using the simulation results. Figure 1 shows the convergence plot of the fair variance strikes and the fair volatility strikes with the number of sampling dates. These results show that in the SVJ model the fair discrete variance strike converges linearly to the fair continuous variance strike with the number of sampling dates, consistent with Proposition 7.

We determine the convergence rate of the fair discrete volatility strike to the fair continuous volatility strike numerically using regression. The regression equation of the volatility strike is

$$\log(K^*_{vol}(n) - K^*_{vol}) = -1.046 \log(n) - 4.428 \qquad R^2 = 0.99 \tag{61}$$

These results show that in SVJ model the fair discrete volatility strike converges linearly to the fair continuous volatility strike with sampling size.

Table 4: Approximation of the fair volatility strike using the convexity correction formula in the SVJ model

n	Convexity correction(cc)	3rd order	4th order	Excess Prob.(p)	$K^*_{vol}(n)(\%)$ using cc	$K^*_{vol}(n)(\%)$ correct value	Diff (%)
Monthly	2.980	12.759	99.433	0.061	10.874	12.756	1.882
Weekly	2.915	12.488	97.879	0.062	10.483	12.313	1.830
Daily	3.039	13.454	122.090	0.072	10.272	12.131	1.859

The first column shows the number of sampling dates in computing the realized variance of one-year maturity volatility swap. The second column shows the convexity correction term (28) with the different number of sampling dates. The third and fourth columns show the 3^{rd} and 4^{th} order term in equation (30). The fifth column shows the excess probability (29). The sixth column column shows the fair volatility strike computed using convexity correction formula. The seventh column shows the fair volatility strike computed using simulation and the last column shows the difference between the fair volatility strike computed using the simulation in Table 3 and the convexity correction formula (28).

Table 4 shows the results of computing the fair volatility strike by the convexity correction formula. The excess probability p from (29) is not equal to zero and the 3^{rd} and 4^{th} order terms in equation (30) are comparable in magnitude with convexity correction term. We can see from the last column that the differences between the exact fair volatility strike computed using numerical integration and the approximation using the convexity correction formula is quite significant, i.e., the convexity correction formula does not provide a good approximation to the fair volatility strike in the SVJ model.

6.3 Discrete strikes and convexity corrections in different models

Next we analyze fair variance strikes and fair volatility strikes in different models with different sampling sizes. Table 5 shows these results. In these results we compute discrete and continuous variance strikes using analytical formulas in all four models. We compute the fair continuous volatility strike using numerical integration in all four models. The discrete volatility strike is computed using numerical integration in the BS and J models and using simulation with control variates in the SV and SVJ models. It can be seen from the results that the fair volatility strike is less than the square root of the fair variance strike since the payoff of the volatility swap is concave in the realized variance. This is true for all models and all sampling sizes except for continuous sampling in the Black-Scholes model in which case they are equal. Even though fair continuous variance strikes are identical in all models, the fair volatility strikes in SVJ model are less compared to the SV and J model, i.e., there is more convexity value in the SVJ model. Fair discrete strikes under monthly and weekly sampling are considerably different than under continuous sampling. Formulas and results derived for idealized contracts with continuous sampling should be applied with caution to instruments which use discrete sampling.

Table 5: Comparison of fair variance strikes and fair volatility strikes in different models

Sampling	$K^*_{var}(\%^2)$				$K^*_{vol}(\%)$			
Size (n)	BS	SV	J	SVJ	BS	SV	J	SVJ
Monthly	13.86	13.92	13.87	14.06	13.54	13.40	12.80	12.76
Weekly	13.33	13.41	13.39	13.44	13.33	13.14	12.56	12.31
Daily	13.27	13.29	13.29	13.30	13.28	13.10	12.50	12.24
Continuous	13.26	13.26	13.26	13.26	13.26	13.09	12.48	12.22

The first column shows the number of sampling dates. Then next four columns show fair variance strikes in the Black-Scholes (BS), the Heston stochastic volatility (SV), the Merton jump-diffusion model (J) and stochastic volatility model with jumps (SVJ) respectively. The last three columns show fair volatility strikes in respective models.

Next we compute variance strikes and volatility strikes with an alternative definition of the realized variance in equation (1). All the results so far were computed for a maturity of one year and with $m = n - 1$ in the realized variance definition specified in equation (1). We have seen that with this definition ($m = n - 1$) discrete variance strikes are different than the continu-

Table 6: Comparison of fair variance strikes with an alternative definition of realized variance and with maturities in the SVJ model

Sampling Size n	1 month		6 months		1 year		30 years	
	$m=n-1$	$m=n$	$m=n-1$	$m=n$	$m=n-1$	$m=n$	$m=n-1$	$m=n$
Monthly			14.59	13.32	14.06	13.46	13.71	13.69
Weekly	14.74	12.76	13.31	13.05	13.44	13.31	13.65	13.64
Daily	12.56	12.35	13.04	12.99	13.30	13.27	13.63	13.63
Continuous		12.29		12.97		13.26		13.63

The first column shows the sampling size. The first row shows the maturity of the variance strike. The columns $m = n-1$ refers to the variance strike computed when $m = n-1$ in the realized variance definition in equation (1). The columns $m = n$ refers to the variance strike computed when $m = n$ in the realized variance definition in equation (1).

ous variance strikes. Table 6 shows the variance strikes computed using two definitions of the realized variance and with different maturities in the SVJ model. The results show that using $m = n$ in the definition of realized variance removes the effect of discrete sampling in the fair variance strike. The effect of discrete sampling is more pronounced for shorter maturity swaps. In practice, the typical maturity of swaps varies from one month to 30 years with one month being most the popular. In practice most of the contracts sampling is done daily or weekly and sometimes monthly. Using these parameters there is a 27 basis point difference between one-month fair variance strikes using discrete sampling (daily) and continuous sampling. The effect of discreteness decreases as maturity increases.

