Open Economy Macroeconomics

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Chapter 1

Business-Cycle Facts Around the World

How volatile is output? Do the components of aggregate demand (consumption, investment, government spending, and exports) move pro or countercyclically? How persistent are movements in aggregate activity? Are economic expansions associated with deficits or surpluses in the trade balance? What about economic contractions? Is aggregate consumption less or more volatile than output? Are emerging countries more or less volatile than developed countries? Does country size matter for business cycles? The answer to these and other similar questions form a basic set of empirical facts about business cycles that one would like macro models of the open economy to be able to explain. Accordingly, the purpose of this chapter is to document these facts using aggregate data on economic activity spanning time and space.
1.1 Measuring Business Cycles

In the theoretical models we study in this book, the basic economic units are the individual consumer, the firm, and the government. The models produce predictions for the consumers’ levels of income, spending, and savings and for firms’ investment and production decisions. To compare the predictions of theoretical models to actual data, it is therefore natural to consider time series and cross-country evidence on per capita measures of aggregate activity. Accordingly, in this chapter we describe the business-cycle properties of output per capita, denoted $y$, total private consumption per capita, denoted $c$, investment per capita, denoted $i$, public consumption per capita, denoted $g$, exports per capita, denoted $x$, imports per capita, denoted $m$, the trade balance, denoted $tb \equiv (x - m)$, and the current account, denoted $ca$.

To compute business-cycle statistics we use annual, cross-country, time-series data from the World Bank’s World Development Indicators (WDI) data base.\(^1\) All time series are expressed in real per capita terms. Only countries with at least 30 uninterrupted years of data for $y$, $c$, $i$, $g$, $x$, and $m$ were considered. The resulting sample contains 120 countries and covers, on average, the period 1965-2010.\(^2\)

A word on the consumption data is in order. The WDI data base contains information on household final consumption expenditure. This time series includes consumption expenditure on non-durables, services, and durables. Typically, business-cycle studies remove expenditures on durables from the definition of consumption. The reason is that from an economic point of view, expenditure on durable consumption goods, such as cars and washing machines, represent an investment in household physical capital. For this reason, researchers often add this component of consumption to the gross investment series. From a statistical point of view, there is also a reason to separate

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\(^1\)The data set is publicly available at databank.worldbank.org. The specific annual data used in this chapter available online at www.columbia.edu/~mu2166/book/

\(^2\)Only 94 countries contained 30 uninterrupted years of current account data.
durables from nondurables and services in the definition of consumption. Expenditures on durables are far more volatile than expenditures on nondurables and services. For example, in the United States, durable consumption is about three times as volatile as output, whereas consumption of nondurable and services is less volatile than output. Even though expenditures on durables represent only 13 percent of total consumption expenditure, the standard deviation of total consumption is 20 percent higher than that of nondurables and services. Unfortunately, the WDI data set does not provide disaggregated consumption data. One should therefore keep in mind that the volatility of consumption reported later in this chapter is likely to be somewhat higher than the one that would result if our measure of consumption excluded expenditures on durables goods.

The focus of our analysis is to understand aggregate fluctuations at business-cycle frequency in open economies. It is therefore important to extract from the raw time series data the associated cyclical component. The existing literature suggests a variety of methods for isolating the cyclical component of a time series. The most popular ones are log-linear detrending, log-quadratic detrending, Hodrick-Prescott (HP) filtering, first differencing, and band-pass filtering. The following analysis uses quadratic detrending, HP filtering, and first differencing.

To extract a log-quadratic trend, we proceed as follows. Let $y_t$ denote the natural logarithm of real output per capita in year $t$ for a given country, $y_t^c$ the cyclical component of $y_t$, and $y_t^s$ the secular (or trend) component of $y_t$. Then we have

$$y_t = y_t^c + y_t^s. \tag{1.1}$$

The components $y_t^c$ and $y_t^s$ are estimated by running the following regression

$$y_t = a + bt + ct^2 + \epsilon_t,$$
and setting 
\[ y_t^c = \epsilon_t \]
and 
\[ y_t^s = a + bt + ct^2. \]

An identical procedure is used to detrend the natural logarithms of consumption, investment, government spending, exports, and imports and the levels of the trade-balance-to-output ratio and the current-account-to-output ratio. The levels of the trade balance and the current account (\(tb\) and \(ca\)) are first divided by the secular component of output (\(e^y_t\)) and then quadratically detrended. We perform the decomposition into cycle and trend for every time series and every country separately.

To illustrate the workings of the log-quadratic filter, we show the decomposition into trend and cycle it delivers for Argentine real GDP per capita over the period 1960-2011. The top panel of figure 1.1 shows with a solid line raw data and with a broken line the estimated quadratic trend, \(y_t^s\). The bottom panel shows the cyclical component, \(y_t^c\). The detrending procedure delivers three well-marked cycles, one from the beginning of the sample until 1980, a second one from 1980 to 1998, and a third one from 1998 to the end of the sample. In particular, the log-quadratic filter succeeds in identifying the two major contractions in postwar Argentina, namely the one associated with the hyperinflation of the late 1980s and the one associated with the demise of the Convertibility Plan in 2001. In the first of these contractions, real GDP per capita fell by about 40 percent from the peak in 1980 to the trough in 1990, giving the 1980s the well-deserved nick name of lost decade.

The behavior of the business-cycle component of real GDP suggests that the Argentine economy has been highly volatile over the past 50 years. The standard deviation of \(y_t^c\) is 10.7 percent per year. The cyclical component is also quite persistent. The serial correlation of \(y_t^c\) is 0.85.

In the next section, we expand this analysis to all macroeconomic aggregates and countries included in our data set.
Figure 1.1: Trend and Cycle of Argentine real per capita GDP

Data Source: WDI Database and authors’ calculations.
1.2 Business-Cycle Facts Around The World

To characterize the average world business cycle, we compute business-cycle statistics for each country in the sample and then take a population-weighted average of each statistic across countries. The resulting average summary statistics appear in table 1.1 under the heading ‘All Countries.’

The table displays standard deviations, correlations with output, and serial correlations. Relative standard deviations are cross-country averages of country-specific relative standard deviations. The table also displays averages of the trade-balance-to-output ratio and the openness ratio, defined as \((x + m)/y\).

According to table 1.1, the world is a pretty volatile place. The average standard deviation of output across all countries is 6.2 percent. To put this number into perspective, we contrast it with the volatility of output in the United States. The standard deviation of the cyclical component of U.S. output is 2.9 percent, less than half of the average volatility of output across all countries in the data set.

**Fact 1.1 (High Global Volatility)** The cross-country average volatility of output is twice as large as its U.S. counterpart.

One statistic in table 1.1 that might attract some attention is that on average across countries private consumption is 5 percent more volatile than output. This fact might seem at odds with the backbone of optimizing models of the business cycle, namely, consumption smoothing. However, recall that the measure of consumption used here includes expenditures on consumer durables, which are highly volatile. The fact that expenditure on durables is highly volatile need not be at odds with consumption smoothing because it represents an investment in household capital rather

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3Country-by-country statistics for a selected number of emerging and rich countries are shown in table 1.8 in the appendix. The online appendix (www.columbia.edu/~mu2166/book/) presents country-by-country statistics for all countries.
Table 1.1: Business Cycles in Poor, Emerging, and Rich Countries

<table>
<thead>
<tr>
<th>Statistic</th>
<th>United States</th>
<th>All Countries</th>
<th>Poor Countries</th>
<th>Emerging Countries</th>
<th>Rich Countries</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_y$</td>
<td>2.94</td>
<td>6.22</td>
<td>6.08</td>
<td>8.71</td>
<td>3.32</td>
</tr>
<tr>
<td>$\sigma_c/\sigma_y$</td>
<td>1.02</td>
<td>1.05</td>
<td>1.12</td>
<td>0.98</td>
<td>0.87</td>
</tr>
<tr>
<td>$\sigma_g/\sigma_y$</td>
<td>1.93</td>
<td>2.26</td>
<td>2.46</td>
<td>2.00</td>
<td>1.73</td>
</tr>
<tr>
<td>$\sigma_i/\sigma_y$</td>
<td>3.52</td>
<td>3.14</td>
<td>3.24</td>
<td>2.79</td>
<td>3.20</td>
</tr>
<tr>
<td>$\sigma_x/\sigma_y$</td>
<td>3.49</td>
<td>3.07</td>
<td>3.08</td>
<td>2.82</td>
<td>3.36</td>
</tr>
<tr>
<td>$\sigma_m/\sigma_y$</td>
<td>3.24</td>
<td>3.23</td>
<td>3.30</td>
<td>2.72</td>
<td>3.64</td>
</tr>
<tr>
<td>$\sigma_{tb/y}$</td>
<td>0.94</td>
<td>2.34</td>
<td>2.12</td>
<td>3.80</td>
<td>1.25</td>
</tr>
<tr>
<td>$\sigma_{ca/y}$</td>
<td>1.11</td>
<td>2.16</td>
<td>2.06</td>
<td>3.08</td>
<td>1.39</td>
</tr>
</tbody>
</table>

**Correlations with $y$**

<table>
<thead>
<tr>
<th>Statistic</th>
<th>United States</th>
<th>All Countries</th>
<th>Poor Countries</th>
<th>Emerging Countries</th>
<th>Rich Countries</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>$c$</td>
<td>0.90</td>
<td>0.69</td>
<td>0.66</td>
<td>0.75</td>
<td>0.76</td>
</tr>
<tr>
<td>$g/y$</td>
<td>-0.32</td>
<td>-0.02</td>
<td>0.08</td>
<td>-0.08</td>
<td>-0.39</td>
</tr>
<tr>
<td>$i$</td>
<td>0.80</td>
<td>0.66</td>
<td>0.60</td>
<td>0.77</td>
<td>0.77</td>
</tr>
<tr>
<td>$x$</td>
<td>-0.11</td>
<td>0.19</td>
<td>0.14</td>
<td>0.35</td>
<td>0.17</td>
</tr>
<tr>
<td>$m$</td>
<td>0.31</td>
<td>0.24</td>
<td>0.14</td>
<td>0.50</td>
<td>0.34</td>
</tr>
<tr>
<td>$tb/y$</td>
<td>-0.51</td>
<td>-0.15</td>
<td>-0.11</td>
<td>-0.21</td>
<td>-0.26</td>
</tr>
<tr>
<td>$tb$</td>
<td>-0.54</td>
<td>-0.18</td>
<td>-0.14</td>
<td>-0.24</td>
<td>-0.25</td>
</tr>
<tr>
<td>$ca/y$</td>
<td>-0.62</td>
<td>-0.28</td>
<td>-0.28</td>
<td>-0.24</td>
<td>-0.30</td>
</tr>
<tr>
<td>$ca$</td>
<td>-0.64</td>
<td>-0.28</td>
<td>-0.28</td>
<td>-0.26</td>
<td>-0.31</td>
</tr>
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**Serial Correlations**

<table>
<thead>
<tr>
<th>Statistic</th>
<th>United States</th>
<th>All Countries</th>
<th>Poor Countries</th>
<th>Emerging Countries</th>
<th>Rich Countries</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0.75</td>
<td>0.71</td>
<td>0.65</td>
<td>0.87</td>
<td>0.76</td>
</tr>
<tr>
<td>$c$</td>
<td>0.82</td>
<td>0.66</td>
<td>0.62</td>
<td>0.74</td>
<td>0.75</td>
</tr>
<tr>
<td>$g$</td>
<td>0.91</td>
<td>0.76</td>
<td>0.71</td>
<td>0.80</td>
<td>0.89</td>
</tr>
<tr>
<td>$i$</td>
<td>0.67</td>
<td>0.56</td>
<td>0.49</td>
<td>0.72</td>
<td>0.67</td>
</tr>
<tr>
<td>$x$</td>
<td>0.75</td>
<td>0.68</td>
<td>0.65</td>
<td>0.74</td>
<td>0.74</td>
</tr>
<tr>
<td>$m$</td>
<td>0.63</td>
<td>0.65</td>
<td>0.61</td>
<td>0.74</td>
<td>0.69</td>
</tr>
<tr>
<td>$tb/y$</td>
<td>0.79</td>
<td>0.61</td>
<td>0.59</td>
<td>0.62</td>
<td>0.69</td>
</tr>
<tr>
<td>$ca/y$</td>
<td>0.79</td>
<td>0.57</td>
<td>0.55</td>
<td>0.52</td>
<td>0.71</td>
</tr>
</tbody>
</table>

**Means**

<table>
<thead>
<tr>
<th>Statistic</th>
<th>United States</th>
<th>All Countries</th>
<th>Poor Countries</th>
<th>Emerging Countries</th>
<th>Rich Countries</th>
</tr>
</thead>
<tbody>
<tr>
<td>$tb/y$</td>
<td>-1.5</td>
<td>-1.3</td>
<td>-1.6</td>
<td>-1.4</td>
<td>-0.0</td>
</tr>
<tr>
<td>$(x + m)/y$</td>
<td>18.9</td>
<td>36.5</td>
<td>32.5</td>
<td>46.4</td>
<td>40.4</td>
</tr>
</tbody>
</table>

Note. The variables $y$, $c$, $g$, $i$, $x$, $m$, $tb \equiv (x - m)$, and $ca$ denote, respectively, output, total private consumption, government spending, investment, exports, imports, the trade balance, and the current account. All variables are expressed in real per capita terms. The variables $y$, $c$, $g$, $i$, $x$, and $m$ are quadratically detrended in logs and expressed in percent deviations from trend. The variables $tb/y$, $g/y$, and $ca/y$ are quadratically detrended in levels. The variables $tb$ and $ca$ are scaled by the secular component of $y$ and quadratically detrended. The sample contains 120 countries and covers, on average, the period 1965-2010 at annual frequency. Moments are averaged across countries using population weights. The sets of poor, emerging, and rich countries are defined as all countries with average PPP converted GDP per capita in U.S. dollars of 2005 over the period 1990-2009 within the ranges 0-3,000, 3,000-25,000, and 25,000-∞, respectively. The lists of poor, emerging, and rich countries are presented in the appendix to this chapter. Data source: World Development Indicators, The World Bank.
than direct consumption. For example, a household that buys a new car every 5 years displays a
choppy path for expenditures on cars, but might choose to experience a smooth consumption of
the services provided by its car.

The government does not appear to smooth its own consumption of goods and services either.
On average, the standard deviation of public consumption is more than twice that of output.

Fact 1.2 (High Volatility Of Government Consumption) *On average across countries govern-
ment consumption is twice as volatile as output.*

Investment, exports, and imports are by far the most volatile components of the national income
and product accounts, with standard deviations around three times as large as those of output.
The trade-balance-to-output ratio and the current-account-to-output ratio are also highly volatile,
with standard deviations of more than 2 percent of GDP.

Fact 1.3 (Global Ranking Of Volatilities) *The ranking of cross-country average standard de-
viations from top to bottom is imports, investment, exports, government spending, consumption,
and output.*

We say that a variable is procyclical when it has a positive correlation with output. Table 1.1
reveals that consumption, investment, exports, and imports are all procyclical. Private consumption
is the most procyclical component of aggregate demand.

Fact 1.4 (Procyclicality Of The Components of Aggregate Demand) *On average consump-
tion, investment, exports, and imports are all positively correlated with output.*

By contrast, the trade balance, the trade-balance-to-output ratio, the current account, and the
current-account-to-output ratio are all countercyclical. This means that countries tend to import
more than they export during booms and to export more than they import during recessions.
Fact 1.5 (Countercyclicality Of The Trade Balance And The Current Account) On average across countries the trade balance, the trade-balance-to-output ratio, the current account, and the current-account-to-output ratio are all negatively correlated with output.

It is worth noting that the government-spending-to-output ratio is roughly acyclical. This empirical regularity runs contrary to the traditional Keynesian stabilization policy prescription according to which the share of government spending in GDP should be increased during contractions and cut during booms.

Fact 1.6 (Acyclicality Of The Share Of Government Consumption in GDP) On average across countries, the share of government consumption in output is roughly uncorrelated with output.

This fact must be qualified along two dimensions. First, here the variable $g$ denotes government consumption of goods. It does not include government investment, which may be more or less procyclical than government consumption. Second, $g$ does not include transfers. To the extent that transfers are countercyclical and directed to households with high propensities to consume—presumably low-income households—total government spending may be more countercyclical than government consumption.

A standard measure of persistence in time series is the first-order serial correlation. Table 1.1 shows that on average across all countries, output is quite persistent, with a serial correlation of 0.71. All components of aggregate demand as well as imports are broadly as persistent as output.

Fact 1.7 (Persistence) The components of aggregate supply (output and imports) and aggregate demand (consumption, government spending, investment, and exports) are all positively serially correlated.
Later in this chapter, we will investigate whether output is a persistent stationary variable or a nonstationary variable. This distinction is important for choosing the stochastic processes of shocks driving our theoretical models of the macro economy.

1.3 Business Cycles in Poor, Emerging, and Rich Countries

An important question in macroeconomics is whether business cycles look differently in poor, emerging, and rich economies. For if this was the case, then a model that is successful in explaining business cycles in, say, rich countries, may be less successful in explaining business cycles in emerging or poor countries. One difficulty with characterizing business cycles at different stages of development is that any definition of concepts such as poor, emerging, or rich country is necessarily arbitrary. For this reason, it is particularly important to be as explicit as possible in describing the classification method adopted.

As the measure of development, we use the geometric average of PPP converted GDP per capita in U.S. dollars of 2005 over the period 1990-2009. Loosely speaking, PPP-converted GDP in a given country is the value of all goods and services produced in that country evaluated at U.S. prices. By evaluating production of goods in different countries at the same prices, PPP conversion makes cross-country comparisons more sensible. To illustrate the concept of PPP conversion, suppose that in a given year country X produces 3 hair cuts and 1 ton of grain and that the unit prices of these items inside country X are, 1 and 200 dollars, respectively. Then, the nonconverted measure of GDP is 203 dollars. Suppose, however, that because a hair cut is not a service that can be easily traded internationally, its price is very different in country X and the United States (few people are willing to fly from one country to another just to take advantage of differences in hair cut prices). Specifically, assume that a hair cut costs 20 dollars in the United States, twenty times more than in country X. Assume also that, unlike hair cuts, grain is freely traded internationally, so its price
is the same in both countries. Then, the PPP-converted measure of GDP in country X is 260. In this example, the PPP adjusted measure is higher than its unadjusted counterpart, reflecting the fact that nontraded services are more expensive in the United States than in country X.

We define the set of poor countries as all countries with annual PPP-converted GDP per capita of up to 3,000 dollars, the set of emerging countries as all countries with PPP-converted GDP per capita between 3,000 and 25,000 dollars, and the set of rich countries as all countries with PPP-converted GDP per capita above 25,000 dollars. This definition delivers 40 poor countries, 58 emerging countries, and 22 rich countries. The lists of countries in each category appear in the appendix to this chapter. The fact that there are fewer rich countries than either emerging or poor countries makes sense because the distribution of GDP per capita across countries is highly skewed to the right, that is, the world is characterized by few very high-income countries and many low-to medium-income countries. Summary statistics for each income group are population-weighted averages of the corresponding country-specific summary statistics.

Table 1.1 shows that there are significant differences in volatility across income levels. Compared to rich countries, the rest of the world is a roller coaster. A simple inspection of table 1.1 makes it clear that the central difference between business cycles in rich countries and business cycles in either emerging or poor countries is that rich countries are about half as volatile as emerging or poor countries. This is true not only for output, but also for all components of aggregate demand.

**Fact 1.8 (Excess Volatility of Poor and Emerging Countries)**  
*Business cycles in rich countries are about half as volatile as business cycles in emerging or poor countries.*

Explaining this impressive fact is perhaps the most important unfinished business in macroeconomics. Are poor and emerging countries more volatile than rich countries because they face more volatile shocks, such as terms of trade, country risk premia, productivity disturbances, or animal
spirits? Or is their elevated instability the result of precarious economic institutions, manifested in, for example, poorly designed monetary and fiscal policies, political distortions, fragile financial systems, or weak enforcement of economic contracts, that tend to exacerbate the aggregate effects of changes in fundamentals? One of the objectives of this book is to shed light on these two non-mutually exclusive views.

A second important fact that emerges from the comparison of business-cycle statistics across income levels is that consumption smoothing is increasing with income per capita. In rich countries consumption is 13 percent less volatile than output, whereas in poor countries it is 12 percent more volatile. In emerging countries, consumption and output are about equally volatile.

**Fact 1.9 (Less Consumption Smoothing in Poor and Emerging Countries)** The relative consumption volatility is higher in poor and emerging countries than in rich countries.

Table 1.1 shows that the trade-balance-to-output ratio is countercyclical for poor, emerging, and rich countries. That is, fact 1.5 holds not only unconditionally, but also conditional on the level of economic development.

An important difference between business cycles in rich countries and the rest of the world that emerges from table 1.1 is that in rich countries the share of government consumption in GDP is significantly more countercyclical than in emerging or poor countries.

**Fact 1.10 (The Countercyclicality of Government Spending Increases With Income)** The share of government consumption is countercyclical in rich countries, but acyclical in emerging and poor countries.

Rich countries appear to deviate less from the classic Keynesian stabilization rule of boosting (reducing) the share of government spending during economic contractions (expansions) than do poor or emerging economies.
1.4 Country Size and Observed Business Cycles

Table 1.2 presents business-cycle facts disaggregated by country size. Countries are sorted into three size categories: small, medium, and large. These three categories are defined, respectively, as all countries with population in 2011 of less than 20 million, between 20 and 80 million, and more than 80 million. The first regularity that emerges from table 1.2 is that conditional on size, rich countries are at least half as volatile as emerging or poor countries. This means that fact 1.8 is robust to controlling for country size. To further characterize the partial correlations of output volatility with economic development and country size, we regress the standard deviation of output per capita of country $i$, denoted $\sigma_{y,i}$, onto a constant, the logarithm of country $i$’s population in 2009, denoted $\ln \text{pop}_i$, the logarithm of country $i$’s average PPP-converted output per capita over the period 1990-2009, denoted $\ln y_i^{PPP}$, and country $i$’s openness share, denoted $xmy_i$. All 120 countries in the sample are included. The regression yields

$$
\sigma_{y,i} = 15.0 -0.08 \ln \text{pop}_i -0.78 \ln y_i^{PPP} +0.86 xmy_i + \epsilon_i \\
t - \text{stat} \ (3.5) \ (-0.4) \ (-2.9) \ (0.9) \\
R^2 = 0.07
$$

This regression shows that both higher income per capita and larger country size tend to be associated with lower output volatility. At the same time, more open economies appear to be more volatile. Note, however, that population and openness are statistically insignificant.

Table 1.2 suggests that the consumption-output volatility ratio falls with income per capita and,
Table 1.2: Business Cycles in Small, Medium, and Large Countries

<table>
<thead>
<tr>
<th></th>
<th>All Countries</th>
<th>Poor Countries</th>
<th>Emerging Countries</th>
<th>Rich Countries</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S</td>
<td>M</td>
<td>L</td>
<td>S</td>
</tr>
<tr>
<td>Standard Deviations</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_y$</td>
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<td>7.92</td>
<td>5.55</td>
<td>8.17</td>
</tr>
<tr>
<td>$\sigma_c/\sigma_y$</td>
<td>1.12</td>
<td>0.96</td>
<td>1.07</td>
<td>1.39</td>
</tr>
<tr>
<td>$\sigma_g/\sigma_y$</td>
<td>2.22</td>
<td>2.21</td>
<td>2.28</td>
<td>2.92</td>
</tr>
<tr>
<td>$\sigma_i/\sigma_y$</td>
<td>3.65</td>
<td>3.23</td>
<td>3.06</td>
<td>4.68</td>
</tr>
<tr>
<td>$\sigma_x/\sigma_y$</td>
<td>2.46</td>
<td>3.29</td>
<td>3.07</td>
<td>2.81</td>
</tr>
<tr>
<td>$\sigma_m/\sigma_y$</td>
<td>2.55</td>
<td>3.12</td>
<td>3.33</td>
<td>2.96</td>
</tr>
<tr>
<td>$\sigma_{tb}/\sigma_y$</td>
<td>4.29</td>
<td>3.64</td>
<td>1.76</td>
<td>5.62</td>
</tr>
<tr>
<td>$\sigma_{ca}/\sigma_y$</td>
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<td>2.97</td>
<td>1.84</td>
<td>4.84</td>
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<td>Correlations with $y$</td>
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<td></td>
</tr>
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<td>$y$</td>
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</tr>
<tr>
<td>$c$</td>
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<td>0.71</td>
<td>0.69</td>
<td>0.58</td>
</tr>
<tr>
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<td>-0.01</td>
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</tr>
<tr>
<td>$i$</td>
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<td>0.66</td>
<td>0.45</td>
</tr>
<tr>
<td>$x$</td>
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<td>0.42</td>
<td>0.08</td>
<td>0.53</td>
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<tr>
<td>$m$</td>
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<td>0.57</td>
<td>0.11</td>
<td>0.53</td>
</tr>
<tr>
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</tr>
<tr>
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<td>-0.15</td>
<td>-0.18</td>
</tr>
<tr>
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<td>-0.30</td>
<td>-0.17</td>
</tr>
<tr>
<td>$ca$</td>
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<td>-0.25</td>
<td>-0.30</td>
<td>-0.23</td>
</tr>
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<td>Serial Correlations</td>
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<td></td>
</tr>
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<td>0.83</td>
<td>0.66</td>
<td>0.76</td>
</tr>
<tr>
<td>$c$</td>
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<td>0.66</td>
<td>0.61</td>
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<tr>
<td>$g$</td>
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<td>0.75</td>
<td>0.61</td>
</tr>
<tr>
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<td>0.66</td>
<td>0.53</td>
<td>0.62</td>
</tr>
<tr>
<td>$x$</td>
<td>0.67</td>
<td>0.75</td>
<td>0.67</td>
<td>0.58</td>
</tr>
<tr>
<td>$m$</td>
<td>0.69</td>
<td>0.70</td>
<td>0.63</td>
<td>0.68</td>
</tr>
<tr>
<td>$tb/y$</td>
<td>0.54</td>
<td>0.58</td>
<td>0.63</td>
<td>0.50</td>
</tr>
<tr>
<td>$ca/y$</td>
<td>0.42</td>
<td>0.50</td>
<td>0.60</td>
<td>0.36</td>
</tr>
<tr>
<td>Means</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$tby$</td>
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<td>-1.5</td>
<td>-0.8</td>
<td>-10.4</td>
</tr>
<tr>
<td>$xmy$</td>
<td>73.9</td>
<td>48.6</td>
<td>29.0</td>
<td>57.7</td>
</tr>
</tbody>
</table>

Note. See table 1.1. The sets of small (S), medium (M), and large (L) countries are defined as countries with 2011 populations of, respectively, less than 20 million, between 20 and 80 million, and more than 80 million.
less strongly, also with country size. This relationship is corroborated by the following regression:

\[
\ln \left( \frac{\sigma_{c,i}}{\sigma_{y,i}} \right) = 1.8 - 0.06 \ln \text{pop}_i - 0.11 \ln y_i^{PPP} + 0.14 x \text{my}_i + \epsilon_i
\]

\[t - \text{stat} \quad (4.1) \quad (-2.7) \quad (-3.8) \quad (+1.4)\]

\[R^2 = 0.19\]

According to this regression, more populous and richer countries tend to have a lower relative volatility of consumption. Taking into account that the volatility of output falls with size and income, this means that the volatility of consumption falls even faster than that of income as size and income increase. These results generalize fact 1.9, according to which consumption smoothing increases with income.

Finally, table 1.2 shows that smaller countries are more open than larger countries. This result holds unconditionally as well as conditional upon the level of income.

### 1.5 Hodrick-Prescott Filtering

We now consider an alternative detrending method developed by Hodrick and Prescott (1997), known as the Hodrick-Prescott, or HP, filter. The HP filter identifies the cyclical component, \( y^c_t \), and the trend component, \( y^s_t \), of a given series \( y_t \), for \( t = 1, 2, \ldots T \), as the solution to the minimization problem

\[
\min_{\{y^c_t, y^s_t\}_{t=1}^T} \left\{ \sum_{t=1}^{T} (y^c_t)^2 + \lambda \sum_{t=2}^{T-1} [(y^s_{t+1} - y^s_t) - (y^s_t - y^s_{t-1})]^2 \right\}
\]  

subject to (1.1). The appendix provides the first-order conditions and solution to this problem. According to this formula, the HP trend is the result of a trade off between minimizing the variance of the cyclical component and keeping the growth rate of the trend constant. This tradeoff is
The larger is $\lambda$ the more penalized are changes in the growth rate of the trend. In the limit as $\lambda$ goes to infinity, the trend component associated with the HP filter coincides with the linear trend. At the other extreme, as $\lambda$ goes to zero, all of the variation in the time series is attributed to the trend and the cyclical component is nil.

Business-cycle studies that use data sampled at an annual frequency typically assume a value of $\lambda$ of 100. Figure 1.2 displays the trend in Argentine real per capita GDP implied by the HP filter for $\lambda$ equal to 100.

The HP filter attributes a significant fraction of the output decline during the lost decade (1980-1989) to the trend. By contrast, the log-quadratic trend is monotonically increasing during this
period, implying that the lost decade was a cyclical phenomenon.

Figure 1.3 displays the cyclical component of Argentine output according to the HP filter ($\lambda = 100$) and the log-quadratic filter. The correlation between the two cyclical components is 0.70, indicating that for the most part they identify the same cyclical movements. However, the two filters imply quite different amplitudes for the Argentine cycle. The standard deviation of the cyclical component of output is 10.8 percent according to the log-quadratic filter, but only 5.7 percent according to the HP filter. The reason for this large reduction in the volatility of the cycle when applying the HP filter is that under this filter the trend moves much more closely with the raw series.
The value of $\lambda$ plays an important role in determining the amplitude of the business cycle implied by the HP filter. Recently, Ravn and Uhlig (2002) have suggested a value of $\lambda$ of 6.25 for annual data. Under this calibration, the standard deviation of the cyclical component of Argentine GDP drops significantly to 3.6 percent. Figure 1.4 displays the actual Argentine GDP and the trend implied by the HP filter when $\lambda$ takes the value 6.25.

In this case, the trend moves much closer with the actual series. In particular, the HP filter now attributes the bulk of the 1989 crisis and much of the 2001 crisis to the trend. This is problematic, especially for the 2001 depression. For this was a V-shaped, relatively short contraction followed by a swift recovery. This suggest that the 2001 crisis was a business-cycle phenomenon. By contrast,
the HP trend displays a significant contraction in 2001, suggesting that the crisis was to a large extent noncyclical. For this reason, we calibrate $\lambda$ at 100 for the subsequent analysis.

Table 1.3 displays business-cycle statistics implied by the HP filter for $\lambda = 100$. The central difference between the business-cycle facts derived from quadratic detrending and HP filtering is that under the latter detrending method the volatility of all variables falls by about a third. In particular, the average cross-country standard deviation of output falls from 6.2 percent under quadratic detrending to 3.8 percent under HP filtering.

In all other respects, the two filters produce very similar business-cycle facts. In particular, facts 1.1-1.10 are robust to applying the HP filter with $\lambda = 100$.

1.6 Growth Rates

Thus far, we have detrended output and all other components of aggregate demand using either a log-quadratic trend or the HP filter. An alternative to these two approaches is to assume that these variables have a stochastic trend. Here, we explore this avenue. Specifically, we assume that the levels of the variables of interest are nonstationary, but that their growth rates are. That is, we assume that the logarithm of output and the components of aggregate demand are integrated of order one.

Table 1.4 displays two statistical tests that provide evidence in favor of modeling these time series as stationary in growth rates and nonstationary in levels. The top panel of the table displays the results of applying the ADF test to the logarithm of real per capita GDP. The ADF test evaluates the null hypothesis that a univariate representation of the time series in question has a unit root against the alternative hypothesis that it does not. The table displays population-weighted cross-country averages of the decision value. The decision value is unity if the null hypothesis is rejected and 0 if it cannot be rejected. The table shows that the null hypothesis is rejected in 30
Table 1.3: HP-Filtered Business Cycles

<table>
<thead>
<tr>
<th>Statistic</th>
<th>All Countries</th>
<th>Poor Countries</th>
<th>Emerging Countries</th>
<th>Rich Countries</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_y )</td>
<td>3.79</td>
<td>4.12</td>
<td>3.98</td>
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<tr>
<td>( \sigma_c/\sigma_y )</td>
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<td>1.09</td>
<td>1.23</td>
<td>0.87</td>
</tr>
<tr>
<td>( \sigma_g/\sigma_y )</td>
<td>2.29</td>
<td>2.53</td>
<td>2.29</td>
<td>1.23</td>
</tr>
<tr>
<td>( \sigma_i/\sigma_y )</td>
<td>3.77</td>
<td>3.80</td>
<td>3.79</td>
<td>3.62</td>
</tr>
<tr>
<td>( \sigma_x/\sigma_y )</td>
<td>3.50</td>
<td>3.47</td>
<td>3.67</td>
<td>3.42</td>
</tr>
<tr>
<td>( \sigma_m/\sigma_y )</td>
<td>3.65</td>
<td>3.70</td>
<td>3.52</td>
<td>3.63</td>
</tr>
<tr>
<td>( \sigma_{tb}/\sigma_y )</td>
<td>1.79</td>
<td>1.64</td>
<td>2.92</td>
<td>0.89</td>
</tr>
<tr>
<td>( \sigma_{ca}/\sigma_y )</td>
<td>1.78</td>
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<td>Correlations with ( y )</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y )</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>( c )</td>
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<td>0.53</td>
<td>0.68</td>
<td>0.82</td>
</tr>
<tr>
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<tr>
<td>( i )</td>
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<tr>
<td>( x )</td>
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<td>0.13</td>
<td>0.30</td>
</tr>
<tr>
<td>( m )</td>
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<td>0.23</td>
<td>0.46</td>
<td>0.58</td>
</tr>
<tr>
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<td>-0.08</td>
<td>-0.34</td>
<td>-0.37</td>
</tr>
<tr>
<td>( tb )</td>
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<td>-0.11</td>
<td>-0.36</td>
<td>-0.36</td>
</tr>
<tr>
<td>( ca/y )</td>
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<td>-0.29</td>
<td>-0.39</td>
<td>-0.38</td>
</tr>
<tr>
<td>( ca )</td>
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<tr>
<td>Serial Correlations</td>
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<td></td>
</tr>
<tr>
<td>( y )</td>
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<td>0.39</td>
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<td>0.55</td>
</tr>
<tr>
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</tbody>
</table>

Note. See table 1.1. The variables \( y \), \( c \), \( g \), \( i \), \( x \), and \( m \) are HP filtered in logs and expressed in percent deviations from trend, and the variables \( tb/y \) and \( ca/y \) are HP filtered in levels and expressed in percentage points of output. The variables \( tb \) and \( ca \) were scaled by the secular component of GDP and then HP-filtered. The parameter \( \lambda \) of the HP filter takes the value 100.
Table 1.4: ADF and KPSS Tests for Output in Poor, Emerging, and Rich Countries

<table>
<thead>
<tr>
<th>Lags</th>
<th>All Countries</th>
<th>Poor Countries</th>
<th>Emerging Countries</th>
<th>Rich Countries</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADF Test</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.5</td>
<td>0.7</td>
<td>0.1</td>
<td>0.3</td>
</tr>
<tr>
<td>1</td>
<td>0.3</td>
<td>0.3</td>
<td>0.0</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.0</td>
<td>0.3</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>0.3</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>AIC for Lag Length</td>
<td>0.8</td>
<td>0.9</td>
<td>1.0</td>
<td>0.5</td>
</tr>
<tr>
<td>KPSS Test</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>1</td>
<td>1.0</td>
<td>1.0</td>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>1.0</td>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>3</td>
<td>0.9</td>
<td>0.9</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>AIC for Lag Length</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Note. See notes to table 1.1. Entries correspond to population-weighted decision values for the ADF and KPSS tests. For each country, a decision value of 1 indicates rejection of the null at 5% confidence level and a decision value of 0 indicates failure to reject the null. The null hypothesis is unit root under the ADF test and all roots within the unit circle in the KPSS test. Decision values are based on an F test. AIC stands for the population weighted cross-country average of the lag length suggested by the Akaike information criterion.
percent of the countries at the lag length of 1 year suggested by the Akaike information criterion (AIC), providing support to the unit-root hypothesis.

The lower panel of table 1.4 displays the results of applying the KPSS test to the logarithm of real output. This test evaluates the null hypothesis that the univariate representation of the logarithm of output has no unit root versus the alternative hypothesis that it does. For the lag length favored by the AIC test, the decision value is unity for virtually all countries, which suggests that the hypothesis of stationarity in levels is strongly rejected.

The results of the ADF and KPSS tests have to be interpreted with caution. The reason is that they both are based on the assumption that the time series in question has a univariate representation. As we will see in the following chapters, in general, theoretical models of the business cycle do not imply that output has a univariate representation.

Table 1.5 displays standard deviations, correlations with output growth, and serial correlations of the growth rates of output, private consumption, government consumption, investment, exports, and imports. Most of the ten business-cycle facts obtained under quadratic detrending also hold true when stationarity is induced by first-differencing the data. For example, world business cycles are highly volatile (fact 1.1). The cross-country average volatility of output growth is twice as large as the volatility of U.S. output growth (not shown). Poor and emerging countries are twice as volatile as rich countries (fact 1.8). The volatility of consumption growth relative to output growth is much higher in emerging and poor countries than in rich countries (fact 1.9). The trade-balance share is negatively correlated with output growth (fact 1.5). Finally, we note that, predictably, the serial correlations of growth rates are much lower than their (detrended) level counterparts.
Table 1.5: First Differenced Business Cycles

<table>
<thead>
<tr>
<th>Statistic</th>
<th>All Countries</th>
<th>Poor Countries</th>
<th>Emerging Countries</th>
<th>Rich Countries</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Standard Deviations</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_{\Delta y}$</td>
<td>4.39</td>
<td>4.94</td>
<td>4.08</td>
<td>2.38</td>
</tr>
<tr>
<td>$\sigma_{\Delta c/\Delta y}$</td>
<td>1.14</td>
<td>1.14</td>
<td>1.34</td>
<td>0.85</td>
</tr>
<tr>
<td>$\sigma_{\Delta g/\Delta y}$</td>
<td>2.14</td>
<td>2.28</td>
<td>2.39</td>
<td>1.17</td>
</tr>
<tr>
<td>$\sigma_{\Delta i/\Delta y}$</td>
<td>3.81</td>
<td>3.80</td>
<td>4.06</td>
<td>3.49</td>
</tr>
<tr>
<td>$\sigma_{\Delta x/\Delta y}$</td>
<td>3.37</td>
<td>3.22</td>
<td>3.98</td>
<td>3.22</td>
</tr>
<tr>
<td>$\sigma_{\Delta m/\Delta y}$</td>
<td>3.60</td>
<td>3.50</td>
<td>3.84</td>
<td>3.76</td>
</tr>
<tr>
<td>$\sigma_{tb/y}$</td>
<td>2.34</td>
<td>2.12</td>
<td>3.80</td>
<td>1.25</td>
</tr>
<tr>
<td>$\sigma_{ca/y}$</td>
<td>2.16</td>
<td>2.06</td>
<td>3.08</td>
<td>1.39</td>
</tr>
<tr>
<td><strong>Correlations with $\Delta y$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta y$</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>$\Delta c$</td>
<td>0.60</td>
<td>0.54</td>
<td>0.64</td>
<td>0.79</td>
</tr>
<tr>
<td>$g/y$</td>
<td>-0.10</td>
<td>-0.02</td>
<td>-0.18</td>
<td>-0.32</td>
</tr>
<tr>
<td>$\Delta i$</td>
<td>0.64</td>
<td>0.59</td>
<td>0.66</td>
<td>0.83</td>
</tr>
<tr>
<td>$\Delta x$</td>
<td>0.21</td>
<td>0.18</td>
<td>0.15</td>
<td>0.42</td>
</tr>
<tr>
<td>$\Delta m$</td>
<td>0.33</td>
<td>0.26</td>
<td>0.40</td>
<td>0.57</td>
</tr>
<tr>
<td>$tb/y$</td>
<td>-0.10</td>
<td>-0.08</td>
<td>-0.20</td>
<td>-0.07</td>
</tr>
<tr>
<td>$ca/y$</td>
<td>-0.07</td>
<td>-0.06</td>
<td>-0.12</td>
<td>-0.07</td>
</tr>
<tr>
<td><strong>Serial Correlations</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta y$</td>
<td>0.29</td>
<td>0.28</td>
<td>0.29</td>
<td>0.32</td>
</tr>
<tr>
<td>$\Delta c$</td>
<td>0.02</td>
<td>-0.03</td>
<td>0.02</td>
<td>0.27</td>
</tr>
<tr>
<td>$\Delta g$</td>
<td>0.18</td>
<td>0.14</td>
<td>0.11</td>
<td>0.48</td>
</tr>
<tr>
<td>$\Delta i$</td>
<td>0.01</td>
<td>-0.01</td>
<td>0.03</td>
<td>0.08</td>
</tr>
<tr>
<td>$\Delta x$</td>
<td>0.07</td>
<td>0.08</td>
<td>-0.00</td>
<td>0.10</td>
</tr>
<tr>
<td>$\Delta m$</td>
<td>0.04</td>
<td>0.08</td>
<td>-0.02</td>
<td>-0.04</td>
</tr>
<tr>
<td>$tb/y$</td>
<td>0.61</td>
<td>0.59</td>
<td>0.62</td>
<td>0.69</td>
</tr>
<tr>
<td>$ca/y$</td>
<td>0.57</td>
<td>0.55</td>
<td>0.52</td>
<td>0.71</td>
</tr>
</tbody>
</table>

Note. See notes to table 1.1. The variables $\Delta y$, $\Delta c$, $\Delta g$, $\Delta i$, $\Delta x$, and $\Delta m$ denote, respectively the log differences of output, consumption, government consumption, investment, exports, and imports. The variables $g/y$, $tb/y$, and $ca/y$ are quadratically detrended in levels. All variables are expressed in percent.
1.7 Duration and Amplitude of Business Cycles in Emerging and Developed Countries

We have documented that emerging countries display significantly more output volatility than developed countries. We now decompose business cycles into contractions and expansions and estimate for each of these phases of the cycle its duration and amplitude. Calderón and Fuentes (2010) adopt a classical approach to characterizing business cycles in emerging and developed countries, consisting in identifying peaks and troughs in the logarithm of real quarterly GDP. They define a peak as an output observation that is larger than the two immediately preceding and succeeding observations. Formally, letting $y_t$ denote the logarithm of real GDP, a peak takes place when $y_t > y_{t+j}$, for $j = -2, -1, 1, 2$. Similarly, a trough is defined as an output observation that is lower than its two immediately preceding and succeeding observations, that is, as a level of $y_t$ satisfying $y_t < y_{t+j}$, for $j = -2, -1, 1, 2$. The duration of a cycle is the period of time between one peak and the next. The duration of a contraction is the period of time between a peak and the next trough. And the duration of an expansion is the period of time that it takes to go from a trough to the next peak. The amplitude of a contraction is the percentage fall in output between a peak and the next trough. The amplitude of an expansion is the percentage increase in output between a trough and the next peak.

Table 1.6 displays the average duration and amplitude of business cycles in two groups of countries, one consisting of 12 Latin America countries and the other of 12 OECD countries. We will identify the former group with emerging countries and the latter with developed countries. The table shows that contractions in emerging and developed countries have equal durations of 3-4 quarters. However, the amplitude of contractions is much larger in emerging countries than in developed countries (6.2 versus 2.2 percent of GDP). Comparing the durations of expansions to that of contractions indicates that expansions are much longer than contractions and that expansions
Table 1.6: Duration and Amplitude of Business Cycles in Emerging and Developed Economies

<table>
<thead>
<tr>
<th>Group of Countries</th>
<th>Duration</th>
<th>Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Contraction</td>
<td>Expansion</td>
</tr>
<tr>
<td>Latin America</td>
<td>3.5</td>
<td>16.0</td>
</tr>
<tr>
<td>OECD</td>
<td>3.6</td>
<td>23.8</td>
</tr>
</tbody>
</table>

Source: Calderón and Fuentes (2010).

Note: The data is quarterly real GDP from 1980:1 to 2006:4. The countries included in the Latin America group are: Argentina, Bolivia, Brazil, Chile, Colombia, Costa Rica, Ecuador, Mexico, Paraguay, Peru, Uruguay, and Venezuela. The countries included in the OECD group are Australia, Canada, France, Germany, Italy, Japan, New Zealand, Portugal, Spain, Sweden, United Kingdom, and the United States.

are relatively shorter in emerging countries than in developed countries (16 versus 23.8 quarters). At the same time, the amplitude of expansions is about the same in both groups of countries (about 20 percent of GDP). Finally, emerging countries are more cyclical than developed countries in the sense that in the former complete cycles are shorter, 20 quarters versus 27 quarters. (A complete cycle is computed as the sum of the average durations of contractions and expansions.) The general pattern that emerges from this classical approach to characterizing business cycles is of emerging countries being more volatile because they display more cycles per unit of time and because they experience deeper contractions.

1.8 Business Cycle Facts With Quarterly Data

Thus far, we have empirically characterized business cycles around the world using annual data. Because annual data on national income and product accounts is readily available, this choice made it possible to derive business cycle facts for a large set of countries and for a relatively long period of time. Many business-cycle studies, however, especially those focused on developed economies, use
quarterly data. For this reason, in this section we characterized business cycle facts using quarterly data.

Gathering data at a quarterly frequency turns out to be much more difficult than doing so at an annual frequency. Most countries have some quarterly data, but often sample periods are short, typically less than 20 years. The problem with so short samples is that it becomes difficult to separate the trend from the cyclical component. For inclusion in the data set, we continue to require that a country has at least 30 years (or 120 quarters) of quarterly data for output, consumption, investment, exports, imports, and public consumption. This restriction reduces significantly the number of countries for which data is available relative to the case of annual data. Specifically, our quarterly panel contains no poor countries, 11 emerging countries, and 17 rich countries. By comparison, our annual panel contains 40 poor countries, 58 emerging countries, and 22 rich countries. The sample period is 1980:Q1 to 2012:Q4 with two exceptions, Uruguay and Argentina.\footnote{The data for Uruguay begins in 1983:Q1 and the time series for private and public consumption in Argentina begin in 1993:Q1.} The data is available online at \url{http://www.columbia.edu/~mu2166/book/}

Table 1.7 displays business-cycle statistics at quarterly frequency for emerging and rich countries and for three different ways to measure the cyclical component, namely, log-quadratic detrending, HP filtering, and first differencing. Overall, the business-cycle facts that emerge from quarterly data are similar to those identified using annual data. In particular, table 1.7 shows that:

a. Investment, government spending, exports, and imports are more volatile than output, and private consumption is about as volatile as output.

b. Consumption, investment, exports, and imports are all procyclical, whereas the trade balance is countercyclical.

c. Output, consumption, investment, exports, and imports are all positively serially correlated.
Table 1.7: Business Cycles in Emerging and Rich Countries, Quarterly Data, 1980Q1-2012Q4

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Log-Quadratic Time Trend</th>
<th>HP Filter</th>
<th>First Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>All</td>
<td>Emerging</td>
<td>Rich</td>
</tr>
<tr>
<td>Standard Deviations</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_y$</td>
<td>3.26</td>
<td>4.27</td>
<td>2.74</td>
</tr>
<tr>
<td>$\sigma_c/\sigma_y$</td>
<td>0.99</td>
<td>1.23</td>
<td>1.01</td>
</tr>
<tr>
<td>$\sigma_g/\sigma_y$</td>
<td>1.46</td>
<td>2.07</td>
<td>1.15</td>
</tr>
<tr>
<td>$\sigma_i/\sigma_y$</td>
<td>3.44</td>
<td>3.67</td>
<td>3.31</td>
</tr>
<tr>
<td>$\sigma_x/\sigma_y$</td>
<td>3.77</td>
<td>3.97</td>
<td>3.67</td>
</tr>
<tr>
<td>$\sigma_m/\sigma_y$</td>
<td>3.52</td>
<td>3.55</td>
<td>3.51</td>
</tr>
<tr>
<td>$\sigma_{tb}/\sigma_y$</td>
<td>1.80</td>
<td>2.93</td>
<td>1.21</td>
</tr>
</tbody>
</table>

Correlations with $y$

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>1.00</td>
</tr>
<tr>
<td>$c$</td>
<td>0.83</td>
</tr>
<tr>
<td>$g/y$</td>
<td>-0.43</td>
</tr>
<tr>
<td>$i$</td>
<td>0.86</td>
</tr>
<tr>
<td>$x$</td>
<td>0.17</td>
</tr>
<tr>
<td>$m$</td>
<td>0.60</td>
</tr>
<tr>
<td>$tb/y$</td>
<td>-0.44</td>
</tr>
<tr>
<td>$tb$</td>
<td>-0.44</td>
</tr>
</tbody>
</table>

Serial Correlations

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Serial Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0.94</td>
</tr>
<tr>
<td>$c$</td>
<td>0.91</td>
</tr>
<tr>
<td>$g$</td>
<td>0.87</td>
</tr>
<tr>
<td>$i$</td>
<td>0.91</td>
</tr>
<tr>
<td>$x$</td>
<td>0.92</td>
</tr>
<tr>
<td>$m$</td>
<td>0.90</td>
</tr>
<tr>
<td>$tb/y$</td>
<td>0.88</td>
</tr>
</tbody>
</table>

Means

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Means</th>
</tr>
</thead>
<tbody>
<tr>
<td>$tb/y$</td>
<td>-0.1</td>
</tr>
<tr>
<td>$(x + m)/y$</td>
<td>43.8</td>
</tr>
</tbody>
</table>

Note. The variables $y$, $c$, $g$, $i$, $x$, $m$, and $tb \equiv (x - m)$, denote, respectively, output, total private consumption, government spending, investment, exports, imports, and the trade balance. All variables are real and per capita. For quadratic detrending or HP filtering the variables $y$, $c$, $g$, $i$, $x$, and $m$ are detrended in logs and expressed in percent deviations from trend. For first differencing, $y$, $c$, $g$, $i$, $x$, and $m$ denote log differences. The variables $tb/y$ and $g/y$ are detrended in levels. The variable $tb$ is scaled by the secular component of $y$ and detrended. The sample contains 11 emerging and 17 rich countries. Moments are averaged across countries using population weights. The sets of emerging and rich countries are defined as all countries with average PPP converted GDP per capita in U.S. dollars of 2005 over the period 1990-2009 within the ranges 3,000-25,000 and 25,000-$\infty$, respectively. Rich Countries: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Hong Kong, Italy, Japan, Netherlands, Norway, Sweden, Switzerland, United Kingdom, United States. Emerging Countries: Argentina, Israel, South Korea, Mexico, New Zealand, Peru, Portugal, South Africa, Spain, Turkey, and Uruguay. The data sources are presented in the appendix to this chapter.
d. Emerging countries are more volatile than rich countries.

e. Consumption is more volatile than output in emerging countries, but less volatile than output in rich countries. And

f. the share of government spending in output is more countercyclical in rich countries than in emerging countries.

As expected, the serial correlation of all macroeconomic indicators is higher in quarterly data than in annual data. Table 1.9 in the appendix presents business cycles statistics for each individual country in the sample.

1.9 Appendix

1.9.1 Countries With At Least 30 Years of Annual Data

The sample consists of 120 countries. There are 22 small poor countries, 11 medium-size poor countries, 7 large poor countries, 41 small emerging countries, 14 medium-size emerging countries, 3 large emerging countries, 14 small rich countries, 5 medium-size rich countries and 3 large rich countries. The individual countries belonging to each group are listed below.

**Small Poor Countries:** Benin, Bhutan, Burkina Faso, Burundi, Central African Republic, Comoros, Gambia, Guyana, Honduras, Lesotho, Malawi, Mali, Mauritania, Mongolia, Niger, Papua New Guinea, Rwanda, Senegal, Sierra Leone, Togo, Zambia, Zimbabwe.


**Large Poor Countries:** Bangladesh, China, Ethiopia, India, Indonesia, Pakistan, Philippines.

**Small Emerging Countries:** Albania, Antigua and Barbuda, Bahrain, Barbados, Bolivia, Botswana, Bulgaria, Chile, Costa Rica, Cuba, Cyprus, Dominica, Dominican Republic, Ecuador, El Salvador,
Fiji, Gabon, Greece, Grenada, Guatemala, Hungary, Israel, Jordan, Malta, Mauritius, Namibia, New Zealand, Panama, Paraguay, Portugal, Puerto Rico, Seychelles, St. Kitts and Nevis, St. Lucia, St. Vincent and the Grenadines, Suriname, Swaziland, Tonga, Trinidad and Tobago, Tunisia, Uruguay.

**Medium Size Emerging Countries:** Algeria, Argentina, Colombia, Iran, South Korea, Malaysia, Morocco, Peru, South Africa, Spain, Syria, Thailand, Turkey, Venezuela.

**Large Emerging Countries:** Brazil, Egypt, Mexico.

**Small Rich Countries:** Austria, Belgium, Denmark, Finland, Hong Kong, Iceland, Ireland, Luxembourg, Macao, Netherlands, Norway, Singapore, Sweden, Switzerland.

**Medium Size Rich Countries:** Australia, Canada, France, Italy, United Kingdom.

**Large Rich Countries:** Germany, Japan, United States.

### 1.9.2 Derivation of the HP Filter

The first-order conditions associated with the problem of choosing the series \( \{y^c_t, y^s_t\}_{t=1}^T \) to minimize (1.2) subject to (1.1) are

\[
y_1 = y^s_1 + \lambda(y^s_1 - 2y^s_2 + y^s_3),
\]

\[
y_2 = y^s_2 + \lambda(-2y^s_1 + 5y^s_2 - 4y^s_3 + y^s_4),
\]

\[
y_t = y^s_t + \lambda(y^s_{t-2} - 4y^s_{t-1} + 6y^s_t - 4y^s_{t+1} + y^s_{t+2}); \quad t = 3, \ldots, T - 2,
\]

\[
y_{T-1} = y^s_{T-1} + \lambda(y^s_{T-3} - 4y^s_{T-2} + 5y^s_{T-1} - 2y^s_T),
\]

and

\[
y_T = y^s_T + \lambda(y^s_{T-2} - 2y^s_{T-1} + y^s_T).
\]
Letting $Y^s \equiv [y_1^s \ y_2^s \ \ldots \ y_T^s]$ and $Y \equiv [y_1 \ y_2 \ \ldots \ y_T]$, the above optimality conditions can be written in matrix form as

$$Y = (I + \lambda A)Y^s,$$

where $I$ is the $T \times T$ identity matrix, and $A$ is the following $T \times T$ matrix of constants

$$A = \begin{bmatrix}
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
-2 & 5 & -4 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\
1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & \ldots & 0 \\
0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & & & & & & & & \vdots \\
0 & \ldots & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 \\
0 & \ldots & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\
0 & \ldots & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 \\
0 & \ldots & 0 & 0 & 0 & 0 & 1 & -4 & 5 & -2 \\
0 & \ldots & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1
\end{bmatrix}.$$ 

Solving for $Y^s$, one obtains:

$$Y^s = (I + \lambda A)^{-1}Y.$$

Finally, letting $Y^c \equiv [y_1^c \ y_2^c \ \ldots \ y_T^c]$ we have that

$$Y^c = Y - Y^s.$$

1.9.3 Country-By-Country Business Cycle Statistics At Annual And Quarterly Frequency
Table 1.8: Business Cycles, Log-Quadratic Detrending, Annual Data, 1965-2010

<table>
<thead>
<tr>
<th>Country</th>
<th>$\sigma_y$</th>
<th>$\sigma_g$</th>
<th>$\sigma_i$</th>
<th>$\sigma_m$</th>
<th>$\sigma_{tb/y}$</th>
<th>Correlations with $y$</th>
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Note. The variables $y$, $c$, $g$, $i$, $x$, and $m$ and $tb \equiv (x - m)$, denote, respectively, output, total private consumption, government spending, investment, exports, imports, and the trade balance. All variables are real and per capita. The variables $y$, $c$, $g$, $i$, $x$, and $m$ are detrended in logs and expressed in percent deviations from trend. The variables $tb/y$ and $g/y$ are detrended in levels. The variable $tb$ is scaled by the secular component of $y$ and detrended. This table includes all countries for which we have not only 30 years of annual data but also 30 years of quarterly data. The country-specific sample periods and data sources are given in the online appendix to this chapter, available at http://www.columbia.edu/~mu2166/book.
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### 1.10 Exercises

**Exercise 1.1 (Business Cycle Regularities in South Korea and the United States)** In this exercise, you are asked to analyze the extent to which the numbered business cycle facts discussed in this chapter apply to (a) South Korea and (b) The United States. To this end compute the relevant business cycle statistics for the following four alternative detrending methods: (a) log-linear detrending; (b) log-quadratic detrending; (c) Hodrick Prescott filtering with $\lambda = 100$; and (d) Hodrick Prescott filtering with $\lambda = 6.25$. Make a two by two graph showing the natural logarithm of real per capita GDP and the trend, one panel per trend. Discuss how the detrending method influences the volatility of the cyclical component of output. Also discuss which detrending method identifies recessions for the U.S. most in line with the NBER business cycle dates. The data should be downloaded from the World Bank’s WDI database. As the sample period for South Korea use 1960 to 2011 and for the United States use 1965-2011. Specifically, use the following time series to construct the required business cycle statistics:

- **GDP per capita (constant LCU)**: NY.GDP.PCAP.KN
- **Household final consumption expenditure, etc. (% of GDP)**: NE.CON.PETC.ZS
- **Gross capital formation (% of GDP)**: NE.GDI.TOTL.ZS
- **General government final consumption expenditure (% of GDP)**: NE.CON.GOVT.ZS
- **Imports of goods and services (% of GDP)**: NE.IMP.GNFS.ZS
- **Exports of goods and services (% of GDP)**: NE.EXP.GNFS.ZS
Chapter 2

An Open Endowment Economy

The purpose of this chapter is to build a canonical dynamic, general equilibrium model of the open economy and contrast its predictions with the empirical regularities documented in chapter 1. The model developed in this chapter is simple enough to allow for a full characterization of its equilibrium dynamics using pen and paper.

2.1 The Model Economy

Consider an economy populated by a large number of infinitely-lived households with preferences described by the utility function

\[ E_0 \sum_{t=0}^{\infty} \beta^t U(c_t), \]  

(2.1)

where \( c_t \) denotes consumption, \( U \) denotes a single-period utility function, assumed to be continuously differentiable, strictly increasing, and strictly concave, and \( E_t \) denotes the mathematical expectation conditional on information available in period \( t \).

Each period, households receive an endowment of goods and have the ability to borrow or lend in a risk-free internationally traded bond. The sequential budget constraint of the representative
household is given by
\[ c_t + (1 + r)d_{t-1} = y_t + d_t, \]
where \( d_{t-1} \) denotes the debt position assumed in period \( t - 1 \) and due in period \( t \), \( r \) denotes the interest rate, assumed to be constant, and \( y_t \) is an exogenous and stochastic endowment of goods. The endowment process represents the sole source of uncertainty in this economy. The above constraint states that the household has two sources of funds, the endowment, \( y_t \), and debt, \( d_t \). It uses those funds to purchase consumption goods, \( c_t \), and pay back the principal and interest on its outstanding debt, \( (1 + r)d_{t-1} \). Households are assumed to be subject to the following sequence of borrowing constraints that prevents them from engaging in Ponzi games:
\[ \lim_{j \to \infty} E_t \frac{d_{t+j}}{(1 + r)^j} \leq 0. \]  
This limit condition states that the household’s debt position must be expected to grow at a rate lower than the interest rate \( r \) in the long run.

The optimal allocation of consumption and debt will always feature the no-Ponzi-game constraint (2.3) holding with strict equality. To see this, suppose, contrary to this claim, that at some \( t \geq 0 \), the optimal allocation \( \{c_{t+j}, d_{t+j}\}_{j=0}^{\infty} \), satisfies
\[ \lim_{j \to \infty} E_t \frac{d_{t+1}}{(1 + r)^j} = -\alpha; \quad \alpha > 0. \]
Then, consider the alternative consumption process
\[ c'_{t+j} = \begin{cases} c_{t+j} + \alpha & \text{if } j = 0 \\ c_{t+j} & \text{if } j > 0 \end{cases}. \]
Clearly, because the period utility function is assumed to be strictly increasing, the process \( \{c'_{t+j}\}_{j=0}^{\infty} \)
must be preferred to the process \( \{c_{t+j}\}_{j=0}^{\infty} \). Now construct the associated alternative debt process, \( \{d'_{t+j}\}_{j=0}^{\infty} \), as follows. For \( j = 0 \), the budget constraint (2.2) dictates

\[
d'_t = (1 + r)d_{t-1} + c'_t - y_t
\]

\[
= (1 + r)d_{t-1} + c_t + \alpha - y_t
\]

\[
= d_t + \alpha.
\]

Note that the right-hand side of the first equality features \( d_{t-1} \) and not \( d'_{t-1} \). This is because we are not disturbing the equilibrium allocation before period \( t \). For \( j = 1 \), the budget constraint (2.2) implies that

\[
d'_{t+1} = (1 + r)d'_t + c'_{t+1} - y_{t+1}
\]

\[
= (1 + r)(d_t + \alpha) + c_{t+1} - y_{t+1}
\]

\[
= d_{t+1} + (1 + r)\alpha.
\]

Continuing with this argument we have that for \( j \geq 0 \)

\[
d'_{t+j} = d_{t+j} + (1 + r)^j \alpha.
\]

We have constructed the alternative process for debt to ensure that it satisfies the sequential budget constraint (2.2) for all dates and states. It remains to check that it also satisfies the no-Ponzi-game
constraint (2.3). To see that this is the case, use the above equation to write

\[
\lim_{j \to \infty} E_t \frac{d_{t+j}}{(1 + r)^j} = \lim_{j \to \infty} E_t \frac{d_{t+j} + (1 + r)^j \alpha}{(1 + r)^j} \\
= \lim_{j \to \infty} E_t \frac{d_{t+j}}{(1 + r)^j} + \alpha \\
= -\alpha + \alpha \\
= 0.
\]

This completes the proof that at the optimal allocation the no-Ponzi-game constraint (2.3) must hold with equality. When the limiting condition (2.3) holds with equality, it is called the transversality condition.

The household chooses processes for \( c_t \) and \( d_t \) for \( t \geq 0 \), so as to maximize (2.1) subject to (2.2) and (2.3). The Lagrangian of this problem in period 0 is given by

\[
\mathcal{L}_0 = E_0 \sum_{t=0}^{\infty} \beta^t \{ U(c_t) + \lambda_t[d_t + y_t - (1 + r)d_{t-1} - c_t] \},
\]

where \( \beta^t \lambda_t \) denotes the Lagrange multiplier associated with the sequential budget constraint in period \( t \). The optimality conditions associated with this problem are (2.2), (2.3) holding with equality,

\[
U'(c_t) = \lambda_t,
\]

and

\[
\lambda_t = \beta(1 + r)E_t \lambda_{t+1}.
\]

The last two conditions are, respectively, the derivatives of the Lagrangian with respect to \( c_t \) and \( d_t \) equalized to zero. Combining these expressions to eliminate \( \lambda_t \) yields the following optimality
condition often referred to as the Euler equation

\[ U'(c_t) = \beta (1 + r) E_t U'(c_{t+1}). \] (2.4)

The interpretation of this expression is that at the margin, the household is indifferent between consuming a unit of good today or saving it and consuming it the next period along with the interest. To see this, note that if the household reduces consumption by one unit in period \( t \), its period-\( t \) utility falls by approximately \( U'(c_t) \). If instead the household invests this unit of consumption in the international bond market, in period \( t + 1 \) it will receive \( 1 + r \) units of consumption, which, if consumed, increase the period-\( t + 1 \) utility by approximately \( (1 + r)U'(c_{t+1}) \). From the perspective of period \( t \) the expected present value of this increase in period-\( t + 1 \) utility equals \( \beta (1 + r) E_t U'(c_{t+1}) \).

### 2.1.1 Equilibrium

All households are assumed to have identical preferences, realizations of the endowment process, and initial asset holdings. Therefore, we can interpret \( c_t \) and \( d_t \) as the aggregate per capital levels of consumption and net foreign liabilities, respectively. Then, a rational expectations equilibrium can be defined as a pair of processes \( \{c_t, d_t\}_{t=0}^{\infty} \) satisfying (2.2), (2.3) holding with equality, and (2.4), given the initial condition \( d_{-1} \) and the exogenous driving process \( \{y_t\}_{t=0}^{\infty} \).

Combining the household’s sequential budget constraint (2.2) and the no-Ponzi-game constraint (2.3) holding with equality yields an We now derive an intertemporal resource constraint. To see this, begin by expressing the sequential budget constraint in period \( t \) as

\[(1 + r)d_{t-1} = y_t - c_t + d_t.\]
Eliminate $d_t$ by combining this expression with itself evaluated one period forward. This yields:

$$(1 + r)d_{t-1} = y_t - c_t + \frac{y_{t+1} - c_{t+1}}{1 + r} + \frac{d_{t+1}}{1 + r}.$$  

Repeat this procedure $s$ times to obtain

$$(1 + r)d_{t-1} = \sum_{j=0}^{s} \frac{y_{t+j} - c_{t+j}}{(1 + r)^j} + \frac{d_{t+s}}{(1 + r)^s}.$$  

Apply expectations conditional on information available at time $t$ and take the limit for $s \to \infty$ using condition (2.3) holding with equality, to get the following intertemporal resource constraint:

$$(1 + r)d_{t-1} = E_t \sum_{j=0}^{\infty} \frac{y_{t+j} - c_{t+j}}{(1 + r)^j}, \quad (2.5)$$  

which must hold for all dates and states.

A key indicator in open economy macroeconomics is the trade balance, which is defined as the difference between exports and imports of goods and services. In the present model, there is a single good. Therefore, in any give period, the country either exports or imports this good depending on whether the endowment is higher or lower than consumption. It follows that the trade balance is given by the difference between output and consumption. Formally, letting $tb_t$ denote the trade balance in period $t$, we have that

$$tb_t \equiv y_t - c_t. \quad (2.6)$$  

Combining this definition with equation (2.5) yields

$$ (1 + r)d_{t-1} = E_t \sum_{j=0}^{\infty} \frac{tb_{t+j}}{(1 + r)^j}, $$
Intuitively, this equation says that the country’s initial net foreign debt position must equal the expected present discounted value of current and future expected trade surpluses. If the economy starts as a net external debtor, i.e., if $d_{t-1}(1 + r) > 0$, then it must run a trade surplus in at least one period, that is, $tb_{t+j}$ must be positive for at least one $j \geq 0$. However, if the economy starts as a net creditor of the rest of the world, i.e., if $d_{t-1}(1 + r) < 0$, then it could in principle run a perpetual trade deficit, that is, $tb_{t+j}$ could in principle be negative for all $j \geq 0$.

At this point, we make two additional assumptions that greatly facilitate the analysis. First we require that the subjective and pecuniary rates of discount, $\beta$ and $1/(1 + r)$, be equal to each other, that is,

$$\beta(1 + r) = 1.$$  

This assumption eliminates long-run growth in consumption when the economy features no stochastic shocks.

Second, we assume that the period utility index is quadratic and given by

$$U(c) = -\frac{1}{2}(c - \bar{c})^2,$$  \hspace{1cm} (2.7)

with $c \leq \bar{c}$. This specification has the appealing feature of allowing for a closed-form solution of the model. After imposing the above two assumptions, our model becomes essentially Hall’s (1978) permanent income model of consumption. In particular, the Euler condition (2.4) now becomes

$$c_t = E_t c_{t+1},$$  \hspace{1cm} (2.8)

which says that consumption follows a random walk: At each point in time, households expect to maintain a constant level of consumption next period. Indeed, households expect all future levels of consumption to be equal to its present level. To see this, lead the Euler equation (2.8) one
period to obtain \( c_{t+1} = E_{t+1}c_{t+2} \). Take expectations conditional on information available at time \( t \) and use the law of iterated expectations to obtain \( E_t c_{t+1} = E_t c_{t+2} \). Finally, using again the Euler equation (2.8) to replace \( E_t c_{t+1} \) by \( c_t \), we can write \( c_t = E_t c_{t+2} \). Repeating this procedure \( j \) times, we can deduce that

\[
E_t c_{t+j} = c_t,
\]

for all \( j \geq 0 \).

To find the closed-form solution for consumption, use the above expression to get rid of expected future consumption in the intertemporal resource constraint (2.5) to obtain (after slightly rearranging terms)

\[
c_t = \frac{r}{1 + r} E_t \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1 + r)^j} - rd_{t-1}.
\]

(2.9)

This expression is quite intuitive. The first term on the right-hand side, is known as (nonfinancial) permanent income, and we denote it by \( y_t^p \),

\[
y_t^p = \frac{r}{1 + r} E_t \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1 + r)^j}.
\]

(2.10)

It is a weighted average of the expected lifetime stream of endowments. To see this, note that the weights, \( \frac{1}{1 + r} \frac{1}{(1 + r)^j} \), for \( j \geq 0 \), are all positive and add up to one, \( \frac{r}{1 + r} \sum_{j=0}^{\infty} \frac{1}{(1 + r)^j} = 1 \). Thus, equation (2.9) states that every period the optimal contingent plan is to spend the permanent income on consumption and interest payments,

\[
c_t + rd_{t-1} = y_t^p
\]

(2.11)

Note that at any date \( t \), the debt level \( d_{t-1} \) is predetermined, and \( E_t y_{t+j} \) for all \( j \geq 0 \) is exogenously given. Therefore, the above expression represents the closed-form solution for \( c_t \). Combining the
above expression with the sequential budget constraint (2.2) yields the closed for solution for the equilibrium level of the country’s external debt

\[ d_t - d_{t-1} = y_t^p - y_t. \]  

(2.12)

Intuitively, the economy borrows from (lend to) the rest of the world when permanent income is higher (lower) than current income.

The current account is defined as the sum of the trade balance and net investment income on the country’s net foreign asset position, \(-rd_{t-1}\). Formally, letting \(ca_t\) denote the current account in period \(t\), we have that

\[ ca_t \equiv tb_t - rd_{t-1}. \]

Combining this expression with the definition of the trade balance given in (2.6) and with the sequential budget constraint (2.2), we obtain the following alternative expression for the current account:

\[ ca_t = -(d_t - d_{t-1}). \]  

(2.13)

This expression, known as the fundamental balance-of-payments identity, says that the current account equals the change in the country’s net foreign asset position. In other words, a current account deficit (surplus) is associated with an increase (reduction) in the country’s external debt of equal magnitude. Combining (2.12) and (2.13) yields

\[ ca_t = y_t - y_t^p, \]  

(2.14)

which says that the country runs current account surpluses (deficits) when current income is higher (lower) than permanent income. Recalling that the current account equals the trade balance plus
interest income, \( ca_t = tb_t - rd_{t-1} \), we can use equation (2.14) to obtain the equilibrium trade balance

\[
tb_t = y_t - y^p_t + rd_{t-1}, \tag{2.15}
\]

which states that the trade balance responds countercyclically (procyclically) to changes in current income if permanent income increases by more (less) than current income in response to increases in current income.

Because expectations of future income, as embedded in the definition of permanent income, feature so prominently in the determination of the current account, the present model is known as the intertemporal approach to the balance of payments. It is clear from the analysis conducted thus far that the equilibrium behavior of the current account and external debt depends crucially on the interaction between current income and permanent income. In turn, this interaction is governed by the properties of the stochastic process followed by the endowment. We address this issue next.

### 2.2 Stationary Income Shocks

Assume that the endowment follows an AR(1) process of the form

\[
y_t = \rho y_{t-1} + \epsilon_t,
\]

where \( \epsilon_t \) denotes an i.i.d. innovation and the parameter \( \rho \in (-1, 1) \) defines the serial correlation of the endowment process. The larger is \( \rho \), the more persistent is the endowment process. Given this autoregressive structure of the endowment, the \( j \)-period-ahead forecast of output in period \( t \) is given by

\[
E_t y_{t+j} = \rho^j y_t.
\]
Using this expression to eliminate expectations of future income from identity (2.9) delivers the following expression for permanent income when the endowment follows an AR(1) process

\[ y_t^p = \frac{r}{1 + r - \rho}y_t. \] (2.16)

If the endowment process is highly persistent \((\rho \rightarrow 1)\), permanent income is close to the endowment itself. This makes sense, because any innovation in the current endowment is expected to affect all future endowments. At the other extreme, if the endowment is highly transitory \((\rho \rightarrow 0)\), only a small fraction \(r/(1 + r)\) of the current endowment is regarded as permanent. Combining the above expression with equation (2.11) we obtain

\[ c_t = \frac{r}{1 + r - \rho}y_t - rd_{t-1}. \] (2.17)

Consider now the effect of an innovation in the endowment. Because \(\rho\) is less than unity, we have that a unit increase in \(y_t\) leads to a less-than-unit increase in consumption. The remaining income is saved to allow for higher future consumption.

Combining equation (2.17) with the definitions of the trade balance and of the current account given above, we can write

\[ tb_t = rd_{t-1} + \frac{1 - \rho}{1 + r - \rho}y_t \] (2.18)

and

\[ ca_t = \frac{1 - \rho}{1 + r - \rho}y_t. \] (2.19)

Note that the current account inherits the stochastic process of the underlying endowment shock. Because the current account equals the change in the country’s net foreign asset position, i.e., \(ca_t = -(d_t - d_{t-1})\), it follows that the equilibrium evolution of the stock of external debt is given
According to this expression, external debt follows a random walk and is therefore nonstationary. A temporary increase in the endowment produces a gradual but permanent decline in the stock of foreign liabilities. Because the long-run behavior of the trade balance is governed by the dynamics of external debt (see equation (2.18)), a temporary increase in the endowment leads to a long-run deterioration in the trade balance.

Let’s examine in more detail the response of the endogenous variables of the model to an unanticipated increase in output. Two polar cases are of interest, purely temporary endowment shocks ($\rho = 0$), and permanent endowment shocks ($\rho \to 1$). When endowment shocks are purely transitory, equation (2.17) implies only a small part of the increase in the endowment, a fraction $r/(1+r)$, is allocated to current consumption. Most of the endowment increase, a fraction $1/(1+r)$, is saved in the form of foreign bonds. As a result the trade balance and the current account both increase by $1/(1+r)$ (see equations (2.18) and (2.19)) and the external debt falls by $1/(1+r)$ (see equation (2.20)). The intuition behind this result is clear. Because income is expected to return quickly to its long-run level, households smooth consumption by eating a tiny part of the current windfall and leaving the rest for future consumption. In this case, the current account plays the role of a shock absorber. The economy saves in response to positive income shocks via current account surpluses and borrows from the rest of the world to finance negative income shocks via current account deficits. Importantly, the current account is procyclical. That is, it improves during expansions and deteriorates during contractions. This prediction of the model is at odds with the data. As documented in chapter 1, in small open economies the current account is countercyclical and not procyclical as predicted by the model studied here.

At the other extreme, when endowment shocks are permanent ($\rho \to 1$), households allocate all
of the increase in output to current consumption (see equation (2.17)). As a result, the trade balance, the current account, and the stock of external debt remain unchanged (see equations (2.18), (2.19), and (2.20)). All remain Intuitively, when endowment shocks are permanent, an increase in output today is accompanied by an increase in output of the same magnitude in all future periods. As a result, the household does not need to save part of the current increase in output to smooth consumption. In this case, the current account does not play the role of a shock absorber. Rather, households adjust consumption up or down in response to output shocks.

The two polar cases suggest that the role of shock absorber of the current account is increasing in the temporariness of the endowment shock. In turn, this implies that the more temporary is the endowment shock, the more volatile is the current account. This implication can be clearly seen from equation (2.19). When $\rho = 0$, the standard deviation of the current account equals $\sigma_y/(1+r)$, which is close to the volatility of the endowment itself, denoted $\sigma_y$, for small values of $r$. On the other extreme, when $\rho \to 1$, the standard deviation of the current account is zero.

The intermediate case of a gradually trend-reverting endowment process, which takes place when $\rho \in (0,1)$, is illustrated in figure 2.1. It displays the economy’s response to a positive endowment shock, assuming that the initial external debt is 0. In response to the endowment shock, consumption experiences a once-and-for-all increase. This expansion is smaller than the initial increase in income. As a result, the trade balance and the current account improve. After their initial increase, these two variables converge gradually to their respective long-run levels. The trade balance converges to a new long-run level lower than the pre-shock one. This is because in the long-run the economy settles at a lower level of external debt, whose service requires a smaller trade surplus. Thus, a positive endowment shock produces a short-run improvement but a long-run deterioration in the trade balance.

Before moving on, let us take stock of the key results derived thus far. The open economy model we are studying captures the essential elements of what has become known as the intertemporal
Figure 2.1: Response to a Positive Endowment Shock

Endowment

\[ y_t \]

Consumption

\[ c_t \]

Trade Balance and Current Account

\[ \text{ca}_t \]
\[ \text{tb}_t \]

External Debt

\[ d_t \]
approach to the current account. In equilibrium, consumption, external debt, the trade balance, and the current account are driven by perceived differences between current and permanent income. A central implication is that external borrowing is guided by the principle ‘finance temporary shocks and adjust to permanent shocks.’ An important weakness of the model is the counterfactual prediction of a procyclical current account and a procyclical trade balance.

Equation (2.14) provides the clue for why the present model fails to deliver a countercyclical current account when the endowment process is AR(1). It says that the current account deteriorates in response to an increase in current income if an only if the increase in current income is smaller than the associated increase in permanent income. But if the endowment process is AR(1), with $0 < \rho < 1$, an increase in $y_t$ is always accompanied by the expectation of a declining path of income, and thus by an increase in permanent income that is less than the increase in current income. Specifically, equation (2.16) shows that an increase in $y_t$ increases $y^p_t$ by a fraction $r/(1 + r - \rho) < 1$. This result suggests that one way to generate a countercyclical current account would be to formulate an endowment process with the property that in response to an increase in current income permanent income increases by more than current income. We turn to this task next.

2.3 Stationary Income Shocks: AR(2) Processes

Suppose the endowment process is autoregressive of order two. Specifically, assume that

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \epsilon_t,$$

(2.21)

where $\epsilon_t$ denotes an i.i.d. shock and the parameters $\rho_1$ and $\rho_2$ are such that this process is stationary, or mean reverting. By mean reversion, we mean that the conditional expectation $E_t y_{t+j}$ exists for
all \( j \geq 0 \) and that
\[
\lim_{t \to \infty} E_t y_{t+j} = 0
\]
for all \( t \geq 0 \) and for any initial condition \((y_{t-1}, y_{t-2})\). To establish the conditions under which this endowment process is mean reverting, let’s begin by writing it as a first-order vector autoregressive process. To this end, define
\[
Y_t = \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}.
\]
Then we can write the endowment process (2.21) as
\[
Y_{t+1} = R Y_t + \begin{bmatrix} \epsilon_{t+1} \\ 0 \end{bmatrix},
\]
with
\[
R \equiv \begin{bmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{bmatrix}.
\]
Then we have that
\[
E_t Y_{t+j} = R^j Y_t. \tag{2.22}
\]
Thus, the mean-reversion condition \( E_t Y_{t+j} = 0 \) is satisfied if and only if both eigenvalues of the matrix \( R \) lie within the unit circle. Therefore, we need that both roots of the characteristic equation
\[
\lambda^2 - \rho_1 \lambda - \rho_2 = 0
\]
be less than unity in absolute value. This latter requirement is satisfied as long as
\[
\gamma_2 < 1 - \gamma_1,
\]
Figure 2.2: Impulse Response of Endowment, AR(2) Process

\[ y_t \]

\(-5\) \hspace{1cm} 0 \hspace{1cm} 5 \hspace{1cm} 10 \hspace{1cm} 15 \hspace{1cm} 20 \hspace{1cm} \text{periods after shock}

\[ \gamma_2 < 1 + \gamma_1, \]

and

\[ \gamma_2 > -1 \]

(see, for example, Zellner, 1971, page 196).

Figure 2.2 presents an impulse response of output to a unit innovation in period 0 when \( \rho_1 > 1 \) and \( \rho_2 < 0 \). In this case, the impulse response is hump-shaped, that is, the peak output response occurs several periods after the shock occurs. (In the figure the peak response is reached three period after the shock.) The case of a hump-shaped path for income is of particular interest because it implies that the current level of output may rise by less than permanent income, that is, the change in \( y_t \) may be less than the change in \( y_t^p \). In this case, the trade balance and the current account would deteriorate in response to an increase in output, bringing the model closer to the
data.

To establish whether there exist parameterizations of the AR(2) endowment process for which the trade balance and the current account respond countercyclically to an endowment shock, let us calculate permanent income. To this end multiply both sides of (2.22) by \((1 + r)^{-j}\) to obtain

\[
\left(\frac{1}{1+r}\right)^j E_t Y_{t+j} = \left(\frac{1}{1+r} R\right)^j Y_t.
\]

Summing this expression we have

\[
\sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j E_t Y_{t+j} = \sum_{j=0}^{\infty} \left(\frac{1}{1+r} R\right)^j Y_t = (I - \frac{1}{1+r} R)^{-1} Y_t = \sum_{j=0}^{\infty} \frac{1}{1+r} \left[ \begin{array}{cc} 1 + r & \rho_2 \\ 1 & 1 + r - \rho_1 \end{array} \right] Y_t.
\]

Now multiplying the first row of this vector by \(r/(1 + r)\), we find permanent income as

\[
y_t^p = \frac{r}{1 + r} E_t \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j} = \frac{r \left[ (1 + r) y_t + \rho_2 y_{t-1} \right]}{(1 + r - \rho_1)(1 + r) - \rho_2}.
\]

Using this expression to replace permanent income in equations (2.11), (2.14), and (2.15) yields the following expressions for consumption, the trade balance, and the current account.

\[
c_t = \frac{r ((1 + r) y_t + \rho_2 y_{t-1})}{(1 + r - \rho_1)(1 + r) - \rho_2 - rd_{t-1}}.
\]
\[ tb_t = y_t - \frac{r ((1 + r)y_t + \rho_2 y_{t-1})}{(1 + r - \rho_1)(1 + r) - \rho_2} + rd_{t-1}, \]

and

\[ ca_t = y_t - \frac{r ((1 + r)y_t + \rho_2 y_{t-1})}{(1 + r - \rho_1)(1 + r) - \rho_2}. \]

When \( \rho_2 = 0 \), that is, when the endowment follows an AR(1) process, these expressions collapse to (2.17), (2.18), and (2.19), respectively. In this case, as we established earlier, there is no chance for an innovation in the endowment to generate a countercyclical response either in the trade balance or in the current account. But when \( \rho_2 \neq 0 \), the dynamic properties of the model can change substantially. To see this, assume that in period \( t \) the economy experiences an unanticipated unit increase in the endowment. To find out whether the trade balance and the current account deteriorate on impact, we have to find out by how much permanent income increases. Equation (2.23) implies that the change in permanent income is equal to \( \frac{r(1+r)}{(1+r-\rho_1)(1+r)-\rho_2} \). Thus permanent income will increase by more than current income if \( r(1+r) > (1+r-\rho_1)(1+r) - \rho_2 > 0 \) (the last inequality is needed for consumption to increase in response to a positive income innovation). Because \( r = \beta^{-1} - 1 > 0 \), and because \( \rho_2 < 1 - \rho_1 \), it follows that a necessary condition for permanent income to increase by more than current income is that \( \rho_1 > 1 \) and \( \rho_2 < 0 \).

The requirement that \( \rho_1 > 1 \) is intuitive because it implies that the impulse response of output is hump-shaped. The requirement that \( \rho_2 < 0 \) ensures that the endowment process is stationary given that \( \rho_1 \) is greater than one. But simply requiring \( \rho_1 > 1 \) and \( \rho_2 < 0 \) does not guarantee a countercyclical response of the trade balance. The necessary and sufficient conditions for this are the restrictions given above and \( \rho_2 > (1 + r)(1 - \rho_1) \). These conditions put a restriction on how negative \( \rho_2 \) can be for a given value of \( \rho_1 > 1 \). The reason why \( \rho_2 \) cannot become too negative is that smaller values of \( \rho_2 \) reduce the response of permanent income to a positive innovation in the endowment. Specifically, the more negative is \( \rho_2 \) the smaller is the hump and the persistence of
the endowment response. We note that for sufficiently large negative values of \( \rho_2 \) (i.e., values close to \(-1\)) the impulse response of the endowment process starts to oscillate. In this case a positive output shock will first drive the endowment up but then down below its average value.

The AR(2) example shows that a higher-order autoregressive endowment process allows the model to predict a countercyclical trade balance response. Notice that in the example the endowment process is stationary, or mean reverting. Hence the example demonstrates that the counterfactual prediction of the model of a procyclical trade balance is a consequence of assuming an AR(1) structure for the endowment process rather than of assuming that the endowment process is stationary. In the next section we show that allowing the endowment to follow a nonstationary process is an alternative mechanism to induce the model to predict that permanent income increases by more than current income.

### 2.4 Nonstationary Income Shocks

Suppose now that the rate of change of output, rather than its level, displays mean reversion. Specifically, let

\[
\Delta y_t = y_t - y_{t-1}
\]  

(2.24)
denote the change in endowment between periods \( t-1 \) and \( t \), and suppose that \( \Delta y_t \) evolves according to the autoregressive process

\[
\Delta y_t = \rho \Delta y_{t-1} + \epsilon_t,
\]

(2.25)

where \( \epsilon_t \) is an i.i.d. shock with mean zero and variance \( \sigma^2_\epsilon \), and \( \rho \in [0, 1) \) is a constant parameter. According to this process specification, the level of income is nonstationary, in the sense that a positive output shock (\( \epsilon_t > 0 \)) produces an increasing expected path of output leading to a permanently higher long-run value. To see this, suppose that \( \Delta y_{t-1} = 0 \) and that \( \epsilon_t > 0 \). Then,
the expected path of output conditional on information available in period $t$ is given by

$$y_t = y_{t-1} + \epsilon_t,$$

$$E_t y_{t+1} = y_{t-1} + (1 + \rho) \epsilon_t,$$

$$E_0 y_{t+2} = y_{t-1} + (1 + \rho + \rho^2) \epsilon_t,$$

$$\ldots$$

$$E_t y_\infty = y_{t-1} + \frac{1}{1 - \rho} \epsilon_t.$$

It follows that

$$y_t < E_t y_{t+1} < E_t y_{t+2} < \cdots < E_t y_\infty.$$

Faced with such an increasing income profile, consumption-smoothing households in period $t$ have an incentive to borrow against future income, thereby producing a countercyclical response in the current account. Figure 2.3 provides a graphical representation of this intuition.
Besides the endowment process, all other aspects of the model economy are as before. In particular, we continue to assume that preferences are described by the utility function (2.7). Therefore, in equilibrium the current account is given by the difference between current income and permanent income, $y_t - y_t^p$. Combining definitions (2.10) and (2.24) we can write

$$y_t - y_t^p = -E_t \sum_{j=1}^{\infty} \frac{\Delta y_{t+j}}{(1 + r)^j}. \quad (2.26)$$

Note that in deriving this expression, we have not used the assumed stochastic properties of output. Therefore, the above expression is valid regardless of whether output follows a stationary or a nonstationary process.

Now combining equations (2.14) and (2.26) yields the following expression for the current account

$$ca_t = -E_t \sum_{j=1}^{\infty} \frac{\Delta y_{t+j}}{(1 + r)^j}. \quad (2.27)$$

This formula states that the current account equals the present discounted value of future expected income decreases. If output is expected to fall over time, then the current account is positive. In this case, households save part of their current income to allow for a smooth path of future consumption. The opposite happens if income is expected to increase over time. In this case, the country runs a current account deficit to finance present spending.

The assumed autoregressive structure for the change in the endowment given in equation (2.25), implies that

$$E_t \Delta y_{t+j} = \rho^j \Delta y_t.$$ 

Using this result to eliminate $E_t \Delta y_{t+j}$ from equation (2.27), we can write the equilibrium current account as

$$ca_t = \frac{-\rho}{1 + r - \rho} \Delta y_t.$$ 

According to this formula, the current account deteriorates in response to a positive innovation in output. This prediction is in line with the cross-country time series evidence presented in chapter 1. We note that the countercyclicality of the current account in the model with nonstationary shocks depends crucially on output changes being positively serially correlated, or $\rho > 0$. When $\rho$ is zero or negative, the current account ceases to be countercyclical. The intuition behind this result is clear. For an unexpected increase in income to induce an increase in consumption larger than the increase in income itself, it is necessary that future income be expected to be higher than current income, which happens only if $\Delta y_t$ is positively serially correlated.

Are implied changes in consumption more or less volatile than changes in output? This question is important because, as we saw in chapter 1, developing countries are characterized by consumption growth being more volatile than output growth. Formally, letting $\sigma_{\Delta c}$ and $\sigma_{\Delta y}$ denote the standard deviations of $\Delta c_t \equiv c_t - c_{t-1}$ and $\Delta y_t$, respectively, we wish to find out conditions under which $\sigma_{\Delta c}$ can be higher than $\sigma_{\Delta y}$ in equilibrium.\footnote{Strictly speaking, this exercise is not comparable to the data displayed in chapter 1, because here we are analyzing changes in the levels of consumption and output (i.e., $\Delta c_t$ and $\Delta y_t$), whereas in chapter 1 we reported statistics pertaining to the growth rates of consumption and output (i.e., $\Delta c_t/c_{t-1}$ and $\Delta y_t/y_{t-1}$).} We start with the definition of the current account

$$ca_t = y_t - c_t - r d_{t-1}.$$ 

Taking differences, we obtain

$$ca_t - ca_{t-1} = \Delta y_t - \Delta c_t - r (d_{t-1} - d_{t-2}).$$
Noting that $d_{t-1} - d_{t-2} = -ca_{t-1}$ and solving for $\Delta c_t$, we obtain:

\[
\begin{align*}
\Delta c_t &= \Delta y_t - ca_t + (1 + r)ca_{t-1} \\
&= \Delta y_t + \frac{\rho}{1 + r - \rho} \Delta y_t - \frac{\rho(1 + r)}{1 + r - \rho} \Delta y_{t-1} \\
&= \frac{1 + r}{1 + r - \rho} \Delta y_t - \frac{\rho(1 + r)}{1 + r - \rho} \Delta y_{t-1} \\
&= \frac{1 + r}{1 + r - \rho} \epsilon_t.
\end{align*}
\]

Thus, the change in consumption is a white noise. This result is not surprising nor is it a consequence of assumed stochastic process for the endowment. Rather it is an implication of the Euler equation (2.8) which states that the level of consumption is a random walk. However, the precise value of the coefficient multiplying $\epsilon_t$ in equation (2.28) does depend on the assumed specification of the endowment process. This coefficient is important for the purpose of the present analysis because it governs the magnitude of the standard deviation of consumption changes. Specifically, equation (2.28) implies that the standard deviation of consumption changes is given by

\[
\sigma_{\Delta c} = \frac{1 + r}{1 + r - \rho} \epsilon_t.
\]

In turn, equation (2.25) implies that $\sigma_{\Delta y} \sqrt{1 - \rho^2} = \epsilon_t$. Then, we can write the ratio of the standard deviation of consumption to the standard deviation of output as

\[
\frac{\sigma_{\Delta c}}{\sigma_{\Delta y}} = \left[ \frac{1 + r}{1 + r - \rho} \right] \sqrt{1 - \rho^2}.
\]

This expression suggests that the persistence of output changes, embodied in the parameter $\rho$, is a key determinant of the relative volatility of consumption changes. When $\rho = 0$, consumption and output changes are equally volatile. This result is intuitive. When $\rho = 0$, we have that
\( y_{t+j} = y_t + \epsilon_{t+1} \ldots \epsilon_{t+j}, \) so \( E_t y_{t+j} = y_t. \) That is, when \( \rho = 0 \) current income is equal to permanent income. Since in the present model the current account equals the difference between current and permanent income, we have that when \( \rho = 0 \) the current account is nil at all times. Households do not need to save or borrow in response to changes in current income because future income is expected to change by exactly the same amount. Now if the current account is zero at all times, consumption must move in tandem with output at all times, implying that consumption and output changes must be equally volatile.

For positive values of \( \rho, \) consumption changes can become more volatile than output changes. To see this, note that the right hand side of (2.29) is increasing in \( \rho \) at \( \rho = 0. \) Since consumption and output changes are equally volatile at \( \rho = 0, \) it follows that there are values of \( \rho \) in the interval \( (0, 1) \) for which the volatility of consumption changes is higher than that of income changes. This property ceases to hold as \( \Delta y_t \) becomes highly persistent. This is because as \( \rho \to 1, \) the variance of \( \Delta y_t \) becomes infinitely large as changes in income become a random walk, whereas, as expression (2.28) shows, \( \Delta c_t \) follows an i.i.d. process with finite variance for all values of \( \rho \in [0, 1). \)

### 2.5 Testing the Intertemporal Approach to the Current Account

Hall (1978) was the first to explore the econometric implication of the simple model developed in this chapter. Specifically, Hall tested the prediction that consumption follows a random walk. Hall’s work motivated a large empirical literature devoted to testing the empirical relevance of the model described above. Campbell (1987), in particular, deduced and tested a number of theoretical restrictions on the equilibrium behavior of national savings. In the context of the open economy, Campbell’s restrictions are readily expressed in terms of the current account. Here we review these restrictions and their empirical validity.
The starting point is equation (2.27), which we reproduce for convenience

\[ ca_t = -\sum_{j=1}^{\infty} (1 + r)^{-j} E_t \Delta y_{t+j}. \]

Recall the intuition behind this expression. It states that the country borrows from the rest of the world (i.e., runs a current account deficit) when income is expected to grow in the future. Similarly, the country chooses to build its net foreign asset position (runs a current account surplus) when income is expected to decline in the future. In this case the country saves for a rainy day. It is important to notice that the derivation of this equation does not require the specification of a particular stochastic process for the endowment \( y_t \).

Consider now an empirical representation of the time series \( \Delta y_t \) and \( ca_t \). Define

\[ x_t = \begin{bmatrix} \Delta y_t \\ ca_t \end{bmatrix}. \]

Consider estimating the following vector autoregression (VAR) in \( x_t \):

\[ x_t = Dx_{t-1} + \epsilon_t. \]

As a first comment on this empirical strategy, we notice that the model is silent on whether \( x_t \) has a VAR representation of this form. An example in which such a representation exists is when \( \Delta y_t \) itself is a univariate AR(1) process like the one assumed in section 2.4.

Let \( H_t \) denote the information contained in the vector \( x_t \). Then, from the above VAR system, we have that the forecast of \( x_{t+j} \) given \( H_t \) is given by

\[ E_t[x_{t+j}|H_t] = D^j x_t. \]
It follows that

\[ \sum_{j=1}^{\infty} (1 + r)^{-j} E_t[\Delta y_{t+j} | H_t] = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left[ I - \frac{D}{1 + r} \right]^{-1} \frac{D}{1 + r} \left[ \begin{array}{c} \Delta y_t \\ ca_t \end{array} \right]. \]

Let

\[ F \equiv - \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left[ I - \frac{D}{1 + r} \right]^{-1} \frac{D}{1 + r}. \]

Now consider running separate regressions of the left- and right-hand sides of equation (2.27) onto the vector \( x_t \). Since \( x_t \) includes \( ca_t \) as one element, we obtain that the regression coefficient for the left-hand side regression is the vector \([0 \ 1]\). The regression coefficients of the right-hand side regression is \( F \). So the model implies the following restriction on the vector \( F \):

\[ F = [0 \ 1]. \]

Nason and Rogers (2006) perform an econometric test of this restriction. They estimate the VAR system using Canadian data on the current account and GDP net of investment and government spending. The estimation sample is 1963:Q1 to 1997:Q4. The VAR system that Nason and Rogers estimate includes 4 lags. In computing \( F \), they calibrate \( r \) at 3.7 percent per year. Their data strongly rejects the above cross-equation restriction of the model. The Wald statistic associated with the null hypothesis that \( F = [0 \ 1] \) is 16.1, with an asymptotic \( p \)-value of 0.04. This \( p \)-value means that if the null hypothesis was true, then the Wald statistic, which reflects the discrepancy of \( F \) from \([0 \ 1]\), would take a value of 16.1 or higher only 4 out of 100 times.

Consider now an additional testable cross-equation restriction on the theoretical model. From equation (2.27) it follows that

\[ E_t ca_{t+1} - (1 + r)ca_t - E_t \Delta y_{t+1} = 0. \] (2.30)
According to this expression, the variable \( c_{a_{t+1}} - (1 + r) c_t - \Delta y_{t+1} \) is unpredictable in period \( t \). In particular, if one runs a regression of this variable on current and past values of \( x_t \), all coefficients should be equal to zero.\(^2\) Nason and Rogers (2006) find that this hypothesis is rejected with a \( p \)-value of 0.06.

This restriction is not valid in a more general version of the model featuring private demand shocks. Consider, for instance, a variation of the model economy where the bliss point, \( \bar{c} \), is a random variable. Specifically, replace \( \bar{c} \) in equation (2.7) by \( \bar{c} + \mu_t \), where \( \bar{c} \) is still a constant, and \( \mu_t \) is an i.i.d. shock with mean zero. In this environment, equation (2.30) becomes

\[
E_t c_{a_{t+1}} - (1 + r) c_t - E_t \Delta y_{t+1} = \mu_t.
\]

Clearly, because in general \( \mu_t \) is correlated with \( c_t \), the orthogonality condition stating that \( c_{a_{t+1}} - (1 + r) c_t - \Delta y_{t+1} \) be orthogonal to variables dated \( t \) or earlier, will not hold. Nevertheless, in this case we have that \( c_{a_{t+1}} - (1 + r) c_t - \Delta y_{t+1} \) should be unpredictable given information available in period \( t - 1 \) or earlier.\(^3\) This orthogonality condition is also strongly rejected by the data. Nason and Rogers (2006) find that a test of the hypothesis that all coefficients are zero in a regression of \( c_{a_{t+1}} - (1 + r) c_t - \Delta y_{t+1} \) onto past values of \( x_t \) has a \( p \)-value of 0.01.

We conclude that the propagation mechanism invoked by the canonical intertemporal model of the current account does not provide a satisfactory account of the observed behavior of current account dynamics, regardless of whether the underlying endowment shock is stationary or nonstationary.

To bring closer together the observed and predicted behavior of the current account and other macroeconomic aggregates, in the following chapters, we will enrich the model’s sources of fluctu-
ations and propagation mechanism.
2.6 Exercises

Exercise 2.1 (Predicted Second Moments) In chapter 1, we showed that two empirical regularities that characterize emerging economies are the countercyclicality of the trade balance-to-output ratio and the fact that consumption growth appears to be more volatile than output growth. In this chapter, we developed a simple small open endowment economy and provided intuitive arguments suggesting that this economy fails to account for these two stylized facts. However, that model does not allow for closed form solutions of second moments of output growth, consumption growth, or the trade balance-to-output ratio. The goal of this assignment is to obtain these implied statistics numerically.

To this end, consider the following parameterization of the model developed in the present chapter:

\[ y_t - \overline{y} = \rho(y_{t-1} - \overline{y}) + \epsilon_t, \]

with \( \rho = 0.9, \overline{y} = 1, \) and \( \epsilon_t \) is distributed normally with mean 0 and standard deviation 0.03. Note that the parameter \( \overline{y}, \) which earlier in this chapter was implicitly assumed to be zero, represents the deterministic steady state of the output process. Assume further that \( r = 1/\beta - 1 = 0.1, d_{-1} = \overline{y}/2, \) and \( y_{-1} = \overline{y}. \)

1. Simulate the economy for 100 years.

2. Discard the first 50 years of artificial data to minimize the dependence of the results on initial conditions.

3. Compute the growth rates of output and consumption and the trade balance-to-output ratio.

4. Compute the sample standard deviations of output growth and consumption growth and the correlation between output growth and the trade balance-to-output ratio. Here, we denote these
three statistics $\sigma_{gy}$, $\sigma_{gc}$, and $\rho_{gy, tby}$, respectively.

5. Replicate steps 1 to 4 1000 times. For each replication, keep record of $\sigma_{gy}$, $\sigma_{gc}$, and $\rho_{gy, tby}$.

6. Report the average of $\sigma_{gy}$, $\sigma_{gc}$, and $\rho_{gy, tby}$ over the 1000 replications.

7. Discuss your results.

Exercise 2.2 (Free Disposal) Consider the small, open, endowment economy with stationary endowment shocks and quadratic preferences analyzed in this chapter. An important implicit assumption of that model is the absence of free disposal of goods. Consider now a variation of the model that allows for free disposal. Specifically, assume that the sequential budget constraint faced by households is of the form

$$d_t = (1 + r)d_{t-1} + y_t - c_t - x_t,$$

where $x_t$ is an endogenous variable determined in period $t$ and subject only to the nonnegativity constraint

$$x_t \geq 0.$$

All other aspects of the model are as presented in the main text. Characterize as many differences as you can between the equilibrium dynamics of the models with and without free disposal.

Exercise 2.3 (An Economy with Endogenous Labor Supply) Consider a small open economy populated by a large number of households with preferences described by the utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, h_t),$$

where $U$ is a period utility function given by

$$U(c, h) = -\frac{1}{2} [(c - \tau)^2 + h^2],$$
where $\bar{c} > 0$ is a satiation point. The household’s budget constraint is given by

$$d_t = (1 + r)d_{t-1} + c_t - y_t,$$

where $d_t$ denotes real debt acquired in period $t$ and due in period $t + 1$, $r > 0$ denotes the world interest rate. To avoid inessential dynamics, we impose

$$\beta(1 + r) = 1.$$

The variable $y_t$ denotes output, which is assumed to be produced by the linear technology

$$y_t = Ah_t.$$

Households are also subject to the no-Ponzi-Game constraint

$$\lim_{j \to \infty} E_t d_{t+j}/(1+r)^j \leq 0.$$

1. Compute the equilibrium laws of motion of consumption, debt, the trade balance, and the current account.

2. Assume that in period 0, unexpectedly, the productivity parameter $A$ increases permanently to $A' > A$. Establish the effect of this shock on output, consumption, the trade balance, the current account, and the stock of debt.

**Exercise 2.4 (An Open Economy With Habit Formation)** Section 2.2 characterizes the equilibrium dynamics of a small open economy with time separable preferences driven by stationary endowment shocks. It shows that a positive endowment shock induces an improvement in the trade balance on impact. This prediction, we argued, was at odds with the empirical evidence presented in chapter 1. Consider now a variant of the aforementioned model economy in which the representative
consumer has time nonseparable preferences described by the utility function

\[-\frac{1}{2} E_t \sum_{j=0}^{\infty} \beta^j [c_{t+j} - \alpha \bar{c}_{t+j-1} - \bar{c}]^2; \quad t \geq 0,\]

where \(c_t\) denotes consumption in period \(t\), \(\bar{c}_t\) denotes the cross-sectional average level of consumption in period \(t\), \(E_t\) denotes the mathematical expectations operator conditional on information available in period \(t\), and \(\beta \in (0, 1), \alpha \in (-1, 1), \) and \(\bar{c} > 0\) are parameters. The case \(\alpha = 0\) corresponds to time separable preferences, which is studied in the main text. Households take as given the evolution of \(\bar{c}_t\). Households can borrow and lend in international financial markets at the constant interest rate \(r\). For simplicity, assume that \((1 + r)\beta\) equals unity. In addition, each period \(t = 0, 1, \ldots\) the household is endowed with an exogenous and stochastic amount of goods \(y_t\). The endowment stream follows an AR(1) process of the form

\[y_{t+1} = \rho y_t + \epsilon_{t+1},\]

where \(\rho \in [0, 1)\) is a parameter and \(\epsilon_t\) is a mean-zero i.i.d. shock. Households are subject to the no-Ponzi-game constraint

\[\lim_{j \to \infty} \frac{E_t d_{t+j}}{(1 + r)^j} \leq 0,\]

where \(d_t\) denotes the representative household’s net debt position at date \(t\). At the beginning of period 0, the household inherits a stock of debt equal to \(d_{-1}\).

1. Derive the initial equilibrium response of consumption to a unit endowment shock in period 0.

2. Discuss conditions (i.e., parameter restrictions), if any, under which a positive output shock can lead to a deterioration of the trade balance.
Exercise 2.5 (Anticipated Endowment Shocks) Consider a small open endowment economy enjoying free capital mobility. Preferences are described by the utility function

\[-\frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t (c_t - \bar{r})^2,\]

with \(\beta \in (0, 1)\). Agents have access to an internationally traded bond paying the constant interest rate \(r^*\), satisfying \(\beta(1 + r^*) = 1\). The representative household starts period zero with an asset position \(b_{-1}\). Each period \(t \geq 0\), the household receives an endowment \(y_t\), which obeys the law of motion, \(y_t = \rho y_{t-1} + \epsilon_{t-1}\), where \(\epsilon_t\) is an i.i.d shock with mean zero and standard deviation \(\sigma_\epsilon\).

Notice that households know already in period \(t - 1\) the level of \(y_t\) with certainty.

1. Derive the equilibrium process of consumption and the current account.

2. Compute the correlation between the current account and output. Compare your result with the standard case in which \(y_t\) is known only in period \(t\).

Exercise 2.6 (Empirical Plausibility of an AR(2) Output Specification) The purpose of this exercise is to obtain econometric estimates of the AR(2) output process given in equation (2.21) and then check whether the estimated values of \(\rho_1\) and \(\rho_2\) satisfy the requirement for permanent income to increase by more than current income in response to an innovation in current income. The satisfaction of this condition guarantees a countercyclical response of the trade balance and the current account to output innovations in the model.

1. Download the quarterly data for chapter 1 posted online. For each country, extract GDP per capita at constant LCU. Denote this series \(\tilde{y}_t\).

2. For each country, obtain a log-quadratically detrended output series, denoted \(\hat{y}_t\), by running
the OLS regression

$$\ln \hat{y}_t = a_0 + a_1 t + a_2 t^2 + \hat{y}_t,$$

where $\hat{y}_t$ is the regression residual.

3. In the model, output is defined in levels. So, for each country, produce the transformed variable

$$y_t \equiv \exp(\hat{y}_t).$$

4. For each country, use the time series $y_t$ to estimate the AR(2) process

$$y_t = \rho_0 + \rho_1 y_{t-1} + \rho_2 y_{t-2} + \epsilon_t$$

by OLS.

5. Ignore the parameter $\rho_0$. Set the interest rate $r$ at 2 percent per quarter. Using the analysis of section 2.3, establish, for each country, whether the condition for permanent income to increase by more than current income in response to an innovation in current income is met. Present your results in the form of a table, with one row for country and columns displaying, in this order, $\rho_1$, $\rho_2$, and yes/no to indicate whether the condition is met or not. Discuss your findings.

6. Change the quarterly interest rate to 1 percent, and recalculate the table. What do you learn and what is the intuition behind your results?

7. Redo the exercise using the annual data for real GDP per capita at constant LCU used in chapter 1 and available online. Make sure to adjust the interest rate in accordance with the change of frequency. Discuss your results.
Exercise 2.7 (Expected Output Changes and Permanent Income) Equation (2.26) expresses the difference between current and permanent income, \( y_t - y_t^p \), as the present discounted value of future expected changes in the endowment. Present a step-by-step derivation of equation (2.26) starting from definitions (2.10) and (2.24). Comment on the cyclical properties of \( y_t - y_t^p \) depending on whether the level or the change of \( y_t \) follows an AR(1) process.

Exercise 2.8 (Impatience and the Current Account) Consider a small open endowment economy populated by a large number of identical consumers with preferences formulated by the utility function

\[
\sum_{t=0}^{\infty} \beta^t \ln(c_t - \bar{c}),
\]

with the usual notation, except that \( \bar{c} > 0 \) denotes a subsistence level of consumption. Consumers have access to the international debt market where the interest rate, denoted \( r \), is positive, constant, and satisfies

\[
\beta(1 + r) < 1.
\]

Consumers start period 0 with an outstanding debt, including interest, of \((1+r)d_{-1}\). It is forbidden to violate the constraint \( \lim_{t \to -\infty} (1+r)^{-j}d_{t+j} \leq 0 \). Each period, everybody receives a positive amount of consumption goods \( y > 0 \), which is nonstorable.

1. State the optimization problem of the representative consumer.

2. Derive the consumer’s optimality conditions.

3. Derive a maximum value of initial debt, \( d_{-1} \), beyond which an equilibrium cannot exist. Assume that \( d_{-1} \) is less than this threshold.

4. Characterize the steady state of this economy. In particular, calculate the steady-state values of consumption, debt, the trade balance, and the current account. Note that in this economy
the steady-state level of external debt is not history dependent. Comment on the factors
determining this property of the model.

5. Derive explicit formulas for the equilibrium dynamic paths of consumption, debt, the trade
balance, and the current account as functions of \( t, d_{-1}, r, \beta, \bar{c}, \) and \( y \).

6. Now assume that in period 0 the outstanding debt, \( d_{-1} \), is at its steady-state level, and that,
unexpectedly, all consumers receive a permanent increase in the endowment from \( y \) to \( y' > y \). Compute the initial response of all endogenous variables. Discuss your result, paying
particular attention to possible differences with the case \( \beta(1 + r) = 1 \).

7. Characterize the economy’s dynamics after period 0.

Exercise 2.9 (Global Approximation of Equilibrium Dynamics) This exercise is concerned
with numerically approximating the equilibrium dynamics of a small open endowment economy by
value-function iterations.

1. Consider an endowment, \( y_t \), following the AR(1) process

\[
y_t - 1 = \rho(y_{t-1} - 1) + \sigma \epsilon_t
\]

where \( \epsilon_t \) is an i.i.d. innovation with mean zero and unit variance, \( \rho \in [0, 1) \), and \( \sigma > 0 \).

Discretize this process by a two-state Markov process defined by the 2-by-1 state vector
\( Y \equiv [Y_1 \ Y_2]' \) and the 2-by-2 transition probability matrix \( \Pi \) with element \( (i, j) \) denoted \( \pi_{ij} \) and
given by \( \pi_{ij} \equiv \text{Prob}\{y_{t+1} = Y_j | y_t = Y_i \} \). To reduce the number of parameters of the Markov
process to two, impose the restrictions \( \pi_{11} = \pi_{22} = \pi \), \( Y_1 = 1 + \gamma \) and \( Y_2 = 1 - \gamma \). Pick \( \pi \)
and \( \gamma \) to match the variance and the serial correlation of \( y_t \). Express \( \pi \) and \( \gamma \) in terms of the
parameters defining the original AR(1) process.
2. Calculate the unconditional probability distribution of $Y$ (this is a 2-by-1 vector).

3. Assume that $\rho = 0.4$ and $\sigma = 0.05$. Evaluate the vector $Y$ and the matrix $\Pi$.

4. Now consider a small open economy populated by a large number of identical households with preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta \frac{c_t^{1-\sigma} - 1}{1-\sigma}.$$ 

Suppose that households face the sequential budget constraint

$$c_t + g + (1 + r)d_{t-1} = y_t + d_t,$$

where $c_t$ denotes consumption in period $t$, $d_t$ denotes one-period debt assumed in period $t$ and maturing in $t + 1$, $g$ denotes a constant level of domestic absorption that yields no utility to households (possibly wasteful government spending), and $r$ denotes the world interest rate, assumed to be constant and exogenous. Households are subject to the no-Ponzi-game constraint $\lim_{j \to \infty} (1 + r)^{-j}d_{t+j} \leq 0$. Express the household’s problem as a Bellman equation. To this end, drop time subscripts and use instead the notation $d = d_{t-1}$, $d' = d_t$, $y = y_t$ and $y' = y_{t+1}$ for all $t$. Denote the value function in $t$ by $v(y, d)$. [Here it suffices to use the notation $y$ and $y'$ because the endowment process is AR(1). Higher-order processes would require an extended notation.]

5. Let $\sigma = 2$, $r = 0.04$, $\beta = 0.954$, and $g = 0.2$. And assume that the endowment process follows the two-state Markov process given in item 3 above. Discretize the debt state, $d$, using 200 equally spaced points ranging from 15 to 19. Calculate the value function and the debt policy function by value function iteration (these are 2 vectors, each of order 400-by-1). Calculate also the policy functions of consumption, the trade balance, and the current account (each of these policy functions is a 400-by-1 vector). Calculate the transition probability matrix of the
state \((y, d)\) (this is a 400-by-400 matrix, whose rows all add up to unity; each row has only 2 nonzero entries).

6. Define the impulse response of the variable \(x_t\) to a one-standard-deviation increase in output as \(E[x_t|y_0 = Y_1] - E[x_t]\) for \(t = 0, 1, 2, \ldots\) (note that these expectations are unconditional with respect to debt; alternatively, we could have conditioned on some value of debt, but we are not pursuing this definition here). Make a figure with 4 subplots (in a 2-by-2 arrangement) showing the impulse responses of output, consumption, the trade balance, and debt for \(t = 0, 1, \ldots, 10\).

7. Plot the unconditional probability distribution of debt.

8. Finally, suppose that government spending, \(g\), increases from 0.2 to 0.22. Plot the resulting unconditional distribution of debt. For comparison superimpose the one corresponding to the baseline case \(g = 0.2\). Provide intuition for the differences you see.

Exercise 2.10 (Determinants of the World Interest Rate) Throughout this chapter, we have studied small open economies in which the world interest rate is given. This exercise aims at illustrating the forces determining this variable.

Consider a two-period world composed of a continuum of countries indexed by \(i \in [0, 1]\). Each country is populated by a large number of identical households with preferences given by

\[
\ln(c_1^i) + \ln(c_2^i)
\]

where \(c_1^i\) and \(c_2^i\) denote consumption of a perishable good in country \(i\) in periods 1 and 2, respectively. Households start period 1 with a nil net debt position. In period 1, they can borrow or lend in the international financial market via a debt instrument, denoted \(d_1^i\), that matures in period 2 and carries the interest rate \(r\). The interest rate \(r\) is exogenous to each country \(i\). In period 1, each
A household receives an endowment of goods $y_1^i = y_1 + \epsilon^i$, where $y_1$ is the world component of the endowment and $\epsilon^i$ is a country-specific component satisfying $\int_0^1 \epsilon^i di = 0$. In period 2, the endowment has no idiosyncratic component and is given by $y_2^i = y_2$. Finally, households are subject to a no-Ponzi-game constraint that forbids them to end period 2 with a positive debt position, that is, they are subject to the constraint $d_2^i \leq 0$, where $d_2^i$ denotes the debt assumed in period 2.

1. Write down and solve the household’s optimization problem in country $i$, given $r$.

2. Derive the equilibrium levels of the trade balance, the current account, and external debt in periods 1 and 2 in country $i$ given $r$.

3. Write down the world resource constraints in periods 1 and 2.

4. Derive the equilibrium level of the world interest rate, $r$.

5. Suppose now that output in period 1 in country $i$ increases by $x > 0$, that is, $\Delta y_1^i = x$. Derive the effect of this shock on the trade balance and the level of external debt in period 1 in country $i$ and on the world interest rate under the following two alternative cases:

   (a) A country-specific endowment shock, $\Delta y_1^i = \Delta \epsilon^i = x$ and $\Delta y_1 = 0$.

   (b) A world endowment shock, $\Delta y_1^i = \Delta y_1 = x$ and $\Delta \epsilon^i = 0$.

   Provide a discussion of your results.

Exercise 2.11 (Leontief Preferences Over Discounted Period Utilities) Consider a perfect-foresight small open economy populated by a large number of identical households with preferences described by the utility function

$$\min_{t \geq 0} \{ \beta^t c_t \},$$
where \( c_t \) denotes consumption in period \( t \), and \( \beta \in (0,1) \) is a parameter. Households have access to the international financial market, where they can borrow or lend at the constant interest rate \( r \). Assume that
\[
\beta(1 + r) = 1 + \gamma,
\]
where \( \gamma > 0 \) is a parameter. Households are endowed with a constant amount of consumption goods denoted \( y \) each period and start period 0 with a level of debt equal to \( d_{-1} > 0 \). Finally, households are subject to a no-Ponzi-game constraint of the form
\[
\lim_{t \to \infty} (1 + r)^{-t} d_t \leq 0,
\]
where \( d_t \) denotes one-period debt acquired in period \( t \) and maturing in \( t+1 \).

1. Formulate the household’s maximization problem.

2. Write down the complete set of optimality conditions.

3. Characterize the equilibrium paths of consumption and debt in this economy. In particular, express the equilibrium levels of \( c_t \) and \( d_t \), for \( t \geq 0 \), in terms of the structural parameters (possibly \( \beta, r, \gamma \), and \( y \)) and the initial condition \( d_{-1} \).

4. What is the equilibrium asymptotic growth rate of the economy’s net asset position? How does it compare to the equilibrium growth rate of consumption?

5. Suppose that in period 0 the economy unexpectedly experiences a permanent increase in the endowment from \( y \) to \( y + \Delta y \), with \( \Delta y > 0 \). Derive the impact response of the trade balance. Briefly discuss your result.

6. Characterize the equilibrium under the assumption that \( \gamma = 0 \).

Exercise 2.12 (An Economy With Labor and Technological Progress) Consider a small open economy inhabited by a large number of identical households with preferences described by the
utility function
\[ \sum_{t=0}^{\infty} \beta^t \ln \left( C_t - \frac{X_t}{2} h_t^2 \right), \]
where \( C_t \) denotes consumption in period \( t \), \( h_t \) denotes hours worked in period \( t \), \( \beta \in (0, 1) \) is a subjective discount factor, and \( X_t \) is an exogenous and deterministic factor governing household and market technological progress, which evolves according to the expression

\[ X_t = \mu X_{t-1}, \]
for all \( t \geq 0 \), where \( \mu > 1 \) is a parameter, and \( X_{-1} = 1 \). Households face the sequential budget constraint

\[ C_t + D_{t-1}(1 + r_{t-1}) = Y_t + D_t, \]
where \( D_t \) denotes the amount of one-period debt assumed in period \( t \), \( r_t \) denotes the interest rate in period \( t \), and \( Y_t \) denotes output in period \( t \), which is produced via the technology

\[ Y_t = X_t h_t. \]

Households are subject to the no-Ponzi-game constraint \( \lim_{j \to \infty} D_{t+j}/\prod_{s=0}^{j-1}(1 + r_{t+s}) \leq 0 \), for all \( t \). The country interest rate, \( r_t \), is the sum of a constant world interest rate, \( r^* \), and a country spread, denoted \( \rho \left( \bar{D}_t/X_t \right) \), where \( \bar{D}_t \) denotes that cross sectional average of \( D_t \). Assume the functional form \( \rho(x) = \gamma \left[ \exp(x - \bar{d}) - 1 \right] \), where \( \gamma, \bar{d} > 0 \) are parameters. Finally, assume that \( 1 + r^* = \mu/\beta \) and that \( (1 + r_{t-1})D_{-1}/X_{-1} = (1 + r^*)\bar{d} \).

1. State the household’s maximization problem.

2. Derive the optimality conditions associated with the household’s problem.

3. Derive the complete set of equilibrium conditions in stationary form by appropriately scaling
variables that display perpetual growth in equilibrium.

4. Derive equilibrium paths of consumption, hours, the trade balance, the current account, and external debt. Provide intuition.
In this chapter, we introduce capital accumulation in the open economy of chapter 2. The purpose of introducing physical capital in the model is twofold. First, an important result derived in the previous chapter is that for the most commonly used stationary specifications of the endowment shock process, namely AR(1) specifications, the simple endowment economy model fails to predict the observed countercyclicality of the trade balance and the current account documented in chapter 1. Here, we show that allowing for capital accumulation can contribute to mending this problem. The reason is that in the augmented model the investment share can be procyclical, which allows for the possibility that the trade balance become countercyclical. Second, the assumption that output is an exogenously given stochastic process, which was maintained throughout the previous chapter, is unsatisfactory if the goal is to understand observed business cycles. For output is perhaps the main variable any theory of the business cycle should aim to explain. In this chapter we provide a partial remedy to this problem by assuming that output is produced with physical capital, which, in turn, is an endogenous variable.
3.1 The Basic Framework

Consider a small open economy populated by a large number of infinitely lived households with preferences described by the utility function

\[ \sum_{t=0}^{\infty} \beta^t U(c_t), \]  

(3.1)

where \( c_t \) denotes consumption, \( \beta \in (0, 1) \) denotes the subjective discount factor, and \( U \) denotes the period utility function, assumed to be increasing and concave. Each period, households face the budget constraint

\[ c_t + i_t + (1 + r)d_{t-1} = y_t + d_t, \]  

(3.2)

The left-hand side displays the uses of wealth, given by purchases of consumption goods, purchases of investment goods, denoted \( i_t \), and payment of principal and interest on debt obligations maturing in \( t \), denoted \( (1 + r)d_{t-1} \). The right-hand side displays the sources of wealth, given by output, denoted \( y_t \), and the acquisition of debt maturing in \( t + 1 \), denoted \( d_t \). Output is produced with physical capital, denoted \( k_t \), using the technology

\[ y_t = A_t F(k_t), \]  

(3.3)

where \( A_t \) is an exogenous and deterministic productivity factor and \( F \) is an increasing and concave production function satisfying the Inada conditions. and \( F \) Capital evolves according to the law of motion

\[ k_{t+1} = k_t + i_t, \]  

(3.4)

For the sake of simplicity, we assume that capital does not depreciate. In later chapters, we relax both the assumption of no depreciation and the assumption of deterministic productivity.
Finally, households are subject to the no-Ponzi-game constraint

$$\lim_{j \to \infty} \frac{d_{t+j}}{(1+r)^j} \leq 0,$$

(3.5)

The Lagrangian associated with the household’s problem is

$$\max_{\{c_t, k_{t+1}, d_t\}} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ U(c_t) + \lambda_t \left[ A_t F(k_t) + d_t - c_t - (k_{t+1} - k_t) - (1+r)d_{t-1} \right] \right\}.$$

The first-order conditions corresponding to this problem are

$$U'(c_t) = \lambda_t,$$

(3.6)

$$\lambda_t = \beta(1+r)\lambda_{t+1},$$

(3.7)

$$\lambda_t = \beta\lambda_{t+1}[A_{t+1} F'(k_{t+1}) + 1],$$

and

$$A_t F(k_t) + d_t = c_t + k_{t+1} - k_t + (1+r)d_{t-1}.$$

Household optimization implies that the borrowing constraint holds with equality

$$\lim_{t \to \infty} \frac{d_t}{(1+r)^t} = 0.$$

As in chapter 2, we assume that

$$\beta(1+r) = 1,$$

to avoid inessential long-run dynamics. This assumption together with the first two of the above
optimality conditions implies that consumption is constant over time,

$$c_{t+1} = c_t; \quad \forall t \geq 0.$$  \hspace{1cm} (3.8)

Note that this result does not require the assumption of quadratic preferences as in the previous chapter. The reason is that in the present model there is no uncertainty. The result $c_t = E_t c_{t+1}$ does not follow for general preferences in the stochastic case.

