Optimal Dynamic Allocation between Taxable and Nontaxable Assets

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ABSTRACT

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This dissertation studies the problem of dynamically trading between taxable and nontaxable assets in order to maximize the expected utility of terminal wealth.

Some properties of the optimal solution are demonstrated and are applied to simplify the representation, analysis and computation of the optimal policy. In particular, the optimal policy can be simply described by one or two target lines in a two-dimensional state space. Here a target line is a curve that consists of the optimal target points on all trading tracks in a certain direction, and a trading track is a set of states with a common characteristic value. The analytical solution for the optimal policy just prior to the termination is also derived. Besides providing a reference for numerical solutions, it can be used throughout the whole time horizon as a good approximate optimal policy.

Several numerical methods are developed for these problems. One method calculates expectation by solving partial differential equations (PDE), and two methods by applying simulation (using point-estimation and regression, respectively). The PDE-based method has the highest computational efficiency and provides the best solution, while the two simulation-based methods can also provide solutions which have investment performance very close to that of PDE-based method.

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To My Parents

Chapter 1

Introduction

Taxes are an important concern to investors and the impacts of taxes are studied in a large pool of literature. However, finding the optimal policy for dynamic investment in the presence of taxes is difficult, because it is a complicated stochastic control problem, for which there is usually no analytical solution.

In recent years, some progress on the numerical solution of tax-aware portfolio optimization models has been accomplished. Considering capital-gains tax and calculating tax for each share of sold stock with its purchase price as the basis (i.e., using the exact tax basis), Dybvig and Koo [5] formulated a dynamic investment problem to maximize the expected utility of final wealth with a fixed horizon, and numerically solved the problem with four periods and one stock.

DeMiguel and Uppal [4] extended the Dybvig and Koo model to include transaction costs, intermediate consumption and labor income, and solved an example with seven periods and two stocks, as well as an example with ten periods and one stock, by applying an efficient nonlinear programming algorithm. However, using the exact tax basis leads to path dependency, and cause the state space to increase exponentially with the number of periods. It is thus not practical to solve problems with more than a small number of periods.

Using the weighted average purchase price as the tax basis, which removes the path dependency, Dammon, Spatt and Zhang ([2] and [3]) provided numerical results for long-term (80 steps) consumption and investment on one or two risky stocks with taxes.[1]

In this dissertation, we consider the numerical solution to the optimal allocation between taxable and nontaxable assets.[2] We focus on a class of problems with the following conditions: constant relative risk aversion (CRRA) type utility function, fixed terminal horizon, with and without the constraint of one-way trading direction from taxable to nontaxable or the reverse. To our knowledge, there has been little research on the numerical solution for this problem.[3] Solving this simple two-assets problem will further our understanding of the impact of taxes and other parameters on the optimal dynamic portfolio strategies, and can serve as a starting point to solve more complicated models.

Unlike the aforementioned research works, we, (1) build a model and conduct computations in a continuous space for the asset price processes (instead of using a

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[1] In the papers [5], [4], [2] and [3], the assets include a riskless bond, in addition to the stock(s).

[2] A nontaxable asset may be a tax-exempt or tax-free asset (e.g., municipal bond), or a asset held in a nontaxable account (such as the Roth IRA).

[3] Huang [6] considered a model with taxable and tax-deferred account, where the tax-deferred account is treated as a nontaxable account (Roth IRA), and there may be multiple assets (not just one asset, as in our model) in each account, but she did not provide numerical results for the optimal policy.

In addition, it is important to note that there are some differences between nontaxable and tax-deferred account: the contribution to a tax-deferred account (such as a 401(k)) is thought of as the net income or gain that will be charged tax, while the asset in a nontaxable account can be transferred from any source of wealth (that could be savings of many years) not just the current income after tax. These two types of accounts are not "equivalent within a constant factor" as treated in [6].

binomial approximation of the asset price processes which may lead to substantial errors); (2) conduct further analysis on the properties of the optimal solution and provide a simple representation of the optimal policy in the form of target line(s) instead of an exhaustive list of decisions at each state.

The rest of this dissertation is organized as follows. In Chapter 2, we present the basic setting of the model and the dynamic programming formulation, demonstrate several properties of the optimal solution for our problem, and represent the problem in a two-dimensional state space, which makes it easier to represent and analyze the optimal policy

In Chapter 3, we study the optimal policies for our problems where trades are made at discrete times. We show the simple structures of the optimal policies, which can be simply described by one or two target lines. As an approximate solution for our problems, we present the analytical solution for the time just prior to termination. We call this the "limiting solution," which is useful to check the numerical solution. Last, we present computational results to illustrate characteristics of the optimal policy (e.g., the shapes of the target lines and their movement in time) and compare performance of the optimal policy and several heuristic policies. One important result is that the myopic policy, which uses the "limiting solution" at all times prior to termination, gives a performance nearly indistinguishable from the optimal policy for two-way trading problems.

In Chapter 4, we introduce the numerical solution methods: a PDE-based (using finite difference) and two simulation-based (using point-estimation and regression, respectively) numerical methods to compute the optimal policy. The key points of the implementation are presented and the computational performance of these numerical methods are compared.

Chapter 2

The model and preliminaries

2.1 The model

2.1.1 Basic setting

In this dissertation, we consider the following situation: there are two assets that can be traded - one is taxable (hereafter we simply call it the stock) and the other is nontaxable (we call it the index fund).[1] A nontaxable asset may be a tax-exempt or tax-free asset (e.g., a municipal bond), or an asset held in a nontaxable account (such as Roth IRA[2]). Let $s(t)$ and $i(t)$ be the price of the stock and index fund at time t, respectively, and assume that they follow geometric Brownian motion diffusion processes:

$$ds(t)/s(t) = \mu_s dt + \sigma_s dz \tag{2.1.1}$$

$$di(t)/i(t) = \mu_i dt + \sigma_i dv \tag{2.1.2}$$

[1] We can imagine a practical situation where an investor has only two funds to invest, one is in a taxable account and the other is in a nontaxable account. Here calling them stock and index fund is just for convenience.

[2] In practice, there are some restrictions on a Roth IRA account, such as maximum annual contributions and penalties for early withdrawals. In this dissertation, we assume that there are no such restrictions on the nontaxable account.

where dv and dz are increments to standard Brownian motions with correlation ρ. We assume that the investor can trade only in discrete time $\{t_k|k = 0, 1, ..., N\}$, and the final time is $T = t_N$.[3] In order to represent the controllable state variables before and after trading, we let t_k^+ denote the time just after decision at time t_k. For brevity, the uncontrolled variables $i(t_k)$ and $s(t_k)$ are denoted by i_k and s_k.

Let $x(t)$ and $y(t)$ be the number of shares (or units) of stock and index fund, respectively, in the investor's account at time t. Obviously $x(t)$ and $y(t)$ remain constant in the period $[t_k^+, t_{k+1}]$ for any $k < N$. We assume that there is a short-sale constraint, that is, the amount of stock and index fund must be nonnegative (i.e., $x(t) \geq 0$ and $y(t) \geq 0$ for all t).

Let $I(t)$ be the dollar value of index fund holdings at time t, i.e., $I(t) = y(t)i(t)$. Between transactions, since $y(t)$ is constant, $I(t)$ is a geometric Brownian motion process with the same parameters as that of $i(t)$.

When shares of stock are sold, some tax may be taken from the sale and the remaining value invested in the index fund. If some units of the index fund are sold, there is no need to pay tax and the entire sales value is invested in the stock. There is no cash flow in or out, and no other transaction costs.

At any trading time t_k, there are two kinds of trades as follows:

- Simple trade: a trade (i.e., an exchange of assets) in which a part of investor's assets is transfered in one direction, either from the taxable to the nontaxable

[3] We use the discrete-time setting in order to make it convenient to represent the problem in a dynamic programming framework and describe the algorithm for a computer. When the time-step is small enough the solution will approach that of a continuous-time problem.

(denoted by "TN") or the reverse (denoted by "NT").

- Compound trade: a series of simple trades.

We call a "one-way trading problem" an investment problem with the restriction that trading is only allowed in a single direction for all trading times, and use the terms "PTN" or "PNT" to denote the one-way trading problem in "TN" or "NT" direction. Correspondingly, we call a "two-way trading problem" a problem without such restriction. It includes two cases: allowing compound trades or not; and they are named as "PCT" and "PNCT", respectively. In problem PNCT, the investor can only make a simple trade (in either direction) at one trading time; while in PCT, the investor can sell and buy stock many times at one trading time.

In the real world, tax laws are complicated, differ across countries, and change over time. Here we consider two simple methods to compute the tax payment, they represent two cases of tax law: allowing reimbursement for loss (RL) and not (NRL). Specifically, given a basis price p and a fixed tax rate α, if d shares of stock are sold at time t, the tax payment will be:

- $\alpha d(s(t) - p)^+$, for the NRL case, or

- $\alpha d(s(t) - p)$, for the RL case.

In the U.S. tax code, there are two allowable methods for setting the basis price: 1) each share of stock uses its purchase price (termed the exact basis price), and 2) all shares of stock use the average purchase price (APP). In this dissertation, we only consider the latter. Let $a(t)$ be the average purchase price of all shares of stock held at time t. Then given $a(t_0)$, $a(t)$ is determined as follows:

- $a(t)$ is constant in the period $[t_k^+, t_{k+1}]$ for any $0 \leq k < N$;
- at time t_k,
  - after a simple trade of selling some shares of stock, $a(t_k^+) = a(t_k)$
  - after a simple trade of buying u shares of stock,

    $a(t_k^+) = \frac{a(t_k)x(t_k)+s(t_k)u}{x(t_k)+u}$,
  - after a compound trade of selling d shares of "old" stock and buying u shares of "new" stock,[4] $a(t_k^+) = \frac{a(t_k)(x(t_k)-d)+s(t_k)u}{x(t_k)-d+u}$.

Using the exact basis price provides investors more flexibility and results in an optimal policy superior to that obtained by using APP. However the computational results of [4] show that the optimal solutions for both situations are very close and we conjecture that this will hold for our problem. In addition, using APP makes the problem path-independent and simplifies the computation considerably. In fact, we will see that using APP allows problems to be represented and solved in a two-dimensional state space.

We define $b(s, p)$ as the realized cash from the sale of one share of stock when the stock price is s and the basis price is p. That is,

- for the NRL case, $b(s, p) = s - \alpha(s-p)^+ = (1-\alpha)s + \alpha \min(s, p)$,

[4] It can be shown that no matter how complicated a compound trade is (i.e., no matter how many simple trades it uses and in what order), the compound trade can always be reduced to a series of two simple trades: first selling stock (a TN trade) and then buying stock (an NT trade).

- for the RL case, $b(s,p) = (1-\alpha)s + \alpha p$.

Hereafter, we call $b(s(t), a(t))$ the "realized price" of stock at time t. For all $x \geq 0$, $xb(s,a) = b(xs, xa)$ in both the NRL and the RL cases.

Let $w(t) = x(t)b(s(t), a(t)) + I(t)$ denote the net wealth at time t if both assets are entirely liquidated, i.e., the "realizable wealth." The investor's objective is to maximize the expected utility of realizable wealth at the final time T, i.e.

$$\max E[U(w(T))],$$

for given utility function $U(w)$. We assume that the utility function is of the constant relative risk aversion (CRRA) type, i.e., for final wealth w, utility is $U(w) = w^\gamma/\gamma$ with $\gamma < 1$ and $\gamma \neq 0$ or $U(w) = \ln(w)$ if $\gamma = 0$.

For the CRRA type of function $U(w)$, we have, for any scaler $\lambda > 0$,

$$U(\lambda w) = \begin{cases} \ln(\lambda) + U(w), & \text{if } \gamma = 0 \\ \lambda^\gamma U(w), & \text{if } \gamma \neq 0 \end{cases}$$

For simplicity, hereafter we present formulation only for problems with utility parameters $\gamma \neq 0$. Similar results hold in all Propositions and Theorems for problems with $\gamma = 0$.

Although we only consider the portfolio optimization problem with two assets, the model can offer advice for some real applications — for example, problem PTN may be applicable to the case where an investor holds a large position in a company stock and wants to liquidate all his holdings to a nontaxable asset by a final date.

In some sense, the tax in this dissertation is referred to as the capital gains tax. A riskless bond with interest rate (or the annual distribution rate, in the case of a bond with coupons or dividends) r in the presence of an income tax rate η can be thought of as a nontaxable asset with interest rate $(1-\eta)r$. Hence, the model of one stock and one riskless bond is just a special case of our model.

2.1.2 Dynamic programming formulation

For later use, we introduce the new variable $X(t) = x(t)a(t)$, the total basis value of the stock at time t. $X(t)$ remains constant in the no-trading period $[t_k^+, t_{k+1}]$ for any $k < N$, since if $x(t) = 0$ then $X(t) = 0$, while if $x(t) > 0$, $x(t)$ and $a(t)$ remain constant in that period.

Let $V_k(x, I, X, s)$ be the value function for pre-decision state $(x(t_k), I(t_k), X(t_k), s_k)$ at time t_k, we can describe our problems in a dynamic programming framework as follows.[5]

(P1):

- at the final stage N,

$$V_N(x, I, X, s) = U(I + b(xs, X))$$

- at stage $k < N$,

$$V_k(x, I, X, s) = \max_{(x', I, X') \in \Omega(x, I, X, s)} \underline{V}_k(x', I', X', s)$$

  with

$$\underline{V}_k(x, I, X, s) = E[V_{k+1}(x, I(t_{k+1}), X, s_{k+1}) | s_k = s, I(t_k^+) = I]$$

[5]We use $X(t)$ instead of $a(t)$ for convenience in representing some properties, especially Theorem 2 in Chapter 3.

Here $\underline{V}_k(x, I, X, s)$ is the indirect value function for the post-decision state $(x(t_k^+), I(t_k^+), X(t_k^+), s_k)$ at time t_k. To distinguish it from the value function $V_k()$, we call $\underline{V}_k()$ the value base function. When the stock price is s, the set of states (x', I', X') reachable from (x, I, X) is denoted by $\Omega(x, I, X, s)$. Depending on the trading restrictions, we formally define four subproblems by the set of reachable states as follows:

1. **Problem PTN:** one-way trading from the taxable to the nontaxable asset.

$$\Omega_{TN}(x, I, xa, s) = \{(x - d, I + db(s, a), (x - d)a) | 0 \le d \le x\}$$

   where the APP basis a is constant for all time.

2. **Problem PNT:** one-way trading from the nontaxable to the taxable asset.

$$\Omega_{NT}(x, I, X, s) = \{(x + u, I - us, X + us) | 0 \le u \le I/s\}$$

3. **Problem PCT:** two-way trading with compound trades.

$$\begin{aligned} \Omega_{CT}(x, I, xa, s) = \{ \quad & (\, x + u - d,\ I - us + db(s, a),\ xa + us - da\,) \\ & |\ \ 0 \le d \le x, \quad 0 \le u \le \frac{I + db(s, a)}{s} \ \} \end{aligned}$$

4. **Problem PNCT:** two-way trading with no compound trades.

$$\Omega_{NCT}(x, I, X, s) = \Omega_{TN}(x, I, X, s) \bigcup \Omega_{NT}(x, I, X, s)$$

To clearly show the effects of the basis price and simplify the expressions, we use xa instead of X in Ω_{TN} and Ω_{CT}. Otherwise we will need to replace a by X/x, and prohibit any d when $x = 0$. In problem PTN, since the basis price $a(t)$ will not change, we do not need the state variable X and a simpler formula applies.

In this dissertation, we only list problem PNCT as an example of a different trading restriction, but do not provide specific analysis for its optimal policy. However, we can show some results such as that when the trading interval is short enough, the optimal policy for PNCT is close to that of PCT.

2.2 Preliminaries

2.2.1 Basic properties

From the model setting, we find some basic properties for our problems. They will be used later to simplify the model, analyze the optimal policy and design numerical solution methods.

Proposition 1. *For problems PTN, PNT, PCT and PNCT, the following properties hold:*

(1) *For any post-trade state* $(x(t_k^+), I(t_k^+), X(t_k^+)) \in \Omega(x(t_k), I(t_k), X(t_k), s)$, $w(t_k^+) = w(t_k)$*;* (i.e., the realizable wealth will not change in trading)*;*

(2) *For any scalar* $\lambda > 0$, $V_k(\lambda x, \lambda I, \lambda X, s) = \lambda^\gamma V_k(x, I, X, s)$*;*[6] (this implies that the optimal policy can be described in a way that is independent of the level of wealth)*;*

(3) *For any scalar* $p > 0$, $V_k(px, I, X, s/p) = V_k(x, I, X, s)$*; furthermore, there exists a function* $J_k(I, X, Y)$ *such that* $J_k(I, X, xs) = V_k(x, I, X, s)$*;*

[6] If $\gamma = 0$, then $V_k(\lambda x, \lambda I, \lambda X, s) = \ln(\lambda) + V_k(x, I, X, s)$.

(4) *$V_k(x, I, X, s)$ is increasing in x, I and X, and more strongly, $J_k(I, X, Y)$ is increasing in I, X and Y.*

Proof:

Here we provide a detailed proof for (1), and give the main points to prove (2), (3) and (4).

(1) We only need to prove $w(t_k^+) = w(t_k)$ in either TN or NT trading, because based on this fact it is easy to see $w(t_k^+) = w(t_k)$ for any sequence of simple trades and then for two-way trading problems.

In TN trading,

$$a(t_k^+) = a(t_k) \quad \text{and} \quad I(t_k^+) = I(t_k) + (x(t_k^+) - x(t_k^+))b(s_k, a(t_k)),$$

$$w(t_k^+) = x(t_k^+)b(s_k, a(t_k^+)) + I(t_k^+) = x(t_k)b(s_k, a(t_k)) + I(t_k) = w(t_k).$$

In NT trading,

$$I(t_k^+) = I(t_k) - s_k(x(t_k^+) - x(t_k)),$$

$$a(t_k^+) = \frac{a(t_k)x(t_k) + s_k(x(t_k^+) - x(t_k))}{x(t_k^+)},$$

$$X(t_k^+) = x(t_k^+)a(t_k^+) = X(t_k) + s_k(x(t_k^+) - x(t_k)),$$

using $X(t_k^+) - s_k x(t_k^+) = X(t_k) - s_k x(t_k)$ and $I(t_k^+) + s_k x(t_k^+) = I(t_k) + s_k x(t_k)$,

$$w(t_k^+) = I(t_k^+) + s_k x(t_k^+) - \alpha f(s_k x(t_k^+) - X(t_k^+)) = w(t_k)$$

with $f(z) = z$ for the RL case, or $f(z) = z^+$ for the NRL case.

(2) We can prove this by coupling two states $(\lambda x, \lambda I, \lambda X, s)$ and (x, I, X, s) with a proportional policy in the same sample path of $s(t)$ and $i(t)$, from the fact that the tax payment (a kind of transaction cost) is proportional to the trade size, and the utility function is homogeneous: $U(\lambda w) = (\lambda)^{\gamma} U(w)$.

(3) From $b(s, a) = pb(s/p, a/p)$ and the process $s(t)/p$ is a geometric Brownian process with the same parameters as that of $s(t)$, we can see x shares of stock with price s and APP a is equivalent to xp shares of stock with price s/p and APP a/p, and hence $V_k(x, I, xa, s) = V_k(px, I, xpa/p, s/p) = V_k(px, I, xa, s/p)$.

(4) We can prove this by the fact that for two states $Z_1 = (I_1, X_1, Y_1)$ and $Z_2 = (I_2, X_2, Y_2)$ with $I_1 \leq I_2$, $X_1 \leq X_2$ and $Y_1 \leq Y_2$, if Z_1 changes to (I_1', X_1', Y_1') then Z_2 can change to (I_2', X_2', Y_2') with $I_1' \leq I_2'$, $X_1' \leq X_2'$ and $Y_1' \leq Y_2'$.

□

Using property (4) of Proposition 1, we can prove that in the RL case, the optimal policy must realize the capital loss on taxable asset, when nontaxable asset is a riskless bond. This result is well known in work relating to investment with taxes, see e.g., Constantinides [1]. It is also true when the nontaxable asset always has a positive return although it may be volatile (though this violates our assumption of GBM on the assets).

2.2.2 Representation in two-dimensional state space

To solve our problem efficiently and represent the optimal policy simply, we can reduce the dimensions of the state space.

In the no-trading period $[t_k^+, t_{k+1}]$, $a(t)$ is constant. Applying property (3) in Proposition 1, we have $V_k(x, I, X, s) = V_k(X, I, X, s/a)$, where there are only three different variables in the right-hand side. We can use the ratio of $s(t)/a(t)$ instead of $s(t)$ and $a(t)$ so that the dimensions of state space is reduced by one.

Since $a(t)$ is meaningful only when $x(t) > 0$, we introduce a new state variable $c(t)$ defined by: $c(t) = s(t)/a(t)$ when $x(t) > 0$ and $c(t) = 1$ when $x(t) = 0$. We call $c(t)$ the "relative price."

To simplify the formulation, we introduce a new function

$$q(c) = b(c, 1).$$

We can see $b(s, p) = pb(s/p, 1) = pq(s/p)$ for both the NRL and the RL cases.

From property (2) in Proposition 1, we can represent our problem with wealth-independent state variables, and reduce the state space by one more dimension. We introduce a new state variable $z(t) = I(t)/w(t)$, the ratio of the nontaxable asset value to the wealth, termed the "N/W ratio."