Next we analyze the accuracy of the convexity correction formula in computing volatility strikes across different models. The difference in the fair volatility strike and the square root of the fair variance strike is called the convexity value. Table 7 shows fair volatility strikes computed using the convexity correction formula and true fair volatility strikes in all models. The convexity correction formula works well in the Black-Scholes model and but tends to perform poorly in the Heston stochastic volatility model and even worse in models with jumps (J and SVJ). It can be seen from the results that the convexity value depends on the model and the sampling size.

Table 7: Comparison of fair volatility strikes and approximations using the convexity correction formula in different models

Sampling	BS $K^*_{vol}(\%)$		SV $K^*_{vol}(\%)$		J $K^*_{vol}(\%)$		SVJ $K^*_{vol}(\%)$	
Size n	cc	true	cc	true	cc	true	cc	true
Monthly	13.54	13.54	13.31	13.40	10.99	12.80	10.87	12.76
Weekly	13.33	13.33	13.12	13.14	10.61	12.56	10.48	12.31
Daily	13.28	13.28	13.09	13.10	10.44	12.50	10.27	12.13

The first column shows the number of sampling dates. Then next two columns show fair volatility strikes in the Black-Scholes (BS) model computed using the convexity correction formula and true fair volatility strikes. The fourth and the fifth columns provide results in the Heston stochastic volatility (SV) and the sixth and seventh columns give results for the Merton jump-diffusion (J) model and last two columns give results for the stochastic volatility model with jumps (SVJ).

7 Conclusion

In this paper we study the pricing of variance swaps and volatility swaps in four financial models. We derive analytical formulas for fair discrete variance strikes in the Black-Scholes model, the Heston stochastic volatility model, the Merton jump-diffusion model and the Bates and Scott stochastic volatility with jumps model. We investigate the effect of discrete sampling and jumps on fair variance strikes. The effect of jumps depends on direction and magnitude of jumps, e.g., the one-month discrete variance strike can be significantly different from the continuous fair variance strike when the underlying has negative jumps. We also present an argument to show that the convexity correction formula to approximate fair volatility strikes may not provide good estimates in jump-diffusion models. We present numerical approaches to compute fair volatility strikes. In particular we compute fair discrete volatility strikes from numerical integration and Monte Carlo simulation techniques. We also show that in all models fair discrete variance and volatility strikes converge linearly to fair continuous variance and volatility strikes, respectively, as the sampling size increases.

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A

Proof of Proposition 1: The Laplace transform of continuous realized variance can be expressed as

$$\begin{aligned} E(\exp(-sV_c(0,T))) &= E(\exp(-s(\sigma^2 + \frac{1}{T}\sum_{i=1}^{N(T)}(\ln(Y_i))^2))) \\ &= \exp(-s\sigma^2)E\Bigg(E\Bigg(\frac{1}{T}\sum_{i=1}^{n}(\ln(Y_i))^2|N(T)=n\Bigg)\Bigg) \qquad \text{(A-1)} \end{aligned}$$

where second equality follows by taking an expectation conditional on a Poisson random variable, $N(T) = n$. Since $\ln(Y_i) \sim N(a, b^2)$, the inner expectation can be computed as:

$$E(\exp(-sV_c(0,T))) = \exp(-s\sigma^2)E\Bigg(\frac{\exp\Big(\frac{-sna^2}{T+2sb^2}\Big)}{(1+\frac{2sb^2}{T})^{\frac{n}{2}}}\Bigg) \qquad \text{(A-2)}$$

We can compute the outer expectation as follows:

$$E(\exp(-sV_c(0,T))) = \exp(-s\sigma^2)\sum_{n=0}^{\infty}\frac{\exp(-\lambda T)(\lambda T)^n}{n!}\Bigg(\frac{\exp\Big(\frac{-sna^2}{T+2sb^2}\Big)}{(1+\frac{2sb^2}{T})^{\frac{n}{2}}}\Bigg) \qquad \text{(A-3)}$$

Simplifying the infinite sum gives

$$\begin{aligned} E(\exp(-sV_c(0,T))) &= \exp(-s\sigma^2)\exp(-\lambda T)\exp\Bigg(\frac{\lambda T\exp(\frac{-sa^2}{T+2sb^2})}{\sqrt{(1+\frac{2sb^2}{T})}}\Bigg) \\ &= \exp\Bigg(-s\sigma^2 + \lambda T\Bigg(\frac{\exp(\frac{-sa^2}{T+2sb^2})}{\sqrt{(1+\frac{2sb^2}{T})}} - 1\Bigg)\Bigg) \qquad \square \end{aligned}$$

Proof of Proposition 2: Applying Itô's lemma in the jump-diffusion model (31) and integrating from t_i to t_{i+1} gives

$$\ln\left(\frac{S_{i+1}}{S_i}\right) = (r - \lambda m - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z_{i+1} + \ln\left(\prod_{j=1}^{n_j} Y_j\right) \qquad \text{(A-4)}$$

where n_j is number of jumps in the stock price during time $t_{i+1} - t_i$. Squaring equation (A-4) on both sides and summing from time 0 to time $n-1$ we get

$$\begin{aligned} \sum_{i=0}^{n-1}\left(\ln\left(\frac{S_{i+1}}{S_i}\right)\right)^2 = \sum_{i=0}^{n-1}\Bigg[&\left(r-\lambda m - \frac{1}{2}\sigma^2\right)^2\Delta t^2 + \sigma^2\Delta t Z_{i+1}^2 + 2\sigma(r-\lambda m - \frac{1}{2}\sigma^2)\Delta t^{\frac{3}{2}}Z_{i+1} \\ &+\left(\sum_{j=1}^{n_j}\ln Y_j\right)^2 + 2\left(r-\lambda m-\frac{1}{2}\sigma^2\right)\Delta t\left(\sum_{j=1}^{n_j}\ln Y_j\right) + 2\sigma\Delta t^{\frac{1}{2}}Z_{i+1}\left(\sum_{j=1}^{n_j}\ln Y_j\right)\Bigg] \end{aligned} \qquad \text{(A-5)}$$