Using this expression, the optimality conditions can be reduced to the following two expressions

$$r = A_{t+1} F'(k_{t+1})$$  \hspace{1cm} (3.9)

and

$$c_t + rd_{t-1} = \frac{r}{1+r} \sum_{j=0}^{\infty} \frac{A_{t+j} F(k_{t+j}) - (k_{t+j+1} - k_{t+j})}{(1+r)^j},$$  \hspace{1cm} (3.10)

for $t \geq 0$.

Equilibrium condition (3.9) states that households invest in physical capital in period $t$ until the expected marginal product of capital in period $t+1$ equals the rate of return on foreign debt. It follows from this equilibrium condition that next period’s level of physical capital, $k_{t+1}$, is an increasing function of the future expected level of productivity, $A_{t+1}$, and a decreasing function of the opportunity cost of holding physical capital, $r$. Formally,

$$k_{t+1} = \kappa \left( \frac{A_{t+1}}{r} \right); \quad \kappa' > 0.$$  \hspace{1cm} (3.11)

To obtain equilibrium condition (3.10), follow the same steps as in the derivation of its counterpart for the endowment economy, equation (2.9). The right-hand side of (3.10) is the household’s nonfinancial permanent income, $y_t^p$. It is a natural generalization of a similar expression obtained
in the endowment economy (see equation (2.10)). In the present environment, nonfinancial permanent income is given by a weighted average of present and future expected output net of investment expenditure. Thus, equilibrium condition (3.10) states that each period households allocate their nonfinancial permanent income to servicing the debt and to consumption.

A perfect-foresight equilibrium is a set of sequences \( \{c_t, d_t, k_{t+1}\}_{t=0}^{\infty} \) satisfying (3.8), (3.10), and (3.11) for all \( t \geq 0 \), given the initial stock of physical capital, \( k_0 \), the initial net external debt position, \( d_{-1} \), and the deterministic sequence of productivity \( \{A_t\}_{t=0}^{\infty} \). To construct a perfect-foresight equilibrium proceed as follows. Given initial conditions \( k_0 \) and \( d_{-1} \) and a deterministic sequence \( \{A_t\} \), use equilibrium condition (3.11) to obtain \( \{k_t\}_{t=1}^{\infty} \). With the path for \( k_t \) in hand, evaluate condition (3.10) at \( t = 0 \) to obtain \( c_0 \). Then use condition (3.8) to find the time path of \( c_t \) for all \( t > 0 \). Finally, solve equation (3.10) evaluated at \( t > 0 \) for \( d_{t-1} \) to obtain the equilibrium sequence for \( d_t \) for any \( t \geq 0 \). We can then determine output from equation (3.3), investment from (3.4), and the marginal utility of consumption, \( \lambda_t \), from (3.6).

The trade balance is given by the difference between output and domestic absorption,

\[
tb_t = y_t - c_t - i_t,
\]

and the current account equals the trade balance plus net investment income,

\[
ca_t = tb_t - rd_{t-1}.
\]

### 3.2 A Steady-State Equilibrium

Suppose that up until period -1 inclusive, the technology factor \( A_t \) was constant and given by \( \bar{A} \). Moreover, assume that for all \( t < 0 \), agents expected \( A_{t+j} \) to be equal to \( \bar{A} \) for all \( j \geq 0 \) with certainty. This assumption gives rise to a steady state for \( t < 0 \) in which all endogenous variables
are constant. We indicate the steady state of a variable by placing a bar over it. The fact that $A_t$ is expected to be equal to $A$ at all times implies, by (3.11), that the capital stock is also constant and given by $k_t = \bar{k} \equiv \kappa(\bar{A}/r)$ for all $t \geq 0$. Similarly, output is constant and given by $y_t = \bar{y} \equiv \bar{A}F(\bar{k})$ for all $t < 0$. Because the capital stock is constant and because the depreciation rate of capital is assumed to be zero, we have by equation (3.4) that investment is also constant and equal to zero, $i_t \equiv \bar{i} = 0$ for all $t < 0$. By the Euler equation (3.8), consumption must also be constant for all $t < 0$. Equilibrium conditions (3.10) and (3.2) then imply that $d_t$ must also be constant for all $t < 0$, at a value denoted $\bar{d}$. Because the current account is defined as the change in the net international asset position ($ca_t = -(d_t) - (d_{t-1})$), we have that in the steady state the current account equals zero, $ca_t = \bar{ca} = 0$, for all $t < 0$. Finally, recalling that the current account is also defined as the sum of the trade balance and net investment income, $ca_t = tb_t - rd_{t-1}$, we have that in the steady state the trade balance must be constant and given by $tb_t = \bar{tb} \equiv r\bar{d}$ for all $t < 0$.

3.3 Adjustment To A Permanent Productivity Shock

Suppose now that in period 0, unexpectedly, the technology factor increases permanently from $\bar{A}$ to $A' > \bar{A}$, that is,

$$A_t = \begin{cases} \bar{A} & \text{for } t \leq -1 \\ A' > \bar{A} & \text{for } t \geq 0 \end{cases}.$$

Figure 3.1 presents the response of the model to this shock. Because $k_0$ and $d_{-1}$ were chosen in period $-1$, when households expected $A_0$ to be equal to $\bar{A}$, we have that $k_0 = \bar{k}$ and $d_{-1} = \bar{d}$. In period 0, investment experiences an increase that raises the level of capital available for production in period 1, $k_1$, from $\bar{k}$ to $k' \equiv \kappa(A'/r) > \kappa(\bar{A}/r) = \bar{k}$. The fact that productivity is constant after period zero implies, by (3.4) and (3.11), that starting in period 1 the capital stock is constant and investment is nil. Thus, $k_t = k'$ for $t \geq 1$, $i_0 = k' - \bar{k} > 0$, and $i_t = 0$, for $t \geq 1$. In words,
Figure 3.1: Adjustment to a Permanent Productivity Increase

\[ A_t, \quad A', \quad \bar{A}, \quad k_t, \quad k', \quad \bar{k}, \quad y_t, \quad i_t, \quad c_t, \quad d_t, \quad r, \quad \rho, \quad \rho_d, \quad \rho_f, \quad \rho_g, \quad \rho_h, \quad \rho_i, \quad \rho_j, \quad \rho_k, \quad \rho_l, \quad \rho_m, \quad \rho_n, \quad \rho_o, \quad \rho_p, \quad \rho_q, \quad \rho_r, \quad \rho_s, \quad \rho_t, \quad \rho_u, \quad \rho_v, \quad \rho_w, \quad \rho_x, \quad \rho_y, \quad \rho_z. \]
investment experiences a one-time increase in period 0 and the capital stock a once-and-for-all increase in period 1. Output increases in period 0, \( y_0 - y_{-1} = A' F(\bar{k}) - \bar{A} F(\bar{k}) > 0 \), and then again in period 1, \( y_1 - y_0 = A' F(k') - A' F(\bar{k}) > 0 \). The one-period lag in the full adjustment of output is due to the fact that it takes one period for investment to become productive capital. Starting in period 1, output is constant over time, \( y_t = A' F(k') \) for \( t \geq 1 \).

Plugging the equilibrium path for the capital stock into the intertemporal resource constraint (3.10) and evaluating that equation at \( t = 0 \) yields
\[
c_0 = -r d + \frac{r}{1+r} \left[ A' F(\bar{k}) - k' + \bar{k} \right] + \frac{1}{1+r} A' F(k').
\]
By equilibrium condition (3.8) \( c_t = c_0 \) for all \( t \geq 0 \). Let this constant level of consumption be denoted \( c' \), so that \( c_t = c' \) for all \( t \geq 0 \). We wish to show that \( c' > \bar{c} \), that is, that consumption increases permanently in response to the permanent positive productivity shock. To this end, rewrite (3.12) as
\[
c' = -r d + A' F(\bar{k}) + \frac{1}{1+r} \left\{ A' \left[ F(k') - F(\bar{k}) \right] - r \left[ k' - \bar{k} \right] \right\}.
\]
Use equation (3.9) to replace \( r \) for \( A' F'(k') \) in the expression within curly brackets to obtain
\[
c' = -r d + A' F(\bar{k}) + \frac{1}{1+r} \left\{ A' \left[ F(k') - F(\bar{k}) \right] - A' F'(k') \left[ k' - \bar{k} \right] \right\}.
\]
Because \( F \) is assumed to be strictly concave and \( k' > \bar{k} \), we have that \( \frac{F(k') - F(\bar{k})}{k' - \bar{k}} > F'(k') \). This means that the expression within curly brackets is strictly positive. Therefore, we have that
\[
c' > -r d + A' F(\bar{k}) > -r d + \bar{A} F(\bar{k}) = \bar{c}.
\]
This establishes that consumption experiences a once-and-for-all increase in period 0.\footnote{We thank Alberto Felettigh for providing this proof. An alternative demonstration of this result is as follows: Consider the following suboptimal paths for consumption and investment: $c^*_t = A'F(k) - rd$ and $i^*_t = 0$ for all $t \geq 0$. Clearly, because $A' > \bar{A}$, the consumption path $c^*_t$ is strictly preferred to the pre-shock path, given by $c \equiv \bar{A}F(k) - rd$. To show that the proposed allocation is feasible, let us plug the consumption and investment paths $c^*_t$ and $i^*_t$ into the sequential budget constraint (3.2) to obtain the sequence of asset positions $d^*_t = \bar{d}$ for all $t \geq 0$. Obviously, $\lim_{t \to \infty} d^*_t/(1+r)^t = 0$, so the proposed suboptimal allocation satisfies the no-Ponzi-game condition (3.5). We have established the existence of a feasible consumption path that is strictly preferred to the pre-shock consumption allocation. It follows that the optimal consumption path must also be strictly preferred to the pre-shock consumption path. This result together with the fact that, from equilibrium condition (3.8), the optimal consumption path is constant starting in period 0, implies that consumption must experience a permanent, once-and-for-all increase in period 0.}

Indeed, consumption initially increases by more than output. To see this use the above inequality and the definition of the steady state to write

$$c_0 - c_{-1} \equiv c' - \bar{c}$$

$$= c' - [\bar{A}F(k) - rd]$$

$$> [A'F(k) - rd] - [\bar{A}F(k) - rd]$$

$$= A'F(k) - \bar{A}F(k)$$

$$= y_0 - y_{-1}.$$ 

The inequality follows from the first inequality in (3.13). The initial overreaction of consumption is due to the fact that output continues to grow after period 0. So, from the perspective of period 0, households observe an increasing path of income over time ($y_t - y_0 = A'F(k') - A'F(k) > 0$ for all $t > 0$). As a consequence, households borrow against future income to finance current consumption. This result resembles what happens in the endowment economy with AR(2) stationary endowment shocks (section 2.3) or with AR(1) nonstationary endowment shocks (section 2.4).

The initial increase in domestic absorption causes the trade balance to deteriorate. To see this, recall that $tb_t = y_t - c_t - i_t$, which implies that $tb_0 - tb_{-1} = (y_0 - y_{-1}) - (c_0 - c_{-1}) - (i_0 - i_{-1})$. We have already shown that $(y_0 - y_{-1}) - (c_0 - c_{-1}) < 0$ and that $i_0 - i_{-1} > 0$. It therefore follows

$$\text{The initial overreaction of consumption is due to the fact that output continues to grow after period 0. So, from the perspective of period 0, households observe an increasing path of income over time ($y_t - y_0 = A'F(k') - A'F(k) > 0$ for all $t > 0$). As a consequence, households borrow against future income to finance current consumption. This result resembles what happens in the endowment economy with AR(2) stationary endowment shocks (section 2.3) or with AR(1) nonstationary endowment shocks (section 2.4).

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that $tb_0 - tb_{-1} < 0$. This result is significantly different from the one obtained in the endowment economy studied in chapter 2 in which a once-and-for-all increase in the endowment leaves the trade balance unchanged.

The trade balance is constant from period 1 onward. To see this, recall that $tb_t = y_t - c_t - i_t$ and that $y_t$, $c_t$, and $i_t$ are all constant from period 1 on. Specifically, we have that $tb_t = tb' \equiv A'F(k') - c'$ for all $t \geq 1$.

The equilibrium stock of external debt, $dt$, is constant starting in period 0. To see this, use the definition of the trade balance and equation (3.2) to write the evolution of debt as $dt = (1 + r)dt_{-1} - tb'$, for $t \geq 1$. Since $r > 0$, it follows that unless $dt = dt_{-1} = tb'/r$, this difference equation will yield a path of debt that grows or falls asymptotically at the rate $r$, violating the transversality condition. The fact that net foreign debt is constant from period 1 on implies that the equilibrium current account is zero, $ca_t = 0$, for all $t \geq 1$.

We next show that the current account deteriorates in period 0, that is, $\Delta ca_0 < 0$. This follows immediately from the definition of the current account, $ca_0 = tb_0 - rd$, and the fact, as we just established, that the trade balance deteriorates in period 0. Indeed, because the current account is nil in the pre-shock steady state ($ca_t = 0$ for $t < 0$), we have that the level of the current account is negative in period 0, $ca_0 < 0$. In turn, this result and the identity $ca_0 = (-d_0) - (-d_{-1})$ together imply that net foreign debt must rise in period zero, that is, $d_0 > \bar{d}$. To service this elevated level of debt the trade balance must increase from period 1 onwards above and beyond the level it had in period $tb_{-1}$ and earlier. That is, the permanent increase in the technology shock first leads to a deterioration of the trade balance and then to an improvement in the trade balance above its pre-shock level.

Let’s take stock of the results obtained thus far. We started with the simple small open economy of chapter 2 and introduced a single modification, namely, capital accumulation. We then showed that the modified model produces very different predictions regarding the initial behavior of the
trade balance in response to a positive permanent increase in productivity. In the present economy, such a shock causes the trade balance to initially deteriorate, whereas in the endowment economy a permanent output shock leaves the trade balance unchanged. What is behind the novel predictions of the present model? An important factor that contributes to causing an initial deterioration of the trade balance is the combination of a demand for goods for investment purposes and a persistent productivity shock. This factor has two implications. One direct implication is that because the positive productivity shock is expected to last investment in physical capital increases. The fact that investment rises in response to the permanent increase in productivity deteriorates the trade balance. This channel is closed in the endowment economy of the previous chapter because investment was by assumption always equal to zero. The second implication is less direct. The combination of capital accumulation and a persistent increase in productivity implies that output increases by more the long run than in period 0. This is so because in the period the productivity shock occurs the capital stock has not yet adjusted and output only increases by the increase in productivity. But in later periods output is higher due to both a higher level of productivity and a higher level of physical capital. Consumption-smoothing households adjust consumption in period 0 taking into account the entire future path of output. Faced with an upward sloping time path of income, households increase consumption in period 0 by more than the increase in output in that period. In the simple endowment economy a once-and-for-all increase in the endowment did not give rise to an upward sloping path of income and hence the consumption response did not exceed the output response in period 0.

The size of the increase in investment and hence the size of the implied decline in the trade balance in response to a positive productivity shock depends on the assumed absence of capital adjustment costs. Note that in response to the increase in future expected productivity, the entire adjustment in investment occurs in period zero. Indeed, investment falls to zero in period 1 and remains nil thereafter. In the presence of costs of adjusting the stock of capital, investment spending
is spread over a number of periods, dampening the increase in domestic absorption in the period the shock occurs. The fact that the productivity shock leads to a time path of income that is increasing is the result of the productivity shock being permanent. To highlight the importance of these two assumptions, namely, absence of adjustment costs, and permanence of the productivity shock, in generating a deterioration of the trade balance in response to a positive productivity shock, in the next two sections we analyze separately an economy with purely temporary productivity shocks and an economy with capital adjustment costs.

### 3.4 Adjustment to Temporary Productivity Shocks

To stress the importance of persistence in productivity movements in inducing a deterioration of the trade balance in response to a positive output shock, it is worth analyzing the effect of a purely temporary shock. Specifically, suppose that up until period -1 inclusive the productivity factor $A_t$ was constant and equal to $\bar{A}$. Suppose also that in period -1 people assigned a zero probability to the event that $A_0$ would be different from $\bar{A}$. In period 0, however, a zero probability event happens, namely, $A_0 = A' > \bar{A}$. Furthermore, suppose that everybody correctly expects the productivity shock to be purely temporary. That is, everybody expects $A_t = \bar{A}$ for all $t > 0$. In this case, equation (3.9) implies that the capital stock, and therefore also investment, are unaffected by the productivity shock. That is, $k_t = \bar{k}$ for all $t \geq 0$, where $\bar{k}$ is the level of capital inherited in period 0. This is intuitive. The productivity of capital unexpectedly increases in period zero. As a result, households would like to have more capital in that period. But $k_0$ is fixed in period zero. Investment in period zero can only increase the future stock of capital. At the same time agents have no incentives to have a higher capital stock in the future, because its productivity is expected to go back down to its historic level $\bar{A}$ right after period 0.

The positive productivity shock in period zero does produce an increase in output in that period,
from $\bar{A}F(\bar{k})$ to $A'F(\bar{k})$. That is

$$y_0 = y_{-1} + (A' - \bar{A})F(\bar{k}),$$

where $y_{-1} \equiv \bar{A}F(\bar{k})$ is the pre-shock level of output. This output increase induces higher consumption. Evaluating equation (3.10) for $t = 0$, recalling that $c_{-1} = -rd + \bar{A}F(\bar{k})$, and that $d_{-1} = d$, we have that

$$c_0 = c_{-1} + \frac{r}{1 + r}(y_0 - y_{-1}).$$

Basically, households invest the entire increase in output in the international financial market and increase consumption by the interest flow associated with that financial investment.

Combining the above two expressions and recalling that investment is unaffected by the temporary shock, we get that the trade balance in period 0 is given by

$$tb_0 - tb_{-1} = (y_0 - y_{-1}) - (c_0 - c_{-1}) - (i_0 - i_{-1}) = \frac{1}{1 + r}(A' - \bar{A})F(\bar{k}) > 0.$$}

This expression shows that the trade balance improves on impact. The reason for this counterfactual prediction is simple: Firms have no incentive to invest, as the increase in the productivity of capital is short lived. At the same time, consumers save most of the purely temporary increase in income in order to smooth consumption over time. As a consequence, domestic absorption increases but by less than the increase in output.

Comparing the results obtained under the two polar cases of permanent and purely temporary productivity shocks, we can derive the following principle:

**Principle I:** The more persistent are productivity shocks, the more likely is an initial deterioration of the trade balance.

We will analyze this principle in more detail in chapters 4 and 5, in the context of models featuring a more flexible notion of persistence.
3.5 Capital Adjustment Costs

Consider now an economy identical to the one analyzed thus far, except that now changes in the stock of capital come at a cost. Capital adjustment costs—in a variety of forms—are a regular feature of business cycle models. A property of most open economy models is that in the absence of adjustment costs investment is excessively volatile. Investment adjustment costs are therefore frequently used to dampen the volatility of investment over the business cycle (see, e.g., Mendoza, 1991; and Schmitt-Grohé, 1998, among many others).

Suppose that the sequential budget constraint is of the form

$$A_t F(k_t) + d_t = (1 + r)d_{t-1} + c_t + i_t + \frac{i_t^2}{2k_t}.$$  (3.14)

Here, capital adjustment costs are given by $i_t^2/(2k_t)$ and are a strictly convex function of investment. Note that the level of this function vanishes at the steady-state value of investment, $i_t = 0$. This means that capital adjustment costs are nil in the steady state. Note further that the slope of the adjustment-cost function, given by $i_t/k_t$, also vanishes in the steady state. As will be clear shortly, this feature implies that in the steady state the relative price of capital goods in terms of consumption goods is unity. As in the economy without adjustment costs, we assume that physical capital does not depreciate, so that the law of motion of the capital stock continues to be given by (3.4).

The household problem then consists in maximizing the utility function (3.1), subject to the law of motion of capital (3.4), to the no-Ponzi-game constraint (3.5), and to the sequential budget constraint (3.14). The Lagrangian associated with this optimization problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ U(c_t) + \lambda_t \left[ A_t F(k_t) + d_t - (1 + r)d_{t-1} - c_t - i_t - \frac{1}{2} \frac{i_t^2}{k_t} + q_t(k_t + i_t - k_{t+1}) \right] \right\}.$$
The variables $\beta^t \lambda_t$ and $\beta^t \lambda_t q_t$ denote Lagrange multipliers on the sequential budget constraint and the law of motion of the capital stock, respectively. The optimality conditions associated with the household problem are (3.4), (3.5) holding with equality, (3.6), (3.7), (3.14),

$$1 + \frac{i_t}{k_t} = q_t,$$

(3.15)

and

$$\lambda_t q_t = \beta \lambda_{t+1} \left[ q_{t+1} + A_{t+1} F'(k_{t+1}) + \frac{1}{2} \left( \frac{i_{t+1}}{k_{t+1}} \right)^2 \right].$$

(3.16)

We continue to assume that $\beta(1+r) = 1$. As in the model without investment adjustment costs, this assumption implies, by optimality conditions (3.6) and (3.7), that $\lambda_t$ and $c_t$ are constant over time. The level of consumption can be found by solving the sequential budget constraint (3.14) forward and using the transversality condition (i.e., the no-Ponzi-game constraint (3.5) holds with equality). This yields

$$c_t = -rd_t + \frac{r}{1+r} \sum_{j=0}^{\infty} A_{t+j} F(k_{t+j}) - i_{t+j} - \frac{1}{2} \frac{(i_{t+j}^2/k_{t+j})}{(1+r)^j}.$$  

This is a familiar expression. Households split their nonfinancial permanent income, given by the second term on the right-hand side, to service their outstanding debt and to consume. The definition of permanent income is adapted to include adjustment costs as one additional component of domestic absorption subtracted from the flow of output.

The variable $q_t$ represents the shadow relative price of capital in terms of consumption goods, and is known as Tobin’s $q$. Optimality condition (3.15) equates the marginal cost of producing a unit of capital, $1 + i_t/k_t$, on the left-hand side, to the marginal revenue of selling a unit of capital, $q_t$, on the right-hand side. If $q_t$ increases, agents have incentives to devote more resources to the production of physical capital, so $i_t$ increases. In turn, the increase in investment raises the
marginal cost of producing capital, \( i_t / k_t \), which tends to restore the equality between the marginal cost and marginal revenue of capital goods.

Using the fact that \((1 + r) \beta = 1\) and that \( \lambda_t \) is constant, optimality condition (3.16) can be written as

\[
(1 + r) q_t = A_{t+1} F'(k_{t+1}) + \frac{1}{2} \left( \frac{i_{t+1}}{k_{t+1}} \right)^2 + q_{t+1}.
\]  (3.17)

The left hand side of this expression is the return of investing \( q_t \) units of goods in bonds, and the right-hand side is the return associated with investing \( q_t \) units of goods in physical capital. Consider first the rate of return of investing in physical capital. Adding one unit to the existing stock costs \( q_t \). The additional unit yields \( A_{t+1} F'(k_{t+1}) \) units of output next period. In addition, an extra unit of capital reduces tomorrow’s adjustment costs by \( (i_{t+1}/k_{t+1})^2/2 \). Finally, the unit of capital can be sold next period at the price \( q_{t+1} \). Alternatively, the agent can engage in a financial investment by purchasing \( q_t \) bonds in period \( t \), which yields a gross return of \((1 + r) q_t\) in period \( t + 1 \). At the optimum both strategies must yield the same return.

**Dynamics of the Capital Stock**

Using the evolution of capital (3.4) to eliminate \( i_t \) from optimality conditions (3.15) and (3.17), we obtain the following two first-order, nonlinear difference equations in \( k_t \) and \( q_t \):

\[
k_{t+1} = q_t k_t
\]  (3.18)

\[
q_t = \frac{A_{t+1} F'(q_t k_t) + (q_{t+1} - 1)^2/2 + q_{t+1}}{1 + r}.
\]  (3.19)

Suppose the technological factor \( A_t \) is constant and equal to \( \bar{A} \). The perfect-foresight solution to these equations is depicted in figure 3.2. The horizontal line \( \bar{K} \bar{K}' \) corresponds to the pairs \( (k_t, q_t) \).
for which $k_{t+1} = k_t$ in equation (3.18). That is,

$$q_t = 1.$$  \hspace{1cm} (3.20)

Above the locus $KK'$, the capital stock grows over time, and below the locus $KK'$ the capital stock declines over time.

The locus $QQ'$ corresponds to the pairs $(k_t, q_t)$ for which $q_{t+1} = q_t$ in equation (3.19). That is,

$$rq_t = \bar{AF}'(q_t k_t) + (q_t - 1)^2/2.$$  \hspace{1cm} (3.21)

For $q_t$ near unity, the locus $QQ'$ is downward sloping. Above and to the right of $QQ'$, $q$ increases over time and below and to the left of $QQ'$, $q$ decreases over time.

Jointly, equations (3.20) and (3.21) determine the steady-state value of the capital stock and
the steady-state value of Tobin’s \( q \). The steady-state value of \( q_t \) is clearly 1. The steady-state value of \( k_t \) is implicitly determined by the expression \( r = AF'(\bar{k}) \). This is the same value obtained in the economy without adjustment costs, and we denoted it \( \bar{k} = \kappa(\bar{A}/r) \), with \( \kappa' > 0 \). This is not surprising, because, as noted earlier, adjustment costs vanish in the steady state.

The system (3.18)-(3.19) is saddle-path stable. The locus \( SS' \) represents the converging saddle path. If the initial capital stock is different from its long-run level, both \( q \) and \( k \) converge monotonically to their steady states along the saddle path. For example, if the economy starts with an initial capital stock below \( \bar{k} \), then the transitional dynamics features a price of capital above average, positive investment, and an increasing stock of capital.

### 3.5.1 A Permanent Technology Shock

Suppose now that in period 0 the technology factor \( A_t \) increases permanently from \( \bar{A} \) to \( A' > \bar{A} \). It is clear from equation (3.20) that the locus \( KK' \) is not affected by the productivity shock. Equation (3.21) shows that the locus \( QQ' \) shifts up and to the right. It follows that in response to a permanent increase in productivity, the long-run level of capital experiences a permanent increase. The price of capital, \( q_t \), on the other hand, is not affected in the long run.

Consider now the transition to the new steady state. Figure 3.2 displays the initial stock of capital, \( \bar{k} \), which was the steady-state value of capital prior to the innovation in productivity. It also displays the new steady-state value of capital, \( k' \). In the period of the shock, the capital stock does not move, since it is predetermined. The price of installed capital, \( q_t \), jumps to the new saddle path, point \( a \) in the figure. This increase in the price of installed capital induces an increase in investment, which in turn makes capital grow over time. After the initial impact, \( q_t \) decreases toward 1. Along this transition, the capital stock increases monotonically towards its new steady-state \( k' \).

The equilibrium dynamics of investment in the presence of adjustment costs are quite different
from those arising in the absence thereof. In the frictionless environment, investment experiences a one-time jump equal to $k' - \bar{k}$ in period zero. Under capital adjustment costs, the initial increase in investment is smaller, as the capital stock adjusts gradually to its long-run level.

The different behavior of investment with and without adjustment costs has consequences for the equilibrium dynamics of the trade balance. In effect, because investment is part of domestic absorption, and because investment tends to be less responsive to productivity shocks in the presence of adjustment costs, it follows that the trade balance falls by less in response to a positive innovation in productivity in the environment with frictions. The following principle therefore emerges:

**Principle II:** The more pronounced are capital adjustment costs, the smaller is the initial trade balance deterioration in response to a positive and persistent productivity shock.

In light of principles I and II derived in this chapter it is natural to ask what the model would predict for the behavior of the trade balance in response to productivity shocks when one introduces realistic degrees of capital adjustment costs and persistence in the productivity-shock process. We address this issue in the next chapter.
3.6 Exercises

Exercise 3.1 (Anticipated Productivity Shocks) Consider a perfect-foresight economy populated by a large number of identical households with preferences described by the utility function

$$\sum_{t=0}^{\infty} \beta^t U(c_t),$$

where $c_t$ denotes consumption, $U$ is a period utility function assumed to be strictly increasing, strictly concave, and twice continuously differentiable, and $\beta \in (0, 1)$ is a parameter denoting the subjective rate of discount. Households are subject to the following four constraints

$$y_t + d_t = (1 + r)d_{t-1} + c_t + i_t,$$

$$y_t = A_t F(k_t)$$

$$k_{t+1} = k_t + i_t,$$

and

$$\lim_{j \to \infty} \frac{d_{t+j}}{(1 + r)^j} \leq 0,$$

given $d_{-1}$, $k_0$, and $\{A_t\}_{t=0}^{\infty}$. The variable $d_t$ denotes holdings of one-period external debt at the end of period $t$, $r$ denotes the interest rate on these debt obligations, $y_t$ denotes output, $k_t$ denotes the (predetermined) stock of physical capital in period $t$, and $i_t$ denotes gross investment, $F$ is a production function assumed to be strictly increasing, strictly concave, and to satisfy the Inada conditions, and $A_t > 0$ is an exogenous productivity factor. Suppose that $\beta(1 + r) = 1$. Assume further that up until period $-1$ inclusive, the productivity factor was constant and equal to $\bar{A} > 0$ and that the economy was in a steady state with a constant level of capital and a constant net debt.
position equal to $\bar{d}$. Suppose further that in period 0 the productivity factor also equals $\bar{A}$, but that agents learn that in period 1 it will jump permanently to $A' > \bar{A}$. That is, in period 0, households know that the path of the productivity factor is given by

$$A_t = \begin{cases} \bar{A} & t \leq -1 \\ \bar{A} & t = 0 \\ A' > \bar{A} & t \geq 1 \end{cases}$$

1. Characterize the equilibrium paths of output, consumption, investment, capital, the net foreign debt position, the trade balance, and the current account.

2. Compare your answer to the case of an unanticipated permanent increase in productivity studied in section 3.3.

3. Now assume that the anticipated productivity shock is transitory. Specifically, assume that the information available to households at $t = 0$ is

$$A_t = \begin{cases} \bar{A} & t \leq -1 \\ \bar{A} & t = 0 \\ A' > \bar{A} & t = 1 \\ \bar{A} & t \geq 2 \end{cases}$$

(a) Characterize the equilibrium dynamics.

(b) Compare your answer to the case of an unanticipated temporary increase in productivity studied in section 3.3.

(c) Compare your answer to the case of an anticipated endowment shocks in the endowment economy studied in exercise 2.5 of chapter 2.
Exercise 3.2 (Adjustment Costs and Temporary Technology Shocks) In the economy with adjustment cost studied in section 3.5, characterize the dynamics triggered by a purely temporary positive technology shock. Specifically, assume that before the shock the economy is in a steady state with capital, debt, and productivity constant at $\bar{k}$, $\bar{d}$, and $\bar{A}$, respectively. Assume that in period 0, unexpectedly, $A_t$ increases to $A' > \bar{A}$, to return permanently to $\bar{A}$ in period 1.

Exercise 3.3 (Balanced Growth) A small open economy is populated by infinitely lived families with preferences given by

$$\sum_{t=0}^{\infty} \beta^t \sqrt{C_t},$$

where $C_t$ denotes consumption of a perishable good in period $t$ and $\beta \in (0, 1)$ is the subjective discount factor. Households can produce goods domestically via the technology

$$Y_t = A_t^{1-\alpha} K_t^\alpha,$$

with $\alpha \in (0, 1)$, where $Y_t$ denotes output, $K_t$ denotes the stock of physical capital, and $A_t$ denotes a technological factor that grows at the constant gross rate $\gamma > 1$, that is,

$$A_{t+1} = \gamma A_t,$$

with $A_0 > 0$ given. The law of motion of the capital stock is given by

$$K_{t+1} = K_t + I_t,$$

with $K_0 > 0$ given, where $I_t$ denotes net investment. Families have access to world financial markets. Each period $t \geq 0$, they can take on one-period debt, denoted $D_t$, that matures in period
The interest, denoted \( r \), is constant and satisfies

\[
\beta(1 + r) = \sqrt{\gamma}.
\]

Households start period 0 with outstanding debt, including interest, of \((1 + r)D_{-1} > 0\). Debt accumulation is subject to the terminal condition \( \lim_{t \to \infty} (1 + r)^{-t}D_t \leq 0 \).

1. Write down the household’s optimization problem.

2. Derive the associated optimality conditions.

3. Characterize the equilibrium dynamics of all variables in the model. Make sure to also consider the equilibrium dynamics of the trade balance and the current account. (You might want to first devise a scaling of variables that makes the equilibrium stationary. After characterizing the equilibrium cast in scaled variables, you can go back to the original variables.)

4. What variables display a trend in equilibrium?

5. Establish whether the equilibrium displays balanced growth. That is, discuss whether key macroeconomic variables, such as consumption, investment, the capital stock, external debt, the trade balance, and the current account, share a common trend with output.

**Exercise 3.4 (Unbalanced Growth)** Consider the same economy as in exercise 3.3, except that the interest rate and the subjective discount factor now satisfy

\[
\beta(1 + r) = 1,
\]

and \( r > 0 \).
1. Characterize the equilibrium dynamics of output, consumption, investment, the trade balance, the current account, and external debt.

2. What variables display a trend in equilibrium?

3. Considering jointly the results obtained in this exercise and in exercise 3.3, what general conclusion can you derive?
Chapter 4

The Small-Open-Economy Real-Business-Cycle Model

In the previous two chapters, we arrived at the conclusion that a model driven by productivity shocks can explain the observed countercyclicality of the trade balance. We also established that two features of the model are important for making this prediction possible. First, productivity shocks must be sufficiently persistent. Second, capital adjustment costs must not be too strong. In this chapter, we extend the model of the previous chapter by allowing for three features that make its structure more realistic: endogenous labor supply and demand, uncertainty in the technology shock process, and capital depreciation. The resulting theoretical framework is known as the Small-Open-Economy Real-Business-Cycle model, or, succinctly, the SOE-RBC model.
4.1 The Model

Consider a small open economy populated by an infinite number of identical households with preferences described by the utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, h_t),$$

(4.1)

where $c_t$ denotes consumption, $h_t$ denotes hours worked, $\beta \in (0, 1)$ is the subjective discount factor, and $U$ is a period utility function, which is assumed to be increasing in its first argument, decreasing in its second argument, and concave. The symbol $E_t$ denotes the expectations operator conditional on information available in period $t$.

The period-by-period budget constraint of the representative household is given by

$$d_t = (1 + r_t) d_{t-1} - y_t + c_t + i_t + \Phi(k_{t+1} - k_t),$$

(4.2)

where $d_t$ denotes the household’s debt position at the end of period $t$, $r_t$ denotes the interest rate at which domestic residents can borrow in period $t$, $y_t$ denotes domestic output, $i_t$ denotes gross investment, and $k_t$ denotes physical capital. The function $\Phi(\cdot)$ is meant to capture capital adjustment costs and is assumed to satisfy $\Phi(0) = \Phi'(0) = 0$ and $\Phi''(0) > 0$. Small open economy models typically include capital adjustment costs to avoid excessive investment volatility in response to variations in the productivity of domestic capital or in the foreign interest rate. The restrictions imposed on $\Phi$ and $\Phi'$ ensure that in the steady state adjustment costs are nil and the relative price of capital goods in terms of consumption goods is unity. Note that here adjustment costs are expressed in terms of final goods. Alternatively, one could assume that adjustment costs take the form of lost capital goods (see exercise 4.9).

Output is produced by means of a linearly homogeneous production function that takes capital
and labor services as inputs,

$$y_t = A_t F(k_t, h_t),$$

(4.3)

where $A_t$ is an exogenous and stochastic productivity shock. This shock represents the single source of aggregate fluctuations in the present model. The stock of capital evolves according to

$$k_{t+1} = (1 - \delta)k_t + i_t,$$

(4.4)

where $\delta \in (0, 1)$ denotes the rate of depreciation of physical capital.

Households choose processes $\{c_t, h_t, y_t, i_t, k_{t+1}, d_t\}_{t=0}^{\infty}$ to maximize the utility function (4.1) subject to (4.2)-(4.4) and a no-Ponzi constraint of the form

$$\lim_{j \to \infty} E_t \frac{d_{t+j}}{\prod_{s=0}^{j}(1 + r_s)} \leq 0.$$  

(4.5)

Use equations (4.3) and (4.4) to eliminate, respectively, $y_t$ and $i_t$ from the sequential budget constraint (4.2). This operation yields

$$d_t = (1 + r_{t-1})d_{t-1} - A_t F(k_t, h_t) + c_t + k_{t+1} - (1 - \delta)k_t + \Phi(k_{t+1} - k_t).$$

(4.6)

The Lagrangian corresponding to the household’s maximization problem is

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \{U(c_t, h_t)$$

$$+ \lambda_t [A_t F(k_t, h_t) + (1 - \delta)k_t + d_t - c_t - (1 + r_{t-1})d_{t-1} - k_{t+1} - \Phi(k_{t+1} - k_t)]\},$$

where $\beta^t \lambda_t$ denotes the Lagrange multiplier associated with the sequential budget constraint (4.6). The first-order conditions associated with the household’s maximization problem are (4.5) holding
with equality, (4.6), and
\[ \lambda_t = \beta (1 + r_t) E_t \lambda_{t+1} \]  \hspace{1cm} (4.7)
\[ U_c(c_t, h_t) = \lambda_t \] \hspace{1cm} (4.8)
\[ -U_h(c_t, h_t) = \lambda_t A_t F_h(k_t, h_t) \] \hspace{1cm} (4.9)
\[ \lambda_t [1 + \Phi'(k_{t+1} - k_t)] = \beta E_t \lambda_{t+1} \left[ A_{t+1} F_k(k_{t+1}, h_{t+1}) + 1 - \delta + \Phi'(k_{t+2} - k_{t+1}) \right]. \] \hspace{1cm} (4.10)

Optimality conditions (4.7), (4.8), and (4.10) are familiar from chapter 3. Optimality condition (4.9) equates the supply of labor to the demand for labor. To put it in a more familiar form, divide (4.9) by (4.8) to eliminate \( \lambda_t \). This yields
\[ -\frac{U_h(c_t, h_t)}{U_c(c_t, h_t)} = A_t F_h(k_t, h_t). \] \hspace{1cm} (4.11)

The left-hand side of this expression is the household’s labor supply schedule. It is the marginal rate of substitution between leisure and consumption, which is increasing in hours worked, holding the level of consumption constant.\(^1\) The right-hand side of (4.11) is the marginal product of labor, which, in a decentralized version of this model equals the demand for labor. The marginal product of labor is decreasing in labor, holding constant the level of capital.

The law of motion of the productivity shock is assumed to be given by the first-order autoregressive process
\[ \ln A_{t+1} = \rho \ln A_t + \eta \epsilon_{t+1}, \] \hspace{1cm} (4.12)
where \( \epsilon_t \) is assumed to be exogenous, stochastic, and i.i.d., with mean zero and unit standard deviation, the parameter \( \eta \) defines the standard deviation of the innovations to productivity, and

\(^1\) A sufficient condition for \(-U_h/U_c\) to be increasing in \( h_t \) holding \( c_t \) constant is \( U_{ch} < 0 \), and the necessary and sufficient condition is \( U_{hh}/U_h > U_{ch}/U_c \).
the parameter $\rho \in (-1, 1)$ measures the serial correlation of the technology shock. According to this expression, the expected value of the productivity shock in period $t + 1$ conditional on information available in period $t$ is a fraction $\rho$ of the current productivity shock,

$$E_t \ln A_{t+1} = \rho \ln A_t. \quad (4.13)$$

More generally, the assumed AR(1) structure of the productivity shock implies that the its expected value $j$ periods ahead conditional on current information is a fraction $\rho^j$ of its present value, $E_t \ln A_{t+j} = \rho^j A_t$. In other words, $\ln A_t$ is always expected to converge to zero at the rate $\rho$.

### 4.1.1 Inducing Stationarity: External Debt-Elastic Interest Rate (EDEIR)

In chapters 2 and 3 we saw that the equilibrium of a small open economy with one internationally traded bond and a constant interest rate satisfying $\beta(1+r) = 1$ features a random walk in consumption, net external debt, and the trade balance. Under perfect foresight, that model predicts that the steady state levels of debt, consumption, and the trade balance depend on initial conditions, such as the initial level of debt itself. This does not mean that the deterministic steady state is indeterminate. Rather, it means that the steady state is history dependent.

The nonstationarity of the small open economy model complicates the task of approximating equilibrium dynamics, because available approximation techniques require stationarity of the state variables. Here, we follow Schmitt-Grohé and Uribe (2003) and induce stationarity by making the interest rate debt elastic.\(^2\)

Specifically, we assume that the interest rate faced by domestic agents, $r_t$, is increasing in the

\(^2\)In section 4.10 we study various alternative ways to induce stationarity.
country’s cross-sectional average of debt, which we denote by $\tilde{d}_t$. Formally, $r_t$ is given by

$$r_t = r^* + p(\tilde{d}_t),$$

(4.14)

where $r^*$ denotes the world interest rate and $p(\cdot)$ is a country-specific interest rate premium. Households take the evolution of $\tilde{d}_t$ as exogenously given. For simplicity, we assume that the world interest rate, $r^*$, is constant. The function $p(\cdot)$ is assumed to be strictly increasing. As we will see shortly, the assumption of a debt-elastic interest rate premium gives rise to a steady state of the model that is independent of initial conditions. In addition, this assumption ensures that a first-order approximation of the equilibrium dynamics converge to the true equilibrium dynamics as the supports of the underlying shocks become small.

The intuition why a debt-elastic interest rate induces stationarity is simple. A growing level of debt causes the country premium to rise inducing households to increase savings, which curbs debt growth. Similarly, if the external debt falls below its steady state level, the country premium falls inducing households to increase consumption and reduce savings, which fosters debt growth.

Here, we have motivated a debt-elastic interest rate on purely technical grounds. However, this feature is also of interest for empirical and theoretical reasons. In chapters 5 and 6, we argue on econometric grounds that data from emerging countries favor a significantly debt-sensitive interest rate. From a theoretical point of view, a debt-elastic interest rate is of interest because it represents a simple way to capture the presence of financial frictions. In chapter 12, we provide micro-foundations to this interpretation in the context of models with imperfect enforcement of international debt contracts.
4.1.2 Equilibrium

Because agents are assumed to be identical, in equilibrium the cross-sectional average level of debt must be equal to the individual level of debt, that is,

\[ \bar{d}_t = d_t. \] (4.15)

Use equations (4.8), (4.14), and (4.15) to eliminate \( \lambda_t, r_t, \) and \( \tilde{d}_t \) from (4.5), (4.6), (4.7), and (4.10) to obtain

\[ d_t = [1 + r^* + p(d_{t-1})]d_{t-1} + c_t + k_{t+1} - (1 - \delta)k_t + \Phi(k_{t+1} - k_t) - A_tF(k_t, h_t). \] (4.16)

\[ U_c(c_t, h_t) = \beta(1 + r^* + p(d_t))E_tU_c(c_{t+1}, h_{t+1}) \] \hspace{1cm} (4.17)

\[ U_c(c_t, h_t)[1 + \Phi'(k_{t+1} - k_t)] = \beta E_tU_c(c_{t+1}, h_{t+1}) \left[ A_{t+1}F_c(k_{t+1}, h_{t+1}) + 1 - \delta + \Phi'(k_{t+2} - k_{t+1}) \right]. \] \hspace{1cm} (4.18)

\[ \lim_{j \to \infty} E_t \frac{d_{t+j}}{\prod_{s=0}^{j}(1 + r^* + p(d_s))} = 0. \] \hspace{1cm} (4.19)

A competitive equilibrium is a set of processes \( \{d_t, c_t, h_t, k_{t+1}, A_t\} \) satisfying (4.11), (4.12), and (4.16)-(4.19), given \( A_0, d_{-1}, \) and \( k_0, \) and the process \( \{\epsilon_t\}_{t=0}^{\infty}. \)

Given the equilibrium processes of consumption, hours, capital, and debt, output is obtained from equation (4.3), investment from equation (4.4), and the interest rate from equation (4.14) evaluated at \( \bar{d}_t = d_t. \) One can then construct the equilibrium process of the trade balance from the definition

\[ tb_t \equiv y_t - c_t - i_t - \Phi(k_{t+1} - k_t), \] (4.20)

where \( tb_t \) denotes the trade balance in period \( t. \) Finally, the current account is given by the sum
of the trade balance and net investment income, that is,

$$ca_t = tb_t - r_{t-1}d_{t-1}. \quad (4.21)$$

Alternatively, one could construct the equilibrium process of the current account by using the fact that the current account measures the change in net foreign assets, that is,

$$ca_t = d_{t-1} - d_t. \quad (4.22)$$

### 4.2 Decentralization

The economy presented thus far assumes that production, employment, and the use of capital are all carried out within the household. Here, we present an alternative formulation in which all of these activities are performed in the marketplace. This formulation is known as the decentralized economy. A key result of this section is that the equilibrium conditions of the decentralized economy are identical to those of the centralized one.

#### 4.2.1 Households in the Decentralized Economy

We assume that each period the household supplies $h_t$ hours to the labor market. We also assume that the household owns shares of a firm that produces physical capital and rents it to firms that produce final goods. Let $w_t$ denote the real wage, $\pi_t$ the profit generated by capital-producing firms, $s_t$ the number of shares of the capital producing firm owned by the household and $p_t^s$ the price of each share. The household takes $w_t$, $\pi_t$, and $p_t^s$ as exogenously given. Its period-by-period budget constraint can then be written as

$$d_t = (1 + r_{t-1})d_{t-1} + c_t + p_t^s(s_t - s_{t-1}) - s_{t-1}\pi_t - w_t h_t. \quad (4.23)$$
The household chooses processes \( \{c_t, h_t, d_t, s_t\}_{t=0}^{\infty} \) to maximize the utility function (4.1) subject to (4.5) and (4.23), taking as given the processes \( \{r_t, w_t, \pi_t, p_t^s\}_{t=0}^{\infty} \) and the initial conditions \((1 + r_{-1})d_{-1}\) and \(s_{-1}\). The first-order conditions associated with the household’s problem are (4.5) holding with equality, (4.7), (4.8), (4.23),

\[
- \frac{U_h(c_t, h_t)}{U_c(c_t, h_t)} = w_t,
\]

and

\[
\lambda_t p_t^s = \beta E_t \lambda_{t+1}[p_{t+1}^s + \pi_{t+1}].
\]

The variable \( p_t^s \) represents a stock market index such as the S&P 500. The above Euler equation can be integrated forward to obtain

\[
p_t^s = E_t \sum_{j=1}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} \pi_{t+j},
\]

which states that the value of the stock market in period \( t \) equals the present discounted value of future expected profits.

### 4.2.2 Firms Producing Final Goods

Firms produce final goods with labor and capital and operate in perfectly competitive markets. The production technology is given by

\[
y_t = A_tF(k_t, h_t).
\]

Profits in period \( t \) are given by

\[
A_tF(k_t, h_t) - w_t h_t - u_t k_t.
\]
Each period $t \geq 0$ the firm hires workers and rents capital to maximize profits. The first-order conditions associated with the firm’s profit maximization problem are

$$A_t F_h(k_t, h_t) = w_t$$  \hspace{1cm} (4.27)

and

$$A_t F_k(k_t, h_t) = u_t.$$  \hspace{1cm} (4.28)

Because the production function is assumed to be homogeneous of degree one, profits are zero at all times. To see this multiply (4.27) by $h_t$, (4.28) by $k_t$ and sum the resulting expressions to obtain $A_t F_h(k_t, h_t) h_t + A_t F_k(k_t, h_t) k_t = w_t h_t + u_t k_t$. By the assumed linear homogeneity of the production function the left hand side of this expression is equal to $A_t F(k_t, h_t)$. It then follows that the total cost of production equals output, or, that profits equal zero.

### 4.2.3 Firms Producing Capital Goods

Firms producing capital invest $i_t$ units of final goods each period and are subject to adjustment costs $\Phi(k_{t+1} - k_t)$ measured in units of final goods. Each period, these firms rent the stock of capital to firms producing final goods at the rental rate $u_t$ per unit. Profits of firms producing capital goods are then given by

$$\pi_t = u_t k_t - i_t - \Phi(k_{t+1} - k_t)$$  \hspace{1cm} (4.29)

The problem of the firm producing capital goods is to choose processes $\{\pi_t, i_t, k_{t+1}\}^\infty_{t=0}$ maximize the present discounted value of profits

$$E_0 \sum_{t=0}^\infty \beta^t \frac{\lambda_t}{\lambda_0} \pi_t$$
subject to the law of motion of the capital stock given in equation (4.4) and the definition of profits given in equation (4.29), taking as given the processes \( \{u_t, \lambda_t\}^\infty_{t=0} \) and the initial condition \( k_0 \). Note that profits are discounted using the factor \( \beta^t \lambda_t / \lambda_0 \), which is the value assigned by households to contingent payments of goods in period \( t \) in terms of units of goods in period 0. This way of discounting makes sense because households own the firms producing capital. Note further that the objective function of the firm is identical to the right-hand side of optimality condition (4.26). This means that the objective of the firm producing capital can be interpreted as maximizing the value of the firm in the stock market.

Using equation (4.4) to eliminate \( i_t \) from equation (4.29) and the resulting expression to eliminate \( \pi_t \) from the firm’s objective function yields

\[
E_0 \sum_{t=0}^{\infty} \beta^t \frac{\lambda_t}{\lambda_0} [(u_t + 1 - \delta)k_t - k_{t+1} - \Phi(k_{t+1} - k_t)]
\]

The optimality condition with respect to \( k_{t+1} \) is then given by

\[
\lambda_t [1 + \Phi'(k_{t+1} - k_t)] = \beta E_t \lambda_{t+1} \left[ u_{t+1} + 1 - \delta + \Phi'(k_{t+2} - k_{t+1}) \right].
\]  

(4.30)

4.2.4 The Decentralized Equilibrium

We can normalize the number of shares to be one per household at all times. Thus, we have

\[
s_t = 1.
\]  

(4.31)

A competitive equilibrium in the decentralized economy is then a set of processes \( \{d_t, \tilde{d}_t, c_t, p_t^s, s_t, r_t, \pi_t, h_t, w_t, \lambda_t, y_t, u_t, k_{t+1}, i_t, A_t\}^\infty_{t=0} \), satisfying (4.3), (4.4), (4.7), (4.8), (4.12), (4.14), (4.15), (4.19), (4.23), (4.24), and (4.26)-(4.31), given \( A_0, d_{-1}, \) and \( k_0 \), and the process \( \{e_t\}_{t=0}^{\infty} \).
It is straightforward to see that the equations included in this definition can be combined to produce all of the equations conforming the equilibrium in the centralized economy defined in section 4.1.2. It can also be readily established that if all of the conditions for an equilibrium in the centralized economy are satisfied, then one can residually construct processes for market prices, profits, and share holdings, namely processes \( \{ w_t, u_t, p_t^k, \pi_t, s_t \}_{t=0}^{\infty} \), so that all of the equilibrium conditions of the decentralized economy listed here are satisfied. This completes the proof that the equilibrium conditions of the centralized and decentralized economies are identical.

### 4.3 Functional Forms

We assume that the period utility function takes the form

\[
U(c, h) = \frac{G(c, h)^{1-\sigma} - 1}{1 - \sigma}, \quad \sigma > 0,
\]

with

\[
G(c, h) = c - \frac{h^\omega}{\omega}, \quad \omega > 1.
\]

The form of the subutility index \( G(c, h) \) is due to Greenwood, Hercowitz, and Huffman (1988) and is typically referred to as GHH preferences. It implies that the labor supply (the marginal rate of substitution between consumption and leisure) is independent of the level of consumption. Specifically, under GHH preferences, equilibrium condition (4.24) becomes

\[
h_t^{\omega-1} = w_t. \tag{4.32}
\]

This labor supply schedule has a wage elasticity of \( 1/(\omega - 1) \) and is independent of \( c_t \). GHH preferences were popularized in the open economy business cycle literature by Mendoza (1991).
The period utility function $U(c, h)$ displays constant relative risk aversion (CRRA) over the subutility index $G(c, h)$. The parameter $\sigma$ measures the degree of relative risk aversion, and its reciprocal, $1/\sigma$, measures the intertemporal elasticity of substitution.

We adopt a Cobb-Douglas specification for the production function,

$$F(k, h) = k^\alpha h^{1-\alpha},$$

with $\alpha \in (0, 1)$. This specification implies a unitary elasticity of substitution between capital and labor. That is, a one percent increase in the wage to rental ratio, $w_t/u_t$, induces firms to increase the capital-labor ratio by one percent. To see this divide equation (4.27) by equation (4.28) and use the Cobb-Douglas form for the production function to obtain

$$\left(\frac{1-\alpha}{\alpha}\right) \frac{k_t}{h_t} = \frac{w_t}{u_t},$$

which implies that in equilibrium the capital-labor ratio is proportional to the wage to rental ratio. The Cobb-Douglas specification of the production function is widely used in the business-cycle literature.

The capital adjustment cost function is assumed to be quadratic,

$$\Phi(x) = \frac{\phi}{2} x^2,$$

with $\phi > 0$. This specification implies that net investment, whether positive or negative, generates resource costs.

Finally, we follow Schmitt-Grohé and Uribe (2003) and assume that the country interest rate premium takes the form

$$p(d) = \psi_1 \left( e^{d - \bar{d}} - 1 \right),$$
where $\psi_1 > 0$ and $\bar{d}$ are parameters. According to this expression the country premium is an increasing and convex function of net external debt.

### 4.4 Deterministic Steady State

Assume that the variance of the innovation to the productivity shock, $\tilde{\eta}$, is nil. We refer to such an environment as a deterministic economy. We define a deterministic steady state as an equilibrium of the deterministic economy in which all endogenous variables are constant over time.

The characterization the deterministic steady state is of interest for two reasons. First, the steady state facilitates the calibration of the model. This is because, to a first approximation, the deterministic steady state coincides with the average position of the model economy. In turn, often several structural parameters of the model are calibrated to match average characteristics of the model economy, such as labor shares, consumption shares, and trade-balance-to-output ratios to their empirical counterparts. Second, the deterministic steady state is often used as a convenient point around which the equilibrium conditions of the stochastic economy are approximated.

For any variable we denote its steady-state value by removing the time subscript. Evaluating equilibrium condition (4.17) at the steady state yields

$$1 = \beta \left[ 1 + r^* + \psi_1 \left( e^{d-\bar{d}} - 1 \right) \right].$$

Assume that

$$\beta (1 + r^*) = 1.$$

In the context of the present model, this assumption is a normalization, and is not necessary to
ensure stationarity. Combining the above two restrictions one obtains

\[ d = d. \]

The steady-state version of (4.18) implies that

\[ 1 = \beta \left[ \alpha \left( \frac{k}{h} \right)^{\alpha-1} + 1 - \delta \right]. \]

This expression delivers the steady-state capital-labor ratio, which we denote by \( \kappa \). Formally,

\[ \kappa \equiv \frac{k}{h} = \left( \frac{\beta^{-1} - 1 + \delta}{\alpha} \right)^{1/(\alpha-1)}. \]

Using this expression to eliminate the capital-labor ratio from equilibrium condition (4.11) evaluated at the steady state, one obtains the following expression for the steady-state level of hours

\[ h = \left[ (1 - \alpha) \kappa^\alpha \right]^{1/(\omega-1)}. \]

Given the steady-state values of labor and the capital-labor ratio, the steady-state level of capital is simply given by

\[ k = \kappa h. \]

Finally, the steady-state level of consumption can be obtained by evaluating equilibrium condition (4.16) at the steady state. This yields

\[ c = -r^* d + \kappa^\alpha h - \delta k. \]

This completes the characterization of the deterministic steady state of the present economy.
4.5 Calibration

An important intermediate step in computing the quantitative predictions of a business-cycle model is to assign values to its structural parameters. There are two main ways to accomplish this step. One is econometric estimation by methods such as the generalized method of moments (GMM), impulse response matching, maximum likelihood, or likelihood-based Bayesian methods. We will explain and apply several of these econometric techniques in later chapters. The second approach, which we study here, is calibration. Almost always, business-cycle studies employ a combination of calibration and econometric estimation.

In general, the calibration method assigns values to the parameters of the model in three different ways: (a) Using sources unrelated to the macro data the model aims to explain. (b) By matching first moments of the data that the model aims to explain. (c) By matching second moments of the data the model aims to explain.

To illustrate how calibration works, we adapt the calibration strategy adopted in Mendoza (1991) to the present model. His SOE-RBC model aims to explain the Canadian business cycle. The time unit in the model is meant to be one year. In the present model, there are 10 parameters that need to be calibrated: $\sigma, \delta, r^*, \alpha, d, \omega, \phi, \psi_1, \rho, \tilde{\eta}$. We separate these parameters into the three calibration categories described above.

(a) Parameters Calibrated Using Sources Unrelated To The Data The Model Aims To Explain

The parameters that fall in this category are the intertemporal elasticity of substitution, $\sigma$, the depreciation rate, $\delta$, and the world interest rate, $r^*$. Based on parameter values widely used in related business-cycle studies, Mendoza sets $\sigma$ equal to 2, $\delta$ equal to 0.1, and $r^*$ equal to 4 percent per year.
(b) Parameters Set to Match First Moments of the Data the Model Aims to Explain

In this category are the capital elasticity of the production function, $\alpha$, and the parameter $d$ pertaining to the country interest-rate premium. The parameter $\alpha$ is set to match the average labor share in Canada of 0.68. In the present model, the labor share, given by the ratio of labor income to output, or $w_t h_t / y_t$, equals $1 - \alpha$ at all times. To see this, note that in equilibrium, $w_t$ equals the marginal product of labor, which, under the assumed Cobb-Douglas production function is given by $(1 - \alpha) y_t / h_t$.

The parameter $d$ is set to match the observed average trade-balance-to-output ratio in Canada of 2 percent. Combining the definition of the trade balance given in equation (4.20) with the resource constraint (4.16) implies that in the steady state

$$tb = r^* d.$$  

This condition states that in the deterministic steady state the country must generate a trade surplus sufficiently large to service its external debt. Dividing both sides by steady-state output and solving for $d$ yields

$$d = \frac{tb}{r^* y},$$

At this point we know that $tb / y = 0.02$ and that $r^* = 0.04$, but $y$ remains unknown. From the derivation of the steady state presented in section 4.4, one can deduce that

$$y = \left[(1 - \alpha) \kappa^{\alpha \omega} \right]^{\frac{1}{\omega - 1}},$$

where $\kappa = [(\alpha / (r^* + \delta))]^{1/(1-\alpha)}$. The only unknown parameter in the expression for $y$, and therefore $\overline{d}$, is $\omega$. Next, we discuss how the calibration strategy assigns values to $\omega$ and the remaining
unknown structural parameters.

(c) Parameters Set To Match Second Moments Of The Data The Model Aims To Explain

This category of parameters contains $\omega$, which governs the wage elasticity of labor supply, $\phi$, which defines the magnitude of capital adjustment costs, $\psi_1$ which determines the debt sensitivity of the interest rate, and $\rho$ and $\tilde{\eta}$ defining, respectively, the persistence and volatility of the technology shock. The calibration strategy for these parameters is to match the following five second moments of the Canadian data at business-cycle frequency: A standard deviation of hours of 2.02 percent, a standard deviation of investment of 9.82 percent, a standard deviation of the trade-balance-to-output ratio of 1.87 percentage points, a serial correlation of output of 0.62, and a standard deviation of output of 2.81 percent.\(^3\) These are natural targets, as their theoretical counterparts are directly linked to the parameters to be calibrated. In practice, this last step of the calibration procedure goes as follows: (i) Guess values for the five parameters in category (c). This automatically determines a value for $d$. (ii) Approximate the equilibrium dynamics of the model. (We will discuss how to accomplished this task shortly.) (iii) Calculate the implied five second moments to be matched in (c). (iv) If the match between actual and predicted second moments is judged satisfactory, the procedure has concluded. If not, try a new guess for the five parameters to be calibrated and return to (i). There is a natural way to update the parameter guess. For instance, if the volatility of output predicted by the model is too low, raise the volatility of the innovation to the technology shock, $\tilde{\eta}$. Similarly, if the volatility of investment is too high, increase the value of $\phi$. And so on. In general, there are no guarantees of the existence of a set of parameter values that will produce an exact match between the targeted empirical second moments and their theoretical counterparts.

\(^3\)The standard deviations of hours, investment, and output are measured in percent because (the cyclical components of) hours, investment, and output are measured as percent deviations of these indicators from trend.
Table 4.1: Calibration of the EDEIR Small Open RBC Economy

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\sigma$</th>
<th>$\delta$</th>
<th>$r^*$</th>
<th>$\alpha$</th>
<th>$d$</th>
<th>$\omega$</th>
<th>$\phi$</th>
<th>$\psi_1$</th>
<th>$\rho$</th>
<th>$\tilde{\eta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>2</td>
<td>0.1</td>
<td>0.04</td>
<td>0.32</td>
<td>0.7442</td>
<td>1.455</td>
<td>0.028</td>
<td>0.000742</td>
<td>0.42</td>
<td>0.0129</td>
</tr>
</tbody>
</table>

So some notion of distance and tolerance is in order. The parameter values that result from this calibration procedure are shown in table 4.1.

It is important to note that the calibration strategy presented here is just one of many possible ones. For instance, we could place $\delta$ in category (b) and add the average investment share as a first moment of the data to be matched. Similarly, we could take the parameter $\omega$ out of category (c) and place it instead in category (a). To assign a value to $\omega$ parameter we could then use existing micro-econometric estimates of the Frisch elasticity of labor supply. Finally, a calibration approach that has been used extensively, especially in the early days of the RBC literature, is to place $\rho$ and $\tilde{\eta}$ into category (a) instead of (c). Under this approach, one uses Solow residuals as a proxy for the productivity shock $A_t$. Then one estimates a univariate representation of the Solow residual to obtain values for $\rho$ and $\tilde{\eta}$.

4.6 Approximating Equilibrium Dynamics

The competitive equilibrium of the SOE-RBC model is described by a system of nonlinear stochastic difference equations. Closed-form solutions to this type of systems are typically unavailable. We therefore must resort to an approximate solution. There exist a number of techniques that have been devised to solve such dynamic systems. The one we study in this section is based on a linear approximation of the equilibrium conditions.

It is important to chose carefully the base of the linearization. It is often appropriate to linearize the system with respect to the logarithm of some variables. This is known as log-linearization, and
is useful for variables whose empirical counterparts are expressed in log (or percent) deviations from trend. In the present SOE-RBC model, this is the case with $y_t$, $c_t$, $h_t$, $k_t$, and $A_t$. For other variables, it is more natural to perform the linearization with respect to their levels, not with respect to their logs. This is the case, for instance, with net interest rates, like $r_t$, or variables that can take negative values, such as $tb_t$, $cat_t$, and $dt_t$, or ratios, like the investment-to-output ratio.

Before performing the linearization of the equilibrium conditions of the SOE-RBC model, we briefly explain how to linearize a function with respect to a mix of bases, the log for some variables and the level for others. As an illustration, consider the expression

$$s_t = E_t m(u_t, v_t, z_{t+1}).$$

We wish to linearize this expression with respect to the logs of $s_t$, $u_t$, and $z_{t+1}$, and with respect to the level of $v_t$. To this end, let $\hat{s}_t \equiv \ln(s_t/s)$, $\hat{u}_t \equiv \ln(u_t/u)$, and $\hat{z}_{t+1} \equiv \ln(z_{t+1}/z)$ denote the log-deviations of $s_t$, $u_t$, and $z_{t+1}$ with respect to their respective deterministic steady-state values, denoted $s$, $u$, and $z$, and let $\hat{v}_t \equiv v_t - v$ denote the deviation of $v_t$ from its steady-state value, denoted $v$. Then, we can write the above expression as

$$se_{\hat{s}_t} = E_t m\left(ue_{\hat{u}_t}, \hat{v}_t + v, ze_{\hat{z}_{t+1}}\right).$$

The linearization results from differentiating the above expression with respect to $\hat{s}_t$, $\hat{u}_t$, $\hat{v}_t$, and $\hat{z}_{t+1}$ around their respective deterministic steady-state values. Note that the deterministic steady-state values of all hatted variables is zero. In performing the differentiation, recall that the expectation operation is an integral, and that the differentiation of an integral with respect to variables appearing in the integrand is the integral of the differentiated integrand. Then the desired linear
approximation is given by

\[ s \tilde{t} = m_u \tilde{u}_t + m_v \tilde{v}_t + m_z \tilde{z}_{t+1}, \]

where \( m_u, m_v, \) and \( m_z \) denote the partial derivatives of \( m(\cdot, \cdot, \cdot) \) with respect to \( u_t, v_t, \) and \( z_{t+1}, \) respectively, evaluated at the steady state \((u, v, z)\). With this background, we now turn to the linearization of the equilibrium conditions of the SOE-RBC model.

We linearize the system with respect to the logs of \( c_t, h_t, k_t, \) and \( A_t, \) and with respect to the level of \( d_t. \) Accordingly, let \( \hat{x}_t \equiv \ln(x_t/x), \) for \( x_t = c_t, h_t, k_t, A_t, \) and \( \hat{d}_t \equiv d_t - d. \) Then, the linearized version of equilibrium conditions (4.11), (4.13), and (4.16)-(4.18) is

\[
\begin{align*}
[\epsilon_{hh} - \epsilon_{ch}] \hat{h}_t + [\epsilon_{hc} - \epsilon_{cc}] \hat{c}_t &= \hat{A}_t + \alpha (\hat{k}_t - \hat{h}_t) \\
E_t \hat{A}_{t+1} &= \rho \hat{A}_t \\
\frac{1}{\gamma} \hat{d}_t &= \frac{1}{\gamma} [\psi_1 d + 1 + r^*] \hat{d}_{t-1} + s_c \hat{c}_t \\
&\quad + \frac{s_i}{\delta} [\hat{k}_{t+1} - (1 - \delta) \hat{k}_t] \\
&\quad - \hat{A}_t - \alpha \hat{k}_t - (1 - \alpha) \hat{h}_t \\
\epsilon_{ch} \hat{h}_t + \epsilon_{cc} \hat{c}_t &= \psi_1 \beta \hat{d}_t + \epsilon_{ch} E_t \hat{h}_{t+1} + \epsilon_{cc} E_t \hat{c}_{t+1} \\
\epsilon_{cc} \hat{c}_t + \epsilon_{ch} \hat{h}_t + \Phi''(0) k (\hat{k}_{t+1} - \hat{k}_t) &= \epsilon_{cc} E_t \hat{c}_{t+1} + \epsilon_{ch} E_t \hat{h}_{t+1} \\
&\quad + \frac{r^* + \delta}{1 + r^*} \left[ E_t \hat{A}_{t+1} + (\alpha - 1)(E_t \hat{k}_{t+1} - E_t \hat{h}_{t+1}) \right] \\
&\quad + \frac{\Phi''(0) k}{1 + r^*} [E_t \hat{k}_{t+2} - E_t \hat{k}_{t+1}] 
\end{align*}
\]

where \( \epsilon_{hh} \equiv U_{hh}/U_h, \epsilon_{ch} \equiv U_{ch}h/U_c, \epsilon_{hc} \equiv U_{hc}c/U_h, \epsilon_{cc} \equiv U_{cc}c/U_c, s_{tb} \equiv r^*d/F(k, h), s_c \equiv c/F(k, h), s_i \equiv \delta k/F(k, h), \) and \( y \equiv F(k, h). \) The linearization uses the particular forms assumed for the production function and the country premium function. Of course, we could have linearized an expanded version of the equilibrium conditions, including equations defining additional macro
indicators of interest. For instance, the system could have included equations (4.3), (4.4), (4.20), and (4.21), jointly defining $y_t$, $i_t$, $t_{bt}$, and $ca_t$.

We now express the set of equilibrium conditions and its linearized version using a more compact notation, which applies to a large class of dynamic stochastic general equilibrium models, not just the SOE-RBC model. Let $y_t$ be a vector collecting the control variables of the model. Control variables in period $t$ are endogenous variables that are determined in period $t$. In the SOE-RBC model, as defined by equations (4.11), (4.13), and (4.16)-(4.18) the vector $y_t$ contains $\ln c_t$ and $\ln h_t$. Let $x^1_t$ denote the vector of endogenous state variables. Endogenous state variables in period $t$ are endogenous variables determined before period $t$. In the SOE-RBC model, $x^1_t$ includes $\ln k_t$ and $d_{t-1}$. Let $x^2_t$ denote the vector of exogenous state variables. Exogenous state variables in period $t$ are exogenous variables that are determined in period $t$ or earlier. In the SOE-RBC model, $x^2_t$ includes a single variable, $\ln A_t$. Let $x_t \equiv [x^1_t \ x^2_t]^\prime$ denote the vector of state variables.

The equilibrium conditions of the model, given by equations (4.11), (4.13), and (4.16)-(4.18), can be written as

$$E_t f(y_{t+1}, y_t, x_{t+1}, x_t) = 0.$$ (4.38)

The law of motion of the exogenous state vector $x^2_t$ is given by

$$x^2_{t+1} = \Lambda x^2_t + \tilde{\eta}\epsilon_{t+1}.$$ (4.39)

where, in general, $\epsilon_t$ is a vector of i.i.d. random variables with mean zero and unit variance, $\Lambda$ is a square matrix with all eigenvalues inside the unit circle, and $\tilde{\eta}$ is a matrix of parameters defining the variance-covariance matrix of innovations to the exogenous state vector. In the SOE-RBC model $\epsilon_t$, $\Lambda$, and $\tilde{\eta}$ are all scalars (with $\Lambda = \rho$).
The deterministic steady state is a pair of constant vectors \( y \) and \( x \) that solves the system

\[
f(y, y, x, x) = 0.
\]

The steady-state vectors \( y \) and \( x \) are assumed to be known. In section 4.4, we derived the steady state of the SOE-RBC model analytically.

We restrict attention to equilibria in which at every date \( t \) the economy is expected to converge to the non-stochastic steady state, that is, we impose

\[
\lim_{j \to \infty} \begin{bmatrix} E_t y_{t+j} \\ E_t x_{t+j} \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix},
\]

(4.40)

This restriction implies that the transversality condition (4.19) is always satisfied.

As mentioned earlier, the representation of an equilibrium given by conditions (4.38)-(4.40) is quite general and applies to a large class of dynamic stochastic general equilibrium models. Thus, the solution technique discussed below is not restricted to the SOE-RBC model.

The first-order Taylor expansion of equation (4.38) is given by

\[
f_{yy} E_t \hat{y}_{t+1} + f_{yy} \hat{y}_t + f_{xx} E_t \hat{x}_{t+1} + f_x \hat{x}_t
\]

(4.41)

where \( \hat{x}_t \equiv x_t - x \) and \( \hat{y}_t \equiv y_t - y \) denote, respectively, the deviations of \( x_t \) and \( y_t \) from their steady state values. The matrices \( f_{yy}, f_{yy}, f_{xx}, \) and \( f_x \) denote, respectively, the partial derivatives of the function \( f \) with respect to \( y', y, x', \) and \( x \) evaluated at the nonstochastic steady state. These matrices are assumed to be known. Except for small models, like the SOE-RBC model studied here, these derivatives can be tedious to obtain by hand. The matlab scripts indicated at the end of this section perform and evaluate these derivatives automatically.
The solution of the linear system (4.41) with the associated exogenous law of motion (4.39), and the terminal condition (4.40) is given by

\[ \hat{x}_{t+1} = h_x \hat{x}_t + \eta \epsilon_{t+1} \]

and

\[ \hat{y}_t = g_x \hat{x}_t. \]

The matrix \( \eta \) is given by

\[ \eta = \begin{bmatrix} \emptyset \\ \tilde{\eta} \end{bmatrix}. \]

The Appendix shows how to obtain the matrices \( h_x \) and \( g_x \), given the matrices \( f_{y'} \), \( f_y \), \( f_{x'} \), and \( f_x \). The Appendix also shows how to compute second moments and impulse response functions predicted by the model.

Matlab code for performing first-order accurate approximations to DSGE models and for computing second moments and impulse response functions is available at \texttt{www.columbia.edu/~mu2166/1st\_order.htm}. Matlab code to solve the specific SOE-RBC EDEIR model studied here is available online at \texttt{www.columbia.edu/~mu2166/book/}.

### 4.7 The Performance of the Model

Having calibrated the model and computed a first-order approximation to the equilibrium dynamics, we are ready to explore its quantitative predictions. As a point of reference, table 4.2 displays empirical second moments of interest from the Canadian economy. The first three columns display the empirical second moments reported by Mendoza (1991). The table shows standard deviations, serial correlations, and contemporaneous correlations of output with output, consumption, invest-
ment, hours, and the trade-balance-to-output ratio. The data is annual, quadratically detrended, and covers the period 1946-1985. Although outdated, we choose to use the empirical moments reported in Mendoza (1991) to preserve coherence with the calibration strategy of subsection 4.5. To gauge the stability of the empirical regularities and the out-of-sample performance of the SOE-RBC model, the middle three columns of table 4.2 display empirical second moments computed using data from 1960 to 2011. Overall, the stylized facts displayed in the table appear to be quite stable across time. In particular, in both samples the ranking of volatilities is $i > y > c > \text{tb}/y$.

All aggregates are positively serially correlated. However, investment has become much more persistent over time. The trade-balance-to-GDP ratio is slightly countercyclical in the early sample but slightly procyclical in the recent one. The Canadian economy appears to have become more volatile, in particular, the volatilities of output and hours worked have increased by 30 and 80 percent, respectively.

Table 4.2 also displays second moments predicted by the SOE-RBC EDEIR model. Comparing the early empirical second moments with their predicted counterparts, it should not come as a surprise that the model does very well at replicating the volatilities of output, hours, investment, and the trade-balance-to-output ratio, and the serial correlation of output. For we calibrated the parameters $\omega$, $\phi$, $\psi_1$, $\rho$, and $\tilde{\eta}$ to match these five moments. But the model performs relatively well along other dimensions. For instance, it correctly implies that consumption is less volatile than output and investment and more volatile than hours and the trade-balance-to-output ratio. Also, the model correctly predicts that the trade balance-to-output ratio is countercyclical. This prediction is of interest because the parameters $\phi$ and $\rho$ governing the degree of capital adjustment costs and the persistence of the productivity shock, which, as we established in the previous chapter, are key determinants of the cyclicality of the trade-balance-to-output ratio, were set independently of the observed cyclical properties of the trade balance. The model does not perform equally well at explaining the comovement of the trade balance with output over the more recent sample. Ex-
Table 4.2: Empirical and Theoretical Second Moments

<table>
<thead>
<tr>
<th>Variable</th>
<th>Canadian Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1946 to 1985</td>
<td>1960 to 2011</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{xt}$</td>
<td>$\rho_{xt,xt-1}$</td>
</tr>
<tr>
<td>$y$</td>
<td>2.81</td>
<td>0.62</td>
</tr>
<tr>
<td>$c$</td>
<td>2.46</td>
<td>0.70</td>
</tr>
<tr>
<td>$i$</td>
<td>9.82</td>
<td>0.31</td>
</tr>
<tr>
<td>$h$</td>
<td>2.02</td>
<td>0.54</td>
</tr>
<tr>
<td>$\frac{tb}{y}$</td>
<td>1.87</td>
<td>0.66</td>
</tr>
<tr>
<td>$\frac{ca}{y}$</td>
<td>1.45</td>
<td>0.32</td>
</tr>
</tbody>
</table>

Note. Empirical moments for the period 1946 to 1985 are taken from Mendoza (1991) and for the period 1960 to 2011 are based on own calculations using data from WDI (GDP, consumption, investment, imports, and exports) and Statistics Canada (hours worked). All empirical second moments based on annual, per capita, and quadratically detrended data. Standard deviations are measured in percentage points. Theoretical moments are produced by running the Matlab code edeir_run.m.
Exercise 4.8 asks you to use the empirical second moments associated with the 1960-2011 sample to recalibrate and evaluate the SOE-RBC EDEIR model.

The model overpredicts the correlations of hours with output. In particular, the predicted correlation is exactly unity. This result is due to the assumed functional form for the period utility index. To see this, note that equilibrium condition (4.11), which equates the marginal product of labor to the marginal rate of substitution between consumption and leisure, can be written as $h_t \omega_t = (1 - \alpha) y_t$. The log-linearized version of this condition is $\hat{\omega}_t h_t = \hat{y}_t$, which implies that $\hat{h}_t$ and $\hat{y}_t$ are perfectly correlated.

Figure 4.1 displays the impulse response functions of a number of variables of interest to a technology shock of size 1 percent in period 0. In response to this innovation, the model predicts an expansion in output, consumption, investment, and hours and a deterioration in the trade-balance-to-output ratio. The level of the trade balance, not shown, also falls on impact. This means that the initial increase in domestic absorption (i.e., the increase in $c_0 + i_0$) is larger than the increase in output. Further, the initial response of consumption is proportionally smaller than that of output, whereas the initial response of investment is about eight times as large as that of output. It follows that in the context of the present SOE-RBC model investment plays a key role in generating a countercyclical initial response of the trade balance.

4.8 The Role of Persistence and Capital Adjustment Costs

In the previous chapter, we deduced that the negative response of the trade balance to a positive technology shock was not a general implication of the neoclassical model. In particular, Principles I and II of the previous chapter state that two conditions must be met for the model to generate a deterioration in the external accounts in response to a mean-reverting improvement in total factor productivity. First, capital adjustment costs must not be too stringent. Second, the productivity
Figure 4.1: Responses to a One-Percent Productivity Shock

Note. To produce this figure, run the Matlab code edeir_run.m.
Figure 4.2: Response of the Trade-Balance-To-Output Ratio to a Positive Technology Shock

shock must be sufficiently persistent. To illustrate this conclusion, figure 4.2 displays the impulse response function of the trade balance-to-GDP ratio to a technology shock of unit size in period 0 under three alternative parameter specifications. The solid line reproduces the benchmark case from figure ???. The broken line depicts an economy where the persistence of the productivity shock is half as large as in the benchmark economy ($\rho = 0.21$). In this case, because the productivity shock is expected to die out quickly, the response of investment is relatively weak. In addition, the temporariness of the shock induces households to save most of the increase in income to smooth consumption over time. As a result, the expansion in aggregate domestic absorption is modest. At the same time, because the size of the productivity shock is the same as in the benchmark economy, the initial responses of output and hours are identical in both economies (recall that, by equation (4.32), $h_t$ depends only on $k_t$ and $A_t$, and that $k_t$ is predetermined in period $t$). The
combination of a weak response in domestic absorption and an initial response in output that is independent of the value of $\rho$, results in an improvement in the trade balance when productivity shocks are not too persistent.

The crossed line depicts the case of high capital adjustment costs. Here the parameter $\phi$ equals 0.084, a value three times as large as in the benchmark case. In this environment, high adjustment costs discourage firms from increasing investment spending by as much as in the benchmark economy. As a result, the response of aggregate domestic demand is weaker, leading to an improvement in the trade balance-to-output ratio.

4.9 The SOE-RBC Model With Complete Asset Markets (CAM)

The SOE-RBC model economy considered thus far features incomplete asset markets. In that model, agents have access to a single financial asset that pays a non-state-contingent rate of return. In the model studied in this section, by contrast, agents are assumed to have access to a complete array of state-contingent claims. As we will see, the introduction of complete asset markets per se induces stationarity in the equilibrium dynamics, so there will be no need to introduce any ad-hoc stationarity inducing feature.

Preferences and technologies are as in the EDEIR model. The period-by-period budget constraint of the household is given by

$$E_t r_{t,t+1} b_{t+1} = b_t + A_t F(k_t, h_t) - c_t - [k_{t+1} - (1 - \delta)k_t] - \Phi(k_{t+1} - k_t),$$

(4.42)

where $r_{t,t+1}$ is a pricing kernel such that the period-$t$ price of a random payment $b_{t+1}$ in period $t + 1$ is given by $E_t r_{t,t+1} b_{t+1}$. To clarify the nature of the pricing kernel $r_{t,t+1}$, define the current
state of nature as $S^t$. Let $p(S^{t+1}|S^t)$ denote the price of a contingent claim that pays one unit of consumption in a particular state $S^{t+1}$ following the current state $S^t$. Then the current price of a portfolio composed of $b(S^{t+1}|S^t)$ units of contingent claims paying in states $S^{t+1}$ following $S^t$ is given by $\sum_{S^{t+1}|S^t} p(S^{t+1}|S^t)b(S^{t+1}|S^t)$. Now let $\pi(S^{t+1}|S^t)$ denote the probability of occurrence of state $S^{t+1}$, given information available at the current state $S^t$. Multiplying and dividing the expression inside the summation sign by $\pi(S^{t+1}|S^t)$ we can write the price of the portfolio as $\sum_{S^{t+1}|S^t} \pi(S^{t+1}|S^t)p(S^{t+1}|S^t)b(S^{t+1}|S^t)\pi(S^{t+1}|S^t)b(S^{t+1}|S^t)$. Now let $r_{t,t+1} \equiv p(S^{t+1}|S^t)/\pi(S^{t+1}|S^t)$ be the price of a contingent claim that pays in state $S^{t+1}|S^t$ scaled by the inverse of the probability of occurrence of the state in which the claim pays. Also, let $b_{t+1} \equiv b(S^{t+1}|S^t)$. Then, we can write the price of the portfolio as $\sum_{S^{t+1}|S^t} \pi(S^{t+1}|S^t)r_{t,t+1}b_{t+1}$. But this expression is simply the conditional expectation $E_tr_{t,t+1}b_{t+1}$.

Note that $E_tr_{t,t+1}$ is the price in period $t$ of an asset that pays 1 unit of consumption goods in every state of period $t+1$. It follows that

$$1 + r_t \equiv \frac{1}{E_tr_{t,t+1}}$$

represents the risk-free real interest rate in period $t$.

Households are assumed to be subject to a no-Ponzi-game constraint of the form

$$\lim_{j \to \infty} E_tr_{t,t+j}b_{t+j} \geq 0,$$

at all dates and under all contingencies. The variable

$$r_{t,t+j} \equiv r_{t,t+1}r_{t+1,t+2}\cdots r_{t+j-1,t+j}$$

represents the pricing kernel such that $E_tr_{t,t+j}b_{t+j}$ is the period-$t$ price of a stochastic payment
To characterize the household’s optimal plan, it is convenient to derive an intertemporal budget constraint. Begin by multiplying both sides of the sequential budget constraint (4.42) by \( r_{0,t} \). Then apply the conditional expectations operator \( E_0 \) to obtain

\[
E_0 r_{0,t} E_t r_{t,t+1} b_{t+1} = E_0 r_{0,t} [b_t + A_t F(k_t, h_t) - c_t - k_{t+1} + (1 - \delta) k_t - \Phi(k_{t+1} - k_t)].
\]

By the definition of the pricing kernel and the law of iterated expectations, we have that \( E_0 r_{0,t} E_t r_{t,t+1} b_{t+1} = E_0 r_{0,t+1} b_{t+1} \). So we can write the above expression as

\[
E_0 r_{0,t+1} b_{t+1} = E_0 r_{0,t} [b_t + A_t F(k_t, h_t) - c_t - k_{t+1} + (1 - \delta) k_t - \Phi(k_{t+1} - k_t)].
\]

Now sum this expression for \( t = 0 \) to \( t = T > 0 \). This yields

\[
E_0 r_{0,T+1} b_{T+1} = b_0 + E_0 \sum_{t=0}^{T} r_{0,t} [A_t F(k_t, h_t) - c_t - k_{t+1} + (1 - \delta) k_t - \Phi(k_{t+1} - k_t)].
\]

Take limit for \( T \to \infty \) and use the no-Ponzi-game constraint (4.43) to obtain

\[
b_0 \geq E_0 \sum_{t=0}^{\infty} r_{0,t} [c_t + k_{t+1} - (1 - \delta) k_t + \Phi(k_{t+1} - k_t) - A_t F(k_t, h_t)]. \tag{4.44}
\]

This expression states that the period-0 value of the stream of current and future trade deficits cannot exceed the value of the initial asset position \( b_0 \).

The household’s problem consists in choosing contingent plans \( \{c_t, h_t, k_{t+1}\} \) to maximize the lifetime utility function (4.1) subject to (4.44), given \( k_0, b_0 \), and exogenous processes \( \{A_t, r_{0,t}\} \).
The Lagrangian associated with this problem is

\[ \mathcal{L} = E_0 \sum_{t=0}^{\infty} \{ \beta^t U(c_t, h_t) + \xi_0 r_{0,t} [A_t F(k_t, h_t) - c_t - k_{t+1} + (1 - \delta)k_t - \Phi(k_{t+1} - k_t)] \} + \xi_0 b_0, \]

where \( \xi_0 > 0 \) denotes the Lagrange multiplier on the time-0 present-value budget constraint (4.44). The first-order conditions associated with the household’s maximization problem are (4.11), (4.18), (4.44) holding with equality, and

\[ \beta^t U(c_t, h_t) = \xi_0 r_{0,t}. \]  
(4.45)

Taking the ratio of this expression to itself evaluated in period \( t + 1 \) yields

\[ \frac{\beta U_c(c_{t+1}, h_{t+1})}{U_c(c_t, h_t)} = r_{t,t+1}, \]

which says that consumers equate their intertemporal marginal rate of substitution of current consumption for consumption in a particular state next period to the price of the corresponding state contingent claim scaled by the probability of occurrence of that state. Rearranging the above expression, taking expectations conditional on information available in period \( t \), and recalling the definition of the risk-free interest rate given above yields

\[ U_c(c_t, h_t) = \beta(1 + r_t) E_t U_c(c_{t+1}, h_{t+1}), \]

which is identical to equation (4.7) in the incomplete asset market version of the model. In other words, the complete asset market model generates a state-by-state version of the Euler equation implied by the incomplete-asset-market model, reflecting the fact that in the present environment consumers have more financial instruments available to diversify risk.

We assume that the economy is small and fully integrated to the international financial market.
Let \( r_{0,t}^* \) denote the pricing kernel prevailing in international financial markets. By the assumption of free capital mobility, we have that domestic asset prices must be equal to foreign asset prices, that is,

\[
    r_{0,t} = r_{0,t}^*
\]

(4.46)

for all dates and states. Foreign households are also assumed to have unrestricted access to international financial markets. Therefore, a condition like (4.45) must also hold abroad. Formally,

\[
    \beta_t U^*_c(c_t^*, h_t^*) = \xi_0 r_{0,t}^* \quad (4.47)
\]

Note that we are assuming that domestic and foreign households share the same subjective discount factor, \( \beta \). Combining (4.45)-(4.47) yields

\[
    U_c(c_t, h_t) = \frac{\xi_0}{\xi_0^*} U^*_c(c_t^*, h_t^*)
\]

for all dates and states. This expression says that under complete asset markets, the marginal utility of consumption is perfectly correlated across countries. The ratio \( \frac{\xi_0}{\xi_0^*} \) reflects differences in per capita wealth between the domestic economy and the rest of the world. Because the present model is one of a small open economy, \( c_t^* \) and \( h_t^* \) are taken as exogenously given. We endogenize the determination of \( c_t^* \) and \( h_t^* \) in exercise 4.9 at the end of this chapter. This exercise analyzes a two-country model with complete asset markets in which one country is large and the other is small.

Because the domestic economy is small, the domestic productivity shock \( A_t \) does not affect the foreign variables, which respond only to foreign shocks. The domestic economy, however, can be affected by foreign shocks via \( c_t^* \) and \( h_t^* \). To be in line with the stochastic structure of the EDEIR model, we shut down all foreign shocks and focus attention only on the effects of innovations in
domestic productivity. Therefore, we assume that the foreign marginal utility of consumption is time invariant and given by $U^*_c(c^*, h^*)$, where $c^*$ and $h^*$ are constants. Let $\psi_{cam} \equiv \xi_0 U^*_{c^*}(c^*, h^*)$. Then, we can rewrite the above expression as

$$U_c(c_t, h_t) = \psi_{cam}. \quad (4.48)$$

This expression reflects the fact that, because domestic consumers have access to a complete set of Arrow-Debreu contingent assets, they can fully diversify domestic risk. Thus, domestic consumers are exposed only to aggregate external risk. We are assuming that aggregate external risk is nil. As a result, by appropriately choosing their asset portfolios, domestic consumers can attain a constant marginal utility of consumption at all times and under all contingencies. Exercise 4.3 at the end of this chapter studies a version of the present model in which $\psi_{cam}$ is stochastic, reflecting the presence of external shocks.

The competitive equilibrium of the CAM economy is a set of processes $\{c_t, h_t, k_{t+1}, A_t\}$ satisfying (4.11), (4.12), (4.18), and (4.48), given $A_0$, $k_0$, and the exogenous process $\{\epsilon_t\}$.

The CAM model delivers stationary processes for all variables of interest. This means that replacing the assumption of incomplete asset markets for the assumption of complete asset markets eliminates the endogenous random walk problem that plagues the dynamics of the one-bond economy. The key feature of the complete asset market responsible for its stationarity property is equation (4.48), which states that with complete asset markets the marginal utility of consumption is constant. By contrast, in the one-bond model, in the absence of any ad-hoc stationarity inducing feature, the marginal utility of consumption follows a random walk. To see this, set $\beta(1 + r_t) = 1$ for all $t$ in equation (4.7).

We now wish to shed light on a question that arises often in models with complete asset markets, namely, what is the current account when financial markets are complete? In the one-bond economy
the answer is simple: the current account can be measured either by changes in net holdings of
the internationally traded bond or by the sum of the trade balance and net interest income paid
by the single bond. Under complete asset markets, there is a large (possibly infinite) number of
state-contingent financial assets, each with different returns. As a result, it is less clear how to
keep track of the country’s net foreign asset position or of its net investment income. It turns out
that there is a simple way of characterizing and computing the equilibrium level of the current
account. Let us begin by addressing the simpler question of defining the trade balance. As in the
one-bond model, the trade balance in the CAM model is simply given by equation (4.20). The
current account can be defined as the change in the country’s net foreign asset position. Let

\[ s_t \equiv E_t r_{t,t+1} b_{t+1} \]

denote the net foreign asset position at the end of period \( t \). Then, the current account is given by

\[ ca_t = s_t - s_{t-1}. \]

Alternatively, the current account can be expressed as the sum of the trade balance and net in-
vestment income. In turn, net investment income is given by the difference between the payoff in
period \( t \) of assets acquired in \( t - 1 \), given by \( b_t \), and the resources spent in \( t - 1 \) on purchases of
contingent claims, given by \( E_{t-1} r_{t-1,t} b_t \). Thus, the current account is given by

\[ ca_t = tb_t + b_t - E_{t-1} r_{t-1,t} b_t. \]

To see that the above two definitions of the current account are identical, use the definition of the
trade balance, equation (4.20), and the definition of the net foreign asset position \( s_t \) to write
Table 4.3: The SOE-RBC Model With Complete Asset Markets: Predicted Second Moments

<table>
<thead>
<tr>
<th>Variable</th>
<th>( \sigma_{x_t} )</th>
<th>( \rho_{x_{t-1},x_t} )</th>
<th>( \rho_{x_{t},\text{GDP}_t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>3.1</td>
<td>0.61</td>
<td>1.00</td>
</tr>
<tr>
<td>( c )</td>
<td>1.9</td>
<td>0.61</td>
<td>1.00</td>
</tr>
<tr>
<td>( i )</td>
<td>9.1</td>
<td>0.07</td>
<td>0.66</td>
</tr>
<tr>
<td>( h )</td>
<td>2.1</td>
<td>0.61</td>
<td>1.00</td>
</tr>
<tr>
<td>( \frac{\bar{\vartheta}}{y} )</td>
<td>1.6</td>
<td>0.39</td>
<td>0.13</td>
</tr>
<tr>
<td>( \frac{\bar{c}}{y} )</td>
<td>3.1</td>
<td>-0.07</td>
<td>-0.49</td>
</tr>
</tbody>
</table>

Note. Standard deviations are measured in percentage points. Matlab code to produce this table is available at http://www.columbia.edu/~mu2166/closing.htm.

Sequential resource constraint (4.42) as

\[ s_t = tb_t + b_t \]

Subtracting \( s_{t-1} \) from both sides of this expression, we have

\[ s_t - s_{t-1} = tb_t + b_t - E_{t-1}r_{t-1,i}b_t. \]

The left-hand side of this expression is our first definition of the current account, and the right-hand side our second definition.

The functions \( U, F, \) and \( \Phi \) are parameterized as in the EDEIR model. The parameters \( \sigma, \beta, \omega, \alpha, \phi, \delta, \rho, \) and \( \bar{\eta} \) take the values displayed in table 4.1. The parameter \( \psi_{cam} \) is set so as to ensure that the steady-state levels of consumption in the CAM and EDEIR models are the same.

Table 4.3 displays unconditional second moments predicted by the SOE-RBC model with complete asset markets. The predictions of the model regarding output, consumption, investment, and
the trade balance are qualitatively similar to those of the (EDEIR) incomplete-asset-market model. In particular, the model preserves the volatility ranking of output, consumption, investment, and the trade balance. Also, the domestic components of aggregate demand are all positively serially correlated and procyclical. Note that the correlation of consumption with output is now unity. This prediction of the CAM model is a consequence of assuming complete markets and GHH preferences. Under complete asset markets the marginal utility of consumption is constant over time, so that up to first order consumption is linear in hours. In turn, with GHH preferences, as we deduced earlier in this chapter, hours are linearly related to output up to first order. A significant difference between the predictions of the complete- and incomplete-asset-market models is that the former implies a highly countercyclical current account, whereas the latter implies an acyclical current account.

4.10 Alternative Ways to Induce Stationarity

The small open economy RBC model analyzed thus far features a debt-elastic country interest-rate premium. As mentioned earlier in this chapter, the inclusion of a debt-elastic premium responds to the need to obtain stationary dynamics up to first order. Had we assumed a constant interest rate, the linearized equilibrium dynamics would have contained an endogenous random walk component and the steady state would have depended on initial conditions. Two problems emerge when the linear approximation possesses a unit root. First, one can no longer claim that when the support of the underlying shocks is sufficiently small the linear system behaves like the original nonlinear system, which is ultimately the focus of interest. Second, when the variables of interest contain random walk elements, it is impossible to compute unconditional first and second moments, such as standard deviations, serial correlations, correlations with output, etc., which are the most common descriptive statistics of the business cycle.
Nonstationarity arises in the small open economy model from three features: an exogenous cost of borrowing in international financial markets, an exogenous subjective discount factor, and incomplete asset markets. Accordingly, in this section we study stationarity inducing devises that consist in altering one of these three features. Our analysis follows closely Schmitt-Grohé and Uribe (2003), but expands their analysis by including two additional approaches to inducing stationarity, a model with an internal interest-rate premium, a model with perpetually-young consumers, and a model in which stationarity is induced by approximating the equilibrium using global methods making agents slightly impatient by making the discount facor $\beta$ smaller than the pecuniary discount factor $1/(1 + r^*)$.

One important question is whether the different stationarity inducing devises affect the predicted business cycle of the small open economy. A result of this section is that, given a common calibration, all models considered deliver similar business cycles.

Before plunging into details, it is important to note that the nature of the non-stationarity that is present in the small open economy model is different from the one that emerges from the introduction of non-stationary exogenous shocks. In the latter case, it is typically possible to find a transformation of variables that renders the model economy stationary in terms of the transformed variables. We will study an economy with non-stationary shocks and provide an example of a stationarity inducing transformation in section 5.2 of chapter 5. By contrast, the nonstationarity that arises in the small open economy model with an exogenous cost of borrowing, an exogenous rate of time preference, and incomplete markets cannot be eliminated by any variable transformations.

The section proceeds by first presenting and calibrating the different stationarity inducing theoretical devises and then comparing the quantitative predictions of the various models.
4.10.1 Internal Debt-Elastic Interest Rate (IDEIR)

The EDEIR model studied thus far assumes that the country interest-rate premium depends upon the cross-sectional average of external debt. As a result, households take the country premium as exogenously given. The model with an internal debt-elastic interest rate assumes instead that the interest rate faced by domestic agents is increasing in the individual debt position, \( d_t \). Consequently, households internalize the effect that their borrowing choices have on the interest rate they face. In all other aspects, the IDEIR and EDEIR models are identical.

Formally, in the IDEIR model the interest rate is given by

\[
r_t = r^* + p(d_t),
\]

where \( r^* \), as before, denotes the world interest rate, but now \( p(\cdot) \) is a household-specific interest-rate premium. Note that the argument of the interest-rate premium function is the household’s own net debt position. This means that in deciding its optimal expenditure and savings plan, the household will take into account the fact that a change in its debt position alters the marginal cost of funds. The only optimality condition that changes relative to the EDEIR model is the Euler equation for debt accumulation, which now takes the form

\[
U_c(c_t, h_t) = \beta [1 + r^* + p(d_t) + p'(d_t)d_t]E_tU_c(c_{t+1}, h_{t+1}).
\]

This expression features the derivative of the premium with respect to debt because households internalize the fact that as their net debt increases, so does the interest rate they face in financial markets. As a result, in the margin, the household cares about the marginal cost of borrowing \( 1 + r^* + p(d_t) + p'(d_t)d_t \) and not about the average cost of borrowing, \( 1 + r^* + p(d_t) \).

The competitive equilibrium of the IDEIR economy is a set of processes \( \{d_t, c_t, h_t, k_{t+1}, A_t\} \)
satisfying (4.11), (4.12), (4.16), (4.18), (4.19), and (4.50), given \( A_0, d_{-1}, \) and \( k_0, \) and the process \( \{\epsilon_t\}. \)

We assume the same functional forms and parameter values as in the EDEIR model (see section 4.3). We note that in the model analyzed here the steady-state level of debt is no longer equal to \( \overline{d}. \) To see this, recall that \( \beta(1 + r^*) = 1 \) and note that the steady-state version of equation (4.50) imposes the following restriction on \( d, \)

\[
(1 + d)e^{d - \overline{d}} = 1,
\]

which does not admit the solution \( d = \overline{d}, \) except in the special case in which \( \overline{d} = 0. \) We set \( \overline{d} = 0.7442, \) which is the value imposed in the EDEIR model. The implied steady-state level of debt is then given by \( d = 0.4045212. \) Intuitively, households internalize that their own debt position drives up the interest rate, hence they choose to borrow less than households in the EDEIR economy, who fail to internalize the dependence of the interest rate on the stock of debt. In this sense, one can say that households in the EDEIR economy overborrow. The fact that the steady-state debt is lower than \( \overline{d} \) implies that the country premium is negative in the steady state. However, the marginal country premium, given by \( p(d_t) + p'(d_t)d_t, \) is nil in the steady state, as it is in the EDEIR economy. Recall that in the EDEIR economy, the marginal and average premia perceived by households are equal to each other and given by \( p(d_t). \) An alternative calibration strategy is to impose \( d = \overline{d}, \) and to adjust \( \beta \) to ensure that equation (4.50) holds in the deterministic steady state. In this case, the country premium vanishes in the steady state, but the marginal premium is positive and equal to \( \psi_1\overline{d}. \)
4.10.2 Portfolio Adjustment Costs (PAC)

In the portfolio adjustment cost (PAC) model, stationarity is induced by assuming that agents face convex costs of holding assets in quantities different from some long-run level. Preferences and technology are as in the EDEIR model. However, in contrast to what is assumed in that model, in the PAC model the interest rate at which domestic households can borrow from the rest of the world is assumed to be constant and equal to the world interest rate, \( r^* \), that is, the country premium is nil at all times. The sequential budget constraint of the household is given by

\[
d_t = (1 + r^*)d_{t-1} - A_t F(k_t, h_t) + c_t + k_{t+1} - (1 - \delta)k_t + \Phi(k_{t+1} - k_t) + \Psi(d_t),
\]

(4.51)

where \( \Psi(\cdot) \) is a convex portfolio adjustment cost function satisfying \( \Psi(d) = \Psi'(d) = 0 \), for some \( d \).

The first-order conditions associated with the household’s maximization problem are identical to those associated with the EDEIR model, except that the Euler condition for debt, equation (4.17), now becomes

\[
U_c(c_t, h_t) = \beta \frac{1 + r^*}{1 - \Psi'(d_t)} E_t U_c(c_{t+1}, h_{t+1}).
\]

(4.52)

This optimality condition implies that the effective interest rate faced by the household, which we denote \( r_t \), is debt elastic and given by

\[
1 + r_t = \frac{1 + r^*}{1 - \Psi'(d_t)}.
\]

(4.53)

Because the portfolio adjustment cost function is convex, the effective interest rate is increasing in the stock of debt. In this regard, the PAC model is a close relative of the EDEIR model, as can be seen by comparing the above Euler equation with its counterpart in the EDEIR model, given by equation (4.17).

The specification adopted here assumes that households directly borrow from abroad. This
setup can be decentralized as follows. Suppose that households face no portfolio adjustment costs and can borrow and lend at the interest rate \( r_t \), which they take as exogenously given and, in particular, as independent of their own debt positions. Their sequential budget constraint is then given by 
\[
\tilde{d}_t = (1 + r_{t-1})\tilde{d}_{t-1} - A_t F(k_t, h_t) + c_t + k_{t+1} - (1 - \delta)k_t + \Phi(k_{t+1} - k_t) - \Pi_t,
\]
where \( \tilde{d}_t \) denotes household debt in period \( t \) and \( \Pi_t \) denotes profit income in period \( t \), which the household takes as exogenously given. The optimality conditions associated with the household problem are identical to those in centralized version of the model, except that the Euler equation now becomes
\[
U_c(c_t, h_t) = \beta(1 + r_t)E_t U_c(c_{t+1}, h_{t+1}).
\]

Assume that financial transactions between domestic and foreign residents are intermediated by domestic financial institutions, or banks. Suppose that there is a continuum of banks of measure one that behave competitively. They capture funds, \( d_t \), from foreign investors at the world interest rate \( r^* \) and lend \( \tilde{d}_t \) to domestic agents at the interest rate \( r_t \). Banks face operational costs, \( \Psi(d_t) \), that are increasing and convex in the volume of intermediation, \( d_t \). Bank profits in period \( t+1 \) are given by 
\[
\Pi_{t+1} = (1 + r_t)\tilde{d}_t - (1 + r^*)d_t.
\]
Banks are subject to the resource constraint \( \tilde{d}_t = d_t - \Psi(d_t) \).

The problem of domestic banks is then to choose \( \tilde{d}_t \) and \( d_t \) to maximize profits subject to the resource constraint, taking \( r_t \) as given. The first-order condition associated with the bank’s profit maximization problem is
\[
1 + r_t = \frac{1 + r^*}{1 - \Psi'(d_t)},
\]
which is identical to equation (4.53). Each period bank profits are distributed to domestic households in a lump-sum fashion. Replacing the expression for bank profits in the household’s budget constraint and using the bank’s resource constraint yields the budget constraint of the centralized economy, equation (4.51). It follows that the equilibrium allocations of the centralized and the decentralized economies are the same.

The competitive equilibrium of the PAC economy is a set of processes \( \{d_t, c_t, h_t, k_{t+1}, A_t\} \) satisfying (4.11), (4.12), (4.18), (4.19), (4.51), and (4.52), given \( A_0, d_{-1}, \) and \( k_0, \) and the process \( \{\epsilon_t\} \).
The world interest rate is assumed to satisfy

$$\beta(1 + r^*) = 1.$$  

This assumption implies that in the steady state, the Euler equation (4.52) becomes

$$\Psi'(d) = 0,$$

where $d$ denotes the steady-state value of debt. The assumptions imposed on the portfolio adjustment cost $\Psi(\cdot)$ imply that the unique solution to the above expression is $d = \bar{d}$. It follows that the steady-state level of debt is independent of initial conditions.

We assume a quadratic form for $\Psi(\cdot)$,

$$\Psi(d) = \frac{\psi_2}{2}(d - \bar{d})^2,$$

where $\psi_2$ and $\bar{d}$ are constant parameters defining the portfolio adjustment cost function. The remaining functional forms and the calibration of common parameters are as in the EDEIR model. We calibrate $\bar{d}$ to 0.7442, which is the same value as in the EDEIR model. This means that the steady-state values of all endogenous variables are the same in the PAC and EDEIR models. We set $\psi_2$ at 0.00074, which ensures that the volatility of the current-account-to-output ratio is the same as in the EDEIR model.

At this point, it might be natural to expect the analysis of an external version of the PAC model in which the portfolio adjustment cost depends on the aggregate level of debt, $\bar{d}_t$, as opposed to the individual debt position $d_t$. However, this modification would fail to render the small open economy model stationary. The reason is that in this case, the optimality condition with respect to debt, given by equation (4.52) in the PAC model, would become $U_c(c_t, h_t) = \beta(1 + r^*)E_tU_c(c_{t+1}, h_{t+1}),$
which, because $\beta(1 + r^*)$ equals one, implies that the marginal utility of consumption follows a random walk, and is therefore nonstationary.

### 4.10.3 External Discount Factor (EDF)

We next study an SOE RBC model in which stationarity is induced by assuming that the subjective discount factor depends upon endogenous variables. Specifically, we consider a preference specification in which the discount factor depends on endogenous variables that are taken as exogenous by individual households. We refer to this environment as the external discount factor (EDF) model.

Suppose that the discount factor depends on the average per capita levels of consumption and hours worked. Formally, preferences are described by

$$
E_0 \sum_{t=0}^{\infty} \theta_t U(c_t, h_t) \quad (4.54)
$$

$$
\theta_{t+1} = \beta(\bar{c}_t, \bar{h}_t)\theta_t \quad t \geq 0, \quad \theta_0 = 1; \quad (4.55)
$$

where $\bar{c}_t$ and $\bar{h}_t$ denote the cross-sectional averages of per capita consumption and hours, respectively, which the individual household takes as exogenously given.

In the EDF model, the interest rate is assumed to be constant and equal to $r^*$. The sequential budget constraint of the household therefore takes the form

$$
d_t = (1 + r^*)d_{t-1} - A_t F(k_t, h_t) + c_t + k_{t+1} - (1 - \delta)k_t + \Phi(k_{t+1} - k_t), \quad (4.56)
$$

and the no-Ponzi-game constraint simplifies to $\lim_{j \to \infty} (1 + r^*)^{-j} E_t d_{t+j} \leq 0$.

The first-order conditions associated with the household’s maximization problem are (4.11), (4.56), and

$$
U_c(c_t, h_t) = \beta(\bar{c}_t, \bar{h}_t)(1 + r^*)E_t U_c(c_{t+1}, h_{t+1}) \quad (4.57)
$$
\[ U_c(c_t, h_t)[1 + \Phi'(k_{t+1} - k_t)] = \beta(\bar{c}_t, \bar{h}_t)E_t U_c(c_{t+1}, h_{t+1}) \left[ A_{t+1} F_k(k_{t+1}, h_{t+1}) + 1 - \delta + \Phi'(k_{t+2} - k_{t+1}) \right] \]  

(4.58)

\[ \lim_{j \to \infty} E_t \frac{d_{t+j}}{(1 + r^*)^j} = 0. \]  

(4.59)

In equilibrium, individual and average per capita levels of consumption and effort are identical. That is,

\[ c_t = \bar{c}_t \]  

(4.60)

and

\[ h_t = \bar{h}_t. \]  

(4.61)

A competitive equilibrium is a set of processes \{d_t, c_t, h_t, \bar{c}_t, \bar{h}_t, k_{t+1}, A_t\} satisfying (4.11), (4.12), and (4.56)-(4.61), given \( A_0, d_{-1}, \) and \( k_0 \) and the stochastic process \{\epsilon_t\}.

We evaluate the model using the same functional forms for the period utility function, the production function, and the capital adjustment cost function as in the EDEIR model. We assume that the subjective discount factor is of the form

\[ \beta(c, h) = \left( 1 + c - \frac{h^\omega}{\omega} \right)^{-\psi_3}, \]  

with \( \psi_3 > 0 \), so that increases in consumption or leisure make households more impatient.

To see that in the EDF model the steady-state level of debt is determined independently of initial conditions, start by noticing that in the steady state, equation (4.57) implies that

\[ \beta(c, h)(1 + r^*) = 1, \]

where \( c \) and \( h \) denote the steady-state values of consumption and hours. Next, notice that, given this result, the steady-state values of hours, capital \( (k) \), and output \( (k^\alpha h^{1-\alpha}) \) can be found in
exactly the same way as in the EDEIR model, with $\beta$ replaced by $(1 + r^\ast)^{-1}$. Notice that $k$ and $h$ depend only on the deep structural parameters $r^\ast$, $\alpha$, $\omega$, and $\delta$. With $h$ in hand, the above expression delivers $c$, which depends only on the deep structural parameters defining $h$ and on $\psi_3$.

Finally, in the steady state, the resource constraint (4.56) implies that the steady state level of debt, $d$, is given by $d = (c + \delta k - k^\alpha h^{1-\alpha})/r^\ast$, which depends only on structural parameters.

The EDF model features one new parameter relative to the EDEIR model, namely the elasticity of the discount factor relative to the composite $1 + c_t - h_t^\omega/\omega$. We set $\psi_3$ to ensure that the steady-state trade-balance-to-output ratio equals 2 percent, in line with the calibration of the EDEIR model. The implied value of $\psi_3$ is 0.11.

Note that in our assumed specification of the endogenous discount factor, the parameter $\psi_3$ governs both the steady-state trade-balance-to-output ratio and the stationarity of the equilibrium dynamics. This dual role may create a conflict. On the one hand, one may want to set $\psi_3$ at a small value so as to ensure stationarity without affecting the predictions of the model at business-cycle frequency. On the other hand, matching the observed average trade-balance-to-output ratio might require a value of $\psi_3$ that does affect the behavior of the model at business-cycle frequency. For this reason, it might be useful to consider a two-parameter specification of the discount factor, such as $\beta(c_t, h_t) = (\tilde{\psi}_3 + c_t - \omega^{-1}h_t^\omega)^{-\psi_3}$, where $\tilde{\psi}_3 > 0$ is a parameter. With this specification, one can fix the parameter $\psi_3$ at a small value, just to ensure stationarity, and set the parameter $\tilde{\psi}_3$ to match the observed trade-balance-to-output ratio.

### 4.10.4 Internal Discount Factor (IDF)

Consider now a variation of the EDF model in which the subjective discount factor depends on the individual levels of consumption and hours worked rather than on the aggregate levels. Specifically,
suppose that preferences are given by equation (4.54), with the following law of motion for $\theta_t$:

$$\theta_{t+1} = \beta(c_t, h_t)\theta_t \quad t \geq 0, \quad \theta_0 = 1.$$  \hspace{1cm} (4.62)

This preference specification was conceived by Uzawa (1968) and introduced in the small-open-economy literature by Mendoza (1991). Under these preferences, households internalize that their choices of consumption and leisure affect their valuations of future period utilities.

Households choose processes $\{c_t, h_t, k_{t+1}, d_t, \theta_{t+1}\}_{t=0}^{\infty}$ so as to maximize the utility function (4.54) subject to the sequential budget constraint (4.56), the law of motion of the discount factor (4.62), and the same no-Ponzi constraint as in the EDF economy. Let $\theta_t \lambda_t$ denote the Lagrange multiplier associated with (4.56) and $\theta_t \eta_t$ the Lagrange multiplier associated with (4.62). The first-order conditions associated with the household’s maximization problem are (4.56), (4.59), and

$$\lambda_t = \beta(c_t, h_t)(1 + \tau_t)E_t \lambda_{t+1} \hspace{1cm} (4.63)$$

$$U_c(c_t, h_t) - \eta_t \beta_c(c_t, h_t) = \lambda_t \hspace{1cm} (4.64)$$

$$-U_h(c_t, h_t) + \eta_t \beta_h(c_t, h_t) = \lambda_t A_t F_k(k_t, h_t) \hspace{1cm} (4.65)$$

$$\eta_t = -E_t U(c_{t+1}, h_{t+1}) + E_t \eta_{t+1} \beta(c_{t+1}, h_{t+1}) \hspace{1cm} (4.66)$$

$$\lambda_t [1 + \Phi'(k_{t+1} - k_t)] = \beta(c_t, h_t)E_t \lambda_{t+1} \left[A_{t+1} F_k(k_{t+1}, h_{t+1}) + 1 - \delta + \Phi'(k_{t+2} - k_{t+1}) \right] \hspace{1cm} (4.67)$$

These first-order conditions are fairly standard, except for the fact that the marginal utility of consumption is not given simply by $U_c(c_t, h_t)$ but rather by $U_c(c_t, h_t) - \beta_c(c_t, h_t) \eta_t$. The second term in this expression reflects the fact that an increase in current consumption lowers the discount factor ($\beta_c < 0$). In turn, a unit decline in the discount factor reduces utility in period $t$ by $\eta_t$. Intuitively,
$-\eta_t$ equals the expected present discounted value of utility from period $t + 1$ onward. To see this, iterate the first-order condition (4.66) forward to obtain $\eta_t = -E_t \sum_{j=1}^{\infty} \left( \frac{\theta_{t+1}}{\eta_{t+1}} \right) U(c_{t+j}, h_{t+j})$. Similarly, the marginal disutility of labor is not simply $U_h(c_t, h_t)$ but instead $U_h(c_t, h_t) - \beta_h(c_t, h_t) \eta_t$.

The competitive equilibrium of the IDF economy is a set of processes $\{d_t, c_t, h_t, k_{t+1}, \eta_t, \lambda_t, A_t\}$ satisfying (4.12), (4.56), (4.59), and (4.63)-(4.67), given the initial conditions $A_0$, $d_{-1}$, and $k_0$ and the exogenous process $\{\epsilon_t\}$.

We pick the same functional forms as in the EDF model. The fact that both the period utility function and the discount factor have a GHH structure implies that, as in all versions of the SOE RBC model considered thus far, the marginal rate of substitution between consumption and leisure depends only on hours worked and is independent of consumption. This yields the, by now familiar, equilibrium condition $h_\omega (1 + r^*) = 1$, which also features in the EDF model. Also, in the steady state, equation (4.67) yields an expression for the capital-labor ratio that is the same as in all versions of the SOE RBC model considered thus far. Finally, the fact that the labor supply schedule and the sequential budget constraint are identical in the EDF and IDF models, implies that $h$, $c$, and $d$ are also equal across the two models. This shows that the IDF model delivers a steady-state value of debt that is independent of initial conditions. Of course, the IDF model includes the variable $\eta_t$, which does not feature in the EDF model. The steady-state value of this variable is given by $-U(c, h)/r^*$.

Finally, we assign the same values to the structural parameters as in the EDF model.

### 4.10.5 The Model With No Stationarity Inducing Features (NSIF)

For comparison with the models studied thus far, we now consider a version of the small open economy RBC model featuring no stationarity inducing features. In this model (a) the discount
factor is constant; (b) the interest rate at which domestic agents borrow from the rest of the world is constant (and equal to the subjective discount rate, $\beta(1 + r^*) = 1$); (c) agents face no frictions in adjusting the size of their asset portfolios; and (d) markets are incomplete, in the sense that domestic households have only access to a single risk-free international bond. Under this specification, the deterministic steady state of consumption depends on the assumed initial level of net foreign debt. Also, up to first order, the equilibrium dynamics contain a random walk component in variables such as consumption, the trade balance, and net external debt.

A competitive equilibrium in the nonstationary model is a set of processes $\{d_t, c_t, h_t, k_{t+1}, A_t\}$ satisfying (4.11), (4.12), (4.18), (4.56), (4.59), and the consumption Euler equation

$$U_c(c_t, h_t) = \beta(1 + r^*)E_tU_c(c_{t+1}, h_{t+1}),$$

given $d_{-1}$, $k_0$, $A_0$, and the exogenous process $\{\epsilon_t\}$.

It is clear from the above consumption Euler equation that in the present model the marginal utility of consumption follows a random walk (recall that $\beta(1 + r^*) = 1$). This property is transmitted to consumption, debt, and the trade balance. Also, because the above Euler equation imposes no restriction in the deterministic steady state, the steady-state values of consumption, debt, and the trade balance are all indeterminate. The model does deliver unique deterministic steady-state values for $k_t$ and $h_t$. We calibrate the parameters $\sigma$, $r^*$, $\omega$, $\alpha$, $\phi$, $\delta$, $\rho$, and $\bar{\eta}$ using the values displayed in tables 4.1.

4.10.6 The Perpetual-Youth Model (PY)

In this subsection, we present an additional way to induce stationarity in the small open economy RBC model. It is a discrete-time, stochastic, small-open-economy version of the perpetual-youth model due to Blanchard (1985). Cardia (1991) represents an early adoption of the perpetual-youth
model in the context of a small open economy. Our model differs from Cardia’s in that we assume a preference specification that allows for an exact aggregation of this model. Our strategy avoids the need to resort to linear approximations prior to aggregation.

The Basic Intuition

The basic intuition behind why the assumption of finite lives by itself helps to eliminate the unit root in the aggregate net foreign asset position can be seen from the following simple example. Consider an economy in which debt holdings of individual agents follow a pure random walk of the form \( d_{s,t} = d_{s,t-1} + \mu_t \). Here, \( d_{s,t} \) denotes the net debt position at the end of period \( t \) of an agent born in period \( s \), and \( \mu_t \) is an exogenous shock common to all agents and potentially serially correlated. This is exactly the equilibrium evolution of debt we obtained in the quadratic-preference, representative-agent economy of chapter 2, see equation (??). We now depart from the representative-agent assumption by introducing a constant and age-independent probability of death at the individual level. Specifically, assume that the population is constant over time and normalized to unity. Each period, individual agents face a probability \( 1 - \theta \in (0,1) \) of dying. In addition, to keep the size of the population constant over time, we assume that \( 1 - \theta \) agents are born each period. Assume that those agents who die leave their outstanding debts unpaid and that newborns inherit no debts. Adding the left- and right-hand sides of the law of motion for debt over all agents alive in period \( t \)—i.e., applying the operator \((1 - \theta) \sum_{s=t}^{\infty} \theta^{t-s}\) on both sides of the expression \( d_{s,t} = d_{s,t-1} + \mu_t \)—yields \( d_t = \theta d_{t-1} + \mu_t \), where \( d_t \) denotes the aggregate debt position in period \( t \). In performing the aggregation, recall that \( d_{t,t-1} = 0 \), because agents are born free of debts. Clearly, the resulting law of motion for the aggregate level of debt is mean reverting at the survival rate \( \theta \). The key difference with the representative agent model is that here each period a fraction \( 1 - \theta \) of the stock of debt simply disappears.

In what follows, we embed this basic stationarity result into the small-open-economy real-
business-cycle model.

**Households**

Each agent maximizes the utility function

$$\frac{-1}{2} E_0 \sum_{t=0}^{\infty} (\beta \theta)^t (x_{s,t} - \overline{\tau})^2$$

with

$$x_{s,t} = c_{s,t} - \frac{h_{s,t}}{\omega},$$

where $c_{s,t}$ and $h_{s,t}$ denote consumption and hours worked in period $t$ by an agent born in period $s$. The parameter $\beta \in (0, 1)$ represents the subjective discount factor, and $\overline{\tau}$ is a parameter denoting a satiation point. The symbol $E_t$ denotes the conditional expectations operator over aggregate states. Following the preference specification used in all of the models studied in this chapter, we assume that agents derive utility from a quasi-difference between consumption and leisure. But we depart from the preference specifications used earlier in this chapter by assuming a quadratic period utility index. As will become clear shortly, this assumption is essential to achieve aggregation in the presence of aggregate uncertainty.

Financial markets are incomplete. Domestic consumers can borrow internationally by means of a bond paying a constant real interest rate. The debts of deceased domestic consumers are assumed to go unpaid. Foreign agents are assumed to lend to a large number of domestic consumers so that the fraction of unpaid loans due to death is deterministic. To compensate foreign lenders for these losses, domestic consumers pay a constant premium over the world interest rate. Specifically, the gross interest rate at which domestic consumers borrow internationally is $(1 + r^*)/\theta$, where $r^*$ denotes the world interest rate. Domestic agents can also lend internationally. The lending contract stipulates that should the domestic lender die, the foreign borrower is relieved of his debt
obligations. Since foreign borrowers can perfectly diversify their loans across domestic agents, they pay a deterministic interest rate. To eliminate pure arbitrage opportunities, domestic consumers must lend at the rate \((1 + r^*)/\theta\). It follows that the gross interest rate on the domestic consumer’s asset position (whether this position is positive or negative) is given by \((1 + r^*)/\theta\).