Now we can represent our problem with a new two-dimensional state $(z(t_k), c(t_k))$ at each stage k. Let $H_k(z, c)$ be the indirect utility value of a state which has one unit wealth, N/W ratio z, and relative price c at stage k. Using $V_k(x, I, X, s) =$

$V_k(X, I, X, c)$ and $w = I + Xq(c) = 1$, we have

$$H_k(z, c) = V_k((1 - z)/q(c), z, (1 - z)/q(c), c)$$

The original problem formulation (P1) in Section 2.1 can be transformed to the following form.

(P2):

- at the final stage N,

$$H_N(z, c) = U(1)$$

- at stage $k < N$

$$H_k(z, c) = \max_{(z', c') \in \Omega'(z,c)} \underline{H}_k(z', c')$$

with

$$\underline{H}_k(z', c') = E[(w_{k+1})^{\gamma} H_{k+1}(z_{k+1}, c' s_{k+1}/s_k)]$$

$$w_{k+1} = \frac{i_{k+1}}{i_k} z' + \frac{q(c' s_{k+1}/s_k)}{q(c')}(1 - z'),$$

$$z_{k+1} = \frac{i_{k+1} z'}{i_k w_{k+1}}$$

The variables w_{k+1} and z_{k+1} are the wealth and the N/W ratio at time t_{k+1} respectively, given a state with post-decision N/W ratio z', relative price c' and wealth $w(t_k) = 1$ at time t_k^+.

The set of states (z', c') reachable from (z, c) is denoted by $\Omega'(z, c)$. It can be defined by using some characteristics of (z, c) in the 2-D state space.

For simple (TN or NT) trading, given an initial state, the set of reachable states is a continuous curve, and the states on this curve share a common characteristic value. In the state space (z, c), it is natural to use c as the characteristic value for TN trading. For NT trading, we use the value K, defined by:

$$K(z, c) = \frac{(c-1)(1-z)}{z(q(c)-1)+1} \tag{2.2.1}$$

as the characteristic value. It is easy to verify that for any two states with the same characteristic value (c or K), one of states can reach the other by a simple (TN or NT) trade.

We call a "trading track" the set of all the states with a given characteristic value (c or K). We let $\mathcal{T}^T(c)$ denote the TN trading track with relative price c, i.e.,

$$\mathcal{T}^T(c) = \{(z, c) : 0 \le z \le 1\},$$

and let $\mathcal{T}^N(K)$ denote the NT trading track with characteristic K, i.e.,

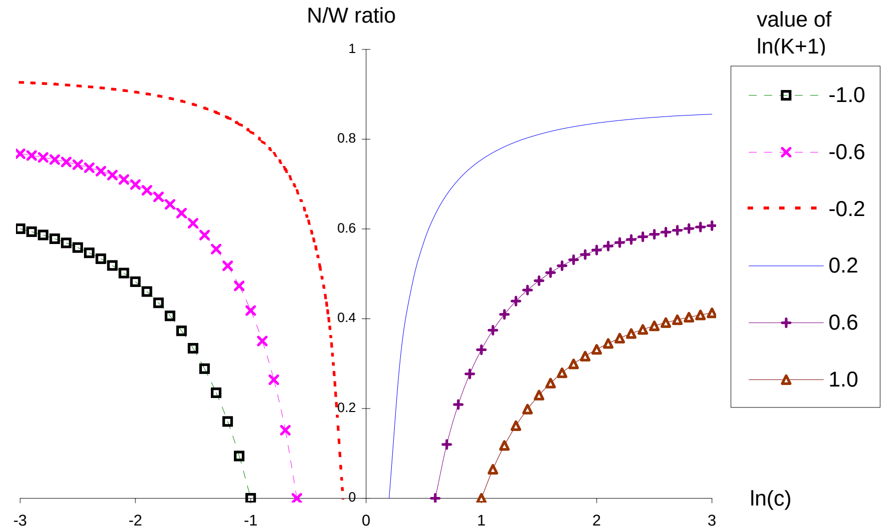
$$\mathcal{T}^N(K) = \{(z, c) : \frac{(c-1)(1-z)}{z(q(c)-1)+1} = K, 0 \le z \le 1 \text{ and } c > 0\},$$

or in the form where z is free:

$$\begin{aligned} \mathcal{T}^N(0) &= \{(z, 1) : 0 \le z \le 1\} \\ \mathcal{T}^N(K) &= \{(z, 1 + \frac{K}{1 - zq(K+1)}) : 0 \le z < m(K)\}, \text{ if } K \ne 0 \end{aligned}$$

where $m(K)$ is the maximal possible z value on the NT trading track with characteristic K. Specifically

$$m(K) = \begin{cases} 1, & \text{if } K = 0 \\ 1/q(K+1), & \text{if } K > 0 \\ (K+1)/q(K+1), & \text{if } K < 0 \end{cases}$$



This figure shows the NT trading tracks for some characteristic K values which are selected such that the value of $\ln(K+1)$ are -1.0, -0.6, -0.2, 0.2, 0.6 and 1.0. The part of a NT trading track down from a state represents the set of all the states that can be reached from this state by a NT trade. Thus a NT trading track also shows how the N/W ratio z and the relative price c change in NT trades.

Figure 2.1: NT Trading tracks

Figure 2.1 shows several NT trading tracks in the space $(z, \ln(c))$.[7] In the NT trading direction, the N/W ratio decreases and the relative price c moves closer to 1, i.e., $\ln(c)$ moves closer to 0.

We also can verify that:

1. In the original state space (x, I, X, s), the K value of a NT trading track passing state (x, I, X, s) is:

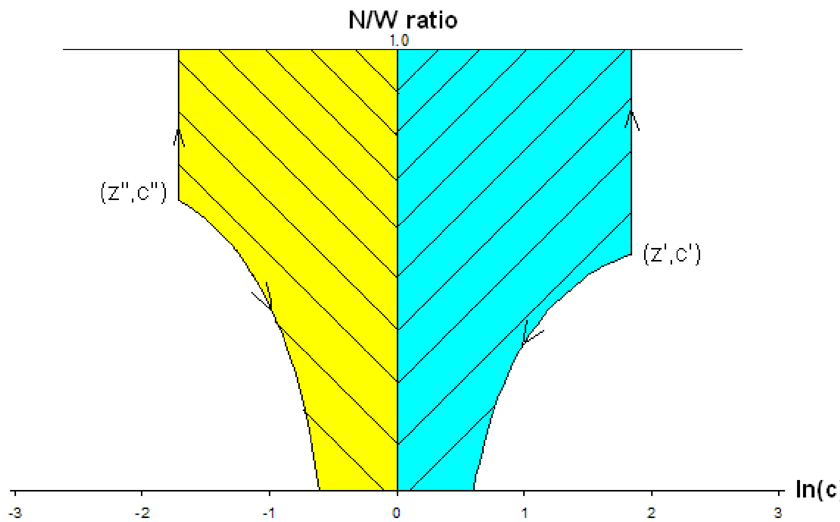
$$K = (xs - X)/(I + X).$$

2. For trading track with characteristic value K, given the N/W ratio z, the

[7] In all figures related to state space in this dissertation except the Appendix, the horizontal axis is measured in logarithm of relative price, i.e., $\ln(c)$.

relative price c is:

$$C(z, K) = 1 + \frac{K}{1 - zq(K+1)} \tag{2.2.2}$$



For problem PTN (or PNT), the set of reachable states for a given state is the part of TN (or NT) trading track moving up (or down) from it. For problem PCT, the set of reachable states for (z', c') with $c' > 1$, is the shaded area on the right side of line $c = 1$ (i.e., $\ln(c) = 0$) bounded by the TN and NT trading tracks; the set of reachable states for (z'', c'') with $c'' < 1$, is the shaded area on the left side of line $c = 1$.

Figure 2.2: Reachable states

From the relation between (x, I, X, s) and (z, c), the set of states reachable from (z, c) for each problem is as follows:

$\Omega'_{TN}(z, c) = \{(y, c) |\ y \geq z\}$,

$\Omega'_{NT}(z, c) = \{(y, 1 + \frac{K(z,c)}{1 - yq(K(z,c)+1)}) |\ 0 \leq y \leq z\}$,

$\Omega'_{NCT}(z, c) = \Omega'_{NT}(z, c) \bigcup \Omega'_{TN}(z, c)$,

$\Omega'_{CT}(z, c) = \{(y, 1 + \frac{K(v,c)}{1 - yq(K(v,c)+1)}) |\ 0 \leq y \leq v, v \geq z\}$.

Figure 2.2 illustrates the reachable states for each problems in the space $(z, \ln(c))$.

Chapter 3

Optimal policies

3.1 Optimal policy at discrete times

In this section, we study the properties of the optimal policies for problems PTN, PNT and PCT. These properties make the computation and representation of optimal policies simpler. The analysis for problem PCT is not as strong as that for one-way trading problems PTN and PNT, for it relies on additional assumptions.

3.1.1 One-way trading problems

For one-way trading problems, we have the following two theorems:

Theorem 1. *In problem PTN, for* $k = 0, 1, 2, ..., N-1$,

(1) $V_k(x, I, xp, s)$ *and* $\underline{V}_k(x, I, xp, s)$ *are concave in* x *and* I*;*

(2) there exists a function $x_k^*(w, s, p)$ *such that the optimal policy for state* (x, I, xp, s) *at stage* k *is: if* $x > x_k^*(I + xb(s, p), s, p)$*, then sell stock to reach the position* $x_k^*(I + xb(s, p), s, p)$*, otherwise remain at* x*.*

Proof: See Appendix.

Theorem 2. *In problem PNT, for* $k = 0, 1, 2, ..., N-1$, *(1) $V_k(x, I, X, s)$ and $\underline{V}_k(x, I, X, s)$ are concave in x, I and X;*

(2) there exists a function $x_k^*(x, I, X, s)$ such that the optimal policy for state (x, I, X, s) at stage k is: if $x < x_k^*(x, I, X, s)$, then buy stock to reach the position $x_k^*(x, I, X, s)$, otherwise remain at x.

Proof: See Appendix.

To represent the optimal policy in (z, c) state space, we define the optimal N/W ratio for each trading track.

- For problem PTN, at stage k for relative price c, let $z_k^T(c)$ be the optimal N/W ratio on TN trading track $\mathcal{T}^T(c)$, i.e.,

$$z_k^T(c) = \arg \max_{0 \leq y \leq 1} \underline{H}_k(y, c)$$

- For problem PNT, at stage k for characteristic value K, let $z_k^N(K)$ be the optimal N/W ratio on NT trading track $\mathcal{T}^N(K)$, i.e.,

$$z_k^N(K) = \arg \max_{0 \leq y \leq m(K)} \underline{H}_k(y, C(y, K))$$

For brevity, we call $z_k^T(c)$ and $z_k^N(K)$ the "target N/W ratio."

From Theorems 1 and 2, we know that, for one-way trading problem, the local optimal state is unique for each trading track, and the optimal policy is of the "threshold" type. Correspondingly, with the characteristic value c or K for each TN or NT trading track, the optimal policy can be represented as follows

- For problem PTN, at stage k, for state (z, c), the optimal action is: if $z < z_k^T(c)$ then increase the N/W ratio to $z_k^T(c)$ (by selling stock), otherwise do not trade.

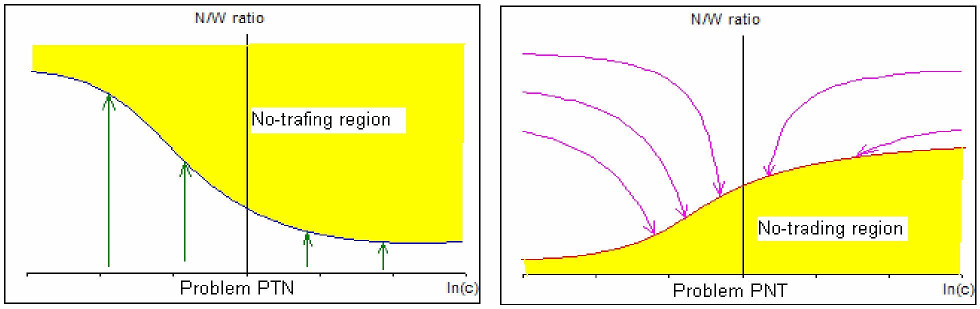


Figure 3.1: Optimal policy for one-way trading problems

- For problem PNT, at stage k, for state (z, c), with characteristic $K = K(z, c)$, the optimal action is: if $z > z_k^N(K)$ then decrease the N/W ratio to $z_k^N(K)$ (by purchasing stock), otherwise do not trade.

The optimal target states (represented, in (z, c) space, as $(z_k^T(c), c)$ for PTN and $(z_k^N(K), C(z_k^N(K), K))$ for PNT) are continuous (since the state and action spaces are continuous) and form a trading/no-trading boundary which separates the state space into "Trading" and "No-trading" regions. If the state is in the trading region, the optimal policy is to trade to move to the boundary; while if it is in the no-trading region, then the optimal policy is to make no trade. Figure 3.1 illustrates this "threshold" optimal policies for problems PTN and PNT.

3.1.2 Problem PCT

For problem PCT, where multiple NT and TN simple trades are allowed at any trading time, we haven't proved the similar concavity of the value base function, or a weaker statement called Condition A that the local optimal states in each NT and TN trading track is unique, for all the times like theorems for one-way trading problems. However, we further our research on the cases where a weaker version of Condition A (which we call Condition A') holds. In our computational experiments, Condition A, which also implies A', holds (within a very small numerical error tolerance) at all times, so we conjecture that it is true. In the later half portion of this subsection, we will analyze the properties of the optimal policy shown in our computational experiments.

Define Condition A as the situation where there is a unique local optimal state in each NT and TN trading track. Specifically, Condition A holding at stage k can be represented in state space (z, c) with value base function $\underline{H}_k(z, c)$ as follows:

- There exists $z_k^T(c)$ for each trading track $\mathcal{T}^T(c)$, such that

  $\underline{H}_k(z_k^T(c), c) = \max_{(y,c)\in\mathcal{T}^T(c)} \underline{H}_k(y, c)$, and

  the local optimality holds for $z_k^T(c)$ on $\mathcal{T}^T(c)$, i.e.,

  for any $z_1 < z_2 < z^T(c)$, $\underline{H}_k(z_1, c) \leq \underline{H}_k(z_2, c)$;

  for any $z_1 > z_2 > z^T(c)$, $\underline{H}_k(z_1, c) \leq \underline{H}_k(z_2, c)$.

- There exists $z_k^N(K)$ for each trading track $\mathcal{T}^N(K)$, such that

  $\underline{H}_k(z_k^N(K), C(z_k^N(K), K)) = \max_{(y,c)\in\mathcal{T}^N(K)} \underline{H}_k(y, c)$, and

  for any two states $(z_1, c_1), (z_2, c_2) \in \mathcal{T}^N(K)$,

  if $z_1 < z_2 < z^N(K)$, or $z_1 > z_2 > z^N(K)$, then $\underline{H}_k(z_1, c_1) \leq \underline{H}_k(z_2, c_2)$.

Define Condition A' as the situation where in each NT or TN trading track, the value base function $\underline{V}()$ or equivalent $\underline{H}()$ will decrease along the trading direction after reaching the maximum value. Specifically, Condition A' holding at stage k

can be represented in the state space (z, c) with the value base function $\underline{H}_k(z, c)$ as follows:

- There exists $z_k^T(c)$ for each trading track $\mathcal{T}^T(c)$, such that

  $\underline{H}_k(z_k^T(c), c) = \max_{(y,c)\in\mathcal{T}^T(c)} \underline{H}_k(y, c)$, and

  for any $z_1 > z_2 > z^T(c)$, $\underline{H}_k(z_1, c) \leq \underline{H}_k(z_2, c)$.

- There exists $z_k^N(K)$ for each trading track $\mathcal{T}^N(K)$, such that

  $\underline{H}_k(z_k^N(K), C(z_k^N(K), K)) = \max_{(y,c)\in\mathcal{T}^N(K)} \underline{H}_k(y, c)$, and

  for any two states $(z_1, c_1), (z_2, c_2) \in \mathcal{T}^N(K)$,

  if $z_1 < z_2 < z^N(K)$, then $\underline{H}_k(z_1, c_1) \leq \underline{H}_k(z_2, c_2)$.

Condition A' is weaker than Condition A because it relaxes the monotonicity requirement on one side of the maximum point on a trading track. Condition A implies Condition A'.

Since state and control spaces are continuous, and the assets' price processes are continuous, $\underline{H}_k(z, c)$ is continuous in the state space (z, c). Then the two sets of target states $\{(z_k^N(K), C(z_k^N(K), K)), \forall K \in (-1, \infty)\}$ and $\{(z_k^T(c), c), \forall c > 0\}$ are two continuous "target lines" in the space (z, c) for NT and TN trading, denoted by "N-line" and "T-line" respectively. Obviously, the two target lines have a cross point at $c = 1$. Applying property (4) in Proposition 1, we have the following result for the T-line in the NRL case.

Proposition 2. *In the NRL case, for problems PTN, PCT and PNCT, the value* *on the T-line is decreasing in c when* $c \leq 1$*, i.e.,* $\underline{H}_k(z_k^T(c_1), c_1) \geq \underline{H}_k(z_k^T(c_2), c_2)$*, if* $0 < c_1 < c_2 \leq 1$*.*

With respect to the optimal policy in PCT, we have:

Theorem 3. *For problem PCT, at stage* k*, if Condition A' holds, then the optimal action for a state is one of following three actions:*

(1) execute a simple trade to make a transition to one of two target lines;

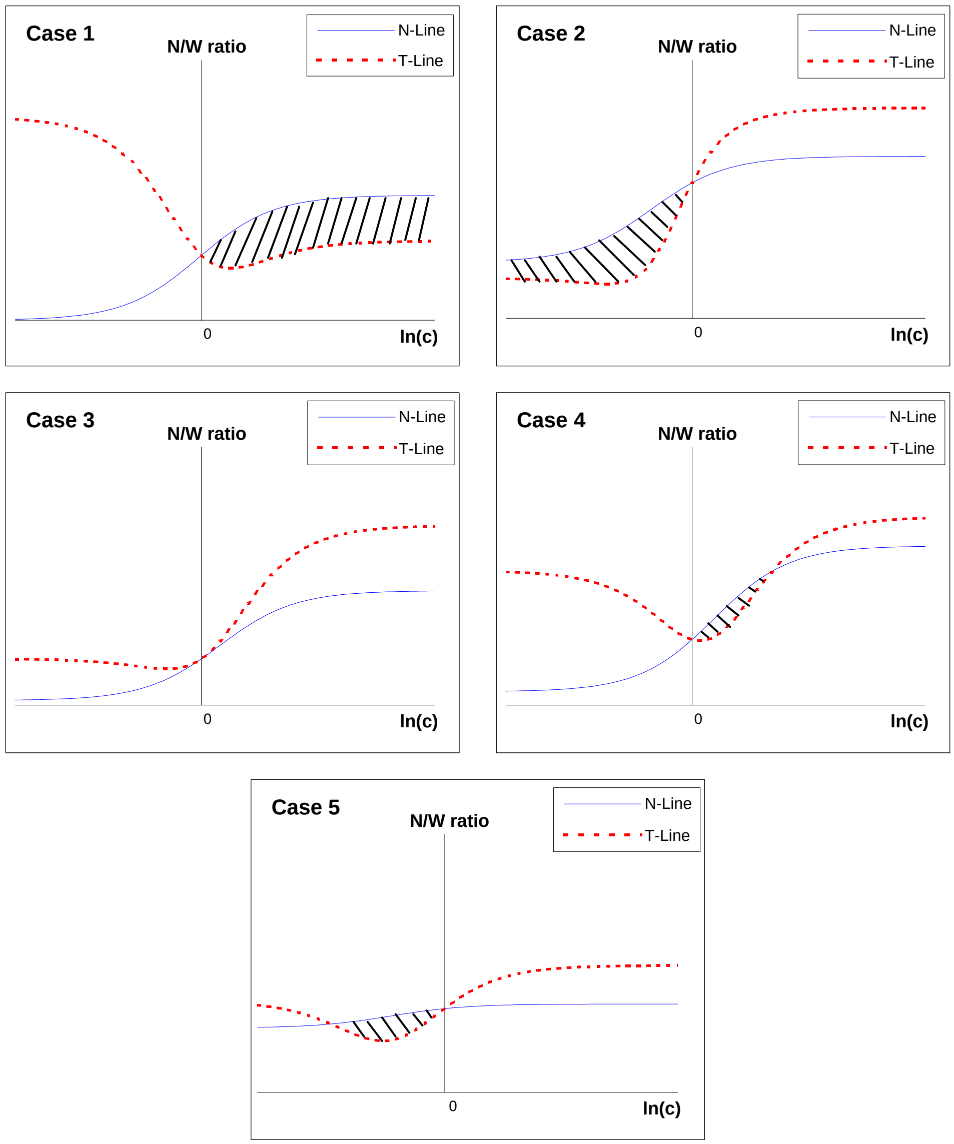
(2) execute a trade to make a transition to a cross point of two target lines;

(3) execute no trade.

Proof: See Appendix.