The fair discrete variance strike can be calculated by dividing equation (A-5) on both sides by $(n-1)\Delta t$ and taking expectation under the risk-neutral measure:

$$\begin{aligned} K^*_{var}(n) = E\Big[V_d(0,n,T)\Big] &= E\Bigg[\Big(r-\lambda m-\frac{1}{2}\sigma^2\Big)^2\Delta t\frac{n}{n-1}+\sigma^2 Z^2_{i+1}\frac{n}{n-1} \\ &\quad +2\sigma(r-\lambda m-\frac{1}{2}\sigma^2)\Delta t^{\frac{1}{2}}Z_{i+1}\frac{n}{n-1}+(\Sigma_{j=1}^{n_j}\ln Y_j)^2\frac{n}{(n-1)\Delta t} \\ &\quad +2(r-\lambda m-\frac{1}{2}\sigma^2)(\Sigma_{j=1}^{n_j}\ln Y_j)\frac{n}{n-1} \\ &\quad +2\sigma Z_{i+1}(\Sigma_{j=1}^{n_j}\ln Y_j)\frac{n}{(n-1)\Delta t^{\frac{1}{2}}}\Bigg] \end{aligned} \tag{A-6}$$

Using properties of the normal and Poisson distributions

$$\begin{aligned} E(Z_i) &= 0 \qquad E[Z^2_{i+1}]=1 \qquad E\Big[\sum_{j=1}^{n_j}\ln Y_j\Big]=a\lambda\Delta t \\ E\Big[\sum_{j=1}^{n_j}\ln Y_j\Big]^2 &= b^2E[n_j]+a^2(E(n_j)^2)=(a^2+b^2)(\lambda\Delta t)+(\lambda\Delta t)^2a^2 \end{aligned}$$

Substituting these in equation (A-6) we get

$$K^*_{var}(n)=(r-\lambda m-\frac{1}{2}\sigma^2)^2\frac{T}{n-1}+\sigma^2\frac{n}{n-1}+\frac{(a^2+b^2)\lambda n}{n-1}+\frac{a^2\lambda^2T}{n-1}+2(r-\lambda m-\frac{1}{2}\sigma^2)\frac{a\lambda T}{n-1} \tag{A-7}$$

The previous expression gives the fair discrete variance strike. Rearranging terms gives (38). □

Proof of Proposition 3: The Laplace transform of discrete realized variance is

$$\begin{aligned} E\Big(\exp(-sV_d(0,n,T))\Big) &= E\Bigg(\exp\Bigg(\frac{-s\sum_{i=0}^{n-1}\Big(\ln\Big(\frac{S_{i+1}}{S_i}\Big)\Big)^2}{(n-1)\Delta t}\Bigg)\Bigg) \\ &= E\Bigg(E\Bigg(\exp\Bigg(\frac{-s\Big(\sum_{i=0}^{n-1}\ln\Big(\frac{S_{i+1}}{S_i}\Big)\Big)^2}{(n-1)\Delta t}\Bigg)\Bigg|N(0),N(t_1)....,N(T)\Bigg) \\ &= E\Bigg(\prod_0^{n-1}E\Bigg(\exp\Bigg(\frac{-s\Big(\ln\Big(\frac{S_{i+1}}{S_i}\Big)\Big)^2}{(n-1)\Delta t}\Bigg)\Bigg|N(0),N(t_1)....,N(T)\Bigg) \end{aligned} \tag{A-8}$$

The third equality follows since a Poisson process has stationary and independent increments where

$$\ln\left(\frac{S_{i+1}}{S_i}\right) = (r - \lambda m - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z_{i+1} + \sum_{j=1}^{n_i}\Big(\ln(Y_j)\Big) \tag{A-9}$$

and n_i is number of jumps in the stock price during the time $t_{i+1} - t_i$. The random variables n_i are independent and identically distributed with Poisson rate $\lambda\Delta t$ for each $i = 0, 1, ..., n-1$. Since $\ln(Y_j) \sim N(a, b^2)$ the distribution of log return given n_i jumps is

$$\ln\left(\frac{S_{i+1}}{S_i}\right) \sim N((r - \lambda m - \frac{1}{2}\sigma^2)\Delta t + an_i, \sigma^2\Delta t + b^2 n_i) \tag{A-10}$$

The inner expectation in equation (A-8) can be solved using property (A-10):

$$\begin{aligned} E\Big(\exp(-sV_d(0,n,T))\Big) &= E\Bigg(\prod_{i=0}^{n-1}\Bigg(\frac{\exp\left(\frac{-s((r-\lambda m-\frac{1}{2}\sigma^2)\Delta t+an_i)^2}{(n-1)\Delta t+2s(\sigma^2\Delta t+b^2n_i)}\right)}{\sqrt{(1+\frac{2s(\sigma^2\Delta t+b^2n_i)}{(n-1)\Delta t})}}\Bigg)\Bigg) \\ &= \Bigg(\sum_{n_i=0}^{\infty}\frac{\exp(-\lambda\Delta t(\lambda\Delta t)^{n_i})}{n_i!}\Bigg(\frac{\exp\left(\frac{-s((r-\lambda m-\frac{1}{2}\sigma^2)\Delta t+an_i)^2}{(n-1)\Delta t+2s(\sigma^2\Delta t+b^2n_i)}\right)}{\sqrt{(1+\frac{2s(\sigma^2\Delta t+b^2n_i)}{(n-1)\Delta t})}}\Bigg)\Bigg)^n \end{aligned} \tag{A-11}$$