The budget constraint of a domestic consumer born in period \(s \leq t\) is

\[
d_{s,t} = \left(\frac{1 + r^*}{\theta}\right) d_{s,t-1} + c_{s,t} - \pi_t - w_t h_{s,t},
\]

(4.69)

where \(\pi_t\) and \(w_t\) denote, respectively, profits received from the ownership of stock shares and the real wage rate. To facilitate aggregation, we assume that agents do not trade shares and that the shares of the dead are passed to the newborn in an egalitarian fashion. Thus, share holdings are identical across agents. Agents are assumed to be subject to the following no-Ponzi-game constraint

\[
\lim_{j \to \infty} E_t \left(\frac{\theta}{1 + r^*}\right)^j d_{s,t+j} \leq 0.
\]

(4.70)

The first-order conditions associated with the agent’s maximization problem are (4.68), (4.69), (4.70) holding with equality, and

\[
- (x_{s,t} - \bar{x}) = \lambda_{s,t},
\]

(4.71)

\[
h_{s,t}^{\omega-1} = w_t,
\]

(4.72)

and

\[
\lambda_{s,t} = \beta(1 + r^*)E_t \lambda_{s,t+1}.
\]

(4.73)

Note that \(h_{s,t}\) is independent of \(s\) (i.e., it is independent of the agent’s birth date). This means
that we can drop the subscript $s$ from $h_{s,t}$ and write

$$h_t^{\omega^{-1}} = w_t.$$  \hfill (4.74)

Use equations (4.68) and (4.72) to eliminate $c_{s,t}$ from the sequential budget constraint (4.69). This yields

$$d_{s,t} = \left( \frac{1 + r^*}{\theta} \right) d_{s,t-1} - \pi_t - \left( 1 - \frac{1}{\omega} \right) w_t h_t + \bar{x} + (x_{s,t} - \bar{x}).$$

To facilitate notation, we introduce the auxiliary variable

$$z_t \equiv \pi_t + \left( 1 - \frac{1}{\omega} \right) w_t h_t - \bar{x},$$  \hfill (4.75)

which is the same for all generations $s$ because both profits and hours worked are independent of the age of the cohort. Then the sequential budget constraint becomes

$$d_{s,t} = \left( \frac{1 + r^*}{\theta} \right) d_{s,t-1} - z_t + (x_{s,t} - \bar{x}).$$  \hfill (4.76)

Now iterate this expression forward, apply the $E_t$ operator, and use the transversality condition (i.e., equation (4.70) holding with equality), to obtain

$$\left( \frac{1 + r^*}{\theta} \right) d_{s,t-1} = E_t \sum_{j=0}^{\infty} \left( \frac{\theta}{1 + r^*} \right)^j \left[ z_{t+j} - (x_{s,t+j} - \bar{x}) \right].$$

Using equations (4.71) and (4.73) to replace $E_t x_{s,t+j}$ yields

$$\left( \frac{1 + r^*}{\theta} \right) d_{s,t-1} = E_t \sum_{j=0}^{\infty} \left( \frac{\theta}{1 + r^*} \right)^j z_{t+j} - \frac{\beta(1 + r^*)^2}{\beta(1 + r^*)^2 - \theta} (x_{s,t} - \bar{x}).$$
Solve for $x_{s,t}$ to obtain

$$x_{s,t} = \bar{x} + \frac{\beta(1 + r^*)^2 - \theta}{\beta \theta (1 + r^*)} (\bar{z}_t - d_{s,t-1}),$$

(4.77)

where

$$\bar{z}_t \equiv \frac{\theta}{1 + r^*} E_t \sum_{j=0}^{\infty} \left( \frac{\theta}{1 + r^*} \right)^j z_{t+j}$$

denotes the weighted average of current and future expected values of $z_t$. It can be expressed recursively as

$$\bar{z}_t = \frac{\theta}{1 + r^*} z_t + \frac{\theta}{1 + r^*} E_t \bar{z}_{t+1}.$$  

(4.78)

We now aggregate individual variables by summing over generations born at time $s \leq t$. Notice that at time $t$ there are alive $1 - \theta$ people born in $t$, $(1 - \theta)\theta$ people born in $t - 1$, and, in general, $(1 - \theta)\theta^s$ people born in period $t - s$. Let

$$x_t \equiv (1 - \theta) \sum_{s=t}^{-\infty} \theta^{t-s} x_{s,t}$$

and

$$d_t \equiv (1 - \theta) \sum_{s=t}^{-\infty} \theta^{t-s} d_{s,t}$$

denote the aggregate levels of $x_{s,t}$ and $d_{s,t}$, respectively. Now multiply (4.77) by $(1 - \theta)\theta^{t-s}$ and then sum for $s = t$ to $s = -\infty$ to obtain the following expression for the aggregate version of equation (4.77):

$$x_t = \bar{x} + \frac{\beta(1 + r^*)^2 - \theta}{\beta \theta (1 + r^*)} (\bar{z}_t - \theta d_{t-1}).$$

(4.79)

In performing this step, keep in mind that $d_{t,t-1} = 0$. That is, consumers are born debt free.

Finally, aggregate the first-order condition (4.71) and the budget constraint (4.76) to obtain

$$-(x_t - \bar{x}) = \lambda_t$$

(4.80)
and
\[ d_t = (1 + r^*)d_{t-1} - z_t + x_t - \bar{\pi}, \quad (4.81) \]

where
\[ \lambda_t \equiv (1 - \theta) \sum_{s=t}^{-\infty} \theta^{t-s} \lambda_{s,t}. \]

denotes the cross-sectional average of marginal utilities of consumption.

**Firms Producing Consumption Goods**

We assume the existence of competitive firms that hire capital and labor services to produce consumption goods. These firms maximize profits, which are given by
\[ A_t F(k_t, h_t) - w_t h_t - u_t k_t, \]

where the function \( F \) and the productivity factor \( A_t \) are as in the EDEIR model. The first-order conditions associated with the firm’s profit-maximization problem are
\[ A_t F_k(k_t, h_t) = u_t \quad (4.82) \]

and
\[ A_t F_h(k_t, h_t) = w_t. \quad (4.83) \]

We assume perfect competition in product and factor markets. Because \( F \) is homogeneous of degree one, firms producing consumption goods make zero profits.
Firms Producing Capital Goods

We assume the existence of firms that buy consumption goods to transform them into investment goods, rent out capital, and pay dividends, $\pi_t$. Formally, dividends in period $t$ are given by

$$\pi_t = u_t k_t - i_t - \Phi(k_{t+1} - k_t). \tag{4.84}$$

The evolution of capital follows the law of motion given in (4.4), which we reproduce here for convenience

$$k_{t+1} = (1 - \delta) k_t + i_t. \tag{4.85}$$

The optimization problem of the capital producing firm is dynamic. This is because investment goods take one period to become productive capital and because of the presence of adjustment costs. The firm must maximize some present discounted value of current and future expected profits. A problem that emerges at this point is what discount factor should the firm use. This issue does not have a clear answer for two reasons: first, the owners of the firm change over time. Recall that the shares of the dead are distributed in equal parts among the newborn. It follows that the firm cannot use as its discount factor the intertemporal marginal rate of substitution of a ‘representative household.’ For the representative household does not exist. Second, the firm operates in a financial environment characterized by incomplete asset markets. For this reason, it cannot use the price of state-contingent claims to discount future profits. For there is no market for such claims.

One must therefore introduce assumptions regarding the firm’s discounting behavior. These assumptions will in general not be innocuous with respect to the dynamics of capital accumulation. With this in mind, we will assume that the firm uses the discount factor $\beta^j \lambda_{t+j}/\lambda_t$ to calculate the period-$t$ value of one unit of consumption delivered in a particular state of period $t + j$. Note
that this discount factor uses the average marginal utility of consumption of agents alive in period \( t + j \) relative to the average marginal utility of consumption of agents alive in period \( t \). Note that we use as the subjective discount factor the parameter \( \beta \) and not \( \beta \theta \). This is because the number of shareholders is constant over time (and equal to unity), unlike the size of a cohort born at a particular date, which declines at the mortality rate \( 1 - \theta \). The Lagrangian associated with the optimization problem of capital goods producers is then given by

\[
\mathcal{L} = E_t \sum_{j=0}^{\infty} \frac{\beta^j \lambda_{t+j}}{\lambda_t} [u_{t+j}k_{t+j} - k_{t+j+1} + (1 - \delta)k_{t+j} - \Phi(k_{t+j+1} - k_{t+j})].
\]

The first-order condition with respect to \( k_{t+1} \) is

\[
\lambda_t [1 + \Phi'(k_{t+1} - k_t)] = \beta E_t \lambda_{t+1} \left[ u_{t+1} + 1 - \delta + \Phi'(k_{t+2} - k_{t+1}) \right].
\] (4.86)

**Equilibrium**

Equations (4.74), (4.75), (4.78)-(4.86) form a system of eleven equations in eleven unknowns: \( x_t, \lambda_t, h_t, w_t, u_t, \pi_t, i_t, k_t, d_t, z_t, \tilde{z}_t \). Here, we reproduce the system of equilibrium conditions for convenience:

\[
h_t^{-1} = w_t,
\]

\[
z_t \equiv \pi_t + \left( 1 - \frac{1}{\omega} \right) w_t h_t - \overline{x},
\]

\[
\tilde{z}_t = \frac{\theta}{1 + r^*} z_t + \frac{\theta}{1 + r^*} E_t \tilde{z}_{t+1},
\]

\[
x_t = \overline{x} + \frac{\beta(1 + r^*)^2 - \theta}{\beta \theta (1 + r^*)} (\tilde{z}_t - \theta d_{t-1}),
\]

\[ -(x_t - \overline{x}) = \lambda_t, \]
\[ d_t = (1 + r^*)d_{t-1} - z_t + x_t - \bar{x}, \]

\[ A_t F_k(k_t, h_t) = u_t, \]

\[ A_t F_h(k_t, h_t) = w_t, \]

\[ \pi_t = u_t k_t - i_t - \Phi(k_{t+1} - k_t), \]

\[ k_{t+1} = (1 - \delta)k_t + i_t, \]

\[ \lambda_t [1 + \Phi'(k_{t+1} - k_t)] = \beta E_t \lambda_{t+1} [u_{t+1} + 1 - \delta + \Phi'(k_{t+2} - k_{t+1})]. \]

It is of interest to consider the special case in which \( \beta(1 + r^*) = 1 \). In this case, the evolution of external debt is given by \( d_t = \theta d_{t-1} + (1 + r^* - \theta)/\theta \bar{z}_t - z_t \). This expression shows that the stock of debt does not follow a random walk as was the case in the representative-agent economy with quadratic preferences of chapter 2. In fact, the (autoregressive) coefficient on past external debt is \( \theta \in (0,1) \). The mean reverting property of aggregate external debt obtains in spite of the fact that individual debt positions follow a random walk. The reason why the aggregate level of external debt is trend reverting in equilibrium is the fact that each period a fraction \( 1 - \theta \in (0,1) \) of the agents die and are replaced by newborns holding no financial assets. As a result, on average, the current aggregate level of debt is only a fraction \( \theta \) of the previous period’s level of debt. This intuition also goes through when \( \beta(1 + r^*) \neq 1 \), although in this case individual levels of debt display a trend in the deterministic equilibrium.

In the deterministic steady state, the aggregate level of debt is given by

\[ d = \frac{\theta(1 - \beta(1 + r^*))}{(1 + r^* - \theta)(\theta - \beta(1 + r^*))} \]

In the special case in which \( \beta(1 + r^*) \) equals unity, the steady-state aggregate stock of debt is nil.
This is because in this case agents, all of whom are born with no debts, wish to hold constant debt levels over time. In this case, the steady state both the aggregate and the individual levels of debt are zero. It can be shown that if $\beta(1 + r^*)$ is less than unity but larger than $\theta$, the steady-state level of debt must be positive.

We adopt the same functional forms for $F$ and $\Phi$ as in the EDEIR model. We calibrate $\omega, \alpha, \phi, \delta, \rho, \beta$ and $\bar{\eta}$ at the values displayed in tables 4.1. Consequently, the steady-state values of hours, capital, output, investment, consumption, and the trade balance are the same as in the EDEIR model. We set $\theta = 1 - 1/75$, which implies a life expectancy of 75 years. Finally, we calibrate $r^*$ and $\bar{x}$ to ensure that in the steady state the trade-balance-to-output ration is 2 percent and the degree of relative risk aversion, given by $-x/(x - \bar{x})$, is 2. This calibration results in an interest rate of 3.7451 percent and a satiation point of 0.6334.

### 4.10.7 Quantitative Results

Table 4.4 displays a number of unconditional second moments of interest implied by the IDF, EDF, EDEIR, IDEIR, PAC, CAM, and PY models. The NSIF model is nonstationary up to first order, and therefore does not have well defined unconditional second moments. The second moments for all models other than the IDEIR and PY models are taken from Schmitt-Grohé and Uribe (2003). We compute the equilibrium dynamics by solving a log-linear approximation to the set of equilibrium conditions. The Matlab computer code used to compute the unconditional second moments and impulse response functions for all models presented in this section is available at www.columbia.edu/~mu2166/closing.htm.

Table 4.4 shows that regardless of how stationarity is induced, the model’s predictions regarding second moments are virtually identical. One noticeable difference arises in the CAM model, the complete markets case, which, as might be expected, predicts less volatile consumption. The low volatility of consumption in the complete markets model introduces a difference between the
Table 4.4: Second Moments Across Models

<table>
<thead>
<tr>
<th></th>
<th>IDF</th>
<th>EDF</th>
<th>IDEIR</th>
<th>EDEIR</th>
<th>PAC</th>
<th>CAM</th>
<th>PY</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Volatilities:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>std($y_t$)</td>
<td>3.1</td>
<td>3.1</td>
<td>3.1</td>
<td>3.1</td>
<td>3.1</td>
<td>3.1</td>
<td>3.1</td>
</tr>
<tr>
<td>std($c_t$)</td>
<td>2.3</td>
<td>2.3</td>
<td>2.5</td>
<td>2.7</td>
<td>2.7</td>
<td>1.9</td>
<td>2.5</td>
</tr>
<tr>
<td>std($i_t$)</td>
<td>9.1</td>
<td>9.1</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9.1</td>
<td>8.7</td>
</tr>
<tr>
<td>std($h_t$)</td>
<td>2.1</td>
<td>2.1</td>
<td>2.1</td>
<td>2.1</td>
<td>2.1</td>
<td>2.1</td>
<td>2.1</td>
</tr>
<tr>
<td>std($\frac{i_t}{y_t}$)</td>
<td>1.5</td>
<td>1.5</td>
<td>1.6</td>
<td>1.8</td>
<td>1.8</td>
<td>1.6</td>
<td>1.5</td>
</tr>
<tr>
<td>std($\frac{c_t}{y_t}$)</td>
<td>1.5</td>
<td>1.5</td>
<td>1.4</td>
<td>1.5</td>
<td>1.5</td>
<td>3.1</td>
<td>1.3</td>
</tr>
<tr>
<td><strong>Serial Correlations:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>corr($y_t$, $y_{t-1}$)</td>
<td>0.61</td>
<td>0.61</td>
<td>0.62</td>
<td>0.62</td>
<td>0.62</td>
<td>0.61</td>
<td>0.62</td>
</tr>
<tr>
<td>corr($c_t$, $c_{t-1}$)</td>
<td>0.7</td>
<td>0.7</td>
<td>0.76</td>
<td>0.78</td>
<td>0.78</td>
<td>0.61</td>
<td>0.74</td>
</tr>
<tr>
<td>corr($i_t$, $i_{t-1}$)</td>
<td>0.07</td>
<td>0.07</td>
<td>0.068</td>
<td>0.069</td>
<td>0.069</td>
<td>0.07</td>
<td>0.064</td>
</tr>
<tr>
<td>corr($h_t$, $h_{t-1}$)</td>
<td>0.61</td>
<td>0.61</td>
<td>0.62</td>
<td>0.62</td>
<td>0.62</td>
<td>0.61</td>
<td>0.62</td>
</tr>
<tr>
<td>corr($\frac{i_t}{y_t}$, $\frac{i_{t-1}}{y_{t-1}}$)</td>
<td>0.33</td>
<td>0.32</td>
<td>0.43</td>
<td>0.51</td>
<td>0.5</td>
<td>0.39</td>
<td>0.34</td>
</tr>
<tr>
<td>corr($\frac{c_t}{y_t}$, $\frac{c_{t-1}}{y_{t-1}}$)</td>
<td>0.3</td>
<td>0.3</td>
<td>0.31</td>
<td>0.32</td>
<td>0.32</td>
<td>-0.07</td>
<td>0.29</td>
</tr>
<tr>
<td><strong>Correlations with Output:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>corr($c_t$, $y_t$)</td>
<td>0.94</td>
<td>0.94</td>
<td>0.89</td>
<td>0.84</td>
<td>0.85</td>
<td>1</td>
<td>0.94</td>
</tr>
<tr>
<td>corr($i_t$, $y_t$)</td>
<td>0.66</td>
<td>0.66</td>
<td>0.68</td>
<td>0.67</td>
<td>0.67</td>
<td>0.66</td>
<td>0.69</td>
</tr>
<tr>
<td>corr($h_t$, $y_t$)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>corr($\frac{i_t}{y_t}$, $y_t$)</td>
<td>-0.012</td>
<td>-0.013</td>
<td>-0.036</td>
<td>-0.044</td>
<td>-0.043</td>
<td>0.13</td>
<td>-0.06</td>
</tr>
<tr>
<td>corr($\frac{c_t}{y_t}$, $y_t$)</td>
<td>0.026</td>
<td>0.025</td>
<td>0.041</td>
<td>0.05</td>
<td>0.051</td>
<td>-0.49</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Note. Standard deviations are measured in percent per year. IDF = Internal Discount Factor; EDF = External Discount Factor; IDEIR = Internal Debt-Elastic Interest Rate; EDEIR = External Debt-Elastic Interest Rate; PAC = Portfolio Adjustment Costs; CAM = Complete Asset Markets; PY = Perpetual Youth Model. Parts of the table are reproduced from Schmitt-Grohé and Uribe (2003).
predictions of this model and those of the IDF, EDF, EDEIR, IDEIR, PAC, and PY models: Because consumption is smoother in the CAM model, its role in determining the cyclicality of the trade balance is smaller. As a result, the CAM model predicts that the correlation between output and the trade balance is positive, whereas the models featuring incomplete asset markets all imply that this correlation is negative.

Figure 4.3 demonstrates that all of the models being compared imply virtually identical impulse response functions to a technology shock. Each panel shows the impulse response of a particular variable in the eight models. For all variables, the impulse response functions are so similar that to the naked eye the graph appears to show just a single line. Again, the only small and barely noticeable difference is given by the responses of consumption and the trade-balance-to-GDP ratio in the complete markets model. In response to a positive technology shock, consumption increases less when markets are complete than when markets are incomplete. This in turn, leads to a smaller decline in the trade balance in the period in which the technology shock occurs.

4.10.8 Inducing Stationarity Through Impatience and Global Solutions

Thus far, we have approximated the SOE-RBC model using a first-order local approximation around the deterministic steady state. Here, we approximate the equilibrium using a global method. We assume that both the subjective and pecuniary discount factors are constant. To induce stationarity, we assume that households are slightly more impatient than indicated by the pecuniary discount factor, that is, we assume that

$$\beta(1 + r^*) < 1.$$ 

In all other respects, the model is exactly as the baseline SOE model of section 4.1. The above restriction eliminates the random walk in debt and consumption. To understand why, assume first that the economy is deterministic. In such an environment, the condition $\beta(1 + r^*) < 1$ induces
Figure 4.3: Impulse Response to a Unit Technology Shock Across Models

Note. Solid line, IDF model; squares, EDF; dashed line, EDEIR model; dash-dotted line, PAC model; dotted line, CAM model; circles, NSIF model; right triangle, IDEIR model; left triangle, PY.
households to accumulate debt in such a way that consumption approaches asymptotically its lowest possible level, given by the disutility of labor, $h\omega/\omega$. Thus, debt converges to a well-defined value given by the present discounted value of the stream of output, $AF(k, h)$, net of capital depreciation, $\delta k$, and consumption, $h\omega/\omega$, that is, $d \to [AF(k, h) - \delta k - h\omega/\omega]/r^*$, where all variables are evaluated at their asymptotic values. Of course, at the limit the household is infinitely unhappy. Thus, when uncertainty is introduced, in the form of stochastic variations in the technology shock $A$, precautionary savings create a well-defined debt distribution to the left of its asymptotic deterministic level. Since this precautionary savings motive cannot be captured by a first-order approximation, it follows that a higher-order approximation, like the global approximation pursued here, is needed to obtain a stationary solution.

The solution algorithm is based on value function iterations over a discretized state space. Specifically, we write the model as the solution to the Bellman equation

$$v(d, k, A) = \max_{\{d', k'\}} \left\{ U \left( AF(k, h) + (1 - \delta)k - k' - \Phi(k' - k) + d' - (1 + r^*)d - \frac{h\omega}{\omega} \right) + \beta E[v(d', k', A')|A] \right\}$$

subject to

$$d' \leq \overline{d},$$

where variables without a subscript or a superscript are dated in period $t$ and primed variables are dated in $t + 1$. The debt constraint places a limit to the level of net external debt, defined by the parameter $\overline{d}$. When $\overline{d}$ is set to a large number so that the debt constraint is never binding the debt constraint serves only as a no-Ponzi-game restriction. But, as it will become clear shortly, a more stringent debt limit is needed for the predictions of the model to be in line with the data. The online materials for this chapter provide the transition probability matrix of the exogenous shock $A$ (file tpm.mat) and the MATLAB script to compute the value function and the associated policy functions (file vfi.m).
Table 4.5: Calibration of the Discount Factor and the Debt Limit

<table>
<thead>
<tr>
<th>Description</th>
<th>$\beta$</th>
<th>$\beta(1 + r^*)$</th>
<th>$d$</th>
<th>$E(\delta b/y)$</th>
<th>$\sigma_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chosen Calibration</td>
<td>0.954</td>
<td>0.9922</td>
<td>1</td>
<td>2.9</td>
<td>8.7</td>
</tr>
<tr>
<td>Natural Debt Limit</td>
<td>0.954</td>
<td>0.9922</td>
<td>9.95</td>
<td>25.8</td>
<td>21.40</td>
</tr>
<tr>
<td>High Patience</td>
<td>0.96</td>
<td>0.9984</td>
<td>1</td>
<td>2.6</td>
<td>9.9</td>
</tr>
<tr>
<td>Data</td>
<td></td>
<td></td>
<td>2.0</td>
<td>9.8</td>
<td></td>
</tr>
</tbody>
</table>

Note. $r^* = 0.04$.

We use the same functional forms for preferences, technologies and adjustment costs as in previous sections and calibrate all parameters, except $\beta$ and $d$, as shown in table 4.1. We discretize the state space with 9 equally spaced points for $\ln A$ from -0.04495 to 0.04495, 70 equally spaced points for $d$, and 30 equally spaced points for $k$. The ranges assigned to the two endogenous states, $d$ and $k$, depend on the values assigned to $\beta$ and $\bar{d}$, as explained below.

Setting $\beta(1 + r^*)$ too close to unity results in a near random walk distribution of debt in equilibrium, which is difficult to approximate. We therefore set $\beta(1 + r^*)$ equal to 0.9922, which, recalling that $r^* = 0.04$, requires setting $\beta$ equal to 0.954. This value of $\beta$ introduces a significant difference of 82 basis points between the subjective and the pecuniary discount rates.

Consider first the case in which the debt limit $d$ is set to a large number so that the borrowing constraint is never binding in equilibrium. Specifically we set the range of $d$ to be $[7.45, 9.95]$. The grid for $k$ is $[2.8, 3.8]$. The top-left panel of figure 4.4 displays the equilibrium probability distribution of $d$. The probability of debt being equal to its upper limit of 9.95 is zero, which implies that the debt limit is never binding in equilibrium. However, the implied average level of external debt, equal to 9.6, turns out to be excessive. Table 4.5 explains why. It shows that in order to support such a high level of debt the economy must generate a trade balance surplus of 25.8 percent of GDP, more than 10 times larger than the one observed in Canada, the economy to which the model is calibrated. Under this calibration, the model is at odds with the data along
Figure 4.4: Probability Distribution of Debt Under Impatience and a Global Approximation

Note. $r^* = 0.04$. 
other dimensions as well. For instance, the table shows that the predicted volatility of investment is 21.4 percent, more than twice what it is in the data. The intuition why investment is so volatile is that at high levels of external debt, the intertemporal marginal rate of consumption substitution $U'(c - h\omega/\omega)/U'(c - h\omega/\omega)$ becomes highly volatile. Recall that this variable represents the kernel households use to discount future returns to capital (see equations (4.8) and (4.10)). In turn, the reason why the intertemporal marginal rate of consumption substitution is highly volatile is that consumption is close to its minimum possible level, given by the disutility of labor $h\omega/\omega$. When $c - h\omega/\omega$ is close to zero, the marginal utility of consumption becomes highly sensitive to changes in $c$ or $h$.

It follows that a tighter limit on external debt is in order. Accordingly, we set $d$ to 1. As shown in table 4.5, this value is low enough to induce an average trade-balance-to-output ratio close to the observed value of 2 percent. The table also shows that with the tighter debt limit, the model improves significantly in explaining the observed volatility of investment (9.8 in the data versus 8.7 in the model). Indeed, the model does a pretty good job at explaining other second moments of interest as well. Table 4.6 displays the standard deviation, serial correlation, and correlation with output of output, consumption, investment, hours, the trade-balance-to-output ratio and the current-account-to-output ratio. For comparison, the table reproduces form table 4.2 the corresponding empirical moments and those implied by the EDEIR model.

The main message of the table is that the model with impatient agents solved using global methods performs as well as the EDEIR model solved with local approximation methods except for the serial correlation of the trade-balance-to-output ratio, which is better captured by the log-linearized EDEIR model. A similar result emerges from figure 4.5, displaying impulse responses to a positive productivity shock. Comparing this figure with figure 4.3, shows that the response of the model with impatient households obtained via a global approximation is similar to that obtained under local approximations and alternative stationarity inducing mechanisms.
Figure 4.5: Impulse Responses to a Positive Productivity Shock Under a Global Approximation
Table 4.6: Model Predictions Under A Global Approximation

<table>
<thead>
<tr>
<th>Variable</th>
<th>Data</th>
<th>Global Solution</th>
<th>EDEIR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_{xt}$</td>
<td>$\rho_{xt,t-1}$</td>
<td>$\rho_{xt,GDP_t}$</td>
</tr>
<tr>
<td>$y$</td>
<td>2.81 0.62 1</td>
<td>3.56 0.68 1</td>
<td>3.08 0.62 1</td>
</tr>
<tr>
<td>$c$</td>
<td>2.46 0.70 0.59</td>
<td>2.96 0.77 0.98</td>
<td>2.71 0.78 0.84</td>
</tr>
<tr>
<td>$i$</td>
<td>9.82 0.31 0.64</td>
<td>8.70 0.09 0.69</td>
<td>9.04 0.07 0.67</td>
</tr>
<tr>
<td>$h$</td>
<td>2.02 0.54 0.80</td>
<td>2.44 0.68 1</td>
<td>2.12 0.62 1</td>
</tr>
<tr>
<td>$\frac{\nu}{y}$</td>
<td>1.87 0.66 -0.13</td>
<td>1.23 -0.46 -0.04</td>
<td>1.78 0.51 -0.04</td>
</tr>
<tr>
<td>$\frac{ca}{y}$</td>
<td>1.21 -0.47 0.05</td>
<td>1.45 0.32 0.05</td>
<td></td>
</tr>
</tbody>
</table>

Note. In the global solution $\beta(1 + r^*)$ takes the value 0.9922 and $d$ is equal to 1. Second moments are computed using the program vfi.m available with the online materials for this chapter. For second moments predicted by the EDEIR model and for empirical second moments, see the note to table 4.2.

Finally, we note that the predictions implied by the global approximation are robust to a wide range of values of $\beta$. Table 4.5 shows that both the trade-balance-to-output ratio and the volatility of investment are little changed if $\beta$ is raised to 0.96, a value quite close to $1/(1 + r^*) = 0.9615$. This value might be empirically appealing because it implies a lower frequency of events in which the debt limit binds, 22 percent with $\beta = 0.96$ versus 31 percent at the baseline value of 0.954 (compare the top-right and the bottom panels of figure 4.4).
4.11 Appendix: First-Order Accurate Approximations to Dynamic General Equilibrium Models

In this appendix, we solve the system

\[ f_y E_t \hat{y}_{t+1} + f_y \hat{y}_t + f_x E_t \hat{x}_{t+1} + f_x \hat{x}_t \]

reproduced from section 4.6. The matrices \( f_y', f_y, f_x', \) and \( f_x \) are assumed to be known. Letting \( A = [f_x', f_y] \) and \( B = -[f_x, f_y] \), we can rewrite the system as

\[
A \begin{bmatrix} E_t \hat{x}_{t+1} \\ E_t \hat{y}_{t+1} \end{bmatrix} = B \begin{bmatrix} \hat{x}_t \\ \hat{y}_t \end{bmatrix}.
\]

Define the vector \( \hat{w}_t \) containing all control and state variables of the system. Formally

\[
\hat{w}_{t+j} \equiv E_t \begin{bmatrix} \hat{x}_{t+j} \\ \hat{y}_{t+j} \end{bmatrix}
\]

for \( j \geq 0 \). Note that this definition implies that

\[
\hat{w}_t \equiv \begin{bmatrix} \hat{x}_t \\ \hat{y}_t \end{bmatrix}.
\]

We can then write the linear system as

\[ A \hat{w}_{t+1} = B \hat{w}_t. \]
In accordance with (4.40), we seek solutions in which

\[ \lim_{j \to \infty} \hat{w}_{t+j} = 0. \]  

(4.87)

This requirement means that at every point in time the vector \( w_t \) is expected to converge to its non-stochastic steady state, \( w \equiv [x' \ y']' \).

The remainder of this section is based on Klein (2000) (see also Sims, 1996). Consider the generalized Schur decomposition of \( A \) and \( B \):

\[ qAz = a \]

and

\[ qBz = b, \]

where \( a \) and \( b \) are upper triangular matrices and \( q \) and \( z \) are orthonormal matrices. Recall that a matrix \( a \) is said to be upper triangular if elements in row \( i \) and column \( j \), denoted \( a(i, j) \) are 0 for \( i > j \). A matrix \( z \) is orthonormal if \( z'z = zz' = I \).

Define

\[ s_t \equiv z'\hat{w}_t. \]

Then we have that

\[ as_{t+1} = bs_t. \]

The ratio \( b(i, i)/a(i, i) \) is known as the generalized eigenvalue of the matrices \( A \) and \( B \). Assume, without loss of generality, that the ratios \( |b(i, i)/a(i, i)| \) are increasing in \( i \). Now partition \( a, b, z, \)
\( \hat{w}_t, \) and \( s_t \) as

\[
a = \begin{bmatrix} a_{11} & a_{12} \\ \emptyset & a_{22} \end{bmatrix}, \quad b = \begin{bmatrix} b_{11} & b_{12} \\ \emptyset & b_{22} \end{bmatrix}; \quad z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}; \quad \hat{w}_t = \begin{bmatrix} \hat{w}^1_t \\ \hat{w}^2_t \end{bmatrix}, \quad s_t = \begin{bmatrix} s^1_t \\ s^2_t \end{bmatrix},
\]

where \( a_{11} \) and \( b_{11} \) are square matrices whose diagonals generate the generalized eigenvalues of \((A, B)\) with absolute values less than one, and \( a_{22} \) and \( b_{22} \) are square matrices whose diagonals generate the generalized eigenvalues of \((A, B)\) with absolute values greater than one. Then we have that

\[ a_{22} s^2_{t+1} = b_{22} s^2_t. \]

The partition of the matrix \( B \) guarantees that all diagonal elements of \( b_{22} \) are nonzero. In addition, recalling that a triangular matrix is invertible if the elements along its main diagonal are nonzero, it follows, that \( b_{22} \) is invertible. So we can write

\[ b_{22}^{-1} a_{22} s^2_{t+1} = s^2_t. \]

By construction, the eigenvalues of \( b_{22}^{-1} a_{22} \) are all less than unity in modulus. To arrive at this conclusion, we use three properties of upper triangular matrices: (a) the inverse of a nonsingular upper triangular matrix is upper triangular; (b) the product of two upper triangular matrices is upper triangular; and (c) the eigenvalues of an upper triangular matrix are the elements of its main diagonal. It follows that the only nonexplosive solution to the above difference equation is

\[ s^2_t = \]
for all \( t \). This result, and the definition of \( s_t^2 \) imply that

\[
z_{12}' \hat{w}_t^1 + z_{22}' \hat{w}_t^2 = 0.
\]

Solving this expression for \( \hat{w}_t^2 \) yields

\[
\hat{w}_t^2 = G \hat{w}_t^1,
\]

where

\[
G \equiv -z_{22}'^{-1} z_{12}'.
\]

The invertibility of \( z_{22}' \) follows from the fact that, being orthonormal, \( z' \) itself is invertible. The condition \( s_t^2 = 0 \) for all \( t \) also implies that

\[
a_{11} s_{t+1}^1 = b_{11} s_t^1.
\]

The criteria used to partition \( A \) and \( B \) guarantee that the diagonal elements of the upper triangular matrix \( a_{11} \) are nonzero. Therefore, \( a_{11} \) is invertible, which allows us to write

\[
s_{t+1}^1 = a_{11}^{-1} b_{11} s_t^1.
\]

Now express \( s_t^1 \) as a linear transformation of \( \hat{w}_t^1 \) as follows:

\[
s_t^1 = z_{11}' \hat{w}_t^1 + z_{21}' \hat{w}_t^2
\]

\[
= (z_{11}' + z_{21}' G) \hat{w}_t^1
\]

\[
= (z_{11}' - z_{21}' z_{22}'^{-1} z_{12}') \hat{w}_t^1
\]

\[
= z_{11}'^{-1} \hat{w}_t^1
\]
The second and third equalities make use of equation (4.88) and identity (4.89), respectively. The last equality follows from the fact that \( z \) is orthonormal.  

Combining this expression with (4.90) yields  
\[
\hat{w}_{t+1}^1 = H \hat{w}_t^1,
\]
where  
\[
H \equiv z_{11} a_{11}^{-1} b_{11} z_{11}^{-1}.
\]

Finally, note that all eigenvalues of \( H \) are inside the unit circle. To see this note that the eigenvalues of \( z_{11} a_{11}^{-1} b_{11} z_{11}^{-1} \) must be same as the eigenvalues of \( a_{11}^{-1} b_{11} \). In turn \( a_{11}^{-1} b_{11} \) is upper triangular with diagonal elements less than one in modulus.

### 4.12 Appendix: Local Existence and Uniqueness of Equilibrium

The analysis thus far has not delivered the matrices \( h_x \) and \( g_x \) that define the first-order accurate solution of the DSGE model. In this section, we accomplish this task and derive conditions under which the equilibrium dynamics are locally unique.

---

To see this, let \( k \equiv z'_{11} - z'_{21} z_{22}^{-1} z'_{12} \). We wish to show that \( k = z_{11}^{-1} \). Note that the orthonormality of \( z \) implies that  
\[
I = z' z = \begin{bmatrix} z_{11}' z_{11} + z_{21}' z_{21} & z_{11}' z_{12} + z_{21}' z_{22} \\ z_{12}' z_{11} + z_{22}' z_{21} & z_{12}' z_{12} + z_{22}' z_{22} \end{bmatrix}.
\]

Use element (2, 1) of \( z' z \) to get \( z_{12}' z_{11} = -z_{22}' z_{21} \). Pre-multiply by \( z_{22}^{-1} \) and post multiply by \( z^{-1}_{11} \) to get \( z_{12}' z_{12}^{-1} z_{12}' = -z_{21}^{-1} z_{11}^{-1} \). Use this expression to eliminate \( z_{12}' z_{12}^{-1} z_{12}' \) from the definition of \( k \) to obtain \( k = [z_{11}' + z_{21}' z_{21}^{-1} z_{11}^{-1}] \). Now use element (1, 1) of \( z' z \) to write \( z_{21}' z_{21} = I - z_{11}' z_{11} \). Using this equation to eliminate \( z_{21}' z_{21} \) from the expression in square brackets, we get \( k = [z_{11}' + (I - z_{11}' z_{11}) z_{11}^{-1}] \), which is simply \( z_{11}^{-1} \). Finally, note that the invertibility of \( z_{11} \) follows from the invertibility of \( z \).
4.12.1 Local Uniqueness of Equilibrium

Suppose that the number of generalized eigenvalues of the matrices $A$ and $B$ with absolute value less than unity is exactly equal to the number of states, $n_x$. That is, suppose that $a_{11}$ and $b_{11}$ are of size $n_x \times n_x$. In this case, the matrix $H$ is also of size $n_x \times n_x$, and the matrix $G$ is of size $n_y \times n_x$. Moreover, since $\hat{w}^1_t$ must be comformable with $H$, we have that $\hat{w}^1_t$ is given by the first $n_x$ elements of $\hat{w}_t$, which exactly coincide with $\hat{x}_t$. In turn, this implies that $\hat{w}^2_t$ must equal $\hat{y}_t$. Defining

\[ h_x \equiv H \]

and

\[ g_x \equiv G, \]

we can then write

\[ \hat{x}_{t+1} = h_x \hat{x}_t \]

and

\[ \hat{y}_t = g_x \hat{x}_t, \]

which is the solution we were looking for. Notice that because $\hat{x}_t$ is predetermined in period $t$, we have that $\hat{y}_t$ and $\hat{x}_{t+1}$ are uniquely determined in period $t$. The evolution of the linearized system is then unique and given by

\[ y_t - y = g_x (x_t - y) \]

\[ x_{t+1} - x = h_x (x_t - x) + \eta \epsilon_{t+1}, \]

where we have set $\sigma$ at the desired value of 1.

Summarizing, the condition for local uniqueness of the equilibrium is that the number of gen-
eralized eigenvalues of the matrices $A$ and $B$ is exactly equal to the number of states, $n_x$.

### 4.12.2 No Local Existence of Equilibrium

Now suppose that the number of generalized eigenvalues of the matrices $A$ and $B$ with absolute value less than one is smaller than the number of state variables, $n_x$. Specifically, suppose that $a_{11}$ and $b_{11}$ are of size $(n_x - m) \times (n_x - m)$, with $0 < m \leq n_x$. In this case, the matrix $H$ is of order $(n_x - m) \times (n_x - m)$ and the matrix $G$ is of order $(n_y + m) \times (n_x - m)$. Moreover, the vectors $\hat{w}_t^1$ and $\hat{w}_t^2$ no longer coincide with $\hat{x}_t$ and $\hat{y}_t$, respectively. Instead, $\hat{w}_t^1$ and $\hat{w}_t^2$ take the form

$$
\hat{w}_t^1 = \hat{x}_t^a
$$

$$
\hat{w}_t^2 = \begin{bmatrix} \hat{x}_t^b \\ \hat{y}_t \end{bmatrix},
$$

where $\hat{x}_t^a$ and $\hat{x}_t^b$ are vectors of lengths $n_x - m$ and $m$, respectively, and satisfy

$$
\hat{x}_t = \begin{bmatrix} \hat{x}_t^a \\ \hat{x}_t^b \end{bmatrix},
$$

The law of motion of $\hat{x}_t$ and $\hat{y}_t$ is then of the form

$$
\hat{x}_{t+1}^a = H\hat{x}_t^a
$$

and

$$
\begin{bmatrix} \hat{x}_t^b \\ \hat{y}_t \end{bmatrix} = G\hat{x}_t^a
$$
This expression states that \( \hat{x}_t^b \) is determined by \( \hat{x}_t^a \). But this is impossible, because \( \hat{x}_t^a \) and \( \hat{x}_t^b \) are predetermined independently of each other. We therefore say that locally there exists no equilibrium.

Summarizing, no local equilibrium exists if the number of generalized eigenvalues of the matrices \( A \) and \( B \) with absolute values less than one is smaller than the number of state variables, \( n_x \).

4.12.3 Local Indeterminacy of Equilibrium

Finally, suppose that the number of generalized eigenvalues of the matrices \( A \) and \( B \) with absolute value less than one is larger than the number of state variables, \( n_x \). Specifically, suppose that \( a_{11} \) and \( b_{11} \) are of size \( (n_x + m) \times (n_x + m) \), with \( 0 < m \leq n_y \). In this case, the matrix \( H \) is of order \( (n_x + m) \times (n_x + m) \) and the matrix \( G \) is of order \( (n_y - m) \times (n_x + m) \). The vectors \( \hat{w}_t^1 \) and \( \hat{w}_t^2 \) take the form

\[
\hat{w}_t^1 = \begin{bmatrix} \hat{x}_t \\ \hat{y}_t^a \end{bmatrix},
\]

\[
\hat{w}_t^2 = \hat{y}_t^b,
\]

where \( \hat{y}_t^a \) and \( \hat{y}_t^b \) are vectors of lengths \( m \) and \( n_y - m \), respectively, and satisfy

\[
\hat{y}_t = \begin{bmatrix} \hat{y}_t^a \\ \hat{y}_t^b \end{bmatrix},
\]

The law of motion of \( \hat{x}_t \) and \( \hat{y}_t \) is then of the form

\[
\begin{bmatrix} \hat{x}_{t+1} \\ \hat{y}_{t+1}^a \end{bmatrix} = H \begin{bmatrix} \hat{x}_t \\ \hat{y}_t^a \end{bmatrix}
\]
These expressions state that one can freely pick \( \hat{y}_t^a \) in period \( t \). Since \( \hat{y}_t^a \) is not predetermined, the equilibrium is indeterminate. In this case, we say that the indeterminacy is of dimension \( m \). The evolution of the system can then be written as

\[
\begin{bmatrix}
  x_{t+1} - x \\
  y_{t+1}^b - y_{t+1}^{ass}
\end{bmatrix} = H \begin{bmatrix}
  x_t - x \\
  y_t^a - y_{t}^{ass}
\end{bmatrix} + \begin{bmatrix}
  \eta \\
  \nu_e & \nu_\mu
\end{bmatrix} \begin{bmatrix}
  \epsilon_{t+1} \\
  \mu_{t+1}
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
  \hat{y}_t^b \\
  y_t^b - y_{t}^{ass}
\end{bmatrix} = G \begin{bmatrix}
  x_t - x \\
  y_t^a - y_{t}^{ass}
\end{bmatrix},
\]

where the matrices \( \nu_e \) and \( \nu_\mu \) allow for nonfundamental uncertainty, and \( \mu_t \) is an i.i.d. innovation with mean \( \emptyset \) and variance covariance matrix equal to the identity matrix.

Summarizing, the equilibrium displays local indeterminacy of dimension \( m \) if the number of generalized eigenvalues of the matrices \( A \) and \( B \) with absolute values less than one exceeds the number of state variables, \( n_x \), by \( 0 < m \leq n_y \).

### 4.13 Appendix: Second Moments

Start with the equilibrium law of motion of the deviation of the state vector with respect to its steady-state value, which is given by

\[
\hat{x}_{t+1} = h_x x_t + \sigma \eta \epsilon_{t+1},
\]

(4.91)
Covariance Matrix of $x_t$

Let

$$\Sigma_x \equiv E\tilde{x}_t\tilde{x}_t'$$

denote the unconditional variance/covariance matrix of $\tilde{x}_t$ and let

$$\Sigma_\epsilon \equiv \sigma^2\eta\eta'.$$

Then we have that

$$\Sigma_x = h_x\Sigma_xh_x' + \Sigma_\epsilon.$$  

We will describe two numerical methods to compute $\Sigma_x$.

**Method 1**

One way to obtain $\Sigma_x$ is to make use of the following useful result. Let $A$, $B$, and $C$ be matrices whose dimensions are such that the product $ABC$ exists. Then

$$\text{vec}(ABC) = (C' \otimes A) \cdot \text{vec}(B),$$

where the vec operator transforms a matrix into a vector by stacking its columns, and the symbol $\otimes$ denotes the Kronecker product. Thus if the vec operator is applied to both sides of

$$\Sigma_x = h_x\Sigma_xh_x' + \Sigma_\epsilon,$$
the result is

\[
\begin{align*}
\text{vec}(\Sigma_x) &= \text{vec}(h_x \Sigma_x h_x') + \text{vec}(\Sigma_\epsilon) \\
&= F \text{vec}(\Sigma_x) + \text{vec}(\Sigma_\epsilon),
\end{align*}
\]

where

\[
F = h_x \otimes h_x.
\]

Solving the above expression for \(\text{vec}(\Sigma_x)\) we obtain

\[
\text{vec}(\Sigma_x) = (I - F)^{-1} \text{vec}(\Sigma_\epsilon)
\]

provided that the inverse of \((I - F)\) exists. The eigenvalues of \(F\) are products of the eigenvalues of the matrix \(h_x\). Because all eigenvalues of the matrix \(h_x\) have by construction modulus less than one, it follows that all eigenvalues of \(F\) are less than one in modulus. This implies that \((I - F)\) is nonsingular and we can indeed solve for \(\Sigma_x\). One possible drawback of this method is that one has to invert a matrix that has dimension \(n_x^2 \times n_x^2\).

**Method 2**

The following iterative procedure, called doubling algorithm, may be faster than the one described above in cases in which the number of state variables \((n_x)\) is large.

\[
\begin{align*}
\Sigma_{x,t+1} &= h_{x,t} \Sigma_{x,t} h_{x,t}' + \Sigma_{\epsilon,t} \\
h_{x,t+1} &= h_{x,t} h_{x,t} \\
\Sigma_{\epsilon,t+1} &= h_{x,t} \Sigma_{\epsilon,t} h_{x,t}' + \Sigma_{\epsilon,t}
\end{align*}
\]
\[ \Sigma_{x,0} = I \]
\[ h_{x,0} = h_x \]
\[ \Sigma_{c,0} = \Sigma_c \]

**Other second moments**

Once the covariance matrix of the state vector, \( x_t \) has been computed, it is easy to find other second moments of interest. Consider for instance the covariance matrix \( E \hat{x}_t \hat{x}'_{t-j} \) for \( j > 0 \). Let \( \mu_t = \sigma \eta \epsilon_t \).

\[
E \hat{x}_t \hat{x}'_{t-j} = E [h^j_x \hat{x}_{t-j} + \sum_{k=0}^{j-1} h^k_x \mu_{t-k} \hat{x}'_{t-j}]
= h^j_x E \hat{x}_{t-j} \hat{x}'_{t-j}
= h^j_x \Sigma_x
\]

Similarly, consider the variance covariance matrix of linear combinations of the state vector \( x_t \). For instance, the co-state, or control vector \( y_t \) is given by \( y_t = y + g_x (x_t - \mu_t) \), which we can write as: \( \hat{y}_t = g_x \hat{x}_t \). Then

\[
E \hat{y}_t \hat{y}'_t = E g_x \hat{x}_t \hat{x}'_t g'_x
= g_x [E \hat{x}_t \hat{x}'_t] g'_x
= g_x \Sigma_x g'_x
\]

and, more generally,

\[
E \hat{y}_{t-j} \hat{y}'_{t-j} = g_x [E \hat{x}_{t-j} \hat{x}'_{t-j} ] g'_x
= g_x h^j_x \Sigma_x g'_x,
\]
for \( j \geq 0 \).

### 4.14 Appendix: Impulse Response Functions

The impulse response to a variable, say \( z_t \) in period \( t + j \) to an impulse in period \( t \) is defined as:

\[
IR(z_{t+j}) \equiv E_t z_{t+j} - E_{t-1} z_{t+j}
\]

The impulse response function traces the expected behavior of the system from period \( t \) on given information available in period \( t \), relative to what was expected at time \( t - 1 \). Using the law of motion \( E_t \hat{x}_{t+1} = h_x \hat{x}_t \) for the state vector, letting \( x \) denote the innovation to the state vector in period 0, that is, \( x = \eta \sigma \epsilon_0 \), and applying the law of iterated expectations we get that the impulse response of the state vector in period \( t \) is given by

\[
IR(\hat{x}_t) \equiv E_0 \hat{x}_t - E_{-1} \hat{x}_t = h_x^t [x_0 - E_{-1} x_0] = h_x^t [\eta \sigma \epsilon_0] = h_x^t x; \quad t \geq 0.
\]

The response of the vector of controls \( \hat{y}_t \) is given by

\[
IR(\hat{y}_t) = g_x h_x^t x.
\]

### 4.15 Appendix: Matlab Code For Linear Perturbation Methods

The program `gx_hx.m` computes the matrices \( g_x \) and \( h_x \) using the Schur decomposition method.

The program `mom.m` computes second moments. The program `ir.m` computes impulse response functions.
4.16 Exercises

Exercise 4.1 (Variation of the Portfolio Adjustment Cost Model) This exercise aims at establishing whether formulating portfolio adjustment costs as a function of the deviation of the household’s debt position from an exogenous reference point, $d_t - \bar{d}$, or as a function of the change in its debt position, $d_t - d_{t-1}$, has consequences for the stationarity of the model.

Consider a small open economy populated by a large number of infinitely lived households with preferences described by the utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t,$$

where $\beta \in (0, 1)$ denotes the subjective discount factor and $c_t$ denotes consumption in period $t$. Each period, households receive an exogenous and stochastic endowment, $y_t$, and can borrow (or lend to) international financial markets at the gross interest rate $1 + r$. Let $d_t$ denote the stock of foreign debt held by households at the end of period $t$. Households are subject to a portfolio adjustment cost of the form $\phi^2 (d_t - d_{t-1})^2$, where $\phi$ is a positive constant. Assume that $\beta (1 + r) = 1$.

1. State the household’s period by period budget constraint.
2. State the household’s utility maximization problem
3. Write the Lagrangian of the household’s problem
4. Define a competitive equilibrium of this economy.
5. Suppose the endowment is non-stochastic and constant, $y_t = y$ for all $t$. Characterize the deterministic steady state. Does it exist? Is it unique?
6. Consider now a temporary endowment shock. Suppose $y_0 > y$ and $y_t = y$ for all $t > 0$ deterministically. Suppose that prior to period 0 the economy was in a deterministic steady
state with \( d_{-1} = d^* \). Is the economy stationary, that is, is \( d_t \) expected to return \( d^* \)? Provide intuition.

**Exercise 4.2 (Variation of the EDF Model)** This exercise analyzes the local stability of the equilibrium of the SOE-EDF model when the household’s subjective discount factor is assumed to be increasing in aggregate consumption, \( \theta'(c_t) > 0 \), as opposed to decreasing, as is assumed in the baseline specification presented in section 4.10.3.

Consider a small open economy populated by infinitely-lived agents. Let \( c_t \) denote consumption in period \( t \). Assume that the discount factor, denoted \( \beta_t \), evolves over time according to \( \beta_{t+1} = \theta(c_t) \beta_t \). Assume that the function \( \theta \) is positive and bounded above by unity. Agents have access to international financial markets where they can borrow or lend at the interest rate \( r > 0 \). Agents choose consumption and external debt, \( d_t \) so as to maximize lifetime utility given by \( \sum_{t=0}^{\infty} \beta_t U(c_t) \), where \( U(.) \) is an increasing and strictly concave function. Agents are endowed with \( y > 0 \) units of goods each period. Agents enter period 0 with a stock \( d_{-1} \) of net foreign debt. Assume that \( \beta_0 = 1 \). Assume that households are subject to some borrowing constraint that prevents them from engaging in Ponzi schemes. Assume that agents fail to internalize that their consumption choices affect their discount factor.

1. Characterize the competitive equilibrium of this economy.

2. Characterize the steady state of this economy. Consider the following two cases: (A.) \( \theta \) is strictly increasing in \( c \) and (B.) \( \theta \) is strictly decreasing in \( c \). What properties does the function \( \theta(.) \) need to have in each case to ensure existence of a steady state. What properties does the function \( \theta \) need to have in each case to ensure that the steady state is unique. Provide an intuitive explanation for your results by comparing them to those you would obtain in an economy in which \( \theta(.) \) is independent of \( c_t \). Which case, (A.) or (B.) is more plausible to you and why?
3. Characterize the local stability of the economy in a small neighborhood around the steady state. Specifically, suppose that $d_{t-1}$ is not equal to the steady state, under what conditions (on the function $\theta$) does there exist a unique perfect foresight equilibrium converging back to the steady state.

4. Assume now, contrary to what was assumed above, that agents internalize that their own consumption choice in period $t$ changes the discount factor, that is, they internalize that $\theta$ depends on $c_t$.

Characterize the competitive equilibrium of this economy. Give an intuitive explanation for the differences in equilibrium conditions in the economy with and without internalization.

5. Characterize the steady state of this economy. Does it exist? Is it unique? Is it the same as in the economy without internalization?

6. Characterize the local stability of the steady state. Specifically, suppose that $d_{t-1}$ is not equal but close to its steady state value. Under what conditions does there exist a unique perfect foresight equilibrium converging back to the steady state. Express your answer in terms of a condition involving the parameter $r$ and the following four elasticities, $\epsilon_\theta \equiv \frac{\theta'(c)c}{\theta(c)}$, $\epsilon_{\theta \theta} \equiv \frac{\theta''(c)c}{\theta(c)}$, $\epsilon_c \equiv \frac{U''(c)c}{U'(c)}$ and $\epsilon_{cc} \equiv \frac{U'''(c)c}{U''(c)}$, evaluated at the steady state value of $c_t$. Discuss how your result differs from that obtained in question 3 above.

Exercise 4.3 [Business Cycles in a Small Open Economy with Complete Asset Markets and External Shocks]

Consider the small open economy model with complete asset markets (CAM) studied in this chapter. Suppose that the productivity factor $A_t$ is constant and normalized to 1. Replace the
equilibrium condition \( U_c(c_t, h_t) = \psi_{cam} \) with the expression

\[
U_c(c_t, h_t) = x_t,
\]

where \( x_t \) is an exogenous and stochastic random variable, which can be interpreted as an external shock. Assume that the external shock follows a process of the form

\[
\hat{x}_t = \rho \hat{x}_{t-1} + \epsilon_t; \quad \epsilon_t \sim N(0, \sigma_{\epsilon}^2),
\]

where \( \hat{x}_t \equiv \ln(x_t/x) \) and \( x \) denotes the non-stochastic steady-state level of \( x_t \). Let \( \rho = 0.9 \) and \( \sigma_{\epsilon} = 0.02 \). Calibrate all other parameters of the model following the calibration of the CAM model presented in the main body of this chapter. Finally, set the steady state value of \( x_t \) in such a way that the steady-state level of consumption equals the level of steady-state consumption in the version of the CAM model studied in the main text.

1. Produce a table displaying the unconditional standard deviation, serial correlation, and correlation with output of \( \hat{y}_t, \hat{c}_t, \hat{i}_t, \hat{h}_t, \) and \( t_h/t_y \).

2. Produce a figure with 5 plots depicting the impulse responses to an external shock (a unit innovation in \( \epsilon_t \)) of \( \hat{y}_t, \hat{c}_t, \hat{i}_t, \hat{h}_t, \) and \( t_h/t_y \).

3. Now replace the values of \( \rho \) and \( \sigma_{\epsilon} \) given above with values such that the volatility and serial correlation of output implied by the model are the same as those reported for the Canadian economy in table 4.2. Answer questions 4.3.a and 4.3.b using these new parameter values.

4. Based on your answer to the previous question, evaluate the ability of external shocks (as defined here) to explain business cycles in Canada.
Exercise 4.4 (A Small Open Economy with an AR(2) TFP Process) In this question you are asked to show that the SOE-RBC model can predict consumption to be more volatile than output when the productivity shock follows a second-order autoregressive process displaying a hump-shaped impulse response. The theoretical model to be used is the External Debt-Elastic Interest Rate (EDEIR) model presented in section 4.1.1 of the current chapter. Replace the AR(1) process with the following AR(2) specification:

\[
\ln A_{t+1} = 1.42 \ln A_t - 0.43 \ln A_{t-1} + \epsilon_{t+1},
\]

where \( \epsilon_t \) is an i.i.d. random variable with mean zero and standard deviation \( \sigma_\epsilon > 0 \). Scale \( \sigma_\epsilon \) to ensure that the predicted standard deviation of output is 3.08, the value predicted by the AR(1) version of this model. Otherwise use the same calibration and functional forms as presented in the chapter. Download the matlab files for the EDEIR model from http://www.columbia.edu/~mu2166/closing.htm. Then modify them to accommodate the present specification.

1. Produce a table displaying the unconditional standard deviation, serial correlation, and correlation with output of output, consumption, investment, hours, the trade-balance-to-output ratio, and the current-account-to-output ratio.

2. Produce a 3 x 2 figure displaying the impulse responses of output, consumption, investment, hours, the trade-balance-to-output ratio, and TFP to a unit innovation in TFP.

3. Compare and contrast the predictions of the model under the AR(1) and the AR(2) TFP processes. Provide intuition.

Exercise 4.5 (Durable Consumption I) Consider a SOE model with non-durable and durable consumption goods. Let \( c_{N,t} \) denote consumption of non-durables in period \( t \) and \( c_{D,t} \) purchases of durables in period \( t \). The stock of durable consumer goods, denoted \( s_t \), is assumed to evolve over
time as \( s_t = (1 - \delta)s_{t-1} + c_{D,t} \), where \( \delta \in (0,1] \) denotes the depreciation rate of durable goods.

Households have preferences over consumption, \( c_t \), of the form \( \sum_{t=0}^{\infty} \beta^t U(c_t) \), where \( U \) is increasing in consumption and concave. Consumption, \( c_t \), is a composite of nondurable consumption and the service flow provided by the stock of consumer durables. Specifically, assume that

\[
c_t = \left[ (1 - \alpha) \frac{1}{\eta} c_{N,t} + \alpha \frac{1}{\eta} s_t \right]^{\frac{1}{1-\eta}},
\]

\( \eta > 0 \), and \( \alpha \in (0,1) \). Households have access to an internationally traded risk-free one-period bond, which pays the interest rate \( r_t \) when held between periods \( t \) and \( t+1 \). The relative price of durables in terms of nondurables is one. The household is subject to a borrowing limit that prevents it from engaging in Ponzi schemes. Output, denoted \( y_t \), is produced with capital according to a production function of the form \( y_t = F(k_t) \), where \( k_t \) denotes physical capital. The capital stock evolves over time as \( k_{t+1} = (1 - \delta_k)k_t + i_t \), where \( i_t \) denotes investment in period \( t \) and \( \delta_k \) is the depreciation rate on physical capital.

1. Describe the household’s budget set.

2. State the optimization problem of the household.

3. Present the complete set of equilibrium conditions.

4. The interest rate is constant over time and equal to \( r_t = r = \beta^{-1} - 1 \). Assume that up to period \(-1\) in the economy was in a steady state equilibrium in which all variables were constant and \( d = \bar{d} > 0 \), where \( d \) denotes net external debt in the steady state.

Find the share of expenditures on durables in total consumption expenditures in the steady state in terms of the parameters \( \delta, r, \alpha, \) and \( \eta \). Suggest a strategy for calibrating those four parameters.
5. Assume that in period 0 the economy unexpectedly receives a positive income shock as a consequence of the rest of the world forgiving part of the country’s net foreign debt. Assume that the positive income shock results in a one percent increase in the consumption of nondurables in period 0. Find the percent increase in purchases of durables and in total consumption expenditures in period 0. Compare your answer to the one you would have obtained if all consumption goods were nondurable.

6. Continuing to assume that consumption of nondurables increased by one percent, find the change in the trade balance in period 0 expressed as a share of steady state consumption expenditures. Is the response of the trade balance countercyclical? Compare your findings to those you would have obtained if all consumption goods were nondurable. How much amplification is there due to the presence of durables.

Exercise 4.6 (Durable Consumption II) Consider an economy populated by a large number of identical households with preferences described by the lifetime utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{(c_{t}^n - \frac{h_t}{\omega}) s_t^\gamma}{1 - \sigma} \right]^{1-\sigma} - 1,$$

where $c_t^n$ denotes consumption of nondurable goods, $h_t$ denotes hours worked, and $s_t$ denotes the stock of durable consumption goods. The parameter $\beta \in (0, 1)$ denotes the subjective discount factor, $\gamma, (\omega - 1), (\sigma - 1) > 0$ are preference parameters, and $E_t$ denotes the expectations operator conditional on information available in period $t$.

The law of motion of the stock of durables is assumed to be of the form

$$s_t = (1 - \delta)s_{t-1} + c_t^d,$$
where $c^d_t$ denotes durable consumption in period $t$, and $\delta \in (0, 1)$ denotes the depreciation rate. The sequential budget constraint of the household is given by

$$d_t = (1 + r_{t-1})d_{t-1} + c^n_t + c^d_t + \frac{\phi^d}{2}(s_t - s_{t-1})^2 + i_t + \frac{\phi^k}{2}(k_{t+1} - k_t)^2 - A_t k_t^\alpha h_t^{1-\alpha},$$

where $d_t$ denotes debt acquired in period $t$ and maturing in period $t+1$, $r_t$ denotes the interest rate on assets held between periods $t$ and $t+1$, $i_t$ denotes gross investment, $k_t$ denotes the stock of physical capital, and $A_t$ represents a technology factor assumed to be exogenous and stochastic. The parameters $\phi^d, \phi^k > 0$ govern the degree of adjustment costs in the accumulation of durable consumption goods and physical capital, respectively. The parameter $\alpha$ resides in the interval $(0, 1)$.

The capital stock evolves over time according to the law of motion

$$k_{t+1} = (1 - \delta)k_t + i_t.$$

Note that we assume that physical capital, $k_t$, is predetermined in period $t$ and that investment, $i_t$, takes one period to become productive capital. By contrast, the stock of consumer durables, $s_t$ is non-predetermined in period $t$, and expenditures in consumer durables in period $t$, $c^d_t$, become productive immediately. Finally, assume that the interest rate is debt elastic,

$$r_t = r^* + \psi \left[ e^{\bar{d}_t - \bar{d}} - 1 \right],$$

where $\bar{d}_t$ denotes the cross-sectional average level of debt per capita, and $r^*, \bar{d}$, and $\psi$ are parameters.

The productivity factor $A_t$ evolves according to the expression

$$\ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1},$$
where $\epsilon_t$ is a white noise with mean zero and variance $\sigma^2_\epsilon$, and $\rho \in (0,1)$ is a parameter. Assume that $\beta(1 + r^*) = 1$.

1. Derive the complete set of equilibrium conditions.

2. Derive the deterministic steady state. Specifically, find analytical expressions for the steady state values of $c_n, h_t, s_t, k_{t+1}, d_t, r_t, i_t, tb_t$, and $ca_t$ in terms of the structural parameters of the model $\sigma, \beta, \delta, \omega, \alpha, \gamma, r^*$, and $\bar{d}$. Here, $tb_t$ and $ca_t$ denote, respectively, the trade balance and the current account.

3. Assume the following parameter values: $\sigma = 2, \delta = 0.1, r^* = 0.04, \alpha = 0.3$, and $\omega = 1.455$. Calibrate $\bar{d}$ and $\gamma$ so that in the steady state the debt to output ratio is 25 percent and the nondurable consumption to output ratio is 68 percent. Report the implied numerical values of $\gamma$ and $\bar{d}$. Also report the numerical steady state values of $r_t, d_t, h_t, k_t, c^n_t, s_t, c^d_t, i_t, tb_t, ca_t$, and $y_t \equiv A_t k^\alpha h^{1-\alpha}$.

4. Approximate the equilibrium dynamics using a first-order perturbation technique. In performing this approximation, express all variables in logs, except for the stock of debt, the interest rate, the trade balance, the current account, the trade-balance-to-output ratio, and the current-account-to-output ratio. You are asked to complete the calibration of the model by setting values for $\psi, \phi^d, \phi^k, \rho$, and $\sigma_\epsilon$ to target key empirical regularities of medium-size emerging countries documented in chapter 1 of Uribe’s Open Economy Macroeconomics textbook. Specifically, the targets are a standard deviation of output, $\sigma_y$, of 8.99 percent, a relative standard deviation of consumption, $\sigma_c/\sigma_y$, of 0.93, a relative standard deviation of gross investment, $\sigma_I/\sigma_y$, of 2.86, a serial correlation of output of 0.84, and a correlation between the trade-balance-to-output ratio and output of -0.24. In general, you will not be able to hit these targets exactly. Instead, you are required to define a distance between the targets and
their corresponding theoretical counterparts and devise a numerical algorithm to minimize it. Define the distance as follows. Let \( z(\psi, \phi^d, \phi^k, \rho, \sigma_\epsilon) \equiv x(\psi, \phi^d, \phi^k, \rho, \sigma_\epsilon) - x^* \), where \( x^* \) is the 5x1 vector of empirical targets (the 5 numbers given above) and \( x(\psi, \phi^d, \phi^k, \rho, \sigma_\epsilon) \) is the 5x1 vector of theoretical counterparts as a function of the parameters. Let \( D(\psi, \phi^d, \phi^k, \rho, \sigma_\epsilon) \equiv \sqrt{z(\psi, \phi^d, \phi^k, \rho, \sigma_\epsilon)'z(\psi, \phi^d, \phi^k, \rho, \sigma_\epsilon)} \) be the distance between the target and its theoretical counterpart. Report (a) the values of \( \psi, \phi^d, \phi^k, \rho, \) and \( \sigma_\epsilon \) that you find and (b) complete the following table:

<table>
<thead>
<tr>
<th>Data</th>
<th>Prediction of the Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_y )</td>
<td>8.99</td>
</tr>
<tr>
<td>( \sigma_c/\sigma_y )</td>
<td>0.93</td>
</tr>
<tr>
<td>( \sigma_i/\sigma_y )</td>
<td>2.86</td>
</tr>
<tr>
<td>corr(( y_t, y_{t-1} ))</td>
<td>0.84</td>
</tr>
<tr>
<td>corr(( tb_t/y_t, y_t ))</td>
<td>-0.24</td>
</tr>
</tbody>
</table>

5. Produce a table displaying the model predictions. The table should contain the unconditional standard deviation, correlation with output, and the first-order serial correlation of output, consumption, investment, consumption of durables, consumption of nondurables, the trade-balance-to-output ratio, and the current-account-to-output ratio. For consumption, consumption of durables, consumption of nondurables, and investment report the standard deviation relative to output. Discuss how well the model is able to explain actual observed second moments that were not targeted in the calibration. Use the second moments reported in table 1.2 of Uribe’s textbook to compare the model’s predictions to actual data.

**Exercise 4.7 (Complete Markets and The Countercyclicality of the Trade Balance)** Consider a small open economy with access to a complete array of internationally traded state contingent claims. There is a single good, which is freely traded internationally. Let \( r_{t,t+1} \) denote the period
The price of a contingent claim that pays one good in a particular state of the world in period $t+1$ divided by the probability of occurrence of that state. The small open economy takes the process for $r_{t,t+1}$ as exogenously given.

Households have preferences over consumption, $c_t$, and hours, $h_t$, given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{(c_t - \frac{h_t}{\omega})^{1-\sigma} - 1}{1 - \sigma} \right]; \quad \sigma, \omega > 1,$$

where $E_0$ denotes the expectations operator conditional on information available in period 0. Households produce goods according to the following production technology

$$A_t k_t^\alpha h_t^{1-\alpha},$$

where $A_t$ denotes an exogenous productivity factor, $k_t$ denotes the capital stock in period $t$, and the parameter $\alpha \in (0,1)$ denotes the elasticity of the production function with respect to capital.

Domestic households are the owners of physical capital. The evolution of capital is given by

$$k_{t+1} = (1 - \delta)k_t + i_t,$$

where $i_t$ denotes investment in physical capital in period $t$ and $\delta \in (0,1)$ denotes the depreciation rate. In period 0, households are endowed with $k_0$ units of capital and hold contingent claims (acquired in period $-1$) that pay $d_0$ goods in period 0.

1. State the household’s period-by-period budget constraint.

2. Specify a borrowing limit that prevents household’s from engaging in Ponzi schemes.

3. State the household’s utility maximization problem. Indicate which variables/processes the
household chooses and which variables/processes it takes as given.

4. Derive the complete set of competitive equilibrium conditions.

5. Let $\tilde{x}_t = \ln x_t / x$ denote the percent deviation of a variable from its non-stochastic steady state value. Assume that in the non-stochastic steady state $r_{0,t} = \beta^t$ and $A_t = 1$. Show that in response to a positive innovation in technology in period $t$, $\tilde{A}_t > 0$, the trade balance will respond countercyclically only if the response in investment in period $t$ is positive. Then find the minimum percent increase in investment in period $t$ required for the trade balance to decline in period $t$ in response to the technology shock. To answer this question use a first-order accurate approximation to the solution of the model. Show that your answer is independent of the expected future value of $A_{t+1}$.

6. Compare and contrast your findings in the previous item to the ones derived in chapter 3 for a model with capital accumulation, no depreciation, no capital adjustment costs, inelastic labor supply, and incomplete markets. In particular, discuss how in that model the sign of the impulse response of the trade balance to a positive innovation in technology shock, $\tilde{A}_t > 0$, depended on the persistence of the technology shock. Give an intuitive explanation for the similarities/differences that you identify.

7. Now find the size of $E_t \tilde{A}_{t+1}$ relative to the size of $\tilde{A}_t$ that guarantees that the trade balance deteriorates in period $t$ in response to a positive innovation in $A_t$ in period $t$. Your answer should be a condition of the form $\tilde{A}_t < M E_t \tilde{A}_{t+1}$, where $M$ is a function of the structural parameters of the model. In particular, it is a function of $\alpha$, $\beta$, $\delta$, and $\omega$. Find the value of $M$ for $\alpha = 1/3$, $\delta = 0.08$, $\beta^{-1} = 1.02$, and $\omega = 1.5$.

8. Discuss to which extent your findings support or contradict Principle I, derived in chapter 3, which states that: “The more persistent are productivity shocks, the more likely is the trade
balance to experience an initial deterioration in response to a positive technology shock.”

9. How would your answers to questions 5 and 7 change if the period utility function was separable in consumption, $c_t$, and hours, $h_t$?

Exercise 4.8 [Calibrating the EDEIR Model Using Canadian Data Over the Period 1960-2011] In section 4.5, we calibrated the EDEIR model using second moments computed using Canadian data over the period 1946-1985. The middle panel of table 4.2 updates the empirical second moments to the period 1960 to 2011. The present exercise uses these empirical regularities to calibrate and evaluate the SOE-RBC model.

1. Calibrate the EDEIR model as follows: Set $\beta = 1/1.04$, $\sigma = 2$, $\omega = 1.455$, $\alpha = 0.32$, $\delta = 0.10$, and $\bar{d} = 0.7442$. Set the remaining four parameters, $\rho$, $\eta$, $\phi$, and $\psi_1$ to match the observed standard deviations and serial correlations of output and the standard deviations of investment and the trade-balance-to-output ratio in Canada over the period 1960-2011. Approximate the equilibrium dynamics up to first order and use a distance minimization procedure similar to the one used in exercise 4.6. Compare the resulting values for $\rho$, $\eta$, $\phi$, and $\psi_1$ with those reported in table 4.1.

2. Compute theoretical second moments and present your findings as in the third panel of table 4.2.

3. Comment on the ability of the model to explain observed business cycles in Canada over the period 1960-2011.

4. Compute the unconditional standard deviation of the productivity shock, $\ln A_t$ under the present calibration. Compare this number to the one corresponding to the 1946-1985 calibration presented in section 4.5. Now do the same with the standard deviation of output. Discuss and interpret your findings.
Exercise 4.9 (A Model of the U.S.-Canada Business Cycle) Consider a world with two economies, Canada and the United States, indexed by \( i = \text{Can, US} \), respectively. Suppose that both economies are populated by a large number of identical households with preferences given by

\[
E_0 \sum_{t=0}^{\infty} \beta^t \left[ c_i^t - (h_i^t)^{\omega} \right]^{1-\sigma} \left( \frac{1}{1-\sigma} - 1 \right),
\]

where \( c_i^t \) and \( h_i^t \) denote, respectively, consumption and hours worked in country \( i \) in period \( t \). In both countries, households operate a technology that produces output, denoted \( y_i^t \), using labor and capital, denoted \( k_i^t \). The production technology is Cobb-Douglas and given by

\[
y_i^t = A_i^t (k_i^t)^{\alpha} (h_i^t)^{1-\alpha},
\]

where \( A_i^t \) denotes a productivity shock in country \( i \), which evolves according to the following AR(1) process:

\[
\ln A_{i+1}^t = \rho^i \ln A_i^t + \eta^i \epsilon_t^{i+1},
\]

where \( \epsilon_t^i \) is an i.i.d. innovation with mean zero and variance equal to one, and \( \rho^i \) and \( \eta^i \) are country-specific parameters. Both countries produce the same good. The evolution of capital obeys the following law of motion:

\[
k_{i+1}^t = k_i^t + \frac{1}{\phi^i} \left[ \left( \frac{i_i^t}{\delta k_i^t} \phi^i \right) - 1 \right] \delta k_i^t,
\]

where \( i_i^t \) denotes investment in country \( i \), and \( \phi^i \) is a country-specific parameter.

Assume that asset markets are complete and that there exists free mobility of goods and financial assets between the United States and Canada, but that labor and installed capital are immobile across countries. Finally, assume that Canada has measure zero relative to the United States, so that the...
latter can be modeled as a closed economy.

Consider the business cycle regularities for Canada for the period 1960 to 2011 shown in exercise 4.8. The following table displays observed standard deviations, serial correlations, and correlations with output for the United States over the period 1960-2011. The source is World Development Indicators. The data are annual and in per capita terms. The series $y$, $c$, and $i$ are in logs, and the series $tb/y$ is in levels. All series were quadratically detrended. Standard deviations are measured in percentage points.

<table>
<thead>
<tr>
<th>Variable</th>
<th>U.S. Data 1960-2011</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_{xt}$</td>
</tr>
<tr>
<td>$y$</td>
<td>2.94</td>
</tr>
<tr>
<td>$c$</td>
<td>3.00</td>
</tr>
<tr>
<td>$i$</td>
<td>10.36</td>
</tr>
<tr>
<td>$tb/y$</td>
<td>0.94</td>
</tr>
</tbody>
</table>

1. Calibrate the model as follows: Assume that the deterministic steady-state levels of consumption per capita are the same in Canada and the United States. Set $\beta = 1/1.04$, $\sigma = 2$, $\omega = 1.455$, $\alpha = 0.32$, and $\delta = 0.10$. Set the remaining six parameters, $\rho^i$, $\eta^i$, and $\phi^i$, for $i = \text{Can,US}$, to match the observed standard deviations and serial correlations of output and the standard deviations of investment in Canada and the United States. Use a distance minimization procedure as in exercise 4.6.

2. Approximate the equilibrium dynamics up to first order. Produce the theoretical counterparts of the two tables showing Canadian and U.S. business-cycle regularities.

3. Comment on the ability of the model to explain observed business cycles in Canada and the United States.

4. Plot the response of Canadian output, consumption, investment, hours, and the trade-balance-to-output ratio to a unit innovation in the Canadian productivity shock. On the same plot,
show the response of the Canadian variables to a unit innovation to the U.S. productivity shock. Discuss the differences in the responses to a domestic and a foreign technology shock and provide intuition.

5. Compare, by means of a graph and a discussion, the predicted responses of Canada and the United States to a unit innovation in the U.S. productivity shock. The graph should include the same variables as the one for the previous item.

6. Compute the fraction of the volatilities of Canadian output and the trade-balance-to-output ratio explained by the U.S. productivity shock according to the present model. To this end, set $\eta_{\text{Can}} = 0$ and compute the two standard deviations of interest. Then, take the ratio of these standard deviations to their respective counterparts when both shocks are active.

7. This question aims to quantify the importance of common shocks as drivers of the U.S.-Canada business cycle. Replace the process for the Canadian productivity shock with the following one:

$$\ln A_{t+1}^{\text{Can}} = \rho^{\text{Can}} \ln A_{t}^{\text{Can}} + \eta^{\text{Can}} \epsilon_{t+1}^{\text{Can}} + \nu \epsilon_{t+1}^{US}.$$  

All other aspects of the model are as before. Recalibrate the model using an augmented version of the strategy described above that includes an additional parameter, $\nu$, and an additional target, the cross-country correlation of output, which in the sample used here is 0.64. Report the new set of calibrated parameters. Compute the variance of Canadian output. Now set $\nu = 0$ keeping all other parameter values unchanged, and recalculate the variance of Canadian output. Explain.

Exercise 4.10 (A EDEIR SOE with GHH Preferences and No Capital)  Consider a small
open economy populated by an infinite number of identical households with preferences of the form
\[
(1 - \sigma)^{-1} \sum_{t=0}^{\infty} \beta^t \left( c_t - \frac{h_t^\alpha}{\omega} \right)^{1-\sigma},
\]
where \( c_t \) denotes consumption of a perishable good in period \( t \), \( h_t \) denotes labor effort in period \( t \), and \( \beta \in (0, 1) \), \( \sigma > 1 \), and \( \omega > 1 \) are parameters. Each household operates a technology that produces consumption goods according to the relationship
\[
y_t = h_t^\alpha,
\]
where \( y_t \) denotes output, and \( \alpha \in (0, 1) \) is a parameter. The household can borrow or lend in international financial markets at the interest rate \( r_t = r^* + \rho(\tilde{d}_t) \), where \( r^* \) denotes the world interest rate and satisfies \( \beta(1 + r^*) = 1 \). The function \( \rho(\tilde{d}_t) \) is a country interest-rate premium in period \( t \), satisfying \( \rho(0) = 0 \), and \( \rho(x) \neq 0 \) for \( x \neq 0 \), where \( \tilde{d}_t \) denotes the cross-sectional average debt holdings in period \( t \) and is taken as given by the individual household. Let \( d_t \) denote the household’s debt holdings in period \( t \) maturing in \( t + 1 \). Households cannot play Ponzi games.

1. Write down the household’s optimization problem.

2. Derive the first-order conditions associated with the household’s optimization problem.

3. Display the complete set of equilibrium conditions.

4. Derive the steady state of the economy. In particular, compute the steady-state values of consumption, hours, output, the trade balance, the current account, and external debt, denoted, respectively, \( c \), \( h \), \( y \), \( tb \), \( ca \), and \( d \).

5. Derive analytically a first-order linear approximation of the equilibrium conditions. Express it as a first-order difference equation in the vector \( [\tilde{d}_{t-1} \, \tilde{c}_t]^\prime \), where \( \tilde{d}_{t-1} \equiv d_{t-1} - d \) and
\[ \dot{c}_t \equiv \ln(c_t/c). \]

6. Derive conditions under which the perfect-foresight equilibrium is locally unique.

**Exercise 4.11 (An SOE-RBC Model with Cobb-Douglas Preferences)** Modify the period utility function of the EDEIR SOE-RBC model of section 4.1 as follows

\[ U(c, h) = \left[ c^{1-\omega} (1-h)^{\omega} \right]^{1-\sigma} - 1 \]

All other features of the model are unchanged.

1. Derive analytically the steady state of the model.

2. Set all parameters of the model as in table 4.1, except for \( \omega \). Calibrate \( \omega \) to ensure that in the deterministic steady state hours equal 1/3 (i.e., to ensure that in the steady state, households spend one third of their time working). Calculate the implied value of \( \omega \).

3. Produce a table of predicted second moments similar to table 4.2. In performing this step, you might find it convenient to use as a starting point the Matlab programs for the EDEIR SOE-RBC model posted online.

4. Compare the predictions of the present model with those of its GHH-preference counterpart.
Chapter 5

Emerging-Country Business Cycles Through the Lens of the SOE-RBC Model

Can the SOE-RBC model of chapter 4 explain business cycles in emerging or poor economies? In chapter 1, we documented that the most striking difference between business cycles in rich and emerging or poor countries is that the latter are twice as volatile as the former (see fact 1.8 in chapter 1). In principle, the SOE-RBC model can account for this difference. All that is needed is to increase the volatility of the productivity shock. After all, the calibration strategy adopted in chapter 4, which is representative of much of the existing related literature, was to set the standard deviation of the exogenous productivity shock to match the observed variance of output. Since not only output, but all components of aggregate demand are more volatile in emerging and poor countries than in rich countries, increasing the volatility of the productivity shock will help in more than one dimension. However, not all volatilities increase by the same proportions as one moves
from rich to emerging or poor economies.

In particular, a second important difference between the group of rich countries and the group of emerging and poor countries is that in the former consumption is less volatile than output, whereas in the latter consumption is at least as volatile as output (see fact 1.9 in chapter 1). The SOE-RBC model of chapter 4 predicts that consumption is less volatile than output. This prediction is in line with the observed relative volatility of consumption in Canada, a rich economy to which the model was calibrated. A natural question is whether there exist calibrations of that SOE-RBC model that can account for the excess volatility of consumption observed in emerging countries.

5.1 Can the SOE-RBC Model Generate Excess Consumption Volatility?

The answer to this question is yes. We note, however, that simply jacking up the volatility of the productivity shock in the SOE-RBC model will not do the job. The reason is that up to first order in models with a single exogenous shock the ratio of any two standard deviations is independent of the standard deviation of the exogenous shock.

The analysis of a small open economy with capital of chapter 3 provides the insight that the response of consumption relative to that of output to a productivity shock depends significantly on the persistence of the productivity shock. Building on this insight, it is natural to explore whether increasing the persistence of the productivity shock will allow the SOE-RBC model of chapter 4 to explain the observed excess volatility of consumption in emerging and poor countries. Figure 5.1 displays the ratio of the volatility of consumption to the volatility of output, $\sigma_c/\sigma_y$, as a function of the persistence of the stationary productivity shock, $\rho$, predicted by the SOE-RBC model of chapter 4, section 4.1.1. For values of $\rho$ larger than 0.88 the volatility of consumption exceeds that of output.
Figure 5.1: The Relative Volatility of Consumption as a Function of the Persistence of the Stationary Technology Shock

Figure 5.2 helps build the intuition behind this result. It displays the impulse response of output to a one-percent increase in productivity for two values of $\rho$, 0.42 (the value used in chapter 4) and 0.99. For the lower value of $\rho$, the impulse response of output to a positive productivity shock is positive on impact and monotonically decreasing. This means that in the period the shock occurs, future output is expected to be lower than current output. Because consumption depends not on current output alone, but on the present discounted value of output, we have that the impact response of consumption is smaller than that of output. By contrast, when the technology shock is highly persistent, the response of output is hump-shaped (see the broken line in figure 5.2). In this case, on impact, output may be smaller than the average of current and future values of output. Consequently the impact response of consumption may exceed that of output, suggesting a higher volatility of consumption relative to output. In turn, the reason why the response of output is humped-shaped has to do with the behavior of investment. If the serial correlation of the
technology shock is small, investment does not react much to innovations in productivity, since they are expected to die out quickly. Thus, the response of output mimics that of technology. However, if the productivity shock is highly persistent, then firms will have an incentive to increase the stock of physical capital, to take advantage of the fact that capital will be highly productive for a number of periods. As a result, output may continue to increase even as TFP falls monotonically back to its steady-state level.

It follows that capital accumulation is crucial for the SOE-RBC model of chapter 4 to capture the excess volatility of consumption characteristic of emerging economies. However, increasing the serial correlation of the stationary productivity shock may come at a cost. Recall, for instance, that in chapter 4, the strategy for calibrating the parameter $\rho$ was to match the observed serial correlation of output. Thus, in principle, there will be a tradeoff between matching the excess
volatility of consumption and matching the serial correlation of output.

A possible solution to this tradeoff is to add an additional shock to the SOE-RBC model. In chapter 2, section 2.4, we suggested that a possible way to induce excess volatility of consumption is to introduce nonstationary shocks. Aguiar and Gopinath (2007) pursue this strategy. Specifically, they specify a small-open-economy version of the closed-economy RBC model with permanent and temporary TFP shocks due to King, Plosser, and Rebelo (1988a,b). Aguiar and Gopinath argue that the SOE-RBC model is an adequate framework for understanding aggregate fluctuations in emerging countries provided it is augmented to allow for both stationary and nonstationary productivity shocks. We present this model next.

5.2 An SOE-RBC Model With Stationary And Nonstationary Technology Shocks

Consider a small open economy populated by a large number of identical households seeking to maximize the utility function

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{[C_t^{\gamma}(1-h_t)^{1-\gamma}]^{1-\sigma} - 1}{1-\sigma}$$

subject to

$$\frac{D_{t+1}}{1+r_t} = D_t + C_t + K_{t+1} - (1-\delta)K_t + \frac{\phi}{2} \left( \frac{K_{t+1}}{K_t} - g \right)^2 K_t - Y_t,$$

and to a no-Ponzi-game constraint of the form

$$\lim_{j \to \infty} E_0 \frac{D_{t+j+1}}{\Pi_{s=0}^{j}(1+r_{t+s})} \leq 0,$$
where

$$Y_t = a_t K_t^\alpha (X_t h_t)^{1-\alpha}$$

denotes output in period $t$. In the above expressions, $C_t$ denotes consumption, $h_t$ denotes hours worked, $K_t$ denotes the stock of physical capital, $D_t$ denotes net external debt, and $r_t$ denotes the interest rate charged by the rest of the world. The parameters $\alpha$, $\beta$, and $\delta$ lie in the interval $(0, 1)$, and the parameters $\gamma$, $\sigma$, $\phi$, and $g$ are positive. As in chapter 4, section 4.1.1, the interest rate is assumed to be debt elastic:

$$r_t = r^* + \psi \left[ e^{\dot{D}_t/X_t d} - 1 \right],$$

where $r^*$, $\psi$, and $d$ are parameters, and $\dot{D}_t$ denotes the cross-sectional average level of external debt per capita in period $t$. In equilibrium, because all households are identical, we have that

$$\dot{D}_t = D_t.$$

This economy is driven by a stationary productivity shock $a_t$ and a nonstationary productivity shock $X_t$. Note that, unlike the SOE-RBC model of chapter 4, the period utility function in the Aguiar and Gopinath model takes a Cobb-Douglas form. Their results, however, are robust to assuming GHH preferences.

The optimality conditions associated with the household’s problem are

$$\frac{1-\gamma}{\gamma} \frac{C_t}{1-h_t} = (1-\alpha)a_t X_t \left( \frac{K_t}{X_t h_t} \right)^\alpha,$$

$$\gamma C_t^{\gamma(1-\sigma)-1} (1-h_t)^{(1-\gamma)(1-\sigma)} = \Lambda_t,$$

$$\Lambda_t = \beta (1+r_t)E_t \Lambda_{t+1},$$
and

\[
\Lambda_t \left[ 1 + \phi \left( \frac{K_{t+1}}{K_t} - g \right) \right] = \beta E_t \Lambda_{t+1} \left[ 1 - \delta + \alpha a_{t+1} \left( \frac{K_{t+1}}{X_{t+1} h_{t+1}} \right)^{\alpha - 1} \right.
\]
\[
+ \phi \frac{K_{t+2}}{K_{t+1}} \left( \frac{K_{t+2}}{K_{t+1}} - g \right) - \frac{\phi}{2} \left( \frac{K_{t+2}}{K_{t+1}} - g \right)^2 \right],
\]

where \( \Lambda_t \) denotes the Lagrange multiplier associated with the sequential budget constraint of the household.

The main difference between the present model and the one studied in chapter 4 is the introduction of the nonstationary productivity shock \( X_t \). Assume that \( X_t \) and \( a_t \) are mutually independent random variables with laws of motion given by

\[
\ln a_t = \rho_a \ln a_{t-1} + \sigma_a \epsilon^a_t
\]

and

\[
\ln \left( \frac{g_t}{g} \right) = \rho_g \ln \left( \frac{g_{t-1}}{g} \right) + \sigma_g \epsilon^g_t,
\]

where

\[
g_t \equiv \frac{X_t}{X_{t-1}}
\]

denotes the gross growth rate of \( X_t \). The parameters \( \rho_a \) and \( \rho_g \) lie in the interval \((-1, 1)\), and \( \sigma_a \) and \( \sigma_g \) are positive. The variables \( \epsilon^a_t \) and \( \epsilon^g_t \) are assumed to be exogenous, mutually independent white noises distributed \( N(0,1) \). The parameter \( g > 0 \) denotes the gross growth rate of productivity in a nonstochastic equilibrium path. Note that the productivity factor \( X_t \) is nonstationary in the sense that it displays both secular growth, at an average rate \( g \), and a random walk component. This last characteristic is reflected in the fact that an innovation in \( g_t \) has a permanent effect on the level of \( X_t \).
Let \( TFP_t \equiv \frac{Y_t}{K_t^{\alpha}h_t^{1-\alpha}} \). Under the present technology specification, we have that

\[
TFP_t = a_t X_t^{1-\alpha}.\tag{5.1}
\]

Clearly, because \( a_t \) is a stationary random variable independent of \( X_t \), total factor productivity inherits the nonstationarity of \( X_t \). And this property will be transmitted in equilibrium to other variables of the model, including consumption, investment, the capital stock, the marginal utility of wealth, and the stock of external debt. Because none of these variables exhibits a deterministic steady state, it is impossible to linearize the model around such point. Fortunately, however, there exists a simple stationary transformation of the variables of the model whose equilibrium behavior is described by a system of equations very similar to the one that governs the joint determination of the original variables. Specifically, let \( c_t \equiv C_t / X_{t-1} \), \( k_t \equiv K_t / X_{t-1} \), \( d_t \equiv D_t / X_{t-1} \), and \( \lambda_t \equiv X_{t-1}^{1+(\sigma-1)\gamma} \Lambda_t \). Then, one can write the system of equilibrium conditions in stationary form as

\[
\begin{align*}
g_t d_{t+1} + 1 + r_t &= d_t + c_t + g_t k_{t+1} - (1 - \delta) k_t + \frac{\phi}{2} \left( \frac{g_t k_{t+1}}{k_t} - g \right)^2 k_t - a_t k_t^\sigma (g_t h_t)^{1-\alpha}, \\
r_t &= r^* + \psi \left[ e^{d_{t+1} - \overline{d}} - 1 \right], \\
\frac{1 - \gamma}{\gamma} \frac{c_t}{1 - h_t} &= (1 - \alpha) a_t g_t \left( \frac{k_t}{g_t h_t} \right)^\alpha, \\
\gamma c_t^{(1-\sigma)-1} (1 - h_t)^{(1-\gamma)(1-\sigma)} &= \lambda_t, \\
\lambda_t &= \beta (1 + r_t) g_t^{(1-\sigma)-1} E_t \lambda_{t+1},
\end{align*}
\]
\[ \lambda_t \left[ 1 + \phi \left( \frac{g_t k_{t+1}}{k_t} - g \right) \right] = \beta g_t^{(1-\sigma)-1} E_t \lambda_{t+1} \left[ 1 - \delta + \alpha a_{t+1} \left( \frac{k_{t+1}}{g_{t+1} h_{t+1}} \right)^{\alpha-1} \right. \\
+ \phi g_{t+1} k_{t+2} \left( \frac{g_{t+1} k_{t+2}}{k_{t+1}} - g \right) - \frac{\phi}{2} \left( \frac{g_{t+1} k_{t+2}}{k_{t+1}} - g \right)^2 \].

This is a system of six stochastic difference equations in the endogenous variables \( d_{t+1}, c_t, k_{t+1}, h_t, \lambda_t, \) and \( r_t \). This system, together with the laws of motion of \( g_t \) and \( a_t \), possesses two properties. First, it has a deterministic steady state that is independent of initial conditions. Second, the rational expectations dynamics of all variables are, up to first order, mean reverting, or stationary.

Recalling that the variable transformations involve scaling by the nonstationary productivity factor, it follows that in this model consumption, output, the capital stock, investment, and net external debt all share the same stochastic trend, \( X_t \). For instance, consumption satisfies \( C_t = c_t X_{t-1} \). Since \( c_t \) is stationary, it follows directly that \( C_t \) carries the same random walk component as \( X_t \). The existence of a common stochastic trend implies that in equilibrium, the shares of consumption, investment, capital, and external debt in GDP are all stationary variables. This property of the model is known as the balanced-growth property.

Aguiar and Gopinath (2007) econometrically estimate the parameters defining the laws of motion of the two productivity shocks as well as the parameter governing the strength of capital adjustment costs. All other parameters of the model are calibrated.

The econometric estimation consists in picking values for \( \sigma_a, \sigma_g, \rho^a, \rho^g, g, \) and \( \phi \) to match the empirical second moments displayed in table 5.3 and the observed average growth rate of GDP. The data are from Mexico and the sample is 1980:Q1 to 2003:Q1. Because the number of estimated parameters (six) is smaller than the number of moments matched (eleven), the estimation procedure uses a weighting matrix following the GMM technique. To perform this estimation, one must assign
values to all non-estimated parameters. Table 5.1 displays these values. And table 5.2 displays the values taken by the estimated parameters.

Table 5.3 displays ten empirical and theoretical second moments for Mexico. The estimated model does a good job at matching all of the moments shown in the table. Of particular relevance is the ability of the model to match the fact that in Mexico, as in most other emerging countries, consumption is more volatile than output. As explained in detail in chapter 2, section 2.4, for the case of nonstationary endowment shocks, the presence of nonstationary productivity shocks plays a key role in making this prediction possible.

It is therefore of interest to calculate the importance of the nonstationary component of productivity in driving movements in total factor productivity implied by the econometrically estimated parameters. To this end, consider the growth rate of total factor productivity, which, from equation (5.1), can be written as

$$\Delta \ln TFP_t = \Delta \ln a_t + (1 - \alpha)g_t.$$  

Because the two terms on the right-hand side of this expression are mutually independent, we can
Table 5.3: Model Fit

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma(y)$</td>
<td>2.40</td>
<td>2.13</td>
</tr>
<tr>
<td>$\sigma(\Delta y)$</td>
<td>1.52</td>
<td>1.42</td>
</tr>
<tr>
<td>$\sigma(c)/\sigma(y)$</td>
<td>1.26</td>
<td>1.10</td>
</tr>
<tr>
<td>$\sigma(i)/\sigma(y)$</td>
<td>4.15</td>
<td>3.83</td>
</tr>
<tr>
<td>$\sigma(nx)/\sigma(y)$</td>
<td>0.80</td>
<td>0.95</td>
</tr>
<tr>
<td>$\rho(y)$</td>
<td>0.83</td>
<td>0.82</td>
</tr>
<tr>
<td>$\rho(\Delta y)$</td>
<td>0.27</td>
<td>0.18</td>
</tr>
<tr>
<td>$\rho(y, nx)$</td>
<td>-0.75</td>
<td>-0.50</td>
</tr>
<tr>
<td>$\rho(y, c)$</td>
<td>0.82</td>
<td>0.91</td>
</tr>
<tr>
<td>$\rho(y, i)$</td>
<td>0.91</td>
<td>0.80</td>
</tr>
</tbody>
</table>

Note. Variables in levels were HP filtered using a parameter of 1600. Growth rates are unfiltered. Source: Aguiar and Gopinath (2007).

ask what fraction of the variance of $\Delta \ln TFP_t$ is explained by $g_t$. It is straightforward to deduce that the variance of $\Delta \ln a_t$ is given by $2\sigma_a^2/(1 + \rho_a)$. At the same time, the variance of $g_t$ is given by $\sigma_g^2/(1 - \rho_g^2)$. Therefore, we have that

$$\frac{\text{var}((1 - \alpha)g_t)}{\text{var}(\Delta \ln TFP_t)} = \frac{(1 - \alpha)^2\sigma_g^2/(1 - \rho_g^2)}{2\sigma_a^2/(1 + \rho_a) + (1 - \alpha)^2\sigma_g^2/(1 - \rho_g^2)} = \frac{(1 - 0.32)^2 \times 0.0213^2/(1 - 0.00^2)}{(1 - 0.32)^2 \times 0.0053^2/(1 + 0.95) + (1 - 0.32)^2 \times 0.0213^2/(1 - 0.00^2)} = 0.8793.$$ (5.2)

That is, the estimated parameters imply that the nonstationary component of productivity explains 88 percent of the variance of the growth rate of total factor productivity. This is an indication that this model can fit the Mexican data best when nonstationary technology shocks play a significant role in moving total factor productivity at business-cycle frequency.

Aguiar and Gopinath (2007) also estimate the present model using quarterly Canadian data.
from 1981:Q1 to 2003:Q2 and find that the nonstationary component explains only 40 percent of movements in total factor productivity. Aguiar and Gopinath conclude that their estimates of the model on Mexican and Canadian data, taken together, suggest that nonstationary productivity shocks are more relevant in emerging economies than in developed ones.

How should we interpret these results? There are three aspects of the econometric estimation that deserve special comments. One is that the data sample, 1980:Q1 to 2003:Q1, is relatively short. Recall that the main purpose of the estimation procedure is to identify the random-walk, or unit-root, component in total factor productivity. It is well known that the only reliable way to disentangle the stationary and nonstationary components of a time series is to use long samples. Short samples can lead to spurious results. In fact, Aguiar and Gopinath (2007) analyze direct evidence on Solow residuals, which in the present model coincide with total factor productivity, for Mexico and Canada over the period 1980-2000, and conclude that it is not possible in that short sample to determine reliably whether the nonstationary component is more important in Mexico or in Canada (see their figure 2).

Figure 5.3 shows why using short samples for estimation may be problematic. It displays the cyclical component of the log of real GDP per capita for seven Latin American countries over the period 1900-2005. In the figure, the cycle is computed as percent deviations of GDP from a cubic trend. The period 1980-2005 contains only between one and a half and two cycles for most of the Latin American economies included in the figure. Doing econometrics with so few cycles is problematic for uncovering virtually any parameter value of a business-cycle model, but particularly for telling apart highly persistent but stationary productivity shocks from nonstationary productivity shocks.

The second difficulty with the econometric strategy pursued in Aguiar and Gopinath is that it allows room only for productivity shocks. This would not be a big problem if no other candidate shocks could be identified as potentially important in driving business cycles in emerging economies.
Figure 5.3: Business Cycles in Latin America: 1900-2005

But this is not the case. For example, a growing number of studies show that world interest-rate shocks and country-spread shocks play an important role in driving business cycles in emerging countries (see, for example, Neumeyer and Perri, 2005; and Uribe and Yue, 2006). Omitting these and other relevant shocks in the econometric estimation necessarily induces a bias in favor of the shocks that are included.

Finally, the present model limits attention to a frictionless neoclassical framework. This might also be an oversimplification. A large body of work points at financial frictions, including default risk and balance-sheet effects, as important propagation mechanisms of business cycles in emerging economies (see part ?? of this book). Omitting these sources of friction might cause a spurious increase in the estimated variance and persistence of the exogenous driving processes.

García-Cicco, Pancrazi, and Uribe (2010) address these concerns by estimating a SOE-RBC model in which stationary and nonstationary productivity shocks compete with interest-rate and country-spread shocks in explaining business cycles. To obtain a reliable measure of the nonstationary component of productivity, they estimate the model using long data samples spanning over 100 years. And to capture the presence of financial frictions, these authors estimate the parameter governing the debt elasticity of the country interest rate. They find that once financial shocks and frictions are taken explicitly into account, the data assigns a small role to permanent technology shocks as drivers of the business cycle. In the next section, we take a closer look at the García-Cicco, Pancrazi, and Uribe (2010) model.

5.3 Letting Technology Shocks Compete With Other Shocks And Frictions

In the model of the previous section, technology shocks monopolize the explanation of the business cycle. In this section, we make stationary and nonstationary technology shocks compete with
interest-rate shocks and other shocks in explaining business cycles in emerging countries. In addition to stationary and nonstationary technology shocks and interest-rate shocks, the competition includes two domestic demand disturbances stemming from shifts in the marginal utility of consumption (preference shocks) and from random changes in aggregate spending (public spending shocks). The nonstructural shocks included in the competition emerge from assuming that the time series used for the econometric estimation of the model may be measured with error. The presentation draws from García-Cicco, Pancrazi, and Uribe (2010), hereafter GPU.

5.3.1 Households

Consider an economy populated by a large number of identical households with preferences described by the utility function

\[ E_0 \sum_{t=0}^{\infty} \nu_t \beta^t \left[ C_t - \omega^{-1} X_{t-1} h_t^\omega \right]^{1-\gamma} - 1 \]

(5.3)

where \( C_t \) denotes consumption, \( h_t \) denotes hours supplied to the labor market, \( \nu_t \) denotes a preference shock, and \( X_t \) denotes a stochastic trend. This preference specification follows Schmitt-Grohé (1998). One might wonder why a stochastic trend appears in the utility function. The technical reason is that, as we will show shortly, this formulation makes it possible for a model with GHH preferences to exhibit balanced growth, that is, an equilibrium in which output, consumption, investment, and the capital stock all grow on average at the same rate and in which hours do not grow in the long run. From an economic point of view, \( X_t \) may reflect the impact of technological progress on household production.

As in the previous section, we use upper case letters to denote variables that contain a trend in equilibrium and lower case letters to denote variables that do not contain a trend in equilibrium.
The laws of motion of $\nu_t$ and $X_t$ are assumed to be

\[
\ln \nu_{t+1} = \rho_\nu \ln \nu_t + \epsilon_\nu^{t+1},
\]

and

\[
\ln(g_{t+1}/g) = \rho_g \ln(g_t/g) + \epsilon_g^{t+1},
\]

where

\[
g_t \equiv \frac{X_t}{X_{t-1}}
\]

denotes the gross growth rate of $X_t$. The innovations $\epsilon_\nu^{t}$ and $\epsilon_g^{t}$ are assumed to be mutually independent i.i.d. processes with mean zero and variances $\sigma_\nu^2$ and $\sigma_g^2$, respectively. The parameter $g$ measures the deterministic gross growth rate of the stochastic trend $X_t$. The parameters $\rho_\nu, \rho_g \in (-1, 1)$ govern the persistence of $\nu_t$ and $g_t$, respectively.

Households face the period-by-period budget constraint

\[
\frac{D_{t+1}^h}{1 + r_t} = D_t^h - W_t h_t - u_t K_t + C_t + S_t + I_t + \phi \left( \frac{K_{t+1}}{K_t} - g \right)^2 K_t - \Pi_t,
\]

(5.4)

where $D_{t+1}^h$ denotes the stock of one-period debt acquired by the household in period $t$ and due in period $t+1$, $r_t$ denotes the country-specific interest rate on debt held between periods $t$ and $t+1$, $I_t$ denotes gross investment, $K_t$ denotes the stock of physical capital owned by the household, $u_t$ denotes the rental rate of capital, and $W_t$ denotes the real wage rate. The variable $S_t$ is meant to capture aggregate shifts in domestic absorption, possibly stemming from unproductive government consumption, and is assumed to be exogenous and stochastic. The variable $\Pi_t$ denotes profits received. Households regard this variable as given. We assume that the detrended component of
$S_t$, denoted

$$s_t \equiv \frac{S_t}{X_{t-1}}, \quad (5.5)$$

obeys the AR(1) process

$$\ln(s_{t+1}/\bar{s}) = \rho_s \ln(s_t/\bar{s}) + \epsilon_{s_{t+1}},$$

where $\bar{s}$ is a parameter. The innovation $\epsilon_{s_t}$ is assumed to be a white noise with mean zero and variance $\sigma_s^2$, and the parameter $\rho_s \in (-1, 1)$ governs the persistence of $s_t$.

The parameter $\phi$ introduces quadratic capital adjustment costs. The capital stock evolves according to the following law of motion:

$$K_{t+1} = (1 - \delta) K_t + I_t, \quad (5.6)$$

where $\delta \in [0, 1)$ denotes the depreciation rate of capital. Consumers are assumed to be subject to a no-Ponzi-scheme constraint of the form $\lim_{j \to \infty} E_t D_{h_{t+1}} Q_j (1 + r_t + s_t) \leq 0$.

The optimization problem of the household consists in choosing processes $\{C_t, h_t, D_{t+1}^h, K_{t+1}, I_t\}$ to maximize the utility function (5.3) subject to (5.4), (5.6), and the no-Ponzi-game constraint, taking as given the processes $\{W_t, u_t, X_t, r_t, \nu_t, s_t\}, \Pi_t,$ and the initial conditions $K_0$ and $D_{h_0}$. Letting $\beta \lambda_t X_{t-1}^{-\gamma}$ denote the Lagrange multiplier associated with the sequential budget constraint, the optimality conditions associated with this problem are (5.4), (5.6), the no-Ponzi-game constraint holding with equality, and

$$\nu_t \left[ C_t / X_{t-1} - \omega^{-1} h_t^\omega \right]^{-\gamma} = \lambda_t,$$

$$\nu_t \left[ C_t / X_{t-1} - \omega^{-1} h_t^\omega \right]^{-\gamma} h_t^{\omega-1} = \frac{W_t}{X_{t-1}} \lambda_t,$$

$$\lambda_t = \beta \frac{1 + r_t}{g_t} E_t \lambda_{t+1}, \quad (5.7)$$
\[ \left[ 1 + \phi \left( \frac{K_{t+1}}{K_t} - g \right) \right] \lambda_t = \frac{\beta}{g_t} E_t \lambda_{t+1} [1 - \delta + u_{t+1} + \phi \left( \frac{K_{t+2}}{K_{t+1}} \right) \left( \frac{K_{t+2}}{K_{t+1}} - g \right) - \phi \left( \frac{K_{t+2}}{K_{t+1}} - g \right)^2]. \]

### 5.3.2 Firms with Working-Capital Constraints

Firms are assumed to operate in perfectly competitive product and factor markets. They produce a single good with a Cobb-Douglas production function that uses capital and labor as inputs and is buffeted by stationary and nonstationary productivity shocks. Formally,

\[ Y_t = a_t K_t^\alpha (X_t h_t)^{1-\alpha}, \tag{5.8} \]

where \( Y_t \) denotes output in period \( t \), \( \alpha \in (0, 1) \) is a parameter, and \( a_t \) represents a stationary productivity shock following an AR(1) process of the form

\[ \ln a_{t+1} = \rho_a \ln a_t + \epsilon_{t+1}^a. \]

The innovation \( \epsilon_{t}^a \) is assumed to be a white noise with mean zero and variance \( \sigma_{a}^2 \), and the parameter \( \rho_a \in [0, 1) \) governs the persistence of \( a_t \).

Following Neumeyer and Perri (2005), Uribe and Yue (2006), and Chang and Fernández (2013), we assume that firms face a working capital constraint. The formulation presented here is a variation of the one developed in Uribe and Yue (2006). Specifically, we assume that for each unit of wage payments firms must hold \( \eta \) units of a non-interest-bearing asset, denoted \( M_t \). Formally, the
working capital constraint takes the form

\[ M_t \geq \eta W_t h_t, \]

where \( M_t \) is the amount of working capital held by the firm in period \( t \). Firms can borrow or lend at the rate \( r_t \) and distribute dividends in the amount

\[ \Pi^f_t = a_t K_t^\alpha (X_t h_t)^{1-\alpha} - u_t K_t - W_t h_t - (M_t - M_{t-1}) + \left( \frac{D_{t+1}^f}{1 + r_t} - D_t^f \right), \]  

(5.9)

where \( D_{t+1}^f \) denotes the amount of discount debt acquired by the firm in period \( t \) and due in period \( t + 1 \). This expression says that distributed dividends equal revenues minus factor costs minus the change in the firm’s holdings of sight deposits, plus the increase in the firm’s borrowing. The firm chooses \( K_t, h_t, M_t, \) and \( D_{t+1}^f \) to maximize the present discounted value of distributed dividends,

\[ E_0 \sum_{t=0}^{\infty} \beta^t X_{t-1}^{-\gamma} \lambda_t \Pi^f_t. \]

This expression uses the household’s marginal utility of wealth to discount profits. This is reasonable, because the firm, and therefore also its profit stream, is assumed to belong to households. The firm is subject to a no-Ponzi-game constraint of the form

\[ \lim_{j \to \infty} E_j \frac{D_{t+j+1}^f - M_{t+j+1}}{\Pi_{t+j}^f} \leq 0. \]

The Lagrangian associated with the firm’s profit maximization problem is given by

\[ \mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t X_{t-1}^{-\gamma} \lambda_t \left[ a_t K_t^\alpha (X_t h_t)^{1-\alpha} - u_t K_t - W_t h_t - M_t + M_{t-1} + \frac{D_{t+1}^f}{1 + r_t} - D_t^f + \xi_t (M_t - \eta W_t h_t) \right], \]

In periods in which \( r_t < 0 \), the presence of this non-interest-bearing asset gives rise to a pure arbitrage opportunity. Thus, the present analysis assumes that \( r_t \geq 0 \). An alternative way to eliminate this type of arbitrage opportunity is to assume that the rate of return on \( M_t \) is the minimum between 0 and \( r_t \). Under this alternative, the term \( M_{t-1} \) in the definition of firm profits must be replaced by \( M_{t-1}(1 + \min(0, r_{t-1})) \).
where $\xi_t\lambda_t\beta^\gamma X_{t-1}$ denotes the Lagrange multiplier on the working capital constraint. The firm’s optimality conditions with respect to $D^f_{t+1}$, $K_t$, $h_t$, and $M_t$ are, respectively,

$$\lambda_t = \beta g_t^{-\gamma}(1 + r_t)E_t\lambda_{t+1}$$

$$\alpha a_t \left( \frac{X_t h_t}{K_t} \right)^{1-\alpha} = u_t,$$

$$(1 - \alpha)a_t X_t \left( \frac{K_t}{X_t h_t} \right)^\alpha = W_t(1 + \eta\xi_t),$$

and

$$\lambda_t(1 - \xi_t) = \beta E_t\lambda_{t+1} g_t^{-\gamma}.$$

Combining the first and the last optimality conditions yields

$$\xi_t = \frac{r_t}{1 + r_t},$$

which states that as long as the opportunity cost of funds is positive ($r_t > 0$) the working capital constraint introduces a distortion that elevates the effective cost of labor. Now combining this expression with the first-order condition with respect to $h_t$ yields

$$(1 - \alpha)a_t X_t \left( \frac{K_t}{X_t h_t} \right)^\alpha = W_t \left[ 1 + \eta \frac{r_t}{1 + r_t} \right].$$

This expression says that in the presence of a working-capital constraint, the total labor cost includes the standard wage component, given by $W_t h_t$, and a financial component, given by $\eta W_t h_t r_t / (1 + r_t)$. Note that an increase in the interest rate acts like an increase in the real wage, thereby inducing firms to reduce employment. This effect is of interest because it introduces a supply-side channel through which changes in the interest rate can affect the economy. In this
way, interest-rate shocks are allowed to directly compete with technology shocks in determining movements in employment and output.

5.3.3 Interest-Rate Shocks

We augment the interest-rate specification of the previous section by introducing interest-rate shocks. Specifically, the domestic interest rate is assumed to be given by

\[ r_t = r^* + \psi \left( e^{\frac{\bar{D}_{t+1} / X_t - \bar{D}}{\bar{Y}_t}} - 1 \right) + e^{\mu_t - 1} - 1, \]  

(5.10)

where \( \bar{D}_{t+1} \) denotes the aggregate level of external debt in period \( t \), and \( \bar{d} \) and \( \bar{Y} \) are parameters. The variable \( \mu_t \) is assumed to be exogenous and stochastic. It is meant to reflect exogenous, random variations in the world interest rate and the country spread. The law of motion of \( \mu_t \) is given by

\[ \ln \mu_{t+1} = \rho_\mu \ln \mu_t + \epsilon_{t+1}^\mu. \]

The innovation \( \epsilon_{t+1}^\mu \) is assumed to be a white noise with mean zero and variance \( \sigma_{\mu}^2 \), and the parameter \( \rho_\mu \in [0, 1) \) governs the persistence of \( \mu_t \).

5.3.4 Equilibrium

Assume that there is a continuum of identical and perfectly competitive domestic financial intermediaries, or banks, that borrow funds in international financial markets and lend them to domestic households and firms. Also, the financial intermediaries accept non-interest-bearing (sight) deposits from firms. The balance sheet of the representative financial intermediary is

\[ \frac{D_{t+1}^h + D_{t+1}^f}{1 + r_t} = \frac{D_{t+1}^f}{1 + r_t} + M_t, \]  

(5.11)
where $D_{t+1}$ denotes the amount of discount debt acquired by the bank in period $t$ maturing in period $t + 1$. The left-hand side of this expression is the bank’s asset portfolio, and the right-hand side represents its liabilities. The bank’s profit in period $t$, denoted $\Pi_t^b$, is given by

$$\Pi_t^b = D_t^h + D_t^f - D_t - M_{t-1}. \quad (5.12)$$

Total profits received by households are given by

$$\Pi_t = \Pi_f^h + \Pi_t^b. \quad (5.13)$$

Combining (5.4), (5.8), (5.9), (5.11)-(5.13) yields the economy’s resource constraint

$$\frac{D_{t+1}}{1 + r_t} = D_t + C_t + S_t + I_t + \frac{\phi}{2} \left( \frac{K_{t+1}}{K_t} - g \right)^2 K_t - Y_t.$$

Because all banks are identical the aggregate level of external debt in period $t$ equals $D_t$, that is,

$$\bar{D}_t = D_t$$

for all $t$.

As in section 5.2, we perform a stationarity inducing transformation by scaling trending variables by $X_{t-1}$. Specifically, define $y_t = Y_t/X_{t-1}$, $c_t = C_t/X_{t-1}$, $s_t = S_t/X_{t-1}$, $d_t = D_t/X_{t-1}$, and $k_t = K_t/X_{t-1}$. Then, a stationary competitive equilibrium is given by a set of processes $\{c_t, h_t, \lambda_t, k_{t+1}, d_{t+1}, i_t, r_t, y_t\}$ satisfying

$$\nu_t [c_t - \omega^{-1} h_t^\omega]^{-\gamma} = \lambda_t,$$
h_{t}^{\omega-1} = (1 - \alpha) a_t g_t^{1-\alpha} \left( \frac{k_t}{h_t} \right)^{\alpha} \left[ 1 + \frac{\eta r_t}{1 + r_t} \right]^{-1},

\lambda_t = \frac{\beta}{g_t} (1 + r_t) E_t \lambda_{t+1},

\left[ 1 + \phi \left( \frac{k_{t+1}}{k_t} g_t - g \right) \right] \lambda_t = \frac{\beta}{g_t} E_t \lambda_{t+1} \left[ 1 - \delta + \alpha a_{t+1} \left( \frac{g_{t+1} h_{t+1}}{k_{t+1}} \right)^{1-\alpha} \right.

+ \phi \frac{k_{t+2}}{k_{t+1}} g_{t+1} \left( \frac{k_{t+2}}{k_{t+1}} g_{t+1} - g \right) - \frac{\phi}{2} \left( \frac{k_{t+2}}{k_{t+1}} g_{t+1} - g \right)^2 \left. \right],

\frac{d_{t+1}}{1 + r_t} g_t = d_t - y_t + c_t + s_t + i_t + \phi \frac{k_{t+1}}{k_t} g_t - g)^2 k_t,

r_t = r^* + \psi \left( e^{\frac{\mu_t - \gamma}{\beta}} - 1 \right) + e^{\mu_t - 1} - 1,

k_{t+1} g_t = (1 - \delta) k_t + i_t,

and

y_t = a_t k_t^{\alpha} (g_t h_t)^{1-\alpha},

given exogenous processes \(a_t, g_t, \nu_t, \mu_t\), and \(s_t\) and initial conditions \(k_0\) and \(d_0\).

### 5.4 Bayesian Estimation On A Century of Data

The econometric estimation uses annual per capita data from Argentina on output growth, consumption growth, investment growth, and the trade-balance-to-output ratio for the period 1900 to 2005. All four of these observable variables are assumed to be measured with error. Specifically,
let the theoretical counterparts of the four observables be the vector

\[ O^*_t = \begin{bmatrix} \Delta \ln Y_t \\ \Delta \ln C_t \\ \Delta \ln I_t \\ TB_t/Y_t \end{bmatrix}, \]

where

\[ TB_t \equiv Y_t - C_t - I_t - S_t - \phi \frac{K_{t+1}}{K_t} \left( \frac{K_{t+1}}{K_t} - g \right)^2 K_t \]

denotes the trade balance. Then, the vector of observables, denoted \( O_t \), is given by

\[ O_t = O^*_t + \begin{bmatrix} \sigma_{g^Y}^\text{me} \epsilon_t^\text{me,g^Y} \\ \sigma_{g^C}^\text{me} \epsilon_t^\text{me,g^C} \\ \sigma_{g^I}^\text{me} \epsilon_t^\text{me,g^I} \\ \sigma_{TB/Y}^\text{me} \epsilon_t^\text{me,TB/Y} \end{bmatrix}, \]

where \( \sigma_{g^Y}^\text{me}, \sigma_{g^C}^\text{me}, \sigma_{g^I}^\text{me}, \sigma_{TB/Y}^\text{me} \) are positive parameters and \( \epsilon_t^\text{me,i} \) is an exogenous i.i.d. disturbance with mean zero and unit variance for \( i = g^Y, g^C, g^I, TB/Y \).

The values assigned to the structural parameters are based on a combination of calibration and econometric estimation. The calibrated parameters are \( g, d/y, \delta, r^*, \alpha, \gamma, \omega \), and \( \bar{\gamma}/\bar{y} \) and are set to match long-run data relations from Argentina or in accordance with related business-cycle studies. Table 5.4 presents the calibrated parameter values. The parameter \( g \) is set to match the average growth rate of per capita GDP in Argentina over the period 1900 to 2005 of 1.07 percent per year. We impose a steady-state trade-balance to output ratio of 0.3 percent, as observed on average in

\footnote{Note that in the theoretical model, the definition of gross investment does not include investment adjustment costs. An alternative definition could include them. This distinction is immaterial in the present context, because up to first order adjustment costs are nil.}
Argentina over the period 1900-2005. We set \( r^* \) to 10 percent per year, and impose the restriction
\[ 1 + r^* = \beta^{-1} g^\gamma. \]
This implies a steady-state interest rate of 10 percent, a value that is empirically plausible for an emerging market economy like Argentina, and a subjective discount factor, \( \beta \), of 0.9286. A further implication of these restrictions is that the steady state of \( d_t \) equals \( \bar{d} \). We restrict \( \bar{y} \) to equal the steady-state value of detrended output, \( y_t \). This restriction and the assumed target for the steady state of the trade-balance-to-output ratio implies a value of \( \bar{d}/\bar{y} \) of 0.037, which coincides with the steady-state debt-to-output ratio. The value assigned to the depreciation rate \( \delta \) implies an average investment share in GDP of 19 percent, which is in line with the average value observed in Argentina over the calibration period. There is no reliable data on factor income shares for Argentina. We therefore set the parameter \( \alpha \), which determines the average capital income share, at 0.32, a value commonly used in the related literature. The parameter \( \gamma \), defining the curvature of the period utility function, takes the value 2, which is standard in related business-cycle studies. The parameter \( \omega \) is calibrated at 1.6, which implies a labor-supply elasticity of \( 1/(\omega - 1) = 1.7 \). Finally, the share of exogenous spending to GDP, \( s/y \), is set at 10 percent, which implies that \( \bar{s}/\bar{y} \)
equals 0.10.

The remaining parameters are estimated using likelihood-based Bayesian techniques on a log-linear approximation of the equilibrium dynamics. The log-linear approximation is computed using the techniques and Matlab code introduced in chapter 4. The estimated parameters consist of thirteen structural parameters and the standard deviations of the four measurement errors. The thirteen structural parameters are the ten parameters defining the stochastic processes of the shocks driving the model (\(\sigma_i\) and \(\rho_i\), for \(i = a, g, \nu, \mu, s\)), the parameter \(\phi\), governing the strength of capital adjustment costs, the parameter \(\psi\), determining the debt elasticity of the country-specific interest rate, and the parameter \(\eta\), defining the size of the working-capital constraint.

Table 5.5 displays salient characteristics of the prior and posterior distributions of the estimated parameters. All prior distributions are assumed to be uniform. For the structural parameters, the supports of the uniform prior distributions are relatively wide. For example, for serial correlations, we allow for the maximum possible range that the parameter can take. Thus, the estimation results, can be interpreted as maximum likelihood estimates. For the prior uniform distributions of the four nonstructural parameters, namely the standard deviations of the measurement errors, we impose upper bounds that imply that measurement errors can account for no more than 6.25 percent of the variance of the corresponding observable.

The statistics pertaining to the posterior distributions were computed using an MCMC chain of length 1 million. The data and Matlab code to reproduce the estimation is available at \texttt{http://www.columbia.edu/~mu2166/book/}.