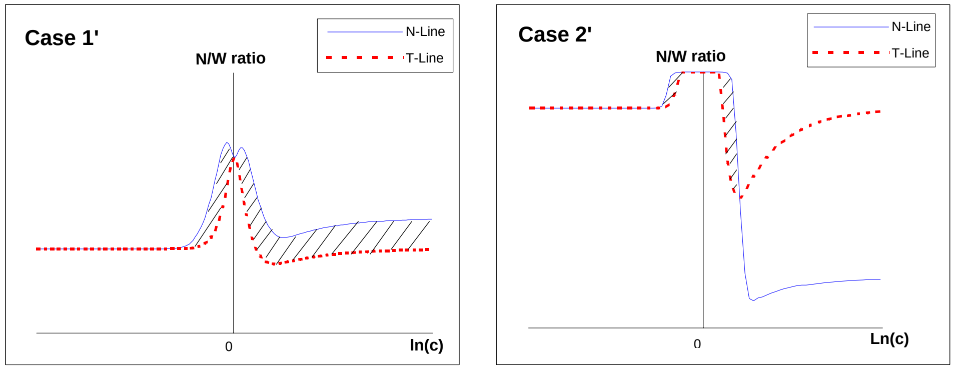
From Theorem 3, we know that, when condition A' holds, the process of finding the optimal target state from a given state can be reduced to selecting the state with maximum value base function ($\underline{H}()$) value among the reachable cross points, the reachable single (NT or TN) trading target states and the original state.

Condition A' significantly simplifies the numerical computation. Otherwise more complex mathematical programming techniques would be needed to calculate the optimal state in the entire set of reachable states for a given state. For example, iterating the search along TN trading and NT trading until a convergent point is reached, and dealing with the potential existence of multiple local optimal states. In addition, Condition A' ensures that the optimal policy can be simply represented with two target lines, compared with an exhaustive list of decisions at each state (z, c).



Under the RL tax rule, there are five possible cases of the position relation between N-line and T-line found in our computation. The shaded areas are the no-trading regions which are bounded by N-line on the top and T-line at the bottom.

Figure 3.2: The position relation of target lines under the RL tax rule



Under the NRL tax rule, there are two possible cases of the position relation between N-line and T-line found in our computation. The shaded areas are the no-trading regions which are bounded by N-line on the top and T-line at the bottom.

Figure 3.3: The position relation of target lines under the NRL tax rule

In our computational experiments, we found that Condition A holds at all times, within a very small numerical error tolerance.[1] In addition, some phenomena hold all the time, which make the computation even simpler. These phenomena are described under different tax rules as follows.

Under the RL tax rule: We found that there are only five possible cases for the position relation between the two target lines as in the following **list L**: (shown in Figure 3.2)

Case 1 T-line is under N-line for $c > 1$ and above N-line for $c < 1$.

Case 2 T-line is above N-line for $c > 1$ and under N-line for $c < 1$.

Case 3 T-line is above N-line for $c > 1$ and $c < 1$.

Case 4 T-line is above N-line for state $c < 1$, and there exists a cross point

[1]Similar small numerical errors also appear in our computational tests for one-way trading problems. These errors affect whether Condition A holds or not. However, Condition A must hold for one-way trading problems, due to the concavity of value base function proved in the Theorems 1 and 2.

(z', c') with $c' > 1$, T-line is under N-line for $1 < c < c'$ and above N-line for $c > c'$.

Case 5 T-line is above N-line for state $c > 1$, and there exists a cross point (z', c') with $c' < 1$, T-line is under N-line for $c' < c < 1$ and above N-line for $c < c'$.

From the five cases, we found: (1) there is at most one cross point besides $c = 1$, and (2) all the cross points (including that on $c = 1$) are not local minimal points in target lines.

Under the NRL tax rule: We found that there are only two possible cases for the relation between the two target lines as in the following **list L'**: (shown in Figure 3.3)

Case 1' T-line is under N-line for $c > 1$ and $c < 1$; and two target lines converge to a constant value as $c \to 0$.

Case 2' There exists a cross point (z', c') with $c' > 1$, T-line is under N-line for $1 < c < c'$ and above N-line for $c > c'$; for $c < 1$, T-line is under N-line and two target lines converge to a constant value as $c \to 0$.

From the two cases, we found: (1) the cross point on $c = 1$ is a local minimal point in target lines, and there are two no-trading regions on the the two sides of this point;[2] (2) when c approaches zero, the two target lines converge to

[2]This fact also implies that in the RNL case optimal policy is not to move back to $c = 1$, i.e., do not sell all taxable assets (if you have any). In addition, we derive in next section that the cross point at $c = 1$ is a minimal point on the target lines for the limiting situation.

a constant value that is the optimal N/W ratio for portfolio optimization without taxes.[3]

In general, these cases show some common characteristics: (1) there are at most one cross point (except the convergence at $c \to 0$ in the NRL case) besides $c = 1$, and the cross point (if it exists) in any half state space (either $c < 1$ or $c > 1$) is a local maximal point in each target line. (2) in a half state space there is at most one no-trading region, and a no-trading region must be adjacent to the cross point at $c = 1$ and bounded by N-line on the top and T-line at the bottom.[4] Hence, the computation, analysis and representation of the optimal policy for them are much simpler, in comparison with the situations where more cross points exist.

For example, in the "Case 1" of the five possible cases under the RL tax rule, the optimal policy is as follows (shown in Figure 3.4):

- when $c > 1$, do the two-way threshold policy, i.e.,
  - if (z, c) is above the N-line, buy taxable asset and reach the state on N-line;
  - if (z, c) is in the no-trading region, do not trade.
  - if (z, c) is below the T-line, sell taxable asset and reach T-line.

[3]An intuitive explanation is: because (i) it is unlikely to transit from a state with $c << 1$ to states with $c > 1$ without trading, i.e., it is unlikely to have $s_{k+1} > a(t_k)$ if $s_k << a(t_k)$, and (ii) the local minimal at $c = 1$ depresses the trading to it, hence the situation at $c << 1$ is similar to portfolio optimization without taxes (and any other transaction costs) since no tax is charged or credited when $c < 1$, and similarly the optimal policy is to keep the N/W ratio to a constant ratio independent of the relative price.

[4]The reason for no-trading is: for any state (z, c) in this area, the only reachable cross point is on $c = 1$, say $(z_1, 1)$, and $\underline{H}(z_1, 1) \leq \underline{H}(z_2, c) \leq \underline{H}(z, c)$, with (z_2, c) being the state on N-line; from Theorem 3, the optimal action for the initial state is not to move.

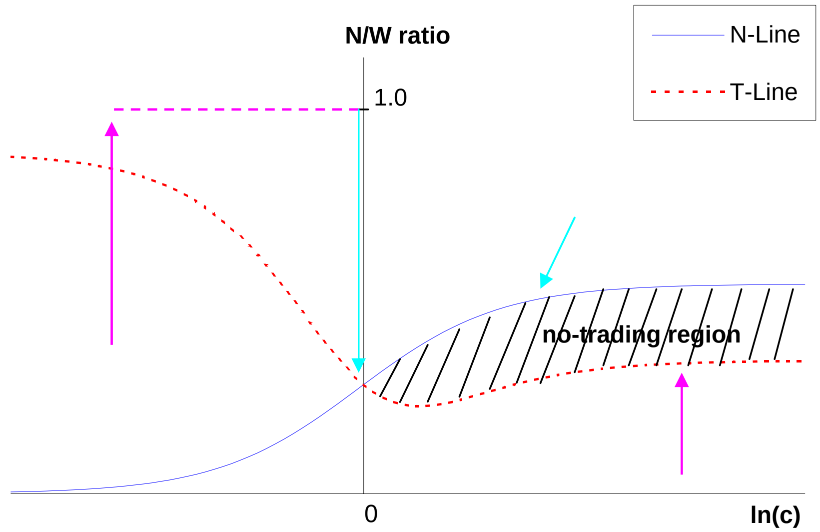


Figure 3.4: The optimal policy for Case 1 of problem PCT under the RL tax rule

- when $c \leq 1$, move to the cross point at $c = 1$, i.e.,

  sell all taxable asset (realize the loss in the RL case), and buy back some of taxable asset at the current price to decrease the N/W ratio to the value of the cross point.

However, we cannot prove that Condition A and the phenomena found in our computational tests hold for all the stages, though our computational tests show that these appear to be true.[5] We only have a definite theoretical result for the stage $N-1$, given in the next theorem.

Theorem 4. *At stage $N-1$, the following hold:*

(1) Condition A holds.

(2) In the RL case, there is no local minimal point in both target lines in the whole

[5] In the Appendix we show that we cannot use the same method for one-way trading problems to prove the concavity (which implies Condition A and some other results in Theorem 4) for PCT at all stages.

space $0 < c < \infty$*, and there are only the five possible cases of the relation of target lines as in list L.*

(3) In the NRL case, there is no local minimal point in both target lines in the half-space $c < 1$ *and the half-space* $c > 1$*; if the cross point at* $c = 1$ *is local minimal in target lines, then there are only two possible cases of the relation of target lines as in list L'.*

Proof: See Appendix.

3.2 Optimal policy just prior to termination

Although we need to use a numerical method to calculate the value function and the optimal policy for each stage, we can derive analytical results for the optimal policy just before the terminal time. These results can be used to verify the numerical solution for short-term problems and also provide a reference for the optimal policy for long-term problems. This analytical result is derived by maximizing the growth rate of expected utility as time approaches the termination (see the Appendix for the detailed derivation). For brevity, we refer to it as the analytical "limiting solution", and denote the time just prior to termination by T^-. To simplify the formula, we introduce the new symbol $\beta = 1 - \alpha$.

3.2.1 Problem PTN

Given the tax rate α and trading track characteristic value c, the optimal target N/W ratio at the time just before termination is as follows:

- For the RL case:

The optimal target N/W ratio is

$$z^T(c) = \frac{c^2\beta^2\sigma_s^2 - q(c)c\rho\beta\sigma_i\sigma_s - q(c)(c\beta\mu_s - q(c)\mu_i)/(1-\gamma)}{\beta^2c^2\sigma_s^2 + q(c)^2\sigma_i^2 - 2q(c)c\rho\beta\sigma_i\sigma_s} \tag{3.2.1}$$

truncated in range $[0, 1]$ if this ratio is out of the range.

In the simple case where index fund is a riskfree bond, i.e., $\sigma_i = 0$, the right-hand side of (3.2.1) is

$$1 - \frac{q(c)(\beta c\mu_s - q(c)\mu_i)}{(1-\gamma)\beta^2c^2\sigma_s^2}$$

When the tax rate $\alpha = 0$ and $q(c) = c$, the last formula gives the optimal ratio of bond value to total wealth for the optimal investment between stock and bond without transaction cost, and the result is the same as the classic Merton solution $1 - \frac{\mu_s - \mu_i}{(1-\gamma)\sigma_s^2}$.

- For the NRL case:

  1. If $c > 1$, the optimal N/W ratio is the same as that for the RL case.
  2. If $c < 1$, the optimal N/W ratio is the same as that for the RL case without taxes, i.e., the tax rate $\alpha = 0$. Specifically it is

     $$\frac{\sigma_s^2 - \rho\sigma_i\sigma_s - (\mu_s - \mu_i)/(1-\gamma)}{\sigma_s^2 + \sigma_i^2 - 2\rho\sigma_i\sigma_s}$$

  3. If $c = 1$,

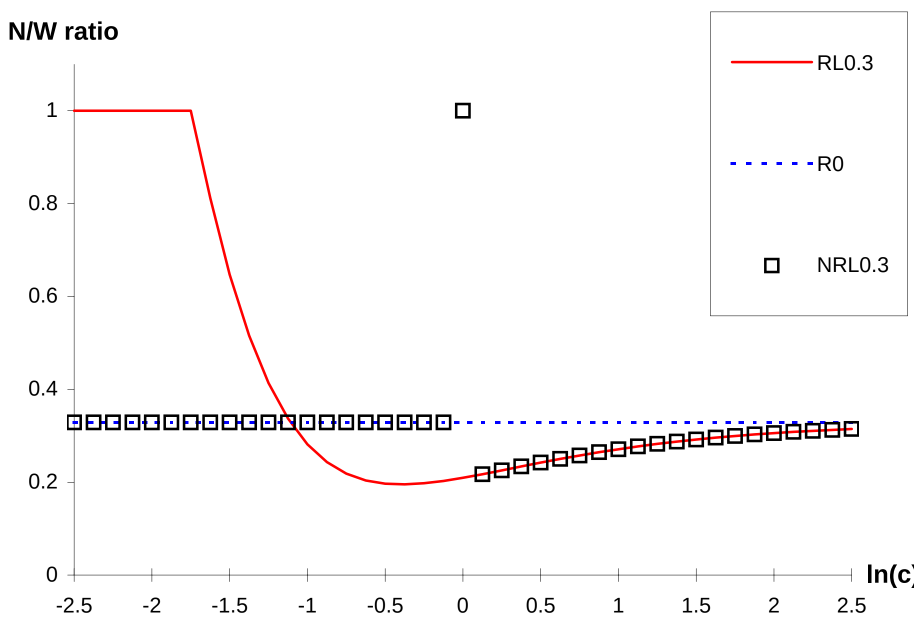
     – if $\sigma_s = 0$, the optimal policy is the same as that for case $c > 1$ or $c < 1$, depending on whether $\mu_s > 0$ or not.

– if $\sigma_s \neq 0$, the optimal N/W ratio is 1.0, i.e., to move all wealth to nontaxable asset.

To see the impact of the tax rate α, we let $Z(\alpha, c)$ be the right-hand side of equation (3.2.1). We find that $Z(\alpha, c)$ has the following property:

$$Z(a_1, c) = Z(a_2, \frac{a_1}{1-a_1}\frac{1-a_2}{a_2}c),$$

so the T-lines for two different tax rates have the same shape in $(z, \ln(c))$ space, and differ only in a shift of $\ln(\frac{a_1}{1-a_1}\frac{1-a_2}{a_2})$.



This figure shows the optimal N/W ratio for TN trading in three cases: with tax rate of 0.3 under the RL or NRL tax rules, respectively, and without taxes. The model's parameters are: $\mu_s = 0.15$, $\mu_i = 0.05$, $\sigma_s = 0.4$, $\sigma_i = 0.1$, $\rho = 0.3$ and utility parameter $\gamma = 0$.

Figure 3.5: Optimal N/W ratios at T^- in TN trading

Figure 3.5 gives an example of optimal target N/W ratios for TN trading just before the terminal time for both the RL and NRL cases. The solid line labeled

"RL0.3" denotes the optimal N/W ratio for the RL case with tax rate $\alpha = 0.3$; the dash line labeled "R0" denotes the optimal N/W ratio for case without taxes, i.e., the tax rate $\alpha = 0$, where there is no difference between the RL and the NRL cases; and the "square" series denotes the optimal N/W ratio for the NRL case with $\alpha = 0.3$, which has the same value as "RL0.3" and "R0" for points $c > 1$ and $c < 1$ respectively, and is discontinuous at point $c = 1$ with value 1.0.

3.2.2 Problem PNT

Given the characteristic value K of the trading track, just prior to the terminal time, the optimal N/W ratio is as follows:

- For the RL case

$$z^N(K) = \frac{\frac{K+1}{1+\beta K}(\beta^2\sigma_s^2 - \rho\beta\sigma_i\sigma_s) - (\beta\mu_s - \mu_i)/(1-\gamma)}{\beta^2\sigma_s^2 + \sigma_i^2 - 2\rho\beta\sigma_i\sigma_s} \tag{3.2.2}$$

  truncated in range $[0, 1]$ if this ratio is out of the range.

  When index fund is riskless, i.e. $\sigma_i = 0$, the right-hand side of (3.2.2) is simply:

$$\frac{K+1}{1+\beta K} - \frac{(\beta\mu_s - \mu_i)}{(1-\gamma)\beta^2\sigma_s^2}$$

- For the NRL case

  1. If $K > 0$, the optimal N/W ratio is the same as that for the RL case.

2. If $K < 0$, the optimal N/W ratio is the same as that for the RL case without taxes. Specifically it is

$$\frac{\sigma_s^2 - \rho\sigma_i\sigma_s - (\mu_s - \mu_i)/(1-\gamma)}{\sigma_s^2 + \sigma_i^2 - 2\rho\sigma_i\sigma_s}$$

   It is the same as that for problem PTN.

3. If $K = 0$, i.e., $c = 1$,

   – if $\sigma_s = 0$, the optimal policy is the same as that for case $K > 0$ or $K < 1$, depending on whether $\mu_s > 0$ or not.

   – if $\sigma_s \neq 0$, the optimal N/W ratio is 1.0, i.e., not to buy any taxable asset.

Remark: The optimal N/W ratios for buying and selling stock in the NT and TN trading, respectively, are the same when $c \leq 1$, in the NRL case.

From the target N/W ratio of each K, i.e., $z^N(K)$, we can find the final relative price $c^*(K)$ by $c^*(K) = C(z^N(K), K)$. From the relation between $z^N(K)$ and $c^*(K)$, we can derive the target N/W ratio for any given final c. This ratio is denoted by $z'^N(c)$. For the RL case, putting $\frac{K+1}{1+\beta K} = \frac{c-\alpha(c-1)z}{q(c)}$ in equation (3.2.2) and noticing that the target N/W ratio is unique for each K, we derive $z'^N(c)$, as follows:

- if $c\beta^2\sigma_s^2 + q(c)\sigma_i^2 - (c+q(c))\rho\beta\sigma_i\sigma_s > 0$

$$z'^N(c) = \frac{c\beta^2\sigma_s^2 - c\rho\beta\sigma_i\sigma_s - q(c)(\beta\mu_s - \mu_i)/(1-\gamma)}{c\beta^2\sigma_s^2 + q(c)\sigma_i^2 - (c+q(c))\rho\beta\sigma_i\sigma_s}$$

  truncated in range $[0, 1]$.

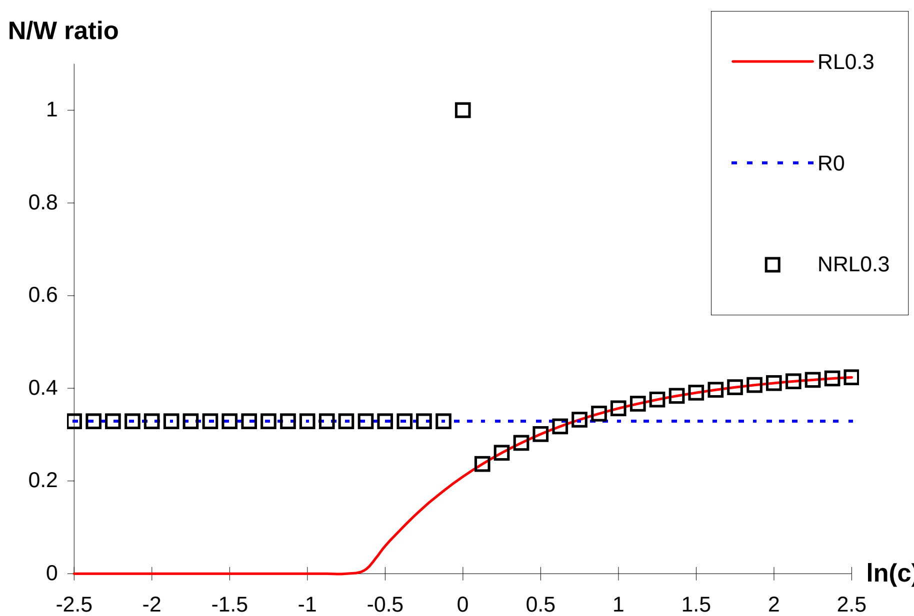
- otherwise

$$z'^N(c) = \begin{cases} 1, & \text{if } \rho\beta\sigma_i\sigma_s - \sigma_i^2 - (\beta\mu_s - \mu_i)/(1-\gamma) > 0 \\ 0, & \text{otherwise} \end{cases}$$

From the process of deriving $z'^N(c)$, we know $z'^N(c)$ is unique for a given c. Thus the curve of set $\{(z'^N(c), c)\}$ will not be the irregular case of N-line in Figure A.1 in the Appendix.

Studying the sign of $dz'^N(c)/dc$, we know $z'^N(c)$ is monotonically increasing or decreasing with c.

Also, we can show that if $z'^N(c) = 1$ at one point c, then $z'^N(c) = 1$ for all c. That is, the N-line cannot have some parts on the boundary $z = 1$ and other parts below the boundary.



This figure shows the optimal N/W ratio for NT trading in three cases: with tax rate of 0.3 under the RL or NRL tax rules, respectively, and without taxes. The model's parameters are: $\mu_s = 0.15$, $\mu_i = 0.05$, $\sigma_s = 0.4$, $\sigma_i = 0.1$, $\rho = 0.3$ and utility parameter $\gamma = 0$.

Figure 3.6: Optimal N/W ratios at T^- in NT trading

Figure 3.6 gives an example of optimal N/W ratios for NT trading just prior to the terminal time for both the RL and NRL cases. The solid line labeled "RL0.3" denotes the N/W ratio for the RL case with tax rate $\alpha = 0.3$; the dash line labeled "R0" denotes the N/W ratio for case without taxes, i.e. tax rate $\alpha = 0$, where there is no difference between the RL and the NRL cases; and the "square" series denotes the N/W ratio for the NRL case with $\alpha = 0.3$, which has the same value as "RL0.3" and "R0" for point $c > 1$ and $c < 1$ respectively, and is discontinuous at point $c = 1$ with value 1.0.

Remark: The optimal N/W ratio just before the terminal time is the limiting result, and in the NRL case it may be not continuous at $c = 1$, for both PTN and PNT. However, for any fixed time-step Δt, at one time-step prior to termination, the optimal N/W ratio curve (i.e., the target line) is continuous in c. There is a peak in the continuous curve at c close to 1, and our computational result verified this phenomenon.