The second equality follows since n_i are independent. □

Proof of Proposition 4: The Laplace transform of $\int_0^T v_s ds$ is given by (Cairns 2000)

$$E_0[e^{-s\frac{1}{T}\int_0^T v_t dt} \mid v(0) = v_0] = \exp[A(T,s) - B(T,s)v_0] \tag{A-12}$$

where

$$\begin{aligned} A(T,s) &= \frac{2\kappa\theta}{\sigma_v^2}\log\left(\frac{2\gamma(s)e^{\frac{(\gamma(s)+\kappa)T}{2}}}{(\gamma(s)+\kappa)(e^{(\gamma(s))T}-1)+2\gamma(s)}\right) \\ B(T,s) &= \frac{2s(e^{(\gamma(s))T}-1)}{T(\gamma(s)+\kappa)(e^{(\gamma(s))T}-1)+2\gamma(s)} \\ \gamma(s) &= \sqrt{\kappa^2 + 2\frac{\sigma_v^2 s}{T}} \end{aligned}$$

From the Laplace transform of $\int_0^T v_s ds$ we can derive the first moment:

$$E\left[\int_t^T v_s ds\right] = -\frac{d}{d\nu}\Big(E^Q[e^{-\nu\int_t^T v_s ds} \mid v(t) = v_t]\Big)\Big|_{(\nu=0)} = \theta(T-t) + \frac{v_t - \theta}{\kappa}(1 - e^{-\kappa(T-t)})$$

which proves (42). $\square$

Proof of Proposition 5:

The discrete variance strike can be derived as follows. Applying Itô's lemma to $\ln(S_t)$ in equation (40) we get

$$d(\ln S_t) = \Big(r - \frac{1}{2}v_t\Big)dt + \sqrt{v_t}\Big(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\Big) \tag{A-13}$$

Integrating equation (A-13) from t_i to t_{i+1} squaring and taking expectations in the risk-neutral measure we get

$$\begin{aligned} E\Big[\ln\Big(\frac{S_{t_{i+1}}}{S_{t_i}}\Big)\Big]^2 &= E\Big[\int_{t_i}^{t_{i+1}}(r - \frac{1}{2}v_t)dt + \int_{t_i}^{t_{i+1}}\sqrt{v_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)\Big]^2 \\ &= E\Big[\Big(r\Delta t - \int_{t_i}^{t_{i+1}}\frac{1}{2}v_t dt\Big)^2 + \Big(\int_{t_i}^{t_{i+1}}\sqrt{v_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)\Big)^2 \\ &\quad +2\Big(r\Delta t - \int_{t_i}^{t_{i+1}}\frac{1}{2}v_t dt\Big)\Big(\int_{t_i}^{t_{i+1}}\sqrt{v_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)\Big)\Big] \end{aligned}$$

Applying Itô's isometry rule on second term and simplifying other terms we get:

$$\begin{aligned} E\Big[\ln\Big(\frac{S_{t_{i+1}}}{S_{t_i}}\Big)\Big]^2 &= (r\Delta t)^2 + \frac{1}{4}E\Big[\Big(\int_{t_i}^{t_{i+1}} v_t dt\Big)^2\Big] - (r\Delta t)E\Big[\Big(\int_{t_i}^{t_{i+1}} v_t dt\Big)\Big] \\ &\quad +E\Big[\int_{t_i}^{t_{i+1}} v_t dt\Big] + 2r\Delta t E\Big[\int_{t_i}^{t_{i+1}}\sqrt{v_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)\Big] \\ &\quad -E\Big[\Big(\int_{t_i}^{t_{i+1}} v_t dt\Big)\Big(\int_{t_i}^{t_{i+1}}\sqrt{v_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)\Big)\Big] \end{aligned} \tag{A-14}$$

The variance process has the following properties:

$$\begin{aligned} E(v_t) &= \exp(-kt)(v_0 - \theta) + \theta \\ E(v_t v_s) &= {\sigma_v}^2 \exp(-k(t+s)\Big(\frac{\exp(ks)-1}{k}(v_0-\theta) + \frac{\exp(2ks)-1}{2k}(\theta)\Big) \\ &\quad + \exp(-k(t+s))(v_0-\theta)^2 + \exp(-kt)(v_0-\theta)\theta + \exp(-ks)(v_0-\theta)\theta + \theta^2 \end{aligned}$$

$$E\Big[\int_{t_i}^{t_{i+1}}\sqrt{v_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)\Big] = 0 \tag{A-15}$$

Using properties (A-15) we solve for the last term in equation (A-14):

$$\begin{aligned} &E\Big[\Big(\int_{t_i}^{t_{i+1}} v_t dt\Big)\Big(\int_{t_i}^{t_{i+1}}\sqrt{v_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)\Big)\Big] \\ &= E\Big[\Big(\int_{t_i}^{t_{i+1}} v_t dt\Big)\Big(\int_{t_i}^{t_{i+1}}\sqrt{v_t}\rho dW_t^1\Big)\Big] + E\Big[\Big(\int_{t_i}^{t_{i+1}} v_t dt\Big)\Big(\int_{t_i}^{t_{i+1}}\sqrt{v_t}\sqrt{1-\rho^2}dW_t^2\Big)\Big] \end{aligned} \tag{A-16}$$