Table 5.6 shows that the estimated model does a good job at matching a number of second moments typically used to characterize business cycles in emerging countries. In particular, the model replicates the excess volatility of consumption relative to output, the high volatility of investment, and a volatility of the trade-balance-to-output ratio comparable to that of output growth. The estimated model also captures the procyclicality of consumption and investment and
Table 5.5: Bayesian Estimation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Prior Distributions</th>
<th>Posterior Distributions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Min</td>
<td>Max</td>
</tr>
<tr>
<td>$\sigma_g$</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>$\rho_g$</td>
<td>-0.99</td>
<td>0.99</td>
</tr>
<tr>
<td>$\sigma_a$</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>$\rho_a$</td>
<td>-0.99</td>
<td>0.99</td>
</tr>
<tr>
<td>$\sigma_\nu$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\rho_\nu$</td>
<td>-0.99</td>
<td>0.99</td>
</tr>
<tr>
<td>$\sigma_s$</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>-0.99</td>
<td>0.99</td>
</tr>
<tr>
<td>$\sigma_\mu$</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>$\rho_\mu$</td>
<td>-0.99</td>
<td>0.99</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>$\psi$</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>$\sigma_{me}^g$</td>
<td>0.0001</td>
<td>0.013</td>
</tr>
<tr>
<td>$\sigma_{me}^{gY}$</td>
<td>0.0001</td>
<td>0.019</td>
</tr>
<tr>
<td>$\sigma_{me}^{gC}$</td>
<td>0.0001</td>
<td>0.051</td>
</tr>
<tr>
<td>$\sigma_{me}^{gI}$</td>
<td>0.0001</td>
<td>0.013</td>
</tr>
</tbody>
</table>

Note. All prior distributions are taken to be uniform. Moments of the posterior distribution are based on a 1,000,000 MCMC chain. The symbol $\sigma_{me}^i$ denotes the standard deviation of the measurement error associated with the observable $i$, for $i = g^Y, g^C, g^I$, and $TB/Y$, where $g^i$ denotes the growth rate of variable $i$, for $i = Y, C, I$, and $TB/Y$ denotes the trade-balance-to-output ratio. The data and Matlab code to reproduce this table are available at http://www.columbia.edu/~mu2166/book/.
Table 5.6: Empirical and Theoretical Second Moments

<table>
<thead>
<tr>
<th>Statistic</th>
<th>$g^Y$</th>
<th>$g^C$</th>
<th>$g^I$</th>
<th>$TB/Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Standard Deviation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Model</td>
<td>6.2</td>
<td>8.9</td>
<td>18.6</td>
<td>4.9</td>
</tr>
<tr>
<td>Data</td>
<td>5.3</td>
<td>7.5</td>
<td>20.4</td>
<td>5.2</td>
</tr>
<tr>
<td></td>
<td>(0.43)</td>
<td>(0.6)</td>
<td>(1.8)</td>
<td>(0.57)</td>
</tr>
<tr>
<td><strong>Correlation with $g^Y$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Model</td>
<td>0.80</td>
<td>0.53</td>
<td>-0.18</td>
<td></td>
</tr>
<tr>
<td>Data</td>
<td>0.72</td>
<td>0.67</td>
<td>-0.035</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.07)</td>
<td>(0.09)</td>
<td>(0.09)</td>
<td></td>
</tr>
<tr>
<td><strong>Correlation with $TB/Y$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Model</td>
<td>-0.37</td>
<td>-0.31</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Data</td>
<td>-0.27</td>
<td>-0.19</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.07)</td>
<td>(0.08)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Serial Correlation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Model</td>
<td>0.04</td>
<td>-0.06</td>
<td>-0.098</td>
<td>0.51</td>
</tr>
<tr>
<td>Data</td>
<td>0.11</td>
<td>-0.0047</td>
<td>0.32</td>
<td>0.58</td>
</tr>
<tr>
<td></td>
<td>(0.09)</td>
<td>(0.08)</td>
<td>(0.10)</td>
<td>(0.07)</td>
</tr>
</tbody>
</table>

Note. Empirical moments are computed using data from Argentina for the period 1900 to 2005. Standard deviations of empirical moments are computed using GMM. Theoretical moments are unconditional moments computed by evaluating the model at the posterior median of the estimated parameters.
5.5 How Important Are Permanent Productivity Shocks?

An important result that emerges from table 5.5 is that the parameters defining the stochastic process of the nonstationary productivity shock are estimated with significant uncertainty. Specifically, the posterior distribution of the standard deviation of innovations to the nonstationary productivity shock, $\sigma_g$, has a median of 0.67 percent but a 95% probability interval that ranges from 0 to 2.1 percent. Similarly, the posterior distribution of the serial correlation of the nonstationary productivity shock, $\rho_g$, has a median of 0.21, but a 95% probability interval that ranges from -0.69 to +0.81. By contrast, the parameters defining the process of the stationary productivity shock are estimated much more tightly. The parameter $\sigma_a$ has a posterior median of 0.032 and a 95-percent probability interval of 0.027 to 0.036, and the parameter $\rho_a$ has a posterior median of 0.84 and a 95-percent probability interval ranging from 0.75 to 0.91.

Consider now computing the share of the variance of the growth rate of total factor productivity explained by nonstationary productivity shocks. That is, consider computing the fraction of the variance of $\Delta \ln TFP_t \equiv \Delta \ln(a_t X_1^{1-\alpha})$ explained by $\Delta \ln(X_1^{1-\alpha})$. Recall from section 5.2 that the SOE-RBC model estimated in Aguiar and Gopinath (2007) using Mexican data from 1980 to 2003 implies that nonstationary productivity shocks explain 88 percent of movements in total factor productivity. How does this share change when one estimates the stochastic trend using a long sample and when productivity shocks are allowed to compete with other shocks and frictions? Evaluating the formula given in (5.2) using the MCMC chain for the posterior estimates of the relevant structural parameters, one can derive a MCMC chain of posterior draws of the share of
Table 5.7: Variance Decomposition

<table>
<thead>
<tr>
<th>Shock</th>
<th>$g^2$</th>
<th>$g^C$</th>
<th>$g^I$</th>
<th>TB/Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nonstationary Tech.</td>
<td>2.6</td>
<td>1.1</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>Stationary Tech.</td>
<td>81.8</td>
<td>42.4</td>
<td>12.7</td>
<td>0.5</td>
</tr>
<tr>
<td>Preference</td>
<td>6.8</td>
<td>27.7</td>
<td>29.1</td>
<td>6.2</td>
</tr>
<tr>
<td>Interest Rate</td>
<td>6.1</td>
<td>25.8</td>
<td>52.0</td>
<td>92.1</td>
</tr>
<tr>
<td>Spending</td>
<td>0.0</td>
<td>0.3</td>
<td>0.3</td>
<td>0.1</td>
</tr>
<tr>
<td>Measurement Error</td>
<td>0.4</td>
<td>0.7</td>
<td>5.2</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Note. Median of 1 million draws from the posterior distribution of the unconditional variance decomposition.

The variance of TFP explained by nonstationary productivity shocks. Using this chain yields

$$\text{posterior median} \left( \frac{\text{var}(\Delta \ln(X_t^{1-\alpha}))}{\text{var}(\Delta \ln TFP_t)} \right) = \text{posterior median} \left( \frac{(1-\alpha)^2 \sigma_g^2}{(1-\rho_g^2)} \right) = 0.024.$$  

That is, nonstationary productivity shocks explain only 2.4 percent of movements in total factor productivity. This result suggests that the long data sample used for the estimation of the model plus the inclusion of additional shocks and financial frictions results in the data favoring stationary productivity shocks over nonstationary productivity shocks as drivers of total factor productivity.

Table 5.7 presents the predicted contribution of each shock to explaining the variances of output growth, consumption growth, investment growth, and the trade-balance-to-output ratio. Three key results emerge from this variance decomposition. First, nonstationary productivity shocks play a negligible role in explaining aggregate fluctuations. They account for less than 5 percent of the variances of all variables considered in the table. This result is in sharp contrast with the one obtained in the model with only productivity shocks and no financial frictions studied in section 5.2. Output growth is driven primarily by stationary productivity shocks, which explain 82 percent of
its unconditional variance.

Second, variations in investment growth and the trade-balance-to-output ratio are mostly accounted for by interest-rate shocks. This source of uncertainty explains 52 percent of the variance of investment growth and 92 percent of the variance of the trade-balance-to-output ratio. Third, consumption growth is driven in roughly equal parts by stationary productivity shocks, interest-rate shocks, and preference shocks with a slightly larger weight on the first shock. In particular, put to choose the data prefer to explain the excess volatility of consumption relative to output observed in Argentina, and typical of many emerging countries, by disturbances other than permanent technology shocks. In section 5.1, we showed that a sufficiently persistent stationary technology shock process can give rise to excess volatility of consumption relative to output. The present estimation delivers this channel by assigning a high posterior value to the serial correlation of the stationary productivity shock of 0.84. This value is about twice as large as the one needed to explain business cycles in Canada, the developed small open economy studied in chapter 4, which does not display excess volatility of consumption. The intuition for why interest rate and preference shocks are also important for delivering excess consumption volatility is that the former change the relative price of present consumption in terms of future consumption and the latter alter the subjective valuation of present consumption relative to future consumption.

The fact that interest rate shocks explain a modest fraction of the variance of output growth, 6.1 percent, indicates that the working-capital friction does not play a central role in the estimated model. Indeed the parameter $\eta$, defining the magnitude of the working-capital friction, is estimated with significant uncertainty. Specifically, the median value of $\eta$ is 0.4, which means that firms hold about 5 months of the wage bill as working capital, but the 95% posterior probability interval ranges from 0.18 to 0.70 (or 2 to 8 months). Moreover, the predictions of the model are virtually unchanged if the model is estimated under the constraint $\eta = 0$ (see García-Cicco, Pancrazi, and Uribe, 2010). Chang and Fernández (2010) find a similar result using quarterly data from Mexico.
5.6 The Role of Financial Frictions

Consider now the importance of the parameter $\psi$, which governs the debt elasticity of the country premium. This elasticity captures, in a reduced-form fashion, financial frictions, which can stem from a variety of sources. For instance, models with imperfect enforcement of international loan contracts à la Eaton and Gersovitz (1981), which will be studied in detail in chapter 12, predict that the country premium increases with the level of external indebtedness. Similarly, models in which international borrowing is limited by collateral constraints, like those studied later in chapter 11, imply a shadow interest premium that is increasing in the level of net external debt. By estimating the parameter $\psi$, we let the data determine the importance of this type of financial friction.

The posterior median estimate of $\psi$, shown in table 5.5, is 1.3. How big a financial friction does this value represent? Consider a partial differentiation of equation (5.10) with respect to $r_t$ and $\tilde{D}_{t+1}$:

$$\Delta r_t = \psi e^{(\tilde{D}_{t+1}/X_t-\bar{d})/\gamma} \frac{\Delta \tilde{D}_{t+1}}{X_t \bar{y}}$$

Assume that debt is at its deterministic steady-state level. That is, set $\tilde{D}_{t+1}/X_t = \bar{d}$. We then have that

$$\Delta r_t = \psi \frac{\Delta \tilde{D}_{t+1}/X_t}{\bar{y}}.$$

Now setting $\psi = 1.3$ (its posterior median estimate) yields

$$\Delta r_t = 1.3 \frac{\Delta \tilde{D}_{t+1}/X_t}{\bar{y}}.$$

This expression indicates that an increase in debt of 1 percent of GDP ($\Delta \tilde{D}_{t+1}/(\bar{y}X_t) = 0.01$), causes an increase of 1.3 percentage points in the interest rate ($\Delta r_t = 0.013$).
This nontrivial debt-elasticity of the interest rate plays an important role in explaining the cyclical behavior of the trade balance. Figure 5.4 displays the empirical and theoretical autocorrelation functions of the trade-balance-to-output ratio. The empirical autocorrelation function, shown with a solid line, is estimated using annual data from Argentina for the period 1900-2005. Broken lines indicate the two-standard-deviation confidence interval. The point estimate of the autocorrelation function starts at about 0.6 and falls gradually toward zero. This pattern is observed more generally in emerging countries (see García-Cicco, Pancrazi, and Uribe, 2010; and Miyamoto and Nguyen, 2013).

The estimated model captures the empirical pattern quite well. The predicted autocorrelation function, shown with a crossed line, lies close to its empirical counterpart and is entirely within the two-standard-error confidence band. For comparison, figure 5.4 also displays, with a circled line, the autocorrelation function of the trade-balance-to-output ratio predicted by a special case
of the present model in which the parameter $\psi$ is calibrated at 0.001. Holding this parameter fixed, the model is then reestimated. In sharp contrast with the predictions of the baseline model, the model without financial frictions (i.e., the model with $\psi = 0.001$) predicts an autocorrelation function that is flat, close to unity, and entirely outside of the confidence band. Furthermore, the estimated model without financial frictions grossly overpredicts the volatility of the trade balance. Specifically, this model implies a standard deviation of the trade-balance-to-output ratio of 32.5 percentage points, six times larger than the observed standard deviation of 5.2 percentage points.

The intuition behind why a small value of the debt elasticity of the interest rate causes the trade-balance-to-output ratio to have an autocorrelation function that is flat and near unity and a standard deviation that is excessively large can be found in the fact that in the absence of financial frictions external debt follows a highly persistent process (a quasi random walk). This is because as $\psi \to 0$, the linearized equilibrium dynamics are governed by an eigenvalue that approaches unity in modulus. If the stock of external debt is extremely persistent, then so is the trade balance, which is determined to a large extent by the need to service external interest obligations. By contrast, when $\psi$ is sufficiently large, as in the baseline model, the stock of external debt follows a less persistent stationary process. In this case, movements in the level of debt cause endogenous, self-stabilizing changes in the country interest-rate premium. For example, if the level of external debt rises too far above its steady-state level, the country premium increases, inducing households to cut spending thereby bringing debt closer to its steady-state level.

The following general result can be established regarding the role of $\psi$ in generating a downward-sloping autocorrelation function of the trade-balance-to-output ratio. Holding all other parameters of the model constant, one can always find a positive but sufficiently small value of $\psi$ such that the equilibrium dynamics are stationary up to first order and the autocorrelation function of the trade-balance-to-output ratio is flat and close to unity. This result implies that, if a particular calibration of a SOE model does not deliver a flat and near-unit autocorrelation function of the
trade-balance-to-output ratio it is because $\psi$ has not been set at a small enough value, given the values assigned to all other parameters of the model.

For example, in the SOE-RBC model of chapter 4, section 4.1, the parameter $\psi$ takes the value $0.0011,^3$ which is about the same as the value considered here, and nevertheless the predicted first-order serial correlation of the trade-balance-to-output ratio is 0.51, significantly below unity (see table 4.2). This discussion motivates the question of what parameters other than the debt elasticity of the country interest rate affect the height and slope of the autocorrelation function of the trade-balance-to-output ratio. We turn to this issue next.

### 5.7 Investment Adjustment Costs and the Persistence of the Trade Balance

In general, given the value of $\psi$ there exist a number of structural parameters that play a role in determining the shape of the autocorrelation function of the trade-balance-to-output ratio. One such parameter is the one governing the degree of capital adjustment costs. Consider again the SOE-RBC model analyzed in chapter 4, section 4.1.1, to which we will refer here as the SGU model. The calibration of this model, shown in table 4.1, is of particular interest for the present discussion because it features a value of $\psi$ of 0.000742 (see footnote 3 for an explanation of why this value has to be adjusted upward to 0.0011 to make it comparable with the GPU model) and predicts an autocorrelation of the trade-balance-to-output ratio of 0.51, significantly below unity. As we have just shown, the GPU model delivers a near unity serial correlation of the trade-balance-to-output ratio for $\psi = 0.001$. It follows that it cannot simply be the size of $\psi$ that determines the persistence

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^3Notice that the specification of the country interest rate premium in chapter 4, section 4.1 makes the interest rate a function of the level of debt as opposed to the level of debt relative to trend output. Therefore, to make the comparison possible, the value of $\psi$ of 0.000742 used in chapter 4 must be multiplied by the level of steady-state output in that model, which equals 1.4865.
of the trade balance.

We argue here that the calibration of the SGU model in chapter 4 features a capital adjustment cost coefficient that is about 20 times smaller than the one estimated for the GPU model. Let \( \phi^{SGU} \) and \( \phi^{GPU} \) be the values of this coefficient in the SGU and GPU models, respectively. From tables 4.1 we have that \( \phi^{SGU} = 0.028 \) and from the reestimation of the GPU model with a fixed value of \( \psi = 0.001 \) (not shown), we find that \( \phi^{GPU} = 2.0 \). The comparison of the degree of capital adjustment costs is more complicated than simply comparing these two number, however, because the SGU and GPU models assume different specifications of the capital-adjustment-cost function. In the SGU model, the capital-adjustment-cost function takes the form \( \frac{\phi}{2}(k_{t+1} - k_t)^2 \), whereas in the GPU model it takes the form \( \frac{\phi}{2}K_t(K_{t+1}/K_t - g)^2 \). In general, using the same value of \( \phi \) in both models introduces different degrees of capital adjustment costs. One can show that in the absence of long-run growth, \( g = 1 \), and holding all other parameters equal across models, both specifications give rise to identical equilibrium dynamics up to first order as long as \( k^{SGU} \phi^{SGU} = \phi^{GPU} \), where \( k^{SGU} \) denotes the steady-state level of capital in the SGU model. Thus if one wished to introduce in the SGU model the same degree of adjustment costs as in the GPU model, one must set \( \phi^{SGU} = \frac{1}{k^{SGU}} \phi^{GPU} \). The calibration of table 4.1 implies that \( k^{SGU} = 3.4 \). Thus, the value of \( \phi^{SGU} \) that makes both models comparable is 0.59. This value is about 20 times larger than the value of 0.028 used to calibrate the SGU model in chapter 4. We conclude that the GPU model estimated with a fixed \( \psi \) of 0.001 features a degree of capital adjustment cost that is about 20 times as large as the one used in the calibration of the SGU model in chapter 4. Moreover, one can show that for values of \( \phi^{SGU} \) ranging from 0.028 to 0.59, the SGU model implies serial correlations of the trade-balance-to-output ratio between 0.51 and 0.97, although the relationship is not monotone. Intuitively, the higher is the size of the adjustment costs, the more persistent is investment. Since the trade-balance-to-output ratio is governed by the sum of the consumption share and the investment share, the persistence of investment is partially transmitted to the trade
balance.
5.8 Exercises

Exercise 5.1 (Explaining the Serial Correlation of Investment Growth) The version of the GPU model estimated in this chapter predicts a negative first-order serial correlation of investment growth of -0.098 (see table 5.6). By contrast, the empirical counterpart is positive and significant, with a point estimate of 0.32. This empirical fact is also observed in other emerging countries over long horizons. For example, Miyamoto and Nguyen (2013) report serial correlations of investment growth greater than or equal to 0.2 for Brazil, Mexico, Peru, Turkey, and Venezuela using annual data covering the period 1900 to 2006.

1. Think of a possible modification of the theoretical model that would result in an improvement of the model’s prediction along this dimension. Provide intuition.

2. Implement your suggestion. Show the complete set of equilibrium conditions.

3. Reestimate your model using the data set for Argentina on which the GPU model of this chapter was estimated.

4. Summarize your results by expanding table 5.6 with appropriate lines containing the predictions of your model.

5. Compare the performance of your model with the data and with the predictions of the version of the GPU model analyzed in this chapter.

Exercise 5.2 (Slow Diffusion of Technology Shocks to the Country Premium, Household Production, and Government Spending)

The model presented in section 5.3 assumes that permanent productivity shocks affect not only the productivity of labor and capital in producing market goods, but also the country premium, home production, and government spending. For instance, the assumption that the country interest
rate depends on $D_{t+1}/X_t$, implies that a positive innovation in $X_t$ in period $t$, causes, all other things equal, a fall in the country premium. In this exercise, we attenuate this type of effect by reformulating the model. Let

$$\bar{X}_t = X_{t-1}^{1-\zeta},$$

with $\zeta \in [0,1)$. Note that the original formulation obtains when $\zeta = 0$. Replace equations (5.3), (5.5), and (5.10), respectively, with

$$E_0 \sum_{t=0}^{\infty} \nu_t \beta^t \left[ C_t - \omega^{-1} X_{t-1} h_t^\omega \right]^{1-\gamma} - 1,$$

$$s_t = \frac{S_t}{X_{t-1}},$$

and

$$r_t = r^* + \psi \left( e^{(D_{t+1} - \bar{X}_t)/\gamma} - 1 \right) + \mu - 1.$$

Keep all other features of the model as presented in section 5.3.

1. Present the equilibrium conditions of the model in stationary form.

2. Using Bayesian techniques, reestimate the model adding $\zeta$ to the vector of estimated parameters. Assume a uniform prior distribution for $\zeta$ with support $[0, 0.99]$ and produce 1 million draws from the posterior distribution of the parameter vector. Present the estimation results in the form of a table like table 5.5. Discuss your findings.

3. Characterize numerically the predictions of the model. In particular, produce tables similar to tables 5.6 and 5.7. Discuss your results and provide intuition.

4. Compute the impulse responses of output, consumption, investment, the trade-balance-to-output ratio, and the country interest rate to a one-percent innovation in $g_t$ for three values
of $\zeta$, namely, 0, its posterior median, and 0.99.

Exercise 5.3 (The Importance of Nonstationary Productivity Shocks in the GPU Model)

The model of section 5.3 introduces three modifications to the SOE-RBC model with stationary and nonstationary technology shocks of section 5.3, namely, a longer sample, additional shocks, and financial frictions. A result of section 5.3 is that once these modifications are put in place, nonstationary productivity shocks cease to play a central role in explaining business cycles. The goal of this exercise is to disentangle which of the three aforementioned modifications is responsible for this result. To this end, estimate, using Bayesian methods, one at the time, the following three variants of the model:


2. Only Technology Shocks: Set to zero the standard deviations of the preference shock, the country-interest-rate shock, and the spending shock.

3. No Financial Frictions: Set $\psi = 0.001$ and $\eta = 0$.

Use the same priors as in the body of the chapter and produce MCMC chains of 1 million draws with an acceptance rate of 25 percent. In each case, report the implied variance decomposition of TFP, output growth, consumption growth, investment growth, and the trade-balance-to output ratio and measures of model fit, using the formats of tables 5.5, 5.6, and 5.7 and equation 5.14. Discuss your results.
Chapter 6

Interest-Rate Shocks

Business cycles in emerging market economies are correlated with the interest rate that these countries face in international financial markets. This observation is illustrated in figure 6.1, which depicts detrended output and the country interest rate for seven developing economies between 1994Q1 and 2001Q4. Periods of low interest rates are typically associated with economic expansions and periods of high interest rates are often characterized by depressed levels of aggregate activity.\(^1\)

Data like those shown in figure 6.1 have motivated researchers to ask what fraction of observed business cycle fluctuations in emerging markets is due to movements in country interest rates. This question is complicated by the fact that the country interest rate is unlikely to be completely exogenous to the country’s domestic conditions.\(^2\) To clarify ideas, let \(R_t\) denote the gross interest rate at which the emerging country borrows in international markets, or the country interest rate. This interest rate can be expressed as \(R_t = R_{us}^{ws}S_t\). Here, \(R_{us}^{ws}\) denotes the gross world interest rate, or the interest rate at which developed countries, like the United States, borrow and lend from one

\(^1\)The estimated correlations (p-values) are: Argentina -0.67 (0.00), Brazil -0.51 (0.00), Ecuador -0.80 (0.00), Mexico -0.58 (0.00), Peru -0.37 (0.12), the Philippines -0.02 (0.95), South Africa -0.07 (0.71).

\(^2\)There is a large literature arguing that domestic variables affect the interest rate at which emerging markets borrow externally. See, for example, Edwards (1984), Cline (1995), and Cline and Barnes (1997).
Figure 6.1: Country Interest Rates and Output in Seven Emerging Countries

another, and $S_t$ denotes the gross country interest-rate spread, or country interest-rate premium. Because the interest-rate premium is country specific, in the data we find an Argentine spread, a Colombian spread, etc. If the country in question is a small player in international financial markets, as many emerging economies are, it is reasonable to assume that the world interest rate, $R^{us}_t$, is exogenous to the emerging country’s domestic conditions. We cannot say the same, however, about the country spread $S_t$. An increase in domestic output, for instance, may induce foreign lenders to lower spreads on believes that the country’s ability to repay its debts has improved.

Interpreting the country interest rate premium as an exogenous variable when in reality it has an endogenous component is likely to result in an overstatement of the importance of interest rates in explaining business cycles. To see why, consider the following example. Suppose that the interest rate premium $S_t$ is purely endogenous. Thus, its contribution to generating business cycles is nil. Assume, furthermore, that $S_t$ is countercyclical, i.e., foreign lenders reduce the country spread in response to expansions in domestic aggregate activity. The researcher, however, wrongly assumes that the interest rate premium is purely exogenous. Suppose now that a domestic productivity shock causes an expansion in output. In response to this output increase, the country interest rate premium falls. The researcher erroneously attributes part of the increase in output to the decline in $S_t$. The right conclusion, of course, is that all of the increase in output is due to the productivity shock.

It follows that in order to quantify the macroeconomic effects of interest rate shocks, the first step is to identify the exogenous components of country spreads. Necessarily, the identification process must combine statistical methods and economic theory. The particular combination adopted in this chapter is taken from Uribe and Yue (2006). A difference between the approaches adopted in this chapter and in chapter 5 is that in the latter inference about the importance of interest-rate shocks in driving business cycles is conducted without using observations on interest rates. There, this choice was motivated by a desire to let interest-rate shocks compete with other shocks that
were treated as latent variables. In such a context, making interest rates observable would have
put this source of uncertainty at a disadvantage. By contrast, in the current chapter observations
on interest rates take center stage.

6.1 An Empirical Model

Consider the first-order vector autoregressive (VAR) system

\[
\begin{bmatrix}
\hat{y}_t \\
\hat{i}_t \\
\hat{tby}_t \\
\hat{R}_t^us \\
\hat{R}_t
\end{bmatrix} = A \begin{bmatrix}
\hat{y}_{t-1} \\
\hat{i}_{t-1} \\
\hat{tby}_{t-1} \\
\hat{R}_{t-1}^us \\
\hat{R}_{t-1}
\end{bmatrix} + B \begin{bmatrix}
\epsilon_y^t \\
\epsilon_i^t \\
\epsilon_{tby}^t \\
\epsilon_{Rus}^t \\
\epsilon_r^t
\end{bmatrix},
\]  

(6.1)

where \( y_t \) denotes real gross domestic output, \( i_t \) denotes real gross domestic investment, \( tby_t \) denotes
the trade balance to output ratio, \( R_t^us \) denotes the gross real U.S. interest rate, and \( R_t \) denotes
the gross real (emerging) country interest rate. A hat on \( y_t \) and \( i_t \) denotes log-deviations from a
linear trend. A hat on \( R_t^us \) and \( R_t \) denotes simply the log. We measure \( R_t^us \) as the 3-month gross
Treasury bill rate divided by the average gross U.S. inflation over the previous four quarters and
the country interest rate \( R_t \) as the sum of J.P. Morgan’s EMBI+ stripped spread and the U.S. real
interest rate. Output, investment, and the trade balance are seasonally adjusted.

To identify the shocks in the empirical model, Uribe and Yue (2006) impose the restriction that
the matrix \( A \) be lower triangular with unit diagonal elements. Because \( R_t^us \) and \( R_t \) appear at the
bottom of the system, this identification strategy presupposes that innovations in world interest
rates (\( \epsilon_{rus}^t \)) and innovations in country interest rates (\( \epsilon_r^t \)) percolate into domestic real variables with
a one-period lag. At the same time, the identification scheme implies that domestic shocks (\( \epsilon_y^t \),
\( \epsilon_i, \) and \( \epsilon^{ty}_{t} \) affect financial markets contemporaneously. This identification strategy is a natural one, for, conceivably, decisions such as employment and spending on durable consumption goods and investment goods take time to plan and implement. Also, it seems reasonable to assume that financial markets are able to react quickly to news about the state of the business cycle. Uribe and Yue (2006) discuss an alternative identification strategy that allows U.S.-interest-rate shocks and country-spread shocks to affect real variables contemporaneously. They find that under this identification an increase in the U.S. interest rate leads to an expansion in domestic economic activity. Because it is difficult to rationalize this result on theoretical grounds, we conclude that it is unlikely that this identification strategy successfully uncovers the true transmission mechanism of interest rate shocks, making the baseline identification preferable.

An additional restriction imposed on the VAR system is that the world interest rate \( R_{us}^{t} \) follows a univariate AR(1) process (i.e., \( A_{4i} = B_{4i} = 0, \) for all \( i \neq 4 \)). Uribe and Yue (2006) adopt this restriction because it is reasonable to assume that disturbances in a particular (small) emerging country will not affect the real interest rate of a large country like the United States.

The country-interest-rate shock, \( \epsilon_r^{t} \), can equivalently be interpreted as a country spread shock. To see this, consider substituting in equation (6.1) the country interest rate \( \hat{R}^t \) using the definition of country spread, \( \hat{S}_{t} \equiv \hat{R}_{t} - \hat{R}_{us}^{t} \). Clearly, because \( \hat{R}_{us}^{t} \) appears as a regressor in the bottom equation of the VAR system, the estimated residual of the newly defined bottom equation, call it \( \epsilon_t^{s} \), is identical to \( \epsilon_t^{r} \). Moreover, the impulse response functions of \( \hat{y}_{t}, \hat{t}_{t}, \) and \( tby_{t} \) associated with \( \epsilon_t^{s} \) are identical to those associated with \( \epsilon_t^{r} \). Therefore, we indistinctly refer to \( \epsilon_t^{r} \) as a country interest rate shock or as a country spread shock.

The restrictions imposed on the matrices \( A \) and \( B \) identify the U.S. interest rate shock \( \epsilon_t^{rus} \) and the country-interest-rate shock \( \epsilon_t^{r} \). The resulting VAR system is known as a structural vector autoregressive (SVAR) system. Uribe and Yue estimate the SVAR system (6.1) equation-by-equation with an intercept and country fixed effects on panel data from Argentina, Brazil, Ecuador, Mexico,
Peru, the Philippines, and South Africa, over the period 1994:Q1 to 2001:Q4. The $\hat{R}_{t}^{gs}$ equation is estimated by OLS over the period 1987:Q1-2002:Q4. Table 6.1 presents the results of the estimation.

The estimated SVAR system can be used to address a number of questions central to disentangling the effects of country-spread shocks and world-interest-rate shocks on aggregate activity in emerging markets: First, how do U.S. interest-rate shocks and country-spread shocks affect real domestic variables such as output, investment, and the trade balance? Second, how do country spreads respond to innovations in U.S. interest rates? Third, how and by how much do country spreads move in response to innovations in emerging-country fundamentals? Fourth, how important are U.S. interest-rate shocks and country-spread shocks in explaining movements in aggregate activity in emerging countries? Fifth, how important are U.S.-interest-rate shocks and country-spread shocks in accounting for movements in country spreads? The next section answers these questions with the help of impulse response functions and variance decompositions.

### 6.2 Impulse Response Functions

Figure 6.2 displays with solid lines the impulse response function implied by the estimated SVAR system (6.1) to a one-percentage-point increase in the country spread shock, $\epsilon_{t}^{r}$. Broken lines depict two-standard-deviation bands. In response to an unanticipated country-spread shock, the country spread itself increases and then quickly falls toward its long-run level. The half-life of the country spread response is about one year. Output, investment, and the trade balance-to-output ratio respond as one would expect. They are unchanged in the period of impact, because of our maintained assumption that external financial shocks take one quarter to affect production and absorption. In the two periods following the country-spread shock, output and investment fall, and subsequently recover gradually until they reach their respective trend levels. The adverse spread
Table 6.1: Parameter Estimates of the VAR System (6.1)

<table>
<thead>
<tr>
<th>Independent Variable</th>
<th>$\hat{y}_t$</th>
<th>$\hat{t}_t$</th>
<th>$tby_t$</th>
<th>$\hat{R}^{us}_t$</th>
<th>$\hat{R}_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{y}_t$</td>
<td>-</td>
<td>2.739</td>
<td>0.295</td>
<td>-</td>
<td>-0.791</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(10.28)</td>
<td>(2.18)</td>
<td></td>
<td>(3.72)</td>
</tr>
<tr>
<td>$\hat{y}_{t-1}$</td>
<td>0.282</td>
<td>-1.425</td>
<td>-0.032</td>
<td>-</td>
<td>0.617</td>
</tr>
<tr>
<td></td>
<td>(2.28)</td>
<td>(-4.03)</td>
<td>(-0.25)</td>
<td></td>
<td>(2.89)</td>
</tr>
<tr>
<td>$\hat{t}_t$</td>
<td>-</td>
<td>-</td>
<td>-0.228</td>
<td>-</td>
<td>0.114</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(-6.19)</td>
<td></td>
<td>(1.74)</td>
</tr>
<tr>
<td>$\hat{t}_{t-1}$</td>
<td>0.162</td>
<td>0.537</td>
<td>0.040</td>
<td>-</td>
<td>-0.122</td>
</tr>
<tr>
<td></td>
<td>(4.56)</td>
<td>(3.64)</td>
<td>(0.77)</td>
<td></td>
<td>(-1.72)</td>
</tr>
<tr>
<td>$tby_{t-1}$</td>
<td>0.267</td>
<td>-0.308</td>
<td>0.317</td>
<td>-</td>
<td>-0.190</td>
</tr>
<tr>
<td></td>
<td>(4.45)</td>
<td>(-1.30)</td>
<td>(2.46)</td>
<td></td>
<td>(-1.29)</td>
</tr>
<tr>
<td>$\hat{R}^{us}_t$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.501</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(1.55)</td>
</tr>
<tr>
<td>$\hat{R}^{us}_{t-1}$</td>
<td>0.0002</td>
<td>-0.269</td>
<td>-0.063</td>
<td>0.830</td>
<td>0.355</td>
</tr>
<tr>
<td></td>
<td>(0.00)</td>
<td>(-0.47)</td>
<td>(-0.28)</td>
<td>(10.89)</td>
<td>(0.73)</td>
</tr>
<tr>
<td>$\hat{R}_t$</td>
<td>-0.170</td>
<td>-0.026</td>
<td>0.191</td>
<td>-</td>
<td>0.635</td>
</tr>
<tr>
<td></td>
<td>(-3.93)</td>
<td>(-0.21)</td>
<td>(3.54)</td>
<td></td>
<td>(-4.25)</td>
</tr>
</tbody>
</table>

$R^2$ | 0.724 | 0.842 | 0.765 | 0.664 | 0.619 |
S.E.  | 0.018 | 0.043 | 0.019 | 0.007 | 0.031 |
No. of obs. | 165 | 165 | 165 | 62 | 160 |

Note: $t$-statistics are shown in parenthesis. The system was estimated equation by equation with an intercept and country fixed effects (not shown). All equations except for the $\hat{R}^{us}_t$ equation were estimated using instrumental variables with panel data from Argentina, Brazil, Ecuador, Mexico, Peru, the Philippines, and South Africa over the period 1994:Q1 to 2001:Q4. The $\hat{R}^{us}_t$ equation was estimated by OLS over the period 1987:Q1-2002:Q4. Source: Uribe and Yue (2006).
Figure 6.2: Impulse Response To A Country-Spread Shock

Notes: (1) Solid lines depict point estimates of impulse responses, and broken lines depict two-standard-deviation error bands. (2) The responses of output and investment are expressed in percent deviations from their respective log-linear trends. The responses of the trade balance-to-GDP ratio and the country interest rate are expressed in percentage point deviations from their respective means. The two-standard-error bands are computed using the delta method.
shock produces a relatively larger contraction in aggregate absorption than in aggregate output. This is reflected in the fact that the trade-balance-to-output ratio improves in the two periods following the shock.

Figure 6.3 displays the response to a one percentage point increase in the U.S. interest rate shock, $\epsilon_t^{us}$. The effects of U.S. interest-rate shocks on domestic variables and country spreads are measured with significant uncertainty, as indicated by the width of the 2-standard-deviation error bands. The point estimates of the impulse response functions of output, investment, and the trade balance, however, are qualitatively similar to those associated with an innovation in the country spread. That is, aggregate activity and gross domestic investment contract, while net exports improve. However, the quantitative effects of an innovation in the U.S. interest rate are much more pronounced than those caused by a country-spread disturbance of equal magnitude. For instance, the trough in the output response is twice as large under a U.S.-interest-rate shock than under a country-spread shock.

It is remarkable that the impulse response function of the country spread to a U.S.-interest-rate shock displays a delayed overshooting. In effect, in the period of impact the country interest rate increases but by less than the jump in the U.S. interest rate. As a result, the country spread initially falls. However, the country spread recovers quickly and after a couple of quarters it is more than one percentage point above its pre-shock level. Thus, country spreads increase significantly in response to innovations in the U.S. interest rate but with a short delay. The negative impact effect is in line with the findings of Eichengreen and Mody (1998) and Kamin and Kleist (1999). We note, however, that because the models estimated by these authors are static in nature, by construction, they are unable to capture the rich dynamic relation linking these two variables. The overshooting of country spreads is responsible for the much larger response of domestic variables to an innovation in the U.S. interest rate than to an innovation in the country spread of equal magnitude.
Figure 6.3: Impulse Response To A U.S.-Interest-Rate Shock

Notes: (1) Solid lines depict point estimates of impulse responses, and broken lines depict two-standard-deviation error bands. (2) The responses of output and investment are expressed in percent deviations from their respective log-linear trends. The responses of the trade balance-to-GDP ratio, the country interest rate, and the U.S. interest rate are expressed in percentage point deviations from their respective means.
We now ask how innovations in output, $\epsilon^y$, impinge upon the variables of our empirical model. The model is vague about the precise nature of output shocks. They can reflect variations in total factor productivity, the terms-of-trade, etc. Figure 6.4 depicts the impulse response function to a one-percent increase in the output shock. The response of output, investment, and the trade balance is very much in line with the impulse response to a positive productivity shock implied by the small open economy RBC model (see figure 4.1 in chapter 4). The response of investment is about three times as large as that of output. At the same time, the trade balance deteriorates significantly by about 0.4 percent and after two quarters starts to improve, converging gradually to its long-run level. More interestingly, the increase in output produces a significant reduction in the country spread of about 0.6 percentage points (or 60 basis points). The half-life of the country spread response is about five quarters. The countercyclical behavior of the country spread in response to output shocks suggests that country interest rates behave in ways that amplify the business-cycle effects of output shocks.

### 6.2.1 Robustness To Expanding the Time and Country Dimensions of the Data

Figures 6.5 and 6.6 display impulse responses to interest-rate, country-spread, and output shocks implied by the empirical model given in equation (6.1) estimated on an expanded data panel. Specifically, the data set now includes eleven additional years of quarterly data, the period 2002:Q1 to 2012:Q4, and nine additional countries, Bulgaria, Chile, Colombia, Hungary, South Korea, Malaysia, Thailand, Turkey, and Uruguay.\(^3\)

The two figures suggest that the results reported by Uribe and Yue (2006) are robust to extending the temporal and geographic coverage of the data. In particular, both country-spread and U.S.-interest-rate shocks cause sizable contractions in output and investment and an improvement in the trade balance-to-GDP ratio. Also, increases in the U.S. interest rate are incorporated into

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\(^3\)The new data set is available online in the file `irs_data_update.mat`. 

Figure 6.4: Impulse Response To An Output Shock

Notes: (1) Solid lines depict point estimates of impulse response functions, and broken lines depict two-standard-deviation error bands. (2) The responses of output and investment are expressed in percent deviations from their respective log-linear trends. The responses of the trade balance-to-GDP ratio and the country interest rate are expressed in percentage point deviations from their respective means.
Figure 6.5: Responses to Country-Spread and U.S.-Interest-Rate Shocks: 1994:Q1 to 2012:Q4 Sample

Notes: (1) Solid lines depict responses to a one-percent country-spread shock, and broken lines depict responses to a one-percent U.S.-interest-rate shock. (2) The responses of output and investment are expressed in percent deviations from their respective log-linear trends. The responses of the trade balance-to-GDP ratio and the country interest rate are expressed in percentage point deviations from their respective means. (3) The countries included in the estimation are Argentina, Brazil, Bulgaria, Chile, Colombia, Ecuador, Hungary, South Korea, Malaysia, Mexico, Peru, South Africa, Thailand, Turkey, and Uruguay.
Figure 6.6: Response to an Output Shock: 1994 to 2012 Sample

Notes: See notes (2) and (3) in figure 6.5.
country interest rates with a delay. Further, a positive output shock causes an expansion in investment, a deterioration of the trade balance-to-GDP ratio, and more importantly, a fall in the country interest rate. A difference between the baseline and expanded data sets is that the latter induces significantly more persistent responses.

6.3 Variance Decompositions

How important are world interest shocks and country spread shocks in explaining movements in output, investment, the trade balance, and the country interest rate itself? And how does the importance of these two shocks vary over forecast horizons? A common way to ascertain the contribution of different driving forces to movements of variables of interest at business-cycle frequency is to compute the variance of the forecast errors attributable to each shock at different horizons.

Forecast errors are computed as follows. In the VAR system (6.1), let \( x_t \equiv [\tilde{y}_t \ tby_t \ \tilde{R}^u_t \ R_t]' \) be the vector of variables and \( \epsilon_t \equiv [\epsilon^y_t \ \epsilon^i_t \ \epsilon^tby_t \ \epsilon^rus_t]' \) the vector of orthogonal disturbances.\(^4\) Then, one can write the MA(∞) representation of \( x_{t+h} \) as

\[
x_{t+h} = \sum_{j=0}^{\infty} C_j \epsilon_{t+h-j},
\]

where

\[
C_j \equiv (A^{-1}B)^j A^{-1}.
\]

The \( h \)-period-ahead forecast of \( x_t \) at time \( t \), is given by the expected value of \( x_{t+h} \) conditional on

\(^4\)We note that the estimates of \( \epsilon^y_t, \epsilon^i_t, \epsilon^tby_t, \) and \( \epsilon^rus_t \) (i.e., the sample residuals of the first, second, third, and fifth equations of the VAR system) are orthogonal to each other by construction. But because \( \tilde{y}_t, \tilde{i}_t, \) and \( tby_t \) are excluded from the \( \tilde{R}^u_t \) equation, we have that in any finite sample the estimates of \( \epsilon^rus_t \) will in general not be orthogonal to the estimates of \( \epsilon^y_t, \epsilon^i_t, \) or \( \epsilon^tby_t \). However, under our maintained assumption that \( \epsilon^rus_t \) is independent of \( \epsilon^y_t, \epsilon^i_t, \) and \( \epsilon^tby_t \), if the model is well specified, this lack of orthogonality should disappear as the sample size increases.
information available at time $t$, that is,

$$E_t x_{t+h} = \sum_{j=h}^{\infty} C_j \epsilon_{t+h-j}.$$ 

The corresponding forecast error, denoted $FE_t^h$, is the difference between the actual value of $x_{t+h}$ and its forecasted value,

$$FE_t^h \equiv x_{t+h} - E_t x_{t+h} = \sum_{j=0}^{h-1} C_j \epsilon_{t+h-j}.$$ 

The variance/covariance matrix of the $h$-period-ahead forecast error is given by

$$FEV^h \equiv \sum_{j=0}^{h-1} C_j \Sigma \epsilon C_j',$$

where $\Sigma$ is the variance/covariance matrix of $\epsilon_t$. Thus, the variance of the $h$-period-ahead forecast error of $x_t$ is simply the vector containing the diagonal elements of $FEV^h$. In turn, the variance-covariance matrix of of the $h$-period-ahead forecasting error of $x_t$ due to a particular shock, say the $i$-th shock, $\epsilon_i t$, is given by

$$FEV_{h,i}^h \equiv \sum_{j=0}^{h-1} (C_j \Lambda_i) \Sigma \epsilon (C_j \Lambda_i)'$$

where $\Lambda_i$ is a $5 \times 5$ matrix with all elements equal to zero except element $(i, i)$, which takes the value one. The matrix $FEV_{h,i}^h$ is the $i$-th component of the orthogonal decomposition of the forecast error variance for $x_t$. This decomposition satisfies the condition $FEV^h = \sum_{i=1}^5 FEV_{h,i}^h$. Then, for any variable of the VAR, say the $k$-th variable, the share of the $h$-period-ahead forecast error variance attributable to $\epsilon_i t$, denoted $SFEV_{k,i}^h$, is given by the ratio of the $k$-th diagonal element of
FEV$^{h,i}$ to the $k$-th diagonal element of $FEV^h$, that is

$$SFEV^h_k = \frac{FEV^{h,i}_{kk}}{FEV^h_{kk}},$$

where the subscript $kk$ denotes element $(k,k)$. As the forecast horizon $h$ approaches infinity, the decomposition of the variance of the forecast error coincides with the decomposition of the unconditional variance of the variable in question.

Figure 6.7 displays the share of the forecast error variance of the variables contained in the SVAR system (6.1) at horizons 1 to 24 quarters ($h = 1, 2, \ldots, 24$), $SFEV^h_k$ for $h = 1, \ldots, 24$, $i = 4, 5$, and $k = 1, \ldots, 5$. Solid lines show the fraction of the variance of the forecast error explained jointly by U.S.-interest-rate shocks and country-spread shocks ($\epsilon^\text{rus}_t$ and $\epsilon^r_t$), given by $SFEV^{h,4}_k + SFEV^{h,5}_k$. Broken lines depict the fraction of the variance of the forecast error explained by U.S.-interest-rate shocks ($\epsilon^\text{rus}_t$), given by $SFEV^{h,4}_k$. Because $\epsilon^\text{rus}_t$ and $\epsilon^r_t$ are orthogonal disturbances, the vertical distance between the solid line and the broken line represents the variance of the forecast error explained by country-spread shocks, given by $SFEV^{h,5}_k$. For the purpose of the present discussion, we associate business-cycle fluctuations with the variance of the forecast error at a horizon of about 20 quarters, or five years. Researchers typically define business cycles as movements in time series at frequencies ranging from 6 quarters to 32 quarters (see, for instance, Stock and Watson, 1999). Our choice of horizon falls in the middle of this window.

The estimated SVAR system (6.1) implies that innovations in the U.S. interest rate, $\epsilon^\text{rus}_t$, explain about 20 percent of movements in aggregate activity (output and investment) in emerging countries at business cycle frequency. At the same time, country-spread shocks, $\epsilon^r_t$, account for 12 percent of aggregate fluctuations in these countries. Thus, around 30 percent of business-cycle fluctuations in emerging economies is explained by disturbances in external financial variables. These disturbances play an even stronger role in explaining movements in international transactions. In effect, U.S.-
Figure 6.7: Variance Decomposition at Different Horizons

Note. Solid lines depict the share of the variance of the h-quarter-ahead forecasting error explained jointly by $\epsilon_t^{rus}$ and $\epsilon_t^r$ for values of h between 1 and 24 quarters. Broken lines depict the share of the variance of the h-quarter-ahead forecasting error explained by $\epsilon_t^{rus}$. 
Table 6.2: Forecast-Error Variance Decomposition

<table>
<thead>
<tr>
<th>Variable</th>
<th>$h = 20$</th>
<th>$h = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\epsilon_t^{rus}$</td>
<td>$\epsilon_t^f$</td>
</tr>
<tr>
<td>$\hat{y}_t$</td>
<td>17</td>
<td>13</td>
</tr>
<tr>
<td>$\hat{i}_t$</td>
<td>20</td>
<td>12</td>
</tr>
<tr>
<td>$tby_t$</td>
<td>23</td>
<td>21</td>
</tr>
<tr>
<td>$\hat{R}_t^{rus}$</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>$\hat{R}_t$</td>
<td>35</td>
<td>53</td>
</tr>
<tr>
<td>$\hat{R}_t - \hat{R}_t^{rus}$</td>
<td>24</td>
<td>61</td>
</tr>
</tbody>
</table>

Note. The table displays the share of the forecast error variance of each variable in the first column explained by $\epsilon_t^{rus}$ and $\epsilon_t^f$ at horizons $h = 20, \infty$ quarters. All shares are expressed in percent and are computed using the estimate of the SVAR system (6.1) given in table 6.1. The decomposition at horizon $\infty$ corresponds to the unconditional variance decomposition.

interest-rate shocks and country-spread shocks are responsible for 44 percent of movements in the trade balance-to-output ratio in the countries included in the panel.

Variations in country spreads are largely explained by innovations in U.S. interest rates and innovations in country-spreads themselves. Jointly, these two sources of uncertainty account for 85 percent of fluctuations in country spreads. Most of this fraction, about 60 percentage points, is attributed to country-spread shocks. This last result concurs with Eichengreen and Mody (1998), who interpret it as suggesting that arbitrary revisions in investors sentiments play a significant role in explaining the behavior of country spreads.

Table 6.2 extends the forecast error variance decomposition to horizon $\infty$ quarters, which, as explained earlier, corresponds to the unconditional variance decomposition. It shows that in the present empirical model, the variance decomposition at forecast horizon 20 quarters is virtually identical to the unconditional variance decomposition.
6.4 An Open Economy Subject To Interest-Rate Shocks

The process of identifying country-spread shocks and U.S.-interest-rate shocks involves a number of restrictions on the matrices defining the VAR system (6.1). To assess the plausibility of these restrictions, it is necessary to use the predictions of some theory of the business cycle as a metric. If the estimated shocks imply similar business cycle fluctuations in the empirical as in theoretical models, we conclude that according to the proposed theory, the identified shocks are plausible.

Accordingly, we assess the plausibility of the identified shocks in four steps: First, we develop a standard model of the business cycle in small open economies. Second, we estimate the deep structural parameters of the model. Third, we feed into the model the estimated version of the fourth and fifth equations of the VAR system (6.1), describing the stochastic laws of motion of the U.S. interest rate and the country spread. Finally, we compare estimated impulse responses (i.e., those shown in figures 6.2 and 6.3) with those implied by the proposed theoretical framework.

The basis of the theoretical model presented here is a standard small open economy like the one studied in chapter 4. The results of this section were first obtained in contributions by Neumeyer and Perri (2005) and Uribe and Yue (2006). The exposition follows closely the latter authors.

The model departs from the canonical version of the SOE-RBC model along four dimensions. To be compatible with the assumption that it takes one period for financial shocks to affect output, investment and the trade balance, households make consumption and labor supply decisions prior to the realization of that period’s world-interest-rate shock and country-spread shock. Thus, innovations in the world interest rate or the country spread are assumed to have allocative effects with a one-period lag. Second, preferences are assumed to feature external habit formation, or catching up with the Joneses as in Abel (1990). This feature improves the predictions of the standard model by preventing an excessive contraction in private non-business absorption in response to external financial shocks. Habit formation has been shown to help explain asset prices and business fluc-
tations in both developed economies (e.g., Boldrin, Christiano, and Fisher, 2001) and emerging countries (e.g., Uribe, 2002). Third, the process of capital accumulation is assumed to be subject to gestation lags and convex adjustment costs. In combination, these two frictions prevent excessive investment volatility, induce persistence, and allow for the observed nonmonotonic (hump-shaped) response of investment in response to a variety of shocks (Uribe, 1997). Fourth, firms are assumed to be subject to a working-capital constraint of the type introduced in section 5.3.2 of chapter 5. This constraint introduces a direct supply side effect of changes in the cost of borrowing in international financial markets and allows the model to predict a more realistic response of domestic output to external financial shocks.

### 6.4.1 Firms and Working-Capital Constraints

Output, denoted $y_t$, is produced by means of a production function that takes labor services, $h_t$, and physical capital, $k_t$, as inputs,

$$ y_t = F(k_t, h_t), \quad (6.2) $$

where the function $F(\cdot, \cdot)$ is assumed to be homogeneous of degree one, increasing in both arguments, and concave. Firms hire labor and capital services from perfectly competitive markets.

The production process is subject to a working-capital constraint that requires firms to hold non-interest-bearing assets to finance a fraction of the wage bill each period. Formally, the working-capital constraint takes the form

$$ m_t \geq \eta w_t h_t; \quad \eta \geq 0, $$

where $m_t$ denotes the amount of working capital held by the representative firm in period $t$, and $w_t$ denotes the wage rate in period $t$. As shown in section 5.3.2 of chapter 5, a working capital
constraint of this type results in the following factor demands by the firm

\[ F_k(k_t, h_t) = w_t \left[ 1 + \eta \left( \frac{R^d_t - 1}{R^d_t} \right) \right] \]  
(6.3)

and

\[ F_k(k_t, h_t) = u_t, \]  
(6.4)

where \( R^d_t \) denotes the gross interest rate at which the firm can borrow in period \( t \) and \( u_t \) denotes the rental rate of capital in period \( t \).

It is clear from the first of these two efficiency conditions that the working-capital constraint distorts the labor market by introducing a wedge between the marginal product of labor and the real wage rate. This distortion is larger the larger the opportunity cost of holding working capital, \((R^d_t - 1)/R^d_t\), and the higher the intensity of the working capital constraint, \( \eta \).

### 6.4.2 Capital Accumulation and Gestation Lags

The process of capital accumulation displays adjustment costs in the form of gestation lags and convex costs as in Uribe (1997). Producing one unit of capital good requires investing 1/4 units of goods for four consecutive periods. Let \( s_{it} \) denote the number of investment projects started in \( t - i \) for \( i = 0, 1, 2, 3 \). Then gross investment in period \( t \), denoted \( i_t \), is given by

\[ i_t = \frac{1}{4} \sum_{i=0}^{3} s_{it}. \]  
(6.5)

In turn, the evolution of \( s_{it} \) is given by

\[ s_{i+1t+1} = s_{it}, \]  
(6.6)
for \( i = 0, 1, 2 \). The stock of capital obeys the law of motion

\[
k_{t+1} = (1 - \delta)k_t + k_t\Phi\left(\frac{s_3}{k_t}\right),
\]

(6.7)

where \( \delta \in (0, 1) \) denotes the rate of depreciation of physical capital. The process of capital accumulation is assumed to be subject to adjustment costs, as defined by the function \( \Phi \), which is assumed to be strictly increasing, concave, and to satisfy \( \Phi(\delta) = \delta \) and \( \Phi'(\delta) = 1 \). These last two assumptions ensure the absence of adjustment costs in the steady state and that the steady-state level of investment is independent of \( \Phi \). The introduction of capital adjustment costs is commonplace in models of the small open economy. As discussed in chapters 3 and 4, adjustment costs are a convenient and plausible way to avoid excessive investment volatility in response to changes in the interest rate faced by the country in international markets.

### 6.4.3 Households and Habit Formation

The economy is populated by a large number of infinitely-lived households with preferences described by the lifetime utility function

\[
E_0 \sum_{t=0}^{\infty} \beta^t U(c_t - \mu \tilde{c}_{t-1}, h_t),
\]

(6.8)

where \( c_t \) denotes consumption in period \( t \), \( \tilde{c}_{t-1} \) denotes the cross-sectional average level of consumption in period \( t - 1 \), and \( h_t \) denotes the fraction of time devoted to work in period \( t \). Households take as given the process for \( \tilde{c}_t \). The single-period utility index \( U(\cdot, \cdot) \) is assumed to be increasing in its first argument, decreasing in its second argument, and concave. The parameter \( \beta \in (0, 1) \) denotes a subjective discount factor, and the parameter \( \mu \) measures the intensity of external habit formation.
Households have access to two types of asset, physical capital and a one-period bond. The capital stock is assumed to be owned entirely by domestic residents. Households have four sources of income: wages, capital rents, profits, and interest income from bond holdings. Each period, households allocate their wealth to purchases of consumption goods, investment goods, and financial assets. The household’s period-by-period budget constraint is given by

\[ d_t = R_{t-1}d_{t-1} + \Psi(d_t) + c_t + i_t - w_t h_t - u_t k_t - \pi_t, \]  

(6.9)

where \( d_t \) denotes the household’s debt position in period \( t \), \( R_t \) denotes the gross interest rate, and \( \pi_t \) denotes profit income, which the household takes as given. We assume that households face costs of adjusting their foreign asset position as in section 4.10.2 of chapter 4. As discussed there, these adjustment costs represent one way to eliminate the familiar unit root present in small open economy models. The portfolio-adjustment cost function \( \Psi(\cdot) \) is assumed to be convex and to satisfy \( \Psi(d) = \Psi'(d) = 0 \), for some \( d > 0 \).

Households choose consumption, labor supply, and investment one period in advance. Therefore, in period \( t \) the variables \( c_t, h_t, \) and \( s_{i,t} \) for \( i = 0, 1, 2, 3 \) are predetermined. In period \( t \), the household chooses \( c_{t+1}, h_{t+1}, s_{i,t+1} \) (\( i = 0, 1, 2, 3 \)), \( d_t \), and \( k_{t+1} \) to maximize the utility function (6.8) subject to the budget constraint (6.9), the laws of motion of investment projects and capital, equations (6.5)-(6.7), and a borrowing constraint of the form

\[ \lim_{j \to \infty} E_t \frac{d_{t+j+1}}{\Pi_{s=0}^{j} R_{t+s}} \leq 0 \]  

(6.10)

that prevents Ponzi schemes. The Lagrangian associated with the household’s optimization problem
can be written as

\[
\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ U(c_t - \mu \tilde{c}_t - 1, h_t) + \lambda_t \left[ d_t - R_{t-1} d_{t-1} - \Psi(d_t) + w_t h_t + u_t k_t + \pi_t - \frac{1}{4} \sum_{i=0}^{3} s_{it} - c_t \right] \\
+ \lambda_t q_t \left[ (1 - \delta) k_t + k_t \Phi \left( \frac{s_{3t}}{k_t} \right) - k_{t+1} \right] + \lambda_t \sum_{i=0}^{2} \nu_{it}(s_{it} - s_{i+1}) \right\},
\]

where \( \beta^t \lambda_t, \beta^t \lambda_t \nu_{it}, \) and \( \beta^t \lambda_t q_t \) are the Lagrange multipliers associated with constraints (6.9), (6.6), and (6.7), respectively. The optimality conditions associated with the household’s problem are (6.6), (6.7), (6.9), (6.10) holding with equality, and

\[
E_t \lambda_{t+1} = U_c(c_{t+1} - \mu \tilde{c}_t, h_{t+1}) \tag{6.11}
\]

\[
E_t[w_{t+1} \lambda_{t+1}] = -U_h(c_{t+1} - \mu \tilde{c}_t, h_{t+1}) \tag{6.12}
\]

\[
\lambda_t \left[ 1 - \Psi'(d_t) \right] = \beta R_t E_t \lambda_{t+1} \tag{6.13}
\]

\[
E_t \lambda_{t+1} \nu_{0t+1} = \frac{1}{4} E_t \lambda_{t+1} \tag{6.14}
\]

\[
\beta E_t \lambda_{t+1} \nu_{1t+1} = \frac{\beta}{4} E_t \lambda_{t+1} + \lambda_t \nu_{0t} \tag{6.15}
\]

\[
\beta E_t \lambda_{t+1} \nu_{2t+1} = \frac{\beta}{4} E_t \lambda_{t+1} + \lambda_t \nu_{1t} \tag{6.16}
\]

\[
\beta E_t \left[ \lambda_{t+1} q_{t+1} \Phi' \left( \frac{s_{3t+1}}{k_{t+1}} \right) \right] = \frac{\beta}{4} E_t \lambda_{t+1} + \lambda_t \nu_{2t} \tag{6.17}
\]

\[
\lambda_t q_t = \beta E_t \left\{ \lambda_{t+1} q_{t+1} \left[ 1 - \delta + \Phi \left( \frac{s_{3t+1}}{k_{t+1}} \right) - \frac{s_{3t+1}}{k_{t+1}} \Phi' \left( \frac{s_{3t+1}}{k_{t+1}} \right) \right] + \lambda_{t+1} u_{t+1} \right\}. \tag{6.18}
\]

It is important to recall that, because of the assumed information structure, the variables \( c_{t+1}, h_{t+1}, \) and \( s_{0t+1} \) all reside in the information set of period \( t \). Equation (6.11) states that in period \( t \) households choose consumption and leisure for period \( t + 1 \) in such as way as to equate the
marginal utility of consumption in period $t + 1$ to the expected marginal utility of wealth in that period, $E_t \lambda_{t+1}$. In general, the marginal utility of wealth will differ from the marginal utility of consumption ($\lambda_t \neq U_c(c_t - \mu \tilde{c}_{t-1}, h_t)$), because current consumption cannot react to unanticipated changes in wealth. Because of the presence of external habit formation, when households observe that their neighbors (the Joneses) increase current consumption, $\tilde{c}_t$, they wake up hungrier in period $t + 1$, which is reflected in a higher marginal utility of consumption. Equation (6.12) defines the household’s labor supply schedule, by equating the marginal disutility of effort in period $t + 1$ to the expected utility value of the wage rate in that period.

Equation (6.13) is an asset pricing relation equating the intertemporal marginal rate of substitution in consumption to the rate of return on financial assets. As discussed in section 4.10.2 of chapter 4, because of the presence of frictions to adjust bond holdings, the relevant rate of return on this type of asset is not simply the market rate $R_t$ but rather the shadow rate of return

$$R^d_t \equiv \frac{R_t}{1 - \Psi'(d_t)}.$$  \hfill (6.19)

Intuitively, when the household’s debt position is, say, above its steady-state level $\overline{d}$, we have that $\Psi'(d_t) > 0$ so that the shadow rate of return is higher than the market rate of return, providing further incentives for households to save, thereby reducing their debt positions.

Equations (6.14)-(6.16) show how to price investment projects at different stages of completion. The price of an investment project in its $i$th quarter of gestation equals the price of a project in the $(i - 1)$th quarter of gestation plus $1/4$ units of goods. Equation (6.17) links the cost of producing a unit of capital to the shadow price of installed capital, or Tobin’s $Q$, $q_t$. Finally, equation (6.18) is a pricing condition for physical capital. It equates the revenue from selling one unit of capital today, $q_t$, to the discounted value of renting the unit of capital for one period and then selling it, $u_{t+1} + q_{t+1}$, net of depreciation and adjustment costs.
6.4.4 Driving Forces

One advantage of limited information methods like the one used by Uribe and Yue (2006) to assess the plausibility of the identified U.S.-interest-rate shocks and country-spread shocks is that one need not feed into the model shocks other than those whose effects one is interested in studying. Accordingly, all that is needed to close the model is to add a law of motion for the country interest rate $R_t$. We use the estimate of the bottom equation of the VAR system (6.1) presented in table 6.1, which we reproduce here for convenience,

$$
\hat{R}_t = 0.635\hat{R}_{t-1} + 0.501\hat{R}_{us,t} + 0.355\hat{R}_{us,t-1} - 0.791\hat{y}_t + 0.617\hat{y}_{t-1} + 0.114\hat{r}_t - 0.122\hat{t}_{t-1},
$$

(6.20)

where $\epsilon_t^r$ is an i.i.d. disturbance with mean zero and standard deviation 0.031. As indicated earlier, the variable $tby_t$ stands for the trade balance-to-GDP ratio and is given by:

$$
tby_t = \frac{y_t - c_t - i_t - \Psi(d_t)}{y_t}.
$$

(6.21)

Because the process for the country interest rate defined by equation (6.20) involves the world interest rate, $R_{us,t}$, which is assumed to be an exogenous random variable, we must also include this variable’s law of motion as part of the set of equations defining the equilibrium behavior of the theoretical model. Accordingly, the model includes the equation

$$
\hat{R}_{us,t} = 0.830\hat{R}_{us,t-1} + \epsilon_{us,t}.
$$

(6.22)

---

5Because the portfolio-adjustment cost $\Psi(d_t)$ is incurred by households, the national income and product accounts would measure private consumption as $c_t + \Psi(d_t)$ and not simply as $c_t$. However, because of the maintained assumption that $\Psi(0) = \Psi(0) = 0$, it follows that $c_t + \Psi(d_t) = c_t$ up to first order.
where \( \epsilon^\text{rus}_t \) is an i.i.d. innovation with mean zero and standard deviation 0.007.

### 6.4.5 Equilibrium

In equilibrium all households consume identical quantities. Thus, individual consumption equals average consumption across households, or

\[
c_t = \bar{c}_t, \quad (6.23)
\]

for \( t \geq 0 \). The resource constraint of the economy is given by

\[
d_t = R_{t-1} d_{t-1} + \Psi(d_t) + c_t + i_t - y_t. \quad (6.24)
\]

The derivation of this constraint combines the steps described in section 4.10.2 of chapter 4 and in section 5.3.2 of chapter 5.

An equilibrium is a set of processes \( c_{t+1}, \bar{c}_t, h_{t+1}, d_t, i_t, k_{t+1}, s_{it+1} \) for \( i = 0, 1, 2, 3, R_t, R^d_t, R^\text{us}_t, w_t, u_t, y_t, tby_t, \lambda_t, q_t, \) and \( \nu_{it} \) (\( i = 0, 1, 2 \)), for \( t \geq 0 \), satisfying conditions (6.2)-(6.7), and (6.11)-(6.24), given \( c_0, y_{-1}, i_{-1}, h_0, k_0, d_{-1}, tby_{-1}, R_{-1}, \) and \( R^\text{us}_{-1}, \) and the processes for the exogenous innovations \( \epsilon^\text{rus}_t \) and \( \epsilon^r_t \). An equilibrium also requires the satisfaction of the no-Ponzi-game constraint (6.10) with equality. The analysis that follows restricts attention to equilibria in which all variables remain in a small neighborhood of their respective deterministic steady states, which guarantees that this condition is met.

### 6.4.6 Estimation By Limited Information Methods

The parameterization of the model combines calibration and econometric estimation. We adopt the following standard functional forms for preferences, technology, capital adjustment costs, and
portfolio adjustment costs,

\[ U(c - \mu \tilde{c}, h) = \frac{[c - \mu \tilde{c} - \omega^{-1}h^\omega]^{1-\gamma} - 1}{1 - \gamma}, \]

\[ F(k, h) = k^\alpha h^{1-\alpha}, \]

\[ \Phi(x) = x - \frac{\phi}{2} (x - \delta)^2; \quad \phi > 0, \]

\[ \Psi(d) = \frac{\psi}{2} (d - \bar{d})^2. \]

The parameterization of the model follows Uribe and Yue (2006). The model has ten parameters, six of which are calibrated and the remaining four are econometrically estimated. The time unit is meant to be one quarter. We set \( \gamma = 2, \omega = 1.455, \) and \( \alpha = 0.32. \) The steady-state real interest rate faced by the small economy in international financial markets is set to 11 percent per year. This value is consistent with an average U.S. interest rate of about 4 percent and an average country premium of 7 percent, both of which are in line with actual data. The depreciation rate is set to 0.025, or 10 percent per year, which is a standard value in business-cycle studies. The steady-state trade-balance-to-GDP ratio, \( tby, \) is set to 2\%. Finally, we impose the restriction \( \beta R = 1. \)

There remain four parameters to assign values to, \( \psi, \phi, \eta, \) and \( \mu. \) Because there exist no available estimates for those parameters for emerging economies, Uribe and Yue estimate them using a limited information approach. The estimation procedure consists in choosing values for these four parameters to minimize the distance between the estimated impulse response functions shown in figures 6.2 and 6.3 and the corresponding theoretical impulse response functions implied by the model. The procedure matches the first 24 quarters of the impulse response functions of 4 variables (output, investment, the trade balance, and the country interest rate), to 2 shocks (the U.S.-interest-rate shock and the country-spread shock). Thus, 4 parameter values are picked to
match $185 = (4 \times 2 \times 24 - 7)$ points. Seven degrees of freedom are lost because the impact effects of both, U.S.-interest rate shocks and country-spread shocks, on output, investment and the trade balance are by construction zero and the impact effect of a unit country-spread shock on the country spread itself is unity. Formally, let $IR^e$ denote the $185 \times 1$ vector of estimated impulse response functions and $IR^m(\psi, \phi, \eta, \mu)$ the corresponding vector of impulse response functions implied by the theoretical model, which are functions of the four parameters to be estimated. Then $(\psi, \phi, \eta, \mu)$ are estimated as

$$\arg\min_{\psi, \phi, \eta, \mu} \{IR^e - IR^m(\psi, \phi, \eta, \mu)\}^T \Sigma^{-1}_{IR^e} [IR^e - IR^m(\psi, \phi, \eta, \mu)],$$

where $\Sigma_{IR^e}$ is a diagonal matrix containing the variance of the empirical impulse response functions along the diagonal. This matrix penalizes those elements of the estimated impulse response functions associated with large error intervals. The resulting parameter estimates are $\psi = 0.00042$, $\phi = 72.8$, $\eta = 1.2$, and $\mu = 0.20$. The implied portfolio adjustment costs are small. For example, a 10 percent increase in $d_t$ over its steady-state value $\bar{d}$ maintained over one year has a resource cost of $4 \times 10^{-6}$ percent of annual GDP. On the other hand, capital adjustment costs appear as more significant. For instance, starting in a steady-state situation, a 10 percent increase in investment for one year produces an increase in the capital stock of 0.88 percent. In the absence of capital adjustment costs, the capital stock would increase by 0.96 percent. The estimated value of $\eta$ implies that firms maintain a level of working capital equivalent to about 3.6 months of wage payments. Finally, the estimated degree of habit formation is modest compared to the values typically used to explain asset-price regularities in closed economies (e.g., Constantinides, 1990). Table 6.3 gathers all parameter values.
Table 6.3: Parameter Values

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R - 1 = \beta^{-1} - 1$</td>
<td>2.77%</td>
<td>Steady-state real country interest rate (quarterly)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>2</td>
<td>Inverse of intertemporal elasticity of substitution</td>
</tr>
<tr>
<td>$\omega$</td>
<td>1.455</td>
<td>$1/(\omega - 1)$ = Labor supply elasticity</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.32</td>
<td>Capital elasticity of output</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.025</td>
<td>Depreciation rate (quarterly)</td>
</tr>
<tr>
<td>$tby$</td>
<td>0.02</td>
<td>Steady-state trade-balance-to-GDP ratio</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>72.8</td>
<td>Capital adjustment cost parameter</td>
</tr>
<tr>
<td>$\psi$</td>
<td>0.00042</td>
<td>Portfolio adjustment cost parameter</td>
</tr>
<tr>
<td>$\eta$</td>
<td>1.20</td>
<td>Fraction of wage bill subject to working-capital constraint</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.20</td>
<td>Habit formation parameter</td>
</tr>
</tbody>
</table>

6.5 Theoretical and Estimated Impulse Responses

Figure 6.8 depicts impulse response functions of output, investment, the trade balance-to-GDP ratio, and the country interest rate. The left column shows impulse responses to a U.S.-interest-rate shock ($\epsilon^{rus}_t$), and the right column shows impulse responses to a country-spread shock ($\epsilon^c_i$). Solid lines display empirical impulse response functions, and broken lines depict the associated two-standard-error bands. This information is reproduced from figures 6.2 and 6.3. Crossed lines depict theoretical impulse response functions.

The model replicates three key qualitative features of the estimated impulse response functions: First, output and investment contract in response to both a U.S.-interest-rate shock and a country-spread shock. Second, the trade balance improves in response to both shocks. Third, the country interest rate displays a hump-shaped response to an innovation in the U.S. interest rate.

---

6The Matlab code used to produce theoretical impulse response functions is available online at [http://www.columbia.edu/~mu2166/uribe_yue_jie/uribe_yue_jie.html](http://www.columbia.edu/~mu2166/uribe_yue_jie/uribe_yue_jie.html).
Figure 6.8: Theoretical and Estimated Impulse Response Functions

Note: The first column displays impulse responses to a U.S. interest rate shock ($\epsilon^{rus}$), and the second column displays impulse responses to a country-spread shock ($\epsilon^r$).
These findings suggest that the scheme used to identify the parameters of the VAR system (6.1) is successful in isolating country-spread shocks and U.S.-interest-rate shocks from the data.

6.6 Theoretical and Estimated Conditional Volatilities

Another way to ascertain whether the U.S.-interest-rate shocks and country-spread shocks identified via the SVAR analysis are plausible is to gauge the ability of the theoretical model to match the conditional standard deviations of the endogenous variables due to these two structural shocks implied by the SVAR. This approach is related to the impulse response analysis of the previous section because the conditional variance of any variable is given by the sum of its squared impulse response to the shock in question scaled by the variance of the shock itself.

Table 6.4 displays standard deviations conditional on U.S.-interest-rate shocks ($\epsilon_{rus}^t$) and country-spread shocks ($\epsilon_r^t$) shocks implied by the empirical SVAR model and the theoretical SOE model. The SOE model does well at capturing the importance of U.S.-interest-rate and country-spread shocks in explaining movements in output and country interest rates. The model also does a satisfactory job at accounting for variations in the trade balance due to U.S.-interest-rate shocks. But the SOE model systematically underpredicts the volatilities of investment and the trade balance caused by country-spread shocks.

The fifth column of the table shows unconditional volatilities implied by the SVAR model. Comparing these volatilities with the conditional volatilities predicted by the SOE model (columns 2 and 4), shows that the two identified shocks jointly explain 30 percent of fluctuations in output ($(1.6^2 + 1.3^2)/3.7^2 = 0.31$). This number is close to the one implied by the SVAR model ($(1.5^2 + 1.3^2)/3.7^2 = 0.29$). On the other hand, the SOE model assigns less importance to the two interest rate shocks in accounting for variations in investment and the trade balance than does the SVAR model, 17% versus 33% and 8% versus 43%, respectively.
Table 6.4: Conditional Volatilities Implied by the SVAR and Theoretical Models

<table>
<thead>
<tr>
<th>Variable</th>
<th>$\epsilon_{t}^{rus}$</th>
<th>$\epsilon_{t}^{r}$</th>
<th>Unconditional</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{y}$</td>
<td>1.5</td>
<td>1.6</td>
<td>1.3</td>
</tr>
<tr>
<td>$\hat{\tau}$</td>
<td>6.4</td>
<td>3.6</td>
<td>5.0</td>
</tr>
<tr>
<td>$tby$</td>
<td>2.1</td>
<td>1.6</td>
<td>2.0</td>
</tr>
<tr>
<td>$\hat{R}^{rus}$</td>
<td>1.3</td>
<td>1.3</td>
<td>0</td>
</tr>
<tr>
<td>$\hat{R}$</td>
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<td>4.7</td>
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</table>

Note. The first two columns display standard deviations conditional on U.S.-interest rate shocks ($\epsilon_{t}^{rus}$), and the last two columns display standard deviations conditional on country-spread shocks ($\epsilon_{t}^{r}$). The fifth column displays unconditional standard deviations implied by the SVAR model. The variables $\hat{y}_{t}$ and $\hat{\tau}_{t}$ are measured in percent deviations from steady state, and $tby_{t}$, $\hat{R}_{t}^{rus}$, and $\hat{R}_{t}$ are measured in percentage point deviations from their respective steady states. Interest rates are annualized.

Overall, the results of this analysis suggest that the two interest-rate shocks ($\epsilon_{t}^{rus}$ and $\epsilon_{t}^{r}$) identified by the SVAR model are sensible and economically important.

6.7 Global Risk Factors And Business Cycles in Emerging Economies

Thus far, we have studied how movements in a risk-free measure of global financial conditions, the real Treasury bill rate, and a measure of country-specific risk, embodied in country spreads, affect business cycles in emerging markets. The analysis has abstracted from other global financial factors that are potentially important for understanding aggregate fluctuations in the developing world. One such factor is given by variations in global risk premia.

Akinci (2013) fills in this gap by expanding the SVAR model of Uribe and Yue (2006) to include

---

7Strictly speaking the real Treasury bill rate is not risk free. The nominal rate of return on Treasury bills can be considered risk-free because the probability that the U.S. government will default on this type of obligations is perceived to be nil. The realized real rate of return on Treasury bills is the difference between the nominal rate of return and realized inflation. Because inflation has an unpredictable component, so does the real rate.
the spread between the U.S. Baa corporate borrowing rate and the 20-year U.S. Treasury bond yield. Baa bonds are debt obligations issued by U.S corporations with medium degree of risk. To form an idea of the relative risk involved in this type of security, we note that the rating agency Moody’s rates corporate bonds on a letter scale ranging from Aaa to C. Historically, the cumulative default rate over 20 years is less than 1 percent for Aaa rated bonds, more than 70 percent for C rated bonds, and 13 percent for Baa rated bonds (Fons, Cantor, and Mahoney, 2002). Thus, the Baa corporate bond spread, is the difference between the return on securities with medium default risk and securities with negligible default risk (Treasury bonds). An increase in the Baa corporate bond spread may reflects an elevated risk of default in THE U.S. corporate sector or changes in market participants’ attitudes toward this type of risk. The present analysis asks how variations in this global risk premium transmit to the real and financial sectors of emerging economies.

The expanded SVAR specification is

\[
A \begin{bmatrix} \tilde{y}_t \\ \tilde{\iota}_t \\ \tilde{tby}_t \\ \tilde{R}^u_{us}_t \\ \tilde{S}^u_{us}_t \\ \tilde{R}_t \\ \end{bmatrix} = B(L) \begin{bmatrix} \tilde{y}_{t-1} \\ \tilde{\iota}_{t-1} \\ \tilde{tby}_{t-1} \\ \tilde{R}^u_{us}_{t-1} \\ \tilde{S}^u_{us}_{t-1} \\ \tilde{R}_{t-1} \\ \end{bmatrix} + \begin{bmatrix} \epsilon^y_t \\ \epsilon^\iota_t \\ \epsilon^{tby}_t \\ \epsilon^{Rus}_t \\ \epsilon^{Sus}_t \\ \epsilon^r_t \\ \end{bmatrix},
\]

where \(S^u_{us}_t\) denotes the U.S. risk premium as measured by the Baa corporate bond spread, and \(\epsilon^{sus}_t\) denotes an innovation to the U.S. risk premium. The SVAR is of order 2, that is, \(B(L) \equiv B_1 + B_2L\) where \(B_1\) and \(B_2\) are 6-by-6 matrices of coefficients and \(L\) denotes the lag operator. The identification scheme follows Uribe and Yue (2006) by imposing that the matrix \(A\) be lower triangular. In addition, the block \([R^u_{us}_t S^u_{us}_t]'\) is assumed to follow a bivariate system. Akinci estimates the SVAR
using a panel containing six countries (Argentina, Brazil, Mexico, Peru, South Africa, and Turkey) using quarterly data from 1994:Q1 to 2011:Q3.

The main result stemming from this estimation is that financial shocks, that is the triplet $[\epsilon_t^{rus}, \epsilon_t^{sus}, \epsilon_t^t]$, play an even larger role than before in explaining business cycles in emerging countries. Jointly, these three shocks explain 42 percent of the variance of output, as compared to 30 percent under the specification that omits global risk-premium shocks. This result reinforces the findings of Uribe and Yue (2006) reported earlier in this chapter. The main difference that emerges when global risk-premium shocks are taken explicitly into account is that they absorb much of the role that was previously assigned to the U.S.-interest-rate shock. In particular, under the present SVAR specification $\epsilon_t^{sus}$ explains 18 percent of the variance of output whereas $\epsilon_t^{rus}$ explains only 6 percent, down from 17 percent in the specification without global risk-premium shocks.

The global risk factor also takes over the role previously played by the U.S. interest rate in driving country spreads. Under the present formulation, $\epsilon_t^{sus}$ explains 18 percent of movements in country spreads, whereas $\epsilon_t^{rus}$ explains only 5 percent, about 20 percentage points less than under the specification without global risk-premium shocks. One channel through which global risk-premium shocks affect real activity is through movements in country spreads. A 1 percentage-point increase in the global risk premium leads to an increase in the country spread of 1.3 percentage points.

Summarizing, this chapter documents the importance of interest rate shocks as drivers of business cycles in emerging markets. This source of uncertainty, whether stemming from movements in the risk free U.S. interest rate, or from elevated risk-premia in the U.S. corporate sector, or from innovations in emerging-country spreads, accounts for more than one third of the variance of output. In chapters 9-12 we analyze how nominal and financial frictions propagate and amplify interest-rate shocks.
Exercise 6.1 (Countercyclical Interest Rate Shocks) Consider a two-period small open endowment economy. Household preferences are given by:

\[ \log c_1 + \log c_2, \]

where \( c_1 \) and \( c_2 \) denote consumption in periods 1 and 2, respectively. Let \( y_1 \) and \( y_2 \) denote the endowments in periods 1 and 2, respectively. Households enter period 1 with zero net foreign assets, \( d_0 = 0 \). Assume free international capital mobility. The world interest rate is \( r^* \) and households are subject to a no-Ponzi-game constraint of the form \( d_2 \leq 0 \), where \( d_t \) for \( t = 1, 2 \) denotes one-period debt assumed in period \( t \) and due in \( t + 1 \).

1. Write down the household’s budget constraints in periods 1 and 2.

2. Derive the household’s intertemporal budget constraint.

3. Write down the household’s utility maximization problem.

4. Characterize the equilibrium allocation of consumption and the trade balance in periods 1 and 2.

5. Now assume that in period 1 the economy is hit by a negative (and purely temporary) endowment shock. Find the change in consumption and the trade balance in period 1. Is the response of the trade balance in period 1 pro- or countercyclical. Explain your findings.

6. As documented earlier in this chapter, in emerging economies the country interest rate tends to be countercyclical. To reflect this regularity assume now that when \( y_1 \) falls, \( r^* \) increases. Let \( \eta = -\frac{d\ln(1+r^*)}{d\ln y_1} \) denote the elasticity of the gross interest rate with respect to the period-1
endowment. Find conditions, in terms of $\eta$, $y_1$, $y_2$, and $r^*$, that guarantee that consumption moves procyclically and the trade balance moves countercyclically in period 1 in response to a period-1 endowment shock. Provide an intuitive explanations for your findings.

Exercise 6.2 (Interest-Rate Shocks, Investment, And the Trade Balance) Consider a two-period, small open economy populated by a large number of households with preferences given by

$$\ln c_1 + \ln c_2,$$

where $c_1$ and $c_2$ denote consumption in periods 1 and 2, respectively, and $\ln$ denotes the natural logarithm. Households start period 1 with a zero net asset position. They have no endowments in either period, but can produce goods in period 2 by operating the technology

$$y_2 = A \sqrt{i_1},$$

where $y_2$ denotes output in period 2, $A > 0$ is a productivity factor, and $i_1$ denotes investment in period 1. In period 1, households can participate in the international financial market, where the interest rate is $r$. They are subject to a no-Ponzi-game constraint of the form $d_2 \leq 0$, where $d_t$, for $t = 1, 2$, denotes one-period debt assumed in period $t$ and maturing in period $t + 1$.

1. Write down the household’s budget constraints in periods 1 and 2.

2. Derive the household’s intertemporal budget constraint.

3. State the household’s maximization problem.

4. Compute the equilibrium values of consumption, investment, the trade balance, the current account, and external debt in periods 1 and 2.
5. Suppose now that the interest rate increases to \( r' > r \). Characterize the effect of this shock on investment and the trade balance in period 1. Is it qualitatively in line with the related SVAR evidence examined in this chapter? Explain.

6. Suppose that in period 1 agents learn that a positive productivity shock elevates \( A \) to \( A' > A \). Analyze the effect of this innovation on the equilibrium levels of investment and the trade balance in period 1. How does this effect relate to the SVAR evidence studied in this chapter?

Exercise 6.3 (An Economy With Time-Varying Interest Rates) A perfect-foresight small open economy is inhabited by identical consumers with preferences of the form

\[
\sum_{t=0}^{\infty} \beta^t \ln(c_t),
\]

where \( c_t \) denotes consumption of a perishable good, and \( \beta \in (0,1) \) denotes the discount factor. Households are subject to the budget constraint

\[
d_t = (1 + r_{t-1})d_{t-1} + c_t - y.
\]

for \( t \geq 0 \), where \( d_t \) denotes one-period debt acquired in period \( t \) and maturing in \( t + 1 \), \( r_t \) denotes the interest paid on \( d_t \), and \( y \) denotes a constant endowment of consumption goods received each period. Households are subject to the no-Ponzi-game constraint

\[
\lim_{t \to \infty} \frac{d_t}{\prod_{s=0}^{t}(1 + r_s)} \leq 0.
\]
The initial stock of debt, $d_{-1}$, is nil. The interest rate is time varying and given by

$$r_t = \begin{cases} 
   r^H & \text{for } t = 0, 2, 4, \ldots \\
   r^L & \text{for } t = 1, 3, 5, \ldots 
\end{cases},$$

where $r^L$ and $r^H$ are parameters satisfying $0 < r^L < r^H$ and

$$\beta \sqrt{(1 + r^H)(1 + r^L)} = 1.$$

1. Derive the equilibrium path of consumption as a function of the structural parameters of the model, $y$, $r^H$, $r^L$, and $\beta$. Discuss its cyclical properties.

2. Characterize the equilibrium behavior of the trade balance, the current account, and external debt.

3. Discuss the macroeconomic effects of the assumed interest-rate variations.

Exercise 6.4 (Inducing Stationarity and Interest-Rate Shocks)  Chapter 4 shows that the business cycle implied by the SOE-RBC model is not affected by the method used to induce stationarity. This result, however, was derived in the context of a model driven by technology shocks. The present exercise aims to establish whether this finding is robust to assuming that business cycles are driven by world-interest-rate shocks.

1. Consider the external debt-elastic interest-rate (EDEIR) model of section 4.1.1 of chapter 4. Shut down the productivity shock by setting $\tilde{\eta} = 0$. Replace equation (4.14) with

$$r_t = r^*_t + p(\tilde{d}_t),$$
and

\[ r_t^* = r^* + \xi(r_{t-1}^* - r^*) + \mu_t, \]

where \( \mu_t \sim N(0, \sigma_\mu^2) \). Set \( \xi = 0.8 \) and \( \sigma_\mu = 0.012 \). Calibrate all other parameters of the model at the values given in table 4.1. Using this version of the EDEIR model, compute the statistics considered in table 4.4 and make a table. Make a figure showing impulse responses of output, consumption, hours, investment, the trade-balance-to-output ratio, and the current-account-to-output ratio implied by the EDEIR model driven by interest-rate shocks. Provide intuition for these results.

2. Now consider the internal discount factor (IDF) model of section 4.10.4. Again, set \( \tilde{\eta} = 0 \). Replace the assumption that \( r_t = r^* \) with

\[ r_t = r^* + \xi(r_{t-1} - r^*) + \mu_t. \]

Calibrate \( \xi, \sigma_\mu \), and all common parameters as in the previous question. Calibrate \( \psi_3 \) as in section 4.10.4. Use the resulting calibrated model to compute unconditional second moments and impulse responses. Provide intuition for your results. To facilitate comparison, place the information generated here in the same table and figure produced in the previous question.

3. Compare the predictions of the EDEIR and IDF models driven by interest rate shocks. Does the stationarity-inducing mechanism make any difference for the business-cycles implied by the SOE model driven by interest-rate shocks?
Chapter 7

Importable Goods, Exportable Goods and the Terms of Trade

Thus far, we have studied models with a single traded good. This good is produced, consumed, sometimes imported, and sometimes exported. In reality, however, countries produce, consume and trade in different goods. For example, Chile’s exports are dominated by copper, but copper represents a small fraction of Chile’s consumption or imports. Similarly, Norway’s exports consist primarily of oil, whereas its imports are mostly consumption goods. Thus, for most countries it makes sense to think separately about importable and exportable goods. An importable good is either an imported good or a good that is produced domestically but is highly substitutable with a good that can be imported. And an exportable good is either an exported good or a good that is sold domestically but is highly substitutable with a good that can be exported.

And there is a third category of goods, known as nontradables, that are neither imported nor exported. Nontradables are exclusively produced and consumed domestically. Personal services, utilities, and local transportation are classic examples of nontradable goods. Transportation costs
are an important factor in determining the tradability of goods and services. Few New Yorkers would travel to New Delhi just to get (import) a haircut because it is much cheaper there. Other factors causing goods to become nontradable include trade barriers, such as tariffs and quotas.

The above three categories of goods give rise to important relative prices. The relative price of exportables in terms of importables is known as the terms of trade. And the relative price of a domestic consumption basket in terms of a foreign consumption basket is known as the real exchange rate. Movements in these relative prices are often considered important determinants of production, consumption, and employment. For example, a drop in the world price of copper can have significant negative consequences for domestic consumption and sectoral employment in Chile. An external crisis that causes a contraction in aggregate demand may cause a greater reduction in employment in the nontraded sector than in the traded sector, simply because production in the former is limited by the size of domestic demand, whereas production in the latter can be directed to either domestic or foreign markets. For this reason, movements in the terms of trade and the real exchange rate have been the subject of much empirical and theoretical investigation.

In the present chapter, we analyze the macroeconomic effects of movements in the terms of trade in the context of an economy with importable and exportable goods. In the next chapter, we add nontraded goods into the picture and analyze the equilibrium determination of the real exchange rate.

### 7.1 A Simple Empirical Model of the Terms of Trade

The terms of trade are defined as the relative price of exports in terms of imports. Letting $P^e_t$ and $P^m_t$ denote, respectively, indices of world prices of exports and imports for a particular country, the
terms of trade for that country are given by

$$\text{tot}_t \equiv \frac{P^x_t}{P^m_t}. $$

The typical emerging country is a small player in the world markets for the goods it exports and imports. It therefore makes sense to assume that the emerging country takes the terms of trade as exogenously given.\(^1\) Thus, variations in the terms of trade can be regarded as an exogenous source of aggregate fluctuations. Accordingly, we postulate that the terms of trade follow a univariate autoregressive process of the form

$$\hat{\text{tot}}_t = \rho \hat{\text{tot}}_{t-1} + \pi \epsilon^t_{t}, $$

(7.1)

where \(\hat{\text{tot}}_t \) denotes the log-deviation of the terms of trade from trend, \(\epsilon^t_t\) is an i.i.d. innovation with mean zero and unit standard deviation, \(\rho \in (-1, 1)\) denotes the serial correlation of the cyclical component of the terms of trade, and \(\pi\) denotes the standard deviation of the innovation in the terms-of-trade process.

We estimate equation (7.1) by OLS using annual data from 51 poor and emerging countries over the period 1980 to 2011.\(^2\) The cyclical component of the terms of trade, \(\hat{\text{tot}}_t\), is obtained by removing a log-quadratic time trend. The estimation is performed by OLS country by country including an intercept (not shown). Table 7.1 displays country-by-country estimates of \(\rho\) and \(\pi\).

---

1 Exercise 7.1 asks you to econometrically test this hypothesis.
2 The data comes from the World Bank’s World Development Indicators data base and is available in the file wdi.xls posted with the materials for chapter 1. The series name is ‘net barter terms of trade index (2000=100)’, and the series code is TT.PRI.MRCH.XD.WD. The time and cross-sectional dimensions of the sample are dictated by the requirement of at least 30 consecutive years of data. The countries included in the sample are Algeria, Argentina, Bolivia, Botswana, Brazil, Burundi, Cameroon, Central African Republic, Chile, China, Colombia, Comoros, Congo, Costa Rica, Cote d’Ivoire, Dominican Republic, Ecuador, Egypt, El Salvador, Ghana, Guatemala, Honduras, India, Indonesia, Jordan, Kenya, South Korea, Madagascar, Malaysia, Mauritania, Mauritius, Mexico, Morocco, Mozambique, Namibia, Pakistan, Paraguay, Peru, Philippines, Rwanda, Senegal, South Africa, Sudan, Thailand, Tunisia, Turkey, Uganda, Uruguay, Venezuela, Zambia, and Zimbabwe. The sample excludes Bangladesh, Gabon, Gambia, Lesotho, Panama, and Swaziland, countries for which the terms-of-trade data appears faulty.
and the associated $R$-squares. The cross-country means of $\rho$ and $\pi$ are 0.50 and 0.10, respectively. The parameter estimates display significant variation across countries. The cross-country standard deviations of $\rho$ and $\pi$ are 0.21 and 0.04, respectively. Also, the fit of the AR(1) process is modest, with a mean $R$-square of 0.30.

The terms-of-trade process evaluated at the mean parameter estimates is

$$\hat{\text{tot}}_t = 0.50 \hat{\text{tot}}_{t-1} + 0.10 \epsilon^{\text{tot}}_t.$$  

This terms-of-trade process implies an unconditional standard deviation of $\hat{\text{tot}}_t$ of $0.1155 = 0.1 / \sqrt{1 - 0.50^2}$, which means that a one-standard-error deviation in the terms of trade places $\hat{\text{tot}}_t$ 12 percent away from its trend value. Terms of trade shocks vanish relatively quickly. Their half life, defined as the value $t$ such that $\rho^t = 1/2$, is just one year.

### 7.2 The Terms-Of-Trade And The Trade Balance: Empirics

The effects of terms-of-trade shocks on the trade balance is an old subject of investigation. Does the trade balance improve or worsen when the terms of trade appreciate? Theoretically, the answer to this question is not unambiguous. Holding constant the quantities of goods imported and exported, an increase in the relative price of exports should improve the trade balance measured in terms of imports or exports. If, in addition, the higher relative price of exports induces domestic firms to produce more exportables, holding demands and production of importables constant, the improvement in the trade balance would be reinforced. On the other hand, there can be consumption substitution and income effects that go in the opposite direction. For example, if consumers substitute importable goods for exportable goods as the latter become more expensive, all other things equal, the trade balance would tend to deteriorate. Also, if the increase in the terms of
Table 7.1: Terms of Trade Process: Country-by-Country Estimates

\[ \hat{\sigma}_{t} = \rho \hat{\sigma}_{t-1} + \pi_{t}^{\text{tot}}, \quad \epsilon_{t}^{\text{tot}} \sim (0, 1) \]

<table>
<thead>
<tr>
<th>Country</th>
<th>( \rho )</th>
<th>( \pi )</th>
<th>( R^{2} )</th>
</tr>
</thead>
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<td>Zimbabwe</td>
<td>0.57</td>
<td>0.05</td>
<td>0.31</td>
</tr>
<tr>
<td>Mean</td>
<td>0.50</td>
<td>0.10</td>
<td>0.30</td>
</tr>
</tbody>
</table>
trade makes households feel richer, the demand for consumption goods will go up. If consumption is mostly concentrated on importable goods, the income effect will tend to deteriorate the trade balance. There may also be intertemporal effects. If an improvement in the terms of trade is perceived as the beginning of further future improvements, aggregate demand may experience a strong expansion, causing the trade balance to deteriorate. But if the terms of trade improvement is expected to die out quickly, households may choose to save much of the increased income it generates, causing the trade balance to improve. Finally, the improvement in the terms of trade could trigger a surge in investment in physical capital, which would also tend to deteriorate the trade balance in the short run. Again, the size of the investment surge will in general depend on the perceived persistence of the terms-of-trade improvement.

Which of these effects dominate? We begin by approaching this question empirically. Consider expanding the univariate autoregressive process (7.1) to include the trade balance. We continue to assume that the terms of trade follow a univariate autoregressive process

\[ \hat{\tau}_{ot} = \rho_1 \hat{\tau}_{ot-1} + u^1_t, \quad (7.2) \]

where \( u^1_t \) is a mean-zero i.i.d. innovation. The trade balance is assumed to evolve according to the following law of motion

\[ \hat{\delta}_t = \alpha_0 \hat{\delta}_t + \alpha_1 \hat{\delta}_{t-1} + \rho_2 \hat{\delta}_{t-1} + u^2_t, \quad (7.3) \]

where \( \hat{\delta}_t \) denotes a detrended measure of the trade balance, \( \alpha_0, \alpha_1, \) and \( \rho_2 \) are parameters, and \( u^2_t \) is a mean-zero i.i.d. innovation. Let \( u_t \equiv [u^1_t u^2_t]' \), and let the 2-by-2 matrix \( \Sigma \) be the variance-covariance matrix of \( u_t \). We estimate the above two equations by OLS equation by equation and country by country including a constant, which we omit for expositional convenience. The trade balance data comes from chapter 1. It is given by the difference between exports and imports and is expressed in units of GDP. As in that chapter, we construct a detrended measure of the trade
balance by first dividing its level by the quadratic trend of output and then removing a quadratic trend from the resulting ratio. As in the estimation of equation (7.1), we use annual data from 51 poor and emerging countries over the period 1980 to 2011 (see footnote 2). The estimation delivers values for $\rho_1$, $\alpha_0$, $\alpha_1$, $\rho_2$, and $\Sigma$ for each of the 51 countries in the panel. The estimates of $\rho_1$ are identical to those obtained under the one-equation model,

$$\rho_1 = \rho,$$

and presented in table 7.1.

The fact that equation (7.3) contains $\hat{\text{tot}}_t$ as a regressor implies that $u_1^t$ and $u_2^t$ are orthogonal

$$E u_1^t u_2^t = 0,$$

which implies that $\Sigma$ is a diagonal matrix. This restriction identifies the structural terms-of-trade shock. Specifically, assume that the nonstructural innovation $u_t$ is a linear combination of two structural orthogonal innovations

$$u_t = \Gamma \begin{bmatrix} \epsilon_1^t \\ \epsilon_2^t \end{bmatrix},$$

where $\Gamma$ is a 2-by-2 matrix of coefficients, $\epsilon_t \equiv [\epsilon_1^t \epsilon_2^t]'$ is a 2-by-1 i.i.d. random disturbance with mean 0 and identity variance-covariance matrix. Then we have that

$$\Sigma = \Gamma \Gamma'.$$

Without loss of generality, assume that $\epsilon_1^t$ is the terms-of-trade shock. This means that $\gamma_{11} \neq 0$ and $\gamma_{22} \neq 0$, where $\gamma_{ij}$ denotes the element $(i, j)$ of $\Gamma$. Then, the fact that $\Sigma$ is diagonal implies
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\[ \gamma_{12} = \gamma_{21} = 0. \]

Therefore, the only innovation that affects the terms of trade contemporaneously is \( \epsilon^1_t \). It is in this sense that the structural shock \( \epsilon^1_t \) represents a terms of trade shock. Further, the condition \( \gamma_{12} = \gamma_{21} = 0 \) identifies \( \gamma_{11} = \sqrt{\sigma_{11}} \) and \( \gamma_{22} = \sqrt{\sigma_{22}} \), where \( \sigma_{ij} \) denotes the \((i,j)\) element of \( \Sigma \).

The estimate of \( \gamma_{11} \) is identical to the estimate of \( \pi \) from the one-equation model (7.1),

\[ \gamma_{11} = \pi, \]

presented in table 7.1.

Rearranging terms in equations (7.2) and (7.3), we obtain the following structural vector autoregression (SVAR) system

\[
\begin{bmatrix}
\hat{\text{tot}}_t \\
\hat{tb}_t
\end{bmatrix}
= h_x
\begin{bmatrix}
\hat{\text{tot}}_{t-1} \\
\hat{tb}_{t-1}
\end{bmatrix}
+ \Pi \epsilon_t,
\]

(7.4)

where

\[
h_x \equiv \begin{bmatrix}
\rho & 0 \\
\alpha_0\rho + \alpha_1 & \rho_2
\end{bmatrix}
\quad \text{and} \quad
\Pi \equiv \begin{bmatrix}
\pi & 0 \\
\alpha_0\pi & \sqrt{\sigma_{22}}
\end{bmatrix}.
\]

We provide country-by-country estimates of \( h_x \) and \( \Pi \) in the file tot.svar.2eqn.mat.

The cross-country mean of the estimated SVAR system is

\[
\begin{bmatrix}
\hat{\text{tot}}_t \\
\hat{tb}_t
\end{bmatrix}
= \begin{bmatrix}
0.50 & 0 \\
-0.02 & 0.57
\end{bmatrix}
\begin{bmatrix}
\hat{\text{tot}}_{t-1} \\
\hat{tb}_{t-1}
\end{bmatrix}
+ \begin{bmatrix}
0.10 & 0 \\
0.008 & 0.032
\end{bmatrix}
\begin{bmatrix}
\epsilon^\text{tot}_t \\
\epsilon^\text{tb}_t
\end{bmatrix},
\]

(7.5)
The impact effect of a terms-of-trade shock on the trade balance is given by

$$\frac{\partial \hat{tb}_t}{\partial \epsilon_{tot}^t} = \pi_{21},$$

(7.6)

where $\pi_{ij}$ denotes element $(i, j)$ of the matrix $\Pi$. It follows that the sign of $\pi_{21}$ determines whether a terms-of-trade shock improves or worsens the trade balance on impact. The dynamic effects (i.e., the effects in the periods following a terms-of-trade shock) depend, in addition, on the values taken by the elements of $h_x$.

The fact that the mean estimate of $\pi_{21}$ is positive, implies that the impact effect of a terms-of-trade shock on the trade balance is positive on average,

$$\frac{\partial \hat{tb}_t}{\partial \epsilon_{tot}^t} = 0.008.$$ 

This expression says that an unexpected increase in the terms of trade equal in magnitude to one standard deviation of $\epsilon_{tot}^t$, that is, an unexpected improvement in the terms of trade of 10 percent ($= \pi \times 100$), improves the trade balance by 0.8 percent of (trend) GDP (or $\pi_{21} \times 100$ percent). The country-by-country estimates of the SVAR system yield 38 out of 51 cases in which $\pi_{21}$ is positive, implying a positive impact response of the trade balance to an increase in the terms of trade in 75 percent of the countries in the panel. Otto (2003), using data from 40 developing countries spanning the period 1960 to 1996, estimates a positive response of the trade balance to an improvement in the terms of trade in 36 out of the 40 countries in his sample.

Figure 7.1 displays the cross-country average impulse responses of the terms of trade and the trade balance to a ten-percent increase in the terms of trade. For each country, the impulse responses are constructed by setting $\epsilon_{tot}^0 = 10/\pi_{11}$ and then tracing the dynamics of $\hat{tot}_t$ and $\hat{tb}_t$ implied by equation (7.4) for years $t = 0, \ldots, 10$. The figure displays the average of these impulse
responses across the 51 countries in the sample. On impact, the trade balance improves by 0.77 percent of trend GDP. The effects of the terms-of-trade innovation vanish relatively quickly. Three years after the shock the trade balance is back to trend. Thus, according to the data analyzed here, the answer to the question posed at the beginning of this section is yes, the trade balance improves when the terms of trade appreciate.

7.3 The Terms of Trade and the Trade Balance: Simple Explanations, Old and New

More than half a century ago, Harberger (1950) and Laursen and Metzler (1950) formalized, within the context of a Keynesian model, the argument that rising terms of trade should be associated with an improving trade balance. This conclusion became known as the Harberger-Laursen-Metzler
(HLM) effect. This view remained more or less unchallenged until the early 1980s, when Obstfeld (1982) and Svensson and Razin (1983), using dynamic optimizing models of the current account, concluded that the effect of terms of trade shocks on the trade balance should depend crucially on the perceived persistence of the terms of trade. In their models a positive relation between terms of trade and the trade balance (i.e., the HLM effect) weakens as the terms of trade become more persistent and may even be overturned if the terms of trade are of a permanent nature. This view became known as the Obstfeld-Razin-Svensson (ORS) effect. Let us look at the HLM and ORS effects in some more detail.

7.3.1 The Harberger-Laursen-Metzler Effect

A simple way to obtain a positive relation between the terms of trade and the trade balance in the context of a Keynesian model is to start with the national accounting identity

\[ y_t = c_t + g_t + i_t + x_t - m_t, \]

where \( y_t \) denotes output, \( c_t \) denotes private consumption, \( g_t \) denotes public consumption, \( i_t \) denotes private investment, \( x_t \) denotes exports, and \( m_t \) denotes imports. Consider the following behavioral equations defining the dynamics of each component of aggregate demand. Public consumption and private investment are assumed to be independent of output. For simplicity, we will assume that these two variables are constant over time and given by

\[ g_t = \bar{g} \]

and

\[ i_t = \bar{i}, \]
respectively, where $\overline{g}$ and $\overline{i}$ are parameters. Consumption is assumed to be an increasing linear function of output

$$c_t = \overline{c} + \alpha y_t,$$

where $\alpha \in (0, 1)$ and $\overline{c} > 0$ are parameters. Imports are assumed to be proportional to output,

$$m_t = \mu y_t,$$

with $\mu \in (0, 1)$. In the jargon of the 1950s, the parameters $\alpha$ and $\mu$ are referred to as the marginal propensities to consume and import, respectively, whereas the term $\overline{c} + \overline{g} + \overline{i}$ is referred to as the autonomous component of domestic absorption. Output and all components of aggregate demand are expressed in terms of import goods. The quantity of goods exported in period $t$ is denoted by $q_t$. Thus, the value of exports in terms of importables, $x_t$, is given by

$$x_t = tot_t q_t,$$

where $tot_t$ denotes the terms of trade. The terms of trade are assumed to evolve exogenously, and the quantity of goods exported, $q_t$, is assumed to be constant and given by

$$q_t = \overline{q},$$

where $\overline{q}$ is a positive parameter. Using the behavioral equations to eliminate $c_t$, $i_t$, $g_t$, $x_t$, and $m_t$ from the national income identity, and solving for output yields

$$y_t = \frac{\overline{c} + \overline{g} + \overline{i} + tot_t \overline{q}}{1 + \mu - \alpha}.$$
The object \( 1/(1+\mu-\alpha) \) is known as the expenditure multiplier. Letting \( t_b \equiv x_t - m_t \) denote the trade balance, we can write

\[
t_b = \frac{1 - \alpha}{1 + \mu - \alpha} q - \frac{\mu(\bar{c} + \bar{f} + \bar{g})}{1 + \mu - \alpha},
\]

which implies that

\[
\frac{\partial t_b}{\partial \text{tot}_t} = \frac{1 - \alpha}{1 + \mu - \alpha} \bar{q}.
\]

According to this expression, an improvement in the terms of trade gives rise to an improvement in the trade balance. This result is known as the HLM effect. The HLM effect is stronger the larger is the volume of exports, \( q \), the smaller is the marginal propensity to import, \( \mu \), and the smaller is the marginal propensity to consume \( \alpha \). The reason the sensitivity of the trade balance with respect to the terms of trade is decreasing in \( \mu \) is that a higher value of \( \mu \) weakens the endogenous expansion in aggregate demand to an exogenous increase in exports, as a larger fraction of income is used to buy foreign goods. Similarly, a larger value of \( \alpha \) reduces the sensitivity of the trade balance to the terms of trade, because it exacerbates the endogenous response of aggregate demand to a terms-of-trade shock through private consumption.

The positive relation between the terms of trade and the trade balance predicted by the present model is qualitatively in line with the empirical estimates of the impact effect of a terms-of-trade shock on the trade balance obtained in the previous section (see figure 7.1). Thus, the data appears to lend support to the HLM effect.

In the context of this model, the magnitude of the effect of a terms-of-trade shock on the trade balance is independent of whether terms of trade shocks are permanent or temporary in nature. The feature of the model driving this result is the assumption that the marginal propensity to consume, \( \alpha \), is independent of the persistence of terms-of-trade shocks. Suppose, by contrast, that the marginal propensity to consume is increasing in the persistence of the terms-of-trade shock.
shock. That is, assume that the marginal propensity to consume is given by \( \alpha(\rho) \), with \( \alpha'(\rho) > 0 \), where, as before, \( \rho \) denotes the persistence of the terms of trade. This assumption is intuitively sound. It makes sense that households save a large fraction of temporary income shocks to smooth consumption over time and a small fraction of persistent income shocks. In this case, we have that

\[
\frac{\partial t_b}{\partial \text{tot}} = \frac{1 - \alpha(\rho)}{1 + \mu - \alpha(\rho)} \tilde{q},
\]

which implies that the sensitivity of the trade balance to terms-of-trade shocks is smaller when the persistence of the latter is higher. The main difference between the Harberger-Laursen-Metzler and the Obstfeld-Razin-Svensson approaches is that in the latter the marginal propensity to consume is endogenously determined by utility-maximizing agents and turns out to be increasing in \( \rho \). We turn to this approach next.

### 7.3.2 The Obstfeld-Razin-Svensson Effect

The ORS effect is cast within a dynamic optimizing theoretical framework that differs fundamentally from the reduced-form Keynesian model we used to derive the HLM effect. Consider the small, open, endowment economy studied in chapter 2. This is an economy inhabited by an infinitely-lived representative household with preferences described by the intertemporal utility function

\[
-\frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t (c_t - \bar{c})^2,
\]

where \( c_t \) denotes consumption, \( \beta \in (0, 1) \) is a subjective discount factor, and \( \bar{c} \) is a positive parameter defining the satiation point. Assume that the consumption good, \( c_t \), is imported, and that each period the household is endowed with 1 unit of exportable goods, from which it derives no utility. As before, let \( \text{tot}_t \) denote the international relative price of exportable goods in terms of
importable goods, or the terms of trade. Then, the household’s unit endowment expressed in terms of importable goods is simply given by \( \text{tot}_t \). The household faces the sequential budget constraint

\[
d_t = (1 + r)d_{t-1} + c_t - \text{tot}_t,
\]

where \( d_t \) denotes the debt position assumed in period \( t \) and due in period \( t+1 \) expressed in terms of import goods and \( r > 0 \) denotes a constant world interest rate and is assumed to satisfy \( \beta(1+r) = 1 \). Finally, the household is subject to the no-Ponzi-game constraint \( \lim_{j \to \infty} (1 + r)^{-j}E_t d_{t+j} \leq 0 \). The economy is small in world product markets, so it takes the evolution of \( \text{tot}_t \) as exogenous. Assume that \( \text{tot}_t \) follows the AR(1) process

\[
\text{tot}_t = \rho \text{tot}_{t-1} + \epsilon^\text{tot}_t,
\]

with \( \rho \in (0, 1) \). The model is therefore identical to the stochastic-endowment economy studied in chapter 2, with \( \text{tot}_t \) taking the place of \( y_t \). We can then use the following result derived in chapter 2 relating the equilibrium trade balance to past debt and the current terms of trade,

\[
\text{tb}_t = rd_{t-1} + \frac{1 - \rho}{1 + r - \rho} \text{tot}_t,
\]

which implies that

\[
\frac{\partial \text{tb}_t}{\partial \text{tot}_t} = \frac{1 - \rho}{1 + r - \rho}.
\] (7.7)

According to this expression, an increase in the terms of trade in period \( t \) produces an improvement in the trade balance in period \( t \). The intuition behind this result is that in response to a mean-reverting increase in export income stemming from an improvement in the terms of trade households consume only part of the additional income and save the rest to smooth consumption over time.
As a result, consumption increases by less than income leading to an improvement in the trade balance.

Importantly, equation (7.7) implies that the effect of terms-of-trade shocks on the trade balance is decreasing in $\rho$. This prediction is known as the ORS effect. Intuitively, consumption-smoothing households have more incentives to save in response to temporary shocks than in response to persistent shocks.

### 7.3.3 Testing for the ORS Effect

Is the ORS effect borne out in the data? If so, we should observe that countries experiencing more persistent terms-of-trade shocks display a weaker response of the trade balance to innovations in the
terms of trade. Figure 7.2 plots the impact effect of a ten-percent terms-of-trade improvement on the trade balance as a function of the estimated persistence of the terms-of-trade shock, $\rho$, for the panel of 51 emerging and poor countries used in this chapter. Each circle in the figure corresponds to the pair $(\rho, \pi_{21}/\pi_{11} \times 10)$ for a particular country. Recall that according to the SVAR model, $\frac{\partial b}{\partial \epsilon_{tot}} = \pi_{21}$ (see equation (7.6)). We scale by $10/\pi_{11}$ to capture the impact response of the trade balance to a ten-percent improvement in the terms of trade in each country. The cloud of circles displays no discernible decreasing pattern, suggesting that the data does not lend strong support to the ORS effect.

7.3.4 The ORS Effect in the SOE-RBC Model

The ORS effect is also present in models with endogenous labor supply and capital accumulation. A simple way to show this is to modify the SOE-RBC model of chapter 4 by assuming again that households do not consume the good they produce. In this case, the productivity shock $A_t$ can be interpreted as a terms-of-trade shock. The trade balance is then given by

$$tb_t = tot_t F(k_t, h_t) - c_t - i_t - \Phi(k_{t+1} - k_t),$$

which is equation (4.20) with $tot_t$ taking the place of $A_t$. The main difference between this framework and the endowment economy analyzed in the previous section for the purpose of understanding the effect of terms-of-trade shocks on the trade balance is the presence of investment in physical capital. Because in the present model economy terms-of-trade shocks are identical to productivity shocks, a persistent increase in the terms of trade induces firms to increase the stock of capital to take advantage of the persistent expected increase in the value of the marginal product of capital (i.e., the marginal product of capital expressed in terms of imports, $tot_{t+j}F_k(k_{t+j}, h_{t+j})$ for $j > 0$). The increase in the desired stock of capital induces a surge in the demand for (imported) investment.
goods, which tends to deteriorate the trade balance. This effect is stronger when the terms-of-trade shock is perceived to be more persistent. Thus, the introduction of capital accumulation into the model strengthens the ORS effect.

One way to visualize the presence of the ORS effect in the SOE-RBC model of chapter 4 is to look at impulse responses. Figure 7.3 displays the impulse responses of the trade balance to a one-percent increase in the terms of trade for three values of $\rho$, 0, 0.25, and 0.5. In constructing the figure, the logarithm of the terms of trade is assumed to follow the univariate AR(1) process given in equation (7.1), and all parameters of the model other than $\rho$ are calibrated using the values shown in table 4.1. The figure shows that the impact response of the trade balance decreases with $\rho$, implying that the more persistent the terms-of-trade shock is the smaller the response of the trade balance is. Actually, the initial response of the trade balance is negative for the largest value of $\rho$ considered, 0.5. In this case, the surge in investment caused by the improvement in the terms of trade is so strong that the trade balance deteriorates. After period 0, the response of the trade balance ceases to be decreasing in $\rho$. In fact, contrary to what the ORS effect states, the largest response is observed for the most persistent specification of the terms of trade.

The different picture that emerges from analyzing the impact response versus the response one or more periods later suggests that a better way to test for the ORS effect is to look at conditional correlations between the terms of trade and the trade balance. Accordingly, figure 7.4 plots with a solid line the predicted conditional correlation of the trade balance with the terms of trade as a function of $\rho$. It shows that this correlation is positive and close to unity for negative values of $\rho$ and falls as this parameter increases, turning negative as $\rho$ approaches unity. Thus, the impact effect of a terms-of-trade shock, shown in figure 7.3, dominates and determines the sign of the conditional correlation. It follows that in the SOE-RBC model, the ORS effect is present not only in the impact effect of a terms of trade shock but more generally in the conditional comovement between the trade balance and the terms of trade.
Figure 7.3: Impulse Response of the Trade Balance to a Terms-of-Trade Shock Implied by the SOE-RBC Model

![Figure 7.3](image)

Note. Response to a one-percent increase in the terms of trade. The calibration of the model is as shown in table 4.1.

Figure 7.4: Correlation-Based Test of the ORS Effect: Conditional Correlation of the Trade Balance with the Terms of Trade and Persistence of Terms-of-Trade Shocks

![Figure 7.4](image)

Note. Each circle corresponds to one country.
To see whether the conditional correlation between the trade balance and the terms of trade also displays an ORS effect in the data, figure 7.4 plots, with circles, country-by-country estimates of the correlation of the trade balance with the terms of trade as a function of the persistence of the terms of trade. In comparing model and data, the first thing to notice is that the model is driven by a single shock, namely, terms-of-trade shocks. Thus, the empirical correlations must be computed conditional on terms of trade shocks only. To this end, we use country-by-country estimates of the SVAR (7.4) and, after shutting off the innovation $\epsilon^b_t$ (by setting $\pi_{22} = 0$), compute the correlation between the trade balance and the terms of trade. Figure 7.4 shows that the cloud of circles does not resemble the declining pattern predicted by the SOE-RBC model. We therefore conclude that the ORS effect is not strongly supported by the data regardless of whether it is measured by the impact effect of a terms-of-trade shock on the trade balance or by the conditional correlation between the terms of trade and the trade balance.

7.4 How Important Are Terms-of-Trade Shocks?

Movements in terms of trade are generally believed to be an important source of business cycles. But how important? This section addresses this question by providing an empirical measure of the contribution of terms of trade shocks to aggregate fluctuations based on an SVAR model. Section 7.8 addresses the same question in the context of a dynamic stochastic general equilibrium (DSGE) model. As we will see, the two approaches arrive at different conclusions.

Consider expanding the SVAR system (7.4) to allow for other macroeconomic indicators of interest. Specifically, in addition to the terms of trade and the trade balance, let us include real GDP per capita, $y_t$, real private consumption per capita, $c_t$, and real gross investment per capita,
\( i_t \). Let
\[
x_t = \begin{bmatrix} x^1_t \\ x^2_t \end{bmatrix}, \quad \text{with } x^1_t = \text{tot}_t \quad \text{and} \quad x^2_t = \begin{bmatrix} \tilde{t}b_t \\ \tilde{y}_t \\ \tilde{c}_t \\ \tilde{i}_t \end{bmatrix}.
\]

A hat on \( \text{tot}_t, \ y_t, \ c_t \), and \( i_t \) denotes log-deviations from a quadratic time trend. As before, we construct \( \tilde{t}b_t \) by first dividing the trade balance by the trend of output and then removing a quadratic trend from this ratio. The proposed SVAR structure is the natural extension of the bivariate SVAR presented in section 7.2. Accordingly, we continue to assume that the terms of trade follow a univariate autoregressive process of order one,
\[
x^1_t = \rho_1 x^1_{t-1} + u^1_t, \quad (7.8)
\]
where \( u^1_t \) is an i.i.d. random variable with mean zero. Let the law of motion of \( x^2_t \) be
\[
x^2_t = \alpha_0 x^1_t + \alpha_1 x^1_{t-1} + \rho_2 x^2_{t-1} + u^2_t, \quad (7.9)
\]
where \( \alpha_0 \) and \( \alpha_1 \) are 4-by-1 vectors of coefficients and \( \rho_2 \) is a 4-by-4 matrix of coefficients. The variable \( u^2_t \) is a 4-by-1 random vector with mean zero and full-rank variance-covariance matrix \( \Sigma \). Let nonstructural shocks \( u^1_t \) and \( u^2_t \) be related to structural innovations as follows
\[
\begin{bmatrix} u^1_t \\ u^2_t \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} \epsilon^1_t \\ \epsilon^2_t \end{bmatrix}.
\]

Without loss of generality, define \( \epsilon^1_t \) as a terms-of-trade shock and \( \epsilon^2_t \) as four other unspecified structural shocks. This means that \( \gamma_{11} \) cannot be nil, for if \( \gamma_{11} \) was zero, we would have that the
terms of trade shock does not affect the terms of trade contemporaneously, which makes no sense. The vector
\[ \epsilon_t \equiv \begin{bmatrix} \epsilon_1^t \\ \epsilon_2^t \end{bmatrix} \]
is assumed to be i.i.d. with mean zero and identity variance-covariance matrix.

The fact that \( x_t^1 \) appears as a regressor in equation (7.9) implies that \( u_t^1 \) and \( u_t^2 \) are orthogonal. Thus, we have that
\[ 0 = E[u_t^1 u_t^2]' = \gamma_{11} E[\epsilon_t^1]^2 \gamma_{21}' + \gamma_{12} E[\epsilon_t^2 \epsilon_t^2]' \gamma_{22}' . \]
The first term on the right-hand side can only be zero if \( \gamma_{21} = 0 \). In turn, this implies that \( u_t^2 = \gamma_{22} \epsilon_t^2 \) and thus \( \Sigma = \gamma_{22} \gamma_{22}' \). Since \( \Sigma \) is full rank, we have that \( \gamma_{22} \) must also be full rank. Now the second term on the right-hand side implies that \( \gamma_{12} \gamma_{22}' = 0 \). Since \( \gamma_{22} \) is full rank, we have that \( \gamma_{12} = 0 \). Summarizing, we have that
\[ u_t^1 = \gamma_{11} \epsilon_t^1, \]
and
\[ u_t^2 = \gamma_{22} \epsilon_t^2. \]

It follows that equation (7.8) is identical to equation (7.1), so that \( \rho_1 = \rho \) and \( \gamma_{11} = \pi \). We have already obtained estimates of \( \rho \) and \( \pi \) for each of the 51 countries in the data panel used in this chapter, see table 7.1. We obtain estimates of \( \alpha_0, \alpha_1, \) and \( \rho_2 \) by OLS equation by equation and country by country. These regressions also deliver an estimate of the matrix \( \Sigma \) for each country. Because the focus of the present analysis is to identify terms-of-trade shocks, the estimate of \( \gamma_{22} \) can be taken to be any matrix such that \( \gamma_{22} \gamma_{22}' = \Sigma \). Without loss of generality, we set \( \gamma_{22} \) equal to the lower-triangular Cholesky decomposition of \( \Sigma \).
Rearranging terms in (7.8) and (7.9) we can then write the SVAR system as

\[ x_t = h_x x_{t-1} + \Pi \epsilon_t, \]

(7.10)

where

\[ h_x \equiv \begin{bmatrix} \rho & 0 \\ \alpha_0 \rho + \alpha_1 & \rho_2 \end{bmatrix} \quad \text{and} \quad \Pi \equiv \begin{bmatrix} \pi & 0 \\ \alpha_0 \pi & \gamma_{22} \end{bmatrix}. \]

Country-by-country estimates of \( h_x \) and \( \Pi \) are available in the file tot_svar_5eqn.mat. These estimates vary substantially across countries, which makes it inadvisable to derive predictions from either the cross-country average of the matrices \( h_x \) and \( \Pi \) or from a panel estimate of the SVAR imposing cross-country constancy of \( h_x \) and \( \Pi \). Shortly, we will argue that both such approaches introduce severe bias in the estimated variance decomposition. Thus, the approach adopted here, as in the previous section, is to first compute predictions of the SVAR model at the country level and then average them across countries.

Figure 7.5 displays the impulse responses of the five variables included in the SVAR system (7.10) to a ten-percent increase in the terms of trade. Impulse responses are computed as averages across countries point by point. The top left panel displays the response of the terms of trade themselves (which is identical to that shown in the top-left panel of figure 7.1) and the top-right panel the response of the trade balance (which is similar, but need not be identical to that shown in the top-right panel of figure 7.1). The novelty of the figure is in the remaining panels. The improvement in the terms of trade causes an expansion in aggregate activity. Specifically, the 10 percent increase in the terms of trade causes an increase of 0.34 percent in GDP. Investment displays a somewhat larger expansion, albeit with a one-year delay. Private consumption contracts on impact and then swiftly returns to its trend path.

A common way to gauge the importance of a particular shock in driving business cycles is
Figure 7.5: Impulse Response to A Ten-Percent Increase in the Terms of Trade: SVAR Evidence

Note. For each variable, its impulse response is computed as the average impulse response across countries period by period.
Table 7.2: Share of Variance Explained by Terms of Trade Shocks: Country-Level SVAR Evidence

<table>
<thead>
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<th>Country</th>
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<th>β</th>
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Mean: 100, 18, 12, 13, 13
Median: 100, 12, 10, 11, 10
Std. Dev.: 0, 15, 12, 12, 11

Using Cross-Country
Mean of α, β, γ, δ: 100, 7, 3, 1, 2
Panel Estimation: 100, 4, 1, 1

Note: Shares are expressed in percent.
to compute the fraction of the variances of indicators of interest explained by it. The variances of the variables included in the SVAR conditional on terms-of-trade shocks are computed as the variances resulting from shutting off all shocks other than the terms-of-trade shock by setting the 4-by-4 matrix $\pi_{22} = \emptyset$. Then, each of these conditional variances is divided by the corresponding unconditional variance as implied by the SVAR model to obtain a share. Table 7.2 displays the share of the variances of the five variables in the SVAR explained by terms-of-trade shocks. The estimates reported in the table indicate that on average terms-of-trade shocks explain between 10 and 12 percent of the variances of output, consumption, investment, and the trade balance. There is a large dispersion of estimates across countries. The cross-country standard deviations of the estimated variance shares are about as high as the cross-country medians. The bottom two rows of the table provide an idea of the downward bias that results from computing variance shares using either a panel estimation of the SVAR or the cross-country mean estimates of $h_x$ and $\Pi$. Under these alternative estimation approaches, the share of the variance of output attributed to terms of trade shocks falls to 3 percent or less.