3.2.3 Problem PCT

Because the analysis is conducted on the last trading action, the situation is the same as at stage $N - 1$, and, from Theorem 4, Condition A and other results of Theorem 4 hold at T^-, though there are some special features of the limiting solution, as the target lines are discontinuous at $c = 1$ and coincide for $c < 1$, for the NRL case.

Now we analyze the optimal policy for problem PCT under different tax rules by studying the relationship between the N-line and the T-line, based on the results from the preceding subsections.

Under the RL tax rule

From the one-way trading problems PTN and PNT, we know that, if the segments on boundaries of $z = 0$ and $z = 1$ are ignored, the T-line is the set of states $\{(z^T(c), c)\}$ with

$$z^T(c) = \frac{c^2\beta^2\sigma_s^2 - q(c)c\rho\beta\sigma_i\sigma_s - q(c)(c\beta\mu_s - q(c)\mu_i)/(1-\gamma)}{\beta^2c^2\sigma_s^2 + q(c)^2\sigma_i^2 - 2q(c)c\rho\beta\sigma_i\sigma_s}$$

and the N-line is the set of states $\{(z'^N(c), c)\}$ with

$$z'^N(c) = \frac{c\beta^2\sigma_s^2 - c\rho\beta\sigma_i\sigma_s - q(c)(\beta\mu_s - \mu_i)/(1-\gamma)}{c\beta^2\sigma_s^2 + q(c)\sigma_i^2 - (c + q(c))\rho\beta\sigma_i\sigma_s}$$

Solving equation $z^T(c) = z'^N(c)$, which is equivalent to a third-order equation, we have three roots:

$$c_0 = 1,$$

$$c_1 = \frac{\alpha}{\alpha - 1},$$

and

$$c_2 = \left(\frac{\alpha}{1-\alpha}\right)\frac{\sigma_i^2\mu_s - \rho\sigma_s\sigma_i\mu_i}{(1-\gamma)\sigma_s^2\sigma_i^2(1-\rho^2) - (\sigma_i^2\mu_s + \sigma_s^2\mu_i) + \rho\sigma_s\sigma_i(\mu_s + \mu_i)}.$$

Since in reality, the tax rate $\alpha \in (0, 1)$, $c_1 < 0$ is invalid, hence there is at most one cross point besides $c = 1$, and this depends on whether $c_2 > 0$ and the validity of z, i.e, the additional cross point can only happen when $c_2 > 0$ and $c_2\beta^2\sigma_s^2 + q(c_2)\sigma_i^2 - (c_2 + q(c_2))\rho\beta\sigma_i\sigma_s > 0$. Similar to Theorem 4, we can show that only the five cases in list L are possible. These five cases are not only possible, but they also exist, for we can select parameters to construct examples of them. The cross point at $c = 1$ has N/W ratio

$$z^T(1) = \frac{\beta^2\sigma_s^2 - \rho\beta\sigma_i\sigma_s - (\beta\mu_s - \mu_i)/(1-\gamma)}{\beta^2\sigma_s^2 + \sigma_i^2 - \rho\beta\sigma_i\sigma_s} \tag{3.2.3}$$

It is the Merton solution (the optimal solution in the situation without taxes and transaction costs) for the stock with μ_s and σ_s being adjusted by a coefficient β. We call this cross point the "tax-adjusted Merton solution."

Under the NRL rule

When $c > 1$, the optimal policy is the same as in the RL case, so there are only two possible cases of the relation between target lines in $c > 1$, as those in list L'. When $c < 1$, the N-line and T-line coincide with a constant N/W ratio, which is the Merton solution for portfolio optimization without taxes, and the optimal policy is to move to the target line.

At $c = 1$, we can derive that the cross point at $c = 1$ is a minimum point in target lines.[6] The reasoning is as follows: (1) If the optimal N/W ratio is $z = 1$, we know the state $(1, 1)$ is a minimum on target lines, because any TN trading can reach this point, and this point must have a smaller value than other points on T-line. (2) If $z \neq 1$, then $\sigma_s = 0$, under the assumption $\mu_s > 0$, we can show that the cross point at $c = 1$ is minimum in target lines by studying the growth rate of expected utility on target lines in range $c \geq 1$ and noting that the growth rates of expected utility on target lines in the half-space $c < 1$ are not less than

[6] In limiting solution, it means, the growth rate of expected utility at this point is minimum in target lines.

that of any state (z, c) at $c = 1$.

3.3 Computational results

Using numerical methods introduced in Chapter 4, we can find optimal solutions for our problems. In this section, we first illustrate the shape of target lines and their movement in time for problems PTN, PNT and PCT, under different tax rules. Then we compare the performance of the optimal policy and several heuristic policies. All of the computational results are obtained with the following parameters: $\mu_s = 0.15$, $\mu_i = 0.05$, $\sigma_s = 0.4$, $\sigma_i = 0.1$, $\rho = 0.3$, utility parameter $\gamma = 0$, tax rate $\alpha = 0.3$, trading interval $\Delta t = 0.1$ year.

3.3.1 Target lines for each problem

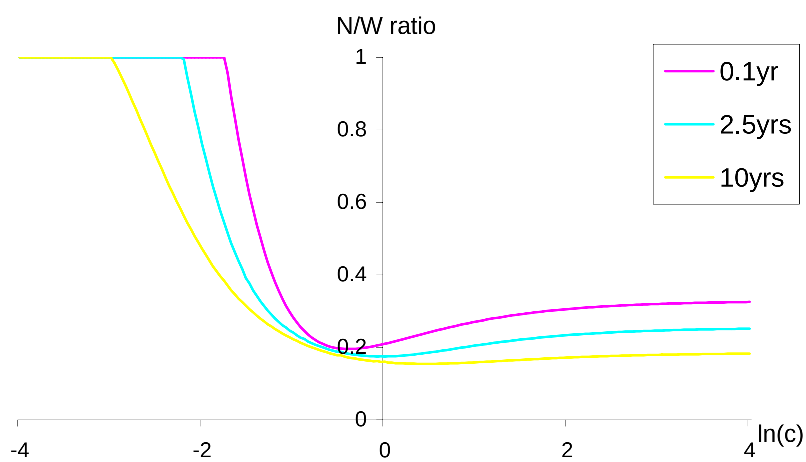


Figure 3.7: T-lines of PTN in the RL case

Figures 3.7 and 3.8 show the target lines for Problem PTN in the RL and NRL cases, respectively. Figures 3.9 and 3.10 show the target lines for Problem

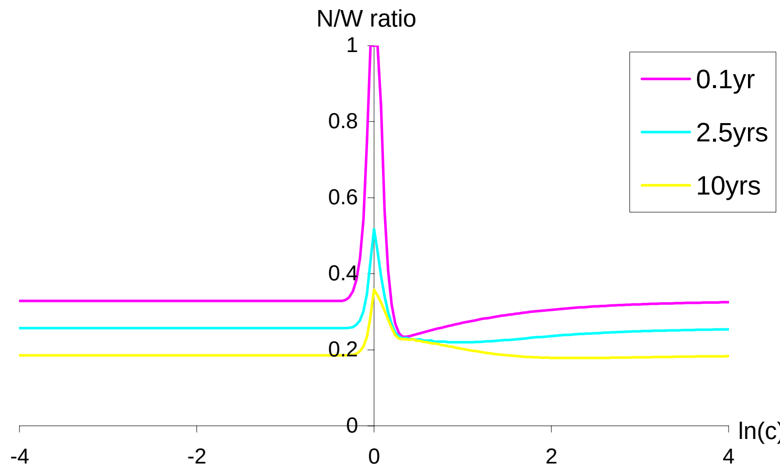


Figure 3.8: T-lines of PTN in the NRL case

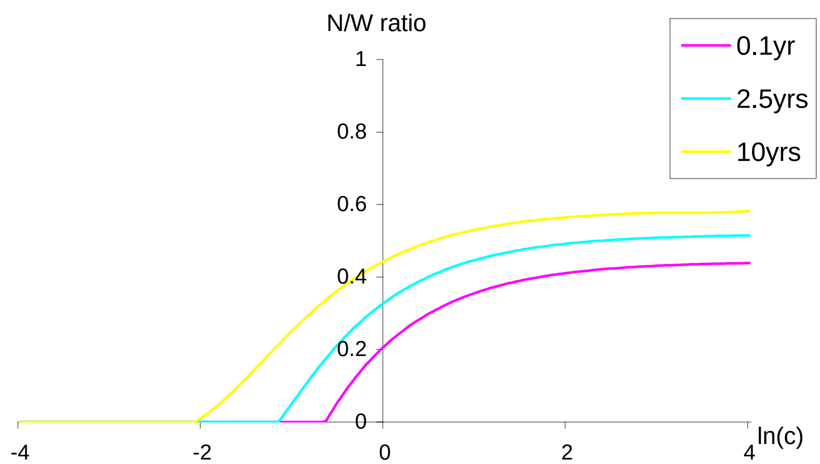


Figure 3.9: N-lines of PNT in the RL case

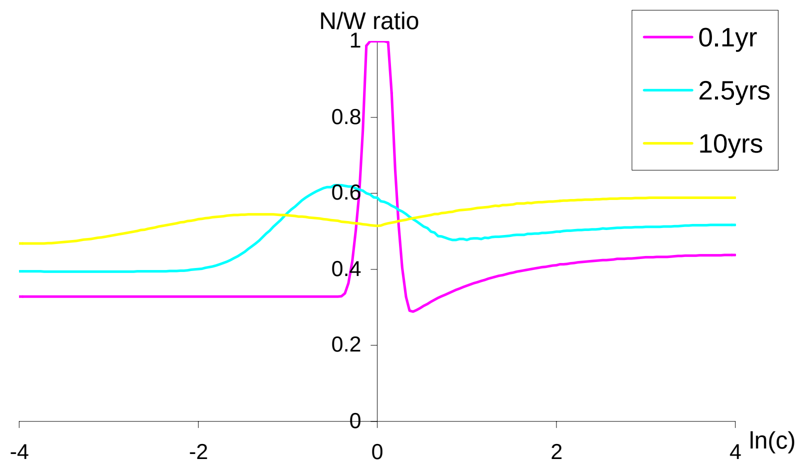


Figure 3.10: N-lines of PNT in the NRL case

PNT in the RL and NRL cases, respectively. The target lines are labeled with the remaining time to termination.

These figures illustrate that: (1) The target line at one time-step (i.e., 0.1 year) before the termination is very close to the analytical limiting solution (comparing the graphs in last section), for both the RL and the NRL cases. (2) As the remaining time increases, the T-line moves down and the N-line moves up.

In the NRL case, there is a peak at $c = 1$ in the target line at time close to termination, and it decreases with time.

Figures 3.11 and 3.12 show the target lines for problem PCT under the RL and the NRL tax rules. The target lines are labeled in the form of "$\mathcal{X}$*time*", where "$\mathcal{X}$" may be "N" or "T" for N-line or T-line and "*time*" is the remaining time to termination.

In Figure 3.11, the target lines belong to Case 1 in list L. In the half-space

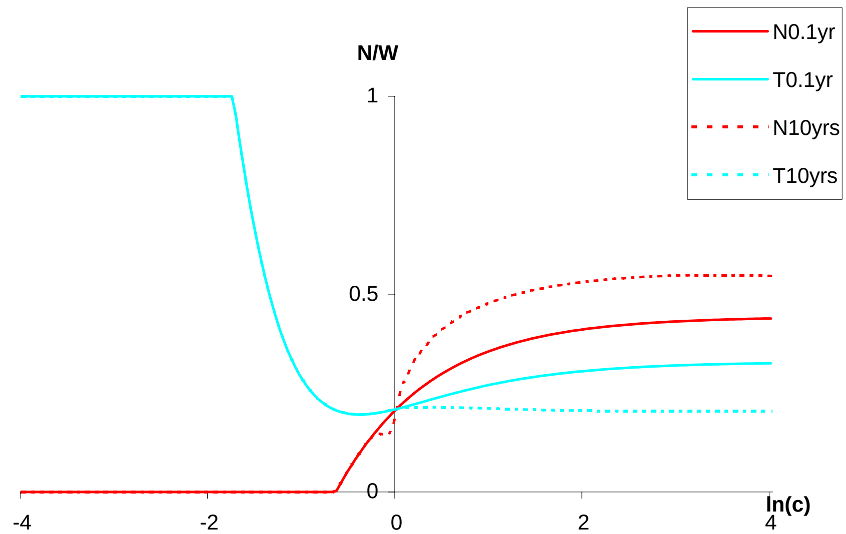


Figure 3.11: Target lines of PCT in the RL case

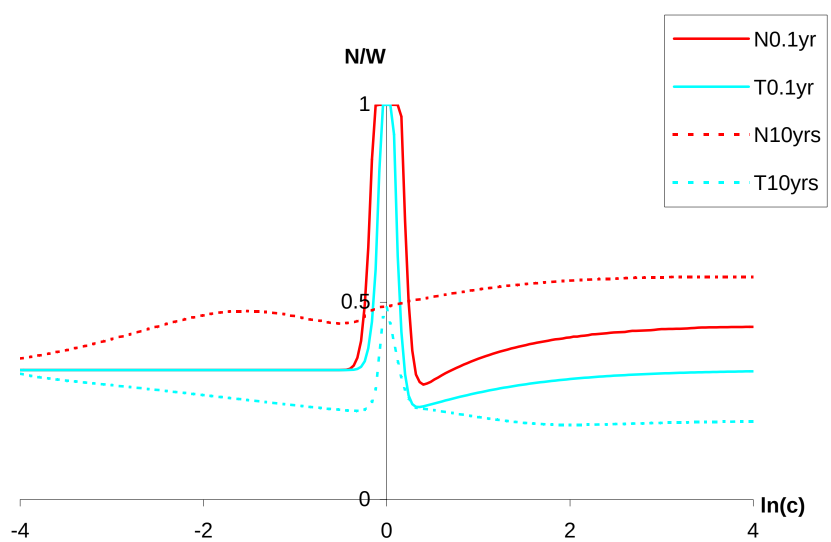


Figure 3.12: Target lines of PCT in the NRL case

$c > 1$, there exists a no-trading region which becomes wider with the increase in remaining time. In the half-space $c < 1$, the two target lines change little with time.

In Figure 3.12, the target lines belong to Case 1' in List L'. There are two no-trading regions on the two sides of $c = 1$, and they become wider with the increase in remaining time. The cross point at $c = 1$ decreases with time.

3.3.2 Policies' performance

To see how much benefit we can obtain from applying the optimal policy, we compare the performance of the optimal policy and the following heuristic policies:

Myopic: uses the analytic limiting solution for all t.

PMerton: uses the Merton solution[7] for all t.

BHs: buy and hold the stock.

BHi: buy and hold the index fund.

We estimated the expected utilities of these policies by applying each of them with 65536 simulated sample paths of the assets in a time horizon of 10 or 30 years, given 1 dollar wealth of the nontaxable asset at the beginning.[8] We measure the performance with certainty equivalent,[9] and show the performance for each policy

[7] That is the optimal proportion of index fund in total nominal wealth for optimal allocation between stock and index fund without taxes and any transaction costs, given in the seminal paper of Merton[8].

[8] In these simulation tests, the standard errors of expected utility of all tested policies are less than 0.002 for 10-year horizon and 0.006 for 30-year horizon.

[9] As $\gamma = 0$ in our test, for an expected utility v, the certainty equivalent is $\exp(v)$. It is measured in dollar.

with different tax rates under the RL tax rule in Tables 3.1 and 3.2 for 10-year and 30-year horizons respectively.

	Tax rate						
Policy	0	0.1	0.2	0.3	0.4	0.5	0.6
Optimal	2.184	2.148	2.107	2.057	1.996	1.920	1.828
Myopic	2.184	2.145	2.102	2.052	1.992	1.918	1.822
PMerton	2.184	2.123	2.052	1.971	1.883	1.786	1.682
BHs	2.021	2.015	1.980	1.926	1.856	1.770	1.669
BHi	1.569	1.569	1.569	1.569	1.569	1.569	1.569

Table 3.1: Performance of policies under different tax rates, 10-year time horizon

	Tax rate						
Policy	0	0.1	0.2	0.3	0.4	0.5	0.6
Optimal	10.431	10.050	9.662	9.220	8.702	8.086	7.282
Myopic	10.431	9.939	9.465	8.991	8.486	7.909	7.164
PMerton	10.431	9.612	8.706	7.753	6.783	5.821	4.888
BHs	8.266	8.095	7.706	7.217	6.654	6.026	5.330
BHi	3.864	3.864	3.864	3.864	3.864	3.864	3.864

Table 3.2: Performance of policies under different tax rates, 30-year time horizon

From Tables 3.1 and 3.2, we see: (1) The certainty equivalent of the optimal policy decreases with the tax rate, and this coincides with our intuition. (2) The difference between the optimal solution and the Merton solution increases with the tax rate; using the Merton solution (i.e., ignoring the tax) results in a significant loss (about 5% in 10 years, 16% to 22% in 30 years) in comparison with the optimal policy with tax rates in the range of 0.3 to 0.4. (3) The difference between optimal and myopic is very small (less than 2.5% even in 30 years) for all tax rates; the myopic policy can be a good approximation in this example.

Remark: Since the shape of target lines is similar to that of the limiting solution, we can use the limiting solution to study the parameters' impact on the optimal policy. This is another use of the limiting solution.

3.4 Concluding remarks

In this chapter, we study a tax-aware portfolio optimization problem, the dynamic allocation between taxable and nontaxable assets. Our main contributions are: (1) We derived several properties of the optimal solution, which make the representation, analysis and computation of optimal policies simpler. Specifically, we show that optimal policies have a simple structure represented by target lines in a reduced two-dimensional state space. (2) We derived the "limiting solution," the analytical solution for the optimal policy just prior to termination, which can be used to verify numerical results and has the potential to generate a near-optimal performance when applied at all trading times.

We also present computational results, which show the optimal policies at different trading times for different problems under different tax rules, as well as the performance of optimal policy and some heuristic policies. In particular, the "myopic" heuristic policy, which uses the "limiting solution" at all trading times, performs remarkably well.

Our results about the optimal solution can be applied to some extended models. For example: (1) All of our theoretical results will hold for the models where assets' processes are geometric Brownian motions with time-varying parameters. (2) For models with utility functions which are concave and increasing with wealth size, Theorems 1 and 2 will still hold. (3) All the results about one-way trading from taxable to nontaxable, will hold for the model where the tax rate depends on holding time.

From our computational experiments for problem PCT, we found that Condition A and some other phenomena about target lines always hold, which makes the computation and analysis of the optimal policy simpler. We have a conjecture about these phenomena, which has however only been proved to hold at the stage next to termination. A further study for other stages is needed, and the results may be relevant to a general principle or analytical method for a large class of stochastic control problems.

In addition, our research may provide a basis for numerically solving problems which have more practical restrictions in the real world, for example, considering restrictions on transfers (such as, the maximum annual contribution, the penalty of early withdrawal, and constraints to wash sell, etc.), dealing with tax-deferred account instead of nontaxable account, and so on. In our view, it is most valuable and challenging to find a good (optimal or near optimal) policy for the tax optimization problem with multiple assets. The satisfactory performance of the myopic policy using the limiting solution suggests to us that it may be a promising approach.

Chapter 4

Computational methods

In this chapter, we present the numerical methods to calculate the optimal policy by using the properties derived in the preceding chapters.

To solve our problems, we developed two kinds of methods: one PDE-based method and two simulation-based methods. They all operate in a dynamic programming framework that works backwards in discrete time, and at each time, selects the optimal action based on expectation of the value on the states at the next time. The main difference between the two kinds of methods is the way the expectation is calculated: one uses the numerical solution of a partial differential equation (PDE), and the others use simulation. The PDE-based method is implemented by using the finite difference approach, and is called the FD method. The two simulation-based methods are implemented by using point-estimation or regression, and are called the PE method or RS method respectively.

The purposes of using multiple numerical methods are: to verify the correctness of computational results, and to compare computational performance and to guide the usage and development of the solution in practice.

Since our problems can be solved in a two-dimensional state space, the PDE-based FD method is more computationally efficient. However, this method is not suitable for solving high-dimensional problems which are more common in practice. Perhaps, using a functional approximation (e.g., the finite element technique) instead of a discrete approximation may be a hopeful way for PDE-based methods to solve high-dimensional problems, but there is little research to date. Considering designing and implementing PDE-based method are not as easy as simulation, we place hopes in the approach of combining functional approximation and simulation. The regression-based simulation method is representative of this kind of methods, and has been applied successfully in some financial applications, e.g., Longstaff and Schwartz [7] and Tsitsiklis and Van Roy [9] for high-dimensional American option pricing. To test the regression-based method on the portfolio optimization, which is a much more complicated problem than pricing American options,[1] we implement the RS method for our problem, also the PE method for comparison.

The structure of this chapter is as follows. We introduce the solution methods in Section 4.1, and then compare the performance of the solution methods in Section 4.2.

[1]Pricing American option requires determining the optimal decision between the choices of exercising or continuing, while portfolio optimization problem requires finding an optimal portfolio, at any state and time.

4.1 Numerical solution methods

In this section, we introduce the numerical solution methods that will be used for our problems. These problems are dynamic programming (DP) problems (though they may be represented in different formulations) and can be solved in a general DP scheme, that is: working backward from the terminal time for each stage, and at each stage executing two critical processes:

1. Computation of expectation: obtain the post-decision values on states, by calculating the expected value resulting from reaching states in next stage.