The 2^{nd} expectation in equation (A-16) is zero and first term can be rewritten using (41) as

$$E\left[\frac{\rho}{\sigma_v}\left(\int_{t_i}^{t_{i+1}} v_t dt\right)\left(v_{t_{i+1}} - v_{t_i} - \int_{t_i}^{t_{i+1}} \kappa(\theta - v_t)dt\right)\right]$$
$$= E\left[\frac{\rho}{\sigma_v}\left(\int_{t_i}^{t_{i+1}} v_t dt\right)\left(v_{t_{i+1}} - v_{t_i}\right)\right] - \frac{\rho}{\sigma_v}\kappa\theta\Delta t E\left(\int_{t_i}^{t_{i+1}} v_t dt\right) + \frac{\rho\kappa}{\sigma_v}E\left(\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} v_t v_s dt ds\right) \tag{A-17}$$

Next we compute the first term in equation (A-17):

$$\begin{aligned} E\left[\left(\int_{t_i}^{t_{i+1}} v_t dt\right)\left(v_{t_i}\right)\right] &= \frac{(1-\exp(-\kappa t_i))\sigma_v^2(v_0-\theta)}{\kappa^2}\Big(+\exp(-\kappa t_i) - \exp(-\kappa t_{i+1})\Big) \\ &+\frac{\sigma_v^2\theta}{2\kappa^2}\Big((\exp(\kappa t_i) - \exp(-\kappa t_i))(-\exp(-\kappa t_{i+1}) + \exp(-\kappa t_i))\Big) \\ &+\frac{\exp(-\kappa t_i)(v_0-\theta)^2}{\kappa}\Big(-\exp(-\kappa t_{i+1}) + \exp(-\kappa t_i)\Big) \\ &+\frac{(v_0-\theta)\theta}{\kappa}\Big(-\exp(-\kappa t_{i+1}) + \exp(-\kappa t_i)\Big) \\ &+\exp(-\kappa t_i)(v_0-\theta)\theta\Delta t + \theta^2\Delta t \end{aligned} \tag{A-18}$$

$$\begin{aligned} E\left[\left(\int_{t_i}^{t_{i+1}} v_{t_{i+1}} v_t dt\right)\right] &= \frac{\exp(-\kappa t_{i+1})\sigma_v^2(v_0-\theta)}{\kappa^2}\Big(\kappa\Delta t - \exp(-\kappa t_i) + \exp(-\kappa t_{i+1})\Big) \\ &+\frac{\exp(-\kappa t_{i+1})\sigma_v^2\theta}{2\kappa^2}\Big(\exp(\kappa t_{i+1}) - \exp(\kappa t_i) + \exp(-\kappa t_{i+1}) - \exp(-\kappa t_i)\Big) \\ &+\frac{\exp(-\kappa t_{i+1})(v_0-\theta)^2}{\kappa}\Big(-\exp(-\kappa t_{i+1}) + \exp(-\kappa t_i)\Big) \\ &+\frac{(v_0-\theta)\theta}{\kappa}\Big(-\exp(-\kappa t_{i+1}) + \exp(-\kappa t_i)\Big) \\ &+\exp(-\kappa t_{i+1})(v_0-\theta)\theta\Delta t + \theta^2\Delta t \end{aligned} \tag{A-19}$$

Subtracting equation (A-18) from (A-19) and simplifying we get

$$\begin{aligned} E\left[\left(\int_{t_i}^{t_{i+1}} v_t dt\right)\left(v_{t_{i+1}} - v_{t_i}\right)\right] &= \frac{\sigma_v^2(v_0-\theta)}{\kappa^2}\Big(\exp(-\kappa t_{i+1})(1+\kappa\Delta t - \exp(\kappa\Delta t))\Big) \\ &-\left(\frac{\sigma_v^2(\theta-2v_0)}{2\kappa^2} + \frac{(v_0-\theta)^2}{\kappa}\right)\left(\exp(-2\kappa t_i)\Big(1-\exp(-\kappa\Delta t)\Big)^2\right) \\ &+(v_0-\theta)\theta\Delta t\Big(\exp(-\kappa t_{i+1})\Big(1-\exp(\kappa\Delta t)\Big)\Big) \end{aligned} \tag{A-20}$$

Summing equation (A-20) from time 0 to time $n-1$ we get

$$
\begin{aligned}
\sum_{i=0}^{n-1} E\left[\left(\int_{t_i}^{t_{i+1}} v_t dt\right)\left(v_{t_{i+1}} - v_{t_i}\right)\right] \;=\;& \frac{\sigma_v^2(v_0-\theta)}{\kappa^2}\Big((1+\kappa\Delta t - \exp(\kappa\Delta t))\Big)\left(\frac{1-\exp(-\kappa T)}{-1+\exp(\frac{\kappa T}{n})}\right) \\
& -\left(\frac{\sigma_v^2(\theta-2v_0)}{2\kappa^2} + \frac{(v_0-\theta)^2}{\kappa}\right)\left(\left(\frac{1-\exp(-2\kappa T)}{1-\exp(-\frac{2\kappa T}{n})}\right)\right. \\
& \left.\Big(1-\exp(-\kappa\Delta t)\Big)^2\right) \\
& +(v_0-\theta)\theta\Delta t\left(\left(\frac{1-\exp(-\kappa T)}{-1+\exp(\frac{\kappa T}{n})}\right)\Big(1-\exp(\kappa\Delta t)\Big)\right)
\end{aligned} \tag{A-21}
$$