The results reported in table 7.2 are in line with the work of Aguirre (2011) who finds that terms of trade shocks explain no more than 5 percent of movements in output at a quarterly frequency over the period 1994:1 to 2009:2 in a panel of 15 emerging countries. Broda (2004) uses annual data from 75 developing countries over the period 1973 to 1996 to study the importance of terms-of-trade shocks conditional on the exchange-rate regime. For floaters, he finds that terms-of-trade shocks explain less than 3 percent of the variance of output, whereas for peggers this fraction rises to 21 percent.\(^3\)

Taken together, the available empirical evidence stemming from SVAR models suggests that the contribution of terms-of-trade shocks to business-cycle fluctuations in emerging and poor economies

\(^3\)Chapter 9 develops an open-economy model with downward nominal wage rigidity that captures Broda’s finding of a larger output effect of terms-of-trade shocks under currency pegs than under (optimal) floats.
is modest and exhibits large cross-sectional variation. What do theoretical models have to say about this? This is the subject of the following sections.

7.5 The MX Model

Earlier in this chapter, in section 7.3.2, we argued that one can use the SOE-RBC model of chapter 4 to provide a story of the contribution of the terms of trade to aggregate fluctuations. We did so by interpreting the productivity shock as a terms-of-trade shock. One unrealistic implication of this way of modeling terms-of-trade shocks is the implicit assumption of an extreme degree of production and absorption specialization, whereby the entire GDP is exported and the totality of domestic absorption (the sum of consumption and investment) is imported. Under this assumption, the trade share, the ratio of exports plus imports to GDP, would be about 200 percent on average across countries. In reality, most countries export only a fraction of GDP and import only a fraction of domestic absorption. Table 1.1 in chapter 1 documents that the average trade share in emerging and poor countries is about 40 percent. In this section, we capture this fact by expanding the SOE-RBC model of chapter 4 to allow for two sectors of production, one producing importable goods and the other producing exportable goods, and by allowing both importable and exportable goods to be useful in the production of final consumption and investment goods. We refer to this model as the importable-exportable model or simply as the MX model.

We continue to maintain the assumption that the country is small in the sense that it takes international prices of goods and financial assets as exogenously given. In particular, the terms of trade continues to be an exogenous variable for the small open economy.
7.5.1 Households

Consider an economy populated by a large number of households with preferences defined over streams of consumption and labor and described by the utility function

\[ E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, h_t^m, h_t^x), \]

where \( c_t \) denotes consumption, \( h_t^m \) denotes hours worked in the importable sector, and \( h_t^x \) hours worked in the exportable sector. Households maximize their lifetime utility subject to the sequential budget constraint

\[ c_t + i_t^m + i_t^x + \Phi(k_{t+1}^m - k_t^m) + \Phi(k_{t+1}^x - k_t^x) + d_t = \frac{d_{t+1}}{1 + r_t} + w_t^m h_t^m + w_t^x h_t^x + u_t^m k_t^m + u_t^x k_t^x, \]

where \( i_t^j, k_t^j, w_t^j, \) and \( u_t^j \) denote respectively, gross investment, the capital stock, the real wage, and the rental rate of capital in sector \( j \), for \( j = m, x \), with the superscript \( m (x) \) denoting the sector producing importable (exportable) goods. The variable \( d_t \) denotes the stock of debt due in period \( t \), and \( r_t \) denotes the interest rate on debt held from period \( t \) to \( t + 1 \). The function \( \Phi(\cdot) \) introduces capital adjustment costs and is assumed to be increasing and convex and to satisfy \( \Phi(0) = \Phi'(0) = 0 \). Consumption, investment, wages, rental rates, debt, and capital adjustment costs are all expressed in units of consumption. The capital stocks obey the familiar laws of motion

\[ k_{t+1}^m = (1 - \delta)k_t^m + i_t^m \quad (7.11) \]

and

\[ k_{t+1}^x = (1 - \delta)k_t^x + i_t^x. \quad (7.12) \]

Two features of this economy aim at slowing down the speed of sectoral reallocation of factors
of production. First, the fact that sector-specific hours worked enter as separate arguments in the utility function allows for imperfect substitutability of labor across different activities. Second, the introduction of two capital adjustment cost functions, one for each sectoral capital stock, implies costs of moving capital from one sector to the other. These assumptions capture the fact that both labor and capital might be sector specific, and that moving these factors of production across sectors might entail costs (e.g., retooling in the case of labor, reconditioning in the case of capital).

Use the laws of motion for capital (7.11) and (7.12) to eliminate $i^m_t$ and $i^x_t$ from the sequential budget constraint. Then, letting $\lambda_t\beta_t$ denote the Lagrange multiplier associated with the household’s budget constraint, we have that the first-order optimality conditions with respect to $c_t$, $h^m_t$, $h^x_t$, $d_{t+1}$, $k^m_{t+1}$, and $k^x_{t+1}$ are, respectively,

$$U_1(c_t, h^m_t, h^x_t) = \lambda_t,$$  \hspace{1cm} (7.13)

$$- U_2(c_t, h^m_t, h^x_t) = \lambda_t w^m_t,$$  \hspace{1cm} (7.14)

$$- U_3(c_t, h^m_t, h^x_t) = \lambda_t w^x_t,$$  \hspace{1cm} (7.15)

$$\lambda_t = \beta (1 + r_t) E_t \lambda_{t+1},$$  \hspace{1cm} (7.16)

$$\lambda_t \left[ 1 + \Phi'(k^m_{t+1} - k^m_t) \right] = \beta E_t \lambda_{t+1} \left[ u^m_{t+1} + 1 - \delta + \Phi'(k^m_{t+2} - k^m_{t+1}) \right],$$  \hspace{1cm} (7.17)

and

$$\lambda_t \left[ 1 + \Phi'(k^x_{t+1} - k^x_t) \right] = \beta E_t \lambda_{t+1} \left[ u^x_{t+1} + 1 - \delta + \Phi'(k^x_{t+2} - k^x_{t+1}) \right].$$  \hspace{1cm} (7.18)

These optimality conditions are similar to those pertaining to the one-sector SOE-RBC model of chapter 4 (see equations (4.7), (4.8), (4.24), and (4.30)), except that now there are sector-specific conditions for capital and labor. It is clear from equations (7.17) and (7.18) that in the steady state the rental rates of capital will be the same in both sectors. This is because of the assumption
of no marginal capital adjustment costs at constant levels of capital $\Phi'(0) = 0$ and a common depreciation rate. This is not the case with labor. In general the model will feature sectoral wage differentials that persist even in the steady state.

7.5.2 Production of Final Goods

The final good is produced using importable and exportable goods as intermediate inputs. Profits are given by

$$A(a^m_t, a^x_t) - p^m_t a^m_t - p^x_t a^x_t,$$

where $A(a^m_t, a^x_t)$ denotes the quantity of final goods produced, $a^m_t$ and $a^x_t$ denote, respectively, the domestic absorptions of importable and exportable goods and $p^m_t$ and $p^x_t$ denote the relative prices of importables and exportables in terms of units of the final good, which is taken to be the numeraire. The function $A(a^m, a^x)$ denotes an aggregation technology and is often referred to as the Armington aggregator after the work of Armington (1969). We assume that the Armington aggregator is increasing, concave, and homogeneous of degree one. Firms in this sector are assumed to behave competitively in intermediate and final goods markets. Then, profit maximization implies that

$$A_1(a^m_t, a^x_t) = p^m_t,$$  \hspace{1cm} (7.19)

and

$$A_2(a^m_t, a^x_t) = p^x_t.$$  \hspace{1cm} (7.20)

These two optimality conditions, together with the assumption of linear homogeneity of the aggregator function, imply that firms make zero profits every period.
7.5.3 Production of Importable and Exportable Goods

Importable and exportable goods are produced with capital and labor via the technologies

\[ y^m_t = F^m(k^m_t, h^m_t) \]  (7.21)

and

\[ y^x_t = F^x(k^x_t, h^x_t), \]  (7.22)

where \( y^j_t \) denotes output in sector \( j = m, x \). The production functions \( F^j, j = m, x \), are assumed to be increasing, concave, and homogeneous of degree one. Profits of firms producing exportable and importable goods are given by

\[ p^j_t F^j(k^j_t, h^j_t) - w^j_t h^j_t - u^j_t k^j_t, \]

for \( j = m, x \). Firms are assumed to behave competitively in product and factor markets. Then, the first-order profit maximization conditions are

\[ p^m_t F^m_1(k^m_t, h^m_t) = u^m_t, \]  (7.23)

\[ p^m_t F^m_2(k^m_t, h^m_t) = w^m_t, \]  (7.24)

\[ p^x_t F^x_1(k^x_t, h^x_t) = u^x_t, \]  (7.25)

and

\[ p^x_t F^x_2(k^x_t, h^x_t) = w^x_t. \]  (7.26)

These efficiency conditions and the assumption of linear homogeneity of the production technologies imply that firms in both sectors make zero profits at all times.
7.5.4 Equilibrium

In equilibrium the demand for final goods must equal the supply of this type of good

\[ c_t + i^m_t + i^x_t + \Phi(k^m_{t+1} - k^m_t) + \Phi(k^x_{t+1} - k^x_t) = A(a^m_t, a^x_t). \]  

(7.27)

Imports, denoted \( m_t \), are defined as the difference between the domestic absorption of importables and output in the importable sector,

\[ m_t = p^m_t(a^m_t - y^m_t). \]  

(7.28)

The price of importables appears on the right-hand side of this definition because \( m_t \) is expressed in units of final goods, whereas \( y^m_t \) and \( a^m_t \) are expressed in units of importable goods. Similarly, exports, denoted \( x_t \), are given by the difference between output in the exportable sector and the domestic absorption of exportables

\[ x_t = p^x_t(y^x_t - a^x_t). \]  

(7.29)

Like imports, exports are measured in terms of final goods. Combining these two definitions, the household’s budget constraint, and the definitions of profits in the final- and intermediate-good markets, and taking into account that firms make zero profits at all times, yields the following expression linking the growth rate of external debt to interest payments and the trade balance

\[ \frac{d_{t+1}}{1 + r_t} = d_t + m_t - x_t. \]  

(7.30)

To ensure a stationary equilibrium process for external debt up to first order, we assume that
the country interest rate premium is debt elastic as in section 4.1.1 of chapter 4,

\[ r_t - r^* = p(d_{t+1}), \]  

(7.31)

where \( r^* \) denotes the world interest rate assumed to be constant. The country premium, \( p(d) \), is assumed to be increasing.

The terms of trade are defined as the relative price of exportable goods in terms of importable goods, that is,

\[ tot_t = \frac{p^x_t}{p^m_t}. \]  

(7.32)

As in the empirical analysis conducted earlier in this chapter, we assume that the country is small in international product markets and therefore takes the evolution of the terms of trade as given. Also in line with that empirical analysis, we assume an AR(1) structure for the law of motion of the logarithm of the terms of trade

\[ \ln \left( \frac{tot_t}{tot} \right) = \rho \ln \left( \frac{tot_{t-1}}{tot} \right) + \pi \epsilon^tot_t, \]  

(7.33)

where \( tot \) denotes the steady state value of \( tot_t \), \( \epsilon^tot_t \) is a white noise with mean zero and unit variance, and \( \rho \in (-1, 1) \) and \( \pi > 0 \) are parameters.

A competitive equilibrium is then a set of processes \( c_t, h^m_t, h^x_t, d_{t+1}, i^m_t, i^x_t, k^m_{t+1}, k^x_{t+1}, a^m_t, a^x_t, p^x_t, y^m_t, y^x_t, p^m_t, r_t, w^m_t, w^x_t, u^m_t, u^x_t, \lambda_t, m_t, x_t, \) and \( tot_t \) satisfying equations (7.11)-(7.33), given initial conditions \( k^m_0, k^x_0, d_0, \) and \( tot_{-1} \), and the stochastic process \( \epsilon^tot_t \).
7.6 Parameterization of the MX Model

In this section we introduce functional forms for preferences and technologies and present a strategy to parameterize the MX model. An innovation of the approach taken here is the use of data on trade in value added and on sectoral GDP to calibrate the share parameter of the Armington aggregator and sectoral relative productivities.

7.6.1 Preferences and Technologies in the MX Model

We assume that the utility function takes a CRRA form in a quasi linear composite of consumption, effort allocated to the importable sector, and effort allocated to the exportable sector,

\[ U(c, h^m, h^x) = \left[ \frac{c - G(h^m, h^x)}{1-\sigma} \right]^{1-\sigma} - 1, \]

where \( \sigma > 0 \) is a parameter. In turn, the composite \( G(h^m, h^x) \) is assumed to be separable in sectoral employment

\[ G(h^m, h^x) = \frac{(h^m)^{\omega_m}}{\omega_m} + \frac{(h^x)^{\omega_x}}{\omega_x}, \]

where \( \omega_m, \omega_x > 0 \) are parameters. This preference specification implies that the income elasticity of the labor supply is zero for both types of labor and that the wage elasticities of labor supply are \( 1/(\omega_m - 1) \) and \( 1/(\omega_x - 1) \) in the importable and exportable sectors, respectively.

We assume Cobb-Douglas technologies for the production of importable and exportable goods

\[ F^m(k^m, h^m) = A^m (k^m)^{\alpha_m} (h^m)^{1-\alpha_m} \]

and

\[ F^x(k^x, h^x) = A^x (k^x)^{\alpha_x} (h^x)^{1-\alpha_x} \]
where $A^m, A^x > 0$ and $\alpha_m, \alpha_x \in (0, 1)$ are parameters.

As in the baseline SOE-RBC model of chapter 4, the debt-elastic interest-rate premium and the capital adjustment cost functions take the forms

$$p(d) = \overline{p} + \psi(e^{d-d} - 1),$$

and

$$\Phi(x) = \frac{\phi}{2}x^2,$$

where $\psi, \phi > 0$, $\overline{p} > 0$, and $\overline{d}$ are parameters. The parameter $\overline{p}$ is new and captures the fact that on average emerging and poor countries pay a positive interest-rate premium. In chapter 4, this parameter is set to 0, because Canada is assumed to be able to borrow at the world interest rate.

We assume that the Armington aggregator takes a constant-elasticity-of-substitution (CES) form

$$A(a^m_t, a^x_t) = \left[\chi (a^m_t)^{1-\mu} + (1 - \chi) (a^x_t)^{1-\mu}\right]^{\frac{1}{1-\mu}},$$

with $\mu > 0$ and $\chi \in (0, 1)$. The parameter $\mu$ represents the intratemporal elasticity of substitution between exportable and importable absorption and is often referred to as the trade elasticity. To see why $\mu$ can be interpreted as an elasticity of substitution, combine the efficiency conditions (7.19) and (7.20) and evaluate the result using the assumed CES functional form of the aggregator function to obtain

$$\frac{a^x_t}{a^m_t} = \left(\frac{1 - \chi}{\chi}\right)^\mu \left(\frac{p^x_t}{p^m_t}\right)^{-\mu}. \quad (7.34)$$

From this expression, it follows that

$$\frac{\partial \ln(a^x_t/a^m_t)}{\partial \ln(p^x_t/p^m_t)} = -\mu.$$
According to this expression, a one-percent increase in the relative price of exportables (i.e., a one percent increase in the terms-of-trade) induces a $\mu$-percent fall in the relative absorption of exportables, as agents substitute importables for exportables in total absorption. As the trade elasticity converges to unity ($\mu \rightarrow 1$), the CES Armington aggregator converges to a Cobb-Douglas form with share parameter $\chi$, that is,

$$
\lim_{\mu \rightarrow 1} A(a^m, a^x) = (a^m)^\chi (a^x)^{1-\chi}.
$$

As the trade elasticity converges to 0, the CES Armington aggregator adopts a Leontief form,

$$
\lim_{\mu \rightarrow 0} A(a^m, a^x) = \min\{\chi a^m, (1 - \chi) a^x\}.
$$

And as the trade elasticity becomes infinite, the CES Armington becomes linear

$$
\lim_{\mu \rightarrow \infty} A(a^m, a^x) = \chi a^m + (1 - \chi) a^x.
$$

Exercise 7.2 asks you to prove these properties.

The parameter $\chi$ is known as the share parameter. The reason is that in the special case in which $\mu = 1$, condition (7.34) implies that

$$
\mu \rightarrow 1 \Rightarrow \chi = \frac{p^m_t a^m_t}{p^m_t a^m_t + p^x_t a^x_t}.
$$

The right hand side of this expression is the share of expenditure on importable goods in total expenditure. When $\mu \neq 1$, the parameter $\chi$ does not have the interpretation of an expenditure share.

The model has 19 structural parameters, $\beta, \delta, r^*, \tau, \rho, \pi, \sigma, \omega_m, \omega_x, \alpha_m, \alpha_x, A^m, A^x, \psi, \bar{d}$,
\(\bar{p}, \phi, \mu,\) and \(\chi\). We set \(\sigma = 2, \omega_m = \omega_x = 1.455, \alpha_m = \alpha_x = 0.32, \delta = 0.1,\) and \(r^* = 0.04,\) as in the calibration of the SOE-RBC model of chapter 4. We assume that emerging and poor countries pay an interest rate premium of 7 percent on average as in the calibration of the emerging-country model of chapter 6. This implies that \(\bar{p} = 0.07.\) We set \(\beta = 1/(1 + r^* + \bar{p}).\) We set \(\text{tot}\) and \(A^m\) equal to unity. The calibration of \(\rho\) and \(\pi\) defining the stochastic process for the terms of trade uses the country-specific estimates presented in table 7.1. Thus, there is one value of \(\rho\) and one value of \(\pi\) for each country. There remain six parameters that need to be assigned values to, \(\mu, \chi, d, A^x, \psi,\) and \(\phi.\) In what follows we describe how to calibrate those six.

7.6.2 Calibration of the Elasticity of Substitution Between Importables and Exportables, \(\mu\)

There is a vast literature on estimating the elasticity of substitution between exportable and importable goods, \(\mu.\) One branch of this literature uses aggregate data at quarterly frequency and estimates \(\mu\) in the context of open-economy DSGE models. This body of work typically estimates \(\mu\) to be below unity. For instance, Corsetti, Dedola, and Leduc (2008), Gust, Leduc, and Sheets (2009), and Justiniano and Preston (2010) all estimate \(\mu\) to lie between 0.8 and 0.86. Miyamoto and Nguyen (2014) estimate \(\mu\) to be 0.4. A second branch of the literature infers the value of \(\mu\) from trade liberalization episodes using average changes in quantities and prices observed over periods of five to ten years. This approach typically yields values of \(\mu\) greater than one, in the neighborhood of 1.5 (see, for example, Whalley, 1985).

It is intuitive that studies based on low frequency data deliver values of \(\mu\) higher than studies based on quarterly data. For it is natural to expect that agents can adjust more fully to relative-price changes in the long run than in the short run. Because our empirical study of the effects of terms-of-trade shocks uses annual data, it is sensible to adopt a value of \(\mu\) in between those stemming from the two aforementioned bodies of work. Accordingly, we set \(\mu\) equal to 1.
7.6.3 Calibration of the Share Parameter, $\chi$

We set $\chi = 0.74$. This number, along with the values assigned to $A^x$ and $\bar{F}$ are inferred from three observable macroeconomic relationships. The first one is the average share of exports in GDP, denoted $s_x$, and defined as

$$s_x = \frac{x}{p^x y^x + p^m y^m},$$

where variables without a subscript denote deterministic steady-state values. The second observable relationship is the average share of exportable output in GDP, denoted $s_{yx}$, and defined as

$$s_{yx} = \frac{p^x y^x}{p^x y^x + p^m y^m}.$$  

And the third observable relationship is the average share of the trade balance in GDP, denoted $s_{tb}$, and given by

$$s_{tb} = \frac{x - m}{p^x y^x + p^m y^m}.$$ 

To see how $\chi$ is related to $s_x$, $s_{yx}$, and $s_{tb}$, start by writing equation (7.34) in steady-state form as

$$\frac{a^x}{a^m} = \left(\frac{1 - \chi}{\chi}\right)^\mu \left(\frac{p^x}{p^m}\right)^{-\mu}.$$  

Let $\Upsilon \equiv p^x a^x / (p^m a^m)$ be the steady-state exportable-to-importable absorption ratio. Then we can solve the above expression for $\chi$ to obtain

$$\chi = \frac{(\Upsilon_{tot}^{\mu - 1})^{-1/\mu}}{1 + (\Upsilon_{tot}^{\mu - 1})^{-1/\mu}}.$$
In turn, we have that

\[
Y = \frac{p^x a^x}{p^m a^m} = \frac{p^x y^x - x}{p^m y^m + m} = \frac{s_{yx} - s_x}{1 - s_{yx} + s_x - s_{tb}}.
\]

It follows from the above two expressions that knowing \( \mu, \text{tot}, s_x, s_{yx}, \) and \( s_{tb} \) yields a value for \( \chi \). At this point, we have values for \( \mu \) and \( \text{tot} \). We set \( s_{tb} \) equal to 1 percent. This value together with the average interest rate of 11 percent yields an average debt-to-output ratio of 10 percent, which is consistent with the average level of net external asset holdings of emerging and developing countries estimated by Lane and Milesi-Ferretti (2007). We now turn to the calibration of \( s_x \) and \( s_{yx} \).

### 7.6.4 Calibration of the Share of Exports of Value-Added in GDP, \( s_x \)

We set \( s_x \) equal to 0.21. Obtaining a value for this share parameter is not straightforward. In the MX model, the country exports and imports only value added. In reality, exports and imports consist not only of value added but also of intermediate goods. To illustrate the difference between gross and value-added exports, consider the example of maquiladoras in Mexico. The Mexican maquiladoras are manufacturing firms that import raw materials and intermediate goods, assemble them, and then export these assembled goods. The value of the export includes the value added in Mexico but also the value of the imported raw materials and the imported intermediate goods. Thus gross exports are larger than exports of value added. As a second example, consider iPhone assemblers in China. According to the OECD (2012), the sale of 10 iPhones by China to the United States generates gross exports for $1,875. Of this gross export, only $65, or 3.5 percent,
is value added of China, corresponding to the assembly of the smart phones. The remaining 96.5 percent are exports of previously imported goods, including $229 in components from the United States (Audio codec, connectivity, GPS, memory, touchscreen controller), $207 from Taiwan (Touch screen, camera), $161 from Germany (baseband, power management, and transceiver components), $800 from Korea for DRAM memory, and $413 from other countries.

While in the model all exports and imports are value added, NIPA data provides only information on gross exports. Fortunately, the OECD in collaboration with the World Trade Organization (WTO) collects data on trade in value added. This trade share data is available (for selected years) in the Trade in Value-Added (TiVA) database. Letting $x_g$ denote gross exports and $y = px + pm$ gross domestic product (i.e., total value added), the gross export share is given by $x_g/y$ and the value-added export share by $x/y$. Table 7.3 presents gross and value-added trade shares for 13 poor and emerging countries in 1995 (the earliest date for which data is available) and in 2008. The 13 countries included in the table represent the intersection of the set of countries included in the OECD-TiVA dataset and the set of 51 countries included in the sample we used to estimate the SVAR model of section 7.4 (see footnote 2 for the complete list of countries). In 1995, gross exports exceeded exports of value added on average by 7 percent of gross domestic product. For example, in Mexico gross exports exceeded value added exports by 8 percentage points of GDP, reflecting the fact that gross exports contained previously imported raw materials and intermediate inputs. The country with the largest difference between gross exports and value added exports is Malaysia, where in 1995 the gross export share was 87 percent of GDP but the value added export share was only 51 percent of GDP.

The table also reports gross and value-added trade shares for the year 2008. Comparing trade shares in 2008 with trade shares in 1995 reveals that the gap between gross export shares and value-added export shares has widened significantly. In 2008, on average gross exports exceeded value added exports by 12 percent of GDP, almost twice the difference observed in 1995. For example,
Table 7.3: Gross and Value-Added Export Shares in 13 Poor and Emerging Countries, 1995 and 2008

<table>
<thead>
<tr>
<th>Country</th>
<th>1995</th>
<th>2008</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x^g$</td>
<td>$x^v$</td>
</tr>
<tr>
<td>Argentina</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>Brazil</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>Chile</td>
<td>28</td>
<td>24</td>
</tr>
<tr>
<td>China</td>
<td>19</td>
<td>16</td>
</tr>
<tr>
<td>India</td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td>Indonesia</td>
<td>24</td>
<td>21</td>
</tr>
<tr>
<td>Korea</td>
<td>27</td>
<td>21</td>
</tr>
<tr>
<td>Malaysia</td>
<td>87</td>
<td>51</td>
</tr>
<tr>
<td>Mexico</td>
<td>27</td>
<td>19</td>
</tr>
<tr>
<td>Philippines</td>
<td>29</td>
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</tr>
<tr>
<td>South Africa</td>
<td>22</td>
<td>19</td>
</tr>
<tr>
<td>Thailand</td>
<td>40</td>
<td>28</td>
</tr>
<tr>
<td>Turkey</td>
<td>17</td>
<td>15</td>
</tr>
<tr>
<td>Mean</td>
<td>27</td>
<td>20</td>
</tr>
</tbody>
</table>

Note. $x^g$ = gross exports, $x$ = exports of value added, and $y$ = value added. Trade shares are in percent. The data source is OECD, TiVA dataset, series codes are EXGR_GDP and FDDVA_GDP, available online at [www.oecd.org/trade/valueadded](http://www.oecd.org/trade/valueadded).
in South Korea only 56 percent of gross exports represent value added exports in 2008, whereas in 1995 the value added component of gross exports was 76 percent.

The cross-country average share of value added in gross exports was 81 percent in 1995. This figure is computed as the cross-country arithmetic mean of the ratio $x/x^g$. We use this number to calibrate the value added export share, $s_x$. We pick the year 1995 instead of 2008 because the former falls in the middle of the sample period used to estimate the SVAR model of section 7.4. In that data sample, the average gross export share across time and countries is 26 percent of GDP, or $x^g/y = 0.26$. Thus, we set

\[
s_x = \frac{x}{y} = \frac{x}{x^g} \frac{x^g}{y} = 0.81 \times 0.26 = 0.21.
\]

7.6.5 Calibration of the Share of Exportable Output in GDP, $s_{yx}$

We set the relative size of the sector producing exportable goods, $s_{yx} \equiv p^x y^x / y$, to 0.47. To calibrate this share, we use data on sectorial GDP. Specifically, the UNCTAD Handbook of Statistics reports panel data on GDP shares for three kinds of activity, agriculture ($s_{agr}^{i,t}$), industry ($s_{mfg}^{i,t}$), and services ($s_{ser}^{i,t}$), where $i$ denotes country and $t$ denotes year.\footnote{The data is available online at unctadstat.unctad.org.} From this source, we extract a data set in which $t$ ranges from 1995 to 2012 and $i$ includes all 51 poor and emerging countries used in the estimation of the SVAR system of section 7.4.\footnote{The beginning of the sample period is dictated by data availability.} The next step is to determine what fraction of these three shares belongs to the exportable sector and what to the importable sector. This task is not straightforward. In reality, no single sector can be viewed as a purely exportable or a purely...
importable sector. For example, in every year, every country imports and exports agricultural goods. So the agricultural sector cannot be categorized as fully importable or fully exportable. To address this issue, we adopt the convention that a fraction $s_{agr,x,i,t}$ of the share of agricultural GDP in country $i$ and year $t$ belongs to the exportable sector, where

$$s_{agr,x,i,t} \equiv \frac{x_{agr,i,t}}{(x_{agr,i,t} + m_{agr,i,t})}$$

denotes the ratio of exports in the agricultural sector to the sum of agricultural exports and agricultural imports in country $i$ in year $t$. We adopt the same convention for the other two sectors, manufacturing and services. Thus, the share of exportable GDP in total GDP in country $i$ and year $t$, $s_{yx,i,t}$, is given by

$$s_{yx,i,t} = s_{agr,x,i,t} s_{agr,x,i,t} + s_{mfg,x,i,t} s_{mfg,x,i,t} + s_{ser,x,i,t} s_{ser,x,i,t}.$$

The export shares, $s_{k,x,i,t}$, for $k = agr, mfg, ser$, is calculated as follows. Table 3.1 of the UNCTAD Handbook of Statistics reports gross exports and gross imports by main SITC Revision 3 product group for the following five groups, (1) All food items, (2) Agricultural raw materials, (3) Fuels, (4) Ores, metals, precious stones and non-monetary gold, and (5) Manufactured goods. We allocate exports and imports of groups (1) and (2) to agricultural activity, and exports and imports of groups (3)-(5) to industry activity. Table 5.1 of the UNCTAD Handbook of Statistics reports exports and imports of services. Then, $s_{yx}$ is given by the average of $s_{yx,i,t}$ across countries $i = 1, \ldots, 51$ and years $t = 1995, \ldots, 2011$.

### 7.6.6 Estimation of the Capital Adjustment Cost Parameter, $\phi$, and the Debt Elasticity of the Interest Rate, $\psi$

The parameters $\phi$ and $\psi$ are estimated country by country by a method-of-moments technique consisting in matching two volatility ratios predicted by the empirical SVAR model of section 7.4. The two targeted volatility ratios are the ratio of the standard deviation of investment to the standard deviation of output and the ratio of the standard deviation of the trade-balance-to-trend-
output-ratio to the standard deviation of output. All standard deviations, empirical and theoretical, are conditional on terms-of-trade shocks. This procedure yields one pair \((\phi, \psi)\) per country. The estimated pairs vary across countries not only because of cross-country variations in the relative conditional volatilities of investment and the trade-balance-to-output ratio, but also because of cross-country variations in the persistence of terms of trade shocks, \(\rho\). Table 7.4 displays the implied values of \(\phi\) and \(\psi\) along with the targeted conditional volatility ratios. The cross-country medians of the estimated values of \(\phi\) and \(\psi\) are 0.9819 and 0.0573, respectively. There is sizable variation in the estimates across countries; the cross-country standard deviations of \(\phi\) and \(\psi\) are, respectively, 4.6205 and 0.6106.

Table 7.5 summarizes the calibration of the MX model.

### 7.7 Response of the MX Model to Terms-of-Trade Shocks

How does the MX model adjust to an innovation in the terms of trade? Figure 7.6 displays cross-country median impulse responses to a ten-percent increase in the terms of trade. Because the calibration of the model features country-specific values for \(\rho\), \(\phi\), and \(\psi\), the impulse response of the model is computed country by country. Then, cross-country medians of impulse responses are computed point by point.

Intuitively, the increase in the terms of trade (i.e., the increase in the relative price of exports in terms of imports) induces a substitution in production away from importable goods and toward exportable goods. As a result, employment, investment, and output increase in the export sector and decline in the import sector. There is also a substitution effect on domestic absorption in favor of importable goods. The elevated demand for importables together with the decline in the production of importables drives up imports. Similarly, the decline in the domestic absorption of exportables together with the increase in the domestic production of exportables results in an
Table 7.4: Country-Specific Estimates of the Capital Adjustment Cost Parameter and the Debt Elasticity of the Interest Rate

<table>
<thead>
<tr>
<th>Country</th>
<th>Estimated Parameters</th>
<th>Targets</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>φ</td>
<td>ψ</td>
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<tr>
<td>Algeria</td>
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<td>Burundi</td>
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<tr>
<td>Cameroon*</td>
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<td>Comoros*</td>
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<tr>
<td>Congo, Dem. Rep.</td>
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<td>Tunisia</td>
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<tr>
<td>Turkey*</td>
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<td>1.2741</td>
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<tr>
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<td>Uruguay</td>
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<td>0.0195</td>
</tr>
<tr>
<td>Zambia</td>
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<tr>
<td>Median</td>
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<td>0.0573</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>4.6205</td>
<td>0.6106</td>
</tr>
</tbody>
</table>

Note. σ_i, σ_y, and σ_tb denote, respectively, the standard deviations of investment, output, and the trade-balance-to-trend-output ratio conditional on terms of trade shocks implied by the SVAR model. A star indicates that even after setting φ equal to 0, the MX model underpredicts the conditional relative volatility of investment.
Figure 7.6: Impulse Responses to a Terms-of-Trade Shock in the MX Model

Note. All variables except for the trade balance and external debt are expressed in percent deviations from steady state. The trade balance and external debt are expressed in level deviations from steady state in percent of steady-state output. Impulse responses are cross-country medians. Countries for which the MX model is unable to match the conditional investment-output volatility ratio were excluded (see note to table 7.4). For each country, impulse responses are produced by running the Matlab script tot_run.m using the country specific calibrations of $\phi$, $\psi$, and $\rho$. 
increase in exports.

Because the improvement in the terms of trade causes both imports and exports to rise, the response of the trade balance is ambiguous. As shown in panel (2,1) of the figure, the trade balance deteriorates on impact. Thus, the model fails to capture the Harberger-Laursen-Metzler effect present in the data (see the discussion in section 7.2).

In the MX model, the appreciation of the terms of trade leads to a boom in aggregate output, consumption, and investment, see panels (3,1), (4,1), and (5,1) of figure 7.6. This prediction of the model is only partially supported by the empirical SVAR model of section 7.4. Figure 7.5 shows that according to the SVAR model on impact the improvement in the terms of trade causes output to increase, but consumption and investment to decline. It follows that the impact response of consumption and investment predicted by the MX model, although quite intuitive, is at odds with the predictions of the empirical SVAR model.
7.8 Terms-Of-Trade Shocks: Less Important in Data than in Theory

How important are terms-of-trade shocks in the MX model? And how does this prediction square with that of the empirical SVAR model? To address these questions we first compute the variance of macroeconomic indicators of interest predicted by the MX model. These variances are conditional on terms-of-trade shocks, since terms-of-trade shocks are the only source of uncertainty included. Next we divide these theoretical conditional variances by the corresponding empirical unconditional variances implied by the SVAR model. We perform these two steps country by country and then compute the cross-country medians.

Table 7.6 shows that according to the MX model, terms of trade shocks explain on average 49 percent of the variance of output, making them a central source of business cycles in emerging and poor countries. This result is in line with the work of Mendoza (1995), who finds, in the context of a three sector SOE-RBC model, that the terms of trade explain 31 percent of the variance of output in developing countries. In the context of a multi-sector SOE-RBC model, Kose (2002) finds that over 80 percent of the variance of output in developing countries is due to variations in world prices.

How do the model-based predictions compare with those stemming from the empirical SVAR model? To address this question, the second column of table 7.6 reproduces from table 7.2 the cross-country median of the share of variances explained by terms of trade shocks according to the empirical SVAR model. The SVAR model predicts that terms-of-trade shocks explain only 10 percent of the variance of output. That is, the contribution of terms-of-trade shocks to output variations is 5 times larger in the theoretical model than it is in the empirical model. This conclusion

---

6Mendoza (1995) reports that the ratio of the standard deviation of output conditional on terms-of-trade shocks to the unconditional standard deviation of output is 0.56. The square of this figure, 0.31, is the share of the variance of output explained by terms-of-trade shocks.
Table 7.6: Share of Variances Explained by Terms-of-Trade Shocks: The MX Model Versus the Empirical SVAR Model

<table>
<thead>
<tr>
<th>Variable</th>
<th>Empirical</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MX Model</td>
<td>SVAR Model</td>
</tr>
<tr>
<td>Terms of Trade</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Trade Balance</td>
<td>53</td>
<td>12</td>
</tr>
<tr>
<td>Output</td>
<td>49</td>
<td>10</td>
</tr>
<tr>
<td>Consumption</td>
<td>50</td>
<td>11</td>
</tr>
<tr>
<td>Investment</td>
<td>44</td>
<td>10</td>
</tr>
</tbody>
</table>

Note. Each entry is the cross-country median of the fraction of the variance of the corresponding indicator explained by terms of trade shocks in percent. In column (1) the numerator of the fraction is the variance conditional on terms-of-trade shocks predicted by the MX model for country specific calibrations of $\phi$, $\psi$, $\rho$, and $\pi$. It is computed by running the Matlab script `tot_runner.m`. The denominator is the unconditional variance implied by the empirical country-specific SVAR model. Column 2 reproduces the row labeled ‘Median’ from table 7.2. Countries for which the MX model is unable to match the conditional investment-output volatility ratio were excluded (see note to table 7.4).
also holds for consumption, investment, and the trade balance. According to the MX model, half of the variance of all three of these aggregates is due to terms of trade shocks, whereas the SVAR attributes only 10 percent of the variances of these three variables to terms of trade shocks.

Table 7.7 presents predictions for each of the 51 countries in the panel. The table shows that according to the empirical SVAR model, terms-of-trade shocks explain more than 10 percent of the variance of output in 50 percent of the countries. By contrast, according to the theoretical MX model, this is the case in 94 percent of the countries.

We conclude that the predictions of the theoretical and empirical models in regard to the importance of terms of trade shocks as a source of business cycles lie far apart.

What could explain the gap between the predictions of the theoretical and empirical models? In the next chapter we introduce nontradable goods. These goods are neither exportable nor importable. Nontradable goods are often estimated to represent a large portion of gross domestic product. Since the nontraded sector is not directly affected by terms of trade shocks, the introduction of nontraded goods might create a buffer that mitigates the effects of terms of trade shocks on aggregate volatility. But this need not be the case, for even in an economy with a large nontraded sector movements in the terms of trade might generate sizable fluctuations in aggregate activity through income and substitution effects.

Exercises 7.3 - 7.5 provide some additional ideas in the context of the MX model. Exercise 7.3 leaves the theoretical model as is and tries to bring the data closer to the model. Specifically, it entertains the hypothesis that commodity prices might be a better measure of the terms of trade than export and import prices, the measure used here, especially for countries whose exports are concentrated in a small number of commodities. For example, for a country like Chile, the world copper price might be a more important source of fluctuations than an index of all Chilean export prices. Exercises 7.4 and 7.5 aim to bring the predictions of the MX model closer to the predictions of the empirical SVAR model. Specifically, exercise 7.4 introduces delays in the reaction
Table 7.7: Share of Variance Explained by Terms of Trade Shocks: Country-Level Predictions

<table>
<thead>
<tr>
<th>Country</th>
<th>$\sigma_t^2$</th>
<th>$\sigma_t\tilde{y}$</th>
<th>$\sigma_t\tilde{c}$</th>
<th>$\sigma_t\tilde{e}$</th>
<th>$\sigma_t\tilde{i}$</th>
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</thead>
<tbody>
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<td>Algeria</td>
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<td>100</td>
<td>1024</td>
<td>74</td>
<td>139</td>
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<td>100</td>
<td>9</td>
<td>25</td>
<td>11</td>
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<td>100</td>
<td>28</td>
<td>2</td>
<td>49</td>
</tr>
<tr>
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<td>100</td>
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<td>Brazil</td>
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<td>100</td>
<td>155</td>
<td>43</td>
<td>39</td>
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<td>100</td>
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<td>Chile</td>
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<td>100</td>
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<tr>
<td>Median</td>
<td>100</td>
<td>100</td>
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</tbody>
</table>

Note. * MX model unable to match the output volatility ratio (see table 7.4). Shares are expressed in percent. See note to table 7.6. Medians and standard deviations exclude starred countries.
Table 7.8: Share of Variances Explained by Terms-of-Trade Shocks In The MX Model: Sensitivity Analysis

<table>
<thead>
<tr>
<th>Parameterization</th>
<th>tot</th>
<th>tb</th>
<th>y</th>
<th>c</th>
<th>i</th>
<th>$s_m + s_x$</th>
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</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>100</td>
<td>53</td>
<td>49</td>
<td>50</td>
<td>44</td>
<td>0.41</td>
</tr>
<tr>
<td>$\mu = 1.5$</td>
<td>100</td>
<td>80</td>
<td>100</td>
<td>98</td>
<td>77</td>
<td>0.59</td>
</tr>
<tr>
<td>$\mu = 0.5$</td>
<td>100</td>
<td>18</td>
<td>10</td>
<td>11</td>
<td>15</td>
<td>0.19</td>
</tr>
<tr>
<td>$\chi = 0.5$</td>
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<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.06</td>
</tr>
<tr>
<td>SVAR Model</td>
<td>100</td>
<td>12</td>
<td>10</td>
<td>11</td>
<td>10</td>
<td>0.41</td>
</tr>
</tbody>
</table>

Note. Variance shares are expressed in percent. See note to Table 7.6. The baseline calibration is $\mu = 1$ and $\chi = 0.7399$. The sum $s_m + s_x \equiv (m + x)/y$ denotes the trade share.

of investment to changes in the terms of trade and exercise 7.5 considers the possibility that the government uses tax policy to isolate the country from fluctuations in world prices. The latter modification could be interpreted as a metaphor for nominal rigidities in prices of traded goods and imperfect exchange-rate pass through.

### 7.8.1 Robustness

Table 7.8 presents a number of robustness checks. It displays the share of the variance of the trade balance, output, consumption, and investment explained by terms-of-trade shocks in the MX model for several variations in the parameter configuration. The parameters being modified are those that are new in the MX model, namely, the elasticity of substitution between exportables and importables, $\mu$, and the share parameter $\chi$. We change one parameter value at the time, keeping all other structural parameters at their baseline values, shown in table 7.5.

Increasing the elasticity of substitution, $\mu$, from its baseline value of 1 to 1.5 drives model and data further apart. Under this parameterization, the MX model attributes 100 percent of variations in output to terms-of-trade shocks, more than 10 times the share implied by the empirical SVAR model. The reason why the MX economy is now more sensitive to variations in the terms of trade is
that, as $\mu$ increases, the economy becomes more open. Table 7.8 shows that the trade share, $s_m + s_x$, increases from 41 to 59 percent. To see why the economy becomes more open to international trade, note that in a steady-state with $tot = 1$, as $\mu$ increases, the marginal productivity of $a_m$ increases relative to that of $a_x$ in the production of the final good $A(a_m, a_x)$. In turn, this increase in relative productivity is due to the fact that the weight on $a_m$ in the Armington aggregator, given by $\chi = 0.7399$, is higher than the weight on $a_x$, which is given by $1 - \chi = 0.2601$. Technically, equation (7.35) implies that $a_m/a_x$ increases with $\mu$. Now if $a_m$ increases relative to $a_x$, the economy imports more and exports more, raising the trade share and hence increasing its exposure to terms of trade shocks.

When the elasticity of substitution, $\mu$, falls from 1 to 0.5, the share of variance of output attributed to terms of trade shocks according to the MX model falls from 49 percent to 10 percent, which is the value predicted by the empirical model. At first glance, it looks as though all it takes to bring the MX model close to the data is a reduction in $\mu$ from 1 to 0.5. However, this conclusion would be misplaced. For lowering $\mu$ results in a drastic reduction in the predicted trade openness from 41 percent to 19 percent, half the value observed in the data. Thus, setting $\mu = 0.5$ results in an economy that is much more closed to international trade than the typical economy contained in our 51-country panel.

Finally, table 7.8 shows that reducing the weight on importables in the Armington aggregator, $\chi$, from its baseline value of 0.7399 to 0.5, leads to a drastic reduction in the share of the variance of output accounted for by terms of trade shocks from 49 percent to 1 percent. Again, one might be tempted to conclude that all it takes to bring data and model closer together is an appropriate adjustment of the share parameter $\chi$. Yet, as in the case of varying $\mu$, this would be the wrong conclusion. For, as shown in the table, lowering the value of $\chi$ comes at the expense of a reduction in trade openness from 41 percent to 6 percent. The intuition for this result is that lessening the productivity of importable goods in the production of final goods by reducing $\chi$ induces agents
to import less. In fact, the considered reduction in $\chi$ is so large that it causes a reversal in the tradability of goods. The good that used to be exported now is imported and vice versa. It follows that lowering $\chi$ does not represent an empirically compelling solution to the disconnect between the predictions of the theoretical and empirical models.

The next chapter investigates whether the shortcomings of the MX model identified here could be due to the model’s empirically unrealistic assumption that all goods are tradable.
7.9 Exercises

Exercise 7.1 (Testing The Univariate Specification For The Terms Of Trade) Throughout the chapter, we have assumed that the typical poor or emerging country is too small to affect its terms of trade. This exercise aims at scrutinizing this assumption. For each of the 51 countries in the panel underlying the empirical analysis of section 7.4, test the null hypothesis that the terms of trade follow a univariate process against the alternative hypothesis that they also depend on one-period lagged values of output, consumption, investment, and the trade balance. Always include a constant in your regressions. Report the countries for which the null hypothesis can be rejected at the 95% confidence level. In completing this exercise, you might find it useful to consult an econometrics textbook, such as Hamilton (1994, section 11.2).

Exercise 7.2 (Special Cases of the CES Armington Aggregator) Consider the CES Armington aggregator

\[ A(a^m, a^x) = \left[ \chi (a^m)^{1-\mu} + (1 - \chi) (a^x)^{1-\mu} \right]^{\frac{1}{1-\mu}}. \]

Show that

\[ \lim_{\mu \to 1} A(a^m, a^x) = (a^m)^{1-\chi}, \]

\[ \lim_{\mu \to 0} A(a^m, a^x) = \min\{\chi a^m, (1 - \chi) a^x\}, \]

and

\[ \lim_{\mu \to \infty} A(a^m, a^x) = \chi a^m + (1 - \chi) a^x. \]

Exercise 7.3 (Export Commodity Prices) The SVAR evidence presented in section 7.4 suggests that the terms of trade play a small role in explaining business cycles in emerging and poor countries. Some have argued that for commodity exporting countries world commodity prices can be an important driver of aggregate activity. To test this hypothesis, consider the case of Chile, a
country for which about half of its exports are concentrated in a single commodity, namely, copper. The present exercise consists in replacing the terms of trade with the real price of copper in the SVAR analysis of section 7.4. To this end, download the time series of the spot price of copper in dollars at the London Market at an annual frequency from the IMF-IFS dataset. Deflate this series using the U.S. CPI index. Remove a quadratic time trend from the natural logarithm of the real copper price. Use the raw data corresponding to Chile contained in the file wdi.xls (posted with the materials for chapter 1) for the remaining four variables of the SVAR. Apply the same detrending and transformations of the variables as in the body of the chapter. Then estimate the SVAR system, equations (7.8) and (7.9), for Chile with your commodity price index taking the place of tot. The sample should be 1980 to 2011. Plot impulse responses of all variables in the SVAR to a ten-percent copper-price increase. Finally, estimate the share of the variance of each variable in the SVAR system explained by copper price shocks. Discuss your findings and compare them to those reported in section 7.4, especially table 7.2, for the case of Chile. Is the price of copper a more important driver of Chilean business cycles than the Chilean terms of trade?

Exercise 7.4 (Time-To-Build And The Effects of Terms of Trade Shocks) An important problem of the MX model of section 7.5 is that it exaggerates the role of the terms of trade relative to the predictions of the empirical SVAR model. Consider the possibility of bringing model and data closer together by introducing delays in the process of capital accumulation. Here, the hope is that if building capital takes time, terms-of-trade shocks may not induce large increases in investment because by the time investment becomes capital, much of the shock might have faded away (recall that the cross-country median value of \( \rho \), implying a half life of terms-of-trade shocks is only one year). Specifically, assume that investment goods take \( i \) years rather than one year to become productive capital, that is, model the law of motion of the capital stocks as

\[
k_{t+1}^i = (1 - \delta)k_t^i + i_{t-i}^i,
\]
for \( j = m, x \). Calibrate the model using the parameter values given in table 7.5 and the cross-country medians of \( \rho, \pi, \phi, \) and \( \psi \), given in tables 7.1 and 7.4. Consider three specifications, \( i = 1, 2, \) and 3. Plot the associated impulse responses of all variables of interest to a one-percent increase in the terms of trade. Explain the results. Now, for each value of \( i \), compute the predicted variances of output, consumption, investment, and the trade balance due to terms of trade shocks. Discuss your answer placing emphasis on the potential of the proposed channel.

**Exercise 7.5 (Tariffs And The Effects Of Terms Of Trade Shocks)** Continuing with the theme of the previous two exercises, one possible explanation behind the mismatch between the predictions of the theoretical and empirical models regarding the importance of terms-of-trade shocks is the presence of trade tariffs. To the extent that trade taxes move systematically in the opposite direction as the terms of trade, the domestic relative price of exportables and importables will be insensitive to movements in their world counterparts, which, in turn, may attenuate the effects of terms-of-trade shocks on domestic activity. Continue to assume that \( p_x^t \) and \( p_m^t \) denote the domestic prices of exportables and importables, respectively, in terms of final goods, and that \( \text{tot}_t \equiv p_x^t / p_m^t \) denotes the domestic terms of trade. The novelty in the present setting is that there is a tax that introduces a wedge, denoted \( \gamma_t \), between the domestic and the foreign terms of trade. Specifically, assume that

\[
\text{tot}_t = \text{tot}_t^* \gamma_t,
\]

where \( \text{tot}_t^* \) denotes the foreign terms of trade. The wedge \( \gamma_t \) is one minus a proportional trade barrier, which might take the form of a combination of import and export taxes. As before, the country takes the evolution of the foreign terms of trade as given. Assume that \( \text{tot}_t^* \) follows the AR(1) process

\[
\ln \left( \frac{\text{tot}_t^*}{\text{tot}_t^*} \right) = \rho \ln \left( \frac{\text{tot}_{t-1}^*}{\text{tot}_t^*} \right) + \pi \epsilon_t^{\text{tot}_t^*},
\]
where $t_0^*$ denotes the steady-state value of $t_0^*$. Assume that the government increases taxes on exports (or reduces taxes on imports) when the foreign terms of trade improves. Specifically, assume that

$$\gamma_t = \left( \frac{t_0^*}{t_0^*} \right)^{-\eta},$$

with $\eta > 0$. Finally, assume that the government rebates the proceeds of trade taxes to households in a lump-sum fashion. Calibrate the model using the values for the structural parameters given in table 7.5 and the cross-country medians of $\rho$, $\pi$, $\phi$, and $\psi$, given in tables 7.1 and 7.4. Consider different values of $\eta$ ranging from 0 to 1. For each value of $\eta$ compute the variances of output, consumption, investment, and the trade balance due to terms-of-trade shocks. Discuss your results.
Chapter 8

Nontradable Goods And The Real Exchange Rate

The MX and SVAR models of chapter 7 appear to bear little connection with each other when it comes to gauging the role of the terms of trade in accounting for business-cycle fluctuations. While in the theoretical model terms-of-trade shocks play a major role, in the empirical model their role is at best modest. We showed that a plausible calibration of the MX model implies that terms-of-trade shocks explain about 50 percent of the observed variance of output, while the estimated SVAR model assigns only 10 percent of the observed variance of output to variations in the terms of trade (see table 7.6).

One unrealistic feature of the MX model is that all goods in the economy are assumed to be internationally traded. Goods are either importable or exportable. It is natural therefore that in this environment the terms of trade, being the relative price of exportables in terms of importables, will play a significant role in the allocation of resources and expenditures across sectors and time. The assumption that all goods are tradable would be an acceptable modeling abstraction if it
was empirically compelling. That is, if most goods could be categorized as either importable or exportable. However, in reality the bulk of goods fall in a different category, known as nontradables. Nontradables are goods whose prices are not equalized across countries because, for various reasons, such as transportation costs and trade barriers, trading them across borders is economically inviable.

In this chapter, we explore the role of nontradable goods with an emphasis on how they affect the transmission of aggregate disturbances, including terms of trade shocks. The introduction of new features into any model increases the chances of explaining facts that the original model was unable to address. At first glance, this looks like a win-win proposition. However, because a richer model makes predictions for a larger set of variables, the number of dimensions along which it can be confronted with the data also increases. Thus, although a richer model makes it easier to explain existing questions, it also introduces the challenge of having to conform with the data across a larger set of predictions. This is the case, for example, with the introduction of nontradables into the MX model. The resulting three-sector model could bring data and model closer together with regards to the importance of terms of trade shocks as a source of fluctuations, but it will also make predictions for new variables, such as the relative price of nontradables in terms of tradables, which will have to be confronted with the data.

8.1 The Real Exchange Rate

The relative price of nontradables in terms of tradables is an important macroeconomic variable. One reason is that it plays a role in determining price-level differences across countries. The relative price of final consumption across countries is known as the real exchange rate. Specifically, the real exchange rate, denoted $RER_t$, is defined as the ratio of the foreign consumer price index to the
domestic consumer price index expressed in a common currency,

\[ \text{RER}_t \equiv \frac{E_t P^*_t}{P_t}, \]

where \( P^*_t \) denotes the nominal price of consumption in the foreign country in units of foreign currency, \( P_t \) denotes the nominal price of consumption in the domestic country in units of domestic currency, and \( E_t \) denotes the nominal exchange rate, defined as the price of one unit of foreign currency in terms of domestic currency. When \( \text{RER}_t \) increases, the domestic economy becomes relatively cheaper than the foreign economy. In this case, we say that the real exchange rate depreciates. Conversely, when \( \text{RER}_t \) decreases, the domestic economy becomes relatively more expensive than the foreign economy, and we say that the real exchange rate appreciates.

The presence of nontradable goods introduces price differences across countries. As a result, variations in the relative price of nontradables lead to variations in the real exchange rate. Movements in the real exchange rate have been the subject of much study in open economy macroeconomics and will feature prominently in the remainder of this chapter as well as in the chapters to come.

### 8.2 The TNT Model

To understand the precise way in which the relative price of nontradables affects the real exchange rate, we begin by studying the workings of a simple perfect-foresight endowment economy with tradable and nontradable goods known as the TNT model. The TNT model is of interest because it allows for an analytical characterization of the equilibrium response of the real exchange rate, the relative price of nontradables, and aggregate activity to various shocks, and because its main predictions extend to more complex economic environments.

Consider an economy populated by identical households with preferences described by the utility
function
\[ \sum_{t=0}^{\infty} \beta^t U(c_t), \]
where the consumption good, \( c_t \), is a composite of importable goods and nontradable goods, obtained via the following aggregation technology

\[ c_t = A(c_m^t, c_n^t), \]

where \( c_m^t \) and \( c_n^t \) denote consumption of importables and nontradables, respectively. Assume that the Armington aggregator \( A(\cdot, \cdot) \) is increasing in both arguments, concave, and homogeneous of degree one. The period budget constraint of the household is given by

\[ c_m^t + p_n^t c_n^t + d_t = d_{t+1} + r + tot_t y^x + p_n y^n, \]

where \( p_n^t \) denotes the relative price of nontradables in terms of importables, \( d_t \) denotes external debt maturing in \( t \) expressed in terms of importables, \( d_{t+1} \) denotes external debt assumed in \( t \) and maturing in \( t + 1 \), \( r > 0 \) denotes the interest rate, \( tot_t \) denotes the terms of trade, defined, as before, as the relative price of exportables in terms of importables, and \( y^x \) and \( y^n \) denote constant endowments of exportables and nontradable goods, respectively. Households derive no utility from consuming exportable goods, therefore the entire endowment of exportables is actually exported. By the same token, all of the consumption of importable goods is actually imported.

Households choose sequences \( \{c_m^t, c_n^t, d_{t+1}\}_{t=0}^{\infty} \) to maximize their lifetime utility subject to the period budget constraint presented above and a no-Ponzi-game constraint of the form

\[ \lim_{j \to \infty} (1 + r)^{-j} d_{t+j} \leq 0, \]
taking as given the price sequences \( \{p_t^n, tot_t\}_{t=0}^\infty \) and the initial debt position \( d_0 \).

Letting \( \beta^t \lambda_t \) denote the Lagrange multiplier associated with the period budget constraint, the first-order optimality conditions of the household’s problem are the period budget constraint itself and

\[
U'(c_t)A_1(c_t^m, c_t^n) = \lambda_t, \tag{8.2}
\]

\[
\lambda_t = \beta(1 + r)\lambda_{t+1}, \tag{8.3}
\]

\[
p_t^n = \frac{A_2(c_t^m, c_t^n)}{A_1(c_t^m, c_t^n)}, \tag{8.4}
\]

and

\[
\lim_{j \to \infty} (1 + r)^{-j}d_{t+j} = 0. \tag{8.5}
\]

Using the fact that the aggregator function \( A \) is increasing, homogeneous of degree one, and concave, we can rewrite the optimality condition (8.4) as

\[
p_t^n = P \left( \frac{c_t^m}{c_t^n} \right), \tag{8.6}
\]

where the function \( P(\cdot) \) is increasing,

\[ P'(\cdot) > 0. \]

Exercise 8.1 asks you to establish this result.

### 8.2.1 The Real Exchange Rate and the Relative Price of Nontradables

In the present model there is a one-to-one relationship between the relative price of nontradables in terms of importables, \( p_t^n \), and the real exchange rate, \( RER_t \). To see this, begin by dividing the numerator and denominator of the right-hand side of equation (8.1) by the domestic nominal price
of the importable good, denoted $P^m_t$, to obtain

$$RER_t = \frac{\mathcal{E}_t P^*_t / P^m_t}{P_t / P^m_t}.$$  

The denominator is the domestic relative price of the final consumption good in terms of importables, which we denote by $p^c_t$. Assume that the law of one price holds for importable goods. This means that when expressed in the same currency, the domestic and foreign prices of importables are equal to each other, that is,

$$P^m_t = \mathcal{E}_t P^m_{t^*}.$$  

Then, the numerator becomes $P^*_t / P^m_{t^*}$. This is the foreign relative price of the final consumption good in terms of importables, which we denote $p^c_{t^*}$. This relative price is taken as given by the domestic small open economy. Further, we assume that $p^c_{t^*}$ is constant and normalized to unity. Then, we can express the real exchange rate as

$$RER_t = \frac{1}{p^c_t}. \quad (8.7)$$

We now wish to link the relative price of final consumption goods, $p^c_t$, to the relative price of nontradables, $p^n_t$. To this end, it is useful to decentralize the production of final consumption goods. This step leaves the equilibrium of the model unchanged. It simply allows us to obtain a market price for the final consumption good. Imagine that the production of final consumption goods is not performed within the household, as assumed thus far, but by firms acting in perfectly competitive markets. Profits of these firms are given by

$$p^c_t A(c^m_t, c^n_t) - c^m_t - p^n_t c^n_t.$$
Firms choose $c^m_t$ and $c^n_t$ to maximize profits, taking $p^e_t$ and $p^n_t$ as given. The first-order optimality condition with respect to $c^m_t$ is

$$p^e_t A_1(c^m_t, c^n_t) = 1.$$ 

Combining this expression with (8.6) and (8.7) and taking into account that $A_1(\cdot, \cdot)$ is homogeneous of degree zero, we can write the real exchange rate as

$$RER_t = A_1 \left( P^{-1}(p^n_t), 1 \right)$$

$$\equiv e(p^n_t).$$

Because $A_1(\cdot, \cdot)$ is decreasing in its first argument and $P(\cdot)$ is increasing, we have that $e(\cdot)$ is decreasing

$$e'(\cdot) < 0.$$ 

This means that the real exchange rate is a decreasing function of the relative price of nontradables. The real exchange rate appreciates if and only if the relative price of nontradables increases and the real exchange rate depreciates if and only if the relative price of nontradables decreases. This relationship between the real exchange rate and the relative price of nontradables is intuitive.

Recall that in the present model the law of one price holds for importables so that importables have the same price domestically and abroad when expressed in the same currency. Therefore, if in the domestic economy nontradables become more expensive in terms of importables, then the domestic consumption basket becomes more expensive relative to the foreign consumption basket. Because of this one-to-one relationship, the relative price of nontradables itself is often referred to as the (inverse of the) real exchange rate. We follow this tradition and use the terms real exchange rate appreciation and increases in the relative price of nontradables interchangeably. Similarly, we use interchangeably the terms real exchange rate depreciation and decrease in the relative price of
nontradables.

8.2.2 The Equilibrium Real Exchange Rate

The defining property of nontradable goods is that their domestic demand must equal their domestic supply, since they can be neither imported nor exported. Therefore, we have that in equilibrium,

\[ c^n_t = y^n_t. \] (8.8)

Combining this market-clearing condition with the period budget constraint yields the following resource constraint in the tradable sector

\[ c^m_t + d_t = \frac{d_{t+1}^{-1}}{1 + r} + \hat{t} t y^x. \] (8.9)

To avoid inessential debt dynamics, assume that the subjective and pecuniary discount factors are equal to each other, that is,

\[ \beta = \frac{1}{1 + r}. \]

This condition together with the Euler equation (8.3) implies that the marginal utility of wealth is constant over time

\[ \lambda_t = \lambda, \]

for all \( t \geq 0 \). In turn, the facts that \( c^n_t \) and \( \lambda_t \) are both constant over time imply, by optimality condition (8.2), that consumption of importable goods is constant over time,

\[ c^m_t = c^m, \]
for all \( t \geq 0 \). Then, iterating the resource constraint (8.9) forward \( j \geq 1 \) times yields

\[
d_t = \frac{d_{t+j}}{(1+r)^j} + \sum_{s=0}^{j-1} \frac{\text{tot}_{t+s} y^x - c^m}{(1+r)^s}.
\]

Taking the limit as \( j \to \infty \), using the transversality condition (8.5), and solving for \( c^m \) yields the following expression for the equilibrium level of consumption of importables

\[
c^m = -\frac{r}{1+r}d_0 + \frac{r}{1+r} \sum_{t=0}^{\infty} \frac{\text{tot}_t y^x}{(1+r)^t}, \tag{8.10}
\]

Intuitively, households consume the annuity value of the lifetime endowment of exportable goods net of interest obligations on their initial external debt. Accordingly, consumption of importables is increasing in the terms of trade and in the endowment of exportables, and decreasing in the initial stock of external debt.

Evaluating optimality condition (8.6) at the equilibrium values of \( c^m_t \) and \( c^n_t \) implies that the relative price of nontradables is constant over time,

\[
p^n_t = p^n,
\]

where

\[
p^n = P \left( \frac{c^m}{y^n} \right). \tag{8.11}
\]

According to this expression, the relative price of nontradables is increasing in the desired demand for importable goods, \( c^m \), and decreasing in the endowment of nontradables, \( y^n \).
8.2.3 Adjustment of the Real Exchange Rate to Terms-of-Trade Shocks

We are now ready to determine the effect of terms-of-trade shocks on the relative price of nontradables and the real exchange rate. Suppose that the terms of trade experience a temporary increase in period 0. Specifically, assume that \( t_{0t} \) increases but \( t_{ot} \) remains unchanged for all \( t > 0 \). From (8.10) and (8.11) we have that

\[
\frac{\partial p^n}{\partial t} \bigg|_{\text{temporary}} = \frac{r}{1 + r y^n} P' \left( \frac{e^m}{y^n} \right) > 0.
\]

Intuitively, the increase in the relative price of the exportable endowment creates a positive income effect. As a result, households increase their demand for all consumption goods, importables and nontradables. Because the supply of nontradables is fixed at \( y^n \), the increase in the demand for nontradables requires an increase in the relative price of nontradables to eliminate the excess demand. Because the relative price of nontradables increases, the real exchange rate, \( RER_t \), appreciates in response to the temporary improvement in the terms of trade (i.e., the domestic economy becomes more expensive relative to the rest of the world).

Consider now a permanent increase in the terms of trade, so that \( t_{ot} \) increases for all \( t \geq 0 \). In this case, equilibrium condition (8.10) implies that

\[
\frac{\partial e^m}{\partial t} \bigg|_{\text{permanent}} = y^x.
\]

In turn, this result and equation (8.11) yield

\[
\frac{\partial p^n}{\partial t} \bigg|_{\text{permanent}} = \frac{y^x}{y^n} P' \left( \frac{e^m}{y^n} \right) > 0.
\]

Clearly, the last two expressions imply that permanent changes in the terms of trade have a larger
effect on the relative price of nontradables than temporary changes in the terms of trade,

$$\left. \frac{\partial p^n}{\partial \text{tot}} \right|_{\text{permanent}} \; > \; \left. \frac{\partial p^n}{\partial \text{tot}} \right|_{\text{temporary}} \; > \; 0.$$ 

This result is intuitive. The more permanent is the increase in the terms of trade, the larger is
the income effect it generates, and therefore the larger the increase in the desired demand for
nontradables, which in turn requires a larger increase in the relative price of nontradables to clear
the market.

### 8.2.4 Adjustment of the Real Exchange Rate to Interest-Rate Shocks

How do interest-rate shocks affect the relative price of nontradables and the real exchange rate?
To answer this question, assume that the interest rate experiences a one-time increase in period 0.
Specifically, assume that the interest rate equals \( r_0 > r \) in period 0 and \( r \) for all \( t > 0 \). Assume
further, for simplicity, that the terms of trade are constant over time, \( \text{tot}_t = \text{tot} \), for all \( t \geq 0 \).
Because the interest rate is constant starting in period 1, we know from the previous analysis that
beginning in period 1 consumption is also constant over time. Specifically,

$$c_t^m = c_1^m,$$

for all \( t \geq 1 \), where

$$c_1^m = -\frac{r}{1 + r} d_1 + \text{tot}_x y^x. \quad (8.12)$$

The resource constraint (8.9) evaluated at \( t = 0 \) implies that

$$c_0^m + d_0 = \frac{d_1}{1 + r_0} + \text{tot}_x y^x. \quad (8.13)$$
Combining (8.2) with the Euler equation (8.3) evaluated at $t = 0$ yields

$$
U'(A(c^m_0, y^n))A_1(c^m_0, y^n) = \beta(1 + r_0)U'(A(c^m_1, y^n))A_1(c^m_1, y^n).
$$

(8.14)

The three-equation system (8.12)-(8.14) is in three unknowns, $c^m_0$, $c^m_1$, and $d_1$. All other variables and parameters in this system are known. Equation (8.14) and the fact that $\beta(1+r_0) > 1$ imply that $c^m_1 > c^m_0$. This is intuitive. A higher interest rate induces households to save more by substituting future consumption for present consumption. We now wish to show that $c^m_0$ falls. We do so by contradiction. Suppose that, contrary to what we wish to show, $c^m_0$ increases. Then, by (8.13), $d_1$ must increase. But if $d_1$ increases, then, by (8.12), $c^m_1$ must fall, which contradicts the finding that $c^m_1 > c^m_0$. This establishes that $c^m_0$ falls, that is,

$$
\frac{\partial c^m_0}{\partial r_0} < 0.
$$

Now this expression together with (8.6) and (8.8) implies that

$$
\frac{\partial p^n_0}{\partial r_0} = P' \left( \frac{c^m_0}{y^n} \right) \frac{1}{y^n} \frac{\partial c^m_0}{\partial r_0} < 0.
$$

This expression says that the real exchange rate depreciates in response to a temporary increase in the interest rate. Intuitively, the increase in the interest rate dampens consumption demand for all goods in period 0 as households’ desired savings increases. Because the supply of nontradables is fixed, the price of nontradables must fall to clear the market.

### 8.2.5 Adjustment of Output Terms-of-Trade Shocks in the TNT Model

In chapter 7, we argued that an important problem of the MX model is that it assigns too large a role to the terms of trade as a driver of the business cycle relative to the predictions stemming
from estimated SVAR models. Could this difficulty be in part due to the fact that in that theoretical environment all goods are tradable? To answer this question in the context of the present environment, consider characterizing how the effect of a terms-of-trade shock on output changes as the share of nontradables in the production of final goods changes. Is this effect smaller the larger the share of nontradables is?

Aggregate output, denoted $y_t$, is the sum of tradable output and nontradable output expressed in terms of final goods,

$$y_t = \frac{P^x_t y^x_t + P^n_t y^n_t}{P^x_t},$$

where $P^x_t$, $P^n_t$, and $P_t$ denote, respectively, the nominal prices of exportable, nontradable, and final goods. Dividing the numerator and denominator by the nominal price of importables gives

$$y_t = \frac{\text{tot}_t y^x_t + p^n_t y^n_t}{p^c_t}.$$

Recalling that $p^c_t = 1/A_1(c^m_t, y^n_t)$ and that $p^n_t = A_2(c^m_t, y^n_t)/A_1(c^m_t, y^n_t)$, we can write

$$y_t = A_1(c^m_t, y^n_t)\text{tot}_t y^x_t + A_2(c^m_t, y^n_t)y^n_t.$$

Adding and subtracting $A_1(c^m_t, y^n_t)c^m_t$ to the right-hand side, and taking into account that $A(\cdot, \cdot)$ is homogeneous of degree one, we can write output as

$$y_t = A(c^m_t, y^n_t) + A_1(c^m_t, y^n_t)(\text{tot}_t y^x_t - c^m_t).$$

The first term on the right-hand side is final consumption, $c_t$. The second term is the trade balance expressed in terms of final consumption goods (recall that $A_1(c^m_t, y^n_t) = 1/p^c_t$). Therefore, the above
expression says that output equals consumption plus the trade balance, or

\[ y_t = c_t + tb_t, \]

where \( tb_t \equiv (\text{tot}_t y^x - c^m_t)/p^c_t \) denotes the trade balance expressed in terms of final goods.

Now assume that there is an unexpected permanent increase in the terms of trade in period 0. Equilibrium conditions (8.10) and (8.15) imply that

\[
\frac{\partial y_0}{\partial \text{tot}} = \frac{y^x}{p^c_t} + A_{11}(c^m, y^n)(\text{tot}y^x - c^m)y^x.
\]

This expression is intuitive. If the trade balance is small, we can forget about the second term on the right-hand side. In this case, a unit increase in the terms of trade increases GDP by the size of exportable output measured in terms of final goods.

Does the presence of nontradables dampen the output effect of a terms-of-trade shock? In particular, how does the output effect of a terms-of-trade shock change with the share of nontradables in total consumption? Assume, for simplicity, that the aggregator function takes the Cobb-Douglas form

\[ A(c^m, y^n) = (c^m)^\alpha(y^n)^{1-\alpha}, \]

with \( \alpha \in (0,1) \). The parameter \( 1-\alpha \) determines the share of domestic expenditure devoted to nontradable goods. To see this, note that under the Cobb-Douglas specification for \( A(\cdot, \cdot) \), the household’s optimality condition (8.4) implies that \( 1 - \alpha = \frac{p^n c^n}{c^m + p^n c^n} \). Now notice that condition (8.10) implies that in equilibrium consumption of importables, \( c^m \), is independent of \( \alpha \). It follows immediately that \( A_1(c^m, y^n) = \alpha(c^m/y^n)^{\alpha-1} \) takes the value 1 when \( 1 - \alpha = 0 \) and the value 0 when \( 1 - \alpha = 1 \). Also, \( A_{11}(c^m, y^n) = \alpha(\alpha-1)(c^m/y^n)^{\alpha-2} \) equals 0 when \( 1 - \alpha = 0 \) and also
when $1 - \alpha = 1$. We then have that

$$\left. \frac{\partial y_0}{\partial \text{tot}} \right|_{1-\alpha=0} = y^x > 0 \quad \text{and} \quad \left. \frac{\partial y_0}{\partial \text{tot}} \right|_{1-\alpha=1} = 0,$$

This expression states that as the share of nontradables in total expenditure increases from zero to 100 percent, the output effect of an improvement in the terms of trade falls from a positive value to zero. The relationship between the share of nontradables in consumption and the effect in the terms of trade is in general nonmonotonic. However, the result obtained here provides some hope that augmenting the MX model to allow for nontradables could reduce the importance of terms of trade shocks and in this way bring the empirical and theoretical models closer together. We turn to this issue next.

### 8.3 Empirical Evidence on the Effects of Terms-of-Trade Shocks on the Real Exchange Rate

The simple endowment economy of section 8.2 predicts that a positive innovation in the terms of trade should be accompanied by an appreciation of the real exchange rate. That is, improvements in the terms of trade are predicted to make the country more expensive. Is this prediction consistent with the data?

To address this question, we expand the SVAR model of section 7.4 in chapter 7 to include the real exchange rate. Specifically, the empirical model is

$$x_t = h_x x_{t-1} + \Pi \epsilon_t.$$
Assume that the vector $x_t$ is given by

$$x_t \equiv \begin{bmatrix} \hat{\text{tot}}_t \\ \hat{\text{tb}}_t \\ \hat{\text{y}}_t \\ \hat{\text{c}}_t \\ \hat{\text{i}}_t \\ \hat{\text{RER}}_t \end{bmatrix},$$

where $\hat{\text{tot}}_t$, $\hat{\text{y}}_t$, $\hat{\text{c}}_t$, $\hat{\text{i}}_t$, and $\hat{\text{RER}}_t$ denote log-deviations of the terms of trade, real output per capita, real private consumption per capita, real gross investment per capita, and the real exchange rate from their respective quadratic time trends. The variable $\hat{\text{tb}}_t$ is constructed by dividing the trade balance by the trend component of output and then removing a quadratic time trend. As the empirical measure of $\text{RER}_t$ we use the bilateral U.S. dollar real exchange rate, defined as

$$\text{RER}_t = \frac{\mathcal{E}_t P_{US}^t}{P_t},$$

where $\mathcal{E}_t$ denotes the dollar nominal exchange rate, given by the domestic-currency price of one U.S. dollar, $P_{US}^t$ denotes the U.S. consumer price index, and $P_t$ denotes the domestic consumer price index.\footnote{An alternative measure of $\text{RER}_t$ is the real effective exchange rate, which is based on the value of a currency against a trade-weighted average of foreign currencies. We do not use this measure because of data availability. In the WDI database, our requirement of at least 30 years of data is satisfied by only 26 of the 51 countries in our panel.} The sample period is 1980 to 2011. Only countries for which at least 30 consecutive years of data for all components of $x_t$ is available were included in the estimation. The resulting sample contains 38 countries.\footnote{The data comes from the World Bank’s World Development Indicators data base and, except for the real exchange rate, is available in the file wdi.xls posted with the materials for chapter 1. In the construction of the empirical measure of $\text{RER}_t$, the series used for the consumer price index is FP.CPI.TOTL and for the nominal exchange rate is PA.NUS.FCRF. The WDI does not provide CPI data for Argentina. The Argentine CPI index was taken from}
elements of the first rows of the matrices $h_t$ and $\Pi$ are nil. The second of these restrictions identifies the terms of trade shock as the first element of the vector $\epsilon_t$.

Figure 8.1 displays the response of the variables included in the vector $x_t$ to a 10 percent improvement in the terms of trade. The responses of variables other than the real exchange rate are similar to those implied by the five-variable SVAR presented in section 7.4 of chapter 7. Specifically, the SVAR implies that a terms-of-trade appreciation improves the trade balance (the HLM effect) and leads to an expansion in aggregate activity. The novelty of the figure is the response of the real exchange rate. The improvement in the terms of trade causes a real-exchange-rate appreciation, that is, the country becomes more expensive vis-à-vis the rest of the world. On impact, the ten-percent improvement in the terms of trade leads to a 1.6 percent real exchange rate appreciation with a half life of about 2 years. The estimated response of the real exchange rate is qualitatively in line with the predictions of the TNT model studied earlier in section 8.2.

The present empirical model implies that terms-of-trade shocks explain a modest fraction of cyclical fluctuations in emerging and poor countries. Table 8.1 displays the share of the variances of output, consumption, investment, the real exchange rate, and the trade balance explained by terms-of-trade shocks for each of the 38 countries included in our panel. Terms-of-trade shocks explain on average between 9 and 14 percent of the variances of all variables included in the SVAR. The estimated importance of terms of trade shocks is virtually identical to that obtained in the four-variable SVAR that excludes the real exchange rate studied in section 7.4 of chapter 7. There is significant cross-country dispersion of the estimated importance of terms of trade shocks. Because of the presence of outliers (e.g., Egypt) the median absolute deviation is a better measure

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INDEC until 2006, and from IPC-7-Provincias from 2007 to 2011 due to systematic underreporting by INDEC during this period. The countries included in the sample are Algeria, Argentina, Bolivia, Botswana, Brazil, Burundi, Cameroon, Central African Republic, Colombia, Congo, Dem. Rep., Costa Rica, Cote d’Ivoire, Dominican Republic, Egypt, Arab Rep., El Salvador, Ghana, Guatemala, Honduras, India, Indonesia, Jordan, Kenya, Korea, Rep., Madagascar, Malaysia, Mauritius, Mexico, Morocco, Pakistan, Paraguay, Peru, Philippines, Senegal, South Africa, Sudan, Thailand, Turkey, and Uruguay.
Figure 8.1: Response of the Real Exchange Rate and Other Variables to An Innovation in the Terms of Trade: SVAR Evidence
Table 8.1: Share of Variance Explained by Terms of Trade Shocks: Country-Level SVAR Evidence

<table>
<thead>
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<th>Country</th>
<th>$t_{ot}$</th>
<th>$t_{by}$</th>
<th>$t_{cy}$</th>
<th>$t_{ci}$</th>
<th>$t_{RER}$</th>
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Note. Shares are expressed in percent.
of dispersion than the standard deviation.\footnote{The median absolute deviation of a variable $x$ is defined as $\text{median}(|x - \text{median}(x)|)$.} The median absolute deviations of the contribution of terms-of-trade shocks to the variances of the variables included in the SVAR is about two thirds as large as the median. The associated standard deviations (not shown) are about twice as large as the median absolute deviations.

In that chapter, we argued that a model with only importable and exportable goods (the MX model), exaggerates the importance of terms of trade shocks, as it predicts that these shocks are responsible for about half of the variance of output, consumption, investment, and the trade balance. In the next section, we expand the MX model to include a nontraded sector and ask whether the addition of this sector can reduce the predicted importance of terms-of-trade shocks to values concurring with the empirical evidence reported here.

### 8.4 The MXN Model

Can the introduction of nontradable goods reconcile the disconnect between the small observed importance of terms of trade shocks documented in the previous section with the large importance assigned to this type of shocks in theoretical models of the open economy like the MX model studied in chapter 7? To address this question, we extend the MX model to include production and absorption of nontradable goods. The resulting model includes three sectors, an importable sector (M), an exportable sector (X), and a nontradable sector (N). We refer to this model as the MXN model. Because the nontraded sector is not directly affected by movements in the relative price of exportables in terms of importables, and because the nontraded sector is large, the hope is that it will act as a buffer that dampens the aggregate effects of terms-of-trade shocks.
8.4.1 Households

The model economy is populated by a large number of identical households with preferences described by the utility function

\[ E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, h_t^m, h_t^x, h_t^n), \]  

(8.16)

where \( c_t \) denotes consumption, \( h_t^m \) denotes hours worked in the importable sector, \( h_t^x \) hours worked in the exportable sector, and \( h_t^n \) hours worked in the nontradable sector. Households maximize their lifetime utility subject to the sequential budget constraint

\[
c_t + i_t^m + i_t^x + i_t^n + \Phi_m(k_{t+1}^m - k_t^m) + \Phi_x(k_{t+1}^x - k_t^x) + \Phi_n(k_{t+1}^n - k_t^n) + p_t^r d_t = \frac{p_t^r d_{t+1}}{1 + r_t} + w_t^m h_t^m + w_t^x h_t^x + w_t^n h_t^n + u_t^m k_t^m + u_t^x k_t^x + u_t^n k_t^n, \]

where \( i_t^j, k_t^j, w_t^j, \) and \( u_t^j \) denote respectively, gross investment, the capital stock, the real wage, and the rental rate of capital in sector \( j \), for \( j = m, x, n \) with the superscripts \( m, x, \) and \( n \) denoting the sector producing, respectively, importable, exportable, and nontraded goods. The functions \( \Phi_j(\cdot) \), \( j = m, x, n \), introduce capital adjustment costs and are assumed to be increasing and convex and to satisfy \( \Phi_j(0) = \Phi'_j(0) = 0 \). The variable \( p_t^r \) denotes the relative price of the tradable composite good in terms of final goods (to be formally defined below), \( d_t \) denotes the stock of debt in period \( t \), denominated in units of the tradable composite good, and \( r_t \) denotes the interest rate on debt held from period \( t \) to \( t + 1 \). Consumption, investment, wages, rental rates, debt, and capital adjustment costs are all expressed in units of final goods.

The capital stocks obey the familiar laws of motion

\[ k_t^m = (1 - \delta) k_{t+1}^m + i_t^m, \]  

(8.17)
June 1, 2015, M. Uribe and S. Schmitt-Grohé

\[ k_{t+1}^x = (1 - \delta) k_t^x + i_t^x, \]  

(8.18)

and

\[ k_{t+1}^n = (1 - \delta) k_t^n + i_t^n. \]  

(8.19)

Using these laws of motion to eliminate \( i_t^m, i_t^x, \) and \( i_t^n \) from the household’s budget constraint and letting \( \lambda_t \) denote the Lagrange multiplier associated with the resulting budget constraint, we have that the first-order optimality conditions with respect to \( c_t, h_t^m, h_t^x, h_t^n, d_{t+1}, k_{t+1}^m, k_{t+1}^x, \) and \( k_{t+1}^n \) are, respectively,

\[
U_1(c_t, h_t^m, h_t^x, h_t^n) = \lambda_t \]

(8.20)

\[-U_2(c_t, h_t^m, h_t^x, h_t^n) = \lambda_t w_t^m \]

(8.21)

\[-U_3(c_t, h_t^m, h_t^x, h_t^n) = \lambda_t w_t^x \]

(8.22)

\[-U_4(c_t, h_t^m, h_t^x, h_t^n) = \lambda_t w_t^n \]

(8.23)

\[ \lambda_t p_t^\tau = \beta (1 + r_t) E_t \lambda_{t+1} p_{t+1}^\tau \]

(8.24)

\[
\lambda_t \left[ 1 + \Phi'_m (k_{t+1}^m - k_t^m) \right] = \beta E_t \lambda_{t+1} \left[ u_{t+1}^m + 1 - \delta + \Phi'_m (k_{t+2}^m - k_{t+1}^m) \right] \]

(8.25)

\[
\lambda_t \left[ 1 + \Phi'_x (k_{t+1}^x - k_t^x) \right] = \beta E_t \lambda_{t+1} \left[ u_{t+1}^x + 1 - \delta + \Phi'_x (k_{t+2}^x - k_{t+1}^x) \right] \]

(8.26)

\[
\lambda_t \left[ 1 + \Phi'_n (k_{t+1}^n - k_t^n) \right] = \beta E_t \lambda_{t+1} \left[ u_{t+1}^n + 1 - \delta + \Phi'_n (k_{t+2}^n - k_{t+1}^n) \right] \]

(8.27)

These optimality conditions are identical to those associated with the MX model, except for three. One is equation (8.23), which equates the marginal disutility of working in the nontraded sector, \(-U_4(c_t, h_t^m, h_t^x, h_t^n)\), to the real wage in that sector expressed in units of utility, \( \lambda_t w_t^n \). The second is the Euler equation for bonds, equation (8.24), which now features the relative price of the tradable composite good in terms of final goods, \( p_t^\tau \). To see why this price appears in the Euler equation
for bonds, note first that in both, the MX and MXN models, debt is denominated in units of the tradable composite good, whereas the household’s budget constraint is expressed in units of the final good. In the MX model, the tradable composite good is the final good, so its relative price is identically equal to one. In the MXN model, however, the tradable composite good is not the final good. The third is the Euler equation (8.27), governing the optimal accumulation of capital in the nontraded sector.

8.4.2 Firms Producing Final Goods

Final goods are produced using nontradable goods and a composite of tradable goods via the technology \( B(a_t^T, a_t^n) \), where \( a_t^T \) denotes the tradable composite good and \( a_t^n \) denotes the nontraded good. The aggregator function \( B(\cdot, \cdot) \) is assumed to be increasing, concave, and homogeneous of degree one. Final goods are sold to households, which then allocate them to consumption or investment purposes. Producers of final goods behave competitively. Their profits are given by

\[
B(a_t^T, a_t^n) - p_t^T a_t^T - p_t^n a_t^n,
\]

where \( p_t^n \) denotes the relative price of nontradable goods in terms of final goods. The firm’s profit maximization conditions are

\[
B_1(a_t^T, a_t^n) = p_t^T \tag{8.28}
\]

and

\[
B_2(a_t^T, a_t^n) = p_t^n. \tag{8.29}
\]

These expressions define the domestic demand functions for nontradables and for the tradable composite good.
8.4.3 Firms Producing the Tradable Composite Good

The tradable composite good is produced using importable and exportable goods as intermediate inputs, via the technology

\[ a_t^\tau = A(a_t^m, a_t^x). \] (8.30)

where \( a_t^m \) and \( a_t^x \) denote the domestic absorptions of importable and exportable goods, respectively. The aggregator function \( A(\cdot, \cdot) \) is increasing, concave, and linearly homogeneous. Profits are given by

\[ p_t^\tau A(a_t^m, a_t^x) - p_t^m a_t^m - p_t^x a_t^x, \]

where \( p_t^m \) denotes the relative price of importable goods in terms of final goods and \( p_t^x \) denotes the relative price of exportable goods in terms of final goods. Firms in this sector are assumed to behave competitively in intermediate and final goods markets. Then, profit maximization implies that

\[ p_t^\tau A_1(a_t^m, a_t^x) = p_t^m \] (8.31)

and

\[ p_t^\tau A_2(a_t^m, a_t^x) = p_t^x. \] (8.32)

These two expressions represent the domestic demand functions for importable and exportable goods.

8.4.4 Firms Producing Importable, Exportable, and Nontradable Goods

Importable, exportable, and nontradable goods are produced with capital and labor via the technologies

\[ y_t^m = A^m F^m(k_t^m, h_t^m) \] (8.33)
\[ y_t^x = A^x F^x(k_t^x, h_t^x), \]  
(8.34)

and

\[ y_t^n = A^n F^n(k_t^n, h_t^n), \]  
(8.35)

where \( y_t^j \) and \( A^j \) denote, respectively, output and a productivity factor in sector \( j = m, x, n \). The production functions \( F^j(\cdot, \cdot), j = m, x, n \), are assumed to be increasing in both arguments, concave, and homogeneous of degree one. Profits of firms producing exportable, importable, or nontraded goods are given by

\[ p_t^j F^j(k_t^j, h_t^j) - w_t^j h_t^j - u_t^j k_t^j, \]

for \( j = m, x, n \). Firms are assumed to behave competitively in product and factor markets. Then, the first-order profit maximization conditions are

\[ p_t^m A^m F_1^m(k_t^m, h_t^m) = u_t^m \]  
(8.36)

\[ p_t^m A^m F_2^m(k_t^m, h_t^m) = w_t^m \]  
(8.37)

\[ p_t^x A^x F_1^x(k_t^x, h_t^x) = u_t^x \]  
(8.38)

\[ p_t^x A^x F_2^x(k_t^x, h_t^x) = w_t^x \]  
(8.39)

\[ p_t^n A^n F_1^n(k_t^n, h_t^n) = u_t^n \]  
(8.40)

\[ p_t^n A^n F_2^n(k_t^n, h_t^n) = w_t^n. \]  
(8.41)

These efficiency conditions represent the sectoral demand functions for capital and labor. Together with the assumption of linear homogeneity of the production technologies, they imply that firms make zero profits at all times.
8.4.5 Market Clearing

In equilibrium the demand for final goods must equal the supply of this type of goods

\[ c_t + i^m_t + i^n_t + \Phi_m(k^{m}_{t+1} - k^m_t) + \Phi_x(k^x_{t+1} - k^x_t) + \Phi_n(k^n_{t+1} - k^n_t) = B(a^n_t, a^n_t). \]  
(8.42)

Also, the demand for nontradables must equal the production of nontradables

\[ a^n_t = y^n_t. \]  
(8.43)

Imports, denoted \( m_t \), are defined as the difference between the domestic absorption of importables, \( a^m_t \), and importable output, \( y^m_t \), or

\[ m_t = p^m_t (a^m_t - y^m_t). \]  
(8.44)

The price of importables appears on the right-hand side of this definition because \( m_t \) is expressed in units of final goods, whereas \( y^m_t \) and \( a^m_t \) are expressed in units of importable goods. Similarly, exports, denoted \( x_t \), are given by the difference between exportable output, \( y^x_t \), and the domestic absorption of exportables, \( a^x_t \),

\[ x_t = p^x_t (y^x_t - a^x_t). \]  
(8.45)

Like imports, exports are measured in terms of final goods.

Combining the above two definitions, the household’s budget constraint, and the definitions of profits in the final- and intermediate-good markets, and taking into account that firms make zero profits at all times, yields the following economy-wide resource constraint

\[ p^x_t \frac{d_{t+1}}{1 + r_t} = p^x_t d_t + m_t - x_t. \]  
(8.46)
To ensure a stationary equilibrium process for external debt, we assume, as in the MX model, that the country interest-rate premium is debt elastic,

\[ r_t = r^* + p(d_{t+1}), \quad (8.47) \]

where \( r^* \) denotes the sum of world interest rate and the constant component of the interest-rate premium, and \( p(d) \) denotes the debt-elastic component of the country interest-rate premium. We assume that \( p(\bar{d}) = 0 \) and \( p'(\bar{d}) > 0 \), for some constant \( \bar{d} \).

The terms of trade, denoted \( tot_t \), are defined as the relative price of exportable goods in terms of importable goods, that is,

\[ tot_t = \frac{p^x_t}{p^m_t}. \quad (8.48) \]

As in the empirical analysis conducted earlier in this chapter, we assume that the country is small in international product markets and therefore takes the evolution of the terms of trade as given. Also in line with the empirical analysis, we assume an AR(1) structure for the law of motion of the logarithm of the terms of trade

\[ \ln \left( \frac{tot_t}{tot} \right) = \rho \ln \left( \frac{tot_{t-1}}{tot} \right) + \pi \epsilon^tot_t, \quad (8.49) \]

where \( \epsilon^tot_t \) is a white noise with mean zero and unit variance, and \( tot > 0 \), \( \rho \in (-1, 1) \), and \( \pi > 0 \) are parameters.

### 8.4.6 Competitive Equilibrium

A competitive equilibrium is then a set of 33 processes \( k^{m}_{t+1}, i^{m}_t, k^{x}_{t+1}, i^{x}_t, k^{n}_{t+1}, i^{n}_t, c_t, h^m_t, h^x_t, h^n_t, \lambda_t, w^{m}_t, w^{x}_t, w^{n}_t, p^m_t, r_t, u^{m}_t, u^{x}_t, u^{n}_t, a^{m}_t, a^{x}_t, a^{n}_t, p^m_t, p^x_t, a^m_t, p^n_t, y^m_t, y^x_t, y^n_t, m_t, x_t, d_{t+1}, \) and \( tot_t \) satisfying equations (8.17) to (8.49), given initial conditions \( k^{m}_0, k^{x}_0, k^{n}_0, d_0, \) and \( tot_{-1} \), and the
stochastic process $\epsilon^t_{\text{tot}}$.

### 8.4.7 Functional Forms

As in the MX model, we assume that the period utility function is CRRA in a quasi linear composite of consumption and labor

$$U(c, h^m, h^x, h^n) = \left[ \frac{c - G(h^m, h^x, h^n)}{1 - \sigma} \right]^{1-\sigma} - 1,$$

where

$$G(h^m, h^x, h^n) = \frac{(h^m)^{\omega_m}}{\omega_m} + \frac{(h^x)^{\omega_x}}{\omega_x} + \frac{(h^n)^{\omega_n}}{\omega_n},$$

with $\sigma, \omega_m, \omega_x, \omega_n > 0$. This specification implies that sectoral labor supplies are wealth inelastic.

The technologies for producing importables, exportables, and nontradables are all assumed to be Cobb-Douglas,

$$F^m(k^m, h^m) = (k^m)^{\alpha_m} (h^m)^{1-\alpha_m},$$

$$F^x(k^x, h^x) = (k^x)^{\alpha_x} (h^x)^{1-\alpha_x},$$

and

$$F^n(k^n, h^n) = (k^n)^{\alpha_n} (h^n)^{1-\alpha_n},$$

where $\alpha_m, \alpha_x, \alpha_n \in (0, 1)$. We assume that the Armington aggregators used in the production of the tradable composite good and the final good take CES forms, that is,

$$A(a^m_t, a^x_t) = \left[ \chi_m (a^m_t)^{1-\frac{1}{\mu_m}} + (1 - \chi_m) (a^x_t)^{1-\frac{1}{\mu_m}} \right]^{1-\frac{1}{\mu_m}},$$

$$B(a^x_t, a^n_t) = \left[ \chi_x (a^x_t)^{1-\frac{1}{\mu_n}} + (1 - \chi_x) (a^n_t)^{1-\frac{1}{\mu_n}} \right]^{1-\frac{1}{\mu_n}},$$
with $\chi_m, \chi_T \in (0, 1)$ and $\mu_{mx}, \mu_{mn} > 0$. The specification of the interest-rate premium and the capital adjustment costs are as in the MX model, that is,

$$p(d) = \psi \left(e^{d - \bar{d}} - 1\right)$$

and

$$\Phi_j(x) = \frac{\phi_j}{2} x^2,$$

with $\psi, \phi_j > 0$, for $j = m, x, n$.

### 8.4.8 Calibration Strategy

The present model is medium scale in size and lies at the intersection of trade and business-cycle analysis. The characterization of the steady state is complex—even numerically. The calibration of the model inherits this complexity.

As usual, we denote the steady-state value of a variable by dropping the time subscript. The equilibrium conditions (8.17)-(8.49) evaluated at the steady state and adopting the assumed functional forms represent a system of 33 equations in 52 unknowns, namely the 33 endogenous variables listed in the definition of equilibrium given in section 8.4.6 and 19 structural parameters, namely, $A_m, A_x, A_n, \delta, \omega_m, \omega_x, \omega_n, \beta, \chi_m, \chi_T, \mu_{mx}, \mu_{mn}, \alpha_m, \alpha_x, \alpha_n, r^*, \bar{d}, \text{tot},$ and $\sigma$.

Therefore, we must add 19 calibration restrictions (which we enumerate in parenthesis). In line with the calibration of the MX model of chapter 7, we set (1) $\sigma = 2$, (2)-(4) $\omega_m = \omega_x = \omega_n = 1.455$, (5) $\delta = 0.1$, (6) $r^* = 0.11$, (7) $\mu_{mx} = 1$, (8) $\text{tot} = 1$, (9) $A_m = 1$, (10) $\beta = 1/(1 + r^*)$, the export share in GDP is 20 percent, or (11) $x/(p^m y^m + p^x y^x + p^n y^n) = 0.2$, the trade balance-to-GDP ratio is 1.
percent, or (12) $\frac{(x - m)}{(p^m y^n + p^x y^x + p^n y^n)} = 0.01$ This adds 12 restrictions, which means that we are still 7 restrictions short. We normalize the productivity factor $A^n$ to unity, which adds the restriction (13) $A^n = 1$. Na (2015) estimates an average labor share for emerging countries of 70 percent, so we impose (14) $\frac{w^m h^m + w^x h^x + w^n h^n}{(p^m y^m + p^x y^x + p^n y^n)} = 0.7$. It is generally assumed that in emerging and poor countries the nontraded sector is more labor intensive than the export or import producing sectors. For instance, Uribe (1997), based on Argentine data, calculates the labor share in the nontraded sector to be 0.75. We follow this calibration and impose the restriction (15) $\frac{w^n h^n}{p^n y^n} = 0.75$. We assume that the importable and exportable sectors are equally labor intensive, that is, we impose (16) $\frac{w^m h^m}{p^m y^m} = \frac{w^x h^x}{p^x y^x}$. We follow the usual practice of proxying the share of nontraded output in total output by the observed share of the service sector in GDP. Using data from UNCTAD’s Handbook of Statistics on sectoral GDP for 51 poor and emerging countries over the period 1995 to 2012, we obtain an average share of services in GDP of slightly above 50 percent (for details on the data, see section 7.6.5 of chapter 7). Thus, we impose the restriction (17) $\frac{p^n y^n}{p^m y^m + p^x y^x + p^n y^n} = 0.5$. In chapter 7, we also document that in emerging and poor countries the exportable and importable sectors are of about the same size. Therefore, we impose the restriction (18) $p^x y^x = p^m y^m$. Finally, Akinci (2011) surveys the literature on estimates of the elasticity of substitution between tradables and nontradables in emerging and poor countries and arrives at a value close to 0.5. Thus we set (19) $\mu_{\tau n} = 0.5$. This completes the calibration strategy of all the parameters appearing in the set of steady-state equilibrium conditions.

The parameters $\pi$, $\rho$, $\psi$, and $\phi_j$, for $j = m, x, n$ do not appear in the steady-state equilibrium conditions, but play a role in the equilibrium dynamics. We calibrate $\pi$ and $\rho$ country by country using the econometric estimates presented in table 7.1 of chapter 7. We use a method of moments to estimate the capital adjustment cost parameters, $\phi_m$, $\phi_x$, $\phi_n$, and the parameter $\psi$ governing
**Table 8.2: Calibration of the MXN Model**

<table>
<thead>
<tr>
<th>Calibrated Structural Parameters</th>
<th>( \sigma )</th>
<th>( \delta )</th>
<th>( r^* )</th>
<th>( \alpha_m, \alpha_x )</th>
<th>( \alpha_n )</th>
<th>( \omega_m, \omega_x, \omega_n )</th>
<th>( \mu_{mx} )</th>
<th>( \mu_{rn} )</th>
<th>Imposed</th>
<th>( A^m )</th>
<th>( A^n )</th>
<th>( \beta )</th>
<th>( \pi )</th>
<th>( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.1</td>
<td>0.11</td>
<td>0.35</td>
<td>0.25</td>
<td></td>
<td>1.455</td>
<td>1</td>
<td>0.5</td>
<td>1</td>
<td>1/((1 + r^*))</td>
<td></td>
<td>**</td>
<td>**</td>
<td>**</td>
</tr>
</tbody>
</table>

**Moment Restrictions**

<table>
<thead>
<tr>
<th>Implied Structural Parameter Values</th>
<th>( \chi_m )</th>
<th>( \chi_T )</th>
<th>( \bar{d} )</th>
<th>( A^x )</th>
<th>( \beta )</th>
<th>( \phi_m )</th>
<th>( \phi_x )</th>
<th>( \phi_n )</th>
<th>( \psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8980</td>
<td>0.4360</td>
<td>0.0078</td>
<td>1</td>
<td>0.9009</td>
<td></td>
<td>**</td>
<td>**</td>
<td>**</td>
<td>**</td>
</tr>
</tbody>
</table>

Note. \( s_n \equiv p^m y^{m}/y \), \( s_x \equiv x/y \), and \( s_{tb} \equiv (x - m)/y \), where \( y \equiv p^m y^m + p^x y^x + p^n y^n \).

\*Country-specific values are given in table 7.1. **Country-specific values are given in table 8.3.

First, McIntyre (2003) estimates, using OECD data over the period 1970 to 1992, that the standard deviation of investment in the traded sector is 1.5 times as large as its counterpart in the nontraded sector. We assume that this relationship also holds for emerging and poor countries and conditional on terms-of-trade shocks. Thus, one of the moment restrictions we impose is \( \sigma_i^{m+i_x}/\sigma_{i_n} = 1.5 \) conditional on terms-of-trade shocks. The other two moment restrictions are the country-by-country empirical estimates of the investment-output volatility ratio and the trade balance-output volatility ratio conditional on terms of trade shocks. The empirical estimates of these volatility ratios are those implied by the country-by-country estimates of the SVAR presented in section 8.3. Thus, this estimation procedure delivers one set of parameters \( \phi_m, \phi_x, \phi_n \), and \( \psi \) for each country.

Tables 8.2 and 8.3 summarize the calibration of the MXN model. Table 8.3 shows that the

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6Even though the number of parameters to be estimated (four) exceeds the number of targeted moments (three), there is no under-identification problem, because no subset of parameters can exactly match the targeted moments.
Table 8.3: Country-Specific Estimates of the Capital Adjustment Cost Parameters and the Debt Elasticity of the Interest Rate

<table>
<thead>
<tr>
<th>Country</th>
<th>$\phi_m$</th>
<th>$\phi_x$</th>
<th>$\phi_n$</th>
<th>$\psi$</th>
<th>$\sigma_i/\sigma_y$ Data</th>
<th>$\sigma_i/\sigma_y$ Model</th>
<th>$\sigma_{ib}/\sigma_y$ Data</th>
<th>$\sigma_{ib}/\sigma_y$ Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algeria</td>
<td>0.01</td>
<td>60.49</td>
<td>32.27</td>
<td>0.01</td>
<td>2.79</td>
<td>2.38</td>
<td>2.10</td>
<td>2.09</td>
</tr>
<tr>
<td>Argentina</td>
<td>0.55</td>
<td>4.87</td>
<td>1.03</td>
<td>3.33</td>
<td>2.01</td>
<td>1.73</td>
<td>0.36</td>
<td>0.36</td>
</tr>
<tr>
<td>Bolivia</td>
<td>15.91</td>
<td>0.00</td>
<td>0.00</td>
<td>0.16</td>
<td>4.28</td>
<td>2.14</td>
<td>0.78</td>
<td>0.78</td>
</tr>
<tr>
<td>Botswana</td>
<td>11.55</td>
<td>0.00</td>
<td>0.00</td>
<td>0.02</td>
<td>5.93</td>
<td>2.43</td>
<td>1.62</td>
<td>1.62</td>
</tr>
<tr>
<td>Brazil</td>
<td>0.00</td>
<td>56.10</td>
<td>0.03</td>
<td>1.16</td>
<td>3.24</td>
<td>2.07</td>
<td>0.48</td>
<td>0.48</td>
</tr>
<tr>
<td>Burundi</td>
<td>15.36</td>
<td>0.00</td>
<td>0.04</td>
<td>0.09</td>
<td>2.52</td>
<td>2.14</td>
<td>0.90</td>
<td>0.89</td>
</tr>
<tr>
<td>Cameroon</td>
<td>0.98</td>
<td>1.07</td>
<td>9.33</td>
<td>0.34</td>
<td>2.14</td>
<td>2.44</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>Central African Republic</td>
<td>17.48</td>
<td>0.00</td>
<td>3.86</td>
<td>0.02</td>
<td>7.78</td>
<td>1.76</td>
<td>1.65</td>
<td>1.65</td>
</tr>
<tr>
<td>Colombia</td>
<td>0.00</td>
<td>26.96</td>
<td>0.00</td>
<td>7.38</td>
<td>3.09</td>
<td>2.04</td>
<td>0.39</td>
<td>0.39</td>
</tr>
<tr>
<td>Congo, Dem. Rep.</td>
<td>0.00</td>
<td>17.96</td>
<td>0.00</td>
<td>8.56</td>
<td>8.18</td>
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<td>Costa Rica</td>
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<td>2.42</td>
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<td>1.50</td>
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<td>Cote d’Ivoire</td>
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<td>19.89</td>
<td>0.00</td>
<td>9.27</td>
<td>3.37</td>
<td>2.02</td>
<td>0.27</td>
<td>0.26</td>
</tr>
<tr>
<td>Dominican Republic</td>
<td>0.00</td>
<td>60.09</td>
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<td>2.86</td>
<td>2.11</td>
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<td>0.41</td>
</tr>
<tr>
<td>Egypt, Arab Rep.</td>
<td>43.20</td>
<td>0.00</td>
<td>27.93</td>
<td>0.05</td>
<td>5.74</td>
<td>2.22</td>
<td>0.97</td>
<td>0.97</td>
</tr>
<tr>
<td>El Salvador</td>
<td>0.00</td>
<td>59.19</td>
<td>0.00</td>
<td>1.59</td>
<td>3.35</td>
<td>2.22</td>
<td>0.60</td>
<td>0.60</td>
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<tr>
<td>Ghana</td>
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<td>61.92</td>
<td>7.14</td>
<td>0.19</td>
<td>9.55</td>
<td>2.47</td>
<td>0.91</td>
<td>0.91</td>
</tr>
<tr>
<td>Guatemala</td>
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<td>0.00</td>
<td>0.01</td>
<td>0.00</td>
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<td>7.02</td>
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<td>1.81</td>
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<tr>
<td>Honduras</td>
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<td>0.00</td>
<td>0.06</td>
<td>6.02</td>
<td>2.31</td>
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<td>1.08</td>
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<td>0.30</td>
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<td>0.76</td>
<td>0.97</td>
<td>1.49</td>
<td>1.29</td>
<td>0.27</td>
<td>0.27</td>
</tr>
<tr>
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<td>27.60</td>
<td>0.01</td>
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<td>4.26</td>
<td>2.01</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>Jordan</td>
<td>2.88</td>
<td>2.88</td>
<td>0.62</td>
<td>0.03</td>
<td>1.04</td>
<td>1.09</td>
<td>1.30</td>
<td>1.23</td>
</tr>
<tr>
<td>Kenya</td>
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<td>54.54</td>
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<td>5.17</td>
<td>2.19</td>
<td>0.71</td>
<td>0.70</td>
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<tr>
<td>Korea, Rep.</td>
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<td>0.00</td>
<td>9.32</td>
<td>0.06</td>
<td>4.86</td>
<td>2.26</td>
<td>0.95</td>
<td>0.95</td>
</tr>
<tr>
<td>Madagascar</td>
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<td>0.03</td>
<td>0.09</td>
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<td>2.18</td>
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<td>0.85</td>
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<tr>
<td>Malaysia</td>
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<td>0.00</td>
<td>0.00</td>
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<td>4.39</td>
<td>2.45</td>
<td>1.68</td>
<td>1.68</td>
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<tr>
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<td>0.02</td>
<td>0.01</td>
<td>4.25</td>
<td>2.40</td>
<td>2.34</td>
<td>2.34</td>
</tr>
<tr>
<td>Mexico</td>
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<td>1.01</td>
<td>0.61</td>
<td>0.24</td>
<td>1.58</td>
<td>1.36</td>
<td>0.40</td>
<td>0.40</td>
</tr>
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<td>Morocco</td>
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<td>0.00</td>
<td>0.60</td>
<td>5.03</td>
<td>2.23</td>
<td>0.67</td>
<td>0.67</td>
</tr>
<tr>
<td>Pakistan</td>
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<td>0.03</td>
<td>0.20</td>
<td>0.05</td>
<td>8.43</td>
<td>2.21</td>
<td>1.16</td>
<td>1.15</td>
</tr>
<tr>
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<td>10.68</td>
<td>4.50</td>
<td>0.72</td>
<td>2.13</td>
<td>1.84</td>
<td>0.62</td>
<td>0.61</td>
</tr>
<tr>
<td>Peru</td>
<td>0.25</td>
<td>14.39</td>
<td>2.61</td>
<td>9.82</td>
<td>2.21</td>
<td>1.91</td>
<td>0.20</td>
<td>0.21</td>
</tr>
<tr>
<td>Philippines</td>
<td>0.10</td>
<td>8.17</td>
<td>1.11</td>
<td>1.12</td>
<td>1.78</td>
<td>1.66</td>
<td>0.44</td>
<td>0.41</td>
</tr>
<tr>
<td>Senegal</td>
<td>70.80</td>
<td>0.00</td>
<td>35.03</td>
<td>0.07</td>
<td>14.21</td>
<td>2.33</td>
<td>0.86</td>
<td>0.85</td>
</tr>
<tr>
<td>South Africa</td>
<td>122.76</td>
<td>0.01</td>
<td>56.35</td>
<td>0.68</td>
<td>2.47</td>
<td>2.11</td>
<td>0.28</td>
<td>0.28</td>
</tr>
<tr>
<td>Sudan</td>
<td>30.29</td>
<td>0.00</td>
<td>0.00</td>
<td>0.49</td>
<td>4.14</td>
<td>1.85</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>Thailand</td>
<td>1.74</td>
<td>1.74</td>
<td>1.32</td>
<td>0.30</td>
<td>0.74</td>
<td>0.78</td>
<td>0.61</td>
<td>0.56</td>
</tr>
<tr>
<td>Turkey</td>
<td>0.00</td>
<td>16.61</td>
<td>0.00</td>
<td>10.74</td>
<td>6.29</td>
<td>2.02</td>
<td>0.31</td>
<td>0.31</td>
</tr>
<tr>
<td>Uruguay</td>
<td>0.40</td>
<td>0.40</td>
<td>2.05</td>
<td>4.72</td>
<td>1.70</td>
<td>1.61</td>
<td>0.27</td>
<td>0.29</td>
</tr>
<tr>
<td>Median</td>
<td>2.31</td>
<td>1.04</td>
<td>0.12</td>
<td>0.27</td>
<td>3.36</td>
<td>2.12</td>
<td>0.64</td>
<td>0.64</td>
</tr>
<tr>
<td>Median Absolute Deviation</td>
<td>2.31</td>
<td>1.04</td>
<td>0.12</td>
<td>0.26</td>
<td>1.42</td>
<td>0.21</td>
<td>0.33</td>
<td>0.33</td>
</tr>
</tbody>
</table>
parameter estimation yields a satisfactory match of the observed and predicted volatility of the trade balance relative to output conditional on terms of trade shocks. Both in the data and in the model the cross-country median of this volatility ratio is 0.64. The matching of the relative volatility of aggregate investment conditional on terms-of-trade shocks is less tight. In the data the median volatility of investment relative to that of output is 3.36 whereas its predicted counterpart is 2.12. One may wonder why the estimated values of $\phi_j$, for $j = m, x, n$ are not lower, given that the model underpredicts the relative volatility of investment. The reason is that, although reducing the values of the $\phi$s raises all sectoral investment volatilities, it need not result in higher aggregate investment volatility. For as the $\phi$s go down in value, sectoral investments become increasingly negatively correlated, conspiring against the volatility of aggregate investment. This negative correlation among sectoral investments is intuitive, because when the relative price of exportables in terms of importables rises (i.e., when the terms of trade appreciate) the value of the marginal product of capital increases in the exportable sector and declines in the importable sector. Thus, an improvement in the terms of trade acts like a positive productivity shock in the exportable sector and a negative productivity shock in the importable sector.

Table 8.3 strongly suggests that there is no such thing as a representative emerging or poor country. It shows that the estimated parameters display significant cross-country variations. The median absolute deviations of all four estimated parameters are as large as the corresponding medians themselves. For this reason, in the analysis that follows, we adopt the approach of deriving model implications country-by-country as opposed to deriving implications for a model economy calibrated to the average parameter estimate.
8.5 Response of the Real Exchange Rate to Terms-of-Trade Shocks in the MXN Model

As explained earlier, the real exchange rate is defined as the ratio of the foreign consumer price index to the domestic consumer price index. Formally,

\[ RER_t = \frac{E_t P^*_t}{P_t}, \]

where \( E_t \) denotes the nominal exchange rate, defined as the domestic-currency price of one unit of foreign currency, \( P^*_t \) denotes the foreign price of consumption, and \( P_t \) denotes the domestic price of consumption. Divide the numerator and denominator by the domestic-currency price of the tradable composite good, denoted \( P^*_\tau \), to get \( RER_t = (E_t P^*_t / P^*_\tau) / (P_t / P^*_\tau) \). We assume that the law of one price holds for importable and exportable goods and that the technology for aggregating importables and exportables into the tradable composite good, \( A(\cdot, \cdot) \), is common across countries. Then, the law of one price must also hold for the tradable composite good, that is, \( E_t P^*_\tau = P^*_\tau \), where \( P^*_\tau \) denotes the foreign price of the tradable composite good. This yields \( RER_t = (P^*_t / P^*_\tau) / (P_t / P^*_\tau) \).

We assume that the terms of trade shocks that are relevant to our small open economy do not affect the relative price of the tradable composite good in terms of consumption goods in the rest of the world. We therefore assume that \( P^*_t / P^*_\tau \) is constant. Without loss of generality, we normalize \( P^*_t / P^*_\tau \) to unity. Finally, noting that \( p^*_\tau \equiv P^*_\tau / P_t \), we have

\[ RER_t = p^*_\tau, \]

which says that the real exchange rate equals the relative price of the tradable composite good in terms of final goods. An increase in \( RER_t \), that is, a depreciation of the real exchange rate, means that tradables become more expensive relative to final goods. Conversely, a decrease in \( RER_t \), that
is, an appreciation of the real exchange rate, means that tradables become less expensive relative to final goods.

Figure 8.2 displays the response of the MXN economy to a ten-percent increase in the terms of trade. The impulse responses shown in the figure are medians across the impulse responses implied by country-specific calibrations of the MXN model. The model does a good job at capturing the response of the real exchange rate to a terms-of-trade shock. Comparing figures 8.1 and 8.2 shows that in both the SVAR and the MXN models an improvement in the terms of trade appreciates the real exchange rate. The explanation behind this result has to do with substitution and income effects. An increase in the relative price of exportables induces a substitution of importable and nontraded absorption for exportable absorption. At the same time, the increase in the price of exportables produces a positive income effect that boosts the domestic demand for all types of goods. Both effects drive up the price of nontradables, because the expansion in the demand for this type of goods must be met by domestic producers, who require a higher price to produce more. The top right panel of figure 8.2 shows that indeed nontradables become more expensive after the positive terms-of-trade shock. In turn, the increase in the price of nontradables translates into an increase in the price of the final good relative to the price of the tradable composite good, that is, $p^\tau_t$ falls.

Intuitively, the increase in the terms of trade produces an expansion in exports. Imports also increase, because as these goods become cheaper relative to exportable goods, consumers increase demand and domestic producers cut back supply. The net effect on the trade balance turns out to be positive. Thus, the MXN model is in line with the Harberger-Laursen-Metzler effect present in the SVAR model and discussed in detail in chapter 7.

In line with the predictions of the SVAR model, the MXN model implies that output expands in response to an improvement in the terms of trade (panel (3,1) of figure 8.2). This expansion is the result of increased activity in the export and the nontraded sectors, which is only partially
Figure 8.2: Response of the MXN Economy to a Ten-Percent Terms-of-Trade Shock

Note. All variables with the exception of the trade balance are expressed in percent deviations from steady state. The trade balance is expressed in level deviations from steady state in percent of steady-state output. Impulse responses are cross-country medians. For each country, impulse responses are produced using the country specific calibrations of $\phi_m$, $\phi_x$, $\phi_n$, $\psi$, and $\rho$. The matlab script plotirbc.m, available online with the materials for this chapter, generates this figure.
offset by a contraction in the importable sector (row 4 of figure 8.2). The similarity between model and data ends at the qualitative level, however. For the increase in output predicted by the MXN model is 10 times larger than that implied by the empirical SVAR model.

The MXN model predicts that both consumption and investment increase in response to the improvement in the terms of trade. This prediction is partly supported by the data. The SVAR predicts that the expansion in consumption and investment is delayed by one period, whereas the MXN model predicts that both variables increase on impact. In addition, as in the case of output, the magnitudes of the theoretical responses are much larger than their empirical counterparts.

Finally, earlier we pointed out that in the MXN model lowering the adjustment cost parameters $\phi_j$, for $j = m, x, n$ does not necessarily result in an increase in aggregate investment volatility because of a negative cross sectoral correlation of investment which soaks up the volatility of each individual component. This effect is patently displayed in the bottom row of figure 8.2. The 10-percent improvement in the terms of trade causes aggregate investment to increase by only 2 percent. However, investment in the exportable sector increase by 70 percent while in the importable sector it decreases by 50 percent. It would be of interest to see whether this prediction of the MXN model is borne out in the data.

8.6 The Terms-of-Trade Disconnect

An important theme of chapter 7 is the finding of a sharp contrast between the relatively small role played by terms of trade shocks as a source of business cycles in empirical SVAR models and the dominant role they play in theory. This result was derived in the context of the MX model, which assumes that all goods are internationally traded. In reality, however, as the evidence presented in section 8.4.8 suggests, about half of value added in any given economy is produced in the nontraded sector. Omitting this large sector from the model could in principle explain the
Table 8.4: Share of Variances Explained by Terms-of-Trade Shocks: The MXN Model Versus the Empirical SVAR Model

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<th>Empirical Model (3)</th>
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<tr>
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</table>

Note. Each entry is the cross-country median of the fraction of the variance of the corresponding indicator explained by terms of trade shocks in percent. Column (1) reproduces from table 7.6 the results for a model without nontraded goods (MX model). Column (2) corresponds to the MXN model. In columns (1) and (2) the numerator of the fraction is the variance conditional on terms-of-trade shocks predicted by the MX and MXN models, respectively, for country-specific calibrations of $\phi_m$, $\phi_x$, $\phi_n$, $\psi$, $\rho$, and $\pi$. The denominator is the unconditional variance implied by the empirical country-specific SVAR model. Column (3) reproduces the row labeled ‘Median’ from table 8.1. The matlab script mxnvar.cbc.m, available online with the materials for this chapter, generates this table.

exaggerated importance the MX model ascribes to terms of trade shocks. One motivation for building the MXN model is to address precisely this concern.

A comparison of the empirical and theoretical impulse responses displayed in figures 8.1 and 8.2, respectively, provides a hint that the introduction of nontradables might not represent a solution to the gap between the model and data. For it shows that the responses of key macro indicators, like output, consumption, and the trade balance, are many times larger in the theoretical model than in the data. Table 8.4 confirms this suspicion. It shows that in the MXN model, terms of trade shocks explain one third of the variances of the trade balance, output and consumption, whereas in the data they explain only 10 percent. That is, the MXN model predicts that terms of trade shocks
### Table 8.5: Share of Variance Explained by Terms of Trade Shocks: Country-Level Predictions of the MXN Model

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Note: The MATLAB script `mxn_varcbc.m`, available online with the materials for this chapter, generates this table.
are three times as important as implied by the empirical model. It is in this precise sense that we say that there is a terms-of-trade disconnect between theory and data. Table 8.5 documents the disconnect country by country.

The introduction of nontradables does help reduce the importance of terms of trade in the model. Comparing columns 1 and 2 of table 8.4 shows that the share of the variance of output due to terms of trade shocks falls from 49 percent in the MX model to 34 percent in the MXN model. A similar pattern holds for the trade balance and consumption. But this improvement is not enough, for the gap between the MXN model and the data is still 200 percent, which is a large number.

The disconnect between the empirical and theoretical models extends, to a lesser degree, to the role of terms-of-trade shocks in explaining fluctuations in the real exchange rate. Table 8.4 shows that in the empirical model terms-of-trade shocks explain on average 14 percent of the variance of the real exchange rate. By contrast, in the theoretical model, terms-of-trade disturbances account on average for only 1 percent of the variance of the real-exchange-rate. It is noteworthy, that while the theoretical model exaggerates the role of terms of trade shocks in explaining variations in aggregate quantities, it underpredicts their role in generating movements in the real exchange rate.

8.7 Summing Up

The conventional wisdom is that terms of trade shocks represent a major source of business cycles in emerging and poor countries. This view is based on the analysis of calibrated business-cycle models. For example, Mendoza (1995) presents a model with a traded and a nontraded sector and finds that for a realistic calibration of the terms-of-trade process, the share of the variance of output due to terms of trade shocks is 31 percent, which is in line with our findings.\footnote{Mendoza reports that the ratio of the standard deviation of output due to terms shocks to the standard deviation of observed output is 0.56. The square of this number, 0.31, is the share of the variance of output due to terms of} Kose
(2002) uses a similar methodology and arrives at even higher numbers. What we have done in this chapter is to confront the predictions coming out of this type of analysis with those stemming from empirical SVAR models. The picture that emerges from this comparison is strikingly different from the conventional view. First, under the present approach, terms of trade play a relatively minor role in the generation of business cycles in emerging countries. Second, there emerges a disconnect between the predictions of theoretical models and those of empirical SVARs, with the former attributing too much importance to terms of trade shocks. The main message of this chapter is therefore that future research should refocus effort toward building models that can explain the relatively minor role played by terms of trade shocks in driving short-run fluctuations.
8.8 Exercises

Exercise 8.1 (Properties of the Armington Aggregator) Assume that the Armington aggregator $A(x, y)$ is increasing in both arguments, homogeneous of degree one, and concave. Show that $A_2(x, y)/A_1(x, y)$ can be written as $P(x/y)$, with $P'(\cdot) > 0$.

Exercise 8.2 (Secular Trend in the Real Exchange Rate) Modify the endowment economy of section 8.2 by assuming that $\beta < 1/(1 + r)$. Characterize the equilibrium dynamics. Provide intuition for the equilibrium behavior of the real exchange rate.

Exercise 8.3 (A Two-Period TNT Economy with Production) Consider a two-period economy populated by identical households with preferences defined by the utility function

$$\ln c_1 + \ln c_2$$

where $c_1$ and $c_2$ denote consumption in periods 1 and 2 respectively. Suppose that consumption is a composite of tradable and nontradable goods given by

$$c_t = \sqrt{c_t^T c_t^N},$$

for $t = 1, 2$, where $c_t^T$ and $c_t^N$ denote consumption of tradables and nontradables in period $t$, respectively. Let $p_t$ denote the relative price of nontradables in terms of tradables. Household are endowed with 2 units of labor per period, which they supply to the market inelastically. Let $w_t$ denote the wage rate in period $t$ expressed in terms of tradables. Households are the owners of firms, from which they receive profits each period. Assume also that households start period 1 without any assets or liabilities and can borrow or lend in the international credit market at a zero interest rate.
Tradable and nontradable goods are produced using labor as the sole input, via the technologies

\[ y^T_t = \sqrt{h^T_t} \]

and

\[ y^N_t = \sqrt{h^N_t}, \]

where \( y^i_t \) and \( h^i_t \) denote output and employment in sector \( i \) in period \( t \) for \( i = T, N \) and \( t = 1, 2 \). Firms in both sectors behave competitively. They pay the wage \( w_t \) per unit of labor and distribute profits to households.