2. Optimization: obtain the pre-decision values on states, by finding the maximal of the post-decision value on all reachable states.

According to the methods to calculate the expectation, we classify the corresponding solution methods into PDE-based or simulation-based.

From Chapter 3, we know optimal policy can be represented by some target lines consisting of optimal target states. Our numerical solution methods are developed to calculate these optimal target states.

4.1.1 PDE-based method

For given k, we define a continuous-time version of the value function

$$V(x, y, X, z, t) = E[V_{k+1}(x, I(t_{k+1}), X, s(t_{k+1}))|I(t) = y, s(t) = z]$$

for t in the no-trading period $[t_k^+, t_{k+1}]$. From

$$\begin{aligned} & E[V_{k+1}(x, I(t_{k+1}), X, s(t_{k+1}))|I(t_k^+) = y, s(t_k^+) = z] \\ = \; & E[E[V_{k+1}(x, I(t_{k+1}), X, s(t_{k+1}))|I(t) = I(t), s(t) = s(t)]|I(t_k^+) = y, s(t_k^+) = z] \end{aligned}$$

we have

$$V(x, y, X, z, t_k^+) = E[V(x, I(t), X, s(t), t)|I(t_k^+) = y, s(t_k^+) = z]$$

which results in a partial differential equation (PDE) $dV/dt = 0$. Specifically,

$$V_t + V_I \mu_i I + V_s \mu_s s + \frac{1}{2} V_{II} \sigma_i^2 I^2 + \frac{1}{2} V_{ss} \sigma_s^2 s^2 + \rho V_{Is} \sigma_i \sigma_s I s = 0 \tag{4.1.1}$$

As $V(x, I, X, s, t_{k+1}) = V_{k+1}(x, I, X, s)$ and $V(x, I, X, s, t_k^+) = \underline{V}_k(x, I, X, s)$, we can use this PDE to calculate $\underline{V}_k()$, given $V_{k+1}()$, the boundary condition at t_{k+1}. However, since the utility function is homogeneous and there is no x-item in (4.1.1), we can construct a new value function with two less state dimensions to improve the efficiency of computation.

Introducing the function $F(u, v, t) = V(1, e^u, 1, e^v, t)$, that implies $V(x, I, X, s, t)$ $= (X)^\gamma F(\ln(I/X), \ln(sx/X), t)$ when $X \neq 0$, we have the following PDE:

$$0 = F_t + F_u(\mu_i - \frac{1}{2}\sigma_i^2) + F_v(\mu_s - \frac{1}{2}\sigma_s^2) + \frac{1}{2} F_{uu} \sigma_i^2 + \frac{1}{2} F_{vv} \sigma_s^2 + \rho F_{uv} \sigma_i \sigma_s \tag{4.1.2}$$

This PDE has constant coefficients and is convenient for computation.

For given k, let $G(u, v)$ be the value $F(u, v, t_{k+1})$ (when $k = N - 1$, $G(u, v)$ is the utility of state (u, v) at final time), and $\underline{F}(u, v) = F(u, v, t_k^+)$. Our problems can be solved by iterating the following process for stage k backward from $N - 1$:

1. Solving PDE: Let $F(u, v, t_{k+1}) = G(u, v)$, calculate $\underline{F}(u, v) = F(u, v, t_k^+)$ with PDE (4.1.2).

2. Optimization: Calculate the related target lines based on $\underline{F}(u, v)$ (and test related conditions if needed), output them, and update $G(u, v)$ according to optimal policy.

Note: since the values of $\underline{F}(\cdot)$ and $G(\cdot)$ are used and updated at each stage, we do not label them with a specific k.

Using software SCiPDE,[2] we apply the finite difference (FD) technique to solve the PDE by setting grid points evenly in the state space (u, v).

We perform the optimization process on a new value base function $\underline{\Theta}(u, v)$ with $\underline{\Theta}(u, v) = \underline{F}(u, v)(\frac{1}{e^u + q(e^v)})^\gamma$, i.e., normalizing $\underline{F}_k(u, v)$ with one unit of wealth. This makes the search for reachable states easier. Correspondingly, for state (u, v), the reachable states is $\{(z, v) : z > u\}$ for the TN trading direction, and $\{(z, \ln(\frac{(e^v - 1)(e^z + 1)}{e^u + 1} + 1)), z < u\}$ for the NT trading direction. After updating $\Theta(u, v)$ with $\max \underline{\Theta}(z, y)$ among states (z, y) reachable from (u, v), we update $G(u, v)$ with $G(u, v) = (e^u + q(e^v))^\gamma \Theta(u, v)$, for the computation of the next iteration.

4.1.2 Simulation-based method

Besides the PDE-based method, we implemented two simulation-based methods for problem PTN: one uses simulation to estimate the values on discrete points and approximate the value base function by interpolation, we denote it "PE" (in short for "point estimation"), the other uses simulation and regression techniques, we denote it "RS" (in short for "regression simulation"). We first introduce the common characteristics and then show some key points of the implementation of each method.

Our simulation-based method is based on formulation (P2) in (z, c) space. For

[2]SCiPDE is a software system produced by SciComp Inc., which generates C code to solve PDE by the FD technique.

problem PTN, the formulation and the computation is simpler than other problems, as it has a simple formula for the set of reachable states. When we obtain the function $\underline{H}_k()$, we know that the optimal policy for state (z, s) at time t_k is to increase z (if possible) to $z_k^*(s) = \arg\max_z \underline{H}_k(z, s)$, that is the optimal N/W ratio.

The DP framework provides a computational outline. The ways to represent and calculate the value function under this framework result in different methods. The following are some key points for implementing the PE and RS methods.

Point-estimation-based method (PE)

We implement the PE method according to the following outline:

1. Initialization: Set $m + 1$ z-nodes $z_0, z_1, ..., z_m$ evenly in the range $[0, 1]$, as $z_i = i/m$; and set $n + 1$ c-nodes $c_0, c_1, ..., c_n$ evenly in the $\ln(c)$ space in the range $[b_l, b_u]$, that is, $c_j = \exp(b_l + (b_u - b_l)j/n)p$. Generate L sample pairs (e^l, f^l), $l = 1$ to L, according to the distribution of $(i_{k+1}/i_k, s_{k+1}/s_k)$.[3] Set $k = N$.

2. If $k = 0$ stop; else $k = k - 1$;

3. For each state point (z_i, c_j) (for $i = 0$ to m and $j = 0$ to n), estimate $\underline{H}_k(z_i, c_j)$ by

$$\underline{H}'_k(z_i, c_j) = \frac{1}{L} \sum_{l=1}^{L} [(w_{k+1}^{il})^\gamma H'_{k+1}(z_i e^l / w_{k+1}^{il}, c_j f^l)]$$

[3]For efficiency, if the process parameters are constant, we can use the same sample set for all stages, otherwise the samples should be generated for each stage.

with $w_{k+1}^{il} = e^l z_i + \frac{q(f^l c_j)}{q(c_j)}(1 - z_i)$.

4. Report computational results for stage k, go to step 2.

In this algorithm, the value base function $\underline{H}_k(z, c)$ is approximated by interpolation on grid points (z_i, c_j)s with value of $\underline{H}'_k(z_i, c_j)$s, and denoted by $\underline{H}'_k(z, c)$. The optimal ratio $z_k^*(c)$ is also calculated on $\underline{H}'_k(z, c)$. In step 3, the function $H'_k(z, c)$ is calculated as follows: if $k = N$, $H'_k(z, c) = U(1)$; else $H'_k(z, c) = \underline{H}'_k(\max\{z, z_k^*(c)\}, c)$.

Regression-based method (RS)

Since our main purpose is to test the performance, and problem PTN only needs to find the optimal z for a fixed c, we implement the regression-based method in the one-dimensional space of c and use interpolation on a set of grid points in z-space.

The outline of the algorithm is as follows:

1. Initialization: Set $m + 1$ z-nodes $z_0, z_1, ..., z_m$ evenly in the range $[0, 1]$, as $z_i = i/m$. Set L c-nodes $c_1, ..., c_L$ evenly in $\ln(c)$ space of range $[b_v, b_u]$. Generate L samples (e^l, f^l) according to distribution of $(i_{k+1}/i_k,\ s_{k+1}/s_k)$ for $l = 1$ to L. Set n basis functions, $\phi_1(c)$,$\phi_2(c)$, ..., $\phi_n(c)$. Set $k = N$.

2. If $k = 0$ stop; else $k = k - 1$;

3. For $i = 0$ to m

   - For $l = 1$ to L, calculate the sample realized value $\chi_k^i(c_l)$ by:

$$\chi_k^i(c_l) = w^\gamma H'_{k+1}(z_i e^l / w, f^l c_l),$$

with $w = e^l z_i + \frac{q(f^l c_l)}{q(c_l)}(1 - z_i)$; the function $H'_{k+1}(z, c)$ will be introduced later.

- Use regression to find a vector $(a_k^{i1}, a_k^{i2}, \ldots, a_k^{in})$ such that $\sum_{l=1}^{L}(\sum_{j=1}^{n} a_k^{ij}\phi_j(c_l) - \chi_k^i(c_l))^2$ is minimized.

4. Report computational results for stage k, go to step 2.

In this algorithm, we use $Q_k^i(c) = \sum_{j=1}^{n} a_k^{ij}\phi_j(c)$ to approximate $\underline{H}_k(z_i, c)$, and then for fixed c, $\underline{H}_k(z, c)$ is approximated by interpolating $Q_k^i(c)$s on z_is, and denoted by $\underline{H}'_k(z, c)$. $z_k^*(c)$ is calculated on this approximated $\underline{H}_k(z, c)$. In step 3, the function $H'(z, c)$ is computed as follows: If $k + 1 = N$, $H'(z, c) = 1$; else $H'(z, c) = \underline{H}'_k(\max\{z, z_k^*(c)\}, c)$.

4.2 Computational performance of all numerical methods on PTN

In this section, we compare the computational performance of all numerical methods on problem PTN (for both RL and NRL cases), on the basis of accuracy of computational solutions (with one time-step and many time-steps) and performance of policies derived from these solutions. Although we do not conduct complete study on computation, these results can show certain characteristics of each solution method.

All the computational results are obtained with the following basic parameters: parameters of assets' price processes $\mu_s = 0.15$, $\mu_i = 0.05$, $\sigma_s = 0.4$, $\sigma_i = 0.1$ and

$\rho = 0.3$, utility parameter $\gamma = 0$, tax rate $\alpha = 0.3$, and basis price $p = 1.0$.[4]

To present our results, we use "**FD**N" to denote the solution of the FD method with a $N \times N$ grid of state points in range $-7 \leq \ln(z) \leq 17$, $-8 \leq \ln(s) \leq 8$, and "**PE**$(n-L)$" to denote the solution of the PE method with $n+1$ points of s in range $-8 \leq \ln(s) \leq 8$ and L samples simulated for each discrete state point, with $m = 50$ for setting z-grids evenly in range [0,1].[5]

For RS methods, we test several sets of basis functions (SBF) with different size n and with different simulated sample size L. The following are some selected RS methods with specific SBF.

RS-l uses power of $\ln(s)$, i.e., SBF=$\{1, \ln(s)^i$ for $i = 1$ to $n-1\}$.

RS-c uses cubic spline series, i.e., SBF=$\{1, \ln(s), \ln(s)^2, \ln(s)^3$ and$(\ln(s) - x_i)^{+3}$ for $i = 1$ to $n-4\}$. The nodes x_is, which are called knot-nodes or simply k-nodes, are set evenly in space of range $-8 \leq \ln(s) \leq 8$.

RS-q uses quadratic spline series, i.e., SBF=$\{1$, $\ln(s)$, $\ln(s)^2$, and $(\ln(s) - x_i)^{+2}$ for $i = 1$ to $n-3\}$, with k-nodes x_is being set densely when s is close to p.[6] It is specifically designed to deal with the peak of target line at $s = p$ in the NRL case.

RS-r uses a special SBF of $\{1, \frac{s}{q(s)}, (\frac{s}{q(s)})^2, (\frac{s}{q(s)})^3, (\frac{s}{q(s)})^4$, and s^i for $i = 1$ to $n-5\}$.

[4]Since the basis price does not change in problem PTN, we use basis price $p = 1$ so that stock price s is also the relative price c in the formulation of Model P2, and we do not use the term "relative price" here.

[5]We found that the results using m greater than 50 are similar, thus here we report solutions with this fixed z-grid.

[6]The k-nodes x_is, for $i = 1$ to $n-3$, are set as: $x_1 = 0$, $x_{2j} = -2^{3-j}$, $x_{2j+1} = -2^{3-j}$ for $j \geq 1$, assuming $p = 1$.

It uses the fact that the growth rate $Q(z)$ at time just before termination can be expanded as $a_1 + a_2 \frac{s}{q(s)} + a_3 (\frac{s}{q(s)})^2 + a_4 s + a_5 s^2$ where a_is are constant, and we use more items to improve its power for representing.

We use the notation "RS-$x(n-i)$" to denote the solution of the RS method "RS-x" using SBF size of n and sample size $L = 2^i$. Like PE solutions, all the solutions for these RS methods reported here use $m = 50$ for z-position setting.

In simulation-related computation, either the computation of PE and RS methods or the evaluation of policies, we found that using the Sobol sequence to generate simulation sample has much higher accuracy and efficiency than plain random numbers. If not mentioned, the simulations reported in this section are implemented with this kind of Quasi-Monte Carlo methods.

4.2.1 Computation with one time-step

At the time one time-step before termination, the computation is based on the accurate value at the final time. Hence we report the results at this time to show the accuracy of each method for one-step computation.

We tested the three kinds of solution methods on two problems: **RL1** and **NRL1**, which both have the same basis parameters and one time-step $dt = 0.01$year to final time, but different tax codes, namely RL and NRL, respectively.

From the computational results on the two problems, we found that:

- With proper computational parameter, all the numerical methods can generate target N/W ratios close to the analytic limiting solution, which is a good reference when time-step dt is very short, though it is not the exact optimal result at time one step before determination.

- With increasing grid density, FD solutions stably converge to some result very close to that of FD1600.
- PE solutions show a tendency of converging to FD1600 when the number of discrete points n and the sample size L are increased.
- The convergence of FD and PE solutions verifies the correctness of these two solution methods presented in different Dynamic Programming formulation with different state space.

From above findings and the best policy performance of FD1600 which will be shown later, we believe that FD1600 is the most accurate solution, though we do not know the exact optimal solution. Hence we analyze the accuracy of solutions via their distances to FD1600, which is measured by the root mean square error (RMSE) on the target N/W ratio on 53 s points evenly distributed in the range $-4.16 \leq \ln(s) \leq 4.16$ for our report.

From the analysis on the RMSE to FD1600 for each solution, we found that:

- The accuracy of RS method largely depends on the set of basis functions (SBF). For example, in the NRL case, with 2^{19} simulation samples, RS-q can provide the correct optimal N/W ratio of 1.0 at $s = p$ and has a RMSE to FD1600 which is less than 0.034, while each of the other tested RS methods has an error greater than 0.5 at $s = p$ and a RMSE to FD1600 around 0.1.

- Comparing PE and RS solutions with the same total number of samples, e.g., comparing PE64-2048 with RS solutions with $L = 2^{17}$ or comparing PE64-8192 with RS solutions with $L = 2^{19}$, generally the RS solutions have larger errors than PE solutions, except in cases of small size of samples for problem NRL1 where there are larger errors than problem RL1.

- With the increase in the total number of samples (and thus also the computation effort), the error of RS solutions seems to reach a limit larger than that of PE solutions. We think that it is caused by the error between the real value base function and the best linear combination of the basis functions. For a SBF with a fixed size of n, this error can not be reduced by increasing the amount of samples. Increasing the size of SBF can improve the representative ability of SBF and reduce the error, but needs an exponential increase of samples to support the computation.

- All the solution methods do not perform as well in the NRL case as they do in the RL case for one-step problem. FD and PE methods (which use discrete approximation on grid points) need more grid nodes to attain a specific accuracy in NRL1 than in RL1. For RS methods, we can easily find a SBF to make error less than 0.01 for problem RL1, while for problem NRL1, the best performer RS-q has an error larger than 0.03 which cannot be improved through increasing the sample size.

  We think the reason is that there is a steep peak on target N/W ratio at small neighborhood of $s = p$, i.e., $s = 1$ in the NRL case.

To show the computational efficiency, in Figures 4.1 and 4.2, we present the pairs of accuracy and computation time for each solution method on problems RL1 and NRL1 respectively. In these two figures, the displayed error for PE is the smallest one among the errors of all PE solutions with the same computation effort. RS-r and RS-q are the the best among all the selected RS methods for problems RL1 and NRL1 respectively. The computation time is measured on a PC with Pentium 4 CPU 3.0GHz.

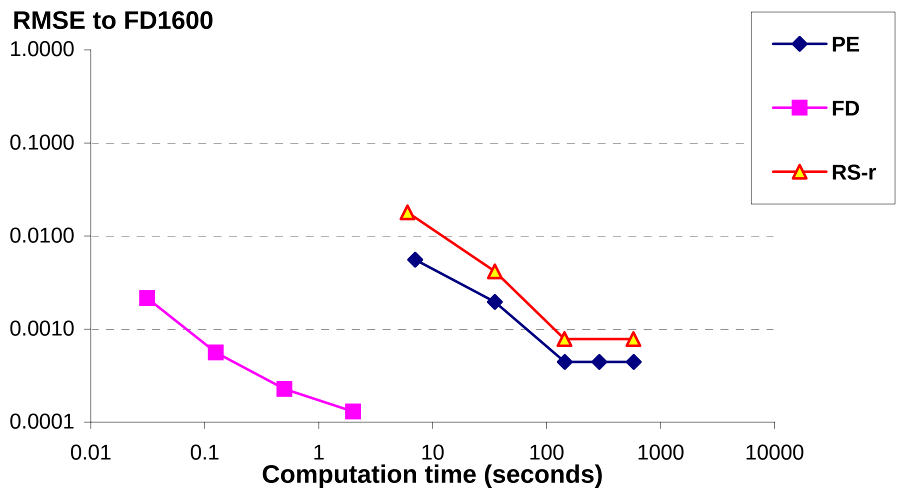


Figure 4.1: Accuracy and computation time of each method in the RL case

4.2.2 Computation with multiple time-steps

All the numerical methods involve computational error, and this error will accumulate along with increase in time-steps. From the accurate results of FD method on problems with one time-step, we are confident of its accuracy for problem with several time-steps and we believe that the FD method have the best performance

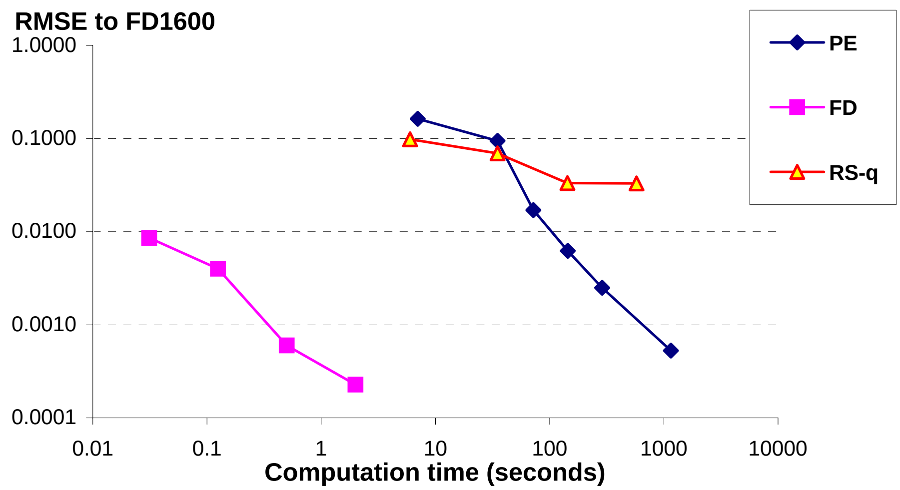


Figure 4.2: Accuracy and computation time of each method in the NRL case

among all methods for problems with many time-steps. Since we do not know the exact solution for problem with many time-steps, we use the RSME, a measure of difference, between FD1600 and other solutions, to make inferences about accumulated errors.

Here we present some computational results for two problems with 100 time-steps, which have the same basis parameters and the length of time-step $dt =$ 0.1year, but different tax codes: RL and NRL. We call them problems **RL2** and **NRL2** respectively.

Tables 4.1 and 4.2 show the RSME of each solution to FD1600 at different time points for problems RL2 and NRL2.

From the computational results, we find:

- Generally, the RSME are kept within a small level (less than 0.1) for all the time for all the methods. The RSME may increase along with the time-steps

timestep	1	2	5	10	20	50	100
FD50	0.00751	0.00554	0.00516	0.01468	0.00791	0.01039	0.01286
FD200	0.00056	0.00078	0.00100	0.00115	0.00099	0.00043	0.00055
FD800	0.00013	0.00038	0.00014	0.00184	0.00087	0.00155	0.00069
PE(32-1024)	0.00556	0.00468	0.00660	0.00854	0.00883	0.00905	0.00811
PE(64-2048)	0.00194	0.00203	0.00200	0.00217	0.00244	0.00215	0.00217
RS-r(8-17)	0.00416	0.03206	0.06061	0.07465	0.08294	0.08578	0.08270
RS-r(8-19)	0.00078	0.02921	0.05716	0.07108	0.07956	0.08361	0.08088
RS-l(12-17)	0.01052	0.04165	0.07462	0.09159	0.10334	0.11076	0.10193
RS-l(16-19)	0.00288	0.02776	0.05372	0.06607	0.07358	0.07149	0.05993
RS-c(16-17)	0.00492	0.01839	0.03894	0.05064	0.05375	0.04568	0.04515
RS-c(16-19)	0.00139	0.01793	0.03821	0.04958	0.05241	0.04405	0.04305

This table shows the root mean square error between FD1600 and other solutions at certain time points (with the number of steps to termination as listed in the first row) for problem RL2 where investor can trade with time-step $dt = 0.1$ year, under the RL tax code.