All terms on right hand side of equation (A-21) are of the order $O(\Delta t)$. We expand the terms to show this:

$$
\begin{aligned}
\Big((1+\kappa\Delta t-\exp(\kappa\Delta t))\Big)\left(\frac{1-\exp(-\kappa T)}{-1+\exp(\frac{\kappa T}{n})}\right) \;&\equiv\; -\left(\frac{(\kappa\Delta t)^2}{2}+\frac{(\kappa\Delta t)^3}{6}+...\right)\left(\frac{1-\exp(-\kappa T)}{(\kappa\Delta t)+\frac{(\kappa\Delta t)^2}{2}+...}\right) \\
&\equiv\; O(\Delta t)
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{1-\exp(-2\kappa T)}{1-\exp(\frac{-2\kappa T}{n})}\right)\Big(1-\exp(-\kappa\Delta t)\Big)^2 \;&\equiv\; \left(\frac{1-\exp(-2\kappa T)}{(2\kappa\Delta t)+\frac{(2\kappa\Delta t)^2}{2}+...}\right)\left((\kappa\Delta t)-\frac{(\kappa\Delta t)^2}{2}+...\right)^2 \\
&\equiv\; O(\Delta t)
\end{aligned}
$$

$$
\Delta t\left(\left(\frac{1-\exp(-\kappa T)}{-1+\exp(\frac{\kappa T}{n})}\right)\Big(1-\exp(\kappa\Delta t)\Big)\right) \;\equiv\; O(\Delta t)
$$

Hence the left hand side of the equation (A-21) is of the order $O(\Delta t)$.

Next we compute the last term in equation (A-17):

$$
\begin{aligned}
\sum_{i=0}^{n-1} E\left[\left(\int_{t_i}^{t_{i+1}} v_t dt\right)^2\right] \;&=\; \sum_{i=0}^{n-1} E\left(\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} v_t v_s dt ds\right) \\
&=\; \sum_{i=0}^{n-1} \mathrm{Var}\left(\int_{t_i}^{t_{i+1}} v_s ds\right) + \sum_{i=0}^{n-1}\left(E\left(\int_{t_i}^{t_{i+1}} v_s ds\right)\right)^2
\end{aligned} \tag{A-22}
$$

Now we compute the both terms on right hand side of equation (A-22)

$$\begin{aligned}
\sum_{i=0}^{n-1} \mathrm{Var}\left(\int_{t_i}^{t_{i+1}} v_s ds\right) &= \sum_{i=0}^{n-1} \frac{\sigma_v^2 e^{-2\kappa(\Delta t)}}{\kappa^3}\left(e^{2\kappa(\Delta t)} - 2e^{\kappa(\Delta t)}\kappa(\Delta t) - 1\right)\left(v_0 - \theta\right)\exp(\frac{-i\kappa T}{n}) \\
&\quad + \frac{\sigma_v^2 e^{-2\kappa(\Delta t)}}{2\kappa^3}(4e^{\kappa(\Delta t)} - 3e^{2\kappa(\Delta t)} + 2e^{2\kappa(\Delta t)}\kappa(\Delta t) - 1)\theta \\
&= \frac{\sigma_v^2 e^{-2\kappa(\Delta t)}}{\kappa^3}\left(e^{2\kappa(\Delta t)} - 2e^{\kappa(\Delta t)}\kappa(\Delta t) - 1\right)\left(v_0 - \theta\right)\left(\frac{1 - \exp(-\kappa T)}{-1 + \exp(\frac{\kappa T}{n})}\right) \\
&\quad + \frac{\sigma_v^2 e^{-2\kappa(\Delta t)}}{2\kappa^3}(4e^{\kappa(\Delta t)} - 3e^{2\kappa(\Delta t)} + 2e^{2\kappa(\Delta t)}\kappa(\Delta t) - 1)\theta n \qquad \text{(A-23)}
\end{aligned}$$

Both terms on the right hand side of the equation are of the order $O(\Delta t)$. We expand the terms to show this:

$$\begin{aligned}
\left(e^{2\kappa(\Delta t)} - 2e^{\kappa(\Delta t)}\kappa(\Delta t) - 1\right)\left(\frac{1 - \exp(-\kappa T)}{-1 + \exp(\frac{\kappa T}{n})}\right) &\equiv \Bigg(1 + (2\kappa\Delta t) + \frac{(2\kappa\Delta t)^2}{2} + ... \\
&\quad -(2\kappa\Delta t)(1 + (\kappa\Delta t) + \frac{(\kappa\Delta t)^2}{2} + ...) - 1\Bigg) \\
&\quad \left(\frac{1 - \exp(-\kappa T)}{-1 + \exp(\frac{\kappa T}{n})}\right) \\
&\equiv \left(\frac{(2\kappa\Delta t)^2}{2} - (2\kappa\Delta t)\frac{(\kappa\Delta t)^2}{2} + ...\right)\left(\frac{1 - \exp(-\kappa T)}{(\kappa\Delta t) + ..}\right) \\
&\equiv O(\Delta t)
\end{aligned}$$

$$\begin{aligned}
(4e^{\kappa(\Delta t)} - 3e^{2\kappa(\Delta t)} + 2e^{2\kappa(\Delta t)}\kappa(\Delta t) - 1) &\equiv \Big(4(1 + \kappa\Delta t + ..) - 3(1 + 2\kappa\Delta t + ..) \\
&\quad +(2\kappa\Delta t)(1 + 2\kappa\Delta t + ..) - 1\Big) \\
&\equiv O(\Delta t)
\end{aligned}$$

Hence the left hand side of the equation (A-23) is of the order $O(\Delta t)$.