1. Write down the household’s utility maximization problem.

2. Derive the optimality conditions associated with the household’s problem.

3. Write down the firms’ profit maximization problems and derive the associated optimality conditions.

4. Write down the market clearing conditions in this economy.

5. Calculate the equilibrium values of consumption of tradables, consumption of nontradables, the wage rate, the relative price of nontradables, the trade balance, and the current account in periods 1 and 2. The answer to this question are 12 numbers, but show your work. Provide intuition for the temporal behavior of the relative price of nontradables and the current account.
Chapter 9

Nominal Rigidity, Exchange Rates, And Unemployment

In this chapter, we build a theoretical framework in which the presence of nominal rigidities can induce an inefficient adjustment to aggregate disturbances. The analysis is guided by two objectives. One is to convey in an intuitive manner how nominal rigidities amplify the business cycle in open economies. The second is to develop a framework from which one can derive quantitative predictions useful for policy evaluation.

To motivate the type of theoretical environment we will study in this chapter, take a look at figure 9.1. It displays the current account, nominal hourly wages, and the unemployment rate in the periphery of Europe between 2000 and 2011. The inception of the Euro in 1999 was followed by massive capital inflows into the region, possibly driven by expectations of a quick convergence of peripheral and central Europe to core Europe (Germany and France). The boom lasted from 2000 to 2008 and was characterized by large current account deficits, spectacular nominal wage increases, and declining rates of unemployment. With the onset of the global crisis of 2008, capital
inflows dried up abruptly (see the sharp reduction in the current account deficit) and the region suffered a severe sudden stop. At the same time, central banks were unable to change the course of monetary policy because the respective countries were either in the eurozone or pegging to the euro. In spite of the collapse in aggregate demand and the lack of a devaluation, nominal wages remained as high as during the boom. Meanwhile, massive unemployment affected all countries in the region. The data in figure 9.1 does not provide any indication of causality. In this chapter, we will interpret episodes of the type illustrated in the figure through the lens of a theoretical model in which, following a negative external shock, the combination of downward nominal wage rigidity and a fixed exchange rate, can cause massive involuntary unemployment.

This chapter builds on Schmitt-Grohé and Uribe (2014).

9.1 An Open Economy With Downward Nominal Wage Rigidity

We develop a model of a small open economy in which nominal wages are downwardly rigid. The model features two types of goods, tradables and nontradables. The economy is driven by two
exogenous shocks, a country-interest-rate shock and a terms-of-trade shock.

9.1.1 Households

Consider an economy populated by a large number of identical households with preferences described by the utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t),$$

(9.1)

where $c_t$ denotes consumption. The period utility function $U$ is assumed to be strictly increasing and strictly concave and the parameter $\beta$, denoting the subjective discount factor, resides in the interval $(0,1)$. The symbol $E_t$ denotes the mathematical expectations operator conditional upon information available in period $t$. The consumption good is a composite of tradable consumption, $c_t^T$, and nontradable consumption, $c_t^N$. The aggregation technology is of the form

$$c_t = A(c_t^T, c_t^N),$$

(9.2)

where $A$ is an increasing, concave, and linearly homogeneous function.

We assume that all of the external liabilities of the household are denominated in foreign currency. This assumption is motivated by the empirical literature on the ‘original sin,’ which documents that virtually all of the debt issued by emerging countries is denominated in foreign currency (see, for example, Eichengreen, Hausmann, and Panizza, 2005). Specifically, households are assumed to have access to a one-period, internationally traded, state non-contingent bond denominated in tradables. We let $d_t$ denote the level of debt assumed in period $t - 1$ and due in period $t$ and $r_t$ the interest rate on debt held between periods $t$ and $t + 1$. The sequential budget constraint of the household is given by

$$P_t^T c_t^T + P_t^N c_t^N + E_t d_t = P_t^T y_t^T + W_t h_t + \Phi_t + \frac{E_t d_{t+1}}{1 + r_t},$$

(9.3)
where $P_T^t$ denotes the nominal price of tradable goods, $P_N^t$ the nominal price of nontradable goods, $\mathcal{E}_t$ the nominal exchange rate defined as the domestic-currency price of one unit of foreign currency, $y_T^t$ the endowment of traded goods, $W_t$ the nominal wage rate, $h_t$ hours worked, and $\Phi_t$ nominal profits from the ownership of firms. The variables $r_t$ and $y_T^t$ are assumed to be exogenous and stochastic. Movements in $y_T^t$ can be interpreted either as shocks to the physical availability of tradable goods or as shocks to the country’s terms of trade.

Households supply inelastically $\bar{h}$ hours to the labor market each period. Thus, hours worked must satisfy $h_t \leq \bar{h}$. In section 9.13 we study the case of an endogenous labor supply.

Households are assumed to be subject to a debt limit

$$d_{t+1} \leq \bar{d},$$

which prevents them from engaging in Ponzi schemes, where $\bar{d}$ denotes the natural debt limit.

We assume that the law of one price holds for tradables. Specifically, letting $P_T^{*t}$ denote the foreign currency price of tradables, the law of one price implies that

$$P_T^t = P_T^{*t} \mathcal{E}_t.$$

We further assume that the foreign-currency price of tradables is constant and normalized to unity, $P_T^{*t} = 1.$\footnote{Exercise 9.6 relaxes this assumption.} Thus, we have that the nominal price of tradables equals the nominal exchange rate,

$$P_T^t = \mathcal{E}_t.$$

Households choose contingent plans $\{c_t, c_T^t, c_N^t, d_{t+1}\}$ to maximize (9.1) subject to (9.2)-(9.4) taking as given $P_T^t$, $P_N^t$, $\mathcal{E}_t$, $W_t$, $h_t$, $\Phi_t$, $r_t$, and $y_T^t$. Letting $p_t \equiv P_N^t / P_T^t$ denote the relative price
of nontradables in terms of tradables and using the fact that $P_t^T = \mathcal{E}_t$, the optimality conditions associated with this problem are (9.2)-(9.4) and

\[
\frac{A_2(c_t^T, c_t^N)}{A_1(c_t^T, c_t^N)} = p_t, \tag{9.5}
\]

\[
\lambda_t = U'(c_t) A_1(c_t^T, c_t^N),
\]

\[
\frac{\lambda_t}{1 + r_t} = \beta E_t \lambda_{t+1} + \mu_t,
\]

\[
\mu_t \geq 0,
\]

\[
\mu_t (d_{t+1} - \bar{d}) = 0,
\]

where $\beta' \lambda_t / P_t^T$ and $\beta' \mu_t$ denote the Lagrange multipliers associated with (9.3) and (9.4), respectively.

Equation (9.5) describes the household’s demand for nontradables as a function of the relative price of nontradables, $p_t$, and the level of tradable absorption, $c_t^T$. Given $c_t^T$, the demand for nontradables is strictly decreasing in $p_t$. This property is a consequence of the assumptions made about the aggregator function $A$ (see exercise 9.1). It reflects the fact that as the relative price of nontradables increases, households tend to consume relatively less nontradables. The demand function for nontradables is depicted in figure 9.2 with a downward sloping solid line. An increase in the absorption of tradables shifts the demand schedule up and to the right, reflecting normality. Such a shift is shown with a dashed downward sloping line in figure 9.2 for an increase in traded consumption from $c_0^T$ to $c_1^T > c_0^T$. It follows that absorption of tradables can be viewed as a shifter of the demand for nontradables. Of course, $c_t^T$ is itself an endogenous variable, which is determined simultaneously with all other endogenous variables of the model.
Figure 9.2: The Demand For Nontradables

\[ A_2(c^T_1, c^N) \]
\[ A_1(c^T_1, c^N) \]

\[ A_2(c^T_0, c^N) \]
\[ A_1(c^T_0, c^N) \]

quantity, \( c^N \)

price, \( p \)

\( c^T_1 > c^T_0 \)
9.1.2 Firms

Nontraded output, denoted $y_t^N$, is produced by perfectly competitive firms. Each firm operates a production technology given by

$$y_t^N = F(h_t),$$

which uses labor services as the sole input. The function $F$ is assumed to be strictly increasing and concave. Firms choose the amount of labor input to maximize profits, given by

$$\Phi_t \equiv P_t^N F(h_t) - W_th_t.$$

The optimality condition associated with this problem is $P_t^N F'(h_t) = W_t$. Dividing both sides by $P_t^T$ and using the facts that $P_t^T = E_t$ and that $h_t = F^{-1}(y_t^N)$ yields a supply schedule of nontradable goods of the form

$$p_t = \frac{W_t/E_t}{F'(F^{-1}(y_t^N))}.$$

This supply schedule is depicted with a solid upward sloping line in figure 9.3. All other things equal, the higher is the relative price of the nontraded good, the larger is the supply of nontradable goods. Also, the higher is the labor cost, $W_t/E_t$, the smaller is the supply of nontradables at each level of the relative price $p_t$. That is, an increase in the nominal wage rate, holding constant the nominal exchange rate, causes the supply schedule to shift up and to the left. Figure 9.3 displays with a broken upward sloping line the shift in the supply schedule that results from an increase in the nominal wage rate from $W_0$ to $W_1 > W_0$, holding the nominal exchange rate constant at $E_0$.

A currency devaluation, holding the nominal wage constant, shifts the supply schedule down and to the right. Intuitively, a devaluation that is not accompanied by a change in nominal wages reduces the real labor cost thereby inducing firms to increase the supply of nontradable goods for any given relative price. To illustrate this effect, assume that the nominal wage equals $W_1$ and the
nominal exchange rate equals $E_0$. The corresponding supply schedule is the upward sloping broken line in figure 9.3. To keep the graph simple, suppose that the government devalues the currency to a level $E_1 > E_0$ such that $W_1/E_1 = W_0/E_0$. Such a devaluation shifts the supply schedule back to its original position given by the solid line.

### 9.1.3 Downward Nominal Wage Rigidity And The Labor Market

The central friction emphasized in this chapter is downward nominal wage rigidity. Specifically, we impose that

$$ W_t \geq \gamma W_{t-1}, \quad \gamma > 0. \tag{9.6} $$

The parameter $\gamma$ governs the degree of downward nominal wage rigidity. The higher is $\gamma$, the more downwardly rigid are nominal wages. This setup nests the cases of absolute downward rigidity,
when $\gamma \geq 1$, and full wage flexibility, when $\gamma = 0$. In section 9.4, we present empirical evidence suggesting that $\gamma$ is close to unity when time is measured in quarters.

The presence of downwardly rigid nominal wages implies that the labor market will in general not clear. Instead, involuntary unemployment, given by $\bar{h} - h_t$, will be a regular feature of this economy. Actual employment must satisfy

$$ h_t \leq \bar{h} \quad (9.7) $$

at all times. We postulate that at any point in time, wages and employment must satisfy the slackness condition

$$ (\bar{h} - h_t) (W_t - \gamma W_{t-1}) = 0. \quad (9.8) $$

This condition states that in periods in which the economy suffers from involuntary unemployment ($h_t < \bar{h}$) the lower bound on nominal wages must be binding ($W_t = \gamma W_{t-1}$). It also states that in periods in which the lower bound on nominal wages is not binding ($W_t > \gamma W_{t-1}$), the economy must be operating at full employment ($h_t = \bar{h}$). Notice that the equilibrium level of employment is always given by the minimum of the demand for labor $h_t$ and the supply of labor $\bar{h}$. Therefore, equilibrium employment is not always demand determined. For example, during booms, employment may be supply determined (at $\bar{h}$). In other words, households are never required to work more hours than they wish to at the going wage and firms are never forced to hire more workers than they desire at the going wage.

### 9.1.4 Equilibrium

In equilibrium, the market for nontraded goods must clear at all times. That is, the condition

$$ c_t^N = y_t^N $$
must hold for all \( t \). Combining this condition, the production technology for nontradables, the household’s budget constraint, and the definition of firm’s profits, we obtain the following market-clearing condition for traded goods:

\[
\begin{align*}
    c_t^T + d_t &= y_t^T + \frac{d_{t+1}}{1+r_t}, \\
    d_{t+1} &\leq \bar{d}, \\
    \mu_t &\geq 0, \\
    \mu_t(d_{t+1} - \bar{d}) &= 0,
\end{align*}
\]

Let

\[
    w_t \equiv \frac{W_t}{\varepsilon_t}
\]

denote the real wage in terms of tradables and

\[
    \epsilon_t \equiv \frac{\varepsilon_t}{\varepsilon_{t-1}}
\]

de the gross devaluation rate of the domestic currency. Then, the wage constraint (9.6) and the slackness condition (9.8) can be expressed in real terms as

\[
    w_t \geq \gamma w_{t-1}/\epsilon_t \quad \text{and} \quad (\bar{h} - h_t)(w_t - \gamma w_{t-1}/\epsilon_t) = 0,
\]

respectively.

A competitive equilibrium is a set of stochastic processes \( \{c_t^T, h_t, w_t, d_{t+1}, p_t, \lambda_t, \mu_t\}_{t=0}^{\infty} \) satisfying

\[
\begin{align*}
    c_t^T + d_t &= y_t^T + \frac{d_{t+1}}{1+r_t}, \\
    d_{t+1} &\leq \bar{d}, \\
    \mu_t &\geq 0, \\
    \mu_t(d_{t+1} - \bar{d}) &= 0, \\
    \lambda_t &= U'(A(c_t^T, F(h_t)))A_1(c_t^T, F(h_t)),
\end{align*}
\]
\[ \frac{\lambda_t}{1 + r_t} = \beta E_t \lambda_{t+1} + \mu_t, \quad (9.14) \]

\[ p_t = \frac{A_2(c_t^T, F(h_t))}{A_1(c_t^T, F(h_t))}, \quad (9.15) \]

\[ p_t = \frac{w_t}{F'(h_t)}, \quad (9.16) \]

\[ w_t \geq \gamma \frac{w_{t-1}}{\epsilon_t}, \quad (9.17) \]

\[ h_t \leq \bar{h}, \quad (9.18) \]

\[ (\bar{h} - h_t) \left( w_t - \gamma \frac{w_{t-1}}{\epsilon_t} \right) = 0, \quad (9.19) \]

given an exchange rate policy \( \{\epsilon_t\}_{t=0}^\infty \), initial conditions \( w_{-1} \) and \( d_0 \), and exogenous stochastic processes \( \{r_t, y_t^T\}_{t=0}^\infty \).

Notice that all markets except the labor market are in equilibrium. One might therefore wonder whether this situation violates Walras’ Law, according to which, if all markets but one can be verified to be in equilibrium, then the remaining market must also be in equilibrium. The answer is that Walras’ Law is not applicable in the current environment, because the present model does not feature a Walrasian equilibrium. A Walrasian equilibrium is built under the assumption that at the price vector submitted by a fictitious auctioneer, all market participants submit notional demand and supplies of final goods and inputs of production. That is, supplies and demands computed under the assumption that at the given price vector (regardless of whether it happens to be the equilibrium price vector or not) the agent could buy or sell any desired quantities of final goods and inputs of production subject only to his budget constraints. But this is not the case under the present non-Walrasian equilibrium. In particular, at any given price vector the household’s labor supply is not its desired supply of labor, \( \bar{h} \), but its realized employment, \( h_t \), reflecting the fact that households internalize the existence of rationing in the labor market. As a result, adding up the
budget constraints of all households and using the fact that all markets but the labor market clear, does not yield the result that the aggregate desired supply of labor equals the aggregate desired demand of labor, or \( \bar{h} = h_t \), but rather the tautology that the desired demand for labor equals the desired demand for labor, \( h_t = h_t \). In other words, in the present model, the fact that in equilibrium all but one market clear does not imply that the remaining market must also clear.

To characterize the competitive equilibrium, one must specify the exchange-rate regime. We will study a variety of empirically realistic exchange-rate policies. We begin with currency pegs, a policy that is frequently observed in the emerging-country world.

### 9.2 Currency Pegs

Countries can find themselves confined to a currency peg in a number of ways. For instance, a country could have adopted a currency peg as a way to stop high or hyperinflation in a swift and nontraumatic way. A classical example is the Argentine Convertibility Law of April 1991, which, by mandating a one-to-one exchange rate between the Argentine peso and the U.S. dollar, painlessly eliminated hyperinflation virtually overnight. Another route by which countries arrive at a currency peg is the joining of a monetary union. Recent examples include emerging countries in the periphery of the European Union, such as Ireland, Portugal, Greece, and a number of small eastern European countries that joined the Eurozone. Most of these countries experienced an initial transition into the Euro characterized by low inflation, low interest rates, and economic expansion.

However, history has shown time and again that fixed exchange rate arrangements are easy to adopt but difficult to maintain. Continuing with the example of the Argentine peg of the 1990s, its initial success in stabilizing inflation and restoring growth turned into a nightmare by the end of the decade. Starting in 1998 Argentina was hit by a string of large negative external shocks, including depressed commodity prices and elevated country premia, which pushed the economy into a deep
deflationary recession. Between 1998 and late 2001, the subemployment rate, which measures the fraction of the population that is either unemployed or involuntarily working part time, increased by 10 percentage points. At the same time, consumer prices were falling at a rate near one percent per year. Eventually, the crisis led to the demise of the peg in December of 2001.

The Achilles' heel of currency pegs is that they hinder the efficient adjustment of the economy to negative external shocks, such as drops in the terms of trade, captured by the variable $y_T$ in our model, or hikes in the country interest-rate, captured by the variable $r_t$. The reason is that such shocks produce a contraction in aggregate demand that requires a decrease in the relative price of nontradables, that is, a real depreciation of the domestic currency, in order to bring about an expenditure switch away from tradables and toward nontradables. In turn, the required real depreciation may come about via a nominal devaluation of the domestic currency or via a fall in nominal prices or both. The currency peg rules out a devaluation. Thus, the only way the necessary real depreciation can occur is through a decline in the nominal price of nontradables. However, when nominal wages are downwardly rigid, producers of nontradables are reluctant to lower prices, for doing so might render their enterprises no longer profitable. As a result, the necessary real depreciation takes place too slowly, causing recession and unemployment along the way.

This narrative goes back at least to Keynes (1925) who argued that Britain's 1925 decision to return to the gold standard at the 1913 parity despite the significant increase in the aggregate price level that took place during World War I would cause deflation in nominal wages with deleterious consequences for unemployment and economic activity. Similarly, Friedman's (1953) seminal essay points at downward nominal wage rigidity as the central argument against fixed exchange rates. This section formalizes this narrative in the context of the dynamic, stochastic, optimizing model developed in the previous section. Later sections use parameterized versions of this model to generate precise quantitative predictions for aggregate activity around external crises and for the welfare costs of currency pegs.
A currency peg is an exchange rate policy in which the nominal exchange rate is fixed. The gross devaluation rate therefore satisfies

$$\epsilon_t = 1,$$

(9.20)

for all $t \geq 0$. Under a currency peg, the economy is subject to two nominal rigidities. One is policy induced: The nominal exchange rate, $E_t$, is kept fixed by the monetary authority. The second is structural and is given by the downward rigidity of the nominal wage $W_t$. The combination of these two nominal rigidities results in a real rigidity. Specifically, under a currency peg, the real wage expressed in terms of tradables, $w_t \equiv W_t/E_t$, is downwardly rigid. Formally, equation (9.17) becomes

$$w_t \geq \gamma w_{t-1}$$

(9.21)

and the slackness condition (9.19) becomes

$$(\bar{h} - h_t) (w_t - \gamma w_{t-1}) = 0.$$

(9.22)

As a result of this real rigidity, in general the labor market is in disequilibrium and features involuntary unemployment. The magnitude of the labor market disequilibrium is a function of the amount by which the past real wage, $w_{t-1}$, exceeds the current full-employment real wage. It follows that under a currency peg $w_{t-1}$ becomes a relevant state variable for the economy.

A competitive equilibrium under a currency peg is a set of stochastic processes $\{c^T_t, h_t, w_t, d_{t+1}, p_t, \lambda_t, \mu_t\}_{t=0}^\infty$ satisfying (9.9)-(9.15), (9.18), (9.21) and (9.22), given initial conditions $w_{-1}$ and $d_0$ and exogenous stochastic processes $\{r_t, y^T_t\}_{t=0}^\infty$. 
9.2.1 A Peg-Induced Externality

Figure 9.4 illustrates the adjustment of the economy to a boom-bust episode under a currency peg. Because in equilibrium $c^N_t = y^N_t = F(h_t)$, the figure plots the demand and supply schedules for nontraded goods in terms of employment in the nontraded sector, so that the horizontal axis measures $h_t$. The intersection of the demand and supply schedules, therefore, indicate the equilibrium demand for labor, given $c^T_t$ and $W_t/E_t$. The figure also shows with a dotted vertical line the labor supply, $\bar{h}$. Suppose that the initial position of the economy is at point $A$, where the labor market is operating at full employment, $h_t = \bar{h}$. Suppose that in response to a positive external shock, such as a decline in the country interest rate, traded absorption increases from $c^T_0$ to $c^T_1 > c^T_0$ causing the demand function to shift up and to the right. If nominal wages stayed unchanged, the new
intersection of the demand and supply schedules would occur at point $B$. However, at point $B$ the demand for labor would exceed the supply of labor $\bar{h}$. The excess demand for labor drives up the nominal wage from $W_0$ to $W_1 > W_0$ causing the supply schedule to shift up and to the left. The new intersection of the demand and supply schedules occurs at point $C$, where all hours of labor are employed and there is no excess demand for labor. Because nominal wages are upwardly flexible, the transition from point $A$ to point $C$ happens instantaneously.

Suppose now that the external shock fades away, and that, therefore, absorption of tradables goes back to its original level $c^T_0$. The decline in $c^T_t$ shifts the demand schedule back to its original position, indicated by the downward sloping solid line. However, the economy does not immediately return to point $A$. Due to downward nominal wage rigidity, the nominal wage stays at $W_1$, and because of the currency peg, the nominal exchange rate remains at $\varepsilon_0$. For simplicity, we draw figure 9.4 assuming that $\gamma$ is unity. As a result, the supply schedule does not move. The new intersection is at point $D$. There, the economy suffers involuntary unemployment equal to $\bar{h} - h^{bust}$. Involuntary unemployment will persist over time unless the government does something to boost the economy. This is because, if $\gamma$ is equal to unity, nominal wages will not fall and the economy will be stuck at point $D$, with permanent involuntary unemployment of $\bar{h} - h^{bust}$. If $\gamma$ is less than unity, nominal wages will gradually return to their initial level $W_0$. In this case, the supply schedule shifts gradually down and to the right, until eventually intersecting the demand schedule at point $A$, where full employment is restored. This entire transition, however, is characterized by involuntary unemployment and a depressed level of activity in the nontraded sector.

The dynamics described above suggest that the combination of downward nominal wage rigidity and a currency peg creates a negative externality. The nature of this externality is that in periods of economic expansion, elevated demand for nontradables drives nominal (and real) wages up. Although this increase in wages occurs in the context of full employment and strong aggregate activity, it places the economy in a vulnerable situation, because in the contractionary phase of
the cycle, downward nominal wage rigidity and the currency peg hinder the downward adjustment of real wages, causing unemployment. Individual agents understand this mechanism, but are too small to internalize the fact that their own expenditure choices collectively exacerbate disruptions in the labor market.

9.2.2 Volatility And Average Unemployment

The present model implies an endogenous connection between the amplitude of the cycle and the average level of involuntary unemployment. The larger the degree of aggregate volatility, the larger the average level of involuntary unemployment. This connection between a second moment (the volatility of the underlying shocks) and a first moment (average unemployment) opens the door to large welfare gains from optimal stabilization policy.

The predicted connection between the volatility of the underlying shocks and the mean level of involuntary unemployment is due to two maintained assumptions: (a) employment is determined as the minimum between the desired demand for labor and the desired supply of labor. And (b) wages are more rigid downwardly than upwardly. The mechanism is as follows: The economy responds efficiently to positive external shocks, as nominal wages adjust upward to ensure that firms are on their labor demand schedules and households are on their labor supply schedules. In sharp contrast, the adjustment to negative external shocks is inefficient, as nominal wages fail to fall forcing households off their labor supply schedules and generating involuntary unemployment. Thus, over the business cycle, the economy fluctuates between periods of full employment (or zero unemployment) and periods of positive unemployment, implying positive unemployment on average. Further, the implied level of unemployment during recessions is larger the larger is the amplitude of the underlying shocks. This is for two reasons. First, the reduction in the demand for labor during contractions is naturally larger the larger is the amplitude of shocks buffeting the economy. Second, the larger the amplitude of the underlying shocks, the larger is the increase
in nominal wages during booms, which exacerbates the negative effects of wage rigidity during contractions. It follows that mean unemployment is increasing in the variance of the underlying shocks.

This prediction represents an important difference with existing sticky-wage models à la Erceg, Henderson, and Levin (2000). In this class of models, assumption (a) does not hold. Instead, it is assumed that employment is always demand determined. As a result increases in involuntary unemployment during recessions are roughly offset by reductions in unemployment during booms. Consequently, the average level of unemployment does not depend in a quantitatively relevant way on the amplitude of the business cycle.

Interestingly, the direct connection between aggregate uncertainty and the average level of unemployment predicted by the present model does not require assumption (b) (downward wage rigidity). It suffices to impose assumption (a), that is, employment is determined as the minimum of the demand and supply of labor. The following example illustrates this point. Consider an economy in which the nominal wage rate is absolutely rigid in both directions. Specifically, suppose that $W_t = \bar{W}$, for all $t$, where $\bar{W}$ is a parameter. Let the exchange-rate regime be a currency peg with $\mathcal{E}_t = \bar{\mathcal{E}}$ for all $t$, where $\bar{\mathcal{E}} > 0$ is a parameter. Suppose that agents have no access to international financial markets, $d_t = 0$ for all $t$. In this environment, consumption of tradables equals the endowment of tradables at all times, $c_t^T = y_t^T$ for all $t$. Suppose that preferences are logarithmic $U(A(c_t^T, c_t^N)) = \ln c_t^T + \ln c_t^N$. Assume that the endowment of tradables, $y_t^T$, can take on the values $\bar{y} + \sigma$ or $\bar{y} - \sigma$ each with probability $1/2$, where $\bar{y} = 1$ and $0 < \sigma < 1$ is a parameter. According to this specification, $E(y_t^T) = 1$ and $\text{var}(y_t^T) = \sigma^2$. Assume that the technology for producing nontradables is $F(h_t) = h_t^\alpha$ and that households supply inelastically one unit of labor each period ($\bar{h} = 1$). Finally, let $\bar{w} \equiv \bar{W}/\bar{\mathcal{E}}$ denote the real wage rate expressed in terms of tradables and suppose that $\bar{w} = \alpha$. This value of $\bar{w}$ is the flexible-wage real wage that would obtain if the endowment took it unconditional mean value, $E(y_t^T) = 1$. The equilibrium conditions associated
with this economy are

\[
\frac{c_T}{c_N} = p_t
\]

\[\alpha p_t (h_t^d)^{\alpha-1} = w_t\]

\[c_T = y_t^T\]

\[c_N^\alpha = h_t^\alpha\]

\[w_t = \bar{w},\]

and

\[h_t = \min\{\bar{h}, h_t^d\},\]

where \(h_t^d\) denotes the desired demand for labor by firms. The first equation is the demand for nontradables, the second is the supply of nontradables, the third and fourth are the market clearing conditions for tradables and nontradables, respectively. The last equation states that employment is determined as the minimum of the supply and demand for labor. It is straightforward to verify that the solution to the above system is

\[h_t = \begin{cases} 
1 - \sigma & \text{if } y_t^T = 1 - \sigma \\
1 & \text{if } y_t^T = 1 + \sigma 
\end{cases}\]

Let \(u_t \equiv \bar{h} - h_t\) denote the unemployment rate. It follows that the equilibrium distribution of \(u_t\) is given by

\[u_t = \begin{cases} 
\sigma & \text{with probability } \frac{1}{2} \\
0 & \text{with probability } \frac{1}{2} 
\end{cases}\]
The unconditional mean of the unemployment rate is then given by

\[ E(u_t) = \frac{\sigma}{2}. \]

This expression shows that the average level of unemployment increases linearly with the volatility of tradable endowment, in spite of the fact that wage rigidity is symmetric.

The assumption that nominal wages are only downwardly inflexible amplifies the effect of volatility on average unemployment. To see this, replace the assumption \( W_t = \bar{W} \) with the assumption \( W_t \geq \bar{W} \). In this case, nominal wages are absolutely inflexible downwardly, but perfectly flexible upwardly. Suppose that all other aspects of the economy are unchanged. In this environment, during booms (i.e., when \( y_t^T = 1 + \sigma \)) the economy is in full employment and the real wage equals \((1 + \sigma)\alpha\), which is larger than \( \bar{w} \), the real wage that prevails during booms when wages are bidirectionally rigid. This makes the economy more vulnerable to negative shocks. Indeed, when \( y_t^T = 1 - \sigma \), the real wage fails to fall causing a contraction in employment to \((1 - \sigma)/(1 + \sigma)\), which is lower than \(1 - \sigma\), the level of employment during contractions in the economy with bidirectional wage rigidity. It follows that the average rate of unemployment equals \( E(u_t) = \sigma/(1 + \sigma) \), which is larger than \( \sigma/2 \), the average rate of unemployment in the economy with two-sided wage rigidity (recall that \( \sigma \) must be less than 1).

In sum, the above example illustrates that the connection between aggregate volatility and the mean level of unemployment does not require the assumption of one-sided (downward) wage rigidity, but is a consequence of the assumption that employment is determined as the minimum between labor demand and labor supply. The example also reveals that downward nominal wage rigidity does exacerbate the increasing relation between the average unemployment rate and the variance of the underlying shocks.
9.2.3 Adjustment To A Temporary Fall in the Interest Rate

This section presents an analytical example showing that when the central bank pegs the domestic currency, a positive external shock can be the prelude to a slump with persistent unemployment. In this example, as in many observed boom-bust cycles in emerging countries, agents borrow internationally to take advantage of temporarily lower interest rates. The resulting capital inflow drives up domestic absorption of tradables and nominal wages. When the interest rate returns to its long-run level, aggregate demand falls and unemployment emerges as real wages—rigid by the combination of nominal wage rigidity and a currency peg—are stuck at a level too high to be consistent with full employment.

Suppose that preferences are given by $U(A(c_t^T, c_t^N)) = \ln c_t^T + \ln c_t^N$ and that the technology for producing nontradable goods is $F(h_t) = h_t^\alpha$, with $\alpha \in (0, 1)$. Suppose that the endowment of tradables is constant over time and given by $y_t^T = y^T$, that $\bar{h} = 1$, and that $\beta(1 + r) = 1$, where $r > 0$ is a parameter. Assume that nominal wages are downwardly rigid and that $\gamma = 1$. Finally, suppose that prior to period 0, the economy had been at a full-employment equilibrium with $d_{t+1} = 0$, $c_t^T = y^T$, $h_t = 1$, $w_t = \alpha y^T$, and $c_t^N = 1$, for $t < 0$.

Consider the adjustment of this economy to a temporary decline in the interest rate. Specifically, suppose that

$$r_t = \begin{cases} 
  r & t < 0 \\
  r - r & t = 0 \\
  r & t > 0
\end{cases}$$

Suppose that the interest-rate shock in period zero comes as a complete surprise, but that from this period onward, agents enjoy perfect foresight, so they know the future paths of $r_t$ and $y_t^T$ with certainty.

To characterize the equilibrium dynamics induced by this positive external shock, we must find
a set of deterministic sequences \( \{c_t^T, h_t, w_t, d_{t+1}, p_t, \lambda_t, \mu_t\}_{t=0}^{\infty} \) satisfying conditions (9.9)-(9.19), with \( \epsilon_t = 1 \) for all \( t \).

Begin by conjecturing that the debt limit (9.10) never binds, that is, \( d_t < \bar{d} \) for all \( t \), and that external debt is constant from period 1 onward, that is, \( d_t = d_1 \), for all \( t \geq 1 \). Once we have found the equilibrium dynamics, we must verify that these two conjectures are satisfied. The first conjecture together with equilibrium condition (9.12) implies that

\[ \mu_t = 0 \]

for all \( t \). Then, by equilibrium conditions (9.13) and (9.14), we have that

\[ c_t^T = c_1^T \]

and

\[ c_1^T = \beta(1+r)c_0^T \]

for all \( t \geq 1 \). Recalling that \( d_0 = 0 \), we have that the resource constraint (9.9) in period 0 is given by

\[ c_0^T = y^T + \frac{d_1}{1+r} \]

By the second conjecture, the resource constraint in period 1 is given by

\[ c_1^T + d_1 = y^T + \frac{d_1}{1+r} \]

Solving the above four expressions yields

\[ c_0^T = y^T \left[ \frac{1}{1+r} + \frac{r}{1+r} \right] > y^T \]
\[ c_t^T = y^T \left[ \frac{1}{1+r} + \frac{r}{1+r} \right] < y^T, \]

and

\[ d_t = y^T \left[ 1 - \frac{1+r}{1+r} \right] > 0, \]

for \( t \geq 1 \). Notice that if \( r = r \), then \( c_1^T = y^T \) and \( d_t = 0 \). However, because \( r < r \), the economy experiences a boom in traded consumption in period 0. This boom is financed with external debt, \( d_1 > 0 = d_0 \). That is, the country experiences a current account deficit, or capital inflows, in period 0. In period 1, consumption falls permanently to a level lower than the one observed prior to the interest rate shock, and the trade balance switches permanently from a deficit to a surplus. This permanent trade-balance surplus is large enough to pay the interest on the external debt generated by the consumption boom of period 0.

The surge in capital inflows in period 0 is accompanied by full employment, that is, \( h_0 = 1 \). To see this, suppose, on the contrary, that \( h_0 < 1 \). Then, by condition (9.16) we have that \( \alpha c_0^T / h_0 = w_0 \). But since \( c_0^T > y^T \) and \( h_0 < 1 \), this expression implies that \( w_0 > \alpha y^T = w_{-1} \), violating the slackness condition (9.19). It follows that \( h_0 = 1 \). The initial rise in capital inflows also elevates the period-0 real wage. Specifically, by (9.16) and the fact that \( h_0 = 1 \), we have that \( w_0 = \alpha c_0^T > \alpha y^T = w_{-1} \). This increase in labor costs results in an increase in the relative price of nontradables, or a real exchange-rate appreciation. This can be see from equation (9.5), which implies that \( p_0 = c_0^T > y^T = p_{-1} \). Graphically, the dynamics described here correspond to a movement from point \( A \) to point \( C \) in figure 9.4.

The elevation in real wages that takes place in period 0 puts the economy in a vulnerable situation in period 1, when the interest rate increases permanently from \( r \) to \( r \). In particular, in period 1, the economy enters in a situation of chronic involuntary unemployment. To see this, note that by (9.15) and (9.16) the full-employment real wage in period 1 is \( \alpha c_1^T \), which is lower...
than $\alpha c_0^T = w_0$. As a result, the lower bound on wage growth must be binding in period 1, that is, condition (9.17) must hold with equality. Recalling that $\epsilon_t = 1$ for all $t$ and that $\gamma = 1$, we then have that $w_1 = w_0$. Combining this expression with (9.15) and (9.16) yields the following expression for the equilibrium level of involuntary unemployment:

$$1 - h_1 = 1 - \frac{1 + r}{1 + r} > 0,$$

This level of unemployment persists indefinitely. To see this, note that at the beginning of period 2, the state of the economy is $\{r_2, y_2^T, d_2, w_1\}$, which, as we have shown, equals $\{r, y^T, d_1, w_0\}$, which, in turn, is the state of the economy observed at the beginning of period 1. A similar argument can be made for periods $t = 3, 4, \ldots$. It follows that for all $t \geq 1$, the unemployment rate is given by

$$1 - h_t = 1 - \frac{1 + r}{1 + r} > 0.$$

Notice that the larger the decline in the interest rate in period 0, the larger is the unemployment rate in periods $t \geq 1$. Figure 9.5 presents a graphical summary of the adjustment process.

It is of interest to compare the dynamic adjustment of the present economy to the one that would take place under flexible wages ($\gamma = 0$). This adjustment is presented with broken lines in figure 9.5. The responses of tradable consumption and external debt are identical whether wages are rigid or not. The reason is that in this example preferences are additively separable in consumption of tradables and nontradables. However, the behavior of the remaining variables of the model is quite different under flexible and sticky wages. Contrary to what happens under sticky wages, the flexible-wage equilibrium is characterized by full employment at all times. Full employment is supported by a permanent fall in nominal (and real) wages in period 1. This decline in labor costs allows firms to lower the relative price of nontradables permanently. In turn, lower relative prices
Figure 9.5: A Temporary Decline in the Country Interest Rate

Country Interest Rate, $r_t$

Consumption of Tradables, $c_t^*$

Debt, $d_t$

Unemployment, $(\bar{h} - h)/\bar{h}$

Real Wage, $w_t$

Real Exchange Rate, $P_t^N/P_t^T$

- currency peg
- flexible wage economy or optimal exchange rate economy
for nontradables induce consumers to switch expenditures from tradables to nontradables.

9.3 Optimal Exchange Rate Policy

In the example we just analyzed, the reason why the economy experiences unemployment when the country interest rate increases is that real wages are too high to clear the labor market. This downward rigidity in real wages is the consequence of downwardly rigid nominal wages and a fixed exchange rate regime. The example suggests that the unemployment problem could be addressed by any policy that results in lower real wages. One way to lower the real wage is to devalue the currency. By making the nominal price of tradables more expensive—recall that $P_t^T = E_t$—a devaluation lowers the purchasing power of wages in terms of tradables. In turn, this erosion in the real value of wages induces firms to hire more workers. In this section, we formalize this intuition by characterizing the optimal exchange-rate policy. We begin by characterizing an exchange-rate policy that guarantees full employment at all times and then show that this policy is indeed Pareto optimal.

9.3.1 The Full-Employment Exchange-Rate Policy

Consider an exchange-rate policy in which each period the central bank sets the devaluation rate to ensure full employment in the labor market, that is, to ensure that

$$h_t = \bar{h},$$

for all $t \geq 0$. We refer to this exchange-rate arrangement as the full-employment exchange-rate policy.

The equilibrium dynamics associated with the full-employment exchange-rate policy are illus-
trated in figure 9.4. Suppose that, after being hit by a negative external shock, the economy is stuck at point \( D \) with involuntary unemployment equal to \( \bar{h} - h_{\text{bust}} \). At point \( D \), the desired demand for tradables is \( c_0^T \), the nominal wage is \( W_1 \), and the nominal exchange rate is \( \mathcal{E}_0 \). Suppose that the central bank were to devalue the domestic currency so as to deflate the purchasing power of nominal wages to a point consistent with full employment. Specifically, suppose that the central bank sets the exchange rate at the level \( \mathcal{E}_1 > \mathcal{E}_0 \) satisfying
\[
(W_1/\mathcal{E}_1)/F'(\bar{h}) = A_2(c_0^T, F(\bar{h})) / A_1(c_0^T, F(\bar{h})).
\]
The devaluation causes the supply schedule to shift down and to the right intersecting the demand schedule at point \( A \), where unemployment is nil \( (h = \bar{h}) \). The devaluation lowers the real cost of labor, making it viable for firms to slash prices. The relative price of nontradables falls from \( p_{\text{boom}} \) at the peak of the cycle, to \( p_0 \) after the negative external shock. This fall in the relative price of nontradables induces households to switch expenditure away from tradables and toward nontradables in a magnitude compatible with full employment.

The full-employment policy amounts to setting the devaluation rate to ensure that the real wage equals the full-employment real wage rate at all times. Formally, the full-employment exchange-rate policy ensures that
\[
w_t = \omega(c_t^T),
\]
where \( \omega(c_t^T) \) denotes the full-employment real wage rate and is given by
\[
\omega(c_t^T) = \frac{A_2(c_t^T, F(\bar{h}))}{A_1(c_t^T, F(\bar{h}))} F'(\bar{h}).
\]
The assumed properties of the aggregator function \( A \) ensure that the function \( \omega(\cdot) \) is strictly increasing in the domestic absorption of tradables, \( c_t^T ; \)
\[
\omega'(c_t^T) > 0.
\]

\(^2\)Exercise 9.4 asks you to prove this statement.
There exists a whole family of full-employment exchange-rate policies. Specifically, combining conditions (9.17) and (9.23), we have that any exchange rate policy satisfying

\[ \epsilon_t \geq \gamma \frac{w_{t-1}}{\omega(c^T_t)} \]  

(9.25)

ensures full employment at all times. To see this, suppose, on the contrary, that the above devaluation policy allows for \( h_t \) to be less than \( \bar{h} \) for some \( t \geq 0 \). Then, by the slackness condition (9.19) we have that

\[ w_t = \gamma \frac{w_{t-1}}{\epsilon_t}. \]  

(9.26)

Solve this expression for \( \epsilon_t \) and use the resulting expression to eliminate \( \epsilon_t \) from (9.25) to obtain \( w_t \leq \omega(c^T_t) \). Now using (9.15) and (9.16) to eliminate \( w_t \) and (9.24) to eliminate \( \omega(c^T_t) \), we can rewrite this inequality as

\[ \frac{A_2(c^T_t, F(h_t))}{A_1(c^T_t, F(h_t))} F'(h_t) \leq \frac{A_2(c^T_t, F(\bar{h}))}{A_1(c^T_t, F(\bar{h}))} F'(\bar{h}). \]

Because the left-hand side of this expression is strictly decreasing in \( h_t \), we have that the only value of \( h_t \) that satisfies the above inequality is \( \bar{h} \). But this is a contradiction, since we started by assuming that \( h_t < \bar{h} \). We have therefore shown that under any exchange rate policy belonging to the family defined by (9.25), unemployment is nil at all dates and states.

A natural question is whether the full-employment exchange-rate policy is the most desirable policy from a social point of view. This question is nontrivial because the welfare of households does not depend directly on the level of employment but rather on the level of consumption of final goods.
9.3.2 Pareto Optimality of the Full-Employment Exchange-Rate Policy

Consider a social planner who wishes to maximize the welfare of the representative household. A relevant question is under what constraints the social planner operates. The answer to this question depends on the issue the researcher is interested in. Here, we consider the problem of a local policymaker who takes as given the international asset market structure. Specifically, we assume that the social planner has access to a single internationally traded state-noncontingent bond denominated in units of tradable goods that pays the exogenously given interest rate $r_t$. Also, we assume that external debt must be bounded above by $\bar{d}$. Further, we assume that the planner takes as given the endowment process $y_t$, the technology for producing nontradables $F(h_t)$, and the time endowment $\bar{h}$. Given these constraints, the social planner picks processes for consumption, hours worked, and net foreign debt to maximize the welfare of the representative household.

We refer to the solution of the social planner’s problem as the Pareto optimal allocation. The key difference between the competitive equilibrium and the Pareto optimal allocation is that the social planner can circumvent the goods and labor markets and impose directly the number of hours each household must work and the quantities of tradables and nontradables it can consume each period. This implies, in particular, that the allocation problem faced by the planner is not affected by the presence of nominal rigidities.

Formally, the social planner’s problem is given by

$$\max_{\{c^T_t, h_t, d_t+1\}_{t=0}^\infty} \mathbb{E}_0 \sum_{t=0}^\infty \beta^t U(A(c^T_t, F(h_t)))$$

subject to

$$h_t \leq \bar{h}, \quad \text{(9.27)}$$

$$c^T_t + d_t = y^T_t + \frac{d_{t+1}}{1 + r_t}, \quad \text{(9.28)}$$
Because the objective function is concave and the constraints define a convex set, the first-order conditions associated with this problem are necessary and sufficient for an optimum. Let $\beta^t \eta_t$, $\beta^t \lambda_t$, and $\beta^t \mu_t$ denote the Lagrange multipliers associated with (9.27), (9.28), and (9.29), respectively.

The first-order conditions with respect to $\eta_t$ and $h_t$ are

$$ h_t \leq \bar{h}, $$

$$ U'(A(c_t^T, F(h_t)))A_2(c_t^T, F(h_t))F'(h_t) = \eta_t, $$

and the associated slackness condition is

$$ \eta_t(\bar{h} - h_t) = 0. $$

Because the functions $U$, $A$, and $F$ are strictly increasing the second optimality condition implies that $\eta_t$ is positive. It then follows from the slackness condition that $h_t = \bar{h}$ for all $t$. In words, the Pareto optimal allocation features full employment at all times.

The first-order conditions of the social planner’s problem with respect $\lambda_t$, $\mu_t$, $c_t^T$, and $d_{t+1}$ are

$$ c_t^T + d_t = y_t^T + \frac{d_{t+1}}{1 + r_t}, $$

$$ d_{t+1} \leq \bar{d}, $$

$$ \lambda_t = U'(A(c_t^T, F(h_t)))A_1(c_t^T, F(h_t)), $$

$$ \frac{\lambda_t}{1 + r_t} = \beta E_t \lambda_{t+1} + \mu_t, $$
with
\[ \mu_t \geq 0, \]
and
\[ \mu_t(d_{t+1} - \bar{d}) = 0. \]

These conditions are identical, respectively, to competitive equilibrium conditions (9.9)-(9.14). This means that the processes \( \mu_t, \lambda_t, c^*_t, d_{t+1}, \) and \( h_t \) that satisfy the conditions for Pareto optimality, also satisfy the competitive equilibrium conditions when the exchange-rate policy belongs to the class of full-employment exchange-rate policies defined by (9.25).

We have therefore established that the real allocation associated with the full-employment exchange-rate policy is Pareto optimal. In other words, any exchange-rate policy that does not induce full employment is welfare dominated by the full-employment exchange-rate policy.

9.3.3 When Is It Inevitable To Devalue?

Because under the optimal exchange policy the real wage is always equal to the full-employment real wage, \( \omega(c^*_t) \), equation (9.25) implies that for all \( t > 0 \) the devaluation rate satisfies
\[ \epsilon_t \geq \gamma \frac{\omega(c^*_{t-1})}{\omega(c^*_t)}; \quad t > 0. \]

Recalling that \( \omega(\cdot) \) is a strictly increasing function of \( c^*_t \), this expression states that optimal devaluations occur in periods of contraction in aggregate expenditure in tradables. A nonstructural econometric analysis of data stemming from this model may lead to the erroneous conclusion that devaluations are contractionary (see, for instance, the empirical literature surveyed in section 3.4 of Frankel, 2011). However, the role of optimal devaluations is precisely the opposite, namely, to prevent the contraction in the tradable sector to spill over into the nontraded sector. It follows
that under the full-employment exchange-rate policy, devaluations are indeed expansionary in the sense that should they not take place, aggregate contractions would be even larger. Thus, under the full-employment exchange rate regime, the present model turns the view that ‘devaluations are contractionary’ on its head and instead predicts that ‘contractions are devaluatory.’

Consider, for example, the case of a temporary decline in the country interest rate studied in section 9.2.3. The equilibrium dynamics under the optimal exchange-rate policy are shown with broken lines in figure 9.5. Recall that under a currency peg, the permanent contraction in tradable consumption that occurs in period 1 causes a permanent increase in involuntary unemployment. Under the optimal exchange-rate policy, the path of tradable consumption is identical, but the permanent contraction in period 1 does not spill over to the nontraded sector or the labor market, which continues to operate under full employment. The monetary authority is able to maintain full employment by devaluing the currency in period 1 (not shown in the figure), which reduces the real cost of labor and makes it optimal for firms not to fire workers.

The full-employment exchange-rate policy completely eliminates any real effect stemming from nominal wage rigidity. Indeed, one can show that the equilibrium under the full-employment exchange rate policy is identical to the equilibrium of an economy with full wage flexibility ($\gamma = 0$). Consider again the case of a temporary decline in the country interest rate studied in section 9.2.3. The equilibrium dynamics under fully flexible wages are identical to those associated with the full-employment exchange-rate policy (shown with broken lines in figure 9.5), except that the decline in the real wage that occurs in period 1 is brought about by a decline in the nominal wage under flexible wages but by a devaluation of the currency under downward wage rigidity and the full-employment exchange-rate policy.

\footnote{See exercise 9.5.}
9.4 Empirical Evidence On Downward Nominal Wage Rigidity

The central friction in the model we have analyzed in this chapter is downward nominal wage rigidity. In this section, we review a body of empirical work suggesting that downward nominal wage rigidity is a widespread phenomenon. This type of nominal friction has been detected in micro and aggregated data stemming from developed, emerging, and poor regions of the world. It has also been found both in formal and informal labor markets. An important byproduct of this review is an estimate of the parameter $\gamma$ governing the degree of nominal wage rigidity in the theoretical model. We will need this parameter value to study the quantitative predictions of the model.

9.4.1 Evidence From Micro Data

A number of studies has examined the rigidity of hourly wages using micro data. Gottschalk (2005), for example, uses panel data from the Survey of Income and Program Participation (SIPP) to estimate the frequency of wage declines, increases, and no changes for male and female hourly workers working for the same employer over the period 1986-1993 in the United States. Table 9.1 shows that over the course of one year only a small fraction of workers experiences a decline in

<table>
<thead>
<tr>
<th>Interviews One Year apart</th>
<th>Males</th>
<th>Females</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decline</td>
<td>5.1%</td>
<td>4.3%</td>
</tr>
<tr>
<td>Constant</td>
<td>53.7%</td>
<td>49.2%</td>
</tr>
<tr>
<td>Increase</td>
<td>41.2%</td>
<td>46.5%</td>
</tr>
</tbody>
</table>


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4The SIPP has been conducted by the Bureau of Labor Statistics since 1983. It is a stratified representative sample of the U.S. population. Individuals are interviewed every four months for a period of 24 to 48 months.
nominal wages, while about half of workers experience no change. The large mass at no changes suggests that nominal wages are rigid. The small mass to the left of zero suggests that nominal wages are downwardly rigid. It is worth noting that the sample period used by Gottschalk comprises the 1990-1991 U.S. recession, for it implies that the observed scarcity of nominal wage cuts took place in the context of elevated unemployment.

Barattieri, Basu, and Gottschalk (2012) report similar findings using data from the 1996-1999 SIPP panel. Figure 9.6 shows that during this period the distribution of nominal wage changes was also truncated to the left of zero. The figure does not show the frequency corresponding to no wage changes. The reason for this omission is that the mass at zero changes is high, so that including it would make the rest of the figure less visible.

Nominal wage cuts were also rare in the United States during the Great Contraction that
started in 2007. Daly, Hobijn, and Lucking (2012) use micro panel data on wage changes of individual workers to construct the empirical distribution of wage changes in 2011.\footnote{The data comes from the Current Population Survey (CPS), which, like the SIPP, is collected by the Bureau of Labor Statistics.} Figure 9.7 displays the empirical distribution of annual nominal wage changes in 2011. The pattern is similar to those displayed above: There is significant mass at no wage changes and more mass to the right of no changes than to the left. To emphasize the asymmetry in the distribution, Daly et al. plot with a dashed line a (symmetric) normal distribution. The figure suggests that during the Great Contraction, despite the fact that unemployment was high and inflation was below its 2 percent target, nominal wage cuts were less frequent than wage increases.

A similar pattern of downward nominal wage rigidity based on microeconomic data is found
in other developed countries. See, for example, Fortin (1996) for Canada, Kuroda and Yamamoto (2003) for Japan, and Fehr and Goette (2005) for Switzerland. Downward nominal wage rigidity is also found in industry-level wage data. See, for example, Holden and Wulfsberg (2008) for evidence from 19 OECD countries over the period 1973-1999.

9.4.2 Evidence From Informal Labor Markets

The evidence referenced above is based on data from formal labor markets in developed economies. However, a similar pattern of asymmetry in nominal wage adjustments emerges in informal labor markets located in poor areas of the world. Kaur (2012), for example, studies the behavior of nominal wages in casual daily agricultural labor markets in rural India. Specifically, she examines market-level wage and employment responses to local rainfall shocks in 500 Indian districts from 1956 to 2008. She finds that nominal wage adjustment is asymmetric. In particular, nominal wages rise in response to positive rain shocks but fail to fall during droughts. In addition, negative rain shocks cause labor rationing and unemployment. Importantly, inflation, which is uncorrelated with local rainfall shocks, moderates these effects. During periods of relatively high inflation, local droughts are more likely to result in lower real wages and less labor rationing. This effect suggests that nominal rather than real wages are downwardly rigid.

9.4.3 Evidence From The Great Depression of 1929

According to the National Bureau of Economic Research, the Great Depression in the United States started in August 1929 and ended in March 1933. By 1931, the economy had experienced an enormous contraction. Employment in the manufacturing sector in 1931 stood 31 percent below its 1929 level. Figure 9.8 shows that despite a highly distressed labor market, nominal wages
remained remarkably firm. Between 1929:8 and 1931:8, the nominal wage rate, shown with a solid line in the figure, fell by only 0.6 percent per year. By contrast, consumer prices, shown with a broken line, fell by 6.6 percent per year over the same period. As a result, in the first two years of the Great Depression, real wages increased by 12 percent in the midst of massive unemployment. In the second half of the depression, nominal wages fell, but nominal prices fell even faster. As a result, by the end of the Great Depression the real wage rate was 26 percent above its 1929 level.

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6The graph shows the nominal wage rate as opposed to average hourly earnings. The problem with the latter series is that it includes compensation for overtime work. Contractions in overtime employment cause drops in average hourly earnings that are not reflective of downward wage flexibility.
The observed resilience of nominal wages in a context of extreme underutilization of the labor force is indicative of downward nominal wage rigidity.

9.4.4 Evidence From Emerging Countries and Inference on $\gamma$

The empirical literature surveyed thus far establishes that nominal wage rigidity is significant and asymmetric. However, because it uses data from developed and poor regions of the world, it does not provide evidence on the importance of downward nominal wage rigidity in emerging countries. In addition, it does not lend itself to calibrating the wage-rigidity parameter $\gamma$, because it does not provide information on the speed of nominal downward wage adjustments. In Schmitt-Grohé and Uribe (2014), we examine data from emerging countries and propose an empirical strategy for identifying $\gamma$. The approach consists in observing the behavior of nominal hourly wages during periods of rising unemployment. We focus on episodes in which an economy undergoing a severe recession keeps the nominal exchange rate fixed. Two prominent examples are Argentina during the second half of the Convertibility Plan (1996-2001) and the periphery of Europe during the great recession of 2008.

Figure 9.9 displays subemployment (defined as the sum of unemployment and underemployment) and nominal hourly wages expressed in pesos for Argentina during the period 1996-2001. The Convertibility Plan was in effect from April 1991 to December 2001 and consisted in a peg of the Argentine peso to the U.S. dollar at a one-for-one rate with free convertibility.

The subperiod 1998-2001 is of particular interest because during that time the Argentine central bank was holding on to the currency peg in spite of the fact that the economy was undergoing a severe contraction and both unemployment and underemployment were in a steep ascent. The contraction was caused by a combination of large adverse external disturbances, including, a collapse in export commodity prices, a 100-percent devaluation in Brazil, Argentina’s main trading partner, in 1999, and a deterioration in international borrowing conditions following the southeast Asian
and Russian financial crises of 1998.

In the context of a flexible-wage model, one would expect that the rise in unemployment would be associated with falling real wages. With the nominal exchange rate pegged, the fall in real wages must materialize through nominal wage deflation. However, during this period, the nominal hourly wage never fell. Indeed, it increased from 7.87 pesos in 1998 to 8.14 pesos in 2001. The model developed in this chapter predicts that with rising unemployment, the lower bound on nominal wages should be binding and therefore $\gamma$ should equal the gross growth rate of nominal wages. We wish to parameterize the model so that one period corresponds to one quarter. An estimate of the parameter $\gamma$ can then be constructed as the average quarterly growth rate of nominal wages over the three-year period considered, that is, $\gamma = (W_{2001}/W_{1998})^{1/12}$. This yields a value of $\gamma$ of 1.0028. This value means that, because of the presence of downward nominal wage rigidity, nominal wages must rise by at least 1.12 percent per year.
In order for this estimate of $\gamma$ to represent an appropriate measure of wage rigidity in the context of the theoretical model, it must be adjusted to account for the fact that the model abstracts from foreign inflation and long-run productivity growth (see exercises 9.6 and 9.7). To carry out this adjustment, we use the growth rate of the U.S. GDP deflator as a proxy for foreign inflation. Between 1998 and 2001, the U.S. GDP deflator grew by 1.77 percent per year on average. We set the long-run growth rate in Argentina at 1.07 percent per year, to match the average growth rate of Argentine per capita real GDP over the period 1900-2005 reported in García-Cicco et al. (2010). The adjusted value of $\gamma$ is then given by $\frac{1.0028}{(1.0107 \times 1.0177)^{1/4}} = 0.9958$.

Finally, we note that during the 1998-2001 Argentine contraction, consumer prices, unlike nominal wages, did fall. The CPI inflation rate was on average -0.86 percent per year over the period 1998-2001. It follows that real wages rose not only in dollar terms but also in terms of CPI units. Incidentally, this evidence provides some support for the assumption, implicit in the theoretical framework developed in this chapter, that downward nominal rigidities are less stringent for product prices than for factor prices.

The second episode from the emerging-market world that we use to document the presence of downward nominal wage rigidity and to infer the value of $\gamma$ is the great recession of 2008 in the periphery of Europe. Table 9.2 presents an estimate of $\gamma$ for twelve European economies that are either on the euro or pegging to the euro. The first two columns of the table show the unemployment rate in 2008:Q1 and 2011:Q2. The starting point of this period corresponds to the beginning of the great recession in Europe according to the CEPR Euro Area Business Cycle Dating Committee. The 2008 crisis caused unemployment rates to rise sharply across all twelve countries. The table also displays the total growth of nominal hourly labor cost in manufacturing, construction and services (including the public sector) over the thirteen-quarter period 2008:Q1-2011:Q2.$^7$ Despite the large surge in unemployment, nominal wages grew in most countries and in

$^7$The public sector is not included for Spain due to data limitations.
Table 9.2: Unemployment, Nominal Wages, and γ: Evidence from the Eurozone

<table>
<thead>
<tr>
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</tr>
<tr>
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</tr>
<tr>
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<tr>
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<td>20.8</td>
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<td>1.009</td>
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<tr>
<td>Slovakia</td>
<td>10.2</td>
<td>13.4</td>
<td>13.3</td>
<td>1.010</td>
<td></td>
</tr>
</tbody>
</table>

Note. $W$ is an index of nominal average hourly labor cost in manufacturing, construction, and services. Unemployment is the economy-wide unemployment rate. Source: Schmitt-Grohé and Uribe (2014).

those in which they fell, the decline was modest. The implied value of $\gamma$, shown in the last column of table 9.2, is given by the average growth rate of nominal wages over the period considered (that is, $\gamma = (W_{2011Q2}/W_{2008Q1})^{1/13}$). The estimated values of $\gamma$ range from 0.996 for Lithuania to 1.028 for Bulgaria.

To adjust $\gamma$ for foreign inflation, we proxy this variable with the inflation rate in Germany. Over the thirteen-quarter sample period considered in table 9.2 inflation in Germany was 3.6 percent, or about 0.3 percent per quarter. To adjust for long-run growth, we use the average growth rate of per capita output in the southern periphery of Europe of 1.2 percent per year or 0.3 percent per quarter.\(^8\) Allowing for these effects suggest an adjusted estimate of $\gamma$ in the interval [0.990, 1.022].

\(^8\)This figure corresponds to the average growth rate of per capita real GDP in Greece, Spain, Portugal, and Italy over the period 1990-2011 according to the World Development Indicators.
9.5 The Case of Equal Intra- And Intertemporal Elasticities of Substitution

For the remainder of this chapter, we assume a CRRA form for the period utility function, a CES form for the aggregator function, and an isoelastic form for the production function of nontradables:

\[
U(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, \quad (9.30)
\]

\[
A(c^T, c^N) = \left[ a(c^T)^{1-\frac{1}{\xi}} + (1-a)(c^N)^{1-\frac{1}{\xi}} \right]^{1-\frac{1}{\xi}}, \quad (9.31)
\]

and

\[
F(h) = h^\alpha,
\]

with \(\sigma, \xi, a, \alpha > 0\). These functional forms are commonplace in the quantitative business-cycle literature.

A case that is of significant interest analytically, computationally, and empirically, is one in which the intra- and intertemporal elasticities of consumption substitution are equal to each other, that is, the case in which

\[
\xi = \frac{1}{\sigma}.
\]

This restriction greatly facilitates the characterization of equilibrium, because it renders the equilibrium processes of external debt, \(d_t\), and consumption of tradables, \(c^T_t\), independent of the level of activity in the nontraded sector. This implication is also of interest because it means that any welfare differences across exchange-rate regimes must be attributable to the effects of exchange-rate policy on unemployment and not to transitional dynamics in external debt. Finally, we will argue in section 9.7 that the case of equal intra- and intertemporal elasticities of substitution is empirically plausible.
To see that setting $\xi = 1/\sigma$ renders the equilibrium levels of external debt and tradable consumption independent of the level of activity in the nontraded sector, note that under this restriction we have that

$$U(A(c^T_t, c^N_t)) = \frac{ac^T_t^{1-\sigma} + (1-a)c^N_t^{1-\sigma} - 1}{1-\sigma},$$

which is additively separable in $c^T_t$ and $c^N_t$. Therefore, equation (9.13) becomes

$$\lambda_t = ac^T_t^{-\sigma},$$

which is independent of $c^N_t$. Thus, the equilibrium processes $\{c^T_t, d_{t+1}, \mu_t, \lambda_t\}_{t=0}^{\infty}$ can be obtained as the solution to the subsystem of equilibrium conditions (9.9)-(9.14). Clearly, this result holds for any exchange-rate policy and for any degree of wage rigidity. In particular, we have that when $\xi = 1/\sigma$, the equilibrium behavior of $d_t$ and $c^T_t$ is the same under a currency peg, under the optimal exchange-rate policy, and under full wage flexibility. We impose this parameter restriction in the quantitative analysis that follows. For an analysis of the case $\sigma \neq 1/\xi$, see Schmitt-Grohé and Uribe (2014).

### 9.6 Approximating Equilibrium Dynamics

The equilibrium dynamics under the optimal exchange rate policy can be characterized as the solution to the following value function problem:

$$v^{OPT}(y^T_t, r_t, d_t) = \max_{\{d_{t+1}, c^T_t\}} \left\{ U(A(c^T_t, F(\bar{h}))) + \beta E_t v^{OPT}(y^T_{t+1}, r_{t+1}, d_{t+1}) \right\}$$

subject to (9.9) and (9.10), where the function $v^{OPT}(y^T_t, r_t, d_t)$ represents the welfare level of the representative agent under the full-employment exchange-rate policy in state $(y^T_t, r_t, d_t)$. To
approximate the solution to this dynamic programming problem, we apply the method of value function iterations over a discretized version of the state space.

We assume that the exogenous driving forces \( y_t \) and \( r_t \) follow a joint discrete Markov process with 21 points for \( y_t \) and 11 points for \( r_t \). In section 9.7.1, we econometrically estimate the parameters defining this process.

We discretize the level of debt with 501 equally spaced points in the interval 1 to 8. The solution of the above dynamic programming problem yields the equilibrium processes for \( d_{t+1} \) and \( c_t \) for all possible states \((y_t, r_t, d_t)\). Given these processes, the equilibrium processes of all other endogenous variables under the optimal exchange-rate policy can be readily obtained. The variable \( h_t \) equals \( \bar{h} \) for all \( t \), \( p_t \) can be obtained from (9.15), and \( w_t \) from (9.16). Finally, if a particular full-employment exchange-rate policy has been chosen from the family defined in equation (9.25) it can readily be backed out.

The maintained parameter restriction \( \xi = 1/\sigma \) implies that the solution for \( d_{t+1} \) and \( c_t \) just described, also applies to the fixed-exchange-rate economy. Under a currency peg, however, the past real wage, \( w_{t-1} \), becomes relevant for the determination of employment, current wages, nontradable output, and the relative price of nontradables. As a result, the fixed-exchange-rate economy carries an additional endogenous state variable, \( w_{t-1} \). To discretize the past real wage, we use 500 points between 0.25 and 6. Points are equally spaced in a logarithmic scale. Computing the equilibrium level of the real wage, \( w_t \), given values for \( w_{t-1} \) and \( c_t \) requires solving a static problem and involves no iterative procedure. Specifically, begin by assuming that \( h_t = \bar{h} \). Then, use (9.24) to obtain the full-employment real wage, \( \omega(c_t) \) (given a value for \( c_t \), this is just a number). Next, check whether the full-employment real wage satisfies the lower bound on nominal wages when \( \epsilon_t = 1 \), that is, whether \( \omega(c_t) \geq \gamma w_{t-1} \). If so, then \( w_t = \omega(c_t) \). Otherwise, the lower bound on nominal wages is binding and \( w_t = \gamma w_{t-1} \). The resulting value of \( w_t \) will in general not coincide exactly with any point in the grid, so pick the closest grid point. Given \( w_t \) and \( c_t \), all other endogenous variables
can be easily obtained. For example, $h_t$ and $p_t$ are the solution to (9.15) and (9.16).

When $\xi \neq 1/\sigma$, approximating the dynamics of the model under a currency peg is computationally more demanding. The reason is that in this case the dynamics of debt and tradable consumption are affected by the level of activity in the nontraded sector. As a result, the equilibrium dynamics of $d_{t+1}$ and $c_{t}^{T}$ can no longer be obtained separately from the dynamics of variables pertaining to the nontraded sector. In addition, because of the distortions created by nominal rigidities, aggregate dynamics cannot be cast in terms of a Bellman equation without introducing additional state variables (such as the individual level of debt, which households perceive as distinct from its aggregate counterpart). In Schmitt-Grohé and Uribe (2014), we show that one can approximate the solution by Euler equation iteration over a discretized version of the state space $(y_{t}^{T}, r_{t}, d_{t}, w_{t-1})$.

9.7 Parameterization of the Model

We calibrate the model at a quarterly frequency. The model contains two types of parameters: structural parameters pertaining to preferences, technologies, and nominal frictions, and parameters defining the stochastic process of the exogenous driving forces. We begin by estimating the latter set of parameters.

9.7.1 Estimation Of The Exogenous Driving Process

We assume that the law of motion of tradable output and the country interest rate is given by the following autoregressive process:

$$
\begin{bmatrix}
\ln y_{t}^{T} \\
\ln \frac{1+r_{t}}{1+r}
\end{bmatrix} = A \begin{bmatrix}
\ln y_{t-1}^{T} \\
\ln \frac{1+r_{t-1}}{1+r}
\end{bmatrix} + \epsilon_{t},
$$

(9.33)
where $\epsilon_t$ is a white noise of order 2 by 1 distributed $N(0, \Sigma)$. The parameter $r$ denotes the deterministic steady-state value of the country interest rate $r_t$. We estimate this system using Argentine data over the period 1983:Q1 to 2001:Q4.

Our empirical measure of $y_t^T$ is the cyclical component of Argentine GDP in agriculture, forestry, fishing, mining, and manufacturing.\textsuperscript{9} As in the empirical business-cycle analysis of chapter 1, we obtain the cyclical component by removing a log-quadratic time trend. Panel (a) of figure 9.10 displays the resulting time series. We measure the country interest rate as the sum of the EMBI+ spread for Argentina and the 90-day Treasury-Bill rate, deflated using a measure of expected dollar inflation.\textsuperscript{10} Specifically, we construct the time series for the quarterly real Argentine interest rate, $r_t$, as $1 + r_t = (1 + i_t)E_t \frac{1}{1 + \pi_{t+1}}$, where $i_t$ denotes the dollar interest rate charged to Argentina in international financial markets and $\pi_t$ is U.S. CPI inflation. For the period 1983:Q1 to 1997:Q4, we take $i_t$ to be the Argentine interest rate series constructed by Neumeyer and Perri (2005).\textsuperscript{11} For the period 1998:Q1 to 2001:Q4, we measure $i_t$ as the sum of the EMBI+ spread and the 90-day Treasury bill rate, which is in line with the definition used in Neumeyer and Perri since 1994:Q2. We measure $E_t \frac{1}{1 + \pi_{t+1}}$ by the fitted component of a regression of $\frac{1}{1 + \pi_{t+1}}$ onto a constant and two lags. This regression uses quarterly data on the growth rate of the U.S. CPI index from 1947:Q1 to 2010:Q2.

\textsuperscript{9}The data were downloaded from www.indec.mecon.ar.
\textsuperscript{10}EMBI+ stands for Emerging Markets Bond Index Plus. The EMBI+ tracks total returns for traded external debt instruments (external meaning foreign currency denominated fixed income) in the emerging markets. Included in the EMBI+ are U.S.-dollar denominated Brady bonds, Eurobonds, and traded loans issued by sovereign entities. Instruments in the EMBI+ must have a minimum face value outstanding of $500 million, a remaining life of 2.5 years or more, and must meet strict criteria for secondary market trading liquidity. The EMBI+ is produced by J. P. Morgan. The time series starts in 1993 or later depending on the country and has a daily frequency. We convert the daily time series into a quarterly time series by taking the arithmetic average of daily observations within each quarter.
\textsuperscript{11}The time series is available online at www.fperri.net/data/neuperri.xls. For the period 1983:Q1 to 1994:Q1 these authors compute the interest rate as the sum of the 90-day U.S. T-bill rate and their own calculation of a spread on Argentine bonds. For the period 1994:Q1 to 1997:Q4 they use the EMBI+ spread.
Figure 9.10: Traded Output and Interest Rate in Argentina, 1983:Q1-2008:Q3

(a) Traded Output

(b) Interest Rate

Note. Traded output is expressed in log-deviations from a quadratic time trend. Source: See the main text.
Our OLS estimates of the matrices $A$ and $\Sigma_\epsilon$ and of the scalar $r$ are

$$A = \begin{bmatrix} 0.79 & -1.36 \\ -0.01 & 0.86 \end{bmatrix}; \quad \Sigma_\epsilon = \begin{bmatrix} 0.00123 & -0.00008 \\ -0.00008 & 0.00004 \end{bmatrix}; \quad r = 0.0316.$$

According to these estimates, both $\ln y^T_t$ and $r_t$ are highly volatile, with unconditional standard deviations of 12.2 percent and 1.7 percent per quarter (6.8 percent per year), respectively. Also, the unconditional contemporaneous correlation between $\ln y^T_t$ and $r_t$ is high and negative at -0.86. This means that periods of relatively low traded output are associated with high interest rates and vice versa. The estimated joint autoregressive process implies that both traded output and the real interest rate are highly persistent, with first-order autocorrelations of 0.95 and 0.93, respectively. Finally, we estimate a steady-state real interest rate of 3.16 percent per quarter, or 13.2 percent per year. This high average value reflects the fact that our sample covers a period in which Argentina underwent a great deal of economic turbulence.

We discretize the joint AR(1) process for $y^T_t$ and $r_t$ given in equation (9.33) using 21 equally spaced points for $\ln y^T_t$ and 11 equally spaced points for $\ln((1 + r_t)/(1 + r))$. The first and last values of the grids for $\ln y^T_t$ and $\ln((1 + r_t)/(1 + r))$ are set to $\pm \sqrt{10}$ times the respective standard deviations ($\pm 0.3858$ and $\pm 0.0539$, respectively). We construct the transition probability matrix of the state $(\ln y^T_t, \ln((1 + r_t)/(1 + r)))$ using the simulation approach proposed in Schmitt-Grohé and Uribe (2009). Briefly, this approach consists in simulating a time series of length 1,000,000 drawn from the system (9.33) and associating each observation in the time series with one of the 231 possible discrete states by distance minimization. The resulting discrete-valued time series is used to compute the probability of transitioning from a particular discrete state in one period to a particular discrete state in the next period. The resulting transition probability matrix captures well the covariance matrices of order 0 and 1.
Table 9.3: Calibration

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<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
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<tr>
<td>$\gamma$</td>
<td>0.99</td>
<td>Degree of downward nominal wage rigidity</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>2</td>
<td>Inverse of intertemporal elasticity of consumption</td>
</tr>
<tr>
<td>$y^T$</td>
<td>1</td>
<td>Steady-state tradable output</td>
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<tr>
<td>$\tilde{h}$</td>
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<td>Labor endowment</td>
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<tr>
<td>$a$</td>
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<td>Share of tradables</td>
</tr>
<tr>
<td>$\xi$</td>
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<td>Elasticity of substitution between tradables and nontradables</td>
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<tr>
<td>$\alpha$</td>
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<td>Labor share in nontraded sector</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.9635</td>
<td>Quarterly subjective discount factor</td>
</tr>
</tbody>
</table>

9.7.2 Calibration Of Preferences, Technologies, and Nominal Rigidities

The values assigned to the structural parameters are shown in table 9.3. We set the parameter $\gamma$ governing the degree of downward nominal wage rigidity at 0.99. This is a conservative value. The estimates of $\gamma$ based on data from Argentina and the periphery of Europe presented in section 9.4 suggest that $\gamma$, after correcting for foreign inflation and long-run growth, lies in the interval $[0.99, 1.022]$. We set $\gamma$ to the lower bound of this interval, which represents the greatest degree of downward wage flexibility consistent with the data used in the estimation.

We normalize the steady-state levels of output of tradables and hours at unity. Then, if the steady-state trade-balance-to-output ratio is small, as is the case in Argentina (see table 1.7 in chapter 1), the parameter $a$ is approximately equal to the share of traded output in total output. We set this parameter at 0.26, which is the share of traded output (as defined above) observed in Argentine data over the period 1980:Q1-2010:Q1. Uribe (1997) presents evidence suggesting that the labor share in the nontraded sector in Argentina is 0.75. Accordingly we set $\alpha$ equal to this value. As in the RBC models studied in chapter 4, we set $\sigma$ equal to 2. As mentioned there, this value is commonly used in business-cycle studies. Using time series data for Argentina over the period 1993Q1-2001Q3, González Rozada et al. (2004) estimate the elasticity of substitution between
traded and nontraded consumption, $\xi$, to be 0.44. This estimate is consistent with the cross-country estimates of Stockman and Tesar (1995). These authors include in their estimation both developed and developing countries. Restricting the sample to include only developing countries yields a value of $\xi$ of 0.43 (see Akinci, 2011). We set $\xi$ equal to 0.5. We pick this particular value for two reasons. First, it is close to the value suggested by existing empirical studies. Second this value is the reciprocal of the one assigned to $\sigma$. As discussed in subsection 9.5, the restriction $\xi = 1/\sigma$ implies that the dynamics of external debt and tradable consumption are independent of the exchange-rate policy or the degree of nominal wage rigidity. This implication greatly facilitates the numerical characterization of the equilibrium dynamics.\footnote{In Schmitt-Grohé and Uribe (2014), we consider the case in which $1/\sigma = 0.2$ and $\xi = 0.44$.}

We set $\bar{d}$ at the natural debt limit, which we define as the level of external debt that can be supported with zero tradable consumption when the household perpetually receives the lowest possible realization of tradable endowment, $y^{T\min}$, and faces the highest possible realization of the interest rate, $r^{\max}$. Formally, $\bar{d} \equiv y^{T\min}(1 + r^{\max})/r^{\max}$. Given our discretized estimate of the exogenous driving process, $\bar{d}$ equals 8.34.

We calibrate the subjective discount factor $\beta$ to match the average external-debt-to-output ratio of 23 percent per year observed in Argentina over the period 1983-2001 (Lane and Milesi-Ferretti, 2007). We set $\beta$ at 0.9635. This value yields an average debt-to-output ratio of 23.2 percent per year under the optimal exchange-rate policy and of 21.5 percent under a currency peg.

9.8 External Crises and Exchange-Rate Policy: A Quantitative Analysis

We are now ready to quantitatively characterize the response of the model economy to a large negative external shock. We have in mind extraordinary contractions like the 1989 or 2001 crises
in Argentina, or the 2008 great recession in peripheral Europe. During the great Argentine crises of 1989 and 2001, for example, traded output fell by about two standard deviations within a period of two and a half years, and the country premium experienced equally large increases. We are particularly interested in contrasting the model economy’s adjustment to this type of external shock under the two polar exchange-rate arrangements we have been considering thus far, a currency peg and the optimal exchange rate policy.

9.8.1 Definition of an External Crisis

We define an external crisis that starts in period \( t \) as a situation in which tradable output is at or above trend in quarter \( t \) and at least two standard deviations below trend in quarter \( t + 10 \).

To characterize the typical behavior of the economy during such episodes, we simulate the model for 20 million quarters and identify windows \((t - 10, t + 30)\) in which movements in traded output conform to the definition of an external crisis. Then, for each variable of interest we average all windows and subtract the respective mean taken over the entire sample of 20 million quarters. The beginning of the typical crisis is normalized at \( t = 0 \).

Figure 9.11 displays the predicted average behavior of the two exogenous variables, traded output and the country interest rate, during a crisis. The downturn in traded output can be interpreted either as a drastic fall in the quantity of tradables produced by the economy or as an exogenous collapse in the country’s terms of trade. The figure shows that at the trough of the crisis (period 10), tradable output is 25 percent below trend. The contraction in tradable output is accompanied by a sharp increase in the interest rate that international financial markets charge to the emerging economy. The country interest rate peaks in quarter 10 at about 14 percentage points per annum (about two standard deviations) above its average value. This behavior of the interest rate is dictated by the estimated high negative correlation between tradable output and country interest rates. Indeed, the typical crisis would look quite similar to the one shown in figure 9.11 if
we had defined a crisis episode as one in which the country interest rate is at or below its average level in period 0 and at least 2 standard deviations above its average level in period 10.

How do the endogenous variables of the model, such as unemployment, real wages, consumption, the trade balance, and inflation, respond to these large negative external shocks? As we will see next, the answer depends crucially on the exchange-rate policy put in place by the monetary authority.

### 9.8.2 Crisis Dynamics Under A Currency Peg

Figure 9.12 depicts with solid lines the response of the endogenous variables to the external crisis defined in subsection 9.8.1 when the exchange-rate policy takes the form of a currency peg.

The large exogenous increase in the country interest rate and the large fall in tradable endowment cause households to sharply reduce consumption of tradable goods. At the trough of the crisis, in quarter 10, tradable consumption is about 33 percent below trend. This adjustment is so pronounced that, in spite of the fact that the endowment of tradables falls significantly during
Figure 9.12: Crisis Dynamics: The Role Of Exchange-Rate Policy

- Unemployment Rate
- Real Wage (in terms of tradable)
- Annualized Devaluation Rate
- Relative Price of Nontradables ($P^N_t/E_t$)
- Annual CPI Inflation Rate
- Consumption of Tradables
- Trade-Balance-To-Output Ratio
- Debt-To-Output Ratio (Annual)

Legend:
- **Currency Peg**
- **Optimal Exchange Rate Policy**
the crisis, the trade balance actually improves. The bottom left panel of figure 9.12 shows that the trade-balance-to-output ratio rises by about 3 percentage points between the beginning and the trough of the crisis. The severity of the contraction in the absorption of tradables is driven primarily by the country-interest-rate hike, which causes a substitution effect against current consumption and a negative wealth effect stemming from an increase in interest payments on the external debt. The elevated cost of debt service causes the country’s external debt to increase during the crisis, in spite of the positive trade balance. The stock of external debt increases not only as a fraction of output (see the bottom right panel of figure 9.12), but also in levels (not shown in figure 9.12).

The contraction in the traded sector spills over to the nontraded sector in ways that can be highly deleterious. The full-employment real wage, \( \omega(c^T_t) \), shown with a broken line in the top right panel of figure 9.12, falls by 66 percent between periods 0 and 10. By contrast, the real wage, \( w_t \), shown with a solid line in the top right panel of Figure 9.12, falls by only 10 percent. The reason for the insufficient downward adjustment in the real wage is, of course, the combination of downward nominal wage rigidity and a currency peg. Recall that the real wage, expressed in terms of tradables, equals the ratio of the nominal wage, \( W_t \), and the nominal exchange rate, \( E_t \). Therefore, a fall in the real wage requires either a fall in nominal wages or a depreciation of the currency (i.e., an increase in \( E_t \)), or a combination of both. Due to downward nominal wage rigidity, nominal wages can fall at most at 1 percent per quarter (since \( \gamma \) equals 0.99). At the same time, because of the currency peg, the nominal exchange rate is constant over time. It follows that the real wage can fall by at most 1 percent per quarter. Thus, real wages fall by only 10 percent between the beginning of the crisis in period 0 and the beginning of the recovery in period 10 more than 50 percentage points short of what would be necessary to ensure full employment.

The sluggish downward adjustment of the real wage causes massive disequilibrium in the labor market. The top left panel of figure 9.12 shows that the rate of involuntary unemployment increases to 25 percent at the trough of the crisis. Further, unemployment is highly persistent. Five years
after the trough of the crisis, the unemployment rate remains more than 7 percentage points above average. The persistence of unemployment is due to the slow downward adjustment of real wages.

In spite of the large contraction in aggregate demand, the relative price of nontradables in terms of tradables, $P_t^N/E_t$, falls little during the crisis. This is because labor costs (the real wage) remain too high making it unprofitable for firms to implement large price cuts. As a consequence of the insufficient fall in the relative price of nontradables, households do not face a strong enough incentive to switch expenditure away from tradables and toward nontradables. Put differently, the combination of downward nominal wage rigidity and a currency peg hinders the ability of the price system to signal to firms and consumers that during the crisis there is a relative aggregate scarcity of tradable goods and a relative aggregate abundance of nontradable goods.

The predictions of our model suggest that a rigid exchange-rate policy whereby the necessary real depreciation is forced to occur via product-price and wage deflation is highly costly in terms of unemployment and forgone consumption of nontradables. As we will see next, the present economy requires large devaluations to bring about full employment.

### 9.8.3 Crisis Dynamics Under Optimal Exchange Rate Policy

We have seen in section 9.3 that there exists a whole family of optimal exchange-rate polices, defined by condition (9.25). Each member of this family supports the Pareto optimal real allocation. However, different members of this family can deliver different outcomes for nominal variables, such as the devaluation rate, price inflation, and wage inflation. Here, we consider the following specification for the optimal exchange-rate policy

$$\epsilon_t = \frac{\omega_t - 1}{\omega(\epsilon_t^*)}.$$  \hspace{1cm} (9.34)
With $\gamma < 1$, this policy clearly belongs to the family defined in condition (9.25). According to this policy, the central bank devalues the domestic currency when the full employment wage falls below the past real wage and revalues the currency when the full employment wage exceeds the past real wage. This policy specification has three interesting properties. First, it ensures that nominal wages are constant at all times. To see this, note that in the Pareto optimal allocation, the real wage equals the full-employment real wage, that is, $w_t = \omega(c_t^T)$. Then, using the fact that $w_t \equiv W_t/E_t$ and that $\epsilon_t = E_t/E_{t-1}$, we can write the above exchange-rate policy as $E_t/E_{t-1} = (W_{t-1}/E_{t-1})/(W_t/E_t)$, which implies that $W_t = W_{t-1}$ for all $t \geq 0$. Thus, the assumed exchange-rate policy stabilizes the nominal price that suffers from downward rigidity. A second property of the assumed optimal exchange-rate policy is that it implies that the nominal price of nontradables is also constant over time. This can be seen from equation (9.16), which states that $p_t F'(h_t) = w_t$. Noticing that under the Pareto optimal allocation $h_t$ is constant and equal to $\bar{h}$, and that $p_t = P_t^N/E_t$, we have that in equilibrium $P_t^N F'(\bar{h}) = W_t$. Since $W_t$ is constant, so is $P_t^N$. A third property of interest is that the assumed optimal exchange-rate policy implies zero inflation and zero devaluation on average. To see this note that the relative price of nontradables, $p_t$, is a stationary variable. That is, it may move over the business cycle, but does not have a trend. Since $p_t$ equals $P_t^N/E_t$ and $P_t^N$ is constant, we have that the nominal exchange rate must also be stationary, that is, $E_t$ does not have a trend. This means that the devaluation rate, $\epsilon_t - 1$, must be zero on average. Finally, because the nominal price of nontradables, $P_t^N$, is constant and the nominal price of tradables, $E_t$, is stationary, it follows that the nominal price of the composite consumption good, $P_t$, must be stationary. This implies, in turn, that the CPI inflation, which is a combination of inflation in tradable and nontradable prices, must be zero on average.

Figure 9.12 displays with broken lines the average response of the economy to the external crisis defined in subsection 9.8.1 under the optimal exchange rate policy. The central difference between the optimal exchange-rate policy and a currency peg is that under the optimal exchange-rate policy
the external crisis does not spill over to the nontraded sector. Indeed, as we saw in section 9.3 the unemployment rate is nil under the optimal exchange-rate policy. The government ensures full employment through a series of devaluations of the domestic currency, which are quite large under the present parameterization of the model. Figure 9.12 shows that the monetary authority devalues the currency at an annualized rate of about 35 percent each quarter for the duration of the contractionary phase of the crisis (quarters 0 to 10). These large devaluations drastically lower the real value of wages in terms of tradables, thereby reducing the labor cost faced by firms. The top right panel of figure 9.12 shows that the real wage, expressed in terms of tradables, falls by more than 60 percent over the first 10 quarters of the crisis. As we will see shortly, this sizable drop in the real value of wages is in line with the Argentine experience at the time of the large devaluation of 2001 that ended the ten-year long exchange-rate peg known as the Convertibility plan (see figure 9.14). In turn, the decline in real labor costs allows firms to lower the relative price of nontradable goods in terms of tradable goods, which results in a large depreciation of the real exchange rate of over 60 percent (see the right panel on the second row of figure 9.12). This sizable change in relative prices induces households to redirect their spending toward nontradable goods. The large nominal and real depreciation of the currency predicted by the model is in line with the empirical findings of Burstein, Eichenbaum, and Rebelo (2005) who report that the primary force behind the observed large drop in the real exchange rate that occurred after the large devaluations in Argentina (2002), Brazil (1999), Korea (1997), Mexico (1994), and Thailand (1997) was the slow adjustment in the nominal price of nontradable goods.

During the crisis, the optimal exchange-rate policy drives the CPI inflation rate to about 10 percent per year (see row 3 column 1 of figure 9.12). By contrast, under the exchange-rate peg the economy experiences deflation of about 6 percent per year. The model thus speaks with a strong voice against allowing the economy to fall into deflation during a crisis.
9.8.4 Devaluations, Revaluations, and Inflation In Reality

The particular optimal exchange-rate policy given in equation (9.34) has the property of inducing zero devaluation and zero inflation on average. This means that the elevated rates of devaluation and inflation predicted during crisis must be followed by revaluations and low inflation in the recovery phase. Row 2 column 1 of figure 9.12 shows that revaluations are predicted to begin as soon as the economy begins to recover. Is this prediction borne out in the data?

Figure 9.13 displays the CPI inflation rate and the devaluation rate against the U.S. dollar for Argentina and an average of Brazil, Chile, Colombia, Mexico, Peru, and Uruguay during the 2008 great contraction. Vertical lines mark the beginning and end of the great contraction in the U.S. according to the NBER. The crisis, which started in the United States in 2008, arrived in South America one year later. All countries in the sample responded to the crisis with sizable devaluations. This response is in line with the predictions of the model under the optimal exchange-rate policy. Both in Argentina and in the group of Latin American countries considered, CPI inflation picked
up during the crisis. At the same time, all countries with the exception of Argentina revalued their currencies as soon as the recovery began. Argentina, by contrast, continued to devalue its currency during the recovery. As predicted by the model, the countries that revalued experienced lower inflation than Argentina.

9.9 Empirical Evidence On The Expansionary Effects of Devaluations

Are devaluations expansionary in the way suggested by the model? Here we examine two episodes pointing in this direction. Both involve countries that during a currency peg are hit by severe negative shocks. After some years of increasing unemployment and general economic duress, these countries decided to abandon the fixed exchange rate regime and to allow their currencies to depreciate. In both cases, the devaluations were followed by a reduction in real wages and an expansion of aggregate employment.

9.9.1 Exiting a Currency Peg: Argentina Post Convertibility

Figure 9.14 displays the nominal exchange rate, subemployment, nominal hourly wages expressed in pesos and in U.S. dollars for Argentina during the period 1996-2006. As discussed in section 9.4.4, the Argentine peso was pegged to the U.S. dollar from April 1991 to December 2001. Since 1998 Argentina was undergoing a severe contraction that had pushed the subemployment rate to 35 percent. In spite of widespread unemployment and a fixed exchange rate, nominal wages did not decline in this period. In 2002 Argentina abandoned the peg devaluing the peso by 250 percent (see the upper left panel of the figure). As shown in the lower right panel of the figure the devaluation coincided with a vertical decline in the real wage by a magnitude proportional to the devaluation. Following the real wage decline, labor market conditions improved quickly. By
Figure 9.14: Nominal Wages and Unemployment in Argentina, 1996-2006

2005 the subemployment rate had fallen by 12 percentage points.

The sizable fall in real wages right after the devaluation of December 2001 suggests that the 1998-2001 period was one of censored wage deflation, which further strengthens the view that nominal wages suffer from downward inflexibility. The fact that nominal wages increased after the devaluation is an indication that the size of the devaluation exceeded the one necessary to restore full employment. More importantly, it suggests that nominal wages are not upwardly rigid. Taken together the dynamics of nominal wages over the period 1998 to 2006 is consistent with the view that nominal wages are downwardly rigid but upwardly flexible.

### 9.9.2 Exiting the Gold Standard: Europe 1929 to 1935

Another piece of indirect historical evidence of the expansionary effects of devaluations is provided by the international effects of the Great Depression of 1929 to 1933. Friedman and Schwartz (1963) observe that countries that left the gold standard early enjoyed more rapid recoveries than countries that stayed on gold longer. The first group of countries was known as the sterling bloc and consisted of the United Kingdom, Sweden, Finland, Norway, and Denmark, and the second group was known as the gold bloc and was formed by France, Belgium, the Netherlands, and Italy. The sterling bloc countries left gold beginning in 1931, whereas the gold block countries stayed on gold much longer, some until 1935.

One can think of the gold standard as a currency union in which members peg their currencies, not to the currency of another country member, but to gold. Thus, abandoning the gold standard is akin to abandoning a currency peg. When the sterling bloc countries left the gold standard, they effectively devalued their currencies, as the price of their currencies in terms of gold went down.

The difference in economic performance was associated with earlier reflation of price levels in the countries leaving gold earlier. Importantly, as pointed out by Eichengreen and Sachs (1986), real wages behaved differently in countries that left the gold standard early and in countries that
Figure 9.15: Changes In Real Wages and Industrial Production, 1929-1935


stuck to it longer. Figure 9.15 shows the change in real wages and in industrial production between 1929 and 1935 in the sterling and gold blocks. It shows that relative to their respective 1929 levels real wages in the sterling bloc countries were lower than real wages in the gold bloc countries. And industrial production in the sterling bloc countries in 1935 exceeded their respective 1929 levels whereas industrial production in the gold bloc countries was below their respective 1929 levels. This suggests two things. First, countries in which real wages increased less showed stronger growth in industrial production. Second, only the countries that devalued showed moderation in real wage growth. Taken together, these two facts suggest that in the great depression years nominal wages were downwardly rigid in Europe and that abandoning a peg during a recession can be expansionary.
9.10 The Welfare Costs of Currency Pegs

Thus far, we have used the theoretical model to compare the performance of currency pegs and the optimal exchange-rate policy over episodes of external crisis. We saw that currency pegs do a poor job at negotiating this type of situations. In this section, we use the theoretical laboratory to compare the performances of currency pegs and the optimal exchange rate policy not just over periods of economic duress, but along the infinite life of households. To this end, we calculate the level of welfare of individual households living in each of the two exchange-rate regimes.

The key variable to understand welfare differences across exchange-rate regimes in the model we are working with is the rate of involuntary unemployment. To see this, note that because of our maintained assumption that the intra- and intertemporal elasticities of substitution are equal to each other ($\sigma = 1/\xi$), the behavior of tradable consumption is identical across exchange-rate regimes (see section 9.5 for a demonstration). Consequently, any welfare differences across exchange-rate regimes must stem from the behavior of nontradable consumption. In turn, since nontradable goods are produced with labor only, all welfare differences must be explained by differences in the predicted dynamics of involuntary unemployment.

We define the welfare cost a currency peg, denoted $\Lambda(y^T_t, r_t, d_t, w_{t-1})$, as the percent increase in the consumption stream of a representative individual living in a currency-peg economy that would make him as happy as living in the optimal-exchange-rate economy. Specifically, $\Lambda(y^T_t, r_t, d_t, w_{t-1})$ is implicitly given by

$$E_t \sum_{s=0}^{\infty} \beta^s \left[ (c^{PEG}_{t+s} + \frac{\Lambda(y^T_t, r_t, d_t, w_{t-1})}{100}) \right]^{1-\sigma} = v^{OPT}(y^T_t, r_t, d_t),$$

where $c^{PEG}_t$ denotes the equilibrium process of consumption in the currency-peg economy, and $v^{OPT}(y^T_t, r_t, d_t)$ denotes the value function associated with the optimal exchange-rate policy, defined
in equation (9.32). Solving for $\Lambda(y^T_t, r_t, d_t, w_{t-1})$, we obtain

$$
\Lambda(y^T_t, r_t, d_t, w_{t-1}) = 100 \left\{ \frac{v^{OPT}(y^T_t, r_t, d_t)(1 - \sigma) + (1 - \beta)^{-1}}{v^{PEG}(y^T_t, r_t, d_t, w_{t-1})(1 - \sigma) + (1 - \beta)^{-1}} \right\}^{1/(1-\sigma)} - 1,
$$

where $v^{PEG}(y^T_t, r_t, d_t, w_{t-1})$ denotes the value function associated with the currency-peg economy and is given by

$$
v^{PEG}(y^T_t, r_t, d_t, w_{t-1}) = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left( \frac{v^{PEG}(y^T_{t+s}, r_{t+s}, d_{t+s}, w_{t+s-1})}{1 - \sigma} - 1 \right).
$$

Table 9.4 reports the median and the mean of $\Lambda(y^T_t, r_t, d_t, w_{t-1})$ along with the average rate of unemployment induced by a currency peg. The distribution of $\Lambda(y^T_t, r_t, d_t, w_{t-1})$ is a function of the equilibrium distribution of the state $(y^T_t, r_t, d_t, w_{t-1})$. In turn, the distribution of $(y^T_t, r_t, d_t, w_{t-1})$, and in particular that of $w_{t-1}$, depends upon the exchange-rate regime in place.\(^{13}\) Because we are interested in the welfare costs of living in a currency peg, we compute the mean and median of the welfare costs of pegs shown in table 9.4 using the equilibrium distribution of the state $(y^T_t, r_t, d_t, w_{t-1})$ induced by the currency peg.

The mean welfare cost of a currency peg is 7.8 percent of the consumption stream. That is, households living in a currency peg economy require on average 7.8 percent more consumption in every date and state in order to be indifferent between staying in the currency-peg regime and switching to the optimal exchange rate regime. This is a big number as welfare costs go in monetary business-cycle theory. Even under the most favorable initial conditions, the welfare cost of a currency peg is large, 4.0 percent of consumption each period. (This figure corresponds to the lower bound of the support of the probability density of $\Lambda(y^T_t, r_t, d_t, w_{t-1})$.)

As mentioned earlier, the welfare cost of pegs is entirely explained by involuntary unemployment. Table 9.4 reports an average rate of unemployment of 11.7 percent under the currency peg. A

\(^{13}\)When $\sigma \neq 1/\xi$, the distribution of $d_t$ also depends upon the exchange-rate regime in place.
Table 9.4: The Welfare Costs of Currency Pegs

<table>
<thead>
<tr>
<th>Parameterization</th>
<th>Welfare Cost of Peg</th>
<th>Average Unemployment Rate Under Peg</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Baseline (γ = 0.99)</strong></td>
<td>Mean  7.8</td>
<td>Median 7.2</td>
</tr>
<tr>
<td><strong>Lower Downward Wage Rigidity</strong></td>
<td>γ = 0.98</td>
<td>Mean  5.7</td>
</tr>
<tr>
<td></td>
<td>γ = 0.97</td>
<td>Mean  3.5</td>
</tr>
<tr>
<td></td>
<td>γ = 0.96</td>
<td>Mean  2.8</td>
</tr>
<tr>
<td><strong>Higher Downward Wage Rigidity</strong></td>
<td>γ = 0.995</td>
<td>Mean 14.3</td>
</tr>
<tr>
<td><strong>Symmetric Wage Rigidity, ( \frac{1}{\gamma} \geq \frac{w_t}{w_{t-1}} \geq \gamma )</strong></td>
<td>γ = 0.99</td>
<td>Mean  3.3</td>
</tr>
<tr>
<td></td>
<td>γ = 0.98</td>
<td>Mean  2.8</td>
</tr>
</tbody>
</table>

Note. The welfare cost of a currency peg is expressed in percent of consumption per quarter, see equation (9.35). The mean and median of the welfare cost of a peg is computed over the distribution of the state \( (y_t, r_t, d_t, w_{t-1}) \) induced by the peg economy.

A back-of-the-envelope calculation can help to understand how unemployment translates into welfare costs. The average fall in nontradable consumption due to unemployment in the nontraded sector is approximately given by the product of the labor elasticity of nontradable output, \( \alpha \), times the average level of unemployment, or \( 0.75 \times 11.7 = 8.8 \) percent per quarter. In turn, the total consumption loss is roughly given by the share of nontradable in total consumption. Which under the present parameterization is about \( 0.75 \times 0.75 \times 8.8 = 6.6 \), which is close to the exact mean welfare cost.

Figure 9.16 displays the unconditional distribution of \( \Lambda(y_t^T, r_t, d_t, w_{t-1}) \). The distribution is skewed to the right, implying that the probability of very high welfare costs is non-negligible. For instance the probability of occurrence of a state associated with welfare costs larger than 10 percent of consumption per quarter is 15 percent, and the probability of occurrence of a state associated with welfare costs larger than 15 percent of consumption per quarter is 1.9 percent. Which states
Figure 9.16: Probability Density Function of the Welfare Cost of Currency Pegs

Note. The welfare cost of a currency peg is defined in equation (9.35). The density function of welfare costs is computed over the distribution of the state \((y^T_t, r_t, d_t, w_{t-1})\) induced by the peg economy.
put the economy in such a vulnerable situation? Figure 9.17 sheds light on what these states are. It displays the welfare cost of currency pegs as a function of the four state variables. In each panel only one state variable is allowed to vary (along the horizontal axis) and the remaining three state variables are fixed at their respective unconditional means. The figure shows that currency pegs are more painful when the country is initially more indebted, when it inherits higher past real wages, when the tradable sector is undergoing a contraction (due, for example, to unfavorable terms of trade), or when the country interest-rate premium is high.

The fact that unemployment is the main source of welfare losses associated with currency pegs suggests that a key parameter determining the magnitude of these welfare losses should be $\gamma$, which governs the degree of downward nominal wage rigidity. The baseline calibration ($\gamma = 0.99$) implies that nominal wages can fall frictionlessly up to four percent per year. In section 9.4, we argue that this is a conservative value in the sense that it allows for falls in nominal wages during crises that are much larger than those observed either in the 2001 Argentine crisis or in the ongoing crisis in peripheral Europe, even after correcting for foreign inflation and long-run growth. We now consider alternative values that allow for lower and higher degrees of downward nominal wage rigidity.

On the more flexible side, we consider values of $\gamma$ that allow for nominal wage declines of up to 16 percent per year. Taking into account that the largest wage decline observed in Argentina in 2001 or in the periphery of Europe since the onset of the great recession was 1.6 percent per year (Lithuania, see table 9.2), it follows that we are considering degrees of wage rigidity substantially lower than those implied by observed wage movements during large contractions. Table 9.4 shows that the mean welfare cost of a currency peg is strictly increasing in the degree of downward nominal wage rigidity. As $\gamma$ falls from its baseline value of 0.99 to the smallest value considered, 0.96, the welfare cost of a peg falls from 7.8 to 2.8 percent of consumption per quarter. This welfare cost is still a large figure compared to existing results in monetary economics. The intuition why currency pegs are less painful when wages are more downwardly flexible is straightforward. A
Figure 9.17: Welfare Cost of Currency Pegs as a Function of the State Variables

Note. In each plot, all states except the one shown on the horizontal axis are fixed at their unconditional mean values. The dashed vertical lines indicate the unconditional mean of the state displayed on the horizontal axis (under a currency peg if the state is endogenous).
negative aggregate demand shock reduces the demand for nontradables which requires a fall in the real wage rate to avoid unemployment. Under a currency peg this downward adjustment must be brought about exclusively by a fall in nominal wages. The less downwardly rigid nominal wages are, the faster the downward adjustment in both the nominal and the real wage is. Therefore the less downwardly rigid nominal wages are, the smaller the resulting level of unemployment is. Table 9.4 confirms this intuition. The average rate of involuntary unemployment falls from 11.7 percent to 4.6 percent as $\gamma$ falls from its baseline value of 0.99 to 0.96.

We also consider a higher degree downward nominal wage rigidity than the one used in the baseline parameterization. Specifically, table 9.4 includes the case $\gamma = 0.995$. This value of $\gamma$ is perhaps of greater empirical interest than the low values just considered because, unlike those, it lies within the range of estimates obtained in section 9.4. This level of $\gamma$ allows nominal wages to fall by up to 2 percent per year, half the fall permitted under the baseline parameterization. The associated welfare costs of pegs are extremely high, 14.3 percent of consumption on average, with an average rate of unemployment of 19.5 percent.

The finding of large welfare costs of currency pegs predicted by the present model economy stand in stark contrast to a large body of work, pioneered by Lucas (1987), suggesting that the costs of business cycles (not just of suboptimal monetary or exchange-rate policy) are minor. Lucas' approach to computing the welfare costs of business cycles consists in first removing a trend from a consumption time series and then evaluating a second-order approximation of welfare using observed deviations of consumption from trend. Implicit in this methodology is the assumption that the trend is unaffected by policy. In the present model, however, suboptimal monetary or exchange-rate policy creates an endogenous connection between the amplitude of the business cycle and the average rate of unemployment (see the analysis in section 9.2.2). In turn, through its effect on the average level of unemployment, suboptimal exchange-rate policy has a significant effect on the average level of consumption. And indeed, as we saw earlier in this section, lower average
consumption is the main reason currency pegs are so costly in the present model. It follows that applying Lucas’ methodology to data stemming from the present model would overlook the effects of policy on trend (or average) consumption and therefore would result in spuriously low welfare costs.

9.11 Symmetric Wage Rigidity

We have shown that the welfare cost of currency pegs is increasing in the degree of downward nominal wage rigidity, governed by the parameter $\gamma$. We can think of this parameter as reflecting the intensive margin of wage rigidity. We now consider tightening the extensive margin of wage rigidity. We do so by imposing an upper bound on the rate at which nominal wages can increase from one period to the next. Specifically, we now assume the following constraint on nominal wage adjustments:

$$\gamma \leq \frac{W_t}{W_{t-1}} \leq \frac{1}{\gamma}.$$

Table 9.4 shows that for the baseline value of $\gamma$ of 0.99, increasing nominal wage rigidity along the extensive margin reduces unemployment and is welfare improving. The rate of involuntary unemployment falls from 11.7 percent under downward nominal wage rigidity to less than half that value under symmetric wage rigidity. This result may seem surprising, for it suggests that less wage flexibility is desirable. However, this prediction of the model is quite intuitive.

As we saw analytically in section 9.2.2, the imposition of upward rigidity in nominal wages alleviates the peg-induced externality. Upward wage rigidity curbs the increase of nominal wages during booms, thereby reducing the required magnitude of wage declines during the contractionary phase of the cycle. As result, recessions bring about less unemployment when wages are bidirectionally rigid.

Consider now increasing wage flexibility along the intensive margin by lowering $\gamma$, but keeping
a symmetric specification of wage rigidity. In the presence of symmetric wage rigidity, lowering $\gamma$ creates a tradeoff. On the one hand, higher downward wage flexibility is desirable because it allows for a more efficient adjustment of real wages during a downturn. On the other hand, higher upward wage flexibility is undesirable because, by allowing for larger wage increasing during booms, it exacerbates the peg-induced externality. Table 9.4 shows that this tradeoff is resolved in favor of higher wage flexibility along the intensive margin. Involuntary unemployment falls from 5.2 to 4.2 percent when $\gamma$ is reduced from 0.99 to 0.98. With lower unemployment the welfare costs of a currency peg are also smaller, 2.8 percent of consumption instead of 3.3 percent.

We close this section by pointing out that the reason why we consider the case of symmetric wage rigidity is for comparison with existing related frameworks, but not because of its empirical relevance. For the evidence presented in section 9.4 speaks clearly in favor of asymmetric specifications. We conclude that of all the parameterizations shown in table 9.4 the ones of greatest empirical relevance are those pertaining to the case in which wages are downwardly rigid and $\gamma$ takes the value 0.99 or 0.995.

### 9.12 The Mussa Puzzle

In an influential empirical study, Mussa (1986) compares the behavior of nominal and real exchange rates under fixed and floating exchange rate regimes. He analyzes data from 13 industrialized countries over the period 1957 to 1984. During the subperiod 1957 to 1970, the countries in the sample pegged their currencies to the U.S. dollar by an exchange-rate agreement known as Bretton Woods. (Recall that prior to 2000, the Euro Area, or eurozone, did not yet exist, so each European country had its own currency.) During the second subperiod, 1973 to 1984, countries in the sample adopted flexible exchange-rate regimes as a consequence of the breakdown of the Bretton Woods.

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14The countries included in the sample are the United Kingdom, West Germany, France, Italy, Japan, Netherlands, Sweden, Switzerland, Austria, Belgium, Denmark, Luxembourg, and Norway.
agreement.

Mussa documents three important facts about nominal and real exchange rates across fixed and floating exchange-rate regimes. First, the variability of the real exchange rate is much higher under flexible exchange rates than under fixed exchange rates. Second, under flexible exchange rates, movements in real exchange rates mimic movements in nominal exchange rates. This fact essentially suggests that under floating exchange rate regimes, observed changes in real exchange rates inherit the stochastic properties of observed changes in nominal exchange rates. Third, the volatility of national inflation rates is broadly the same under floating and fixed exchange-rate regimes.

The reason why these facts are often referred to as a puzzle is that they suggest that relative prices depend on the behavior of nominal prices. At the time of Mussa’s writing, the dominant paradigm for understanding short-run fluctuations was the flexible-price, neoclassical, real-business-cycle framework treated in Part ?? of this book. In a neoclassical world, real variables, including relative prices, are determined by real factors, such as technologies, preferences, and real disturbances. In this type of environment, nominal exchange regime neutrality holds, in the sense that the exchange-rate regime can have effects on other nominal variables, but does not matter for real allocations. Stockman (1988), for example, shows the difficulties faced by flexible-price models to capture the Mussa facts.

Since the publication of Mussa’s work in 1986, nominal rigidities have found their way into the standard paradigm, changing researchers’ views on the ability of monetary policy in general, and exchange-rate policy in particular, to shape the course of real variables. An early analysis of the Mussa puzzle within the context of a sticky-price model is Monacelli (2004).

We wish to ascertain whether the predictions of the theoretical model analyzed in this chapter are consistent with the empirical regularities documented by Mussa. To this end, let’s define the
Table 9.5: Real and Nominal Exchange Rates Under Fixed And Floating Exchange-Rate Regimes

<table>
<thead>
<tr>
<th>Peg</th>
<th>Float</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Optimal</td>
</tr>
<tr>
<td>(\text{std}(\epsilon_t^{RER}))</td>
<td>12.0</td>
</tr>
<tr>
<td>(\text{std}(\epsilon_t))</td>
<td>0</td>
</tr>
<tr>
<td>(\text{corr}(\epsilon_t^{RER}, \epsilon_{t-1}^{RER}))</td>
<td>0.18</td>
</tr>
<tr>
<td>(\text{corr}(\epsilon_t, \epsilon_{t-1}))</td>
<td>–</td>
</tr>
<tr>
<td>(\text{corr}(\epsilon_t^{RER}, \epsilon_t))</td>
<td>–</td>
</tr>
<tr>
<td>(\text{std}(\pi_t))</td>
<td>13.2</td>
</tr>
</tbody>
</table>

Note. Standard deviations are expressed in percent per year. The optimal floating exchange-rate policy is given by \(\epsilon_t = w_{t-1}/\omega(c_t^T)\), and the suboptimal floating exchange-rate policy is given by \(\epsilon_t = \omega(c_t^T)/w_{t-1}\).

real depreciation rate, denoted \(\epsilon_t^{RER}\), as the gross growth rate of the real exchange rate. That is,

\[
\epsilon_t^{RER} = \frac{RER_t}{RER_{t-1}},
\]

where \(RER_t\) denotes the real exchange rate as defined in equation (??), which we reproduce here for convenience

\[
RER_t = \frac{\mathcal{E}_t \mathcal{P}_t^*}{P_t}.
\] (??)

Table 9.5 compares the behavior of \(\epsilon_t^{RER}\) under a currency peg and under the optimal exchange-rate policy given in equation (9.34). The first of Mussa’s facts means that the standard deviation of \(\epsilon_t^{RER}\) is larger under flexible exchange rate regimes than under currency pegs. The first line of the table 9.5 shows that the predicted standard deviation of the real depreciation rate is much larger under the optimal floating exchange rate policy than under the peg. This prediction of the model is consistent with Mussa’s first fact. Lines 2 through 5 of the table show that under the optimal flexible exchange rate regime, the nominal and real exchange rates have similar standard deviation.
and first-order serial correlations, and are highly positively contemporaneously correlated. This finding suggests that the model captures Mussa’s second observation, namely, that under flexible exchange rates the real exchange rate shares, to a large extent, the stochastic properties of the nominal exchange rate. Finally, the last line of the table shows that the predicted volatility of CPI inflation, denoted $\pi_t \equiv P_t/P_{t-1}$, is the same under the peg and the optimal floating regime, which concurs with Mussa’s third fact.

### 9.12.1 Was Post-Bretton-Woods Exchange-Rate Policy Optimal?

At this point, it is important to clarify a common misconception. Many empirical studies classify exchange-rate regimes into fixed or floating, and then derive stylized facts associated with each regime. This practice is problematic because in reality there is not just one floating exchange-rate regime but an infinite family. And importantly, different floating exchange-rate regimes can induce different real allocations and, in particular, different nominal and real exchange-rate dynamics. To illustrate this point, we consider an alternative floating exchange-rate policy that does not belong to the class of optimal exchange-rate policies given in (9.25). Specifically, assume that the central bank sets the nominal exchange rate according to the rule $\epsilon_t = \omega(c^T_{t})/w_{t-1}$. This policy could be named the ‘anti’ optimal floating exchange-rate regime, as it revalues when the optimal rule, given in (9.34), calls for devaluations and vice versa. Table 9.5 shows that under this alternative floating exchange-rate policy, the model fails to capture all three of the Mussa facts. It follows that seen through the lens of the present model, Mussa’s facts can be interpreted as suggesting that during the early post-Bretton-Woods period, overall, the countries in the sample adopted floating regimes that gave rise to exchange-rate and inflation dynamics that are consistent with the ones associated with optimal exchange-rate policy.
9.13 Endogenous Labor Supply

We now relax the assumption of an inelastic labor supply schedule. Specifically, we consider a period-utility specification of the form

\[ U(c_t, \ell_t) = \frac{c_t^{1-\sigma} - 1}{1 - \sigma} + \varphi \frac{\ell_t^{1-\theta} - 1}{1 - \theta}, \]

where \( \ell_t \) denotes leisure in period \( t \), and \( \varphi \) and \( \theta \) are positive parameters. Under this specification, the household’s optimization problem features a new first-order condition determining the desired amount of leisure

\[ \varphi (\ell^v_t) - \theta = w_t \lambda_t, \]

where \( \ell^v_t \) denotes the desired or voluntary amount of leisure. The above expression is a notional labor supply. It is notional in the sense that the worker may not be able to work the desired number of hours. We assume that households are endowed with \( \bar{h} \) hours per period. Let \( h^v_t \) denote the number of hours households desire work (the voluntary labor supply). The (voluntary) labor supply is the difference between the the endowment of hours and voluntary leisure, that is,

\[ h^v_t = \bar{h} - \ell^v_t. \]

As before, households may not be able to sell all of the hours they supply to the labor market. Let \( h_t \) denote the actual number of hours worked. Then, we impose

\[ h^v_t \geq h_t. \]
This expression can also be interpreted as a ‘no slavery’ condition, as it states that no one can be forced to work longer hours than they wish. We impose the following slackness condition:

\[
(h_v^t - h_t) \left( w_t - \gamma \frac{w_{t-1}}{\epsilon_t} \right) = 0.
\]

(9.40)

Expressions (9.37)-(9.40) are the counterparts of conditions (9.7) and (9.19) in the baseline economy. All other conditions describing aggregate dynamics are as before.

An important remaining issue is how to evaluate welfare in the present environment. This issue is not trivial because now leisure has two components, voluntary leisure, \( \ell_v^t \), and involuntary leisure, or, synonymously, involuntary unemployment, which we denote by \( u_t \). Involuntary leisure is given by the difference between the number of hours the household voluntarily supplies to the market, \( h_v^t \), and the number of hours the household is actually employed, \( h_t \), that is,

\[
u_t = h_v^t - h_t.
\]

(9.41)

How should voluntary and involuntary leisure enter in the utility function? One possibility is to assume that voluntary and involuntary leisure are perfect substitutes. In this case, the second argument of the period utility function is \( \ell_t = \ell_v^t + u_t \). However, there exists an extensive empirical literature suggesting that voluntary and involuntary leisure are far from perfect substitutes. For instance, Krueger and Mueller (2012), using longitudinal data from a survey of unemployed workers in New Jersey, find that despite the fact that the unemployed spend relatively more time in leisure-related activities, they enjoy these activities to a lesser degree than their employed counterparts and thus, on an average day, report higher levels of sadness than the employed. Similarly, Winkelmann and Winkelmann (1998), using longitudinal data of working-age men in Germany, find that, after controlling for individual fixed effects and income, unemployment has a large non-
pecuniary detrimental effect on life satisfaction. Another source of non-substitutability between voluntary and involuntary leisure stems from the fact that the unemployed spend more time than the employed looking for work, an activity that they perceive as highly unsatisfying. Krueger and Mueller (2012), for example, report that the unemployed work 391 minutes less per day than the employed but spend 101 minutes more per day on job search. In addition, these authors find that job search generates the highest feeling of sadness after personal care out of 13 time-use categories.

Based on this evidence, it is important to consider specifications in which voluntary and involuntary leisure are imperfect substitutes in utility. Specifically, we model leisure as

$$\ell_t = \ell^v_t + \delta u_t.$$  \hspace{1cm} (9.42)

The existing literature strongly suggests that $\delta$ is less than unity. However, estimates of this parameter are not available. For this reason, we consider three values of $\delta$, 0.5, 0.75, and 1.

We calibrate the remaining new parameters of the model as follows: we assume that under full employment households spend a third of their time working. In addition, we adopt a Frisch wage elasticity of labor supply of 2, which is on the high end of available empirical estimates from micro and aggregate data (see, for example, Blundell and MaCurdy, 1999; Justiniano, Primiceri, and Tambalotti, 2010; and Smets and Wouters, 2007). Finally, we normalize the number of hours worked under full employment at unity so as to preserve the size of the nontraded sector relative to the traded sector as in the baseline economy. This calibration strategy yields $\varphi = 1.11$, $\bar{h} = 3$, and $\theta = 1$.

Table 9.6 shows that the average rate of involuntary unemployment rises from 11.7 to 30.9 percent as the labor supply elasticity increases from 0 to 2. The reason why involuntary unemployment is much larger on average with an elastic labor supply specification is that during slumps households experience a negative income effect, which induces them to increase their supply of labor. However,
Table 9.6: Endogenous Labor Supply And The Welfare Costs of Currency Pegs

<table>
<thead>
<tr>
<th>Parameterization</th>
<th>Welfare Cost of Peg</th>
<th>Average Unemployment Rate Under Peg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline (inelastic labor supply)</td>
<td>7.8  7.2</td>
<td>11.7</td>
</tr>
<tr>
<td>Endogenous Labor Supply</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\delta = 0.5$</td>
<td>16.5  15.2</td>
<td>30.9</td>
</tr>
<tr>
<td>$\delta = 0.75$</td>
<td>8.2  7.5</td>
<td>30.9</td>
</tr>
<tr>
<td>$\delta = 1$</td>
<td>1.7  1.5</td>
<td>30.9</td>
</tr>
</tbody>
</table>

Note. The welfare cost of a currency peg is expressed in percent of consumption per quarter. Unemployment rates are expressed in percent.

during slumps employment is determined by the demand for labor and not affected by the shift in labor supply. It follows that all of the increase in the labor supply contributes to increasing involuntary unemployment. During booms, regardless of the labor supply elasticity, employment is determined by the intersection of the labor supply and the labor demand schedules, and involuntary unemployment is nil. Thus, with an elastic labor supply the unemployment problem becomes worse during contractions but stays the same during booms. On net, therefore, unemployment must be higher in the economy with an endogenous labor supply. It is important to note that the behavior of unemployment is independent of the assumed value of $\delta$, the parameter governing the relative valuation of voluntary and involuntary unemployment. Technically, this is because $\delta$ does not appear in any equilibrium condition of the model, but affects only the welfare consequences of unemployment. Intuitively, the reason why unemployment is independent of $\delta$ is that the household takes the number of hours worked as exogenously given. Then $\delta$ simply measures how differently households feel about voluntary and involuntary leisure.

Table 9.6 also shows the welfare cost of currency pegs relative to the optimal exchange-rate policy implied by the endogenous-labor-supply model. The welfare cost of a currency peg depends significantly on the degree of substitutability between voluntary and involuntary leisure, measured
by the parameter $\delta$. The more substitutable voluntary and involuntary leisure are, i.e., the larger $\delta$ is, the lower the welfare cost of currency pegs are. This result should be expected. Consider the case in which voluntary and involuntary unemployment are perfect substitutes ($\delta = 1$). In this case, pegs reduce welfare because involuntary unemployment reduces the production and hence consumption of nontradable goods. However, unemployment increases leisure one for one and in this way increases utility, greatly offsetting the negative welfare effect of lower nontradable consumption. As $\delta$ falls, the marginal contribution of involuntary unemployment to total leisure, and therefore welfare, also falls. For a value of $\delta$ of 0.75, for instance, the welfare cost of currency pegs is 8.2 percent per period, which is higher than in the case with inelastic labor supply. With $\delta$ equals 0.5, the welfare cost currency pegs rises to 16.5 percent of consumption per period.

It follows that allowing for endogenous labor supply increases the average rate of unemployment caused by the combination of a currency peg and downward nominal wage rigidity, and may increase or decrease the welfare cost of currency pegs depending on how enjoyable involuntary leisure is assumed to be.

### 9.14 Production in the Traded Sector

Thus far, we have assumed that the supply of tradables, $y^T_t$, is exogenous. This section relaxes this assumption by considering a specification in which tradables are produced with labor. Specifically, suppose that

$$y^T_t = e^{z_t} (h^T_t)^{\alpha_T},$$

where $y^T_t$ denotes production of tradable goods, $h^T_t$ denotes labor employed in the traded sector, and $\alpha_T \in (0, 1)$ is a parameter. The variable $z_t$ is assumed to be exogenous and stochastic. One can interpret $z_t$ either as a productivity shock in the traded sector or as a disturbance in the country’s terms of trade. As before, assume that firms are perfectly competitive in product and
labor markets. Further, we assume that labor is perfectly mobile across sectors. We make this assumption to create a sharp contrast with the baseline formulation in which labor is completely immobile across sectors. A more realistic formulation would be one in which, in the short run, labor does move across sectors, but sluggishly. The assumption of free labor mobility across sectors implies that wages are equalized across sectors.

Firms in the traded sector choose labor to maximize profits, which are given by

\[ P_t^T e^{z_t} (h_t^T)^{\alpha_T} - W_t h_t^T. \]

The first-order condition associated with the firm’s profit maximization problem is

\[ \alpha_T P_t^T e^{z_t} (h_t^T)^{\alpha_T - 1} = W_t. \] (9.43)

Letting \( h_t^N \) denote hours employed in the nontraded sector, total hours worked, denoted by \( h_t \), are then given by

\[ h_t = h_t^T + h_t^N. \]

All other conditions of the model are as in the baseline formulation.

In Schmitt-Grohé and Uribe (2014) we assume that \( z_t \) and \( r_t \) follow the joint stochastic process given in equation (9.33), with \( z_t \) taking the place of \( \ln y_t^T \) and calibrate the parameter \( \alpha_T \) at 0.5. We find that average unemployment under a currency peg continues to be high. This result might appear counterintuitive because one might think that during contractions the unemployment created in the non-traded sector could be absorbed by the traded sector. However, because nominal wages are also downwardly rigid in the traded sector, firms there have no incentives to hire more workers during a contraction. At the same time, the model predicts that, although high, the unemployment rate under a currency peg is lower in the economy with production in the traded
sector than in the economy with an exogenous tradable output. The reason is that employment in the traded sector acts as a stabilizer of the wage rate during booms, thereby attenuating the negative externality caused by the combination of downward nominal wage rigidity and a currency peg. To see this, consider a decline in the country interest rate that raises the desired absorption of tradable and nontradable goods. This shock causes the demand for labor to increase in the nontraded sector driving up wages. This increase in wages induces firms in the traded sector to reduce employment. In turn, these freed up hours dampen the increase in wages required to clear the labor market. This dampening effect is beneficial because it means that once the boom is over the economy enters its way down to trend with lower real wages making the downward wage rigidity less stringent.

9.15 Product Price Rigidity

Consider now the case of product price rigidity. We first analyze the case of downward rigidity and then introduce symmetric price rigidity. For now, we assume that nominal wages are fully flexible.

Suppose that the nominal price of nontradables is subject to the following constraint

\[ P_t^N \geq \gamma_p P_{t-1}^N, \]

where \( \gamma_p \) is a parameter governing the degree of downward nominal price rigidity. Dividing both sides of this expression by the nominal exchange rate, \( \mathcal{E}_t \), yields

\[ p_t \geq \frac{\gamma_p}{e_t} p_{t-1}. \]  \hfill (9.44)

This expression replaces condition (9.17) of the economy with downward nominal wage rigidity.