Table 4.1: RSME between FD1600 and other solutions on several time points for problem RL2

timestep	1	2	5	10	20	50	100
FD50	0.01116	0.02751	0.03125	0.01676	0.01248	0.01233	0.01096
FD200	0.00699	0.00533	0.00576	0.00291	0.00291	0.00237	0.00219
FD800	0.00228	0.00123	0.00544	0.00272	0.00134	0.00064	0.00049
PE(32-1024)	0.16280	0.14551	0.13258	0.10954	0.09244	0.07130	0.06295
PE(64-2048)	0.09439	0.06819	0.05134	0.04665	0.04103	0.03701	0.03576
PE(128-2048)	0.01703	0.01295	0.01043	0.00853	0.00709	0.00577	0.00477
PE(512-2048)	0.00250	0.00244	0.00233	0.00233	0.00214	0.00214	0.00188
RS-q(12-17)	0.07478	0.07641	0.06622	0.06805	0.06531	0.03675	0.01196
RS-q(16-17)	0.06946	0.06833	0.06307	0.07979	0.05586	0.04509	0.01763
RS-q(16-19)	0.03321	0.02342	0.09353	0.03423	0.03313	0.02409	0.01523

This table shows the root mean square error between FD1600 and other solutions at certain time points (listed in the first row) for problem NRL2 where investor can trade with time-step $dt = 0.1$ year, under the NRL tax code.

Table 4.2: RSME between FD1600 and other solutions on several time points for problem NRL2

at beginning, but when they reach to some point (usually within 10 steps), they will not increase significantly and may even decrease. For the NRL case, RSME of almost all the methods even have a decreasing trend along with time. The reason may be that the optimal N/W ratios approach a limit value with increases of time to termination, and each solution also has a trend to reach a limit and thus the errors will not enlarge when the number of time-steps is more than a certain number.

- For the RL case, RS methods have bigger errors than other methods, and the errors become serious from the second last time-step while the errors of other methods are not sensitive at this time-point. Among RS methods, RS-r method has the largest error since the second last step, although it has the smallest error for the latest step. It means that an RS method performing well for one step problem may not perform well for multi-steps problem.

- For FD and PE solutions, the RSME do not increase significantly along with time. The RSME are very small for all FD solutions, especially FD200 and FD800. For the NRL case, FD and PE methods need a denser grid than for the RL case to reach the same error level.

4.2.3 Performance of policies

Each numerical solution produces a policy and implementing the policy starting from a initial state will result in a corresponding expected utility of the final wealth. We can use simulation to obtain the estimated expected utility at some states for

	Starting stock prices				
Policy	0.0408	0.2019	1.0000	4.9530	24.5325
RS-l(12-17)	1.56401	1.65947	1.97088	2.11522	2.14970
RS-l(16-19)	1.56831	1.66479	1.97108	2.11524	2.14970
RS-r(8-17)	1.56465	1.66339	1.97102	2.11522	2.14970
RS-r(8-19)	1.56460	1.66364	1.97104	2.11524	2.14970
RS-c(16-17)	1.56800	1.66661	1.97118	2.11524	2.14970
RS-c(16-19)	1.56800	1.66681	1.97120	2.11524	2.14970
PE(32-1024)	1.56831	1.66878	1.97131	2.11522	2.14968
PE(64-2048)	1.56831	1.66881	1.97131	2.11524	2.14970
FD50	1.56831	1.66878	1.97129	2.11518	2.14962
FD200	1.56831	1.66881	1.97133	2.11526	2.14970
FD1600	1.56831	1.66881	1.97133	2.11526	2.14970
Myopic	1.56831	1.59966	1.96272	2.09535	2.12510
A0.5	1.42307	1.63948	1.88534	1.99783	2.02693
BH1.0	1.56831	1.56831	1.56831	1.56831	1.56831
BH0.5	1.41690	1.64897	1.92595	2.05410	2.08700
BH0	1.21480	1.57359	1.91986	2.01244	2.01611

This table shows the certainty equivalent of the estimated expected utility for conducting each policy listed in the first column. The results are estimated with simulation on a time horizon of 10 years, starting from the states with the stock price s in the row under “Starting stock prices” given the initial realizable wealth of 1 dollar of stock and basis price of 1. The model setting is the same as that for RL2.

Table 4.3: Policies’ performance on RL2 by simulation

each policy, and then evaluate the performances of all the policies and the solution methods.[7]

We test the performances of the policies generated by the solution of different methods for preceding problems RL2 and NRL2, and show relevant results in Tables 4.3 and 4.4 respectively.

In the Tables 4.3 and 4.4, data in rows headed by the title of policy are the certainty equivalent of the estimated expected utility of implementing that policy from each states with a certain stock price s at beginning, given initial realizable wealth of 1 dollar of stock. They were estimated by simulation with 262144 sample

[7]For problem PTN, the policy is to move to the target line of N/W ratio if possible. Our numerical solutions provide target N/W ratios on c-grid points. We use interpolation on these grid points to approximate the target line for testing the policies of the numerical solutions.

policy	Starting stock prices				
	0.0408	0.2019	1.0000	4.9530	24.5325
RS-q(12-17)	2.15416	2.09883	1.89417	2.11051	2.14914
RS-q(16-17)	2.15416	2.09900	1.89455	2.11051	2.14914
RS-q(16-19)	2.15448	2.10003	1.89402	2.11095	2.14960
PE(32-1024)	2.15308	2.08352	1.84611	2.10762	2.14962
PE(64-2048)	2.15450	2.09872	1.88942	2.11089	2.14968
PE(128-2048)	2.15472	2.10061	1.89513	2.11114	2.14968
PE(512-2048)	2.15474	2.10066	1.89532	2.11117	2.14968
FD50	2.15459	2.10009	1.89459	2.11106	2.14960
FD200	2.15474	2.10066	1.89529	2.11117	2.14968
FD800	2.15474	2.10066	1.89532	2.11117	2.14968
FD1600	2.15474	2.10068	1.89534	2.11117	2.14968
Myopic	2.12853	2.07402	1.56831	2.09279	2.12508
A0.5	2.03157	1.99093	1.84914	1.99663	2.02691
BH1.0	1.56831	1.56831	1.56831	1.56831	1.56831
BH0.5	2.09064	2.03267	1.88263	2.05269	2.08700
BH0	2.00856	1.94381	1.73060	1.99875	2.01587

This table shows the certainty equivalent of the estimated expected utility for conducting each policy listed in the first column. The results are estimated with simulation on a time horizon of 10 years, starting from the states with the stock price s in the row under "Starting stock prices" given the initial realizable wealth of 1 dollar of stock and basis price of 1. The model setting is the same as that for NRL2.

Table 4.4: Policies' performance on NRL2 by simulation

paths generated by Sobol sequence. The results of using Sobol sequence are much more accurate than that of using plain random numbers. For example, the exact expected utility of policy "holding all the wealth on index fund" is 0.45, the simulation result of using Sobol is 0.45000011, while the result of using plain random numbers is 0.45064688. Although we do not know the standard error of simulation results of using Sobol, we can get some estimation or reference from result of using plain random numbers. From simulation on the same test sets, we found that all the standard errors of results using plain random numbers are less than 0.001.

Besides the policies produced by each numerical methods, we also test following heuristic policies.

Dynamic policies: "Myopic" and "A0.5," which use the limiting solution and constant 1/2 respectively as the target N/W ratio for all the time.

Buy-and-hold policies: BH0, BH0.5 and BH1.0, which respectively put 0, 50 and 100 percent of wealth on nontaxable asset at beginning, and make no trade until the final time.

From the Tables 4.3 and 4.4, we can see that:

- FD1600 has the highest performance in all tested states, FD200 performs almost as well as FD1600.

- The difference on the policy performance of all solutions are much smaller than that on the target N/W ratio. For example, RS-l(12-17) has a RMSE range from 0.01 to 0.11 on the 100 time-points, while the certainty equivalent of its policy has a relative error less than 0.5% to that of FD1600. A solution with RMSE less than 0.01 on 100 time-points will have a performance close to that of FD1600 within a 0.005% relative error, e.g. PE(32-1024) in RL2, and PE(512-2048) in NRL2. This, in some sense, means the objective value (the expected utility) is insensitive to the policy.

- Using moderate grid density and sample size, the simulation-based methods can produce rather good policies whose performances are close to that of FD1600 with a relative error of less than 0.1%, which is much better than the best of heuristic policies.

- Among the Heuristic policies, "Myopic" shows a performance not far from that of FD1600 in most tested states, except at $s = 1$ in the NRL case, where the optimal N/W ratio changes a lot with time. The difference between the performance of FD1600 and policy "Myopic" shows the benefit of using time-dependent policy.

4.3 Concluding remarks

In this chapter, we developed one PDE-based (using finite difference) and two simulation-based (using point-estimation and regression, respectively) numerical methods for dynamic allocation between taxable and nontaxable assets. The computational performance of these numerical methods are also compared.

Because our problem can be reduced in a two-dimensional state space, the PDE-based FD method has the best performance (with the highest accuracy and lowest computation effort). However this method may suffer from the curse of dimension for problems with a high dimensional state space, which are common in practice. In addition, a well-defined PDE may not always be available for all problems. For example, when our problems are represented in the formulation of model P2, we can not have a PDE for the NRL case, because the function $q(c)$ does not have a continuous derivative at $c = 1$. Hence it is worth developing other numerical methods.

In our computational tests, the PE method is used as a reference to compare with the RS method. Since the PE method uses discrete approximation for the value function on grid points, as the FD method, it still cannot deal with high-dimensional problems, besides it uses more computation on expectation than the

FD method. Using binomial or multinomial process to approximate the process of asset's price can be seen as a simplification of quasi-simulation. A solution method based on this kind of approximation, which we call a lattice method, is a simplified PE method. Lattice methods have similar computational features and complexity as FD methods, but they would have bigger errors than FD methods and the PE methods which use large size of samples. When the PDE is unavailable and a quick solution is needed, a lattice method may be acceptable.

The essence of the RS method is to use a linear combination of a set of basis functions (SBF). Hence its performance is determined by the SBF. However there lacks a guideline or a systematic method to find a good SBF. Although we tried many sets of basis functions, we still could not find a good one that makes the RS method perform better than the plain PE method for multi-steps problems. In comparison with PE solutions, all of our tested RS solutions have the following weaknesses: (1) with the same computation effort, the solutions of RS method have bigger errors for almost all the cases;[8] (2) the RS method has a slower (or no) trend of error reduction with increase of sample size; (3) the RS method has a bad control on accumulated error with increase of steps.

When we do not know the characteristics of a value function well, spline series is a good choice for SBF. The spline type of SBFs performed well in most cases in our tests. In addition to the high accuracy and policy's performance among all the RS solutions, spline type of SBFs have some other strengths: (1) they

[8]The unique counter-example is the RS-q solution on one-step problem with small samples.

have a good control on accumulated error for multi-period problems; (2) we can set appropriate k-nodes according to some known features of the approximated function, as we did for RS-q for problem NRL2, and this work is easier than designing special basis functions. However, using spline series as SBF still cannot deal with high-dimensional problems, as it needs to set k-nodes for each dimension.

Finally, it is worth pointing out that we should not disregard the RS method, as it has a potential for high dimensional problem, just in our 2-D problems it failed to performs as well as the FD and PE methods. There is also a need to point out that, even if there are some obvious differences between the solutions (on the target N/W ratio) provided by all these methods, the performances of the policies derived from these solutions are very close, sometimes almost indistinguishable.

Chapter 5

Summary — some words to investors

In this dissertation, I study dynamic investment policies in the presence of taxes. An investment policy describes what to do (e.g., how much of each asset to buy or sell) in different situations, and the term "dynamic" means that it involves a sequence of decisions over time. As a starting point to solve more complicated models, I study a simpler case, the dynamic allocation of two assets: one taxable and one nontaxable asset.

In my research, I derived properties of the optimal policy, and demonstrated its simple structure. Here I illustrate the optimal policy for a typical situation where the taxable asset is stock and the nontaxable asset is a riskfree bond. The Merton policy, which is optimal in dynamic asset allocation without taxes, is to maintain a constant portfolio (i.e., the assets' proportion in total wealth) at all times. For our model, the optimal policy has the following features:

- When there is an embedded capital loss, i.e., the stock price is lower than the basis price, the optimal policy is to sell all stock (i.e., realize the tax credit) and then buy stock (which sets the basis price to the current market price). The final portfolio is optimal for the situation where the mean return and volatility of stock are adjusted by a coefficient of (1-tax rate) in the trading environment with no tax and no transaction cost, i.e., the Merton model. We call this optimal portfolio the "tax-adjusted Merton solution," which is first mentioned in subsection 3.2.3, the "cross point" at $c = 1$. The reason for realizing the loss is that the tax credit from realizing the loss outweighs the present value of the extra tax payment in the future caused by the lower basis.

- When there is an embedded capital gain, i.e., the stock price is higher than the basis price, the optimal policy can be described by the critical prices to sell or buy stock. The optimal policy is to sell stock if the stock price is greater than the critical sell price, to buy stock if the stock price is less than the critical buy price, and do nothing when the stock price is between the critical buy and sell price. If the policy is to do a trade (either to buy or to sell stock), the final position is determined by the stock price, basis price and the fraction of stock in total wealth, and is usually not the tax-adjusted Merton solution. The Merton policy maintains a unique portfolio (i.e., any deviation in portfolio will be adjusted back to this portfolio) and obtains the best diversification benefit. However, in the presence of taxes, there exists a region where it is optimal not to trade. The reason is that in the presence of taxes, we need to compare the tax payment with the diversification benefit, and we only trade when the diversification benefits outweighs the tax cost.

To better understand these features, in the following tables, I present some numerical results. These numerical results are computed by the "limiting solution." Since the policy of the "limiting solution " is very close to the optimal policy, we don't distinguish these terms in what follows. In the computations, we assume the stock has a mean annual return of 10% and a volatility (standard deviation) of 25%, the riskfree bond has an annual return of 5% and the trading interval is one year.

Table 5.1 illustrates the trading strategy for a single sample path. We assume an initial wealth of $10,000, and a fraction in stock of 65.3% (the tax-adjusted Merton solution) at time 0. At time 1, since the stock price is lower than the basis price, the optimal action is to realize the loss and buy stock so that the fraction in stock is the tax-adjusted Merton solution. At time 2, the stock price increases, and is larger than basis price but less than the critical buy price, so the optimal action is to buy stock. At times 5 and 6, the stock prices are higher than the critical sell price, so the optimal actions are to sell stock; after trading the portfolios are be far from the tax-adjusted Merton solution. At other times, the stock price is higher than basis price (i.e., there is a capital gain) but in the range between critical buy price and sell price, so the optimal action is not to trade.

Tables 5.2, 5.3 and 5.4 show the parameters' impact on the optimal policy, the tax-adjusted Merton solution (shown in the form of the nominal fraction in stock) and the critical prices to buy and sell. These numerical results are calculated under

time	stock price	state variables before/after decision				crtcl price buy/sell	action
		#stock	#bond	basis price	stock fraction		
1	98.2	65.3/67.1	34.7/33.3	100.0/98.2	63.7%/65.3%	103.3/226.2	realize loss
2	101.2	67.1/67.2	33.3/33.2	98.2/98.2	64.9%/64.9%	101.4/222.1	buy stock
3	131.1	67.2/67.2	33.2/33.2	98.2/98.2	69.5%/69.5%	104.7/243.0	no trade
4	152.8	67.2/67.2	33.2/33.2	98.2/98.2	71.6%/71.6%	108.3/264.1	no trade
5	289.1	67.2/67.0	33.2/33.5	98.2/98.2	82.0%/81.8%	112.0/285.3	sell stock
6	320.6	67.0/66.7	33.5/34.1	98.2/98.2	82.6%/82.3%	116.7/310.8	sell stock
7	241.2	66.7/66.7	34.1/34.1	98.2/98.2	76.9%/76.9%	122.7/343.2	no trade
8	356.1	66.7/66.7	34.1/34.1	98.2/98.2	82.4%/82.4%	127.2/366.6	no trade

This table shows a sample path of states under the optimal policy in eight annual time steps. The risk aversion parameter is zero, and the tax rate is 30%.

Table 5.1: A sample path of states under the optimal policy

the assumption that at time 0, the market prices of the stock and bond are \$100, and the initial portfolio is the tax-adjusted Merton solution. The critical prices to sell or buy are for the optimal policy at time 1.

tax rate	tax-adjusted Merton sol.	buy price	sell price
0	80.00%	105.13	105.13
0.1	79.01%	103.72	106.53
0.2	75.00%	103.20	115.79
0.3	65.31%	103.28	226.18
0.4	44.44%	103.88	778.16
0.5	0.00%	N/A	N/A

This table shows the tax-adjusted Merton solution (represented in the form of the nominal fraction in stock) at the current time and the critical sell and buy prices at the next trading time. The risk aversion parameter is zero and the basis price is \$100 at time 0.

Table 5.2: Tax-adjusted Merton solution and critical prices to buy and sell under different tax rates

Table 5.2 shows the impact of tax rates. When the tax rate is greater than 50% the optimal portfolio is to hold only the bond, so the critical prices are meaningful when the tax rate is less than 50%. From this table we can see: 1) when the tax rate is zero, the optimal portfolio is unique (i.e., the Merton solution), and the critical buy and sell prices are the same; 2) when the tax rate is greater than zero, the critical prices are different for buying and selling. The critical buy price is not sensitive to the tax rate. The critical sell price increases with the tax rate, because it is optimal to wait longer before selling as the tax payment (the cost of selling) increases; 3) the range of stock prices for not trading (i.e., the range between the critical buy and sell prices) increases with the tax rate. The no-trading region can be quite large. For example, for a tax rate of 30%, stock is not sold until the stock price exceeds the critical sell price of $226.18.

risk aversion	tax-adjusted Merton sol.	buy price	sell price
0	65.31%	103.28	226.18
-1	32.65%	104.48	127.46
-2	21.77%	104.73	121.59
-3	16.33%	104.84	119.68
-4	13.06%	104.91	118.73
-5	10.88%	104.95	118.18
-6	9.33%	104.97	117.81
-7	8.16%	104.99	117.55
-8	7.26%	105.01	117.35
-9	6.53%	105.02	117.20
-10	5.94%	105.03	117.08

This table shows the tax-adjusted Merton solution (represented in the form of the nominal fraction in stock) at the current time and critical sell and buy prices at the next trading time. The tax rate is 30% and the basis price is $100 at time 0.

Table 5.3: Tax-adjusted Merton solution and critical prices to buy and sell under different risk aversion parameters

Table 5.3 shows the impact of the risk aversion parameter. From this table we can see that when the investor becomes more risk averse (i.e., the risk aversion parameter decreases), the range of stock prices for not trading becomes more narrow. More risk averse investors trade more frequently to maintain a sufficiently diversified portfolio.

basis price	buy price	sell price
100	103.28	226.18
90	99.67	244.43
80	96.06	258.65
70	92.45	270.42
60	88.84	280.44
50	85.23	289.13
40	81.62	296.74
30	78.01	303.43
20	74.40	309.34
10	70.80	314.55

This table shows the critical sell and buy prices at time 1, for different basis prices given at time 0. The tax rate is 30% and the risk averse parameter is zero. At time 0 the nominal fraction in stock is the tax-adjusted Merton solution, 65.31%.

Table 5.4: Critical prices to buy and sell under different basis prices

Table 5.4 shows the impact of the basis price on the critical prices. As the basis price decreases, the critical buy price decreases and the critical sell price increases, i.e., the no-trading region increases. When the capital gain is larger, the cost for selling stock is larger, so it is optimal to wait longer to sell stock.

In summary, the optimal trading strategy in the presence of taxes is quite different than with taxes. This thesis derives theoretical results showing the structure of optimal trading strategies and develops algorithms for computing these strategies in the presence of taxes.

Appendix A

Proofs and Derivations for Chapter 3

A.1 The proofs of theorems for one-way trading problems

Lemma A.1.1. *Let $S \subset R^m$ be a convex set, $P \subset R^{n+m}$ a convex set such that for any $x \in S$, set $Q(x) = \{u : (u, x) \in P\}$ is not empty. If the function $f(u)$ is concave in u, then the function $f^*(x) \equiv \max_{u \in Q(x)} f(u)$ is concave in x.*

Proof:

Let x_1, $x_2 \in S$, and for $i = 1, 2$, let $u_i = \arg\max_{u \in Q(x_i)} f(u)$, that is, $f^*(x_i) = f(u_i)$ and $u_i \in Q(x_i)$ which also implies $(u_i, x_i) \in P$.