$$\begin{aligned}
\sum_{i=0}^{n-1}\left(E\left(\int_{t_i}^{t_{i+1}} v_s ds\right)\right)^2 &= \sum_{i=0}^{n-1}\left(\theta\Delta t + (1 - \exp(-\kappa\Delta t))\frac{E(v_{t_i}) - \theta}{\kappa}\right)^2 \\
&= n(\theta\Delta t)^2 + (1 - \exp(-\kappa\Delta t))\frac{2\theta\Delta t(v_0 - \theta)}{\kappa}\sum_{i=0}^{n-1}\exp(\frac{-i\kappa T}{n}) \\
&\quad + \frac{(1 - \exp(-\kappa\Delta t))^2}{\kappa^2}\sum_{i=0}^{n-1} E(v_{t_i} - \theta)^2 \qquad \text{(A-24)}
\end{aligned}$$

$$\sum_{i=0}^{n-1} E(v_{t_i} - \theta)^2 = \sum_{i=0}^{n-1} E(v_{t_i})^2 - n\theta^2 - 2\theta(v_0 - \theta)\sum_{i=0}^{n-1}\exp(\frac{-i\kappa T}{n}) \tag{A-25}$$

$$\begin{aligned}\sum_{i=0}^{n-1} E(v_{t_i})^2 \quad &= \quad \left((v_0-\theta)^2 - \frac{(v_0-\theta)\sigma_v^2}{\kappa} - \frac{\sigma_v^2\theta}{2\kappa}\right)\sum_{i=0}^{n-1}\exp(\frac{-i2\kappa T}{n}) \\ &\quad + \left(2(v_0-\theta)\theta + \frac{(v_0-\theta)\sigma_v^2}{\kappa}\right)\sum_{i=0}^{n-1}\exp(\frac{-i\kappa T}{n}) + n(\theta^2 + \frac{\sigma_v^2\theta}{2\kappa}) \end{aligned} \tag{A-26}$$

All terms on right hand side of equation (A-24) are of the order $O(\Delta t)$. We expand the terms to show this:

$$\begin{aligned}(1-\exp(-\kappa\Delta t))\Delta t\sum_{i=0}^{n-1}\exp(\frac{-i\kappa T}{n}) \quad &\equiv \quad (1-\exp(-\kappa\Delta t))\Delta t\left(\frac{1-\exp(-\kappa T)}{1-\exp(-\kappa\Delta t)}\right) \\ &\equiv \quad O(\Delta t)\end{aligned}$$

$$\begin{aligned}(1-\exp(-\kappa\Delta t))^2\sum_{i=0}^{n-1}E(v_{t_i}-\theta)^2 \quad &\equiv \quad (1-\exp(-\kappa\Delta t))^2\left(\sum_{i=0}^{n-1}E(v_{t_i})^2\right. \\ &\qquad \left. -n\theta^2 - 2\theta(v_0-\theta)\sum_{i=0}^{n-1}\exp(\frac{-i\kappa T}{n})\right) \\ &\equiv \quad O(\Delta t)\end{aligned}$$

Using equations (A-25) and (A-26) we can compute (A-24). Using equations (A-24) and (A-23) we can compute (A-22). The fair discrete variance strike is the expectation of the discrete realized variance

$$K_{var}^*(n) = E[V_d(0,n,T)] = E_0^Q\left[\frac{\sum_{i=0}^{n-1}(\ln(\frac{S_{i+1}}{S_i}))^2}{(n-1)\Delta t}\right]$$

Dividing equation (A-14) on both sides by $(n-1)\Delta t$ and summing from time 0 to time $n-1$

we get

$$E[V_d(0,n,T)] = \frac{r^2T}{n-1} + \frac{\sum_{i=0}^{n-1} E\left(\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} v_t v_s dt ds\right)}{4(n-1)\Delta t} - \frac{rE[\int_0^T v_t dt]}{(n-1)} + \frac{nE[\int_0^T v_t dt]}{T(n-1)} + \frac{\rho\kappa\theta E[\int_0^T v_t dt]}{(n-1)\sigma_v} - \frac{\sum_{i=0}^{n-1} E\left(\rho\kappa\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} v_t v_s dt ds\right)}{\sigma_v(n-1)\Delta t} - \frac{\sum_{i=0}^{n-1} E\left[\rho\left(\int_{t_i}^{t_{i+1}} v_t dt\right)\left(v_{t_{i+1}} - v_{t_i}\right)\right]}{\sigma_v(n-1)\Delta t} \tag{A-27}$$

This equation can be simplified as

$$E[V_d(0,n,T)] = \frac{1}{T}E[\int_0^T v_t dt] + \frac{E[\int_0^T v_t dt]}{T(n-1)} + \frac{r^2T}{n-1} - \frac{rE[\int_0^T v_t dt]}{(n-1)} + \frac{\rho\kappa\theta E[\int_0^T v_t dt]}{(n-1)\sigma_v} + \frac{\sum_{i=0}^{n-1} E\left(\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} v_t v_s dt ds\right)}{(n-1)\Delta t}\left(\frac{1}{4} - \frac{\rho\kappa}{\sigma_v}\right) - \frac{\sum_{i=0}^{n-1} E\left[\rho\left(\int_{t_i}^{t_{i+1}} v_t dt\right)\left(v_{t_{i+1}} - v_{t_i}\right)\right]}{\sigma_v(n-1)\Delta t} \tag{A-28}$$

The last term in equation (A-28) is given by equation (A-21). The second last term in equation (A-28) is given by equation (A-22). All terms except the first one on right hand side in expression (A-28) is of the order $O(\Delta t)$ or $O(\frac{1}{n})$. The first term on the right hand side is the fair continuous variance strike, i.e., $\frac{1}{T}E[\int_0^T v_t dt] = K_{var}^*$. It is given by equation (42). Hence, the discrete variance strike can be represented as:

$$K_{var}^*(n) = K_{var}^* + g(r,\rho,\sigma_v,\kappa,\theta,n) \tag{A-29}$$

where

$$g(r,\rho,\sigma_v,\kappa,\theta,n) = \frac{r^2T}{n-1} + \frac{1}{T}E\left(\int_0^T v_t dt\right)\left(\frac{1}{(n-1)} - \frac{rT}{(n-1)} + \frac{\rho\kappa\theta T}{(n-1)\sigma_v}\right) + \frac{\sum_{i=0}^{n-1} E\left(\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} v_t v_s dt ds\right)}{(n-1)\Delta t}\left(\frac{1}{4} - \frac{\rho\kappa}{\sigma_v}\right) - \frac{\sum_{i=0}^{n-1} E\left[\rho\left(\int_{t_i}^{t_{i+1}} v_t dt\right)\left(v_{t_{i+1}} - v_{t_i}\right)\right]}{\sigma_v(n-1)\Delta t} \tag{A-30}$$

$$g(r,\rho,\sigma_v,\kappa,\theta,n) = O\left(\frac{1}{n}\right) \tag{A-31}$$

Hence

$$K^*_{var}(n) \to E\left[\frac{1}{T}\int_0^T v_t dt\right] = K^*_{var} \quad \text{as } \Delta t \to 0 \qquad \square \tag{A-32}$$

Proof of Proposition 7: The discrete variance strike can be derived as follows. Applying Itô's lemma to $\ln(S_t)$ in equation (46) and integrating from t_i to t_{i+1} gives

$$\ln\left(\frac{S_{i+1}}{S_i}\right) = \int_{t_i}^{t_{i+1}} (r - \lambda m - \frac{1}{2}v_t)dt + \int_{t_i}^{t_{i+1}} \sqrt{v_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2) + \ln\left(\prod_{j=1}^{n_j} Y_j\right) \tag{A-33}$$

where n_j is number of jumps in the stock price during time $t_{i+1} - t_i$. Squaring equation (A-33), summing from time 0 to time $n-1$, dividing on both sides by $(n-1)\Delta t$ and taking expectation under the risk-neutral measure we get

$$\begin{aligned}
&E\left[\sum_{i=0}^{n-1}\frac{1}{(n-1)\Delta t}\left(\ln\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)\right)^2\right] \\
&= E\sum_{i=0}^{n-1}\frac{1}{(n-1)\Delta t}\Bigg[\left((r-\lambda m)\Delta t - \int_{t_i}^{t_{i+1}}\frac{1}{2}v_t dt\right)^2 + \left(\int_{t_i}^{t_{i+1}}\sqrt{v_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)\right)^2 \\
&+2\left((r-\lambda m)\Delta t - \int_{t_i}^{t_{i+1}}\frac{1}{2}v_t dt\right)\left(\int_{t_i}^{t_{i+1}}\sqrt{v_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)\right) \\
&+\left(\sum_{j=1}^{n_j}\ln Y_j\right)^2 + 2\left((r-\lambda m)\Delta t - \int_{t_i}^{t_{i+1}}\frac{1}{2}v_t dt\right)\left(\sum_{j=1}^{n_j}\ln Y_j\right) \\
&+2\left(\int_{t_i}^{t_{i+1}}\sqrt{v_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)\right)\left(\sum_{j=1}^{n_j}\ln Y_j\right)\Bigg]
\end{aligned} \tag{A-34}$$

The first two lines of expressions on the right hand side of equation (A-34) can be computed using Proposition 5 and it is equal to

$$\frac{1}{T}E\left[\int_0^T v_t dt\right] + g(r-\lambda m,\rho,\sigma_v,\kappa,\theta,n) \tag{A-35}$$

and using equation(A-31) we get

$$g(r-\lambda m,\rho,\sigma_v,\kappa,\theta,n) = O\left(\frac{1}{n}\right) \tag{A-36}$$

The last two lines of expressions on the right hand side of equation (A-34) can be computed using Proposition 2 and it is equal to

$$\lambda(a^2+b^2) + \frac{\lambda(a^2+b^2) + \lambda^2 a^2 T + \lambda a\Big(2(r-\lambda m)T - E[\int_0^T v_t dt]\Big)}{n-1} \tag{A-37}$$

where $E[\int_0^T v_t dt]$ is given by equation (42). Hence using equations (A-34), (A-35) and (A-35) the fair discrete variance strike is given by:

$$\begin{aligned} K^*_{var}(n) &= E\Big[V_d(0,n,T)\Big] = E\Big[\sum_{i=0}^{n-1}\frac{1}{(n-1)\Delta t}\ln\Big(\frac{S_{t_{i+1}}}{S_{t_i}}\Big)\Big]^2 \\ &= \frac{1}{T}E\Big[\int_0^T v_t dt\Big] + \lambda(a^2+b^2) + g(r-\lambda m, \rho, \sigma_v, \kappa, \theta, n) \\ &\quad + \frac{\lambda(a^2+b^2) + \lambda^2 a^2 T + \lambda a\Big(2(r-\lambda m)T - E[\int_0^T v_t dt]\Big)}{n-1} \end{aligned} \tag{A-38}$$

Hence, the fair discrete variance strike can be represented as:

$$K^*_{var}(n) = K^*_{var} + h(r, \rho, \sigma_v, \kappa, \theta, m, b, n) \tag{A-39}$$

where

$$K^*_{var} = \theta + \frac{v_0 - \theta}{\kappa T}(1 - e^{-\kappa T}) + \lambda(a^2+b^2) \tag{A-40}$$

and

$$\begin{aligned} h(r, \rho, \sigma_v, \kappa, \theta, m, b, n) &= g(r-\lambda m, \rho, \sigma_v, \kappa, \theta, n) \\ &\quad + \frac{\lambda(a^2+b^2) + \lambda^2 a^2 T + \lambda a\Big(2(r-\lambda m)T - E[\int_0^T v_t dt]\Big)}{n-1} \end{aligned} \tag{A-41}$$

$$h(r, \rho, \sigma_v, \kappa, \theta, m, b, n) = O\Big(\frac{1}{n}\Big) \tag{A-42}$$