Define the full-employment relative price of nontradables, denoted \( \rho(e_t^T) \), as the value of \( p_t \) that
induces households to voluntarily demand the full-employment level of nontradable output, \( F(\bar{h}) \), given their consumption of tradables, \( c^T_t \). By equation (9.15), we have that \( \rho(c^T_t) \) is given by

\[
\rho(c^T_t) \equiv \frac{A_2(c^T_t, F(\bar{h}))}{A_1(c^T_t, F(h))}.
\]

Given the assumed properties of the aggregator function \( A(\cdot, \cdot) \), we have that \( \rho(c^T_t) \) is increasing in \( c^T_t \). Intuitively, households have an incentive to consume relatively more tradables only if non-tradables become more expensive. Also, since \( A_1 \) is increasing in its second argument and \( A_2 \) is decreasing in its second argument, equation (9.15) implies that \( p_t \) equals \( \rho(c^T_t) \) if and only if \( h_t \) equals \( \bar{h} \). Therefore, we can postulate the following slackness condition

\[
(\bar{h} - h_t) \left( p_t - \frac{\gamma_p}{\epsilon_t} p_{t-1} \right) = 0, \tag{9.45}
\]

which replaces condition (9.19) of the economy with downward nominal wage rigidity. The new slackness conditions states that if the economy experiences involuntary unemployment \( (h_t < \bar{h}) \), then the price of nontradables must be stuck at its lower bound. By the properties of \( \rho(c^T_t) \), this means that in this situation \( p_t \) must exceed its full-employment level. The above slackness condition also states that should the lower bound on the price of nontradables not bind, then the economy must have full employment. This, in turn, means that in these circumstances \( p_t \) must equal its full-employment value, \( \rho(c^T_t) \).

An equilibrium in the economy with downward nominal price rigidity is then a set of stochastic processes \( \{c_t^T, h_t, d_{t+1}, p_t, \lambda_t, \mu_t\}_{t=0}^\infty \) satisfying (9.9)-(9.15), (9.18), (9.44), and (9.45), given an exchange-rate policy \( \{\epsilon_t\}_{t=0}^\infty \), initial conditions \( d_0 \) and \( p_{-1} \), and exogenous stochastic processes \( \{r_t, y^T_t\}_{t=0}^\infty \).

Notice that the real wage \( w_t \) does not enter in any equilibrium condition. Also, the labor demand
schedule, \( p_t F'(h_t) = w_t \), is no longer part of the set of equilibrium conditions. This is because firms are off their labor demand schedule when the economy suffers from involuntary unemployment, as they are rationed in product markets and thus employment is indirectly determined by the demand for nontradable goods. In such periods the real wage falls to zero and the value of the marginal product of labor exceeds the real wage, \( p_t F'(h_t) > w_t \).

Consider now the workings of this economy under a currency peg, that is, assume that the exchange-rate policy takes the form \( \epsilon_t = 1 \) for all \( t \). Figure 9.18 illustrates the economy’s adjustment to a negative external shock. Before the shock the demand for traded consumption is equal to \( c_T^0 \).

Figure 9.18: Adjustment to a Negative External Shock with Downward Price Rigidity Under A Currency Peg

and the demand schedule for nontraded goods is given by the solid downward sloping line. The

\[ c_T^1 < c_T^0; \quad \gamma_p = 1 \]

\[ 15 \] In a model with an endogenous labor supply as the one implied by the utility function (9.36), the real wage will be positive and equal to a value that ensures that households are on their labor supply schedule. Firms, however, would continue to be quantity rationed during these periods and off their labor demand schedules.
economy is at point A and enjoys full employment. Suppose that the negative external shock lowers traded consumption from $c^T_0$ to $c^T_1 < c^T_0$. Consequently the demand schedule shifts down and to the left, as depicted by the broken downward sloping line.

Under price flexibility, the new equilibrium would be at point C. In this equilibrium, the relative price of nontradables, $p$, falls from $\rho(c^T_0)$ to $\rho(c^T_1)$. All of this adjustment occurs via a fall in the nominal price of nontradables, $P^N$, since the nominal exchange rate is constant. However, under downward nominal price rigidity, $P^N$ cannot fall (for this illustration, we assume $\gamma_p = 1$). As a result, the relative price of nontradables is stuck at $\rho(c^T_0)$, and the equilibrium is at point B. At this point, the economy suffers from involuntary unemployment in the amount $\bar{h} - h^{bust}$. Also firms are rationed in product markets, in the sense that at the going price, they would like to sell more units than are demanded by consumers.

Notice that here a fall in wages would not solve the unemployment problem, since firms cannot sell more than $F(h^{bust})$ units of goods in the market. Note also that the unemployment problem is more severe under price rigidity than under wage rigidity for identical values of $\gamma$ and $\gamma_p$. We saw in previous sections that under downward nominal wage rigidity, the equilibrium is somewhere between points B and C at the intersection of the broken demand schedule and the supply schedule (not shown). There, the economy experiences less unemployment and lower product prices than under downward price rigidity.

The peg-induced externality analyzed in section 9.2.1 for the case of downward wage rigidity is also present under downward price rigidity. To see this, consider a positive external shock that increases the desired consumption of tradables. This shocks pushes the demand schedule for nontradables up and to the right. The new equilibrium features full employment, a higher nominal price of nontradables, and no rationing in the goods market. On the surface, no problems arise in the adjustment process. However, the increase in the nominal price of nontradables makes the economy weaker. For once the positive external shock fades away, the nominal price of nontradables will have
to fall to induce households to consume a quantity of nontradables compatible with full employment of the labor force. But this fall in the nominal price does not take place quickly enough if prices are rigid, and involuntary unemployment emerges. Collectively, households would be better off if they limited the initial expansion in the demand for nontradables. And they understand this. But each household is too small to affect the initial rise in prices by curbing its individual expenditure. There lies the peg induced externality.

The optimal exchange-rate policy under price rigidity is quite similar to its counterpart under wage rigidity. Indeed, as exercise 9.10 asks you to demonstrate, under certain conditions, any exchange-rate policy that is optimal under downward nominal wage rigidity is also optimal under downward nominal price rigidity. In response to negative external shocks, the monetary authority can preserve full-employment (point $C$ in figure 9.18), by devaluing the currency. In this way, the monetary authority can bring down the relative price of nontradables from $\rho(c_{0}^{T})$ to $\rho(c_{1}^{T})$. The required depreciation is larger the larger the contraction in the demand for tradable goods caused by the negative external shock. We then have that, as in the case of downward nominal wage rigidity, contractions are devaluatory under the optimal exchange-rate policy.

To quantify the consequences of a currency peg for unemployment and welfare, we calibrate the model using the same parameter values as in the case of downward nominal wage rigidity, see table 9.3, with the difference that now wages are fully flexible ($\gamma = 0$) and that the nominal price of nontradables is downwardly rigid. For comparison with the case of wage rigidity, we set $\gamma_{p} = 0.99$. Table 9.7 shows that the mean unemployment rate under a currency peg is 14.1 percent, which is higher than the mean unemployment rate under wage rigidity. This finding confirms the intuition build around figure 9.18. With higher average unemployment, the welfare costs of currency pegs under downward price rigidity are also larger, with a mean of 9.9 percent of consumption per period as compared to 7.8 percent of consumption under downward wage rigidity.

Unlike the empirical literature on wage rigidity, the empirical literature on nominal price rigidi-
ties has not drawn attention to asymmetries in product price adjustments (see, for example, Nakamura and Steinsson, 2008). This suggests, that the case of greater empirical relevance may be one in which price rigidity is symmetric. For this reason, we now consider an economy in which nominal prices of nontradables are subject to the constraint

$$\gamma_p \leq \frac{P_t^N}{P_{t-1}^N} \leq \frac{1}{\gamma_p}.$$  

As in the case of nominal wage rigidity increasing the degree of price rigidity along the extensive margin (from downward to downward and upward rigidity) lowers unemployment and the welfare costs of currency pegs. Table 9.7 shows that unemployment and the welfare costs of currency pegs fall by more than half as upward rigidity is added. The intuition behind this result is that, as in the case of wage rigidity, hindering price increases ameliorates the externality created by currency pegs.

<table>
<thead>
<tr>
<th>Parameterization</th>
<th>Welfare Cost of Peg Mean</th>
<th>Welfare Cost of Peg Median</th>
<th>Average Unemployment Rate Under Peg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline (wage rigidity, $\gamma = 0.99$ and $\gamma_p = 0.$)</td>
<td>7.8</td>
<td>7.2</td>
<td>11.7</td>
</tr>
<tr>
<td>Nominal Price Rigidity ($\gamma = 0$, $\gamma_p = 0.99$)</td>
<td>9.9</td>
<td>9.0</td>
<td>14.1</td>
</tr>
<tr>
<td>Downward Price Rigidity, $P_t^N/P_{t-1}^N \geq \gamma_p$</td>
<td>4.4</td>
<td>3.9</td>
<td>6.6</td>
</tr>
<tr>
<td>Symmetric Price Rigidity, $1/\gamma_p \geq P_t^N/P_{t-1}^N \geq \gamma_p$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note. The welfare cost of a currency peg is expressed in percent of consumption per quarter. Unemployment rates are expressed in percent.
9.16 Exercises

**Exercise 9.1 (The Demand Schedule Of Nontradables)** In section 9.1, we assume that the aggregator function, $A(c^T, c^N)$, given in (9.2), is increasing, concave, and linearly homogeneous.

1. Show that these assumptions are sufficient to ensure that the demand schedule of nontradables, given in equation (9.5) and depicted in figure 9.2, is downward sloping in the space $(c^N, p)$, holding constant $c^T$.

2. Show that the aforementioned assumptions about the aggregator function $A(c^T, c^N)$ are sufficient to guarantee that increases (decreases) in $c^T$ shift the demand schedule up and to the right (down and to the left).

3. Assume that the aggregator function takes the Cobb-Douglas form $A(c^T, c^N) = \sqrt{c^T c^N}$. Find the demand function of nontradables.

4. Now assume the CES form $A(c^T, c^N) = \left[a(c^T)^{1-\frac{1}{\xi}} + (1-a)(c^N)^{1-\frac{1}{\xi}}\right]^{\frac{1}{1-\frac{1}{\xi}}}$. Derive the demand function of nontradables. Interpret the parameter $\xi$.

**Exercise 9.2 (Unwanted Positive Shocks)** Show that in the example of section 9.2.3, the fall in the interest rate is welfare decreasing under downward nominal wage rigidity but welfare increasing under flexible wages. How can this be?

**Exercise 9.3 (Is More Wage Rigidity desirable?)** Modify the example of section 9.2.3 to allow for wage rigidity in both directions, downwardly and upwardly. Characterize the economy’s response to a temporary decline in the interest rate. Show that welfare is higher under full wage rigidity than under downward wage rigidity. Provide intuition.

**Exercise 9.4 (Properties of the Full-Employment Real Wage)** Show that $\omega'(c^T_t)$ is positive.
Exercise 9.5 (Pareto Optimality of the Flexible-Wage Equilibrium) Demonstrate that when nominal wages are fully flexible, the competitive equilibrium is Pareto optimal for any exchange-rate policy.

Exercise 9.6 (Foreign Inflation) Assume that the foreign price of tradable goods, $P_t^{T*}$ grows at the deterministic gross rate $\pi^*$, that is $P_{t+1}^{T*}/P_t^{T*} = \pi^*$. How do the equilibrium conditions (9.9)-(9.19) change by the introduction of this assumption.

Exercise 9.7 (Trend Growth) Continue to assume, as in exercise 9.6, that the foreign price of tradables grows at the rate $\pi^*$. In addition, assume that the production of nontradables is given by $Y_t^N = X_tF(h_t)$ and that the endowment of tradables is given by $Y_t^T = X_t^T T_t$, where $X_t$ is a deterministic trend that grows at the gross rate $g$, that is $X_{t+1}/X_t = g$. Again, show how equilibrium conditions (9.9)-(9.19) are modified by the imposition of this assumption. Hint: As in the model of chapter 5, you will have to transform some variables appropriately to make them stationary. The equilibrium conditions should include only stationary variables.

Exercise 9.8 (Exchange-Rate Policy and GDP in Terms of Tradables) Take a look at the two bottom panels of figure 9.12 showing the behavior of the trade-balance-to-output ratio and the debt-to-output ratio predicted by the model of section 9.1 during an external crisis under a currency peg and under the optimal exchange-rate policy. Note that these responses differ across the two exchange-rate regimes, in spite of the fact that the responses of the levels of the trade balance and the external debt are identical across exchange-rate policies, due to the assumption $\sigma = 1/\xi$. Of course, all of the differences must be due to the fact that output (measured in terms of tradables) behaves differently across exchange-rate regimes. Explain analytically the nature of these differences. Consider in particular the cases $\xi = 1$ and $\xi > 1$, under the maintained assumption $\sigma = 1/\xi$. 
Exercise 9.9 (The CPI Index) Show that if the technology for producing the composite consumption given in (9.2) is of the CES form, then the consumption price level, \( P_t \), can be expressed as a CES function of the nominal prices of tradable and nontradables, \( E_t \) and \( P_t^N \), respectively.

Exercise 9.10 (Optimal Exchange-rate Policy Under Price and Wage Stickiness) Show that, in the context of the model developed in this chapter, the families of optimal exchange-rate policies are identical under downward price rigidity and downward wage rigidity provided that \( \gamma = \gamma_p \), that the economy was in full employment in period \(-1\), and that the sources of uncertainty are stochastic disturbances in \( r_t \) and \( y_T^T \). Show that this result would fail to obtain in the presence of productivity shocks in the nontraded sector.

Exercise 9.11 (Sudden Stops) Consider an open economy that lasts for only two periods, denoted 1 and 2. Households are endowed with 10 units of tradables in period 1 and 13.2 units in period 2 (\( y_1^T = 10 \) and \( y_2^T = 13.2 \)). The country interest rate is 10 percent, or \( r = 0.1 \), the nominal exchange rate, defined as the price of foreign currency in terms of domestic currency, is fixed and equal to 1 in both periods (\( E_1 = E_2 = 1 \)). Suppose that the foreign-currency price of tradable goods is constant and equal to one in both periods, and that the law of one price holds for tradable goods in both periods. Nominal wages are downwardly rigid. Specifically, assume that the nominal wage, measured in terms of domestic currency, is subject to the constraint

\[
W_t \geq W_{t-1}
\]

for \( t = 1, 2 \), with \( W_0 = 8.25 \). Suppose the economy starts period 1 with no assets or debts carried over from the past \( (d_1 = 0) \). Households are subject to the no-Ponzi-game constraint \( d_3 \leq 0 \).

Suppose that the household’s preferences are defined over consumption of tradable and nontrad-
able goods in periods 1 and 2, and are described by the following utility function,

\[ \ln C^T_1 + \ln C^N_1 + \ln C^T_2 + \ln C^N_2, \]

where \( C^T_i \) and \( C^N_i \) denote, respectively, consumption of tradables and nontradables in period \( i = 1, 2 \). Let \( p_1 \) and \( p_2 \) denote the relative prices of nontradables in terms of tradables in periods 1 and 2, respectively. Households supply inelastically \( \bar{h} = 1 \) units of labor to the market each period. Finally, firms produce nontradable goods using labor as the sole input. The production technology is given by

\[ y^N_t = h^\alpha_t \]

for \( t = 1, 2 \) where \( y^N_t \) and \( h_t \) denote, respectively, nontradable output and hours employed in period \( t = 1, 2 \). The parameter \( \alpha \) is equal to 0.75.

1. Compute the equilibrium levels of consumption of tradables and the trade balance in periods 1 and 2.

2. Compute the equilibrium levels of employment, nontradable output, and the relative price of nontradables in periods 1 and 2.

3. Suppose now that the country interest rate increases to 32 percent. Calculate the equilibrium levels of consumption of tradables, the trade balance, consumption of nontradables, the level of unemployment, and the relative price of nontradables in periods 1 and 2. Provide intuition.

4. Given the situation in the previous question, calculate the minimum devaluation rates in periods 1 and 2 consistent with full employment in both periods. To answer this question, assume that the nominal exchange rate in period 0 was also fixed at unity. Explain.

5. Continue to assume that \( W_0 = 8.25 \) and that the interest rate is 32 percent. Assume also that
the government is not willing to devalue the domestic currency, so that \( E_1 = E_2 = 1 \). Instead, the government chooses to apply capital controls in period 1. Specifically, let \( d_2/(1+r_1) \) denote the amount of funds borrowed in period 1, which generate the obligation to pay \( d_2 \) in period 2. Suppose that in period 1 the government imposes a proportional tax/subsidy \( \tau_1 \) on borrowed funds, so that the amount received by the household is \((1 - \tau_1)d_2/(1 + r_1)\). Suppose that this tax/subsidy is rebated/financed in a lump-sum fashion. Calculate the Ramsey optimal level of \( \tau_1 \).

Exercise 9.12 (Productivity Shocks in the Nontraded Sector) Consider an economy like the one developed in section 9.1, in which the nontraded good is produced with the technology \( y_t^N = e^{z_t} h_t^\alpha \), where \( z_t \) denotes an exogenous and stochastic productivity shock. Assume that \( z_t \) evolves according to the law of motion \( z_t = \rho z_{t-1} + \mu_t \), where \( \rho \in [0, 1) \) is a parameter and \( \mu_t \) is an i.i.d. disturbance with mean zero and standard deviation \( \sigma_\mu \). Suppose that the endowment of tradables is constant and equal to \( y_T > 0 \) and that the interest rate is constant and equal to \( r \). Assume that \( r \) satisfies \( \beta(1 + r) = 1 \). Assume that the period utility function and the aggregator function are given by (9.30) and (9.31), respectively, with \( \xi = 1/\sigma < 1 \). Suppose that \( d_0 = 0 \) and that the economy was operating at full employment in \( t = -1 \).

1. Find the equilibrium process of consumption of tradables, \( c_t^T \).

2. Derive the optimal devaluation rate, \( \epsilon_t \), as a function of present and past values of the productivity shock.

3. Provide a graphical analysis of the effect of an increase in productivity \((z_0 > z_{-1})\) under the optimal exchange-rate policy and under a currency peg. Provide intuition.

4. Suppose that the monetary authority follows the optimal exchange-rate policy that makes the domestic currency as strong as possible relative to the foreign currency at all times. Find
the unconditional correlation between the net devaluation rate, $\ln \epsilon_t$ and the growth rate of productivity, $z_t/z_{t-1}$. Provide intuition.

5. How would the sign of the correlation obtained in the previous item and the intuition behind it change in the case $\xi = 1/\sigma > 1$. 
Chapter 10

Exchange Rate Policy And Capital Controls

In chapter 9, we studied a model with nominal rigidities in which the nominal exchange rate can be used to bring about the Pareto optimal allocation. We established that the optimal exchange-rate policy calls for devaluations when the economy is hit by negative external shocks. And when these shocks are large, so are the required devaluations. However, for many emerging countries, such as those that are part of a currency union, devaluations may not be an option. This chapter explores the potential of other nonmonetary policies to address the distortions created by nominal rigidities when the exchange-rate regime is suboptimal. We begin by studying tax policies that can bring about the first-best (or Pareto optimal) allocation and then analyze optimal capital control policy.

We analyze these questions in the context of the model developed in chapter 9, in which nominal frictions take the form of downwardly rigid nominal wages. We will assume that the nominal exchange rate is fixed. But the insights of this chapter apply more broadly to any suboptimal exchange-rate regime.
10.1 First-Best Fiscal Policy Under Fixed Exchange Rates

Because wage rigidity creates a distortion in the labor market, and because nontradable goods are labor intensive (in the model as well as in the data), it is reasonable to begin by studying fiscal instruments directly targeted to the labor market or the market for nontraded goods as vehicles to remedy this source of inefficiency.

We start by studying optimal labor subsidy schemes, which are perhaps the most direct way to address the distortions created by the combination of wage rigidity and suboptimal exchange-rate policy. Contributions in this area include Schmitt-Grohé and Uribe (2012, 2014) and Farhi, Gopinath, and Itskhoki (2014). The treatment here follows the former authors.

10.1.1 Labor Subsidies

The reason why in the model of chapter 9 negative external shocks cause involuntary unemployment is that the combination of downward nominal wage rigidity and a currency peg prevents the real wage from falling to the level compatible with full employment. In these circumstances, a labor subsidy would reduce the firm’s perceived labor cost thereby increasing the demand for labor. Specifically, suppose that the government subsidizes employment at the firm level at the proportional rate $s^h_t$. Profits expressed in terms of tradable goods are then given by

$$p_t F(h_t) - (1 - s^h_t) w_t h_t.$$  

The firm’s optimality condition becomes

$$p_t = (1 - s^h_t) \frac{w_t}{F'(h_t)}.$$
This expression states that for any given real wage, $w_t$, the larger the subsidy, $s^h_t$, is, the lower is the marginal cost of labor perceived by the firm, $(1 - s^h_t)w_t$. Therefore, for any given relative price, $p_t$, the larger the subsidy is, the larger the number of hours the firm is willing to hire.

Figure 10.1 illustrates how labor subsidies of this form can bring about the efficient allocation. Consider a situation in which an external shock, such as an increase in the country interest rate, brings the economy from an initial situation with full employment, point A, to one with involuntary unemployment in the amount $\bar{h} - h_{bust}$, point B. The labor subsidy causes the labor supply schedule to shift down and to the right, as shown by the dashed upward sloping line. The new intersection of the demand and supply schedules is at point C, where full employment is restored. Unlike what happens under the optimal exchange-rate policy, under the optimal labor subsidy, the real wage does not fall during the crisis. Specifically, the real wage received by the household remains constant.
at \( w_0 = W_0/\mathcal{E}_0 \). Once the negative external shock dissipates, i.e., once the interest rate falls back to its original level, the fiscal authority can safely remove the subsidy, without compromising its full employment objective.

This graphical analysis suggests that a labor subsidy can support full employment at all times. Let’s now establish this result more formally. A relevant question is how the government should finance this subsidy. It turns out that in the present model the government can tax any source of income in a nondistorting fashion. Suppose, for instance, that the government levies a proportional tax, \( \tau_t \), on all sources of household income, wage income, profits, and the tradable endowment. In this case, the government budget constraint takes the form

\[
sh_t w_t h_t = \tau_t \left( y^T_t + w_t h_t + \phi_t \right),
\]

where \( \phi_t \) denotes profits expressed in terms of tradables. Implicit in this budget constraint is the simplifying assumption that the government issues no debt, i.e., that it follows a balanced-budget rule. This assumption entails no loss of generality. The left-hand side of this expression represents the government’s outlays, consisting in subsidies to firms. The right-hand side represents tax revenues. Given the level of labor subsidies, \( s^h_t \), the income tax rate, \( \tau_t \), adjusts endogenously to guarantee the government’s budget constraint holds period by period. To see that the income tax, \( \tau_t \), is nondistorting, consider the household’s budget constraint, which now takes the form

\[
c^T_t + p_t c^N_t + d_t = (1 - \tau_t)(y^T_t + w_t h_t + \phi_t) + \frac{d_{t+1}}{1 + r_t}.
\]

Let’s inspect each source of household income separately. Because the endowment of tradable goods, \( y^T_t \), is assumed to be exogenous, it is not affected by taxation. Similarly, profit income from the ownership of firms, \( \phi_t \), is taken as given by individual households. Consequently, the imposition
of profit taxes at the household level is non-distorting. Finally, notice that households either supply \( \bar{h} \) hours of work inelastically, in periods of full-employment, or are rationed in the labor market, in periods of unemployment. In any event, households take their employment status as given. As a result, taxes do not alter households’ incentives to work. (Exercise 10.1 asks you to demonstrate that income taxes continue to be nondistorting even when the labor supply is endogenous.) It follows that the first-order conditions associated with the household’s utility-maximization problem are the same as those given in chapter 9.1 for an economy without taxation of household income.

Combining the government budget constraint, the household budget constraint, the definition of firm’s profits, and the market clearing condition in the nontraded sector, \( c_t^N = F(h_t) \), yields the resource constraint (9.9). An equilibrium under a currency peg \( (\epsilon_t = 1) \) with labor subsidies is then given by a set of processes \( \{c_t^T, h_t, w_t, d_{t+1}, p_t, \lambda_t, \mu_t\}_{t=0}^{\infty} \) satisfying

\[
c_t^T + d_t = y_t^T + \frac{d_{t+1}}{1 + r_t}, \tag{10.1}
\]

\[
d_{t+1} \leq \bar{d}, \tag{10.2}
\]

\[
\mu_t \geq 0, \tag{10.3}
\]

\[
\mu_t (d_{t+1} - \bar{d}) = 0, \tag{10.4}
\]

\[
\lambda_t = U'(A(c_t^T, F(h_t)))A_1(c_t^T, F(h_t)), \tag{10.5}
\]

\[
\frac{\lambda_t}{1 + r_t} = \beta E_t \lambda_{t+1} + \mu_t, \tag{10.6}
\]

\[
p_t = A_2(c_t^T, F(h_t)) \frac{A_1(c_t^T, F(h_t))}{A_1(c_t^T, F(h_t))}, \tag{10.7}
\]

\[
p_t = (1 - s_t^h) \frac{w_t}{F'(h_t)}, \tag{10.8}
\]
\begin{equation}
    w_t \geq \gamma w_{t-1}, \tag{10.9}
\end{equation}

\begin{equation}
    h_t \leq \bar{h}, \tag{10.10}
\end{equation}

\begin{equation}
    (\bar{h} - h_t) (w_t - \gamma w_{t-1}) = 0, \tag{10.11}
\end{equation}

given a labor-subsidy policy, \( \{s^h_t\}_{t=0}^{\infty} \), initial conditions \( w_{-1} \) and \( d_0 \), and exogenous stochastic processes \( \{r_t, y^T_t\}_{t=0}^{\infty} \). This set of equilibrium conditions is identical to that pertaining to an economy with a currency peg but without labor subsidies given in section 9.2 with the exception that here the labor subsidy \( s^h_t \) creates a wedge between the marginal cost of production of nontradables and the relative price of nontradables (equation 10.8). The benevolent government will manipulate this wedge to undo the real wage rigidity created by the combination of downward nominal wage rigidity and a currency peg.

Consider a policymaker who wishes to set the labor subsidy \( s^h_t \) in a Ramsey optimal fashion. The optimization problem faced by this policymaker is to maximize

\begin{equation}
    E_0 \sum_{t=0}^{\infty} U(A(c^T_t, F(h_t))) \tag{10.12}
\end{equation}

subject to the complete set of equilibrium conditions (10.1)-(10.11). To see that the Ramsey-optimal labor subsidy policy supports the Pareto optimal allocation, consider the less restricted problem of maximizing (10.12) subject to only three constraints, namely (10.1), (10.2), and (10.10). This less restricted optimization problem is the optimization problem of the Pareto planner and therefore yields the Pareto optimal allocation, which, among other things, is characterized by full employment at all times, \( h_t = \bar{h} \) for all \( t \) (see chapter 9, section 9.3.2). To establish that this allocation is indeed the solution to the Ramsey optimization problem, we must show that the remaining constraints of the Ramsey problem, namely (10.3)-(10.9) and (10.11), are also satisfied. To see this, notice
that equations (10.3)-(10.6) are first-order conditions of the less restricted optimization problem, so they are always satisfied. Also, since the Pareto optimal allocation implies full employment at all times, the slackness condition (10.4) is always satisfied. Now set $p_t$ to satisfy (10.7). Then, for every $t \geq 0$, given $w_{t-1}$, set $w_t$ at any arbitrary value satisfying (10.9). Finally, set the labor subsidy $s^h_t$ to satisfy equation (10.8). This completes the proof that the Pareto optimal allocation solves the optimization of the Ramsey planner. In other words, the Ramsey optimal labor subsidy policy supports the Pareto optimal allocation.

**Equivalence of Labor Subsidies and Devaluations**

How does the optimal labor subsidy in the present economy compare to the optimal devaluation policy studied in chapter 9? Combine equilibrium conditions (10.7) and (10.8) and evaluate the result at the optimal allocation to obtain $w_t(1 - s^h_t) = \frac{A_2(c^T_t, F(\bar{h}))}{A_1(c^T_t, F(\bar{h}))}F'(\bar{h})$. As in chapter 9, define

$$\omega(c^T_t) \equiv \frac{A_2(c^T_t, F(\bar{h}))}{A_1(c^T_t, F(\bar{h}))}F'(\bar{h}).$$

Then we can write $w_t = \omega(c^T_t)/(1 - s^h_t)$. Finally, combine this expression with equilibrium condition (10.9) to get

$$\frac{1}{1 - s^h_t} \geq \frac{\gamma w_{t-1}}{\omega(c^T_t)}$$

(10.13)

Any subsidy policy satisfying this condition is Ramsey and Pareto optimal. As in the case of optimal exchange-rate policy, there is a whole family of labor subsidy policies that support the Pareto optimal allocation. Furthermore, comparing this expression with condition (9.25), we obtain the following equivalence result:

If the process for the devaluation policy $\epsilon_t$ is optimal in the economy with no labor subsidies, then the process $s^h_t \equiv (\epsilon_t - 1)/\epsilon_t$ is optimal in the economy with a fixed exchange rate.
This relationship between the optimal exchange-rate policy and the optimal labor subsidy policy as alternative ways of achieving the first-best allocation is useful to gauge the magnitude of the labor subsidy necessary to preserve full employment during crises. In chapter 9, we found that during a large crisis like the one observed in Argentina in 2001, the model predicts optimal devaluations of between 30 and 40 percent per year, or between 7 and 9 percent per quarter, for about two and a half years (see figure 9.12). Using the formula given above, the implied optimal labor subsidy required to prevent unemployment ranges form 6.5 to 8 percent. These are large numbers. Consider a labor share of 75 percent of GDP and a share of nontradables of 75 percent of GDP as well. Then, the budgetary impact of a labor subsidy of 6.5 to 8 percent is 3.5 to 4.5 percent of GDP.

Finally, we note that a property of the optimal labor subsidies characterized here is that there is a sense in which they are good for only one crisis. Specifically, suppose the fiscal authority grants a labor subsidy during a crisis and keeps it in place once the crisis is over. When the next crisis comes, the old subsidy does not help at all to avoid unemployment. The reason is that the recovery after the first crisis causes nominal wages to increase, placing the economy in a vulnerable situation to face the next downturn. The new crisis would then require another increase in labor subsidies. This logic leads to a process for labor subsidies converging to one hundred percent. To avoid this situation, the policymaker must remove the subsidy as soon as the crisis is over. In this way, the recoveries occur in the context of nominal wage stability, and the optimal subsidy policy is stationary. Formally, the stationary optimal labor-subsidy policy takes the form

$$\frac{1}{1 - s^h_t} = \max \left\{ 1, \frac{\gamma w_{t-1}}{\omega(c^t_e)} \right\}$$

which clearly belongs to the family of optimal labor-subsidy policies given in equation (10.13).
10.1.2 Sales Subsidies

Another fiscal alternative to achieve the Pareto optimal allocation under a currency peg is to subsidize sales in the nontraded sector. Let $s_t^N$ be a proportional subsidy on sales in the nontraded sector. Then, profits of a representative firm in the nontraded sector are given by

$$(1 + s_t^N)p_tF(h_t) - \frac{W_t}{\xi_t}h_t.$$ 

The profit-maximization condition of the firm becomes

$$p_t = \frac{1}{1 + s_t^N}\frac{W_t}{\xi_t}F'(h_t).$$

According to this expression, an increase in the sales subsidy increases the marginal revenue of the firm. Like a wage subsidy, a sales subsidy shifts the supply schedule down and to the right. The graphical analysis is therefore qualitatively identical to that used to explain the workings of the optimal wage subsidy shown in figure 10.1. Clearly the following equivalence result obtains: If the labor subsidy process $s_t^h$ is Pareto optimal under a currency peg without sales taxes, then the sales tax process $s_t^N = s_t^h/(1 - s_t^h)$ is Pareto optimal under a currency peg without labor subsidies.

10.1.3 Consumption Subsidies

A third fiscal instrument that can be used to ensure full employment at all times in a pegging economy with downward nominal wage rigidity is a proportional subsidy to the consumption of nontradables. Specifically, assume that the after-subsidy price of nontradable goods faced by consumers is $(1 - s_t^N)p_t$. The subsidy on nontraded consumption makes nontradables less expensive relative to tradables. It can therefore be used by the government during a crisis to facilitate an expenditure switch toward nontraded consumption and away from tradable consumption. With
subsidies on nontraded consumption, the demand schedule is given by

\[(1 - s^c_N)p_t = \frac{A_2(c_t^T, F(h_t))}{A_1(c_t^T, F(h_t))}.

Figure 10.2 illustrates how the consumption subsidy can be used to ensure the efficient functioning of the labor market. Suppose a negative external shock reduces the desired demand for tradable goods shifting the demand schedule for nontradables down and to the left as indicated by the downward sloping broken line. As discussed before, in the absence of any intervention, the pegging economy would be stuck at the inefficient point B, with involuntary unemployment. The introduction of the subsidy to nontraded consumption shifts the demand schedule back up and to the right. If the magnitude of the subsidy is chosen appropriately, the demand schedule will cross the supply
schedule exactly at point $A$, where the labor market returns to full employment. Note that the real exchange rate does not adjust to the external shock. However, the after-tax real exchange rate experiences a depreciation when the consumption subsidy is implemented. This perceived depreciation boosts the demand for nontradables thereby preventing a spillover to the nontraded sector of a contraction originating in the traded sector.

This fiscal instrument is equivalent to a labor subsidy in the following sense: If the labor subsidy $s^h_t$ implements the Pareto optimal allocation under a currency peg, then so does the consumption tax $s^{CN}_t = s^h_t$. Taken together the results of this section show that the optimal devaluation policy is equivalent to a currency peg with optimal labor subsidies or optimal sales subsidies in the nontraded sector or optimal subsidies to the consumption of nontradables.

A criticism that can be raised against all three of the fiscal stabilization schemes considered here is that the implied tax policies inherit the stochastic properties of the underlying sources of uncertainty (i.e., $r_t$ and $y^T_t$). This means that tax rates must change at business-cycle frequencies. To the extent that changes to the tax code are subject to legislative approval, the long and uncertain lags involved in this process might render the implementation of the optimal tax policy impractical.

### 10.2 Capital Controls

Fixed-exchange rate arrangements are often part of broader economic reform programs that include liberalization of international capital flows. For small emerging economies, such a policy combination has been a mixed blessing. A case in point is the periphery of the European Union. Figure 10.3 displays the current-account-to-GDP ratio, an index of nominal hourly wages in Euros, and the rate of unemployment for eleven peripheral members of the European Union from 2000 to 2011. All of these countries allow for free capital mobility among one another and with the rest of Europe as part of the Union membership. In addition, all of the eleven countries were either
on the euro (Cyprus, Greece, Ireland, Portugal, Slovakia, Slovenia, and Spain) or pegging to the euro (Bulgaria, Estonia, Lithuania, and Latvia) during the great contraction. In the early 2000s, these countries enjoyed large capital inflows, which, through their expansionary effect on domestic absorption, led to sizable appreciations in hourly wages. With the onset of the global recession in 2008, however, capital inflows dried up and aggregate demand collapsed. At the same time nominal wages remained at the level they had achieved at the peak of the boom. The combination of depressed levels of aggregate demand and high nominal wages was associated with a massive increase in involuntary unemployment. In turn, local monetary authorities were unable to reduce real wages via a devaluation because of their commitment to the currency union.

Viewed through the lens of the model with nominal rigidities studied earlier in this chapter and in the previous chapter, the type of empirical evidence presented above suggests the possibility that countries might benefit from adopting prudential capital controls, that is, from taxing net capital inflows during booms and subsidizing them during contractions. Capital controls essentially represent a wedge between the interest rate at which the rest of the world is willing to lend to domestic residents and the interest rate effectively paid by these agents. In other words, by using capital controls, the government can control the interest rate. Therefore, raising capital controls during booms can curb the expansion in aggregate demand and in this way slow nominal wage growth. In turn, this would allow the economy to enter the contractionary phase of the cycle with a lower level of wages, which would result in less unemployment. The reduction in unemployment could be even larger if during the contraction the policy authority subsidized net capital inflows. Thus, capital controls can be viewed as one way out of Mundell’s (1963) trilemma of international finance, according to which a country cannot simultaneously have a fixed exchange rate, free capital mobility, and an independent interest-rate policy. For, by breaking free capital mobility, capital controls allow a country to pursue an independent interest rate policy.

Imposing capital controls, however, comes at a cost. The reason is that capital controls distort
Figure 10.3: Boom-Bust Cycle in Peripheral Europe, 2000-2011
Figure 10.3: continued

the real interest rate perceived by private domestic agents, thereby introducing inefficiencies in the intertemporal allocation of consumption of tradable goods. Thus, the government contemplating the imposition of capital controls faces a tradeoff between intertemporal distortions, caused by capital controls themselves, and static distortions, caused by the combination of downward nominal wage rigidity and suboptimal exchange-rate policy. The remainder of this chapter is devoted to analyzing this tradeoff from a Ramsey perspective. The analysis follows Schmitt-Grohé and Uribe (2012, 2014). See also Farhi and Werning (2012).

10.2.1 Capital Controls As A Distortion To The Interest Rate

We embed capital controls into the small open economy with downward nominal wage rigidity developed in chapter 9. Throughout the analysis, we assume that the nominal exchange rate is constant.

Let $\tau^d_t$ denote a tax on net foreign debt in period $t$. Then, the household’s sequential budget constraint is given by

$$c_t^T + p_t c_t^N + d_t = (1 - \tau_t)(y_t^T + w_t h_t + \phi_t) + \frac{(1 - \tau^d_t)d_{t+1}}{1 + r_t}.$$ 

The government intervention in the international financial market through the capital control variable $\tau^d_t$ alters the effective gross interest rate paid by the household from $1 + r_t$ to $(1 + r_t)/(1 - \tau^d_t)$. The rate $\tau^d_t$ can take positive or negative values. When it is positive, the government discourages external borrowing by raising the effective interest rate. In this case, we say that the government imposes capital controls. When $\tau^d_t$ is negative, the government subsidizes international borrowing by lowering the effective interest rate. As we will see shortly, a benevolent government will make heavy use of cyclical adjustments in capital controls to stabilize consumption and employment.

The variable $\tau_t$ denotes a proportional tax rate (or subsidy rate if negative) on personal income.
In section 10.1.1, we showed that $\tau_t$ is a nondistorting tax instrument in the present economy. The government is assumed to set $\tau_t$ so as to balance its budget period by period. Specifically, $\tau_t$ is used to rebate or finance any revenue or deficit generated by capital controls. Assuming that the government starts out with no public debt outstanding, its balanced-budget rule implies that, given $\tau^d_t$, the income tax rate $\tau_t$ is set residually to satisfy

$$\tau^d_t \frac{d_{t+1}}{1+r_t} + \tau_t (y^T_t + w_t h_t + \phi_t) = 0.$$ 

### 10.2.2 Equilibrium Under Capital Controls And A Currency Peg

The introduction of capital controls alters only one of the equilibrium conditions of the economy with a currency peg given in section 9.2, namely the Euler equation (9.14) stemming from the household’s choice of foreign debt. This optimality condition now becomes

$$\lambda_t \frac{1}{1+r_t} = \beta E_t \lambda_{t+1} + \mu_t.$$ 

It is clear from this expression that the gross interest rate that is relevant for the household is not $1 + r_t$ but $(1 + r_t)/(1 - \tau^d_t)$.

A competitive equilibrium under a fixed exchange-rate regime with capital controls is then a set of processes $\{c^T_t, d_{t+1}, h_t, w_t, \lambda_t, \mu_t\}_{t=0}^{\infty}$ satisfying

$$c^T_t + d_t = y^T_t + \frac{d_{t+1}}{1+r_t},$$

$$\frac{A_2(c^T_t, F(h_t))}{A_1(c^T_t, F(h_t))} F'(h_t) = w_t,$$ 

$$h_t \leq \bar{h},$$
\( w_t \geq \gamma w_{t-1}, \quad \text{(10.17)} \)

\( d_{t+1} \leq \bar{d}, \quad \text{(10.18)} \)

\[ \lambda_t = U'(A(c_t^T, F(h_t)))A_1(c_t^T, F(h_t)), \quad \text{(10.19)} \]

\[ \frac{\lambda_t(1 - \tau^d_t)}{1 + r_t} = \beta E_t \lambda_{t+1} + \mu_t, \quad \text{(10.20)} \]

\( \mu_t \geq 0, \quad \text{(10.21)} \)

\( \mu_t(d_{t+1} - \bar{d}) = 0, \quad \text{(10.22)} \)

\( (h_t - \bar{h})(w_t - \gamma w_{t-1}) = 0, \quad \text{(10.23)} \)

given exogenous stochastic processes \( \{y_t^T, r_t\}_t=0^\infty \), initial conditions \( d_0 \) and \( w_{-1} \), and a capital-control policy \( \{\tau^d_t\}_t=0^\infty \).

### 10.3 Optimal Capital Controls Under Fixed Exchange Rates

Assume that the government is benevolent in the sense that it chooses the capital control policy to maximize the lifetime welfare of the representative household subject to the complete set of equilibrium conditions given above. This type of optimization problem is known as the Ramsey problem, its solution as the Ramsey optimal equilibrium, and the resulting policy as the Ramsey optimal policy. As it will become clear shortly, in the present model the Ramsey optimal capital-control policy turns out to be time consistent, that is, in any given period and taking the fixed exchange rate as given, the government has no incentive to deviate from the Ramsey-optimal capital-control policy.

The Ramsey planner’s optimization problem consists in choosing processes \( \{\tau^d_t, c_t^T, d_{t+1}, h_t, w_t, \lambda_t, \mu_t\}_t=0^\infty \) to maximize (10.12) subject to conditions (10.14)-(10.23).
The strategy we adopt to characterize the Ramsey equilibrium is to first solve a simplified Ramsey optimization problem and then show that the solution to this problem also solves the original Ramsey optimization problem. Specifically, we first drop conditions (10.19)-(10.23) from the set of constraints of the Ramsey planner’s problem and then show that the solution to the resulting less constrained Ramsey problem satisfies the omitted constraints. This second step ensures that the solution to the less constrained Ramsey-planner problem is indeed the Ramsey-optimal equilibrium.

To see that the solution to the less constrained Ramsey problem also satisfies the constraints omitted from the original Ramsey problem, suppose that the processes \( \{c^T_t, h_t, w_t\} \) solve the less constrained problem. Now pick \( \lambda_t \) to satisfy (10.19). Next, set \( \mu_t = 0 \) for all \( t \). This implies that (10.21) and (10.22) are satisfied. Then pick \( \tau^d_t \) to satisfy (10.20). It remains to show that the slackness condition (10.23) is satisfied when evaluated at the allocation that solves the less constrained Ramsey problem. To see that this is the case, consider the following proof by contradiction. Suppose, contrary to what we wish to show, that the allocation that solves the less constrained Ramsey problem is such that \( h_t' < \bar{h} \) and \( w_t' > \gamma w_{t'-1} \) at some date \( t' \geq 0 \). Consider now a small increase in hours only at date \( t' \) from \( h_t' \) to \( \tilde{h} \leq \bar{h} \), holding the processes \( \{c^T_t, d_{t+1}\}_{t=0}^{\infty} \) unchanged. Clearly, this perturbation in hours does not violate the resource constraint (10.14) because \( h_t \) does not appear in this equation. From (10.15) we have that the real wage in period \( t' \) falls to \( \tilde{w} \equiv \frac{A_2(c^T_{t'}, F(\tilde{h}))}{A_1(c^T_{t'}, F(\tilde{h}))} F'(\tilde{h}) < w_{t'} \). Because \( A_1, A_2, \) and \( F' \) are continuous functions, the wage lower bound (10.17) is satisfied provided the increase in hours is sufficiently small (recall that we are assuming that prior to the perturbation, this lower bound holds with strict inequality). In period \( t' + 1 \), the wage lower bound (10.17) is also satisfied because \( \tilde{w} < w_{t'} \). We have therefore

\[ \text{Note that in states in which the Ramsey allocation calls for setting } d_{t+1} < \bar{d}, \mu_t \text{ must be chosen to be zero. However, in states in which the Ramsey allocation yields } d_{t+1} = \bar{d}, \mu_t \text{ need not be chosen to be zero. In these states, any positive value of } \mu_t \text{ could be supported in the decentralization of the Ramsey equilibrium. Of course, in this case, } \tau^d_t \text{ will depend on the chosen value of } \mu_t. \text{ In particular, } \tau^d_t \text{ will be strictly decreasing in the arbitrarily chosen value of } \mu_t. \]
established that the perturbation is feasible. Finally, the perturbation is clearly welfare increasing because it raises the consumption of nontradables in period $t'$ without affecting the consumption of tradables in any period or the consumption of nontradables in any period other than $t'$. It follows that an allocation that does not satisfy the slackness condition (10.23) cannot be a solution to the less constrained Ramsey problem. This completes the proof that the solution to the less constrained problem is indeed the Ramsey optimal equilibrium.

One can write the Ramsey-optimal allocation recursively as the solution to the following Bellman equation problem:

$$v(y_t^T, r_t, d_t, w_{t-1}) = \max \left[ U(A(c_t^T, F(h_t))) + \beta E_t v(y_{t+1}^T, r_{t+1}, d_{t+1}, w_t) \right]$$

subject to (10.14)-(10.18), where $v(y_t^T, r_t, d_t, w_{t-1})$ denotes the value function of the representative household. This representation is quite useful for computational purposes, and we will exploit it in the quantitative analysis presented later in the chapter.

Note that the capital control rate $\tau^d_t$ does not appear in this problem. However, from the arguments presented above, we have that it is readily backed up as

$$\tau^d_t = 1 - \beta(1 + r_t) \frac{E_t U''(A(c_{t+1}^T, F(h_{t+1})) A_1(c_{t+1}^T, F(h_{t+1}))}{U'(A(c_t^T, F(h_t))) A_1(c_t^T, F(h_t))},$$

where $c_t^T$ and $h_t$ are evaluated at the solution of the above Bellman equation problem.

### 10.4 The Optimality of Prudential Capital-Control Policy

We now present an analytical example showing the prudential nature of optimal capital controls. In this example, following a temporary decline in the interest rate, the Ramsey government introduces capital controls to discourage capital inflows. It does so in order to attenuate the
impact on future unemployment once the interest rate goes back up to its long-run level.

The example is a continuation of the one analyzed in chapter 9, section 9.2.3. There, we studied the response to a temporary decline in the interest rate under a currency peg and under the optimal exchange rate policy, assuming free capital mobility in both cases. Here, we characterize the response of the economy under a currency peg and Ramsey optimal capital controls.\(^2\)

Preferences are given by \(U(c_t) = 2 \ln(c_t)\) and \(A(c^T_t, c^N_t) = \sqrt{c^T_t c^N_t}\). The technology for producing nontradable goods is \(F(h_t) = h_t^\alpha\), with \(\alpha \in (0, 1)\). The economy starts period zero with no outstanding debt, \(d_0 = 0\). The endowment of tradables, \(y^T > 0\), is constant over time. The real wage in period \(-1\) equals \(\alpha y^T\). The economy is subject to a temporary interest rate decline in period zero. Specifically, \(r_t = r\) for all \(t \neq 0\), and \(r_0 = \bar{r} < r\). This interest-rate shock is assumed to be unanticipated. Assume that \(\beta(1+r) = 1\), \(\gamma = 1\), \(\bar{h} = 1\), and \(\alpha > r\).\(^3\) Finally, assume that the economy was at a full-employment equilibrium in periods \(t < 0\), with \(d_t = 0\), \(c^T_t = y^T\), \(c^N_t = 1\), and \(h_t = 1\).

Recall that aggregate dynamics under free capital mobility and a currency peg are given by

\[
c^T_0 = y^T \left[ \frac{1}{1 + \bar{r}} + \frac{r}{1 + r} \right] > y^T,
\]

\[
c^T_t = y^T \left[ \frac{1}{1 + \bar{r}} + \frac{r}{1 + r} \frac{1 - \bar{r}}{1 + r} \right] < y^T; \quad t \geq 1,
\]

\[
d_t = y^T \left[ 1 - \frac{1 + \bar{r}}{1 + r} \right] > 0; \quad t \geq 1,
\]

\[
h_0 = 1,
\]

\(^2\)In the context of the present model, capital controls would be superfluous under the optimal exchange-rate policy, since the latter achieves the first-best allocation even under free capital mobility. In chapter 12, section 12.12, we will study a version of the present model with imperfect enforcement of international debt contracts. In that environment, both the exchange rate and capital controls play nontrivial roles in the Ramsey equilibrium.

\(^3\)Exercise 10.4 deals with the case \(\alpha < r\).
Figure 10.4 depicts these equilibrium dynamics with solid lines. Under free capital mobility and a fixed exchange rate, the fall in the interest rate in period 0 causes an expansion in the consumption of tradables financed by external debt and an increase in real wages and the relative price of nontradables (not shown). In period 1, the country interest rate rises to its long-run value and consumption of tradables falls permanently to a level below the endowment. The resulting trade surplus is used to pay the interest on the debt incurred in period 0. The fall in aggregate demand in period 1 puts downward pressure on real wages. However, because nominal wages are downwardly rigid and the nominal exchange rate is fixed, the real wage cannot fall, causing disequilibrium in the labor market and the emergence of permanent involuntary unemployment.

We conjecture that under Ramsey optimal capital controls and a currency peg the equilibrium allocation is given by

\[ c_t^T = y^T; \quad t \geq 0, \]
\[ h_t = 1; \quad t \geq 0, \]
\[ d_{t+1} = 0; \quad t \geq 0, \]

and

\[ \tau_t^d = \begin{cases} 1 - \frac{1+r}{1+r} > 0 & \text{for } t = 0 \\ 0 & \text{for } t \geq 1 \end{cases}. \]

Figure 10.4 displays these equilibrium dynamics with crossed broken lines. The Ramsey optimal capital control policy taxes capital inflows in period 0 by setting \( \tau_0^d > 0 \), which raises the cost of external borrowing above \( r \). Indeed, the Ramsey planner finds it optimal to fully undo the temporary decline in the world interest rate. The effective interest rate faced by domestic households, \( (1+r_t)/(1-\tau_t^d) - 1 \), equals \( r \) even in period 0. In this way, the Ramsey planner curbs the boom
Figure 10.4: Adjustment Under Ramsey Optimal Capital Control Policy To a Temporary Interest Rate Decline

Country Interest Rate, $r_t$

Consumption of Tradables, $c^T_t$

Debt, $d_t$

Unemployment, $(\bar{h} - h)/\bar{h}$

Real Wage, $w_t$

Capital Control Tax, $\tau^d_t$

Time, $t$

Time, $t$

Time, $t$

Time, $t$

-3 -2 -1 0 1 2 3

-3 -2 -1 0 1 2 3

-3 -2 -1 0 1 2 3

-3 -2 -1 0 1 2 3

-3 -2 -1 0 1 2 3

-3 -2 -1 0 1 2 3

Free Capital Mobility

Ramsey Optimal Capital Controls
in aggregate demand and limits the appreciation of real wages in period 0. Consumption is fully smoothed over time and as a result the labor market is unaffected by the temporary decline in the interest rate.

In the absence of downward nominal wage rigidity, it would be optimal for the economy to take advantage of the temporary fall in the interest rate and expand the absorption of tradables in period zero. Thus, by imposing capital controls at \( t = 0 \), the planner distorts the intertemporal allocation of consumption of tradables. In isolation, this distortion is, of course, welfare decreasing. However, the gains stemming from avoiding unemployment in periods \( t \geq 1 \) more than offset these welfare losses. In fact, in the present example, the tradeoff between distortions in the allocation of tradable consumption and unemployment is resolved entirely in favor of the latter. In general, the resolution of the tradeoff will involve some unemployment and some increase in the absorption of tradables (see exercise 10.4).

We now show formally the validity of the conjectured Ramsey optimal equilibrium. The optimal capital control policy is the solution to the problem of maximizing the value function (10.24) subject to (10.14)-(10.18). The appendix to this chapter shows that under Ramsey optimal capital controls, beginning in period \( t = 1 \) all variables are constant, that is, \( c_t^T = c_1^T, h_t = h_1, d_{t+1} = d_1, w_t = w_1 \) for all \( t \geq 1 \). Because \( c_t^T = y^T \) for all \( t \), welfare in the conjectured Ramsey-optimal equilibrium is

\[
\nu^{opt} = \frac{1}{1 - \beta} \ln y^T. \tag{10.26}
\]

To see that the conjecture is correct, consider first an alternative solution in which \( c_0^T > y^T \). In this case, \( d_1 > 0 \) and therefore \( c_1^T < y^T \). The surplus \( y^T - c_1^T \) is necessary to service the debt. The full-employment wage in period 0 is \( \alpha c_0^T > \alpha y^T \equiv w_{-1} \). It follows that \( h_0 = 1 \) and \( w_0 = \alpha c_0^T \). In period 1, the full-employment wage rate is \( \alpha c_1^T \), which is clearly less than \( w_0 \). As a result, we have that in period 1 the lower bound on wages binds, that is, \( w_1 = w_0 \). Equation (10.15) then implies
that \( h_t = c_T^t / c_0^T < 1 \) and that \( \ln c_t^N = \alpha (\ln c_1^T - \ln c_0^T) \) for all \( t \geq 1 \). Lifetime utility is then given by

\[
\tilde{v} = \frac{1 - \frac{1 + \alpha}{1 + \beta}}{1 - \beta} \ln c_0^T + \frac{1 + \alpha}{1 - \beta} \ln c_1^T. \tag{10.27}
\]

Because, by assumption, \( \alpha > r \), the coefficient on \( \ln c_0^T \) is negative and the coefficient on \( \ln c_1^T \) is positive. This means that moving from \( c_0^T = c_1^T = y^T \) to \( c_0^T > y^T > c_1^T \) must be welfare decreasing. It follows that \( v^{opt} > \tilde{v} \). Therefore, we have established that \( c_0^T > y^T \) is not Ramsey optimal.

The intuition for why the parameter \( \alpha \), governing labor productivity, is relevant for establishing this result is that if labor productivity is too low, then it would not pay for the planner to avoid unemployment by raising capital controls, because the additional output of nontradables resulting from higher employment is too little. In this case, the planner prefers to preserve intertemporal efficiency in the allocation of tradables at the expense of some unemployment in the nontraded sector (see exercise 10.4).

We must now check that \( c_0^T < y^T \) is not Ramsey optimal. To this end, begin by noticing that if \( c_0^T < y^T \), then \( d_1 < 0 \), and therefore \( c_t^T > y^T \), for all \( t \geq 1 \). In this case, the full-employment real wage in period 0 is \( \alpha c_0^T < \alpha y^T = w_{-1} \), which implies the existence of involuntary unemployment in period 0. Equation (10.15) then implies that \( h_0 = c_0^T / y^T < 1 \). By a similar logic, there is full employment starting in period 1, \( h_t = 1 \) for \( t \geq 1 \). Lifetime welfare is then given by

\[
\hat{v} = (1 + \alpha) \ln c_0^T + \frac{\beta}{1 - \beta} \ln c_1^T - \alpha \ln y^T.
\]

Now, combine the sequential budgets constraint (10.14) evaluated at \( t = 0 \) and \( t = 1 \), given, respectively, by \( c_0^T = y^T + d_1 / (1 + r) \) and \( c_1^T = y^T - r / (1 + r) d_1 \), to obtain

\[
c_1^T = \left( 1 + r \frac{1 + r}{1 + r} \right) y^T - r \frac{(1 + r)}{(1 + r)} c_0^T. \tag{10.28}
\]
Using this expression to eliminate $c_1^T$ from lifetime welfare, we obtain

$$\dot{v} = (1 + \alpha) \ln c_0^T + \frac{\beta}{1 - \beta} \ln \left[ \left( 1 + \frac{r}{1 + r} \right) y^T - \frac{r(1 + r)}{(1 + r)} c_0^T \right] - \alpha \ln y^T.$$ \(\text{Notice that } \dot{v} = v^{opt} \text{ when } c_0^T = y^T. \text{ Moreover, the derivative of } \dot{v} \text{ with respect to } c_0^T \text{ is positive for any } c_0^T \leq y^T. \text{ This implies that } \dot{v} < v^{opt} \text{ for any } c_0^T < y^T. \text{ We have therefore established the validity of the conjectured Ramsey optimal equilibrium.} \)

Finally, the capital control policy that supports the Ramsey equilibrium can be read off the household’s Euler equation (10.25) evaluated at $c_0^T = c_1^T = y^T$, which yields

$$\tau_0^d = 1 - \frac{1 + r}{1 + r} > 0$$

for $t = 0$ and

$$\tau_t^d = 0$$

for $t \geq 1$. These two expressions underline the prudential nature of optimal capital controls under fixed exchange rates. The government imposes restrictions to international capital inflows in period 0, when the country faces particularly favorable borrowing conditions from the rest of the world and relaxes them as soon as the cost of foreign funds goes up.

### 10.5 Optimal Capital Controls During a Boom-Bust Episode

To gauge the role of optimal capital controls in a more realistic economy, we now characterize the Ramsey-optimal equilibrium in the calibrated economy of chapter 9 (section 9.7). This is a stochastic environment characterized by random disturbances to the country interest rate, $r_t$, and to the endowment of tradables, $y_t^T$. We estimated the joint law of motion of these two exogenous
driving forces using data from Argentina over the period 1983:Q1 to 2001:Q4. The functional forms for preferences and technologies are as in chapter 9, section 9.5, and the calibration of the structural parameters of the model is taken from table 9.3 in that chapter. Under this more realistic stochastic structure, there exists no closed form solution to the Ramsey optimal capital control problem. Therefore, we resort to numerical methods to approximate the solution to the value-function problem of maximizing (10.24) subject to (10.14)-(10.18). We apply a value-function-iteration procedure over a discretized state space. The discretization is described in chapter 9, section 9.6.

We define a boom-bust episode as a situation in which tradable output, $y_t^T$, is at or below trend in period 0, at least one standard deviation above trend in period 10, and at least one standard deviation below trend in period 20. To characterize the typical boom-bust cycle, we simulate the model economy for 20 million periods and select all subperiods that satisfy the definition of a boom-bust episode. We then average across these episodes.

Figure 10.5 depicts the model’s predictions during a boom-bust cycle. Solid lines correspond to the economy with free capital mobility (i.e., no capital controls) and broken lines to the economy with optimal capital controls. The two top panels of the figure display the dynamics of the two exogenous driving forces, tradable output and the country interest rate. By construction, $y_t^T$ and $r_t$ are unaffected by capital controls.

The middle left panel of the figure shows that the Ramsey government uses capital controls in a prudential fashion during boom-bust episodes. It increases them from about half a percent at the beginning of the episode to almost 3 percent at the peak of the cycle and then relaxes them drastically during the contraction to such a degree that at the bottom of the crisis, it actually subsidizes capital inflows at a rate of about 2 percent.

The increase in capital controls during the expansionary phase of the cycle puts sand in the wheels of capital inflows, thereby curbing the boom in tradable consumption (see the middle right
Figure 10.5: Prudential Policy For Peggers: Boom-Bust Dynamics With and Without Capital Controls

- **Traded Output,** $y_T^*$
- **Annualized Interest Rate,** $r_t$
- **Capital Control Rate,** $\tau^d_t$
- **Traded Consumption,** $c_T^*$
- **Unemployment Rate,** $1 - h_t$
- **Consumption,** $c_t$

*No Capital Controls*  
*Optimal Capital Controls*
panel of figure 10.5). In the contractionary phase, the fall in capital controls fosters tradable absorption, thereby inducing a soft landing. Thus, the main effect of optimal capital controls is to produce a much smoother path of tradable consumption than in the economy with free capital mobility. Because unemployment depends directly upon variations in the level of tradable absorption through the latter’s role as a shifter of the demand schedule for nontradables, and because optimal capital controls stabilize the absorption of tradables, unemployment is also stable over the boom-bust cycle. Specifically, as can be seen from the bottom left panel of figure 10.5, in the absence of capital controls unemployment increases sharply by about 20 percentage points during the recession. By contrast, under optimal capital controls the rate of unemployment is virtually zero. It follows that the Ramsey planner’s tradeoff between distorting the intertemporal allocation of tradable consumption and reducing unemployment is overwhelmingly resolved in favor of the latter.

10.6 Level And Volatility Effects of Optimal Capital Controls Under A Currency Peg

Under fixed exchange rates, Ramsey optimal capital controls are prudential not only during boom-bust episodes but also unconditionally over the business cycle. Table 10.1 displays unconditional first and second moments of macroeconomic indicators of interest for the economies with free capital mobility (FCM) and optimal capital controls (OCC). The correlation between $\tau_i^d$ and $y_i^T$ is 0.7 and the correlation between $\tau_i^d$ and $r_i$ is -0.9. This means that the government tends to restrict (facilitate) capital inflows when the country is hit by positive (negative) shocks. This prudential implementation of capital controls is reflected in an implied correlation between the capital-control tax rate and output of 0.7.

The optimal capital control policy is highly effective in reducing involuntary unemployment.
Table 10.1: Optimal Capital Controls And Currency Pegs: Level and Volatility Effects

<table>
<thead>
<tr>
<th>Capital Control Tax Rate $100 \times \tau_t^d$</th>
<th>Country Interest Rate $400 \times r_t$</th>
<th>Effective Interest Rate $400 \times \left(\frac{1+r_t}{1-\tau_t^d} - 1\right)$</th>
<th>Unemployment Rate $100 \times \left(\frac{h-h_t}{h}\right)$</th>
<th>Growth in Traded Consumption $400 \times \ln(c_t^T/c_{t-1}^T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mean</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FCM</td>
<td>0</td>
<td>12.7</td>
<td>12.7</td>
<td>11.8</td>
</tr>
<tr>
<td>OCC</td>
<td>0.6</td>
<td>12.6</td>
<td>15.1</td>
<td>0.4</td>
</tr>
<tr>
<td><strong>Standard Deviation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FCM</td>
<td>0</td>
<td>7.1</td>
<td>7.1</td>
<td>10.4</td>
</tr>
<tr>
<td>OCC</td>
<td>2.4</td>
<td>7.1</td>
<td>5.0</td>
<td>1.5</td>
</tr>
<tr>
<td><strong>Correlation with $y_t^T$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FCM</td>
<td>–</td>
<td>-0.8</td>
<td>-0.8</td>
<td>-0.6</td>
</tr>
<tr>
<td>OCC</td>
<td>0.7</td>
<td>-0.8</td>
<td>0.3</td>
<td>-0.2</td>
</tr>
<tr>
<td><strong>Correlation with $r_t$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FCM</td>
<td>–</td>
<td>1.0</td>
<td>1.0</td>
<td>0.7</td>
</tr>
<tr>
<td>OCC</td>
<td>-0.9</td>
<td>1.0</td>
<td>-0.3</td>
<td>0.2</td>
</tr>
<tr>
<td><strong>Correlation with GDP</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FCM</td>
<td>–</td>
<td>-0.8</td>
<td>-0.8</td>
<td>-1.0</td>
</tr>
<tr>
<td>OCC</td>
<td>0.7</td>
<td>-0.8</td>
<td>0.2</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

Note. FCM stands for free capital mobility ($\tau_t^d = 0$ for all $t$), and OCC stands for optimal capital controls. GDP stands for gross domestic product and is expressed in terms of units of the composite good.
The average unemployment rate falls from 11.8 percent under free capital mobility to 0.4 percent under optimal capital controls. The virtually complete eradication of unemployment is achieved through a drastic reduction in the volatility of the shifter of the demand for nontradables, namely, the domestic absorption of tradables. The standard deviation of the growth rate of consumption of tradables falls from 23.2 percent under free capital mobility to only 4.9 percent under optimal capital controls. In turn, this smoothing of tradable consumption is engineered via a time varying wedge between the country interest rate, $r_t$ and the effective interest rate perceived by households, $(1 + r_t)/(1 - \tau^d_t) - 1$. The size of the wedge is governed by the size of the capital control tax rate, $\tau^d_t$. Indeed, capital controls are so strongly prudential that even though the country interest rate is markedly countercyclical (with a correlation of -0.8 with output), the effective interest rate under optimal capital controls is procyclical (with a correlation of 0.2 with output).

The strong interference of the Ramsey planner with the intertemporal allocation of tradable consumption represents a large deviation from the first-best allocation. Recall that given the assumption of equality of the inter- and intratemporal elasticities of substitution ($1/\sigma = \xi$), the path of traded consumption under a peg with free capital mobility coincides with the path associated with the first-best allocation. Thus, as in the simple analytical example of section 10.4, in the present economy the tradeoff between an inefficient intertemporal allocation of tradable consumption and unemployment is resolved in favor of eliminating unemployment.

### 10.7 Overborrowing Under Fixed Exchange Rates

The present model predicts that economies with free capital mobility and a fixed exchange rate overborrow in international financial markets. This prediction is evident from figure 10.6, which shows the unconditional distribution of external debt in the calibrated fixed-exchange-rate economy of the previous section under free capital mobility (solid line) and under optimal capital controls.
The average level of external debt is 22.4 percent of output under free capital mobility and -14.0 percent of output under optimal capital controls. The Ramsey planner induces a lower average level of external debt by taxing borrowing at a positive rate. Table 10.1 shows that the effective interest rate (i.e., the after-tax interest rate) is 2.5 percentage points higher than the pre-tax interest rate. It follows that pegging economies with free capital mobility accumulate inefficiently large amounts of external debt.

The reason why the average external debt is lower under optimal capital controls than under free capital mobility has to do with the fact that external debt has a higher variance under optimal capital controls ($1.6^2$ versus $0.65^2$). A more volatile process for external debt requires centering the debt distribution further away from the natural debt limit, for precautionary reasons (see
figure 10.6). But why does the Ramsey planner like wide swings in the external debt position? The answer is that such variations are necessary to insulate the domestic absorption of tradable goods from exogenous disturbances buffeting the economy (recall that the main role of capital controls in a peg economy is to smooth the consumption of tradables). If \( c^T_t \) does not move much over the business cycle, the resource constraint of the economy dictates that disturbances in \( y^T_t \) or \( r_t \) must be met by compensating movements in \( d_{t+1} \). In the peg economy with free capital mobility, \( c^T_t \) is more responsive to variations in \( y^T_t \) or \( r_t \), so \( d_{t+1} \) need not adjust so much. Put differently, in the peg economy with optimal capital controls, external debt plays the role of a shock absorber to a much larger extent that it does in the peg economy with free capital mobility.

We close this section by commenting on the commonly held view that imposing capital controls amounts to making the current account more closed. This need not be the case. Indeed, in the present model, optimal capital controls play the opposite role. Under optimal capital controls, the economy makes more heavy use of the current account to smooth consumption than it does under free capital mobility. The volatility of the current account, given by the standard deviation of \( d_t/(1 + r_{t-1}) - d_{t+1}/(1 + r_t) \), is 50 percent higher under optimal capital controls than under free capital mobility. It follows that, far from insulating the economy from world financial markets, optimal capital controls foster international asset transactions over the business cycle.

### 10.8 The Welfare Cost of Free Capital Mobility In Fixed-Exchange-Rate Economies

The model studied in this chapter predicts that the combination of a fixed exchange rate and free capital mobility entails excessive external debt and unemployment. Both of these factors tend to depress consumption and therefore reduce welfare. In this section, we put a number to these welfare losses.
The welfare cost of free capital mobility in a fixed-exchange-rate economy conditional on a particular state \(\{y_t^T, r_t, d_t, w_{t-1}\}\), denoted \(\Lambda^{FCM}(y_t^T, r_t, d_t, w_{t-1})\), is defined as the permanent percent increase in the lifetime consumption stream required by an individual living in a fixed-exchange-rate economy with free capital mobility to be as well off as an individual living in a fixed-exchange-rate economy with optimal capital controls. Formally, \(\Lambda^{FCM}(y_t^T, r_t, d_t, w_{t-1})\) is implicitly given by

\[
E_t \sum_{s=0}^{\infty} \beta^s \left[ \frac{c_{t+s}^{FCM} \left( 1 + \frac{\Lambda^{FCM}(y_t^T, r_t, d_t, w_{t-1})}{100} \right) }{1 - \sigma} \right]^{1-\sigma} - 1 = v^{OCC}(y_t^T, r_t, d_t, w_{t-1}),
\]

where \(c_t^{FCM}\) denotes the equilibrium process of consumption in the currency-peg economy with free capital mobility (and is identical to \(c_t^{PEG}\) in the notation of chapter 9.10) and \(v^{OCC}(y_t^T, r_t, d_t, w_{t-1})\) denotes the value function associated with the optimal capital-control policy in the fixed-exchange-rate economy and is given by

\[
v^{OCC}(y_t^T, r_t, d_t, w_{t-1}) \equiv E_t \sum_{s=0}^{\infty} \beta^s \left[ \frac{c_{t+s}^{OCC} \left( 1 - \sigma \right) + (1 - \beta)^{-1} \left[ v^{FCM}(y_t^T, r_t, d_t, w_{t-1}) \right]^{1/(1-\sigma)} }{1 - \sigma} \right]^{1-\sigma} - 1,
\]

where \(c_t^{OCC}\) denotes the equilibrium process of consumption in the fixed-exchange-rate economy with optimal capital controls. Solving for \(\Lambda^{FCM}(y_t^T, r_t, d_t, w_{t-1})\) yields

\[
\Lambda^{FCM}(y_t^T, r_t, d_t, w_{t-1}) = 100 \left\{ \frac{v^{OCC}(y_t^T, r_t, d_t, w_{t-1})(1 - \sigma) + (1 - \beta)^{-1}}{v^{FCM}(y_t^T, r_t, d_t, w_{t-1})(1 - \sigma) + (1 - \beta)^{-1}} \right\}^{1/(1-\sigma)} - 1
\]

where \(v^{FCM}(y_t^T, r_t, d_t, w_{t-1})\) denotes the value function associated with free capital mobility and a fixed exchange rate and is given by

\[
v^{FCM}(y_t^T, r_t, d_t, w_{t-1}) \equiv E_t \sum_{s=0}^{\infty} \beta^s \left[ \frac{c_{t+s}^{FCM} \left( 1 - \sigma \right) + (1 - \beta)^{-1} \left[ v^{POO}(y_t^T, r_t, d_t, w_{t-1}) \right]^{1/(1-\sigma)} }{1 - \sigma} \right]^{1-\sigma} - 1.
\]
This value function is identical to $v^{PEG}(y^T_t, r_t, d_t, w_{t-1})$ introduced in chapter 9.10. Because the state vector is stochastic, the conditional welfare cost measure, $\Lambda^{FCM}(y^T_t, r_t, d_t, w_{t-1})$, is itself stochastic. We wish to compute the unconditional mean of $\Lambda^{FCM}(y^T_t, r_t, d_t, w_{t-1})$. This requires knowledge of the unconditional probability distribution of the state vector $(y^T_t, r_t, d_t, w_{t-1})$. The distribution of the endogenous elements of the state vector, namely $d_t$ and $w_{t-1}$, depends on the exchange-rate policy (a currency peg) and on the capital control regime. Because we are analyzing the welfare gains of switching from a peg with free capital mobility to a peg with optimal capital controls, the relevant probability distribution is the one associated with the peg economy under free capital mobility. Let $\lambda^{FCM}$ denote the unconditional mean of $\Lambda^{FCM}(y^T_t, r_t, d_t, w_{t-1})$ and $\pi^{FCM}(y^T_t, r_t, d_t, w_{t-1})$ the unconditional probability of the state vector $(y^T_t, r_t, d_t, w_{t-1})$ under a peg with free capital mobility. Then $\lambda^{FCM}$ is given by

$$\lambda^{FCM} = \sum_{\{y^T_t, r_t, d_t, w_{t-1}\}} \pi^{FCM}(y^T_t, r_t, d_t, w_{t-1}) \Lambda^{FCM}(y^T_t, r_t, d_t, w_{t-1}),$$

(10.30)

where the sum is over all points in the discretized four dimensional state space. The calibrated economy yields

$$\lambda^{FCM} = 3.65.$$ 

This figure means that for a an economy with a fixed exchange rate the average welfare gains of switching from free capital mobility to optimal capital controls are large. The representative household living in a peg economy with free capital mobility requires an increase of 3.65 percent in its entire consumption stream to be indifferent between continuing to live in that economy and switching to a peg economy with optimal capital controls.

An important fraction of the welfare cost of free capital mobility is accounted for by the transitional dynamics put in motion as policy switches from free capital mobility to Ramsey optimal
capital controls. Recall that the peg economy with free capital mobility is on average substantially more indebted than the peg economy with optimal capital controls. The transition from a free capital mobility regime to the optimal capital control regime, therefore, requires a significant amount of deleveraging. In turn, deleveraging requires households to temporarily cut consumption of traded goods making it less enticing to switch from free capital mobility to optimal capital controls. To quantify the welfare impact of these transitional dynamics, one can compute the unconditional welfare cost of free capital mobility in a peg economy. Formally, this welfare measure, which we denote $\lambda^{FCMU}$, is given by

$$E \sum_{t=0}^{\infty} \beta^t U \left( c_{t}^{FCM} \left( 1 + \frac{\lambda^{FCMU}}{100} \right) \right) = E \sum_{t=0}^{\infty} \beta^t U (c_{t}^{OCC})$$

where the unconditional expectation on the left side is computed using the distribution of the state vector $\{y_t^T, r_t, d_t, w_{t-1}\}$ under free capital mobility and the expectation on the right side using the distribution under optimal capital controls. Solving for $\lambda^{FCMU}$ gives

$$\lambda^{FCMU} = 100 \left\{ \left[ \frac{E c^{OCC}}{E c^{FCM}} \right]^{\frac{1}{1-\sigma}} - 1 \right\}.$$ 

This welfare measure is useful for answering the following question: Suppose there are two countries, A and B. Both countries have a fixed-exchange-rate regime in place. Country A operates under free capital mobility, whereas country B applies optimal capital controls. Not knowing the state of either economy, by how much would the consumption stream in country A have to be increased for individuals to be indifferent between being born in countries A or B? The calibrated economy implies that

$$\lambda^{FCMU} = 13.0.$$ 

This figure suggests that unconditionally the welfare gains of optimal capital controls are enormous.
Much of these gains are explained by the fact that unemployment is much higher in the peg economy with free capital mobility than in the peg economy with optimal capital controls (see table 10.1).

10.9 Are Observed Capital Controls Prudential?

Until recently, capital controls were generally considered a bad idea. For, it was argued, they hinder the efficient allocation of capital across countries, thereby reducing growth potential. However, the fact that many of the external crises that occurred in the past two decades began with large waves of capital inflows (e.g., the crises in Southeast Asia, Russia, and Latin America in the late 1990s and early 2000s, and more recently in the periphery of Europe), has made many in academia and policy circles look at capital controls with more benign eyes. The most salient indication of this change of sentiment is given by the IMF, which, after a long-standing negative view on capital controls, now accepts them as one more tool for macroeconomic stabilization (see International Monetary Fund, 2011).

This change of mind has spurred theoretical work on the cyclical properties of optimal capital controls. The present chapter, for example, has been devoted to showing that the combination of a nominal rigidity and suboptimal monetary policy (such as a currency peg), gives rise to an externality that can be dealt with through optimal capital control policy. Another literature, studied in more detail in chapter 11, motivates the use of capital controls in the context of models with financial frictions. In this class of model, international borrowing is limited by collateral constraints. In turn, the value of collateral is assumed to depend on some price (e.g., the relative price of real estate) that individual households take as given. During booms, this price goes up, expanding the value of collateral and inducing households to borrow and spend excessively. Similarly, during recessions the price that determines the value of collateral falls, causing deleveraging and an excessive contraction in aggregate spending. Thus, the model features a pecuniary externality that
exacerbates booms and busts. In this context, capital controls can be useful as a means to induce households to internalize the pecuniary externality. In both theories summarized above, optimal capital controls are prudential in nature. The policy authority should impose restrictions on international capital flows during booms and relax them during contractions, rather than passively waiting until the crisis occurs to pick up the broken pieces.

A natural question is whether in reality capital controls are procyclical, or prudential, as suggested by theory. Fernández, Rebucci, and Uribe (2013) address this empirical question. Their starting point is a data set of capital control indices constructed by Schindler (2009) covering 91 countries during the period 1995 to 2005. The Schindler index is based on information on capital controls provided by the IMF’s Annual Report on Exchange Arrangements and Exchange Restrictions (AREAER). The index covers six asset categories (equity, bonds, money markets, mutual funds, financial credit, and foreign direct investment) and distinguishes between controls on capital inflows and controls on capital outflows. The index takes on 13 equally spaced values between 0 (no restrictions) and 1 (restrictions on all types of transactions).

Fernández, Rebucci, and Uribe (2013) extend the Schindler data set to cover the period 1995 to 2011. They find that capital controls are remarkably stable. The average standard deviation of the capital control index across all countries is 0.07 for inflows and 0.06 for outflows. These standard deviations are tiny. Recall that the index ranges from 0 to 1 in steps of 0.083. The second main finding reported in Fernández, Rebucci, and Uribe is that, contrary to what the recent theories of macro prudential policy suggest, capital controls are virtually acyclical. Specifically, the average correlation between the capital control index and output is -0.01 for inflows and -0.03 for outflows.

Because the second moments reported above are unconditional, they may not fully represent the prudential content of observed capital control policy. The reason is that policymakers may be willing to put the capital-control machinery to work only in response to large deviations of aggregate activity from trend. Under this conjecture, unconditional moments may be dominated
by normal, relatively small, fluctuations that do not trigger movements in international capital restrictions. Motivated by this possibility, Fernández, Rebucci, and Uribe examine the behavior of capital controls conditional on the economy experiencing a boom or a bust in aggregate activity. They define a boom (bust) as a situation in which output is above (below) trend for at least three consecutive years.

Figure 10.7 displays the average deviation of output (starred lines) and capital controls (solid lines) from their respective trends across all booms (left column) and busts (right column) in the sample. It also shows (broken lines) a two-standard-deviation band around the average behavior of capital controls. The main message of the figure is that capital controls are virtually flat during booms or busts in aggregate activity. On average, policymakers do not seem to use capital controls in a consistent fashion (either prudentially or nonprudentially). Fernández, Rebucci, and Uribe show that this result is robust to disaggregating the data by level of economic development, exchange-rate regime, level of external indebtedness, or asset category. They also find that the result holds if one defines booms and busts not in terms of output, but in terms of the current account or the real exchange rate.

Fernández, Rebucci, and Uribe offer two explanations for the lack of cyclicality observed in actual capital-control policy. One possible explanation is that theory is ahead of actual policy making. Capital controls should be procyclical, but policymakers are not yet aware or fully convinced of the benefits of applying capital controls in a prudential fashion. Under this interpretation, one should expect that as time goes by and the new theories penetrate policy-making spheres, observed capital controls will become more procyclical. The second explanation offered by Fernández, Rebucci, and Uribe is that policy making may be running ahead of theory. Policy authorities may have information about effects of capital controls that are not incorporated in existing theories. These theoretically unaccounted effects may call for the optimal capital control policy to be unresponsive over the business cycle, in line with the empirical evidence. Under this interpretation, one should
Figure 10.7: Boom-Bust Episodes and Capital Controls

Note. Booms (busts) are defined as periods longer than or equal to three years in which the output gap is always positive (negative). Capital controls and the output gap are expressed in deviations from trend and averaged across episodes. Output gaps are in percent. Source: Fernández, Rebucci, and Uribe (2013).
expect that over time, as academic research assimilates the feedback received from economic practitioners, new models will begin to play down the convenience of applying capital controls in a prudential fashion. Time will tell which, if any, of these two explanations ends up being vindicated.
Appendix: Equilibrium for $t \geq 1$ in Section 10.4

We wish to show that in the economy analyzed in section 10.4, in which the nominal exchange rate is constant and capital controls are set in a Ramsey optimal fashion, the equilibrium allocation features constant values for consumption, debt, hours, and wages for $t \geq 1$.

The optimal capital control problem for $t \geq 1$ can be written as follows

$$\max_{\{c_t^T, h_t, w_t, d_{t+1}\}} \sum_{t=1}^{\infty} \beta^t [\ln c_t^T + \alpha \ln h_t]$$

subject to

$$c_t^T + d_t = y^T + \frac{d_{t+1}}{1 + r},$$

$$\frac{\alpha c_t^T}{h_t} = w_t,$$

$$h_t \leq 1,$$

$$w_t \geq w_{t-1},$$

and

$$d_{t+1} \leq \bar{d},$$

given $d_1$ and $w_0$.

Consider next the less restrictive problem of maximizing (10.31) subject to (10.32), (10.34), and (10.36), given $d_1$. It is straightforward to see that the solution of this problem is $c_t^T = c^{T*} = y^T - \frac{r}{1+r} d_1$, $d_{t+1} = d_1$, and $h_t = 1$ for all $t \geq 1$. For this solution to comply with equilibrium condition (10.33), the wage rate must satisfy $w_t = \alpha c^{T*}$ for all $t \geq 1$. In turn, for equilibrium
condition (10.35) to hold, we need that

\[ \alpha c^*_T \geq w_0. \]  

(10.37)

Therefore, if this condition holds, the solution to the less constrained problem is also the solution to the original Ramsey problem, and the Ramsey optimal allocation implies constant paths for consumption, debt, hours, and wages, which is what we set out to show.

Now assume that condition (10.37) is not satisfied, that is, assume that

\[ \alpha c^*_T < w_0. \]  

(10.38)

Use equation (10.33) to eliminate \( h_t \) from the utility function and from (10.34). Then, we can rewrite the Ramsey problem as

\[ \max \{ c^*_T, w_t, d_t + 1 \} \]

\[ \sum_{t=1}^{\infty} \beta^t [(1 + \alpha) \ln c^*_T + \alpha \ln \alpha - \alpha \ln w_t] \]  

(10.39)

subject to (10.32), (10.35), (10.36), and

\[ w_t \geq \alpha c^*_T, \]  

(10.40)

given \( d_1 \) and \( w_0 \). Constraint (10.40) guarantees that \( h_t \) is always less than or equal to 1.

Consider the less restrictive problem consisting in dropping (10.40) from the above maximization problem. Since the indirect utility function (10.39) is separable in consumption of tradables and wages and since the only constraint in the less restrictive problem that features wages, namely, equation (10.35), contains neither consumption of tradables nor debt, we can separate the less
restricted problem into two independent problems. One is

$$\max_{\{c_t, d_{t+1}\}_{t=1}^\infty} \sum_{t=1}^\infty \beta^t [(1 + \alpha) \ln c_t^T + \alpha \ln \alpha]$$

subject to (10.32) and (10.36). The solution of this problem is $c_t^T = c^T*$ and $d_{t+1} = d_1$ for all $t \geq 1$. The second problem is

$$\max_{\{w_t\}_{t=1}^\infty} \sum_{t=1}^\infty \beta^t [-\alpha \ln w_t]$$

subject to (10.35), given $w_0$. The solution to this problem is $w_t = w_0$ for all $t \geq 1$.

It remains to show that the solution to these two problems satisfy the omitted constraint (10.40). To see that this is indeed the case, note that $w_t = w_0 > \alpha c_t^T = \alpha c_t^T$, where the inequality follows from (10.38).

We have therefore shown that in the Ramsey equilibrium all variables are constant for $t \geq 1$. 
10.10 Exercises

Exercise 10.1 [Labor Subsidies] Modify the model of section 10.1.1 by assuming that households have preferences of the type given in (9.36). Show that a labor subsidy at the firm level financed by a proportional income tax at the household level (i.e., the fiscal scheme studied in section 10.1.1) can support the Pareto optimal allocation.

Exercise 10.2 Optimal Lump-Sum Transfers and Hand-to-Mouth Consumers Consider an economy in which the government can participate in the international financial market but households cannot. Assume further that the only fiscal policy instrument available to the government are lump-sum taxes or transfers. The exchange-rate regime is a currency peg. The government sets the level of external debt and lump-sum taxes or transfers in a Ramsey optimal fashion. All other aspects of the model are as in section 9.1. Show that the equilibrium real allocation is identical to the one obtained in section 10.3 under Ramsey optimal capital control policy.

Exercise 10.3 Equivalence Between Capital Controls and Consumption Taxes Show that the allocation under Ramsey optimal capital controls characterized in section 10.3 can be replicated with a Ramsey optimal consumption tax scheme in which the tax rate on next period’s consumption is determined in the current period.

Exercise 10.4 (Interest-Rate Shocks, Capital Controls, And Unemployment) Consider the analytical example of section 10.4. In that example, in response to a temporary decline in the interest rate from $r$ to $\bar{r}$, the Ramsey-optimal capital-control policy induces a constant path for tradable consumption and ensures full employment at all times. The Ramsey planners achieves this allocation by imposing capital controls in period 0 to make the effective interest rate perceived by domestic households, $(1 + \bar{r})/(1 - \tau^d_0)$, insensitive to the change in the world interest rate. These results hinge on the assumption $\alpha > r$. Redo the analysis of section 10.4 under the assumption that
$\alpha < r$. Provide intuition.
Chapter 11

Overborrowing

Business cycles in emerging countries are characterized by booms and contractions that are larger in amplitude than those observed in developed countries. One possible explanation for this phenomenon is simply that emerging countries are subject to larger shocks. Chapters 4, 6, and ?? are devoted to evaluating this hypothesis. A second explanation holds emerging countries suffer from more severe economic and political distortions than do developed countries, which amplify the cycles caused by aggregate shocks. The overborrowing hypothesis belongs to this line of thought. It argues that during booms emerging economies borrow and spend excessively (i.e., inefficiently), which exacerbates the expansionary phase of the cycle. During downturns, the argument continues, countries find themselves with too much debt and are forced to engage in drastic cuts in spending, which aggravates the contraction.

We will analyze in some detail two very different theories of overborrowing. One stresses the role of policy credibility. It shows that policies that in principle are beneficial to the economy, can have deleterious effects if the policymaker who is in charge of implementing it lacks credibility. The second theory we will discuss argues that there exist pecuniary externalities in the market for external funds. In this branch of the overborrowing literature, debt limits faced by individual agents
depend upon variables that are exogenous to them but endogenous to the economy. For instance, the willingness of foreign lenders to provide funds to a given emerging economy might depend upon the country’s aggregate level of external debt, or on the value of nontradable output. These variables (and therefore the cost of external borrowing) are not controlled by individual agents, but do depend on their collective behavior. This externality can give rise to inefficient borrowing. A common theme in much of the overborrowing literature is the need for regulatory government intervention. We will keep this issue in mind as we present the different theories.\footnote{We note, however, that some theories argue that regulatory policy may be the cause rather than the remedy to overborrowing. McKinnon’s (1973) model of deposit guarantees, for example, has been intensively used to understand overborrowing in the aftermath of financial liberalization in the Southern Cone of Latin America in the 1970s. In McKinnon’s model, deposit guarantees induce moral hazard, as banks tend to undertake immoderately risky projects and depositors have less incentives to monitor the quality of banks’ loan portfolios. As a result, deposit guarantees open the door to excessive lending and increase the likelihood of generalized bank failures.}

### 11.1 Imperfect Policy Credibility

The imperfect-credibility hypothesis is due to Calvo (1986, 1987, 1988). Its basic premise is quite simple. Suppose that the government, possibly with good intentions, announces the removal of a consumption tax. The public, however, believes that the policy will be abandoned after a certain period. That is, they interpret the policy reform as being temporary. As a result, they take advantage of what they perceive to be a temporarily lower consumption tax and increase spending while the policy lasts. In the aggregate, the spending boom is financed by current account deficits. At some point, the euphoria ends, either because the tax cut is abandoned or because agents convince themselves that it is indeed permanent, spending collapses, and the current account experiences a (possibly sharp) reversal.

The imperfect-credibility theory of overborrowing can be applied to a variety of policy environments. Calvo (1986), for instance, studies the consequences of a temporary inflation stabilization program. In this case, the consumption tax takes the form of inflation. To visualize inflation as
a consumption tax, imagine an economy in which households face a cash-in-advance constraint, whereby purchases of consumption goods require holding money. Because inflation erodes the real value of money, it acts as an indirect tax on consumption. Calvo (1987) applies the imperfect-credibility hypothesis to shed light on the dynamics of balance-of-payment crises. Here, the consumption tax also takes the form of inflation, and lack of credibility stems from the fact that agents disbelieve that the government will be able to cut deficits to a level consistent with low inflation in the long run. The particular application we analyze in this chapter is a trade reform and is due to Calvo (1988).

Consider a perfect-foresight economy populated by a large number of infinitely lived households with preferences described by the utility function

\[ \sum_{t=0}^{\infty} \beta^t U(c_t), \]

where \( c_t \) denotes consumption in period \( t \), \( \beta \in (0,1) \) denotes a subjective discount factor, and \( U \) denotes a period utility function assumed to be increasing and strictly concave. Consumption goods are not produced domestically, and must be imported from abroad. Each period, households receive a constant endowment of goods, \( y > 0 \). This endowment is not consumed domestically, but can be exported. For simplicity, we assume that the relative price of exportables in terms of importables, the terms of trade, is constant and normalized to unity. Households start each period with a stock of debt, \( d_t^h \), carried over from the previous period. Debt is denominated in units of importable goods and carries a constant interest rate \( r > 0 \). In addition, households receive a lump-sum transfer \( x_t \) from the government each period. Households use their financial and nonfinancial and income to purchase consumption goods, \( c_t \), and to increase their asset position. Imports are
subject to a proportional tariff $\tau_t$. The household’s sequential budget constraint is then given by

$$d^h_t = (1 + r)d^h_{t-1} - y - x_t + c_t(1 + \tau_t).$$

To prevent Ponzi games, households are subject to the following borrowing constraint:

$$\lim_{t \to \infty} \frac{d^h_{t+j}}{(1 + r)^j} \leq 0.$$ 

The fact that the period utility function is increasing implies that in the optimal plan the no-Ponzi-game constraint must hold with equality. Combining the sequential budget constraint and the no-Ponzi-game constraint holding with equality yields the following intertemporal budget constraint:

$$(1 + r)d^h_{t-1} = \sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t [y + x_t - c_t(1 + \tau_t)].$$

To avoid inessential long-run dynamics, we assume that the subjective and pecuniary discount rates are identical, that is

$$\beta(1 + r) = 1.$$ 

The consumer’s problem consists in choosing a sequence $\{c_t\}_{t=0}^{\infty}$ to maximize his lifetime utility function subject to this intertemporal budget constraint. Letting $\lambda_0$ denote the Lagrange multiplier associated with the intertemporal budget constraint, the optimality conditions associated with this problem are the intertemporal budget constraint and

$$U'(c_t) = \lambda_0(1 + \tau_t).$$ (11.1)

Note that $\lambda_0$ is determined in period zero but is constant over time.
11.1.1 The Government

Like households, the government has access to the international financial market. The government’s sources of income are import tariffs, \( \tau_t c_t \), and the issuance of new public debt, \( d_t^g - d_{t-1}^g \). Government spending stem from interest payments, \( rd_{t-1}^g \), and lump-sum transfers to households, \( x_t \). The resulting sequential budget constraint of the government is given by

\[
d_t^g = (1 + r)d_{t-1}^g - \tau_t c_t + x_t.
\]

The variable \( \tau_t c_t - x_t \) represents the primary fiscal surplus in period \( t \), and the variable \( -rd_{t-1}^g + \tau_t c_t - x_t \) represents the secondary fiscal surplus. We assume that fiscal policy is such that in the long run the government’s debt position does not grow in absolute value at a rate larger than the interest rate. That is, we assume that the fiscal policy ensures that

\[
\lim_{t \to \infty} \frac{d_t^g}{(1 + r)^t} = 0.
\]

This condition together with the government’s sequential budget constraint implies the following intertemporal government budget constraint:

\[
(1 + r)d_{-1}^g = \sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t (\tau_t c_t - x_t).
\]

This constraint states that the present discounted value of current and future primary fiscal surpluses must equal the government’s initial debt position.

Let \( d_t \equiv d_t^h + d_t^g \) denote the country’s net foreign debt position. Then, combining the intertemporal budget constraints of the household and the government yields the following intertemporal
economy-wide resource constraint:

\[(1 + r)d_{-1} = \sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t (y - c_t). \tag{11.2} \]

According to this expression, the present discounted value of the stream of current and future trade surpluses must equal the country’s initial net debt position.

A competitive equilibrium is a scalar \(\lambda_0\) and a sequence \(\{c_t\}_{t=0}^{\infty}\) satisfying (11.1) and (11.2), given the initial debt position \(d_{-1}\) and a sequence of import tariffs \(\{\tau_t\}_{t=0}^{\infty}\) specified by the government.

The focal interest of our analysis is to compare the economic effects of two polar tariff regimes. In one, the government implements a credible permanent tariff reform. In the second, the government implements a temporary tariff reform. We begin with the analysis of a permanent tariff reform.

### 11.1.2 A Credible Permanent Trade Reform

Suppose that in period 0 the government unexpectedly implements a permanent tariff reform consisting in lowering \(\tau_t\) from its initial level, which we denote by \(\tau^H\), to a new level \(\tau^L < \tau^H\). Formally, the time path of \(\tau_t\) is given by

\[\tau_t = \tau^L; \quad t \geq 0.\]

It follows directly from the efficiency condition (11.1) that in this case consumption is constant over time. Let the equilibrium level of consumption be denoted by \(\overline{c}\). The intertemporal resource constraint (11.2) then implies that \(\overline{c}\) is given by

\[\overline{c} = y - rd_{-1}.\]

According to this expression, each period, households consume their endowment net of interest payments on the country’s net external debt position per capita. In turn, the level of external debt
per person, \( d_t \), is constant over time and equal to \( d_{-1} \). Every period the country generates a trade surplus, \( y - \bar{c} \), which is just large enough to pay the interest accrued on the external debt.

Importantly, the equilibrium level of consumption is independent of the level at which the government sets the import tariff. Any constant tariff path, implemented unexpectedly at time 0, gives rise to the same level of consumption. Similarly, the trade balance, \( y - \bar{c} \), the net foreign asset position, \( d_{-1} \), and the current account, \( y - \bar{c} - rd_{-1} \), are all unaffected by the permanent trade liberalization.

The intuition behind this result is as follows. A constant tariff is in effect a constant tax on consumption. Since consumption is taxed at the same rate in all periods, households have no incentive to substitute expenditure intertemporally.

### 11.1.3 A Temporary Tariff Reform

Suppose now that in period 0 the government unexpectedly announces and implements a temporary trade liberalization. The announcement specifies that the tariff is reduced from \( \tau^H \) to \( \tau^L \) between periods 0 and \( T - 1 \), and permanently increased back to \( \tau^H \) in period \( T \). Formally, the announced path of \( \tau_t \) is given by

\[
\tau_t = \begin{cases} 
\tau^L & \text{for } 0 \leq t \leq T - 1 \\
\tau^H & \text{for } t \geq T 
\end{cases}, \tag{11.3}
\]

with \( \tau^L < \tau^H \). Here, \( T > 0 \) denotes the length of the commercial liberalization policy.

It is clear from equation (11.1) that consumption is constant over the periods \( 0 \leq t \leq T - 1 \) and \( t \geq T \). We can therefore write

\[
c_t = \begin{cases} 
c^1 & \text{for } 0 \leq t \leq T - 1 \\
c^2 & \text{for } t \geq T 
\end{cases}, \tag{11.4}
\]
where \( c^1 \) and \( c^2 \) are two scalars determined endogenously. Because the period utility function is concave, it follows from the efficiency condition (11.1) that \( c^1 \) is greater than \( c^2 \). Intuitively, households substitute consumption in the low-tariff period for consumption in the high-tariff period. We can then write

\[
c^1 = (1 + \kappa)c^2; \quad \text{for some } \kappa > 0. \tag{11.5}
\]

The parameter \( \kappa \) is an increasing function of the intertemporal tariff distortion \((1 + \tau^H)/(1 + \tau^L)\). For example, if the period utility function is of the CRRA form, \( U(c) = c^{1-\sigma}/(1 - \sigma) \), then we have that \( 1 + \kappa = \left(\frac{1 + \tau^H}{1 + \tau^L}\right)^{1/\sigma} \). Of course, the consumption stream must respect the intertemporal resource constraint (11.2). This restriction implies the following relationship between \( c^1 \) and \( c^2 \):

\[
y - rd_{-1} = (1 - \beta^T)c^1 + \beta^Tc^2. \tag{11.6}
\]

The left-hand side of equation (11.6) is \( \bar{c} \), the level of consumption that results under a credible permanent trade reform. We can therefore write the (11.6) as

\[
\bar{c} = (1 - \beta^T)c^1 + \beta^Tc^2.
\]

Because \( \beta^T \in (0, 1) \), it follows from this expression that \( \bar{c} \) is a weighted average of \( c^1 \) and \( c^2 \). We therefore have that

\[
c^1 > \bar{c} > c^2.
\]

This means that if the economy was at a steady state \( \bar{c} \) before period \( t \), then the announcement of the temporary tariff reform causes consumption to rise to \( c^1 \) at time 0, stay at that elevated level until period \( T - 1 \), and then fall in period \( T \) to a new long-run level \( c^2 \), which is lower than the pre-reform level of consumption.
The temporary tariff reform induces households to engage in a consumption path that has the same present discounted value as the one associated with a permanent tariff, but is less smooth. Therefore, because households have concave preferences, it must be the case that the time-varying tariff regime must result in lower welfare than the constant-tariff regime. To state this more formally, consider the problem of a benevolent social planner trying to design a tariff policy that maximizes welfare subject to the constraint that the policy must belong to the family defined in (11.3). The planner’s objective function is to choose scalars $c^1$ and $c^2$ to maximize the household’s lifetime utility function evaluated at the consumption path given in (11.4), subject to the lifetime resource constraint (11.6). That is, the planner’s problem is:

$$\max_{c^1, c^2} \left[(1 - \beta^T)U(c^1) + \beta^T U(c^2)\right]$$

subject to

$$-rd_{-1} + y = (1 - \beta^T)c^1 + \beta^T c^2.$$ 

Because the period utility function $U(\cdot)$ is assumed to be strictly concave, the objective function is strictly concave in $c^1$ and $c^2$. Also, because the constraint is linear in $c^1$ and $c^2$, it describes a convex set of feasible pairs $(c^1, c^2)$. It follows that the first-order conditions of this problem are necessary and sufficient for a maximum. These conditions are:

$$U'(c^1) = \lambda,$$

and

$$U'(c^2) = \lambda,$$

---

2 We omit the multiplicative factor $(1 - \beta)^{-1}$ from the objective function because it is a constant and therefore does not affect the solution.
where \( \lambda \) denotes the Lagrange multiplier on the constraint. Clearly, the solution to this problem is \( c_1 = c_2 \). It then follows immediately from (11.1) that the set of tariff schemes that implements the planner’s problem solution satisfies \( \tau^L = \tau^H \). Consequently, a temporary trade liberalization policy of the type studied here with \( \tau^L < \tau^H \) is welfare dominated by a constant-tariff regime.

Consider the behavior of the trade balance, the current account, and the net foreign asset position induced by the temporary commercial liberalization policy. Assume for simplicity that the initial net debt position is nil \( (d_{-1} = 0) \). In the pre-reform equilibrium, \( t < 0 \), agents expect tariffs to be constant forever. Thus, we have that before the trade reform consumption equals output, \( \overline{\tau} = y \), the trade balance is zero, \( y - \overline{\tau} = 0 \), and so is the current account, \( y - \overline{\tau} - rd_{-1} = 0 \). Recalling that \( c^1 > \overline{\tau} = y > c^2 \), we have that for \( t \geq 0 \), the path of the trade balance, which we denote by \( tb_t \), is given by

\[
\begin{align*}
    tb_t &= \begin{cases} 
    y - c^1 < 0 & \text{for } 0 \leq t < T, \\
    y - c^2 > 0 & \text{for } t \geq T.
    \end{cases}
\end{align*}
\]

Thus, the initial consumption boom causes a trade balance deficit that lasts for the duration of the tariff cut. When the policy is abandoned, consumption falls and the trade balance displays a sharp reversal from deficit to surplus. The evolution of the net foreign debt position is given by

\[
d_t = (c^1 - y) \sum_{j=0}^{t} (1 + r)^j > 0; \quad 0 \leq t < T.
\]

This expression states that the foreign asset position is negative and deteriorates over time at an increasing rate until period \( T \). The current account, denoted by \( ca_t \) and given by the change in the asset position, \(-(d_t - d_{t-1})\), evolves according to the following expression:

\[
ca_t = (y - c^1)(1 + r)^t; \quad 0 \leq t < T.
\]
According to this expression, the current account is negative and deteriorates exponentially. The paths of the current account and the asset position are unsustainable in the long run. Therefore, in period $T$, the household cuts consumption to a point at which external debt stops growing and the current account experiences a sudden improvement to a balanced position. Formally, we have

$$d_t = d_{T-1}; \quad t \geq T,$$

and

$$ca_t = 0; \quad t \geq T.$$

The level of consumption that results after the demise of the trade reform, $c^2$, therefore, satisfies

$$d_{T-1} = (1 + r)d_{T-1} + c^2 - y. \quad (11.7)$$

Solving for $c^2$, we obtain

$$c^2 = y - rd_{T-1} < y - rd_{-1} = y = \bar{c}.$$

The inequality follows from the fact that $d_{T-1}$ is negative while $d_{-1}$ was assumed to be zero. The above expression shows, again, that after the collapse of the trade reform consumption falls below its pre-reform level. The reason why consumption must fall so sharply is that the initial boom in private consumption is entirely financed by external borrowing. As a result, in the new steady state the country must redirect resources away from consumption and toward servicing the increased external debt.

The model described in this section is one of overborrowing in the sense that it delivers an initial phase ($0 \leq t < T$) in which households embark in a socially inefficient spending spree with external debt growing at an increasing rate and the external accounts displaying widening imbalances. Note
that in this model overborrowing ends in a sudden stop in period $T$. In this period, the country experiences a reversal of the current account and a sharp contraction in domestic absorption.

It is worth noting that in this particular model the terms ‘temporariness’ and ‘imperfect credibility’ are equivalent in a specific sense. To illustrate this idea, suppose that instead of announcing a temporary tariff cut, in period 0 the government announces a permanent tariff cut, but that the public disbelieves the announcement. Specifically, the government announces a permanent tariff reduction from $\tau^H$ to $\tau^L$ but the public believes that the tariff will return to $\tau^H$ in period $T$. Clearly, the dynamics between periods 0 and $T-1$ are identical as the ones associated with the temporary trade liberalization. This is because consumption decisions during the period $0 \leq t < T$ are based on expectations of a future tariff increase regardless of whether the government explicitly announces it or simply the public believes it will take place. Suppose now that in period $T$, contrary to the public’s expectations, the government maintains the trade reform ($\tau_t = \tau^L$ for $t \geq T$), and, moreover, that the policy becomes credible to the public. From the point of view of period $T$, the situation is one in which the tariff is constant forever. It follows from our previous analysis that consumption must be constant from $t = T$ on. A constant level of consumption can only be sustained by a constant level of assets.\footnote{To see this, note that the difference equation $d_t = (1+r)d_{t-1} - y + c$, for $t \geq T$, implies a path of $d_t$ that converges to plus or minus infinity at a rate larger than $r$ (implying a violation of the no-Ponzi-game constraint in the first case and a suboptimal accumulation of wealth in the second) unless $c$ is such that $(1+r)d_{T-1} - y + c$ equals $d_{-T}$. That is, a constant path of consumption is sustainable over time only if the asset position is also constant over time.} Thus, consumption must satisfy $d_{T-1} = (1+r)d_{T-1} - y + c$, where $c$ is the level of consumption prevailing for $t \geq T$. Comparing this expression with (11.7), it follows that $c$ must equal $c^2$. This result establishes that the equilibrium path of consumption is identical whether the policy is temporary or permanent but imperfectly credible.

The equivalence between temporariness and lack of credibility is not always valid. Exercise 4 illustrates a case in which lack of credibility can be much more costly that temporariness. The exercises augments the model discussed in this section to allow for nontradable goods and down-
ward nominal wage rigidity. In this setting, changes in the import tariff distort the relative price of nontradables relative to tradables. The difference between temporariness and lack of credibility arises in period $T$. In this period, given the relative price of nontradables, the demand for nontradables collapses under both temporariness and lack of credibility. This contraction in the desired demand for nontradables is driven by a negative wealth effect caused by the overborrowing that takes place between periods 0 and $T$. Under temporariness, the tariff increase that takes place in period $T$ makes nontradables relatively cheaper, generating a positive substitution effect which tends to offset the negative aforementione wealth effect. By contrast, under lack of credibility, the tariff never increases, since the trade liberalization policy was indeed permanent. As a result, the offsetting substitution effect does not take place, and the demand for nontradables falls more sharply than under policy temporariness. Because wages are downwardly rigid, the contraction in the demand for nontradables causes unemployment. Since the demand shift is larger under lack of credibility, we have that the increase in unemployment is more pronounced under lack of credibility than under temporariness.

11.2 Financial Externalities

We now study a class of models in which individual agents face a financial friction that depends upon an endogenous variable that individuals take as exogenous. In this class of models there is an externality because individual agents do not internalize the fact that their behavior collectively determines the strength of the financial friction.

The financial friction typically takes the form of a collateral constraint imposed by foreign lenders. And the variable in question takes various forms. It could be a stock, such as external debt per capita, or a flow, such as output, or a relative price, such as the price of real estate in terms of consumption goods. A theme of this section is that the nature of this variable is a key
determinant of whether the model will or will not generate overborrowing.

11.2.1 The No Overborrowing Result

Imagine a theoretical environment in which foreign lenders impose a borrowing constraint on emerging countries. Consider the following two alternative specifications of such constraint: \( d_t \leq \bar{d} \) and \( D_t \leq \bar{d} \), where \( d_t \) denotes the individual household’s net external debt in period \( t \), \( D_t \) denotes the average net external debt across all households in the economy in period \( t \), and \( \bar{d} \) is a constant. The key difference between these two formulations is that the consumer internalizes the first borrowing constraint, because it involves its own level of debt, but does not internalize the second one, because it features the cross-sectional average of debt, which is out of his control. These two specifications are meant to capture two distinct lending environments: One in which foreign lenders base their loan decisions on each borrower’s capacity to repay, and one in which foreign lenders regard the emerging country as a single investment opportunity and look only at aggregate variables in deciding whether to lend or not. The question is whether the two formulations give rise to different equilibrium distributions of debt, and in particular whether the one based on aggregate variables delivers more frequent crises (i.e., episodes in which the debt limit binds). The present analysis is based on Uribe (2006).

Consider first the case of an aggregate borrowing constraint. The economy is assumed to be populated by a large number of households with preferences described by the utility function

\[ E_0 \sum_{t=0}^{\infty} \theta_t U(c_t, h_t), \]  

(11.8)

where \( c_t \) denotes consumption, \( h_t \) denotes hours worked, and \( \theta \) denotes a subjective discount factor. All of the results in this chapter go through for any specification of \( \theta \) that is exogenous to the
household. For concreteness, we adopt the specification studied in section (4.10.3) and assume that

$$\theta_{t+1} = \beta(C_t, H_t)\theta_t$$

for $t \geq 0$ and $\theta_0 = 1$. The variables $C_t$ and $H_t$ denote, respectively, the cross sectional averages of consumption and hours, and the function $\beta$ is assumed to be decreasing in its first argument and increasing in its second argument. Households take the evolution of $C_t$ and $H_t$ as exogenous. In equilibrium, we have that $C_t = c_t$ and $H_t = h_t$, because all households are assumed to be identical. Output is produced using hours as the sole input with the technology $e^{z_t}F(k^*, h_t)$, where $z_t$ is a productivity shock assumed to be exogenous and stochastic, and $k^*$ is a fixed factor of production, such as land. The function $F$ is assumed to be increasing, concave, and to satisfy the Inada conditions. Households have access to a single risk-free bond denominated in units of consumption that pays the interest rate $r_t$ when held between periods $t$ and $t+1$. The interest rate $r_t$ is country specific and may differ from the world interest rate, which we assume to be constant and denoted by $r$. In the present model, the country premium, given by $r_t - r$, is endogenously determined. The sequential budget constraint is then given by

$$d_t = (1 + r_{t-1})d_{t-1} + c_t - e^{z_t}F(k^*, h_t).$$

In addition, households face a no-Ponzi-game constraint of the form

$$\lim_{j \to \infty} \frac{d_{t+j}}{\prod_{s=0}^{j-1}(1 + r_{t+s})} \leq 0 \quad (11.9)$$

for all $t$. The household chooses processes $\{c_t, h_t, d_t\}$ to maximize its utility function subject to the sequential budget constraint and the no-Ponzi-game constraint. The first-order conditions of this problem are the sequential budget constraint, the no-Ponzi-game constraint holding with equality,
and

\[
- \frac{U_h(c_t, h_t)}{U_c(c_t, h_t)} = e^{zt} F_h(k^*, h_t) \quad (11.10)
\]

\[
U_c(c_t, h_t) = \beta(C_t, H_t)(1 + r_t)E_t U_c(c_{t+1}, h_{t+1})
\]

Debt accumulation is subject to the following aggregate constraint:

\[ D_t \leq \overline{d}. \]

Because all agents are identical, we have that in equilibrium \( D_t \) must equal \( d_t \). When the borrowing constraint does not bind \( (d_t < \overline{d}) \), domestic agents borrow less than the total amount of funds foreign lenders are willing to invest in the domestic economy. As a result, in this case the domestic interest rate equals the world interest rate \( (r_t = r) \). By contrast, when the borrowing constraint binds, households compete for a fixed amount of loans \( \overline{d} \). At the world interest rate \( r \), the demand for loans exceeds \( \overline{d} \), and consequently the domestic interest rate \( r_t \) rises to a point at which everybody is happy holding exactly \( \overline{d} \) units of loans. The quantity \( (r_t - r)d_t \), represents a pure financial rent. It is important to specify who appropriates this rent. There are two polar cases. In one, the financial rent is appropriated by domestic banks, which then distribute it to households in a lump-sum fashion. In this case, the emergence of rents does not represent a loss of resources for the country. In the second polar case, the rent is appropriated by foreign banks. In this case, the financial rent generates a resource cost for the domestic economy. Intermediate cases are also possible. We first study the case in which the financial rent is appropriated domestically.

A stationary competitive equilibrium in the economy with an aggregate borrowing constraint is a set of processes \( \{d_t, r_t, c_t, h_t\} \) satisfying the following conditions:

\[
- \frac{U_h(c_t, h_t)}{U_c(c_t, h_t)} = e^{zt} F_h(k^*, h_t), \quad (11.11)
\]
\[ U_c(c_t, h_t) = \beta(c_t, h_t)(1 + r_t)E_tU_c(c_{t+1}, h_{t+1}), \tag{11.12} \]
\[ d_t = (1 + r)d_{t-1} + c_t - e^{zt}F(k^*, h_t), \tag{11.13} \]
\[ d_t \leq \bar{d}, \tag{11.14} \]
\[ r_t \geq r, \tag{11.15} \]

and
\[ (d_t - \bar{d})(r_t - r) = 0. \tag{11.16} \]

The last of these expressions is a slackness condition stating that if the borrowing constraint is not binding then the domestic interest rate must equal the world interest rate, and that if the domestic interest rate is strictly above the world interest rate the borrowing constraint must be binding. Note that the resource constraint (11.13) features the world interest rate \( r \) and not the domestic interest rate \( r_t \). This is because we are assuming that the financial rent that emerges when the borrowing constraint is binding stays within the country.

Consider now an environment in which the borrowing constraint is imposed at the level of the individual household. In this case, the household’s optimization problem consists in maximizing the utility function (11.8) subject to the constraints

\[ d_t = (1 + r)d_{t-1} + c_t - e^{zt}F(k^*, h_t) \]

and
\[ d_t \leq \bar{d}. \]

In this environment, household take explicitly into account the borrowing limit. Letting \( \theta_t \lambda_t \) and \( \theta_t \lambda_t \mu_t \) denote the Lagrange multipliers on the sequential budget constraint and the borrowing
constraint, respectively, we have that the optimality conditions associated with the household’s problem are the above two constraints, the labor efficiency condition (11.10), and

\[ U_c(c_t, h_t) = \lambda_t \]

\[ \lambda_t - \lambda_t \mu_t = \beta(C_t, H_t)(1 + r_t)E_t \lambda_{t+1} \]

\[ \mu_t \geq 0 \]

\[ (d_t - \bar{d}) \mu_t = 0. \]

The first and second of these expressions states that in periods in which the borrowing constraint binds (i.e., when \( \mu_t > 0 \)), the marginal utility of an extra unit of debt is not given by the marginal utility of consumption \( U_c(c_t, h_t) \) but by the smaller value \( U_c(c_t, h_t)(1 - \mu_t) \). The reason behind this is that in this case an extra unit of debt tightens the borrowing constraint even more and therefore carries a shadow punishment, given by \( U_c(c_t, h_t) \mu_t \) utils.

A competitive equilibrium in the economy with an individual borrowing constraint is a set of processes \( \{d_t, \mu_t, c_t, h_t\} \) satisfying the following conditions:

\[ - \frac{U_h(c_t, h_t)}{U_c(c_t, h_t)} = e^{zt} F_h(k^*, h_t) \quad (11.17) \]

\[ U_c(c_t, h_t)(1 - \mu_t) = \beta(c_t, h_t)(1 + r_t)E_t U_c(c_{t+1}, h_{t+1}) \quad (11.18) \]

\[ d_t = (1 + r)d_{t-1} + c_t - e^{zt} F(k^*, h_t). \quad (11.19) \]

\[ d_t \leq \bar{d} \quad (11.20) \]

\[ \mu_t \geq 0 \quad (11.21) \]
We wish to establish that the equilibrium behavior of external debt, consumption, and hours is identical in the economy with an aggregate borrowing constraint and in the economy with an individual borrowing constraint. That is, we want to show that the processes \( \{d_t, c_t, h_t\} \) implied by the system (11.11)-(11.16) is identical to the one implied by the system (11.17)-(11.22). To this end, we will show that by performing variable transformations, the system (11.17)-(11.22) can be written exactly like the system (11.11)-(11.16). Define the shadow interest rate \( \tilde{\sigma}_t \) as

\[
1 + \tilde{\sigma}_t = \frac{1 + r}{1 - \mu_t}.
\]

(11.23)

Note that \( \mu_t \) must be nonnegative and less than unity. This last property follows from the fact that, from equation (11.18), a value of \( \mu_t \) greater than or equal to unity would imply that the marginal utility of consumption \( U_c(c_t, h_t) \) is infinite or negative. It follows that \( \tilde{\sigma}_t \) is greater than or equal to \( r \). Furthermore, it is straightforward to see that \( \mu_t > 0 \) if and only if \( \tilde{\sigma}_t > r \) and that \( \mu_t = 0 \) if and only if \( \tilde{\sigma}_t = r \). We can therefore write the system (11.17)-(11.22) as

\[
- \frac{U_h(c_t, h_t)}{U_c(c_t, h_t)} = e^{zt} F_h(k^*, h_t)
\]

(11.24)

\[
U_c(c_t, h_t) = \beta(c_t, h_t)(1 + \tilde{\sigma}_t) E_t U_c(c_{t+1}, h_{t+1})
\]

(11.25)

\[
d_t = (1 + r) d_{t-1} + c_t - e^{zt} F(k^*, h_t)
\]

(11.26)

\[
d_t \leq \bar{d}
\]

(11.27)

\[
\tilde{\sigma}_t \geq r
\]

(11.28)

\[
(d_t - \bar{d})(\tilde{\sigma}_t - r) = 0.
\]

(11.29)
The systems (11.11)-(11.16) and (11.24)-(11.29) are identical, and must therefore deliver identical equilibrium processes for \( d_t, c_t, \) and \( h_t. \) This result demonstrates that whether the borrowing constraint is imposed at the aggregate or the individual level, the real allocation is the same. In other words, the imposition of the borrowing constraint at the aggregate level generates no overborrowing. What happens is that the market interest rate in the economy with the aggregate borrowing limit conveys exactly the same signal as the Lagrange multiplier \( \mu_t \) in the economy with the individual borrowing constraint.

Resource Costs

When rents from financial rationing are appropriated by foreign lenders, the equilibrium conditions of the economy with the aggregate borrowing constraint are as before, except that the resource constraint becomes

\[
d_t = (1 + r_{t-1})d_{t-1} + c_t - e^\sigma F(k^*, h_t).
\]

The fact that the domestic interest rate, \( r_t \geq r, \) appears in the resource constraint implies that when the borrowing constraint is binding and \( r_t > r, \) the country as a whole loses resources in the amount \( (r_t - r)d_t. \) These resources represent pure rents paid to foreign lenders.

When rents are appropriated by foreign lenders, it is no longer possible to compare analytically the dynamics of external debt in the economies with the aggregate debt limit and in the economy with the individual debt limit. We therefore resort to numerical methods to characterize competitive equilibria. Preferences and technologies are parameterized as follows: \( U(c, h) = \left[ c - \omega^{-1} h^\omega \right]^{1-\sigma} / (1 - \sigma), \) \( \beta(c, h) = \left[ 1 + c - \omega^{-1} h^\omega \right]^{-\psi}, \) and \( F(k^*, h) = k^\alpha h^{1-\alpha}, \) where \( \sigma, \omega, \psi, k^*, \) and \( \alpha \) are fixed parameters. Table 11.1 displays the values assigned to these parameters. The time unit is meant to be one year. The values for \( \alpha, \omega, \sigma, \) and \( r \) are taken from Schmitt-Grohé and Uribe (2003). The parameter \( \psi \) is set to induce a debt-to-GDP ratio, \( d/y, \) of 50 percent in
Table 11.1: Parameter Values

<table>
<thead>
<tr>
<th>σ</th>
<th>ω</th>
<th>ψ</th>
<th>α</th>
<th>r</th>
<th>κ</th>
<th>k*</th>
<th>π_{11} = π_{22}</th>
<th>z^1 = -z^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.455</td>
<td>0.0222</td>
<td>0.32</td>
<td>0.04</td>
<td>7.83</td>
<td>78.3</td>
<td>0.71</td>
<td>0.0258</td>
</tr>
</tbody>
</table>

the deterministic steady state. The calibrated value of κ is such that in the economy without the debt limit, the probability that \( d_t \) is larger than κ is about 15 percent. I set the level of the fixed factor of production \( k^* \) so that its market price in terms of consumption goods in the deterministic steady state is unity. The productivity shock is assumed to follow a two-state symmetric Markov process with mean zero. Formally, \( z_t \) takes on values from the set \( \{ z^1, z^2 \} \) with transition probability matrix \( \Pi \). I assume that and \( z^1, z^2 \) and \( \Pi \) satisfy \( z^1 = -z^2 \) and \( \pi_{11} = \pi_{22} \). I set \( \pi_{11} \) equal to 0.71 and \( z^1 \) equal to 0.0258. This process displays the same serial correlation (0.58) and twice as large a standard deviation (2.58 percent) as the one estimated for Canada by Mendoza (1991).

The choice of a process for the productivity shock that is twice as volatile as the one observed in a developed small open economy like Canada reflects the view that the most salient distinction between business cycles in developed and developing countries is, as argued in chapter 1, that the latter are about twice as volatile as the former.

The model is solved using the Chebyshev parameterized expectations method. The state space is discretized using 1000 points for the stock of debt, \( d_t \). The parameterization of expectations uses 50 coefficients. We compute the equilibrium for three model economies: An economy with no debt limit, an economy with a debt limit and financial rents accruing to domestic residents, and an economy with a debt limit and financial rents flowing abroad.\(^4\) The Matlab code that implements the numerical results reported in this section are available at [http://www.columbia.](http://www.columbia.)

\(^4\)The procedure approximates the equilibrium with reasonable accuracy. The DenHaan-Marcet test for 5-percent left and right tails yields (0.047,0.046) for the economy without a debt limit, (0.043,0.056) for the economy with a debt limit and rents owned domestically, and (0.048,0.056) for the economy with a debt limit and rents flowing abroad. This test was conducted using 1000 simulations of 5000 years each, dropping the first 1000 periods.
Figure 11.1 displays with a solid line the equilibrium probability distribution of external debt in the economy with an aggregate debt limit and financial rents from rationing accruing to domestic agents. According to the no-overborrowing result obtained earlier, this economy is identical to the one with a household-specific debt limit. The figure shows with a dash-crossed line the distribution of debt in the economy with an aggregate debt limit and financial rents accruing to foreign lenders. As a reference, the figure also displays, with a dashed line, the debt distribution in an economy without a debt limit (except, of course, for the natural debt limit that prevents Ponzi schemes). The main result conveyed by the figure is that the no overborrowing result is robust to allowing for financial rents to belong to foreign lenders. Specifically, the distribution of debt is virtually

[Figure 11.1: Equilibrium Distribution of External Debt]
unaffected by whether financial rents are assumed to flow abroad or stay within the country’s limits.\footnote{This no-overborrowing result is robust to imposing a more stringent debt limit. We experimented lowering the value of $\kappa$ by 25 percent, from 7.8 to 5.9. This smaller value of the debt limit is such that in the unconstrained economy the probability that $a_t$ is larger than $\kappa$ is about 30 percent. Under this parameterization, we continue to find no overborrowing. Specifically, the debt distribution in the economy with an aggregate borrowing limit and rents accruing to foreign lenders is virtually identical to the distribution of debt in the economy with an aggregate debt limit and rents accruing to domestic households, which, as demonstrated earlier, is identical to the debt distribution in the economy with an individual borrowing limit.}

The reason behind this result is that the resource cost incurred by the economy when financial rents belong to foreigners is fairly small, about 0.008 percent of annual GDP. This implication, in turn, is the result of two properties of the equilibrium dynamics. First, the economy seldom hits the debt limit. The debt constraint binds on average less than once every one hundred years. This is because agents engage in precautionary saving to mitigate the likelihood of finding themselves holding too much debt in periods in which the interest rate is above the world interest rate. Second, when the debt limit does bind, it produces a country interest-rate premium of less than 2 percent on average, and the external debt is about 40 percent of GDP when the economy hits the debt limit. This observation implies that the average cost of remitting financial rents abroad is less than $0.008 = 40 \times 0.02 \times 100^{-1}$ percent of GDP per year.