For any $\lambda \in (0, 1)$, we have $\lambda u_1 + (1 - \lambda)u_2 \in Q(\lambda x_1 + (1 - \lambda)x_2)$, because $(\lambda u_1 + (1 - \lambda)u_2, \lambda x_1 + (1 - \lambda)x_2) = \lambda(u_1, x_1) + (1 - \lambda)(u_2, x_2) \in P$, from the convexity of P.

Then

$$\begin{aligned} f^*(\lambda x_1 + (1-\lambda)x_2) &\geq f(\lambda u_1 + (1-\lambda)u_2) \\ &\geq \lambda f(u_1) + (1-\lambda) f(u_2) \\ &= \lambda f^*(x_1) + (1-\lambda) f^*(x_2) \end{aligned}$$

where the first inequality follows by $\lambda u_1 + (1-\lambda)u_2 \in Q(\lambda x_1 + (1-\lambda)x_2)$, the second inequality follows by the fact that $f(u)$ is concave in u.

Therefore, $f^*(x)$ is concave in x.

□

Proof of Theorem 1:

Note: In problem PTN, the APP $a(t) = p$ for all the stages, so we neglect the item "xp" in the state in the proof for simplicity.

(1) Concavity can be proved by induction on k.

Assuming $V_{k+1}(x, I, s)$ is concave in x and I, then for any positive r, $V_{k+1}(x, rI, s)$ is still concave in x and I.

From $\underline{V}_k(x, I, s) = E[V_{k+1}(x, \frac{i_{k+1}}{i_k} I, s_{k+1}) | s_k = s]$, and $V_{k+1}(x, \frac{i_{k+1}}{i_k} I, s_{k+1})$ is concave in x and I for each sample $(\frac{i_{k+1}}{i_k}, s_{k+1})$, we have $\underline{V}_k(x, I, s)$ is concave in x and I.

For given state (x, I, s) the reachable state is $\{(z, y, s) : 0 \leq z \leq x, y = I + (x - z)b(p, s)\}$ and the set $\{(z, y, x, I) : 0 \leq z \leq x, y = I + (x - z)b(p, s)\}$ is convex. From the concavity of $\underline{V}_k()$, and using Lemma A.1.1, we know $V_k(x, I, s)$ is concave in x and I.

By inducting, since $V_N(x, I, s) = U(I + xb(s, p))$ is concave in x and I, we know $V_k(x, I, s)$ and $\underline{V}_k(x, I, s)$ are concave in x and I, for $k = 0, 1, 2, ..., N-1$.

(2) From the concavity of $\underline{V}_k(x, I, s)$ and the reachable states have a linear relation between x and I, we know the optimal state must be the unique local optimal in $\underline{V}_k()$, and the optimal policy is of the threshold type. The detailed proof follows.

Define

$$F_k(z, w, s, p) = \underline{V}_k(z, w - zb(s, p), zp, s)$$

and

$$x_k^*(w, s, p) = \arg \max_{0 \le z \le w/b(s,p)} F_k(z, w, s, p).$$

From the concavity of $\underline{V}_k(x, I, xp, s)$, we know that $F_k(z, w, s, p)$ is concave in z and then for $0 \le x \le \frac{w}{b(s,p)}$,

$$\max_{0 \le z \le x} F_k(z, w, s, p) = \begin{cases} F_k(x_k^*(w, s, p), w, s, p), & \text{if } x \ge x_k^*(w, s, p) \\ F_k(x, w, s, p), & \text{otherwise} \end{cases}$$

Since

$$\begin{aligned} V_k(x, I, xp, s) &= \max_{0 \le u \le x} \underline{V}_k(x - u, I + ub(s, p), (x - u)p, s) \\ &= \max_{0 \le z \le x} F_k(z, I + xb(s, p), s, p), \end{aligned}$$

the optimal policy for state (x, I, xp, s) is: if $x > x_k^*(I + xb(s, p), s, p)$ then sell stock to reach the position $x_k^*(I + xb(s, p), s, p)$, otherwise remain at x.

□

Proof of Theorem 2:

(1) The proof of concavity is similar to Theorem 1.

The concavity of $\underline{V}_k()$ is easy to prove by expectation of concave function $V_{k+1}()$.

The concavity of $V_k()$ for $k < N$ can be proved by Lemma A.1.1, on the fact that for given state (x, I, X, s), the reachable state is $\{(z, y, A, s) : x \le z \le x + I/s, y = I + (z - x)s, A = X + (z - x)s\}$, and the set $\{(z, y, A, x, I, X) : x \le z \le x + I/s, y = I + (z - x)s, A = X + (z - x)s\}$ is convex.

The concavity of $V_N()$ need more words as follows.

- for the NRL case,

$$V_N(x, I, X, s) = U(I + xb(s, X/x)) = U(I + \alpha \min(X, xs) + \beta xs).$$

Given 2 states (x_1, I_1, X_1, s) and (x_2, I_2, X_2, s), for any $\lambda \in [0, 1]$,

$$\lambda X_1 + (1 - \lambda)X_2 \ge \lambda \min(X_1, x_1 s) + (1 - \lambda) \min(X_2, x_2 s)$$

and

$$(\lambda x_1 + (1 - \lambda)x_2)s \ge \lambda \min(X_1, x_1 s) + (1 - \lambda) \min(X_2, x_2 s),$$

so

$$\min(\lambda X_1 + (1 - \lambda)X_2, (\lambda x_1 + (1 - \lambda)x_2)s) \ge \lambda \min(X_1, x_1 s) + (1 - \lambda) \min(X_2, x_2 s).$$

Hence $V_N(x, I, X, s)$ is concave in x, I and X by the following derivation.

$$\begin{aligned}
& V_N(\lambda x_1 + (1-\lambda)x_2, \lambda I_1 + (1-\lambda)I_2, \lambda X_1 + (1-\lambda)X_2, s) \\
= \;& U(\lambda I_1 + (1-\lambda)I_2 + \beta(\lambda x_1 + (1-\lambda)x_2)s \\
& +\alpha \min(\lambda X_1 + (1-\lambda)X_2, (\lambda x_1 + (1-\lambda)x_2)s)) \\
\geq \;& U(\lambda I_1 + (1-\lambda)I_2 + \beta(\lambda x_1 + (1-\lambda)x_2)s \\
& +\alpha(\lambda \min(X_1, x_1 s) + (1-\lambda)\min(X_2, x_2 s)) \\
= \;& U(\lambda(I_1 + \beta x_1 s + \alpha \min(X_1, x_1 s)) \\
& +(1-\lambda)(I_2 + \beta x_2 s + \alpha \min(X_2, x_2 s))) \\
\geq \;& \lambda U(I_1 + \beta x_1 s + \alpha \min(X_1, x_1 s)) \\
& +(1-\lambda)U(I_2 + \beta x_2 s + \alpha \min(X_2, x_2 s)) \\
= \;& \lambda V_N(x_1, I_1, X_1, s) + (1-\lambda)V_N(x_2, I_2, X_2, s)
\end{aligned}$$

Where the first inequality follows by the fact that $U(w)$ is increasing in w, the second inequality by the concavity of $U(w)$.

- for the RL case,

$$V_N(x, I, X, s) = U(I + xb(s, X/x)) = U(I + \alpha X + \beta xs)$$

$V_N(x, I, X, s)$ is concave in x, I and X from the concavity of $U()$.

(2) From the concavity of $\underline{V}_k(x, I, X, s)$ and the reachable states is a search line with a linear relation between x, I and X, we know the optimal state must be the unique local optimal in $\underline{V}_k()$ along the search line, and the optimal policy

is of the threshold type. The main points of the proof are: let

$$x_k^*(x, I, X, s) = \arg \max_{\max\{X/s-x,0\} \leq z \leq I/s+x} \underline{V}_k(z, I + xs - zs, X - xs + zs, s),$$

and $\underline{V}_k(z, I + xs - zs, X - xs + zs, s)$ is concave in z from the concavity of $\underline{V}_k(x, I, X, s)$.

Since only NT trading is allowed, the optimal policy for state (x, I, X, s) is: if $x < x_k^*(x, I, X, s)$ then buy stock to reach the position $x_k^*(x, I, X, s)$, otherwise remain at x.

□

Remark Lemma A.1.1 is very general. For our proofs to Theorem 1 and Theorem 2, we only apply Lemma A.1.1 for cases where set P is defined by a system of linear inequalities, i.e., it can be represented in form of $Au + Bx \leq b$ where A and B are matrices, and b a vector.

However, for problem PCT, we cannot get the similar result for the concavity of value functions for the all stages. Because for given state (x, I, X, s) the set of reachable states is $\{(z, y, A, s) : z = x + u - d, y = I - us + db(s, X/x), A = X + us - dX/x, 0 \leq d \leq x, 0 \leq u \leq \frac{xI+dX}{xs}\}$, and due to the item X/x, the set $P = \{(z, y, A, x, I, X) : z = x + u - d, y = I - us + db(s, X/x), A = X + us - dX/x, 0 \leq d \leq x, 0 \leq u \leq \frac{xI+dX}{xs}\}$ may not be convex.

For example, let $p_1 = (z_1, y_1, A_1, x_1, I_1, X_1) = (z_1, I_1 + (x_1 - z_1)b(s, X_1/x_1), X_1 - (x_1 - z_1)X_1/x_1, x_1, I_1, X_1)$ with $z_1 \in (0, x_1)$, $p_2 = (z_2, y_2, A_2, x_2, I_2, X_2) = (x_2, I_2, X_2, x_2, I_2, X_2)$, we can verify $p_1 \in P$ and $p_2 \in P$, however $p' = (p_1 + p_2)/2$ may not be $\in P$. To see this, let $a_1 = X_1/x_1$, $a_2 = X_2/x_2$ and $s > a_1 > a_2$, and define a set of new variables v' by $v' = (v_1 + v_2)/2$ with v being any symbol of x, I, X, z, y and A. Condition $p' \in P$ requires that exist $u, d \geq 0$ such that the following conditions hold

$$z' = x' + u - d \tag{A.1.1}$$

$$y' = I' - us + db(s, X'/x') \tag{A.1.2}$$

$$A' = X' + us - dX'/x' \tag{A.1.3}$$

$$d \leq x' \tag{A.1.4}$$

$$u \leq \frac{I' + dX'/x'}{s} \tag{A.1.5}$$

Equations equations (A.1.1) and (A.1.3) can be simplified as

$$x_1 - z_1 = 2(d - u) \tag{A.1.6}$$

$$(x_1 - z_1)a_1 = 2(dX'/x' - us) \tag{A.1.7}$$

Then

$$u = \frac{(x_1 - z_1)(a_1 - X'/x')}{2(X'/x' - s)}.$$

From $a_1 > a_2$ and $a_1, a_2 < s$, we have $X'/x' = \frac{a_1x_1+a_2x_2}{x_1+x_2} < a_1$ and $X'/x' < s$, and then we have $u < 0$, the condition $u \geq 0$ is not satisfied, and $p' \notin P$.

A.2 The proofs of theorems for PCT

We obtained our results from studying the position relation between N-line and T-line.

For simplicity, we study a certain time stage k, and neglect the subscript k. All the following results are based on assuming that condition A' holds at stage k.

Since each state point $(z^N(K), C(z^N(K), K))$ or $(z^T(c), c)$ represent one characteristic value K or c, so neither "N-line" nor "T-line" will cross itself, and each of them separate state space (z, c) into two part.

For simplicity, we informally introduce "up" and "down" for direction of increasing or decreasing z in state space (z, c) and then the relation "above" and "under". Because TN trading increases N/W ratio and NT trading decreases N/W ratio, we know each target line separates space (z, c) into two parts: one is "above" the line and one is "under" the line.

From Condition A', we know

Fact 1: the value of $\underline{H}(z, c)$ will decrease for both NT trades from states "under" N-line and TN trades from states "above" T-line.

Since $\mathcal{T}^N(0) = \mathcal{T}^T(1) = \{(z, 1) : 0 \leq z \leq 1\}$, the trading track for NT and TN trading are the same line segment, but with different direction, hence we have

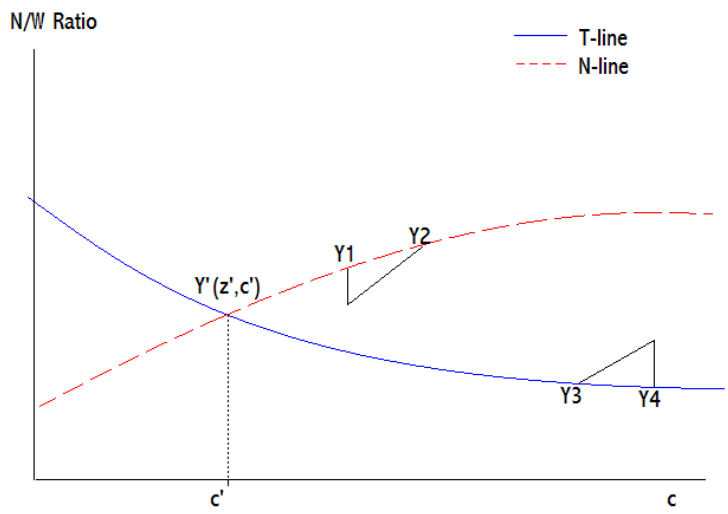
Fact 2: the two target lines must cross at $c = 1$, $z^N(0) = z^T(1)$.

For brevity, denote $\underline{H}(z^T(c), c)$ by $\underline{H}^T(c)$, and $\underline{H}(z^N(K), C(z^N(K), K))$ by $\underline{H}^N(K)$. From continuity of state and action space and distribution of assets' process, $\underline{H}^T(c)$ and $\underline{H}^N(K)$ are continuous.

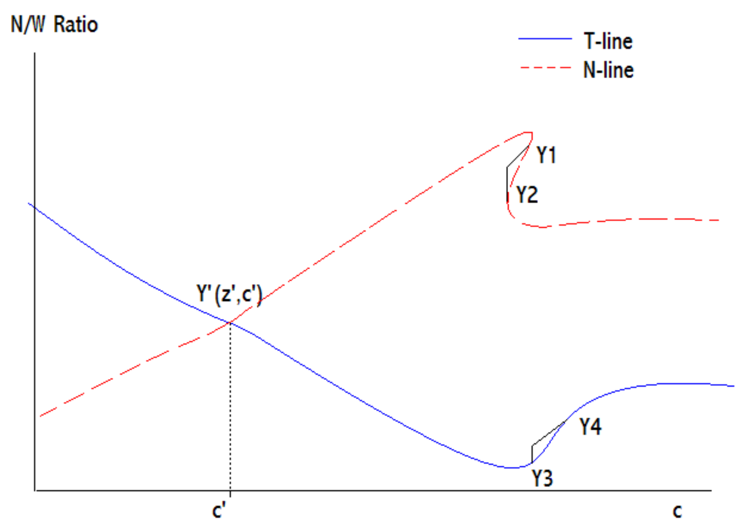
Lemma A.2.1. *If condition A' holds, then*

(1) for segment of N-line above T-line, $\underline{H}^N(K)$ is increasing with K if $K > 0$, or is decreasing with K if $K < 0$;

(2) for the segment of T-line under N-line, $\underline{H}^T(c)$ is increasing with c if $c > 1$, or is decreasing with c if $c < 1$.



regular Target lines



irregular Target lines

Figure A.1: Illustration of N-line and T-line for proof of Lemma A.2.1.

Proof:

We only study the states with $c > 1$, that also implies $K > 0$, on the following two cases: (1) the segment of N-line which is above the T-line and (2) the segment of T-line which is under the N-line. The results for other cases with $c < 1$ can be derived similarly.

(1) The segment of N-line which is above the T-line

For two states on N-line, $Y1 = (z_1, c_1) = (z^N(K_1), C(z^N(K_1), K_1))$ and $Y2 = (z_2, c_2) = (z^N(K_1+\delta), C(z^N(K_1+\delta), K_1+\delta))$, with small $\delta > 0$, there always exists a path from $Y2$ to $Y1$ by NT trading first and then TN trading in case $c_1 < c_2$ (as in the regular case, the upper graph of Fig. A.1) or by TN trading first and then NT trading in case $c_2 < c_1$ (as in the irregular case, the lower part in Fig. A.1) or just a simple TN trading for case $c_2 = c_1$ (not show in graph), because TN trading decrease K value when $c > 1$.

When δ is small enough, the path will not cross T-line, i.e. keep in the area above T-line, from Fact 1, we know $\underline{H}()$ is decreasing along this path, and then $\underline{H}(Y2) \geq \underline{H}(Y1)$.

(2) The segment of T-line which is under the N-line

Similarly, for two states on T-line, $Y3 = ((z^T(c_3), c_3)$ and $Y4 = (z^T(c_3+\delta), c_3+\delta)$ with small $\delta > 0$, there always exists a path from $Y4$ to $Y3$ by TN trading first and then NT trading in case $K(Y3) < K(Y4)$ (as in the regular case in Fig. A.1) or by NT trading first and then TN trading in case $K(Y3) > K(Y4)$ (as in the irregular case) or just a simple NT trading for case $K(Y3) = K(Y4)$ (not show in graph), because NT trading decreased c value when $K > 0$.

When δ is small enough the path will not cross N-line, i.e. keep in the area under N-line, from Fact 1, we know $\underline{H}()$ is decreasing along this path, and then $\underline{H}(Y4) \geq \underline{H}(Y3)$.

$\square$

Proof of Theorem 3:

Because of the local optimality of the optimal state on trading track, if the optimal action is "to trade", it cannot stop at the state that is not on target lines, and must stop on one of target lines. Hence, we only need to prove that if the optimal action is "to trade" then if it is to perform a compound trade it must stop at a cross point.

Assume the optimal action for a state (z_0, c_0) is to perform a compound trade that stops at (z_2, c_2) and (z_2, c_2) is not a cross point of two target lines.

If $c_2 = 1$, the optimal state on line $\{(z, 1) : 0 \leq z \leq 1\}$ is the cross point of the two target lines, then it contradicts the assumption. Hence we consider the case $c_2 > 1$ or $c_2 < 1$.

Since any compound trade can be looked as a series of two simple trades: a TN trade first and then a NT trade, we let the compound trade from (z_0, c_0) to (z_2, c_2) be made of a TN trade from (z_0, c_0) to (z_1, c_1) and then a NT trade from (z_1, c_1) to (z_2, c_2). We know (z_2, c_2) must be on the N-line and above T-line, otherwise it can be improved more. We also know $c_1 = c_0$ because $c_1 \neq c_0$ is possible only when $(z_1, c_1) = (1, 1)$ and that will cause $c_2 = 1$.

Now we consider a new path: a TN trade from (z_0, c_0) to $(z_1 - \delta, c_0)$ and then a NT trade to state (z_3, c_3) on N-line.

Because any trade keeps state variable c on the same side of 1, i.e., if $c_0 > 1$ then $c_2 > 1$, and if $c_0 < 1$ then $c_2 < 1$. We study the following two cases of the initial state: $c_0 > 1$ and $c_0 < 1$.

In case $c_0 > 1$, $K(z_1, c_0) < K(z_1 - \delta, c_0)$, thus $K(z_2, c_2) < K(z_3, c_3)$. When δ is small enough, (z_3, c_3) is still above T-line, we know $\underline{H}(z_2, c_2) \leq \underline{H}(z_3, c_3)$ from Lemma A.2.1, hence the new path is better than optimal, and contradicts the assumption.

Similarly, in case $c_0 < 1$, $K(z_1, c_0) > K(z_1 - \delta, c_0)$, thus $K(z_2, c_2) > K(z_3, c_3)$. When δ is small enough, (z_3, c_3) is still above T-line, we know $\underline{H}(z_2, c_2) \leq \underline{H}(z_3, c_3)$ from Lemma A.2.1, hence the new path is better than optimal, and contradicts the assumption.

$\square$

Proof of Theorem 4:

From Theorem 1 and 2, we know $\underline{V}_{N-1}(x, I, X, s)$ is concave in all NT and TN trading tracks, so Condition A holds at stage $N - 1$.

A state in (z, c) is corresponding to a state in space (x, I, X, s) with wealth $w = I + b(xs, X) = 1$ given a certain s. For the RL case, all states in (z, c) are in the plane of $I + (1 - \alpha)xs + \alpha X = 1$, for the NRL case, all states with $c \geq 1$ are in the plane of $I + (1 - \alpha)xs + \alpha X = 1$ and all states with for $c \leq 1$ are in the plane of $I + xs = 1$; all the target lines are in these planes. From Theorem 2 we know that $\underline{V}_{N-1}(x, I, X, s)$ is concave in x, I and X, hence $\underline{V}_{N-1}(x, I, X, s)$ is concave on the two linear planes of $I + (1 - \alpha)xs + \alpha X = 1$ and $I + xs = 1$, and is concave on the tangent line on any point of the target lines which are in one of these planes. Hence there are no local minimals on target lines in whole space $0 < c < \infty$ for the RL case and in half space of $c < 1$ or $c > 1$ for the NRL case.

For the RL case, from an analysis similar to Lemma A.2.1, we can know that besides at $c = 1$ there is at most one cross point in whole space, the cross point on $c = 1$ is not minimal on target lines, and there are only the five cases of the relation of target lines as in list L, other cases would cause a cross point being minimal on target lines.

For the NRL case, from Proposition 2 we know that for $c \leq 1$, value on T-line is decreasing in c and the relation between two target lines must be the case that N-line is above T-line and two lines converge as $c \to 0$. When the cross point at $c = 1$ is local minimal, we can know that there are only two possible cases in the half space $c > 1$ as in List L', otherwise there would exist a cross point which is minimal in target lines in half space $c > 1$.

$\square$

A.3 Derivation of optimal policy at time just before terminal time

Here we derive the optimal target N/W ratio for two trading direction, given the state $(x(t), I(t), a(t), s)$ is (x, I, p, s). It is easy to represent the result for each trading track characterized by c or K.

For short writing, we let $\beta = 1 - \alpha$.

Given $s(t)$ and $i(t)$, random variables $s(t+dt)$ and $i(t+dt)$ can be represented as:

$$s(t+dt) = s(t) + ds = s(t) + s(t)(\mu_s dt + \sigma_s \sqrt{dt} z) \tag{A.3.1}$$

and

$$i(t+dt) = i(t) + di = i(t) + i(t)(\mu_i dt + \sigma_i \sqrt{dt} v) \tag{A.3.2}$$

where z and v are random variables with standard normal distribution and correlate with parameter ρ.

To prove the results for the NRL case in both PNT and PTN problems, we need the following lemma about some limits related to normal distribution.

Lemma A.3.1. *For any integer $i \geq 0$, real number $d, \sigma > 0$, and real number μ,*

$$\lim_{dt \to 0} \frac{\int_{-\infty}^{\frac{-d-\mu dt}{\sigma\sqrt{dt}}} (\sqrt{dt} z)^i e^{-z^2/2} dz}{dt} = 0$$

Proof:

There exists $\eta > 0$ such that if $dt < \eta$ then $dt < 1$ and $\frac{-d-\mu dt}{\sigma\sqrt{dt}} < -1$. For example, $\eta = \min\{1, \frac{d^2}{\sigma^2}\}$ if $\mu = 0$, or $\eta = \min\{1, (\frac{d}{2\sigma})^2, \frac{\sigma^2}{\mu^2}\}$ if $\mu \neq 0$.

When $dt < \eta$, there are the following three cases to analyze:

- for $i = 0$ or 1,

  $|\int_{-\infty}^{\frac{-d-\mu dt}{\sigma\sqrt{dt}}} (\sqrt{dt} z)^i e^{-z^2/2} dz| \leq |\sqrt{dt}^i \int_{-\infty}^{\frac{-d-\mu dt}{\sigma\sqrt{dt}}} z e^{-z^2/2} dz| < e^{\frac{-d\mu}{\sigma^2}} e^{\frac{-d^2}{2\sigma^2 dt}}$.

- for $i = 2$,

$$|\int_{-\infty}^{\frac{-d-\mu dt}{\sigma\sqrt{dt}}}(\sqrt{dt}z)^2e^{-z^2/2}dz| \leq dt(\frac{d+\mu dt}{\sigma\sqrt{dt}}e^{\frac{-d\mu}{\sigma^2}}e^{\frac{-d^2}{2\sigma^2 dt}} + |\int_{-\infty}^{\frac{-d-\mu dt}{\sigma\sqrt{dt}}}e^{-z^2/2}dz|)$$
$$\leq \frac{d+|\mu|+\sigma}{\sigma}e^{\frac{-d\mu}{\sigma^2}}e^{\frac{-d^2}{2\sigma^2 dt}}.$$

- for $i > 2$,

$$|\int_{-\infty}^{\frac{-d-\mu dt}{\sigma\sqrt{dt}}}(\sqrt{dt}z)^ie^{-z^2/2}dz| \leq \sqrt{dt}^i|\int_{-\infty}^{0}z^ie^{-z^2/2}dz| = C_i\sqrt{dt}^i.$$

  for given i, C_i is a certain number.

Calculating the limit and using the fact that $\lim_{dt\to 0}\frac{e^{\frac{-b}{dt}}}{dt} = 0$, for $b > 0$, we get our result.

$\square$

A.3.1 Problem PTN

Assume at time $t = T - dt$, the state $(x(t), I(t), s(t))$ is (x, I, s) with wealth $w = I + xb(s,p)$, let u shares of stock are sold to index, and no trade be made thereafter, then the final wealth at time T is

$$\begin{aligned} w(T) &= (x-u)b(s(t+dt),p) + (I+ub(s,p))i(t+dt)/i(t) \\ &= w(t) + (x-u)(b(s(t+dt),p) - b(s,p)) \\ &\quad +(I+ub(s,p))(i(t+dt)-i(t))/i(t) \end{aligned}$$

In the following, we will present the analytical results about how to choose feasible u to maximize the growth rate of expected utility, $Q(u)$, for RL and NRL case respectively. $Q(u)$ is defined as $Q(u) = \lim_{dt\to 0}\frac{E[U(w(T))-U(w(t))]}{dt}$ if the limit exists.

The RL case

In the RL case , $b(c,p) = \beta c + \alpha p$, for any c, then $b(s(t+dt),p) - b(s,p) = \beta ds$.

Applying Ito Lemma, we have the growth rate of expected utility

$$\begin{aligned} Q(u) &= \lim_{dt\to 0} E[U(w(T)) - U(w(t))]/dt \\ &= U'(w(t))[(x-u)\beta s\mu_s + (I + ub(s,p))\mu_i] \\ &\quad + \frac{1}{2}U''(w(t))[\beta^2 s^2\sigma_s^2(x-u)^2 + (I+ub(s,p))^2\sigma_i^2 \\ &\qquad + 2\rho\beta s(I+ub(s,p))(x-u)\sigma_i\sigma_s] \end{aligned}$$

Since $U(w)$ is concave, $U''(w(t)) < 0$, $Q(u)$ is concave. To maximize $Q(u)$, let $dQ/du = 0$, i.e.

$$\begin{aligned} 0 &= U'(w(t))(-\beta s\mu_s + b(s,p)\mu_i) \\ &\quad + U''(w(t))[\beta^2 s^2\sigma_s^2(u-x) + b(s,p)(I+ub(s,p))\sigma_i^2 \\ &\qquad + \rho\beta s(xb(s,p) - I - 2ub(s,p))\sigma_i\sigma_s] \end{aligned}$$

Using $w(t) = xb(s,p) + I$, and $U'(w)/U''(w) = w/(\gamma-1)$, we know that the above is equivalent to

$$(x-u)(\beta^2 s^2\sigma_s^2 + b(s,p)^2\sigma_i^2 - 2\rho b(s,p)\beta s\sigma_i\sigma_s) =$$

$$w(t)((\beta s\mu_s - b(s,p)\mu_i)/(1-\gamma) + b(s,p)\sigma_i^2 - \rho\beta s\sigma_i\sigma_s)$$

Therefore, the optimal policy is to sell some stock (if possible) so that the N/W ratio, the portion of the nontaxable asset value in the total wealth, reach the following ratio

$$\frac{I + ub(s,p)}{w} = \frac{\beta^2 s^2\sigma_s^2 - \rho b(s,p)\beta s\sigma_i\sigma_s - b(s,p)(\beta s\mu_s - b(s,p)\mu_i)/(1-\gamma)}{\beta^2 s^2\sigma_s^2 + b(s,p)^2\sigma_i^2 - 2\rho b(s,p)\beta s\sigma_i\sigma_s}$$

truncated in range $[0, 1]$ if this ratio is out of the range.

The NRL case

In the NRL case, $b(s, p) = s - \alpha(s-p)^+$. We have $b(s(t+dt), p) - b(s, p) = ds + \alpha((s-p)^+ - (s+ds-p)^+)$.

Define $dw = w(T) - w(t)$, that is $dw = (x-u)(b(s(t+dt), p) - b(s, p)) + (I + ub(s, p))di/i(t)$, then

$$\begin{aligned} Q(u) &= \lim_{dt \to 0} E[U(w(T)) - U(w(t))]/dt \\ &= \lim_{dt \to 0} \frac{U'(w(t))E[dw] + \frac{1}{2}U''(w(t))E[dw^2] + h(dt)}{dt} \end{aligned}$$

where $h(dt)$ represents all the items of $E[dw^i]$, for $i > 2$, in the Taylor series for $E[U(w(T))]$ expanded from $w(t)$.

$dw = w(T) - w(t) = (x-u)(b(s(t+dt), p) - b(s, p)) + (I + ub(s, p))di/i(t)$, and all the expectations $E[dw^i]$, for $i > 0$, can be calculated with representing $(dw)^i$ as the function of normal distributed random variables of z and v, which determine $s(t+dt)$ and $i(t+dt)$ as defined in (A.3.1) and (A.3.2).

Denote $\delta = s - p$, we derive the optimal ratios for the following three cases.

1. If $\delta > 0$, $b(s(t+dt), p) - b(s, p) = \beta ds + \alpha(ds + \delta)\mathbf{1}\{ds < -\delta\}$.

   - If $\sigma_s \neq 0$, the indicator $\mathbf{1}\{ds < -\delta\} = \mathbf{1}\{z < \frac{-\delta/s - \mu_s dt}{\sigma_s \sqrt{dt}}\}$. Using Lemma A.3.1, we have

     – $\lim_{dt \to 0} \frac{E[dw]}{dt} = (x-u)\beta s\mu_s + (I + ub(s, p))\mu_i$,

- $\lim_{dt\to 0} \frac{E[dw^2]}{dt} = \beta^2 s^2 \sigma_s^2 (x-u)^2 + (I + ub(s,p))^2 \sigma_i^2 + 2\rho\beta s(I + ub(s,p))(x-u)\sigma_i\sigma_s$,
- and $\lim_{dt\to 0} \frac{E[dw^n]}{dt} = 0$, for all $n > 2$.

- If $\sigma_s = 0$, $\lim_{dt\to 0} \mathbf{1}\{ds < -\delta\} = 0$, same results will be derived easily.

So we have the same equation of $Q(u)$ and then have the same optimal target ratio as that for the RL case.

2. If $\delta < 0$, $b(s(t+dt),p) - b(s,p) = ds - \alpha(ds+\delta)\mathbf{1}\{ds > -\delta\}$, similarly, we can derive that the optimal target ratio is the same as that for RL case but with the tax rate is zero or say without tax.

3. If $\delta = 0$, i.e., $s = p$, we have $b(s(t+dt),p) - b(s,p) = ds - \alpha ds^+$

   - If $\sigma_s = 0$,

     if $\mu_s > 0$, $b(s(t+dt),p) - b(s,p) = \beta\mu_s dt$, and then we have the same optimal ratio as that for the RL case.

     if $\mu_s \leq 0$, $b(s(t+dt),p) - b(s,p) = \mu_s dt$, and then we have the optimal ratio the same as that for the RL case without tax.

     Therefore the optimal policy is the same as that for above case $\delta > 0$ or $\delta < 0$, depending on $\mu_s > 0$ or not.

- If $\sigma_s \neq 0$,

$$\begin{aligned} E[ds^+]/p &= \int_{-\mu_s\sqrt{dt}/\sigma_s}^{\infty} (\mu_s dt + \sigma_s\sqrt{dt}z)(1/\sqrt{2\pi})e^{-z^2/2}dz \\ &= \int_{-\mu_s\sqrt{dt}/\sigma_s}^{0} (\mu_s dt + \sigma_s\sqrt{dt}z)(1/\sqrt{2\pi})e^{-z^2/2}dz \\ &\quad +\mu_s dt/2 + \sigma_s\sqrt{dt/2\pi} \end{aligned}$$

Since $\lim_{dt\to 0}[\int_{-\mu_s\sqrt{dt}/\sigma_s}^{0}(\mu_s dt + \sigma_s\sqrt{dt}z)(1/\sqrt{2\pi})e^{-z^2/2}dz]/dt = 0$, we have $\lim_{dt\to 0}\frac{E[ds^+]}{dt} = \infty$, $\lim_{dt\to 0}\frac{E[b(s(t+dt),p)-b(s,p)]}{dt} = -\infty$, and then

$$\lim_{dt\to 0}\frac{E[dw]}{dt} = (x-u)(-\infty) + (I + ub(s,p))\mu_i.$$

Similarly, we can have $\lim_{dt\to 0}\frac{E[dw^2]}{dt} = c$, with c is a constant, and $\lim_{dt\to 0}\frac{E[dw^n]}{dt} = 0$, for $n > 2$.

So the growth rate of expected utility is

$$Q(u) = U'(w(t))[(x-u)(-\infty) + (I + ub(s,p))\mu_i] + \frac{1}{2}U''(w(t))c$$

Since $U'() > 0$, maximizing $Q(u)$ needs to minimize $x-u$, the optimal policy is to sell out all the stock. The N/W ratio at point $s = p$ is 1.0.

Remark: The intuition behind the mathematical result is: when $s(t) > p$, as t is extremely close to terminal time T, it is unlikely that $s(T) < p$, so the optimal N/W ratio should be the same as that for the RL case; similarly when $s(t) < p$, it is unlikely that $s(T) > p$, and the realized price $b(s,p) = s$, the optimal N/W ratio should be the same as that without tax. When $s(t) = p$ and $\sigma_s \neq 0$, $b(s(T),p) - p$, the earning on realized value of stock, is not symmetric for $s(T)$ on both side of p: specifically, for small $\delta > 0$, $[b(p-\delta,p)-p]+[b(p+\delta,p)-p]<0$, so any holding of stock will cause the loss of expected value, and the optimal policy is not to hold any stock.

A.3.2 Problem PNT

Assume at time $t = T - dt$, the state $(x(t), I(t), a(t), s)$ is (x, I, p, s) with wealth $w = I + xb(s,p)$, let u share of stock is bought from index fund, and no trade be made thereafter, the final wealth at time T is

$$\begin{aligned} w(T) &= (x+u)b(s(t+dt), \frac{xp+us}{x+u}) + (I-us)i(t+dt)/i(t) \\ &= w + (x+u)b(s(t+dt), \frac{xp+us}{x+u}) - xb(s,p) - us \\ &\quad +(I-us)[i(t+dt)-i(t)]/i(t) \end{aligned}$$

Similar to the problem PTN, the optimal policy is choosing feasible u to maximize the growth rate of expected utility, which is defined as:

$Q(u) = \lim_{dt\to 0} \frac{E[U(w(T))-U(w(t))]}{dt}$ if the limit exists.

The RL case

In the RL case, $b(c,a) = \beta c + \alpha a$ for any c and a.

$$\begin{aligned} w(T) &= w + (x+u)\beta s(t+dt) + \alpha(xp+us) - x(\beta s + \alpha p) - us \\ &\quad +(I-us)(i(t+dt)-i(t))/i(t) \\ &= w + (x+u)\beta ds + (I-us)di/i(t) \end{aligned}$$

Applying Ito Lemma, we have

$$\begin{aligned} Q(u) &= \lim_{dt\to 0} E[U(w(T)) - U(w(t))]/dt \\ &= U'(w(t))[(x+u)\beta s\mu_s + (I-us)\mu_i] \\ &\quad + \frac{1}{2}U''(w(t))[\beta^2 s^2\sigma_s^2(x+u)^2 + (I-us)^2\sigma_i^2 \\ &\qquad + 2\rho\beta s(I-us)(x+u)\sigma_i\sigma_s] \end{aligned}$$

Since $U(w)$ is concave, $U''(w) < 0$, $Q(u)$ is concave. To maximize $Q(u)$, let $dQ/du = 0$, i.e.

$$\begin{aligned} 0 &= U'(w)(\beta s\mu_s - s\mu_i) \\ &\quad + U''(w)[\beta^2 s^2\sigma_s^2(u+x) + s(us-I)\sigma_i^2 + \rho\beta s(I - xs - 2us)\sigma_i\sigma_s] \end{aligned}$$

From $U'(w) = wU''(w)/(\gamma-1)$, above equation is reduced to

$$w(\beta\mu_s - \mu_i)/(1-\gamma) = (x+u)s(\beta^2\sigma_s^2 + \sigma_i^2 - 2\rho\beta\sigma_i\sigma_s) + (I+sx)(\rho\beta\sigma_i\sigma_s - \sigma_i^2)$$

Using $I - us = I + sx - (x+u)s$, we have that the optimal policy is to buy some stock (if possible) so that the N/W ratio reaches the following ratio

$$\frac{I-us}{w} = \frac{(I+sx)/w)(\beta^2\sigma_s^2 - \rho\beta\sigma_i\sigma_s) - (\beta\mu_s - \mu_i)/(1-\gamma)}{\beta^2\sigma_s^2 + \sigma_i^2 - 2\rho\beta\sigma_i\sigma_s}$$

With characteristic $K = \frac{xs-X}{I+X}$,

$$\frac{I+sx}{w} = \frac{I+sx}{I+xb(s,p)} = \frac{(K+1)(I/X+1)}{I/X+\beta s/p+\alpha} = \frac{K+1}{1+\beta K}$$

The optimal N/W ratio is

$$\frac{\frac{K+1}{1+\beta K}(\beta^2\sigma_s^2 - \rho\beta\sigma_i\sigma_s) - (\beta\mu_s - \mu_i)/(1-\gamma)}{\beta^2\sigma_s^2 + \sigma_i^2 - 2\rho\beta\sigma_i\sigma_s}$$

which is determined by K, the characteristic value of trading track.

The NRL case

In the NRL case, $b(s(t+dt), \frac{xp+us}{x+u}) = s(t+dt) - \alpha \max\{s(t+dt) - \frac{xp+us}{x+u}, 0\}$, and $(x+u)b(s(t+dt), \frac{xp+us}{x+u}) = (x+u)s(t+dt) - \alpha \max\{(x+u)s(t+dt) - (xp+us), 0\}$.

Similar to problem PTN, let $dw = w(T) - w(t)$

$$
\begin{aligned}
Q(u) &= \lim_{dt \to 0} E[U(w(T)) - U(w(t))]/dt \\
&= \lim_{dt \to 0} \frac{U'(w(t))E[dw] + \frac{1}{2}U''(w(t))E[dw^2] + h(dt)}{dt}
\end{aligned}
$$

There are three cases to analyze: $s > p$, $s = p$ and $s < p$. Because when $x = 0$, after buying $u > 0$ share of stock, the new basis price is s, we can think in this case $p = s$.[1] Therefore when we consider the cases of $s > p$ and $s < p$, we assume $x > 0$, and the three cases is equivalent to $x(s-p) >, =$ or < 0,

1. If $s > p$, $b(s,p) = \beta s + \alpha p$.

$$dw = (x+u)ds - \alpha \max\{(x+u)ds, -x(s-p)\} + (I - us)di/i(t)$$

$$= \beta(x+u)ds + \alpha[(x+u)ds + x(s-p)]\mathbf{1}\{(x+u)ds < x(p-s)\} + (I-us)di/i(t)$$

- If $\sigma_s \neq 0$, the indicator $\mathbf{1}\{(x+u)ds < x(p-s)\} = \mathbf{1}\{z < \frac{x(p-s)-(x+u)s\mu_s dt}{(x+u)s\sigma_s\sqrt{dt}}\}$.

  Using Lemma A.3.1 and $x(s-p) > 0$, we have

  - $\lim_{dt\to 0} \frac{E[dw]}{dt} = (x+u)\beta s \mu_s + (I - us)\mu_i$,
  - $\lim_{dt\to 0} \frac{E[dw^2]}{dt} = \beta^2 s^2 \sigma_s^2 (x+u)^2 + (I-us)^2\sigma_i^2 + 2\rho\beta s(I-us)(x+u)\sigma_i\sigma_s$,
  - and $\lim_{dt\to 0} \frac{E[dw^n]}{dt} = 0$, for all $n > 2$.

[1] It is also the reason why we define APP $a(t) = s(t)$ when $x(t) = 0$ in Section 2.1.

- If $\sigma_s = 0$, $\lim_{dt\to 0} \mathbf{1}\{(x+u)ds < x(p-s)\} = 0$, same results will be derived easily.

So we have the same equation of $Q(u)$ and then have the same optimal target ratio as that for the RL case.

2. If $s < p$, $b(s,p) = s$.

$dw = (x+u)ds - \alpha[(x+u)ds + x(s-p)]\mathbf{1}\{(x+u)ds > x(p-s)\} + (I-us)di/i(t)$

similarly, we can derive that the optimal target ratio is the same as that for RL case but with the tax rate is zero or say without tax.

3. If $s = p$,

$$dw = (x+u)ds - \alpha(x+u)ds^+ + (I-us)di/i(t)$$

- If $\sigma_s = 0$, $ds = \mu_s dt$,

  if $\mu_s > 0$, $dw = \beta(x+u)\mu_s dt + (I-us)di/i(t)$, and then we have the optimal policy the same as that for the RL case.

  if $\mu_s \leq 0$, $dw = (x+u)\mu_s dt + (I-us)di/i(t)$, and then we have the optimal policy the same as that for the RL case without tax.

  Therefor the optimal policy is the same as that for above case $\delta > 0$ or $\delta < 0$, depending on $\mu_s > 0$ or not.

- If $\sigma_s \neq 0$,

  like in problem PTN, $\lim_{dt\to 0} \frac{E[ds^+]}{dt} = \infty$, and then

$$\lim_{dt\to 0} \frac{E[dw]}{dt} = (x+u)(-\infty) + (I-us)\mu_i.$$

Similarly, we can have $\lim_{dt\to 0}\frac{E[dw^2]}{dt} = c$, with c is a constant, and $\lim_{dt\to 0}\frac{E[dw^n]}{dt} = 0$, for $n > 2$.

So the growth rate of expected utility is

$$Q(u) = U'(w(t))[(x+u)(-\infty) + (I - us)\mu_i] + \frac{1}{2}U''(w(t))c$$

Since $U'() > 0$, maximizing $Q(u)$ needs to minimize $x + u$, the optimal policy is not to buy any amount of stock. The N/W ratio at point $s = p$ is 1.

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