**The Role of Asset Prices**

Thus far, we have limited attention to a constant debt limit. In practice, debt limits take the form of collateral constraints limiting the size of debt to a fraction of the market value of an asset, such as land or structures. Theoretically, this type of borrowing limit have been shown to help explain observed macroeconomic dynamics during sudden stops, as they tend to exacerbate the contractionary effects of negative aggregate shocks. This is because states in which the collateral constraint is binding are associated with sharp declines in stock prices and fire sales of collateral (see, for example, Mendoza, 2010). The central question for the issues being analyzed in this section is whether these fire sales are more or less severe when the collateral constraint is imposed at an
aggregate level as opposed to at the level of the individual borrower. To model a time-varying collateral constraint, assume that output is produced via an homogeneous-of-degree-one function \( F \) that takes labor and land as inputs. Formally,

\[
y_t = e^{zt} F(k_t, h_t).
\]

Suppose further that the aggregate per capita supply of land is fixed and given by \( k^* > 0 \). Unlike in the previous section, here we will assume that there is a market for land. Let \( q_t \) denote the market price of land in terms of consumption goods. The sequential budget constraint of the household is given by

\[
d_t = (1 + r_{t-1})d_{t-1} + c_t + q_t(k_{t+1} - k_t) - e^{zt} F(k_t, h_t) \tag{11.30}
\]

Note that in period \( t \), the household’s land holdings, \( k_t \), are predetermined. Each period \( t \geq 0 \), the household chooses the amount of land, \( k_{t+1} \), that it will use in production in period \( t + 1 \).

Consider first the case in which foreign investors impose a collateral constraint at the country level of the form

\[
D_t \leq \kappa q_t k^*.
\]

Individual households do not internalize this constraint. The household’s optimization problem consists in choosing processes \( \{c_t, h_t, d_t, k_{t+1}\} \) to maximize the utility function (11.8) subject to the sequential budget constraint (11.30) and the no-Ponzi-game constraint (11.9). The first-order conditions associated with this optimization problem with respect to consumption, hours, and debt are identical to those of its no-land-market counterpart. In the present environment, however, there is an additional first-order condition. It is an Euler equation associated with the use of land. This
condition, evaluated at equilibrium quantities, is given by

\[ q_t = E_t \{ \Lambda_{t,t+1} [q_{t+1} + e^{z_{t+1}} F_k(k^*, h_{t+1})] \}, \tag{11.31} \]

where

\[ \Lambda_{t,t+j} = \frac{U_c(c_{t+j}, h_{t+j})}{U_c(c_t, h_t)} \prod_{s=0}^{j-1} \beta(c_{t+s}, h_{t+s}) \tag{11.32} \]

is a stochastic discount factor given by the equilibrium marginal rate of consumption substitution between periods \( t \) and \( t + j \). Iterating this expression forward yields

\[ q_t = E_t \sum_{j=1}^{\infty} \Lambda_{t,t+j} e^{z_{t+j}} F_k(k^*, h_{t+j}). \tag{11.33} \]

Intuitively, this expression states that the price of land equals the present discounted value of its future expected marginal products.

A stationary competitive equilibrium with an aggregate collateral constraint and financial rents accruing domestically is given by a set of stationary stochastic processes \( \{d_t, r_t, c_t, h_t, q_t, \Lambda_{t,t+1}\}_{t=0}^{\infty} \) satisfying (11.11)-(11.13), (11.15), (11.31), (11.32), and

\[ d_t \leq \kappa q_t k^*, \tag{11.34} \]

and

\[ (r_t - r)(d_t - \kappa q_t k^*) = 0, \tag{11.35} \]

Consider now the case in which the collateral constraint is imposed at the level of each borrower. That is,

\[ d_t \leq \kappa q_t k_{t+1}. \]
The household internalizes this borrowing limit. However, as first noted by Auernheimer and García-Saltos (2000), the presence of the price of land on the right-hand side of this borrowing constraint introduces an externality. This is because the individual household, being a price taker, takes as exogenous the evolution of the price of land, $q_t$.

In this case, all external loans are extended at the world interest rate $r$. The pricing equation for land takes the form

$$q_t \left[ 1 - \kappa \left( \frac{1}{1 + r} - \frac{1}{1 + \tilde{r}_t} \right) \right] = E_t \{ \Lambda_{t,t+1} [q_{t+1} + e^{z_{t+1}} F_k(k^*, h_{t+1})] \}, \quad (11.36)$$

where $\tilde{r}_t \geq r$ denotes the shadow interest rate as defined by equation (11.23). Iterate this expression forward to obtain

$$q_t = E_t \sum_{j=1}^{\infty} \Lambda_{t,t+j} \frac{e^{z_{t+j}} F_k(k^*, h_{t+j})}{\prod_{s=0}^{j-1} \left[ 1 - \kappa \left( \frac{1}{1 + r} - \frac{1}{1 + \tilde{r}_{t+s}} \right) \right]}. \quad (11.36)$$

Comparing this expression with its counterpart in the economy with an aggregate borrowing constraint (equation (11.33)), we observe that the fact that the shadow value of collateral, given by $1/(1 + r) - 1/(1 + \tilde{r}_t)$, is nonnegative implies, ceteris paribus, that the individual agent discounts future marginal products of land less heavily in the economy with the internalized borrowing constraint than in the economy with an aggregate borrowing constraint. This is because when the collateral constraint is internalized the individual agent values the financial service provided by land, namely, collateral. Note that the above expression for $q_t$ does not state that the price of land should be higher in periods in which the shadow interest rate $\tilde{r}_t$ exceeds $r$. Indeed, we will see shortly that the real value of land falls dramatically during such periods. The above expression does state that when $\tilde{r}_t$ is larger than $r$, all other things equal, the value of land is likely to be higher in the economy in which the collateral constraint is internalized than in an economy in which it is not. In turn, the fact that in the economy with an internalized collateral constraint the value
of collateral is higher than it is in an economy with an aggregate collateral constraint suggests that borrowing is less limited when the collateral constraint is internalized. Thus, intuitively one should not expect the no-overborrowing result of the previous section to be overturned by the introduction of a time-varying collateral constraint of the type considered in this section.

A stationary competitive equilibrium with an individual collateral constraint and financial rents accruing domestically is given by a set of stationary stochastic processes \( \{d_t, c_t, h_t, d_t, \tilde{r}_t, q_t, \Lambda_{t,t+1}\} \) satisfying (11.24)-(11.26), (11.28), (11.32), (11.34), (11.36), and

\[
(\tilde{r}_t - r)(d_t - \kappa q_t k^*) = 0,
\]

(11.37)

The only difference between the equilibrium conditions of the economy with an aggregate collateral constraint and those of the economy with an individual collateral constraint is the Euler condition for land (equation (11.31) versus equation (11.36)).

To ascertain whether the imposition of an aggregate collateral constraint induces external overborrowing, we compute equilibrium dynamics numerically. I calibrate the economy as in the previous subsection, except for the parameter \( \kappa \), which now takes the value 0.1.\(^6\) The model is solved using the Chebyshev parameterized expectations method.\(^7\)

The top-left panel of figure 11.2 displays the unconditional distribution of external debt. A solid line corresponds to the economy with an internal collateral constraint, and a dashed line corresponds to the economy with an aggregate collateral constraint. The distribution of debt is virtually identical in the economy with an individual collateral constraint and in the economy with

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\(^6\)In the deterministic steady state we have that \( q_t = 1 \), so that \( \kappa q_t k^* = 7.83 \), which is the value assigned to \( \kappa \) in the economy with the constant debt limit.

\(^7\)The DenHaan-Marcet test for 5-percent left and right tails yields (0.043,0.061) for the economy with an individual collateral constraint, and (0.048,0.06) for the economy with an aggregate collateral constraint. This test is conducted using 5000 simulations of 5000 years each, dropping the first 1000 periods. The Matlab code that implements the numerical results reported in this section are available at [http://www.columbia.edu/~mu2166/overborrowing/overborrowing.html](http://www.columbia.edu/~mu2166/overborrowing/overborrowing.html).
Figure 11.2: Equilibrium Under a Time-Varying Collateral Constraint

Note: ‘Indiv CC’ stands for Individual Collateral Constraint, and ‘Agg CC’ stands for Aggregate Collateral Constraint.
an aggregate collateral constraint. Similarly, as shown in the top-right and bottom-left panels of the figure, whether the collateral constraint is imposed at the individual or the aggregate levels appears to make no difference for the equilibrium dynamics of stock prices or consumption. Note that when the stock of debt is high agents engage in fire sales of land resulting in sharp declines in its market price, $q_t$. But the collapse of land prices is quantitatively similar in the two economies. The contraction in real estate prices is caused by the increase in interest rates as the economy approaches the debt limit. Formally, the fire sale of land is driven by a drop in the stochastic discount factor $\Lambda_{t,t+1}$. When $\Lambda_{t,t+1}$ falls, future expected marginal products of land are discounted more heavily, depressing the value of the asset. To see that in a crisis the stochastic discount factor falls, note that the Euler equation for bond holdings is given by

$$1 = (1 + r_t)E_t\Lambda_{t,t+1},$$

when the borrowing constraint is imposed at the aggregate level. When it is imposed at the individual level, $\tilde{r}_t$ replaces $r_t$. During a crisis, $r_t$ and $\tilde{r}_t$ increase, generating expectations of a decline in $\Lambda_{t,t+1}$. Finally, in line with the intuition developed earlier in this section, land prices are indeed higher in the economy with an individual debt limit, but this difference is quantitatively small. The fact that land prices are slightly higher in the economy with the individual borrowing limit means that in this economy the value of collateral is on average higher than in the economy with an aggregate borrowing limit. This allows households in the economy with the individual borrowing limit to actually hold on average more debt than households in the economy with an aggregate borrowing limit (see the top left panel of the figure 11.2). It follows that the current model predicts underborrowing. The intuition behind this result is that in the economy with the individual borrowing limit, households internalize the fact that holding land relaxes the borrowing limit. As a result, the demand for land is higher in the economy with an individual debt limit than
in the economy in which the collateral service provided by land is not internalized. This causes land prices to be higher in the economy with the internalized borrowing constraint allowing households to borrow more. We will return to the issue of underborrowing in section 11.2.3.

It follows from the analysis of this section, that the no-overborrowing result is robust to allowing for a debt limit that is increasing in the market value of a fixed factor of production. The reason why in this class of models households do not have a larger propensity to borrow under an aggregate debt limit is that the market and social prices of international liquidity are identical (in the case of a constant debt limit) or almost identical (in the case of a debt limit that depends on the price of land). Two features of the economies studied in this section are crucial in generating the equality of market and social prices of debt. First, when the borrowing limit is internalized the shadow price of funds, given by the pseudo interest rate $\tilde{r}_t$, is constant and equal to the world interest rate $r$ except when the debt ceiling is binding. Importantly, the shadow price of funds equals the world interest rate even as households operate arbitrarily close to the debt ceiling. Second, in the economy with the individual debt constraint, when the debt ceiling binds, it does so for all agents simultaneously. This property is a consequence of the assumption of homogeneity across economic agents. The absence of either of the abovementioned two features may cause the market price of foreign funds to be below the social price, thereby inducing overborrowing. The next section explores this issue in more detail.

11.2.2 The Case of Overborrowing

The present section provides three variations of the environment studied thus far that give rise to overborrowing. In the first example, agents are heterogeneous in endowments, in the second agents face a debt-elastic interest rate, and in the third the collateral is a flow that depends on the relative price of nontradables, which, in turn, is a continuous function of aggregate spending.
Heterogeneous Agents

The following model, taken from Uribe (2007), describes a situation in which overborrowing occurs because debt limits do not bind for all agents at the same time. The model features a two-period, endowment economy without uncertainty. The country faces a constant debt ceiling $\kappa$ per capita. There is a continuum of agents of measure one, and agents are heterogeneous. The central result obtains under a variety of sources of heterogeneity, such as differences in endowments, preferences, or initial asset positions. Here, I assume that agents are identical in all respects except for their period-2 endowments. Specifically, in period 1 all households receive the same endowment $y$, whereas in period 2 half of the households receive an endowment $y^a > y$ and the other half receive a smaller endowment $y^b < y^a$.

Agents receiving the larger future endowment have a stronger incentive to borrow in period 1 to smooth consumption over time. Suppose that in the absence of a debt ceiling households with high expected endowment consume $c^a > y + \kappa$ units in period 1 and the rest of the households consume $c^b < y + \kappa$ units. Figure 11.3 depicts the equilibrium in the absence of a debt constraint. In the unconstrained equilibrium, aggregate external debt per capita equals $d^u = (c^a + c^b)/2 - y$.

When the borrowing ceiling $\kappa$ is imposed at the level of each individual household, half of the households—those with high period-2 endowment—are constrained and consume $y + \kappa$ units, whereas the other households are unconstrained and consume $c^b$. Aggregate external debt per capita equals $d^i = (\kappa + c^b - y)/2 < d^u$. Clearly, we also have that $d^i < \kappa$.

Now suppose that the debt ceiling is imposed at the aggregate level. Two alternative situations are possible. One is that the aggregate debt limit is not binding. This case takes place when in the absence of a debt constraint debt per capita does not exceed the ceiling $\kappa$. That is, when $d^u \leq \kappa$. In this case, the equilibrium interest rate equals the world interest rate $r$, and consumption of each agent equals the level attained in the absence of any borrowing constraint. External debt is given
by $d^a = d^u > d^i$.

Alternatively, if the aggregate level of external debt in the unconstrained environment exceeds the ceiling (i.e., if $d^a > \kappa$), then the economy is financially rationed, the domestic interest rate exceeds the world interest rate, and aggregate borrowing per capita is given by $d^a = \kappa > d^i$.

It follows that, regardless of whether the aggregate debt limit is binding or not, external borrowing is higher when the debt ceiling is imposed at the aggregate level. That is, the combination of heterogeneous consumers and a debt limit imposed at the aggregate level induces overborrowing in equilibrium. Overborrowing occurs because of a financial externality. Specifically, the group of more frugal consumers provides a financial service to the group of more lavish consumers by placing comparatively less pressure on the aggregate borrowing constraint. This service, however, is not priced in the competitive equilibrium.\footnote{Interestingly, economic heterogeneity, although of a different nature, is also the root cause of overborrowing in the dual-liquidity model of emerging-market crisis developed by Caballero and Krishnamurthy (2001). In their model, there is heterogeneity in the provision of liquidity across assets. Some assets are recognized as liquid collateral by both domestic and foreign lenders, while other assets serve as collateral only to domestic lenders. Caballero and}
This overborrowing result relies on the absence of a domestic financial market. When the debt limit $\kappa$ is imposed at the individual level, intertemporal marginal rates of substitution are not equalized across households. Suppose that a domestic financial market existed in which frugal households (households with relatively low future endowments) could borrow externally and lend internally to lavish households. In this case, in equilibrium, intertemporal marginal rates of substitution would be equal across households and the consumption allocation would equal the one emerging under an aggregate collateral constraint. It follows that with a domestic financial market, the overborrowing result disappears. This reemergence of the no-overborrowing result relies, however, on the assumption that foreign lenders will keep the external debt limit on lavish agents equal to $\kappa$. Alternatively, foreign lenders may realistically impose that total debt held by lavish households (i.e., the sum of external and domestic debt) be limited by $\kappa$. In this case, a domestic financial market would be ineffective in eliminating overborrowing.

A comment on the concept of overborrowing when agents are heterogeneous is in order. The term overborrowing has a negative connotation, referring to a suboptimal amount of external financing. In the models with homogeneous agents and a constant debt limit studied earlier in this chapter, one can safely interpret any excess external debt in the economy with an aggregate debt limit over the economy with an individual debt limit as suboptimal, or overborrowing. This is because the competitive equilibrium associated with the economy featuring an individual debt limit coincides with the optimal allocation chosen by a social planner that internalizes the debt limit. When agents are heterogeneous it is not necessarily the case that the debt distribution associated with the economy featuring an aggregate debt limit is less desirable than the one implied by the economy with an individual debt limit. To see this, suppose, for instance, that in the economy analyzed in this section the social planner cared only about the well being of agents with high

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Krishnamurthy show that in financially underdeveloped economies this type of heterogeneity produces an externality whereby the market price of international liquidity is below its social marginal cost.
period-2 endowments. In this case, the social planner would favor the equilibrium associated with an aggregate borrowing limit over the one associated with an individual debt limit.

**Debt-Elastic Country Premium**

In chapter 4, section 4.1.1, we studied a way to induce stationarity in the small-open-economy business-cycle model consisting in making the country interest rate a function of the cross-sectional average of external debt per capita. In this case, agents do not internalize that their individual borrowing contributes to increasing the cost of external funds. As a result, an externality emerges whereby the economy assumes more external debt than it is socially optimal. Formally, let the country interest rate be given by an increasing function of aggregate external debt of the form \( r_t = r + \rho(D_t) \), with \( \rho' > 0 \). Because individual households take the evolution of the aggregate debt position, \( D_t \), as exogenous, they do not internalize the dependence of the interest rate on their individual debt positions. The reason why the cost of funds is debt elastic is unspecified in this simple setting, but it could be due to the presence of default risk as in models of sovereign debt. We study this class of models in chapter 12. We assume that all households are identical. Therefore, in equilibrium, we have that \( d_t = D_t \), where \( d_t \) is the individual level of debt held by the representative household. Let \( d^* > 0 \) denote the steady-state value of debt in this economy. Then \( d^* \) must satisfy the condition

\[
1 = (1 + r + \rho(d^*))\beta, \tag{11.38}
\]

where, as usual, \( \beta \) is a constant subjective discount factor. This steady-state condition arises in virtually all formulations of the small open economy with utility-maximizing households (e.g., Schmitt-Grohé and Uribe, 2003). Assume now that the debt-elastic interest-rate schedule is imposed at the level of each individual household, so that \( r_t = r + \rho(d_t) \). Let \( d^{**} > 0 \) denote the steady-state
level of external debt in this economy. It can be shown that $d^{**}$ is determined by the condition

$$1 - \beta d^{**} \rho'(d^{**}) = (1 + r + \rho(d^{**})) \beta. \quad (11.39)$$

Comparing equations (11.38) and (11.39), it is clear that because $\rho' > 0$, we have that $d^* > d^{**}$.

That is, the economy with the financial externality generates overborrowing. Auernheimer and García-Saltos (2000) derive a similar result in a model in which the interest rate depends on the leverage ratio. That is, $r_t = \rho \left( \frac{D_t}{q_t k_t} \right)$ in the case of an aggregate debt limit, and $r_t = \rho \left( \frac{d_t}{q_t q_t k_{t-1}} \right)$ in the case of an individual debt limit. As noted earlier, in the Auernheimer-García-Saltos model an externality emerges even in this latter case, because agents do not internalize the effect that their borrowing behavior has on the price of land, $q_t$. We note that in the economy with the aggregate debt limit the market price of foreign funds, $r + \rho(d_t)$, is strictly lower than the social cost of foreign funds, given by $r + \rho(d_t) + d_t \rho'(d_t)$. This discrepancy, which is key in generating overborrowing, is absent in the economy of the previous sections.

**Nontraded Output As Collateral**

Consider now a variation of the model analyzed thus far in which the object that serves as collateral is output. Here, the main mechanism for inducing overborrowing is the real exchange rate, defined as the relative price of nontradable goods in terms of tradables. Suppose that due to a negative shock, aggregate demand falls causing the relative price of nontradables to collapse. In this case, the value of nontradable output in terms of tradable goods falls. Since this variable serves as collateral, the borrowing constraint tightens exacerbating the economic contraction. This mechanism was
first studied by Korinek (2010), in the context of a three-period model. Bianchi (2010), extends the Korinek model to an infinite-horizon quantitative setting. Our exposition follows Bianchi’s formulation.

Consider a small open endowment economy in which households have preferences of the form

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t),$$  
(11.40)

with the usual notation. The period utility function takes the form $U(c) = (c^{1-\sigma} - 1)/(1 - \sigma)$. The consumption good is assumed to be a composite of tradable and nontradable consumption as follows:

$$c_t = A(c^T_t, c^N_t) \equiv \left[ \omega c^T_{t}^{1-1/\eta} + (1 - \omega) c^N_{t}^{1-1/\eta} \right]^{1/(1-1/\eta)},$$  
(11.41)

where $c^T_t$ denotes consumption of tradables and $c^N_t$ denotes consumption of nontradables. Households are assumed to have access to a single, one-period, risk-free, internationally-traded bond that pays the constant interest rate $r$. The household’s sequential budget constraint is given by

$$d_t = (1+r)d_{t-1} + c^T_t - y^T_t + p^N_t (c^N_t - y^N_t),$$  
(11.42)

where $d_t$ denotes the amount of debt assumed in period $t$ and maturing in $t+1$, $p^N_t$ denotes the relative price of nontradables in terms of tradables, and $y^T_t$ and $y^N_t$ denote the endowments of tradables and nontradables, respectively. Both endowments are assumed to be exogenous and stochastic. The borrowing constraint takes the form

$$d_t \leq \kappa^T y^T_t + \kappa^N p^N_t y^N_t,$$  
(11.43)

where $\kappa^T, \kappa^N > 0$ are parameters. Households internalize this borrowing limit. However, just as
in the case in which the value of land is used as collateral, this borrowing constraint introduces an externality à la Auernheimer and García-Saltos (2000), because each individual household takes the real exchange rate \( p_t^N \) as exogenously determined, even though their collective absorptions of nontradable goods is a key determinant of this relative price.

Households choose a set of processes \( \{c_t^T, c_t^N, c_t, d_t\} \) to maximize (11.40) subject to (11.41)-(11.43), given the processes \( \{p_t^N, y_t^T, y_t^N\} \) and the initial debt position \( d_{-1} \). The first-order conditions of this problem are (11.41)-(11.43) and

\[
G(c_t^T, c_t^N) = \lambda_t,
\]

\[
p_t^N = \frac{1 - \omega}{\omega} \left( \frac{c_t^T}{c_t^N} \right)^{1/\eta},
\]

\[
\lambda_t = \beta(1 + r) E_t \lambda_{t+1} + \mu_t,
\]

\[
\mu_t \geq 0,
\]

and

\[
\mu_t (d_t - \kappa^T y_t^T - \kappa^N p_t^N y_t^N) = 0,
\]

where \( G(c_t^T, c_t^N) \equiv \omega U'(A(c_t^T, c_t^N)) \left( \frac{A(c_t^T, c_t^N)}{c_t^T} \right)^{1/\eta} \) denotes the marginal utility of tradable consumption, and \( \lambda_t \) and \( \mu_t \) denote the Lagrange multipliers on the sequential budget constraint (11.42) and the collateral constraint (11.43), respectively. As usual, the Euler equation (11.44) equates the marginal benefit of assuming more debt with its marginal cost. During tranquil times, when the collateral constraint does not bind, the benefit of increasing \( d_t \) by one unit is the marginal utility of tradables \( G(c_t^T, c_t^N) \), which in turn equals \( \lambda_t \). The marginal cost of an extra unit of debt is the present discounted value of the payment that it generates in the next period, \( \beta(1 + r) E_t \lambda_{t+1} \). During financial crises, when the collateral constraint binds, the marginal utility of increasing debt
is unchanged, but the marginal cost increases to \( \beta (1 + r) E_t \lambda_{t+1} + \mu_t \), reflecting a shadow penalty for trying to increase debt when the collateral constraint is binding.

In equilibrium, the market for nontradables must clear. That is,

\[
c_t^N = y_t^N.
\]

Then, a competitive equilibrium is a set of processes \( \{c_t^T, d_t, \mu_t\} \) satisfying

\[
G(c_t^T, y_t^N) = \beta (1 + r) E_t G(c_{t+1}^T, y_{t+1}^N) + \mu_t,
\]

\[
d_t = (1 + r) d_{t-1} + c_t^T - y_t^T
\]

\[
d_t \leq \kappa^T y_t^T + \kappa^N \left( \frac{1 - \omega}{\omega} \right) c_t^{1/\eta} y_t^{1-1/\eta}
\]

\[
\mu_t \left[ d_t - \kappa^T y_t^T - \kappa^N \left( \frac{1 - \omega}{\omega} \right) c_t^{1/\eta} y_t^{1-1/\eta} \right] = 0,
\]

\[
\mu_t \geq 0,
\]

given processes \( \{y_t^T, y_t^N\} \) and the initial condition \( d_{-1} \).

The fact that \( c_t^T \) appears on the right-hand side of the equilibrium version of the collateral constraint (11.47) means that during contractions in which the absorption of tradables falls the collateral constraint endogenously tightens. Individual agents do not take this effect into account in choosing their consumption plans. This is the nature of the financial externality in this model. A benevolent government would be interested in designing policies that induce households to internalize the financial externality.
The Socially Optimal Equilibrium

The socially optimal policy aims at attaining the allocation implied by an optimization problem that takes into account the dependence of the value of collateral upon the aggregate consumption of tradables. Such allocation is the solution to the following social planner’s problem:

$$\max_{\{c_t^T, d_t\}} E_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t^T, y_t^N))$$

subject to the equilibrium conditions (11.46) and (11.47), which we reproduce here for convenience:

$$d_t = (1 + r)d_{t-1} + c_t^T - y_t^T$$

$$d_t \leq \kappa T y_t^T + \kappa N \left( \frac{1 - \omega}{\omega} \right) c_t^{1/\eta} y_t^{N^{1-1/\eta}}.$$  

The right-hand side of this expression contains the equilibrium level of consumption of tradables, $c_t^T$, whose evolution is taken as endogenous by the social planner but as exogenous by the individual household. Also, the social planner internalizes the fact that in the competitive equilibrium the market for nontraded goods clears at all times. This internalization implies that the endogenous variable $c_t^N$, is replaced everywhere by the exogenous variable $y_t^N$.

The first-order conditions associated with this problem are the above two constraints and

$$G(c_t^T, y_t^N) + \mu_t \Gamma(c_t^T, y_t^N) = \beta(1 + r)E_t \{ [G(c_{t+1}^T, y_{t+1}^N) + \mu_{t+1} \Gamma(c_{t+1}^T, y_{t+1}^N)] \} + \mu_t$$ (11.50)

$$\mu_t \left[ d_t - \kappa T y_t^T - \kappa N \left( \frac{1 - \omega}{\omega} \right) c_t^{1/\eta} y_t^{N^{1-1/\eta}} \right] = 0$$ (11.51)

and

$$\mu_t \geq 0,$$ (11.52)
where $\mu_t$ denotes the Lagrange multiplier on the borrowing constraint (11.47), and $\Gamma(c_t^T, y_t^N) \equiv \frac{\kappa N}{\eta} \left( \frac{1-\omega}{\omega} \right) \left( \frac{y_t^N}{c_t^T} \right)^{1-1/\eta}$ denotes the amount by which an extra unit of tradable consumption relaxes the borrowing constraint (11.47). In periods in which the borrowing constraint is binding, the marginal utility of tradable consumption from a social point of view is given by the sum of the direct marginal utility $G(c_t^T, y_t^N)$ and the factor $\mu_t \Gamma(c_t^T, y_t^N)$ reflecting the fact that an extra unit of consumption of tradables rises the relative price of nontradables, thereby making the collateral constraint marginally less tight. The social-planner equilibrium is then given by a set of processes $\{c_t^T, d_t, \mu_t\}$ satisfying (11.46), (11.47), and (11.50)-(11.52).

Shortly, we will compare the macroeconomic dynamics induced by the competitive equilibrium and the social-planner equilibrium. Before doing so, however, we wish to address the issue of how to implement the latter. That is, we wish to design fiscal instruments capable of supporting the social-planner allocation as the outcome of a competitive equilibrium.

**Optimal Fiscal Policy**

The competitive equilibrium conditions and the social-planner equilibrium conditions differ only in the Euler equation (compare equations (11.45) and (11.50)). The question we entertain here is whether there exists a fiscal-policy scheme that induces households to internalize their collective effect on the credit limit and thereby makes the competitive-equilibrium conditions identical to the social-planner equilibrium conditions. It turns out that such optimal fiscal policy takes the form of a proportional tax on external debt and a lump-sum transfer that rebates the entire proceeds from the debt tax equally among households. Specifically, let $\tau_t$ denote the tax on debt and $s_t$ denote the lump-sum transfer. Then, the budget constraint of the household is given by

$$d_t = (1 + r)(1 + \tau_{t-1})d_{t-1} + c_t^T - y_t^T + p_t^N (c_t^N - y_t^N) - s_t.$$
Households choose processes \(\{c_T^t, c_N^t, c_t, d_t\}\) to maximize (11.40) subject to this budget constraint and conditions (11.41) and (11.43), given the processes \(\{p_t^N, y_T^t, y_N^t, \tau_t, s_t\}\) and the initial debt position \(d_{-1}\). The Euler equation associated with this problem is of the form \(G(c_T^t, c_N^t) = \beta(1 + r)(1 + \tau_t)E_t G(c_{t+1}^T, c_{t+1}^N)\). This expression says that households choose a level of consumption at which the marginal benefit of an extra unit of debt, given by the current marginal utility of tradable consumption \(G(c_T^t, c_N^t)\), equals the marginal cost of debt, given by the expected present discounted value of the after-tax interest payment measured in terms of utils, \(\beta(1 + r)(1 + \tau_t)E_t G(c_{t+1}^T, c_{t+1}^N)\). It follows that the higher the tax rate on debt, the more costly it is for households to accumulate debt. Of course, this tax cost of debt is only an individual perception aimed at distorting household’s spending behavior. In the aggregate, the government rebates the tax to households in a lump-sum fashion by adopting a balanced-budget rule of the form \(s_t = \tau_{t-1}(1 + r)d_{t-1}\). It is straightforward to establish that a competitive equilibrium in this economy is given by processes \(\{c_t^T, d_t, \mu_t\}\) satisfying (11.46)-(11.49), which are common to both the competitive equilibrium without taxes and the social planner’s equilibrium, and

\[
G(c_T^t, y_N^t) = \beta(1 + r)(1 + \tau_t)E_t G(c_{t+1}^T, y_{t+1}^N) + \mu_t, \tag{11.53}
\]

given a fiscal policy \(\tau_t\), exogenous processes \(\{y_T^t, y_N^t\}\), and the initial condition \(d_{-1}\). Thus, the competitive equilibrium conditions in the economy with taxes and the social planner’s equilibrium conditions differ only in their respective Euler equations, given by (11.53) and (11.50). The optimal fiscal policy then is the tax process \(\{\tau_t\}\) that makes these two Euler equations equal to each other. It is easy to establish that the optimal tax process is given by

\[
\tau_t = \frac{\beta(1 + r)E_t \mu_{t+1} \Gamma(c_{t+1}^T, y_{t+1}^N) - \mu_t \Gamma(c_t^T, y_t^N)}{\beta(1 + r)E_t G(c_{t+1}^T, y_{t+1}^N)}. \tag{11.54}
\]
According to this expression, in tranquil periods, in which the borrowing constraint does not bind \((\mu_t = 0)\) and is not expected to bind in the next period \((\mu_{t+1} = 0 \text{ in all states of period } t+1)\), the government does not tax external debt \((\tau_t = 0)\). In periods of uncertainty, when the collateral constraint does not bind in the current period but has some probability of binding in the next period \((\mu_t = 0 \text{ and } \mu_{t+1} > 0 \text{ in some states of } t+1)\), the government taxes debt holdings to discourage excessive spending and external borrowing.

**Quantitative Predictions**

The numerical exercise conducted by Bianchi (2010) considers an approximately optimal tax policy in which the government levies a tax on debt equal to the one prescribed by the optimal tax policy only in periods in which the borrowing constraint is not binding in the current period but does bind in at least one of the possible states of period \(t+1\). When the borrowing constraint does bind in the current period or when it does not bind either currently or in any state of the next period, the tax rate is set at zero. The model is calibrated at an annual frequency. The output process is assumed to follow a bivariate AR(1) process. An empirical measure of traded output is defined as GDP in the manufacturing, agriculture, and mining sectors, and nontraded output as the difference between GDP and GDP in the traded sector. Using data from the World Development Indicators from 1965 to 2007, the output process is estimated to be

\[
\begin{bmatrix}
    y_T^t \\
    y_N^t
\end{bmatrix}
= 
\begin{bmatrix}
    0.90 & 0.46 \\
    -0.45 & 0.22
\end{bmatrix}
\begin{bmatrix}
    y_T^{t-1} \\
    y_N^{t-1}
\end{bmatrix}
+ 
\begin{bmatrix}
    0.047 & 0 \\
    0.035 & 0.022
\end{bmatrix}
\begin{bmatrix}
    \epsilon_1^t \\
    \epsilon_2^t
\end{bmatrix},
\]

with \(\epsilon_i^t \sim N(0,1)\) for \(i = 1, 2\).\(^9\) The remaining parameters are calibrated as follows: \(\kappa_T = \kappa_N = 0.32\), \(\beta = 0.91\), \(\omega = 0.31\), \(\eta = 0.83\), \(\sigma = 2\), and \(r = 0.04\).

\(^9\)Unfortunately, the 2010 version of Bianchi’s paper does not specify how the output series were detrended, and in which units they are expressed.
Bianchi finds that the optimal debt accumulation decision is quite different when the borrowing constraint binds and when it does not. Let the optimal debt accumulation rule be \( d_t = D(d_{t-1}, y_t^T, y_t^N) \) in the competitive equilibrium (CE) and \( d_t = D^s(d_{t-1}, y_t^T, y_t^N) \) in the social-planner (SP) equilibrium. The functions \( D \) and \( D^s \) are increasing in the level of debt \( (D_1 > 0, D_1^s > 0) \) when the borrowing constraint is not binding but are decreasing in the level of debt \( (D_1 < 0, D_1^s < 0) \) when the borrowing constraint is binding. This means that the CE and SP economies experience capital outflows \( (d_t - d_{t-1} < 0) \) when the borrowing constraint binds.

There are also significant differences between the debt-accumulation behavior in the CE and SP economies: First, \( D \) is larger than \( D^s \), especially when the borrowing limit is not binding in either the CE or SP economies or when it is binding in the CE but not in the SP economy. Second, given the levels of sectoral output, the borrowing constraint binds at a higher level of debt in the SP economy than in the CE economy. Third, under the present calibration, there is a 15 percent probability that debt in the CE is higher than the upper bound of the support of the stationary distribution of debt in the SP economy. These predictions of the model point at a significant degree of overborrowing. The required tax on external debt that corrects this inefficiency, is 5 percent on average, and is increasing in the level of debt.

Finally, the presence of overborrowing in this economy has quantitatively important effects on the frequency and size of economic crises. The probability of a crisis, defined as a situation in which the borrowing constraint is binding and capital outflows \( (d_t - d_{t-1} < 0) \) are larger than one standard deviation, is 5.5 percent in the CE and 0.5 percent in the SP economy. Moreover, the initial contraction in consumption in a crisis is higher in the CE economy than in the SP economy.

But these crises do not appear to inflict too much pain to the representative household. Bianchi finds that on average the representative household of the CE economy requires an increase of only one tenth of one percent of its consumption stream to be as happy as the household of the SP economy. This modest welfare cost of overborrowing is due to the fact that in this economy output
is assumed to be exogenous. The next section shows that relaxing this strong assumption can lead to a surprising result.

11.2.3 The Case of Underborrowing

One lesson we can derive from the models of overborrowing based on collateral constraints is that the assumption of what precise object is used as collateral (a constant, the market value of some asset, traded output, nontraded output, etc.) can have crucial consequences for whether or not the presence of the borrowing constraint causes overborrowing, defined as a situation in which the competitive equilibrium produces more debt accumulation than the social planner equilibrium. In this section we stress this idea by considering a modification of Bianchi’s (2010) model due to Benigno, Chen, Otrok, Rebucci, and Young (2009), hereafter BCORY, that under plausible calibrations can produce underborrowing.

The key modifications introduced by BCORY (2009) is to assume that the labor supply is endogenous and that nontradables are produced using labor—recall that Bianchi’s model assumes that output is exogenous. This simple and realistic extension can fundamentally alter the nature of the financial externality. To see this, suppose that, as in the Bianchi (2010) model, borrowing is limited by income. From a private point of view, a way to relax the borrowing constraint is to supply more hours of work, thereby increasing income at any given wage rate. In equilibrium, however, this can have a deleterious effect on the country’s ability to borrow. For an increase in the labor supply causes the supply of nontradables to rise, which, if the elasticity of substitution of tradables for nontradables is low (less than unity), may in turn cause a fall in the value of nontraded income in terms of tradables, tightening the borrowing limit.

Formally, the representative household is assumed to have preferences described by the utility
function
\[ E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, h_t), \] (11.55)
where \( h_t \) denotes hours worked in period \( t \). The period utility function, given by
\[ U(c, h) = \left( \frac{c - \frac{h^\delta}{\sigma}}{1 - \sigma} \right)^{1-\sigma} - 1, \]
is assumed to be increasing in consumption and decreasing in hours worked. As before, the consumption good is a composite of tradables and nontradables and the aggregation technology is given by equation (11.41). The sequential budget constraint is of the form
\[ d_t = (1 + r) d_{t-1} + c_T^T - y_T^T + p_t^N c_N^T - w_t h_t, \] (11.56)
where \( w_t \) denotes the real wage expressed in terms of tradables. The endowment of tradables, \( y_T^T \), is assumed to be exogenous and stochastic. Borrowing in international markets is limited by income:
\[ d_t \leq \kappa (y_T^T + w_t h_t), \] (11.57)
where \( \kappa > 0 \) is a parameter. Note that, unlike the Bianchi (2010) model, this borrowing limit contains on its right-hand side a variable that is endogenous to households, namely, \( h_t \). In particular, in the present environment households internalize the benefit that working longer hours has on their ability to borrow.

The household’s optimization problem consists in choosing processes \( \{c_t, c_T^T, c_N^T, h_t, d_t\} \) to maximize the utility function (11.55) subject to the aggregation technology (11.41), the sequential budget constraint (11.56), and the borrowing constraint (11.57). The first-order conditions associated with this problem are identical to those derived for the Bianchi (2010) model, except for the
emergence of a labor supply expression of the form:

\[-U_h(c_t, h_t) = \lambda_t(w_t + \kappa \mu_t/\lambda_t),\]

were, as before, \(\lambda_t\) denotes the Lagrange multiplier associated with the sequential budget constraint, and \(\mu_t\) denotes the Lagrange multiplier associated with the borrowing constraint. This expression shows that periods in which the borrowing constraint is binding \((\mu_t > 0)\), the shadow real wage, given by \(w_t + \kappa \mu_t/\lambda_t\), exceeds the market wage rate \(w_t\), giving households an extra incentive to supply hours to the labor market. This incentive to work originates, as explained earlier, in the fact that at the going wage rate, an increase in hours raises labor income thereby enlarging the value of collateral and relaxing the borrowing limit.

Firms produce nontradables using a linear technology that takes labor as the sole input of production. Specifically, output of tradables is given by \(y_t^N = h_t\). The problem of the firm consists in choosing \(h_t\) to maximize profits, given by \(p_t^N y_t^N - w_t h_t\), subject to the production technology. We assume that free entry guarantees zero profits in the nontraded sector. This means that

\[p_t^N = w_t\]

at all times.

It is of interest to examine the shape of the borrowing limit, now not from the household’s perspective, but from an equilibrium perspective. Noting that the zero-profit condition implies that \(w_t h_t = p_t^N y_t^N\) at all times, we can write equation (11.56) as:

\[d_t \leq \kappa(y_t^T + p_t^N y_t^N),\]

which is identical to the borrowing constraint assumed by Bianchi (2010), given by equation (11.43),
when $\kappa_T = \kappa_N = \kappa$. Note that both in the present model and in Bianchi’s, households take the variables that appear on the right-hand side of this expression as exogenous. The key difference between the two models is that whereas in the present model the above expression holds only in equilibrium, in Bianchi’s model it holds both in equilibrium and at the level of the individual household.

To understand why the incentives of households to work and in that way relax the borrowing constraint can be counterproductive in equilibrium, we can rewrite the borrowing constraint (11.43) in yet another way by using the zero-profit condition $w_t = p_t^N$ to get rid of $w_t$, the efficiency condition $p_t^N = \frac{1 - \omega}{\omega} \left( \frac{y_t^T}{c_t^N} \right)^{1/\eta}$ to get rid of $p_t^N$, the market-clearing condition $y_t^N = c_t^N$ to get rid of $c_t^N$, and the technological relation $y_t^N = h_t$ to get rid of $y_t^N$. This yields the following equilibrium borrowing limit

$$d_t \leq \kappa \left( y_t^T + \frac{1 - \omega}{\omega} c_t^{T^{1/\eta} h_t^{1-1/\eta}} \right)$$

This expression says that, unlike the perception of the individual household, i.e., that an increase in hours expands the borrowing limit, in equilibrium increasing hours might actually shrink the borrowing possibility set. This will be the case when tradables and nontradables are poor substitutes, or, more precisely, when the elasticity of substitution between these two types of good, $\eta$, is less than unity. In this case, an increase in hours produces a one-to-one increase in the supply of nontradables which drives the relative price of this type of goods down so much that the value of nontraded output ends up falling. BCORY (2009) argue that this is indeed a realistic possibility. Existing estimates of the elasticity of substitution $\eta$ for emerging countries yield values ranging from slightly below 0.5 to about 0.83 (see, for instance, Ostry and Reinhart, 1992; and Neumeyer and González-Rozada, 2003).

BCORY (2009) set $\eta$ at 0.76 and calibrate the rest of the parameters of the model using values that are plausible for the typical emerging economy. They then compute both the competitive
equilibrium and the social planner’s equilibrium. The latter is, as before, the solution to a maximization problem in which the endogeneity of the wage rate, \( w_t \), on the right-hand side of (11.57) is internalized. BCORY (2009) find that the competitive equilibrium yields a lower average debt-to-output ratio and a smaller probability that the borrowing constraint will bind than the social planner’s equilibrium. That is, the model generates underborrowing.

### 11.2.4 Discussion

We close this section by reiterating that in models in which borrowing is limited by a collateral constraint, the emergence of overborrowing depends crucially on what objects are assumed to serve as collateral. This adds an element of discretionality to the analysis because in the class of models reviewed here the collateral constraint is assumed in an ad hoc fashion. This conclusion suggests two priorities for future research in this area. One is empirical and should aim at answering a simple question: what do foreign lenders accept as collateral from borrowers residing in emerging countries? The second line of research suggested as important by the current state of the overborrowing literature is theoretical. It concerns the production of microfoundations for the type of borrowing constrains analyzed in this chapter. The hope here is that through such foundations we will be able to narrow the type of items that can sensibly be included on the right-hand side of the borrowing constraint.
Exercise 11.1 (The Temporariness Hypothesis With Downward Nominal Wage Rigidity)

Consider a small open perfect-foresight economy populated by a large number of identical infinitely lived consumers with preferences described by the utility function

$$\sum_{t=0}^{\infty} \beta^t \ln c_t,$$

where $c_t$ denotes consumption and $\beta \in (0, 1)$ denotes the subjective discount factor. Consumption is a composite good made of imported and nontradable goods, denoted $c_t^M$ and $c_t^N$ respectively, via the aggregator function

$$c_t = \sqrt{c_t^M c_t^N}.$$

The sequential budget constraint of the representative household is given by

$$d_{t}^h = (1 + r)d_{t-1}^h + (1 + \tau_t)(c_t^M + p_t c_t^N) - w_t h_t - y - x_t,$$

where $d_t^h$ denotes debt acquired in period $t$ and maturing in $t + 1$, $h_t$ denotes hours worked, $\tau_t$ is a proportional consumption tax, $w_t$ denotes the real wage in terms of importables, $p_t$ denotes the relative price of nontradables in terms of importables, $y = 1$ is an endowment of exportable goods, $x_t$ denotes a lump-sum transfer received from the government, and $r$ denotes the real interest rate. Debt is denominated in terms of importables. The terms of trade are assumed to be constant and normalized to unity. Households are subject to the no-Ponzi-game constraint

$$\lim_{j \to \infty} \frac{d_{t+j}^h}{(1 + r)^j} \leq 0.$$
Assume that the household’s initial debt position is nil ($d_{-1}^h = 0$). Households supply inelastically 1 unit of labor to the market each period. Suppose that the law of one price holds for importables, so that $P_t^M = P_t^{M*}E_t$, where $P_t^M$ denotes the domestic-currency price of importables, $E_t$ denotes the nominal exchange rate, defined as the price of foreign currency in terms of domestic currency, and $P_t^{M*}$ denotes the foreign-currency price of importables. Assume that $P_t^{M*}$ is constant and equal to unity for all $t$.

Firms in the nontraded sector produce goods by means of the linear technology $y_t^N = h_t$, where $y_t^N$ denotes output of nontradables. Firms are price takers in product and labor markets and there is free entry, so that all firms make zero profits at all times.

The government starts period 0 with no debt or assets outstanding and runs a balanced budget period by period, that is, $x_t = \tau_t (c_t^M + p_t c_t^N)$. The monetary authority pegs the exchange rate at unity, so that $E_t = 1$ for all $t$. Finally, assume that $1 + r = \beta^{-1} = 1.04$.

1. Suppose that nominal wages are flexible and that before period 0 the economy was in a steady state with constant consumption of importables and nontradables and no external debt.

(a) Compute the equilibrium paths of $c_t^M$, $w_t$, $W_t$, $p_t$, the trade balance, and the current account under two alternative tax policies:

Policy 1: $\tau_t = 0, \forall t$

Policy 2: $\tau_t = \begin{cases} 
0 & 0 \leq t \leq 11 \\
0.3 & t \geq 12 
\end{cases}$

(b) Compute the welfare cost of policy 2 relative to policy 1, defined as the percentage increase in the consumption stream of a consumer living under policy 2 required to make him as well off as living under policy 1. Formally, the welfare cost of policy 2 relative to policy
1 is given by $\lambda \times 100$, where $\lambda$ is implicitly given by

$$\sum_{t=0}^{\infty} \beta^t \ln[c^2_t (1 + \lambda)] = \sum_{t=0}^{\infty} \beta^t \ln c^1_t,$$

where $c^1_t$ and $c^2_t$ denote consumption in period $t$ under policies 1 and 2, respectively.

2. Now answer question 1 under the assumption that wages are downwardly inflexible. Specifically, assume that $W_t \geq W_{t-1}$ for all $t \geq 0$. Begin by computing $W_{-1}$ under the assumption that before period 0 the economy was in a steady state with constant consumption of tradables and nontradables, full employment, no debt, and a nominal exchange rate equal to unity.

3. How would your answers to questions 1 and 2 change if $\tau_t$ was a tax only on consumption of importables (i.e., an import tariff)? Provide a quantitative and intuitive answer.

4. Continue to assume that $\tau_t$ is a tax on consumption of imported goods only. In section 11.1, in the context of a model with tradable goods only, we derived the result that policy 2 induces the same equilibrium paths of consumption, $c_t$, as policy 1 in the presence of imperfect credibility, that is, when the government announces and successfully implements policy 1 but households believe that policy 2 is in place and begin to believe in the permanence of the tariff cut only after period 11. How does this result change in the present model with tradables and nontradables? Consider separately the cases of flexible and downwardly rigid nominal wages.
Chapter 12

Sovereign Debt

Why do countries pay their international debts? This is a fundamental question in open-economy macroeconomics. A key distinction between international and domestic debts is that the latter are enforceable. Countries typically have in place domestic judicial systems capable of punishing defaulters. Thus, one reason why residents of a given country honor their debts with other residents of the same country is because creditors are protected by a government able and willing to apply force against delinquent debtors. At the international level the situation is quite different. For there is no such thing as a supranational authority with the capacity to enforce financial contracts between residents of different countries. Defaulting on international financial contracts appears to have no legal consequences. If agents have no incentives to pay their international debts, then lenders should have no reason to lend internationally to begin with. Yet, we do observe a significant amount of borrowing and lending across nations. It follows that international borrowers must have reasons to repay their debts other than pure legal enforcement.

Two main reasons are typically offered for why countries honor their international debts: economic sanctions and reputation. Economic sanctions may take many forms, such as seizures of
debtor country’s assets located abroad, trade embargoes, and import tariffs and quotas. \footnote{The use of force by one country or a group of countries to collect debt from another country was not uncommon until the beginning of the twentieth century. In 1902, an attempt by Great Britain, Germany, and Italy to collect the public debt of Venezuela by force prompted the Argentine jurist Luis-María Drago, who at the time was serving as minister of foreign affairs of Argentina, to articulate a doctrine stating that no public debt should be collected from a sovereign American state by armed force or through the occupation of American territory by a foreign power. The Drago doctrine was approved by the Hague Conference of 1907.} Intuitively, the stronger is the ability of creditor countries to impose economic sanctions, the weaker the incentives for debtor countries to default.

A reputational motive to pay international debts arises when creditor countries have the ability to exclude from international financial markets countries with a reputation of being defaulters. Being isolated from international financial markets is costly, as it precludes the use of the current account to smooth consumption over time in response to aggregate domestic income shocks. As a result, countries may choose to repay their debts simply to preserve access to international financing.

This chapter investigates whether the existing theories of sovereign debt are capable of explaining the observed levels of sovereign debt. Before plunging into theoretical models of country debt, however, we will present some stylized facts about international lending and default that will guide us in evaluating the existing theories.

### 12.1 Empirical Regularities

In this section, we take a look at the observed patterns of sovereign defaults and their relation to macroeconomic indicators of interest. We draw on existing empirical research and also provide some new evidence.

#### 12.1.1 Frequency And Length of Defaults

How often do countries default? How long do countries take to resolve their debt disputes? To address these questions, we must first establish empirical definitions of default and of its resolution.
Much of the data on sovereign default is produced by credit rating agencies, especially Standard and Poor’s. Standard and Poor’s defines default as the failure to meet a principal or interest payment on the due date (or within a specified grace period) contained in the original terms of a debt issue (Beers and Chambers, 2006). This definition includes not only situations in which the sovereign simply refuses to pay interest or principal, but also situations in which it forces an exchange of old debt for new debt with less-favorable terms than the original issue or it converts debt into a different currency of less than equivalent face value.

A country is considered to have emerged from default when it resumes payments of interest and principal including arrears. But defaults often involve a debt renegotiation that culminates in a restructuring of debt contracts that may include the swap of old debt for new debt. For this reason, when such a settlement occurs and the rating agency concludes that no further near-term resolution of creditors’ claims is likely, the sovereign is regarded as having emerged from default (Beers and Chambers, 2006). This definition of reemergence from default clearly requires a value judgment as it involves the expectation on the part of the rating agency that no further disputes will emerge in relation to the default in question. Such a judgment call may or may not turn out to be correct. For example, all rating agencies concluded that Argentina emerged from the 2001 default in 2005 when it restructured its 81.8 billion dollar debt by issuing new instruments involving a haircut of 73 percent. However, in 2014 a small group of holdouts (i.e., bondholders that did not participate in the debt restructuring) won a law suit against Argentina in the United States. The country’s refusal to comply with this ruling put it back into default status.

Table 12.1 displays empirical probabilities of default for 9 emerging countries over the period 1824-2014. On average, the probability of default is about 3 percent per year. That is, countries defaulted on average once every 33 years. This empirical probability is computed by dividing the number of years in which default events occurred by the number of years in the sample (191 years). A related measure replaces the denominator in this ratio by the number of years in the sample in
<table>
<thead>
<tr>
<th>Country</th>
<th>Probability of Default</th>
<th>Years in State of Default per Default Episode</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>all years</td>
<td>years not in default</td>
</tr>
<tr>
<td>Argentina</td>
<td>0.026</td>
<td>0.035</td>
</tr>
<tr>
<td>Brazil</td>
<td>0.037</td>
<td>0.047</td>
</tr>
<tr>
<td>Chile</td>
<td>0.016</td>
<td>0.020</td>
</tr>
<tr>
<td>Colombia</td>
<td>0.037</td>
<td>0.058</td>
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<tr>
<td>Egypt</td>
<td>0.011</td>
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</tr>
<tr>
<td>Mexico</td>
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<td>0.056</td>
</tr>
<tr>
<td>Philippines</td>
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<td>0.006</td>
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<tr>
<td>Turkey</td>
<td>0.032</td>
<td>0.037</td>
</tr>
<tr>
<td>Venezuela</td>
<td>0.053</td>
<td>0.079</td>
</tr>
<tr>
<td>Average</td>
<td>0.029</td>
<td>0.039</td>
</tr>
</tbody>
</table>

Note: The sample includes only emerging countries with at least one external-debt default or restructuring episode between 1824 and 1999. The 2014 selective default of Argentina with 1 percent of the holdout investors that did not enter the debt restructuring of 2005 and 2010 is not counted as a default event. Source: Own calculations based on table 12.19 and Reinhart, Rogoff, and Savastano (2003), table 1.
which the country is not in default status. This measure, which by construction delivers higher
default probabilities, is given in the second column of the table. On average the default probability
for the 9 countries included in the table conditional on the country being in no-default standing
is 4 percent. The difference between these two default probabilities can be sizeable (as large as 2
percent) because when a country defaults it may remain in default status for a significant number
of years.

The average number of years countries are in state of default or restructuring after a default
event is 11 years. If one assumes that while in state of default countries have limited access to
fresh funds from international markets, one would conclude that default causes countries to be in
financial autarky for about a decade. But the connection between being in state of default and
being in financial autarky should not be taken too far. For being in state of default with one set
of lenders, does not necessarily preclude the possibility of obtaining new loans from other lenders
with which the borrower has no unpaid debts. In this case, the period of financial autarky would
be shorter than the period of being in state of default. The converse can also be true. Suppose that
foreign lenders choose to punish defaulters by excluding them from financial markets even after
the delinquent country has come to an agreement with its creditors. In this case, the period of
financial autarky could last longer than the period in default status. We discuss empirical estimates
of exclusion periods in section 12.2.1.

The information on frequency of default and length of default state provided in table 12.1 spans
a long period of time (from 1824 to 2014). In the past decades, however, international financial
markets have experienced enormous changes, including an expansion in the set of sovereigns with
access to international credit markets and the participation of small lenders. For this reason, it is
of interest to ask if the frequency and length of sovereign defaults have changed. Table 12.19 in
the appendix displays data on the beginning and end of default episodes for all defaulters between
1975 and 2014. During this period 93 sovereigns defaulted at least once and there were a total
of 147 default episodes. This means that the empirical probability of default over the 40-year period 1975-2014 equals 147/(40 × 93) or around 4 percent. Comparing this number with the one corresponding to the period 1824 to 2014 suggests that the default frequency has increased from 3 defaults per century per country to 4 defaults per country, conditional on the country having defaulted at least once. Table 12.19 also reveals that the average length of a default episode in the more recent sample period is 8 years, 3 years shorter than during the entire sample. We therefore conclude that in recent years, sovereign defaults have become more frequent but shorter. Table 12.2 summarizes these results.

Figure 12.1 displays the empirical probability distribution of the length of default episodes over the period 1975 to 2014. The distribution is strongly skewed to the right, which means that there are some default episodes that took very long to be resolved. For example, table 12.19 documents that out of the 147 defaults recorded over the period 1975-2014, nine lasted longer than 30 years. The skewness of the distribution is reflected in a median of default length significantly lower than the mean, 5 versus 8 years. As pointed out by Tomz and Wright (2013), the shape of the empirical probability distribution resembles an exponential distribution. Assuming that the length of default episodes is a good proxy for the time of exclusion from international financial markets (for further discussion of this assumption, see section 12.2.1), this suggests modeling the

---


<table>
<thead>
<tr>
<th>Period</th>
<th>Probability of Default per year</th>
<th>Years in State of Default per Default Episode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1824-2014</td>
<td>0.029</td>
<td>11</td>
</tr>
<tr>
<td>1975-2014</td>
<td>0.040</td>
<td>8</td>
</tr>
</tbody>
</table>

Source: Tables 12.1 and 12.19.
probability of a defaulter regaining access to financial markets as constant over time. As we will see in section 12.6, this is precisely the way reentry is modeled in most quantitative models of sovereign default.

### 12.1.2 Haircuts

How large are defaults? Most existing theoretical models of default assume that when the country defaults it does so on the entire stock of outstanding external debt. In reality, however, this is not the case. Typically, countries default on a fraction of their outstanding debts. The resulting losses inflicted to creditors are called haircuts.

Sturzenegger and Zettelmeyer (2008) measure haircuts as the percentage difference between the present values of old and new instruments discounted at market rates prevailing immediately after the debt exchange. They estimate haircuts for all major debt restructurings that occurred between
1998 and 2005. They find that haircuts are on average 40 percent. That is, after the default the creditor expects to receive a stream of payments with present discounted value 40 percent lower than prior to the default. At the same time, they report a high dispersion of the size of haircuts ranging from 13 percent (Uruguay, 2003) to 73 percent (Argentina, 2005).

Other studies find similar results. Cruces and Trebesch (2013) use the same methodology to measure haircuts as Sturzenegger and Zettelmeyer but greatly expand the data set to include all debt restructurings with foreign banks and bond holders that took place between 1970 and 2010. The resulting data set covers 180 default episodes in 68 countries. They find that the average haircut is 37 percent, with a standard deviation of 22 percent. Benjamin and Wright (2008) use a constant rate of 10 percent to discount pre- and post-restructuring payments in their computation of haircuts. Their data set includes 90 default episodes in 73 countries over the period 1989 to 2006. Like the other two studies, these authors find an average haircut of 40 percent with a large associated dispersion.

12.1.3 Debt And Default

Table 12.3 displays average debt-to-GNP ratios over the period 1970-2000 for a number of emerging countries that defaulted upon or restructured their external debt at least once between 1824 and 1999. The table also displays average debt-to-GNP ratios at the beginning of default or restructuring episodes. The data suggest that at the time of default debt-to-GNP ratios are significantly above average. In effect, for the countries considered in the sample, the debt-to-GNP ratio at the onset of a default or restructuring episode was on average 14 percentage points above normal times. The information provided in the table is silent, however, about whether the higher debt-to-GNP ratios observed at the brink of default episodes result from a contraction in aggregate activity or from a faster-than-average accumulation of debt in periods immediately preceding default or both.
Table 12.3: Debt-to-GNP Ratios Among Defaulters

<table>
<thead>
<tr>
<th>Country</th>
<th>Average Debt-to-GNP Ratio</th>
<th>Debt-to-GNP Ratio in Year of Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argentina</td>
<td>37.1</td>
<td>54.4</td>
</tr>
<tr>
<td>Brazil</td>
<td>30.7</td>
<td>50.1</td>
</tr>
<tr>
<td>Chile</td>
<td>58.4</td>
<td>63.7</td>
</tr>
<tr>
<td>Colombia</td>
<td>33.6</td>
<td></td>
</tr>
<tr>
<td>Egypt</td>
<td>70.6</td>
<td>112.0</td>
</tr>
<tr>
<td>Mexico</td>
<td>38.2</td>
<td>46.7</td>
</tr>
<tr>
<td>Philippines</td>
<td>55.2</td>
<td>70.6</td>
</tr>
<tr>
<td>Turkey</td>
<td>31.5</td>
<td>21.0</td>
</tr>
<tr>
<td>Venezuela</td>
<td>41.3</td>
<td>46.3</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td><strong>44.1</strong></td>
<td><strong>58.1</strong></td>
</tr>
</tbody>
</table>

Notes: The sample includes only emerging countries with at least one external-debt default or restructuring episode between 1824 and 1999. Debt-to-GNP ratios are averages over the period 1970-2000. Debt-to-GNP ratios at the beginning of a default episode are averages over the following default dates in the interval 1970-2002: Argentina 1982 and 2001; Brazil 1983; Chile 1972 and 1983; Egypt 1984; Mexico 1982; Philippines 1983; Turkey 1978; Venezuela 1982 and 1995. Colombia did not register an external default or restructuring episode between 1970 and 2002.

Source: Own calculations based on Reinhart, Rogoff, and Savastano (2003), tables 3 and 6.
Table 12.4: Country Premia Among Defaulters

<table>
<thead>
<tr>
<th>Country</th>
<th>Average Country Spread</th>
<th>All Years Not in Default</th>
<th>Years In Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argentina</td>
<td>15.8</td>
<td>7.43</td>
<td></td>
</tr>
<tr>
<td>Brazil</td>
<td>5.61</td>
<td>5.61</td>
<td></td>
</tr>
<tr>
<td>Chile</td>
<td>1.44</td>
<td>1.44</td>
<td></td>
</tr>
<tr>
<td>Colombia</td>
<td>3.70</td>
<td>3.70</td>
<td></td>
</tr>
<tr>
<td>Egypt</td>
<td>2.46</td>
<td>2.46</td>
<td></td>
</tr>
<tr>
<td>Mexico</td>
<td>3.47</td>
<td>3.47</td>
<td></td>
</tr>
<tr>
<td>Philippines</td>
<td>3.49</td>
<td>3.49</td>
<td></td>
</tr>
<tr>
<td>Turkey</td>
<td>4.10</td>
<td>4.10</td>
<td></td>
</tr>
<tr>
<td>Venezuela</td>
<td>9.24</td>
<td>9.23</td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>5.5</td>
<td>4.5</td>
<td></td>
</tr>
</tbody>
</table>

Notes: The sample includes only emerging countries with at least one external-debt default or restructuring episode between 1824 and 1999. Country spreads are measured using the EMBI Global index, produced by J.P. Morgan, and expressed in percent, and are averages through 2013, with varying starting dates as follows: Argentina 1994; Brazil 1995; Chile 2000; Colombia 1998; Egypt 2002; Mexico 1994; Philippines 1998; Turkey 1997; Venezuela 1994. Average country spreads are computed over all available periods (column ‘All Years’) and over all periods during which the country is not in default (column ‘Years Not in Default’). Start and end dates of default episodes are taken from Table 12.19 in the appendix to this chapter.

12.1.4 Country Premia

Table 12.4 displays average country premia over a period starting on average in 1996 and ending in 2013. The country premium, or country spread, is the difference between the interest rate at which a country borrows in international financial markets and the interest rate at which developed countries borrow from one another. The average country premium for the 9 countries included in the table is 5.5 percent per year.

Country spreads are much higher during periods in which countries are in default status than
during periods of good financial standing. For this reason, it is of interest to examine country spreads conditional on countries not being in default status. The sample period covered in table 12.4 does contain a number of default events. Specifically, Argentina was in default from 2001 to 2005 and Venezuela was in default from 1995 to 1998 and then again in 2005. Thus, column 2 of the table reports average spreads conditional on the country not being in default status. The corrected spreads are uniformly higher. For example, the Argentine spread was about half the size during periods not in default than over the entire sample (7.4 versus 15.8 percent).

12.1.5 Country Spreads And Default Probabilities: A Sample Mismatch Problem

Country spreads reflect default probabilities. In a world in which lenders are risk neutral, spreads should be on average equal to the probability of default. With partial default, spreads should be lower than default probabilities. Spreads in excess of default probabilities are hence an indication of risk aversion on the part of foreign lenders. For this reason, it is of interest to characterize empirically the spread-default-frequency differential. Later in section 12.9 we will revisit this issue from a theoretical perspective.

Computing the spread-default-frequency differential is not an easy task. The problem is that spreads and default data come in different sample sizes. For most countries data on interest rate spreads is available only since the early 1990s. By contrast, data on default events is quite long, going back to the early 19th century. As a result, often the empirical regularities involving interest rate spreads and default frequency are based on a mix of short samples for the former and long samples for the latter. This is the case, for instance, when we compare the default frequencies reported in table 12.1 with the average country spreads reported in table 12.4.

This sample mismatch can sometimes lead to spurious conclusions regarding the relation between default frequency and average country spreads. For example, for Colombia table 12.1 shows
a default frequency of 5.8 percent and table 12.4 indicates that the average country spread is 3.52 percent, that is, the default frequency exceeds the country premium. The high default frequency obtains because Colombia defaulted quite frequently, 7 times, between 1824 and 1935, but has not defaulted since then. At the same time, EMBI spread data for this country covers only the period 1997 to 2013, which contains no defaults and which is preceded by more than 60 years of good financial standing. It is reasonable to suspect that the sample mismatch introduces a negative bias in the spread-default-frequency differential. A similar problem is likely to affect Chile and Mexico. These two countries were quite unstable and displayed frequent economic crises until the end of the 1980s. Since then, however, both have implemented a number of structural reforms, including trade, fiscal, and monetary policy. Progress along this dimension has been widely recognized, as reflected, for instance, in both countries’ accession to the Organization for Economic Cooperation and Development (OECD). Not surprisingly, country spreads have been quite low in Chile and Mexico since the early 1990s, which is precisely the period spanned by the EMBI data set. On the other hand, measures of default frequency based on long samples put in one bag periods of high and low instability. As a result, both Chile and Mexico display smaller country spreads than default probabilities (1.5 versus 2 percent for Chile and 3.5 versus 5.6 percent for Mexico). Again, in both of these countries, the sample mismatch is likely to bias downwardly the spread-default-frequency differential.

To provide a partial correction to this problem, table 12.5 displays empirical default frequencies computed over a country-specific sample given by the period for which both the EMBI and default event data is available. Under this correction the average country spread is on average more than twice as large as the default frequency, 4.5 versus 2.2 percent. A weakness of this way of comparing spreads and default frequencies is the scarcity of data. It is therefore important to continue to reassess the validity of the stylized fact that country premia are significantly larger than default probabilities as more data become available.
### Table 12.5: Default Probability And Country Spreads Over A Common Sample

<table>
<thead>
<tr>
<th>Country</th>
<th>Spread (percent)</th>
<th>Probability (percent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argentina</td>
<td>7.43</td>
<td>6.7</td>
</tr>
<tr>
<td>Brazil</td>
<td>5.61</td>
<td>0</td>
</tr>
<tr>
<td>Chile</td>
<td>1.44</td>
<td>0</td>
</tr>
<tr>
<td>Colombia</td>
<td>3.70</td>
<td>0</td>
</tr>
<tr>
<td>Egypt</td>
<td>2.56</td>
<td>0</td>
</tr>
<tr>
<td>Mexico</td>
<td>3.47</td>
<td>0</td>
</tr>
<tr>
<td>Philippines</td>
<td>3.49</td>
<td>0</td>
</tr>
<tr>
<td>Turkey</td>
<td>4.10</td>
<td>0</td>
</tr>
<tr>
<td>Venezuela</td>
<td>9.23</td>
<td>13.3</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td><strong>4.5</strong></td>
<td><strong>2.2</strong></td>
</tr>
</tbody>
</table>

Source. Own calculations based on tables 12.4 and 12.19.

### 12.1.6 Do Countries Default In Bad Times?

An important question in the theoretical literature on default is whether countries tend to dishonor their debt obligations during economic expansions or contractions. The reason is that different models produce opposite predictions in this regard. As a preview of the theory to come, consider the following two simple examples. Suppose first that a country that wishes to smooth consumption signs a contract with foreign investors that is state contingent. Specifically, suppose that the contract specifies that the country receives a transfer from the rest of the world if domestic output is below average and makes a payment to the rest of the world if domestic output is above average. Clearly, under this contract, incentives to default are highest when output is above average, since these are the states in which the country must make payments. The second example is one in which the country cannot sign state-contingent contracts. Instead, suppose that the sovereign borrows internationally and has to pay next period the amount borrowed plus a fixed amount in interest...
regardless of the state of the domestic economy next period. In this case, the incentive to default is likely to be strongest when output is low, because the cost of sacrificing consumption to service the debt is higher when consumption is already low due to the weak level of domestic output. Which of these two examples is favored by the data?

Figure 12.2 displays the typical behavior of output around the default episodes listed in table 12.19. Output is measured as percent deviations of real GDP per capita from a log-quadratic trend. The figure displays a window of 3 years before and after a default episode. The year of default is normalized to 0. The typical behavior of output around a default episode is captured by computing the median of output period by period across the default episodes. The figure shows that defaults typically occur after long and severe economic contractions. Specifically, the typical country experiences a 6.5 percent contraction in output per capita in the 3 years leading up to default. This result suggests that the answer to the question posed in the title of this subsection is yes, countries default in bad times.

Some authors, however, arrive at a different conclusion. For example, Tomz and Wright (2007, 2013), using data from 1820 to 2005, argue that countries do default during bad times, but that the supporting evidence is weak. Tomz’s and Wright’s argument is based on two observations. First, they find that at the time of default, output is only about 1.5 percent below trend. This finding is in line with the results shown in figure 12.2. However, note that by the time the economy reaches the period of default, period 0 in the figure, it has contracted by more than 5 percent between periods -3 and -1. The second reason why Tomz and Wright argue that the evidence that countries default in bad times is weak is their finding that only 60 percent of the default episodes occur when output is below trend. This finding is also corroborated in the sample considered here. However, if
Note: Output is measured as real per capita GDP, log quadratically detrended at annual frequency. Median across the default episodes listed in table 12.19. Countries with less than 30 consecutive years of output data were excluded, resulting in 105 default episodes over the period 1975-2014. Source: Output is from World Development Indicators, and default dates are from table 12.19.
one asks the question of what fraction of the countries were contracting (i.e., experiencing output growth below trend growth) at the time of default, the answer is quite different. We find that 75 percent of the default episodes occur at a time in which output growth is below trend. This finding further strengthens the conclusion that defaults occur in times of significant economic distress.

Note that the graph in figure 12.2 flattens one period after default. This means that output growth returns to its long-run trend one year after default. Levy-Yeyati and Panizza (2011) were the first to identify the growth recovery regularity. We note, however, that even three years after the default, the level of output remains 4 percent below trend. The broad picture that emerges from figure 12.2 is that default marks the end of a large contraction, and the beginning of a growth recovery, albeit not the beginning of a recovery in the level of output.

Figure 12.3 shows that the conclusion that countries default in bad times extends to variables other than output. The figure displays the cyclical components of private consumption, gross investment, the trade-balance-to-GDP ratio, and the real effective exchange rate. Private consumption contracts by as much as output (about 6 percent) in the run up to default, and investment experiences a fall 3 times as large as output. At the same time, the trade balance is below average up until the year of default, when it experiences a reversal of about 2 percent. The bottom right panel displays the behavior of the real exchange rate. This variable is defined in such a way that an increase means a real appreciation (i.e., the economy in question becomes more expensive than its trading partners). The figure shows that the real exchange rate depreciates significantly, by more than 4 percent, in the year of default. After the default, the real exchange rate begins to gradually appreciate.
Figure 12.3: Consumption, Investment, The Trade Balance, and The Real Exchange Rate Around Default Episodes

Note: The data is annual. Consumption, investment, and the real exchange rate are log-quadratically detrended. The trade-balance-to-output ratio is linearly detrended. Median across all default episodes. Countries with less than 30 consecutive years of data were excluded. Data sources: Default dates are from table 12.19 and all other variables from World Development Indicators. An increase in the real exchange rate indicates a real appreciation of the domestic currency.
12.2 The Cost of Default: Empirical Evidence

The empirical literature has identified three main sources of costs associated with sovereign default: exclusion from financial markets, output losses, and international trade reductions.

12.2.1 Default and Exclusion From Financial Markets

Do defaulting sovereigns lose access to credit markets upon default? And, if so, how long does it take them to regain access? One possible approach to addressing this question is to assume that countries are excluded from credit markets as long as they are in default status. Under this approach, exclusion from credit markets begins in the year the default occurs and reaccess takes place in the year after the country reemerges from default. According to table 12.2, for the sample 1975-2014, the average length of exclusion from international financial markets was 8 years. As mentioned earlier, however, this approach is a bit naive, because in principle nothing prevents foreign lenders from extending fresh loans to countries before or after their default disputes are settled. For instance, it took Argentina 11 years to settle its 1982 default (see table 12.19). However, according to Gelos, Sahay, and Sandleris (2011) the country began to receive fresh external funds already in 1986. At the same time, and according to the same sources, Chile was in default status from 1983 to 1990, but was able to reaccess international credit markets only in 1994.

Thus, the arrival of new funds and the end of default status do not seem to always match, suggesting the importance of directly measuring the date in which countries are able to borrow again after a default. Gelos, Sahay, and Sandleris (2011) undertake this approach systematically by examining micro data on sovereign bond issuance and public syndicated bank loans over the period 1980-2000. As in most of the related literature, including this chapter, their measure of the year of default follows Beers and Chambers (2006). Their innovation is to measure resumption of access to credit markets as the first year after default in which the government borrows either in
the form of bonds or syndicated loans and the stock of debt increases. The latter requirement is aimed at avoiding counting as reaccess cases in which the country is simply rolling over an existing debt. It is not clear, however, that this requirement is appropriate. For the rolling over of existing debt is actually a manifestation of participation in international capital markets. Exclusion from financial markets is measured as the number of years between default and resumption. Gelos et al. find that the mean exclusion period is 4.7 years — about half the length obtained by using the date of settlement as a proxy for reaccess. However, this result is likely to be downwardly biased by the authors’ decision to include only default episodes associated with resumptions happening within the sample. That is, countries that defaulted between 1980 and 2000 and had not regained access to credit markets by 2000 were excluded from the calculations. The bias is likely to be strong because the sample includes equal numbers of resumptions and no resumptions before 2000.\footnote{The authors also report a significant drop in the exclusion period from 4 to 2 years between the 1980s and the 1990s. This result is likely to be particularly affected by the bias reported in the body of the text.} We therefore interpret the results reported in Gelos et al. as suggesting that the period of exclusion from financial markets resulting from sovereign default is at least 4 years.

Using data from 1980 to 2005, Richmond and Dias (2009) also study the issue of exclusion after default. Their methodology differs from Gelos et al. (2011) in three aspects. First, Richmond and Dias measure net borrowing using aggregate data. Second, and more importantly, they exclude from the definition of reaccess situations in which an increase in borrowing reflects the capitalization of arrears.\footnote{It is not clear that increases in net foreign government debt due to the capitalization of arrears is not the reflection of reaccess. For such situations may be the result of a successful negotiation between the lender and the debtor culminating in the ability of the latter to tap international markets again.} Third, they count years of exclusion starting the year the country emerged from default, as opposed to the year in which the country entered into default. Their definition of reaccess distinguishes the cases of partial and full reaccess. Partial reaccess is defined as the first year with positive aggregate flows to the public sector after the country has emerged from default. Full reaccess is defined as the first year in which debt flows exceeds 1 percent of GDP. They find that
on average countries regain partial access to credit markets 5.7 years after emerging from default and full access 8.4 years after emerging from default. To make these numbers comparable, with those reported by Gelos et al., we add the average number of years a country is in default, which according to table 12.2 is 8 years. This adjustment is reasonable because Richmond and Dias find that only a small fraction (less than 10 percent in their sample) of default episodes were associated with reaccess while the country was in default. Thus, according to the adjusted estimate, countries regain partial access to international credit markets 13.7 years after default and full access 16.4 years after default.

Cruces and Trebesch (2013) extend and combine the data and criteria for market exclusion of Gelos et al. (2011) and Richmond and Dias (2009). Their data on reaccess covers the period 1980 to 2010. They find that on average countries regain partial access 5.1 years after emerging from default and full access 7.4 years after emerging from default. Introducing the same adjustment we applied to the Richmond and Dias estimates to measure exclusion from the beginning of the default episode, the estimates of Cruces and Trebesch imply that it takes defaulters on average 13.1 years to regain partial access and 15.4 years to gain full access after default. Table 12.6 summarizes the results of the estimates of exclusion we have discussed.

The studies covered above also analyze the determinants of the length of exclusion after default. Gelos et al. (2011) find that the frequency of default is not a significant determinant of the length of exclusion. That is, markets seem to punish more or less equally one-time and serial defaulters. This finding lends support to an assumption along these lines maintained by most existing theories of sovereign default. Gelos et al. also find that defaults that resolve quickly do not result in significant exclusion. Richmond and Dias (2009) find that excusable defaults (such as those following a natural disaster) are associated with reduced exclusion periods. Cruces and Trebesch (2013) provide evidence suggesting that lenders may use exclusion as a punishment, by documenting that restructurings involving higher hair cuts (i.e., higher losses to creditors) are associated with significantly
### Table 12.6: Estimates Of Years Of Exclusion From Credit Markets After Default

<table>
<thead>
<tr>
<th>Measure</th>
<th>Partial Reaccess (flows&gt; 0)</th>
<th>Full Reaccess (flows&gt; 1%GDP)</th>
<th>Sample Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of Default Status (table 12.19)*</td>
<td>8</td>
<td>1975-2014</td>
<td></td>
</tr>
<tr>
<td>First Issuance of New Debt</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>- Richmond and Dias (2009)**</td>
<td>5.7</td>
<td>8.4</td>
<td>1980-2005</td>
</tr>
<tr>
<td>- Adjusted Richmond and Dias (2009)*</td>
<td>13.7</td>
<td>16.4</td>
<td>1980-2005</td>
</tr>
<tr>
<td>- Cruces and Trebesch (2013)**</td>
<td>5.1</td>
<td>7.4</td>
<td>1980-2010</td>
</tr>
<tr>
<td>- Adjusted Cruces and Trebesch (2013)*</td>
<td>13.1</td>
<td>15.4</td>
<td>1980-2010</td>
</tr>
<tr>
<td>Average</td>
<td>9.8</td>
<td>15.9</td>
<td></td>
</tr>
</tbody>
</table>

Note. Reaccess is measured in years. after the beginning of default (*) or in years after the end of default (**). Averages are taken over single-star lines.

longer periods of capital market exclusion.

We close this discussion by pointing out that the existing attempts to measure exclusion do not incorporate distinctly supply and demand determinants of external credit. Since observed variations in debt are equilibrium outcomes, the lack of increase in debt need not be a reflection of supply restrictions related to sanctions. In this regard, the studies reviewed above may incorporate an upward bias in the estimates of the numbers of years a defaulter is excluded from international financial markets.

#### 12.2.2 Output Costs Of Default

A standard assumption in theoretical models of sovereign debt is that default entails an output loss (see section 12.6). This assumption helps the model economy to sustain a higher level of external debt in equilibrium. A number of authors have attempted to empirically estimate this cost (see, for example, Chuhan and Sturzenegger, 2005; Borensztein and Panizza, 2009; De Paoli,
Hoggarth, and Saporta, 2011; and Levy Yeyati and Panizza, 2011). The typical approach is to use cross country panel data to run a standard growth regression augmented with variables capturing default. Borensztein and Panizza (2009), for example, estimate the following regression using data from 83 countries over the period from 1972 to 2000,

$$\text{Growth}_{it} = \alpha + \beta X_{it} + \gamma \text{Default}_{it} + \sum_{j=0}^{3} \delta_j \text{DefaultB}_{it-j} + \epsilon_{it}$$

where $\text{Growth}_{it}$ denotes the growth rate of real per capital GDP in country $i$ from year $t-1$ to year $t$ in percent, $X_{it}$ denotes a vector of controls typically used in growth regressions, $\text{Default}_{it}$ is a dummy variable taking the value 1 if country $i$ is in default in year $t$ and zero otherwise, $\text{DefaultB}_{it}$ is a dummy variable taking the value 1 if country $i$ entered in default in period $t$, and $\epsilon_{it}$ is an error term. As in table 12.19, the dates of entering and exiting default are based on data from Standard and Poor’s.

Borensztein and Panizza estimate $\gamma = -1.184$ and $\delta_i = -1.388, 0.481, 0.337, 0.994$ for $i = 0, 1, 2, 3$, respectively, with $\gamma$ and $\delta_0$ significant at confidence level of 5% or less. This estimate implies that the beginning of a default is accompanied by a 2.6 percent fall in the growth rate of output ($\gamma + \delta_0$). Subsequently, the growth rate recovers. However, the level of output never recovers, implying that default is associated with a permanent loss of output.

Figure 12.4 displays with a broken line the logarithm of per capita output after a default implied by the Borensztein and Panizza regression. In the figure, the default is assumed to last for 5 years, the median length of the defaults reported in table 12.19. The long-run growth rate is assumed to be 1.5 percent, and the default date is normalized to 0. For comparison, the figure displays with a

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6 The variables included in $X_{it}$ are investment divided by GDP, population growth, GDP per capita in 1970, percentage of the population that completed secondary education, total population, lagged government consumption over GDP, an index of civil rights, the change in terms of trade, trade openness (defined as exports plus imports divided by GDP), a dummy variable taking a value of one in presence of a banking crisis, and three regional dummies (for sub-Saharan Africa, Latin America and Caribbean, and transition economies).
solid line the trajectory of output absent a default. After an initial fall, output gradually regains its long-run growth rate of 1.5 percent. However, the level of output remains forever 5.5 percent below the pre-default trajectory.

Taken at face value, the above regression results suggest an enormous cost of default. But they are likely to represent an upwardly biased estimate of the output cost of default for two reasons. First, the regression may include an insufficient number of lags in the default variables Default\(_{it}\) and Default\(_{B_{it}}\). The former variable actually appears only contemporaneously, and the latter with 3 lags. To the extent that the coefficients associated with these variables are positive, the gap between the no-default trajectory and the default trajectory could be narrowed. Thus, adding more lags could be important even if they are individually estimated with low significance. Second, and more fundamentally, output growth and the default decision are endogenous variables, which may introduce a bias in the coefficients of the default variables. For instance, if defaults tend to occur during periods of low growth, then the estimated coefficient in the default variables may be
negative even if default had no effect on growth. Thus, as stressed by Borensztein and Panizza (2009) and others, regression results of the type presented here should be interpreted as simply documenting a partial correlation between output growth and default.

Zarazaga (2012) proposes a growth accounting approach to gauge the output loss associated with default. He documents that the Argentine defaults of 1982 and 2001 were both characterized by a peak in the capital-output ratio in the run up to the default, followed by a significant decline in the years after the default (see figure 12.5). For instance, by 2002 the capital-output ratio had reached 1.9. Indeed, Zarazaga argues that a value of around 1.9 or higher is a normal long-run level for the capital-output ratio in emerging economies. Thus, his argument goes, absent any crisis, the capital output ratio should have remained at 1.9 or higher after 2002. However, after the
default the capital-output ratio fell, reaching a trough of 1.35 in 2007. What is the fall in output per person associated with this loss of capital per unit of final production? Zarazaga assumes a production function of the form \( y_t = k_t^{0.4} h_t^{0.6} \), where \( y_t \) denotes output, \( k_t \) denotes physical capital, and \( h_t \) denotes employment. This technology implies that the output-to-worker ratio, \( y_t/h_t \), is linked to the capital-to-output ratio, \( k_t/y_t \), by the relationship \( y_t/h_t = (k_t/y_t)^{2/3} \). This means that if the capital-output ratio had not fallen between 2002 and 2007, that is, if \( k_t/y_t \) had not fallen from 1.9 to 1.35, output per worker in 2007 would have been 26 percent higher than it actually was \( ([1.9/1.35]^{2/3} - 1) \times 100 \). Thus, on average, between 2002 and 2007, output per worker was 13\((= 26/2)\) percent lower than it would have been had the capital output ratio not fallen. If one ascribes all of the fall in the capital-to-output ratio observed between 2002 and 2007 to the sovereign default of 2002, then one would conclude that the output cost of the default was 13 percent each year between 2002 and 2007. Further, because it takes time for the capital-output ratio to recover its long-run level, the 13 percent loss should continue for more years. Assuming that the recovery is as fast as the decline, the cost should be extended for another 5 years, that is, until 2012. In summary, this accounting suggests that the Argentine default of 2002 had an output cost of 13 percent per year per worker for 10 years, which is quite large.

The 1982 default is not as clear cut as the default of 2001 because the capital-output ratio did not decline until late in the exclusion period (1982 to 1993). The capital-output ratio stayed at around 1.9 until 1990 and then fell to about 1.45 by 1996. It then took until 2002 to reach 1.9 again. Calculating the output loss following the same strategy applied to the 2002 default, we find that the output loss was 0 percent between 1982 and 1990 (8 years) and around 10 percent between 1990 and 2002 (12 years). On average, the output loss associated with the 1982 default, was therefore 6 percent of output per worker per year for 20 years, which is also a large number.

It is important to keep in mind that these cost estimates hinge on the assumption that the entire decline in the capital-output ratio was caused by the defaults. To the extent that the fall in
the capital-output ratio was in part driven by factors other than default, the cost estimates must be interpreted as an upper bound.

12.2.3 Trade Costs of Default

Default episodes are also associated with disruptions in international trade. Rose (2005) investigates this issue empirically. The question of whether default disrupts international trade is of interest because if for some reason trade between two countries is significantly diminished as a result of one country defaulting on its financial debts with other countries, then maintaining access to international trade could represent a reason why countries tend to honor their international financial obligations. Rose estimates an equation of the form

$$\ln T_{ijt} = \beta_0 + \beta X_{ijt} + \sum_{m=0}^{M} \phi_m ACRED_{ijt-m} + \epsilon_{ijt},$$

where $T_{ijt}$ denotes the average real value of bilateral trade between countries $i$ and $j$ in period $t$. Rose identifies default with dates in which a country enters a debt restructuring deal with the Paris Club. The Paris Club is an informal association of creditor-country finance ministers and central bankers that meets to negotiate bilateral debt rescheduling agreements with debtor-country governments. The regressor $ACRED_{ijt}$ is a proxy for default. It is a binary variable taking the value 1 if one of the countries in the pair $(i, j)$ is involved with the other in a Paris-Club debt-restructuring deal in period $t$ and zero otherwise. The main focus of Rose’s work is the estimation of the coefficients $\phi_m$.

Rose’s empirical model belongs to the family of gravity models. The variable $X_{ijt}$ is a vector of regressors including (current and possibly lagged) characteristics of the country pair $ij$ at time $t$ such as distance, combined output, combined population, combined area, sharing of a common language, sharing of land borders, being co-signers of a free trade agreement, and having had a
colonial relationship. The vector $X_{ijt}$ also includes country-pair-specific dummies and current and lagged values of a variable denoted $IMF_{ijt}$ that takes the values 0, 1, or 2, respectively, if neither, one, or both countries $i$ and $j$ engaged in an IMF program at time $t$.

The data set used for the estimation of the model covers all bilateral trades between 217 countries between 1948 and 1997 at an annual frequency. The sample contains 283 Paris-Club debt-restructuring deals. Rose finds sensible estimates of the parameters pertaining to the gravity model. Specifically, countries that are more distant geographically trade less, whereas high-income country pairs trade more. Countries that share a common currency, a common language, a common border, or membership in a regional free trade agreement trade more. Landlocked countries and islands trade less, and most of the colonial effects are large and positive. The inception of IMF programs is associated with a cumulative contraction in trade of about 10 percent over three years.

Default, as measured by the dummy variable $ACRED_{ijt}$, has a significant and negative effect on bilateral trade. Rose estimates the parameter $\phi_m$ to be on average about -0.07, i.e., $\sum_{t=0}^{M} \frac{\phi_m}{1+M} = -0.07$, and the lag length, $M$, to be about 15 years. This means that entering in a debt restructuring agreement with a member of the Paris Club leads to a decline in bilateral trade of about 7 percent per year for about 16 years. For instance, if trade in period -1 was 100 and the country enters a restructuring agreement with the Paris Club in period 0, then its trade in periods 0 to 15 will be on average 93. Thus, the cumulative effect of default on trade is more than one year worth of trade in the long run. Based on this finding, Rose concludes that one reason why countries pay back their international financial obligations is fear of trade disruptions in the case of default.

Do the estimated values of $\phi_m$ really capture the effect of trade sanctions imposed by creditor countries to defaulting countries? Countries undergoing default or restructuring of their external financial obligations typically are subject to severe economic distress, which may be associated with a general decline in international trade that is unrelated to trade sanctions by the creditor country. If this is indeed the case, then the coefficients $\phi_m$ would be picking up the combined effects of
trade sanctions and of general economic distress during default episodes. To disentangle these two effects, Martínez and Sandleris (2011) estimate the following variant of Rose’s gravity model:

$$\ln T_{ijt} = \beta_0 + \beta X_{ijt} + \sum_{m=0}^{M} \phi_m ACRED_{ijt-m} + \sum_{m=0}^{M} \gamma_m DEBTOR_{ijt-m} + \epsilon_{ijt},$$

where $DEBTOR_{ijt-m}$ is a binary variable taking the value one if either country $i$ or country $j$ is a debtor country involved in a debt-restructuring deal in the context of the Paris Club in period $t$, and zero otherwise. Notice that, unlike variable $ACRED_{ijt}$, variable $DEBTOR_{ijt}$ is unity as long as one of the countries is a debtor involved in a restructuring deal with the Paris Club, regardless of whether or not the other country in the pair is the restructuring creditor. This regressor is meant to capture the general effect of default on trade with all trading countries, not just with those with which the debtor country is restructuring debt through the Paris Club.

In this version of the gravity model, evidence of trade sanctions would require a point estimate for $\sum_{m=0}^{M} \phi_m$ that is negative and significant, and evidence of a general effect of default on trade would require a negative and significant estimate of $\sum_{m=0}^{M} \gamma_m$. Martínez and Sandleris estimate $\sum_{m=0}^{15} \gamma_m$ to be -0.41. That is, when a country enters in default its international trade falls by about 40 percent over 15 years with all countries. More importantly, they obtain a point estimate of $\sum_{m=0}^{15} \phi_m$ that is positive and equal to 0.01. The sign of the point estimates are robust to setting the number of lags, $M$, at 0, 5, or 10. This result would point at the absence of trade sanctions if creditor countries acted in isolation against defaulters. However, if creditors behaved collectively by applying sanctions to defaulters whether or not they are directly affected, then the $\gamma_m$ coefficients might in part be capturing sanction effects.

Martínez and Sandleris control for collective-sanction effects by estimating two additional vari-
ants of the gravity model. One is of the form

$$\ln T_{ijt} = \beta_0 + \beta X_{ijt} + \sum_{m=0}^{M} \phi_m CRED_{ijt-m} + \sum_{m=0}^{M} \gamma_m DEBTOR_{ijt-m} + \epsilon_{ijt},$$

where $CRED_{ijt}$ is a binary variable that takes the value 1 if one of the countries in the pair $ij$ is a debtor in a debt restructuring deal with the Paris Club in period $t$ and the other is a creditor, independently of whether or not it is re-negotiating with the debtor country in the pair. Evidence of trade sanctions would require $\sum_{m=0}^{M} \phi_m$ to be negative and significant. The point estimate of $\sum_{m=0}^{M} \phi_m$ turns out to be sensitive to the lag length considered. At lag lengths of 0, 5, and 10 years the point estimate is positive and equal to 0.09, 0.19, and 0.01, respectively. But when the lag length is set at 15 years, the point estimate turns negative and equal to -0.19.

The third variant of Rose’s gravity model considered by Martínez and Sandleris aims at disentangling the individual and collective punishment effects. It takes the form:

$$\ln T_{ijt} = \beta_0 + \beta X_{ijt} + \sum_{m=0}^{M} \phi_m ACRED_{ijt-m} + \sum_{m=0}^{M} \xi_m NACRED_{ijt-m} + \sum_{m=0}^{M} \gamma_m NOTCRED_{ijt-m} + \epsilon_{ijt}.$$  

Here, $ACRED_{ijt}$, $NACRED_{ijt}$, and $NOTCRED_{ijt}$ are all binary variables taking the values 1 or 0. The variable $NACRED_{ijt}$ takes the value 1 if one of the countries in the pair $ij$ is a defaulter negotiating its debt in the context of the Paris Club in period $t$ and the other country is a nonnegotiating Paris Club creditor (a nonaffected creditor). The variable $NOTCRED_{ijt}$ takes the value 1 if one of the countries in the pair $ij$ is a defaulter negotiating its debt in the context of the Paris Club in period $t$ and the other country is not a creditor. In this variant of the model, evidence of individual and collective trade sanctions would require both $\sum_{m=0}^{M} \phi_m$ and $\sum_{m=0}^{M} \xi_m$ to be negative and significant. The cumulative effect of default on trade between defaulters and
nonaffected creditors, given by $\sum_{m=0}^{M} \xi_{m}$, is consistently negative and robust across lag lengths. Specifically, it takes the values -0.0246, -0.2314, -0.4675, and -0.5629, at lag lengths of 0, 5, 10, and 15 years, respectively. However, the cumulative effect of default on trade between defaulters and directly affected creditors, given by $\sum_{m=0}^{M} \phi_{m}$, is again sensitive to the specified lag length, taking positive values at short and medium lag lengths and turning negative at long lag lengths. Specifically, the point estimate is 0.0631, 0.0854, 0.0119, and -0.3916 at lag lengths of 0, 5, 10, and 15, respectively.

We interpret the work of Martínez and Sandleris as suggesting that the importance of trade sanctions as a cost of default depends crucially upon one’s beliefs regarding the magnitude of the delay with which creditors are able or willing to punish defaulting debtors. If one believes that a reasonable period over which creditors apply trade sanctions to defaulting debtors is less than a decade, then the gravity model offers little evidence of trade sanctions to defaulters. Virtually all of the observed decline in the bilateral trade of debtors after a default episode can be attributed to economic distress and not to punishment inflicted by creditors. However, if one believes that creditors have good memory and are capable of castigating defaulting debtors many years (more than a decade) after a default episode, then the gravity model identifies a significant punishment component in the observed decline in bilateral trade following default episodes of about 50 percent of the trade volume cumulated over 15 years.

### 12.3 Default Incentives With State-Contingent Contracts

The focus of this section is to analyze the structure of international debt contracts when agents have access to state-contingent financial instruments but may lack commitment to honor debt obligations. The material in this section draws from the influential work of Grossman and Van Huyck (1988).
Consider a one-period economy facing a stochastic endowment given by

\[ y = \bar{y} + \epsilon, \]

where \( \bar{y} > 0 \) is a constant and \( \epsilon \) is a mean-zero random variable with density \( \pi(\epsilon) \) defined over the interval \([\epsilon_L, \epsilon_H]\). Thus, \( \bar{y} \) is the mean of the endowment process and \( \epsilon \) is an endowment shock satisfying

\[ \int_{\epsilon_L}^{\epsilon_H} \epsilon \pi(\epsilon) d\epsilon = 0. \tag{12.1} \]

Assume that before \( \epsilon \) is realized, the country can buy insurance against endowment shocks. This insurance is sold by foreign investors and takes the form of state-contingent debt contracts. Specifically, these debt contracts stipulate that the country must pay \( d(\epsilon) \) units of goods to foreign lenders after the realization of the shock. The objective of the country is to pick the debt contract \( d(\epsilon) \) optimally. This state-contingent payment can take positive or negative values. In states in which \( d(\epsilon) \) is negative, the country receives a payment from foreign lenders, and in states in which \( d(\epsilon) \) is positive, the country makes payments to foreign lenders. Foreign lenders are assumed to be risk neutral, to operate in a perfectly competitive market, and to face an opportunity cost of funds equal to zero. These assumptions imply that debt contracts carrying an expected payment of zero are sufficient to ensure the participation of foreign investors. Formally, the zero-expected-profit condition, known as the participation constraint, can be written as

\[ \int_{\epsilon_L}^{\epsilon_H} d(\epsilon) \pi(\epsilon) d\epsilon = 0. \]

The country seeks to maximize the welfare of its representative consumer, which is assumed to be of the form

\[ \int_{\epsilon_L}^{\epsilon_H} u(c(\epsilon)) \pi(\epsilon) d\epsilon, \tag{12.2} \]
where $c(\epsilon)$ denotes consumption, and $u(\cdot)$ is a strictly increasing and strictly concave utility index. For the remainder of this section, we will use the terms country and household indistinctly. The household’s budget constraint is given by

$$c(\epsilon) = \bar{y} + \epsilon - d(\epsilon). \quad (12.3)$$

We are now ready to characterize the form of the optimal external debt contract. We begin by considering the case in which the country can commit to honor its promises.

### 12.3.1 The Optimal Debt Contract With Commitment

Let’s assume that after the realization of the endowment shock $\epsilon$, the household honors any promises made before the occurrence of the shock. In this case, before the realization of the shock, the household’s problem consists in choosing a state-contingent debt contract $d(\epsilon)$ to maximize

$$\int_{\epsilon_L}^{\epsilon_H} u(c(\epsilon))\pi(\epsilon)d\epsilon$$

subject to the budget constraint

$$c(\epsilon) = \bar{y} + \epsilon - d(\epsilon) \quad (12.3)$$

and the participation constraint

$$\int_{\epsilon_L}^{\epsilon_H} d(\epsilon)\pi(\epsilon)d\epsilon = 0. \quad (12.1)$$

The Lagrangian associated with this problem can be written as

$$\mathcal{L} = \int_{\epsilon_L}^{\epsilon_H} [u(\bar{y} + \epsilon - d(\epsilon)) + \lambda d(\epsilon)]\pi(\epsilon)d\epsilon,$$
where \( \lambda \) denotes the Lagrange multiplier associated with the participation constraint (12.1). Note that \( \lambda \) is not state contingent. The first-order conditions associated with the representative household’s problem are (12.1), (12.3), and

\[
u'(c(\epsilon)) = \lambda.
\]

Because the multiplier \( \lambda \) is independent of \( \epsilon \), this expression implies that consumption is also independent of \( \epsilon \). That is, the optimal debt contract achieves perfect consumption smoothing across states of nature. Multiplying both sides of the budget constraint (12.3) by \( \pi(\epsilon) \) and integrating over the interval \([\epsilon_L, \epsilon_H]\) yields

\[
c(\epsilon) \int_{\epsilon_L}^{\epsilon_H} \pi(\epsilon) d\epsilon = \bar{y} \int_{\epsilon_L}^{\epsilon_H} \pi(\epsilon) d\epsilon + \int_{\epsilon_L}^{\epsilon_H} \epsilon \pi(\epsilon) d\epsilon - \int_{\epsilon_L}^{\epsilon_H} d(\epsilon) \pi(\epsilon) d\epsilon.
\]

In this step, we are using the fact that \( c(\epsilon) \) is independent of \( \epsilon \). Since \( \int_{\epsilon_L}^{\epsilon_H} \pi(\epsilon) d\epsilon = 1 \), \( \int_{\epsilon_L}^{\epsilon_H} \epsilon \pi(\epsilon) d\epsilon = 0 \), and \( \int_{\epsilon_L}^{\epsilon_H} d(\epsilon) \pi(\epsilon) d\epsilon = 0 \), we have that

\[
c(\epsilon) = \bar{y}.
\]

That is, under the optimal contract consumption equals the average endowment in all states. It then follows from the budget constraint (12.3), that the associated debt payments are exactly equal to the endowment shocks,

\[
d(\epsilon) = \epsilon.
\]

Under the optimal contract, domestic risk-averse households transfer all of their income uncertainty to risk-neutral foreign lenders. They do so by receiving full compensation from foreign investors for any realization of the endowment below average and by transferring to foreign investors any amount of endowment in excess of the mean. Thus, net payments to the rest of the world move one for one with the endowment, that is,

\[
d'(\epsilon) = 1.
\]
The derivative of the debt contract with respect to the endowment shock is a convenient summary of how much insurance the contract provides. A unit slope is a benchmark that we will use to ascertain how much protection from output fluctuations the country can achieve through the optimal debt contract under alternative environments varying in the amount of commitment the country has to honor debt, and on the ability of foreign lenders to punish defaulters.

12.3.2 The Optimal Debt Contract Without Commitment

In the economy under analysis, there are no negative consequences for not paying debt obligations. Moreover, debtors have incentives not to pay. In effect, in any state of the world in which the contract stipulates a payment to foreign lenders (i.e., in states in which the endowment is above average), the debtor country would be better off defaulting and consuming the resources it owes. After consuming these resources, the world simply ends, so debtors cannot be punished for having defaulted.

The perfect-risk-sharing equilibrium we analyzed in the previous subsection was built on the premise that the sovereign can resist the temptation to default. What if this commitment to honoring debts was absent? Clearly, in our one-shot world, the country would default in any state in which the contract stipulates a payment to the rest of the world. It then follows that any debt contract must include the additional incentive-compatibility constraint

$$d(\epsilon) \leq 0$$

for all $\epsilon \in [\epsilon^L, \epsilon^H]$. The representative household’s problem then is to maximize

$$\int_{\epsilon^L}^{\epsilon^H} u(c(\epsilon))\pi(\epsilon)d\epsilon$$
subject to

\[ c(\epsilon) = \bar{y} + \epsilon - d(\epsilon), \quad (12.3) \]

\[ \int_{\epsilon}^{\epsilon_H} d(\epsilon) \pi(\epsilon) d\epsilon = 0, \quad (12.1) \]

\[ d(\epsilon) \leq 0. \quad (12.4) \]

Restrictions (12.1) and (12.4) state that debt payments must be zero on average and never positive. The only debt contract that can satisfy these two requirements simultaneously is clearly

\[ d(\epsilon) = 0, \]

for all \( \epsilon \). This is a trivial contract stipulating no transfers of any sort in any state. It follows that under lack of commitment international risk sharing breaks down. No meaningful debt contract can be supported in equilibrium. As a result, the country is in complete financial autarky and must consume its endowment in every state

\[ c(\epsilon) = \bar{y} + \epsilon, \]

for all \( \epsilon \). This consumption profile has the same mean as the one that can be supported with commitment, namely, \( \bar{y} \). However, the consumption plan under commitment is constant across states, whereas the one associated with autarky inherits the volatility of the endowment process. It follows immediately that risk-averse households (i.e., households with concave preferences) are worse off in the financially autarkic economy. Formally, by the definition of concavity, we have that

\[ u \left( \int_{\epsilon}^{\epsilon_H} (\bar{y} + \epsilon) \pi(\epsilon) d\epsilon \right) > \int_{\epsilon}^{\epsilon_H} u(\bar{y} + \epsilon) \pi(\epsilon) d\epsilon. \] Put differently, commitment is welfare increasing.

Because in the economy without commitment international transfers are constant (and equal to zero) across states, we have that

\[ d'(\epsilon) = 0. \]
This result is in sharp contrast with what we obtained under full commitment. In that case, the derivative of debt payments with respect to the endowment shock is unity at all endowment levels.

12.3.3 Direct Sanctions

Suppose that foreign lenders (or their representative governments) could punish defaulting sovereigns by seizing national property, such as financial assets or exports. One would expect that this type of actions would deter borrowers from defaulting at least as long as debt obligations do not exceed the value of the seizure. What is the shape of the optimal debt contract that emerges in this type of environment?

We model direct sanctions by assuming that in the case of default lenders can seize $k > 0$ units of goods from the delinquent debtor. It follows that the borrower will honor all debts not exceeding $k$ in value. Formally, this means that the incentive-compatibility constraint now takes the form

$$d(\epsilon) \leq k. \quad (12.5)$$

Under commitment, the optimal debt contract stipulates a maximum payment of $\epsilon^H$. This means that if $k \geq \epsilon^H$, the optimal contract under commitment can be supported in an environment without commitment but with sanctions. At the opposite extreme, if $k = 0$, we are in the case with no commitment and no sanctions, and no payments can be supported in equilibrium, which results in financial autarky. Here, our interest is to characterize the optimal debt contract in the intermediate case

$$k \in (0, \epsilon^H).$$
The representative household’s problem then consists in maximizing

$$\int_{\epsilon L}^{\epsilon H} u(c(\epsilon))\pi(\epsilon) d\epsilon \quad (12.2)$$

subject to

$$c(\epsilon) = \bar{y} + \epsilon - d(\epsilon), \quad (12.3)$$

$$\int_{\epsilon L}^{\epsilon H} d(\epsilon)\pi(\epsilon) d\epsilon = 0, \quad (12.1)$$

$$d(\epsilon) \leq k. \quad (12.5)$$

The Lagrangian associated with this problem can be written as

$$\mathcal{L} = \int_{\epsilon L}^{\epsilon H} \{ u(\bar{y} + \epsilon - d(\epsilon)) + \lambda d(\epsilon) + \gamma(\epsilon)[k - d(\epsilon)] \} \pi(\epsilon) d\epsilon,$$

where $\lambda$ denotes the Lagrange multiplier associated with the participation constraint (12.1) and $\gamma(\epsilon)\pi(\epsilon)$ denotes the Lagrange multiplier associated with the incentive-compatibility constraint (12.5) in state $\epsilon$ (there is a continuum of such multipliers one for each possible value of $\epsilon$). The first-order conditions associated with the representative household’s problem are (12.1), (12.3), (12.5), and

$$u'(c(\epsilon)) = \lambda - \gamma(\epsilon), \quad (12.6)$$

$$\gamma(\epsilon) \geq 0, \quad (12.7)$$

and the slackness condition

$$(k - d(\epsilon))\gamma(\epsilon) = 0. \quad (12.8)$$

In states in which the incentive-compatibility constraint does not bind, i.e., when $d(\epsilon) < k$, the
slackness condition (12.8) states that the Lagrange multiplier $\gamma(\epsilon)$ must vanish. It then follows from optimality condition (12.6) that the marginal utility of consumption equals $\lambda$ for all states of nature in which the incentive-compatibility constraint does not bind. This means that consumption is constant across all states in which the incentive-compatibility constraint does not bind.

In turn, the budget constraint (12.3) implies that across states in which the incentive-compatibility constraint does not bind, payments to foreign lenders must differ from the endowment innovation $\epsilon$ by only a constant. Formally, we have that

$$d(\epsilon) = \bar{d} + \epsilon,$$

for all $\epsilon$ such that $d(\epsilon) < k$, where $\bar{d}$ is an endogenously determined constant.

Based on our analysis of the case with commitment, in which payments to the rest of the world take place in states of nature featuring positive endowment shocks, it is natural to conjecture that the optimal contract will feature the incentive-compatibility constraint binding at relatively high levels of income and not binding at relatively low levels of income. To see that this is indeed the case, let us show that if the incentive-compatibility constraint is not binding for some $\epsilon'$, then it is not binding for all $\epsilon'' < \epsilon'$. Formally, we wish to show that if $d(\epsilon') < k$ for some $\epsilon' \in (\epsilon^L, \epsilon^H)$, then $d(\epsilon'') < d(\epsilon')$ for any $\epsilon'' \in [\epsilon^L, \epsilon')$. The proof is by contradiction. Let $\epsilon'' \in [\epsilon^L, \epsilon')$. Suppose that $d(\epsilon'') \geq d(\epsilon')$. Then, by the budget constraint (12.3), we have that $c(\epsilon'') = \bar{y} + \epsilon'' - d(\epsilon'') < \bar{y} + \epsilon' - d(\epsilon') = c(\epsilon')$. It follows from the strict concavity of the utility function that $u'(c(\epsilon'')) > u'(c(\epsilon'))$. Then by optimality condition (12.6) we have that $\gamma(\epsilon'') < \gamma(\epsilon')$. But $\gamma(\epsilon') = 0$ by the slackness condition (12.8) and the assumption that $d(\epsilon') < k$. So we have that $\gamma(\epsilon'') < 0$, which contradicts optimality condition (12.7).
It follows that there exists an $\bar{\epsilon}$ such that

\[
    d(\epsilon) = \begin{cases} 
        \bar{d} + \epsilon & \text{for } \epsilon < \bar{\epsilon} \\
        k & \text{for } \epsilon > \bar{\epsilon}
    \end{cases}
\]

(12.9)

We will show shortly that the debt contract described by this expression is indeed continuous in the endowment shock. That is, we will show that

\[
    d(\bar{\epsilon}) = \bar{d} + \bar{\epsilon} = k.
\]

(12.10)

We will also show that this condition implies that the constant $\bar{d}$ is indeed positive. This means that under the optimal debt contract without commitment but with direct sanctions the borrower enjoys less insurance than in the case of full commitment. This is because in relatively low-endowment states (i.e., states in which the incentive-compatibility constraint does not bind) the borrower must pay $\bar{d} + \epsilon$, which is a larger sum than the one that is stipulated for the same state in the optimal contract with full commitment, given simply by $\epsilon$.

To see that if condition (12.10) holds, i.e., if the optimal debt contract is continuous, then $\bar{d}$ is positive, write the participation constraint (12.1), which indicates that debt payments must be nil on average, as

\[
    0 = \int_{\epsilon_L}^{\bar{\epsilon}} (\bar{d} + \epsilon) \pi(\epsilon)d\epsilon + \int_{\epsilon}^{\epsilon_H} k\pi(\epsilon)d\epsilon
    = \int_{\epsilon_L}^{\bar{\epsilon}} (\bar{d} + \epsilon)\pi(\epsilon)d\epsilon + \int_{\bar{\epsilon}}^{\epsilon_H} (\bar{d} + \bar{\epsilon})\pi(\epsilon)d\epsilon
    = \bar{d} + \int_{\epsilon_L}^{\bar{\epsilon}} \epsilon\pi(\epsilon)d\epsilon + \int_{\bar{\epsilon}}^{\epsilon_H} \bar{\epsilon}\pi(\epsilon)d\epsilon
    = \bar{d} - \int_{\bar{\epsilon}}^{\epsilon_H} (\epsilon - \bar{\epsilon})\pi(\epsilon)d\epsilon.
\]
Since $\bar{\epsilon} < \epsilon^H$, we have that$^7$

$$\bar{d} > 0.$$ 

In showing that $\bar{d}$ is positive, we made use of the conjecture that the debt contract is continuous in the endowment, that is, that $\bar{d} + \bar{\epsilon} = k$. We are now ready to establish this result. Using (12.9) to eliminate $d(\epsilon)$ from (12.3) and (12.1), the optimal contract sets $\bar{\epsilon}$ and $\bar{d}$ to maximize

$$\int_{\bar{\epsilon}}^{\epsilon^H} u(\bar{y} - \bar{d})\pi(\epsilon)d\epsilon + \int_{\bar{\epsilon}}^{\epsilon^H} u(\bar{y} + \epsilon - k)\pi(\epsilon)d\epsilon$$

subject to

$$\int_{\epsilon^L}^{\bar{\epsilon}} (\bar{d} + \epsilon)\pi(\epsilon)d\epsilon + [1 - F(\bar{\epsilon})]k = 0,$$

where $F(\bar{\epsilon}) \equiv \int_{\epsilon^L}^{\bar{\epsilon}} \pi(\epsilon)d\epsilon$ denotes the probability that $\epsilon$ is less than $\bar{\epsilon}$. Now differentiate (12.11) with respect to $\bar{d}$ and $\bar{\epsilon}$ and set the result equal to zero. Also, differentiate (12.12). The resulting expressions are, respectively,

$$- u'(\bar{y} - \bar{d})F(\bar{\epsilon})d\bar{d} + [u(\bar{y} - \bar{d}) - u(\bar{y} + \bar{\epsilon} - k)]\pi(\bar{\epsilon})d\bar{\epsilon} = 0$$

and

$$[(\bar{\epsilon} - k + \bar{d})\pi(\bar{\epsilon})d\epsilon + F(\bar{\epsilon})d\bar{d} = 0.$$  

Combining equations (12.13) and (12.14) we obtain the optimality condition

$$- u'(\bar{y} - \bar{d})[k - \bar{d} - \bar{\epsilon}] + [u(\bar{y} - \bar{d}) - u(\bar{y} + \bar{\epsilon} - k)] = 0.$$  

Conditions (12.12) and (12.15) represent a system of two equations in the two unknowns, $\bar{\epsilon}$ and $\bar{d}$. 

$^7$To see that $\bar{\epsilon} < \epsilon^H$, show that if $\bar{\epsilon} = \epsilon^H$, then (12.1) and (12.9) imply that $d(\epsilon) = \epsilon$, which violates the incentive-compatibility constraint (12.5) for all $\epsilon > k$. 

Clearly, equation (12.15) is satisfied for any pair \((\bar{\epsilon}, \bar{d})\) such that \(\bar{d} + \bar{\epsilon} = k\). That is, any continuous contract from the family defined in (12.9) satisfies the optimality condition (12.15). We now need to show that there exists a continuous contract that satisfies (12.12). To this end, replace \(\bar{d}\) in (12.12) by \(k - \bar{\epsilon}\) to obtain

\[
k = \int_{\bar{\epsilon}}^{\epsilon} (\bar{\epsilon} - \epsilon) \pi(\epsilon) d\epsilon.
\]

The function \((\bar{\epsilon} - \epsilon)\pi(\epsilon)\) is nonnegative for \(\epsilon \leq \bar{\epsilon}\), which means that the right-hand side of this expression is a continuous, nondecreasing function of \(\bar{\epsilon}\). Moreover, the right-hand side takes the value 0 at \(\bar{\epsilon} = \epsilon_L\) and the value \(\epsilon_H\) for \(\bar{\epsilon} = \epsilon_H\). Since the sanction \(k\) belongs to the interval \((0, \epsilon_H)\), we have that there is at least one value of \(\bar{\epsilon}\) that satisfies the above expression. We have therefore established that there exists at least one continuous debt contract that satisfies the two optimality conditions (12.12) and (12.15). Clearly, if the density function \(\pi(\epsilon)\) is strictly positive for all \(\epsilon \in (\epsilon_L, \epsilon_H)\), then there is a unique continuous contract that satisfies both optimality conditions.

Our analysis shows that the case with no commitment and direct sanctions falls in between the case with full commitment and the case with no commitment and no direct sanctions. In particular, payments to foreign creditors increase one-for-one with the endowment shock for \(\epsilon < \bar{\epsilon}\) (as in the case with full commitment), and are independent of the endowment shock for \(\epsilon\) larger than \(\bar{\epsilon}\) (as in the case without commitment and no direct sanctions), that is,

\[
d'(\epsilon) = \begin{cases} 
1 & \epsilon < \bar{\epsilon} \\
0 & \epsilon > \bar{\epsilon}
\end{cases}.
\]

Note also that if the sanction is sufficiently large—specifically, if \(k > \epsilon^H\)—then \(\bar{\epsilon} > \epsilon^H\) and the optimal contract is identical to the one that results in the case of full commitment. By contrast, if creditors are unable to impose sanctions \((k = 0)\), then \(\bar{\epsilon} = \epsilon^L\) and the optimal contract stipulates financial autarky as in the case with neither commitment nor direct sanctions. It follows, perhaps
Figure 12.6: Consumption Profiles Under Full Commitment and No Commitment With Direct Sanctions

Note: $c^c(\epsilon)$ and $c^s(\epsilon)$ denote the levels of consumption in state $\epsilon$ under commitment and sanctions, respectively, $y(\epsilon) \equiv \bar{y} + \epsilon$ denotes output, and $\epsilon$ denotes the endowment shock.

paradoxically, that the larger the ability of creditors to punish debtor countries in case of default, the higher the welfare of the debtor countries themselves.

Finally, it is of interest to compare the consumption profiles across states in the model with commitment and in the model with direct sanctions and no commitment. Figure 12.6 provides a graphical representation of this comparison. In the model with commitment, consumption is perfectly smooth across states and equal to the average endowment. As mentioned earlier, in this case the risk-averse debtor country transfers all of the risk to risk neutral lenders. In the absence of commitment, consumption smoothing is a direct function of the ability of the lender to
punish debtors in the case of default. Consumption is flat in low-endowment states (from $\epsilon^L$ to $\bar{\epsilon}$) and increasing in the endowment in high-endowment states (from $\bar{\epsilon}$ to $\epsilon^H$). The reduced ability of the risk averse agent to transfer risk to risk neutral lenders is reflected in two features of the consumption profile. First, the profile is no longer flat across all states of nature. Second, the flat segment of the consumption profile is lower than the level of consumption achieved under full commitment. This means that in the case with sanctions but no commitment, although households are protected by a safety net ($\bar{y} - \bar{d}$) below which consumption cannot fall no matter how severe the contraction of their income is, this safety net is more precarious than the one provided by full commitment ($\bar{y} > \bar{y} - d$).

Finally, a consequence of our maintained assumption that asset markets are complete is that the punishment is never materialized in equilibrium. The incentive-compatibility constraint ensures that the amount of payments contracted in each state of nature never exceeds the possible punishment. As a result, in equilibrium, the debtor never has an incentive to default.

### 12.3.4 Reputation

Suppose now that creditors do not have access to direct sanctions to punish debtors who choose to default. Instead, assume that creditors have the ability to exclude delinquent debtors from financial markets. Because financial autarky entails the cost of an elevated consumption volatility, financial exclusion has the potential to support international lending. Debtor countries pay their obligations to maintain their performing status.

Clearly, the model we have in mind here can no longer be a one-period model like the one studied thus far. Time is at the center of any reputational model of debt. Accordingly, we assume that the debtor country lives forever and each period receives an endowment equal to $\bar{y} + \epsilon$, where $\bar{y}$ is a positive constant and $\epsilon$ is a mean-zero random variable with a time-invariant density $\pi(\epsilon) > 0$ over the continuous support $[\epsilon^L, \epsilon^H]$. For simplicity, we assume that the country cannot transfer
resources intertemporally via a storage technology or financial markets.

If the country defaulted in the present or in the past, it is considered to be in bad financial standing. Suppose that foreign lenders punish countries that are in bad financial standing by perpetually excluding them from international financial markets. Let \( v^b(\epsilon) \) denote the welfare of a country that is in bad financial standing. Then, we have that \( v^b(\epsilon) \) is given by

\[
v^b(\epsilon) \equiv u(\bar{y} + \epsilon) + \beta \int_{\epsilon_L}^{\epsilon_H} v^b(\epsilon') \pi(\epsilon') d\epsilon'
\]

\[
= u(\bar{y} + \epsilon) + \frac{\beta}{1 - \beta} \int_{\epsilon_L}^{\epsilon_H} u(\bar{y} + \epsilon') \pi(\epsilon') d\epsilon'.
\] (12.16)

Consider designing a debt contract with the following characteristics: (1) Payments are state contingent, but time independent. That is, the contract stipulates that in any state \( \epsilon \), the country must pay \( d(\epsilon) \) to foreign lenders independently of history. (2) The contract is incentive compatible, that is, in every state and date, the country prefers to pay its debt rather than default. (3) The contract satisfies the participation constraint (12.1) period by period. That is, for each date, the expected value of payments to foreign lenders across states must be equal to zero.

In any date and state, the country can be either in good or bad financial standing. We denote by \( v^g(\epsilon) \) the welfare of a country that enters the period in good standing and by \( v^c(\epsilon) \) the welfare of a country that enters the period in good standing and chooses to honor its external obligations in that period. Then, we have that

\[
v^c(\epsilon) \equiv u(\bar{y} + \epsilon - d(\epsilon)) + \beta \int_{\epsilon_L}^{\epsilon_H} v^g(\epsilon') \pi(\epsilon') d\epsilon'
\] (12.17)

and

\[
v^g(\epsilon) = \max\{v^b(\epsilon), v^c(\epsilon)\}.
\]
The incentive-compatibility constraint requires that the country prefer to honor its debt, or, formally, that

\[ v^c(\epsilon) \geq v^b(\epsilon). \]  \hspace{1cm} (12.18)

The above two expressions then imply that

\[ v^g(\epsilon) = v^c(\epsilon). \]

This equation implies that in equilibrium countries never default. As we will see shortly, this will not be the case in models with incomplete financial markets. The above expression can be used to eliminate \( v^g(\epsilon) \) from (12.17) to get

\[ v^c(\epsilon) = u(\bar{y} + \epsilon - d(\epsilon)) + \beta \int_{\epsilon_L}^{\epsilon_H} v^c(\epsilon') \pi(\epsilon') d\epsilon'. \]

Iterating this expression forward, yields

\[ v^c(\epsilon) = u(\bar{y} + \epsilon - d(\epsilon)) + \frac{\beta}{1 - \beta} \int_{\epsilon_L}^{\epsilon_H} u(\bar{y} + \epsilon' - d(\epsilon')) \pi(\epsilon') d\epsilon'. \]  \hspace{1cm} (12.19)

We can then rewrite the incentive-compatibility constraint (12.18) as

\[ u(\bar{y} + \epsilon - d(\epsilon)) + \frac{\beta}{1 - \beta} \int_{\epsilon_L}^{\epsilon_H} u(\bar{y} + \epsilon' - d(\epsilon')) \pi(\epsilon') d\epsilon' \geq u(\bar{y} + \epsilon) + \frac{\beta}{1 - \beta} \int_{\epsilon_L}^{\epsilon_H} u(\bar{y} + \epsilon') \pi(\epsilon') d\epsilon'. \]  \hspace{1cm} (12.20)

This restriction must hold for every state \( \epsilon \).

Before continuing with the characterization of the optimal debt contract, it is instructive to see what factors make the first-best contract, \( d(\epsilon) = \epsilon \), fail in the present environment without commitment. Evaluating the above incentive-compatibility constraint at the first-best contract,
and rearranging terms, one obtains

\[ u(\bar{y} + \epsilon) - u(\bar{y}) \leq \frac{\beta}{1 - \beta} [u(\bar{y}) - Eu(\bar{y} + \epsilon)], \]

where \( Eu(\bar{y} + \epsilon) \equiv \int_{\tilde{\epsilon}}^{\epsilon} u(\tilde{y} + \epsilon) \pi(\epsilon) d\epsilon \) denotes the expected value of the period utility under financial autarky. The left-hand side of this expression measures the short-run gains of defaulting, whereas the right-hand side, which is positive because of the assumption of strict concavity of the period utility index, measures the long-run costs of default. The short-run gains have to do with the extra utility derived from above-average realizations of the current endowment, and the long-run costs of default are associated with the lack of consumption smoothing that defaulters must endure under financial autarky.

The above expression shows that the first-best debt contract may not be implementable on reputational grounds alone, because it may violate the incentive-compatibility constraint in certain states. In general, incentives to default are stronger the larger the current realization of the endowment, \( \epsilon \). In particular, under the first-best debt contract, default could take place in states in which the endowment is above average (\( \epsilon > 0 \)). Also, the more impatient the debtor is (i.e., the lower \( \beta \) is), the larger the incentive to default. Intuitively, an impatient debtor does not place much value on the fact that future expected utility is higher under the first-best contract than under financial isolation. In addition, all other things equal, less risk-averse countries have stronger incentives to default on the first-best debt contract. It follows from this analysis that the first-best contract is in general not incentive compatible in the absence of commitment even if creditors could use financial isolation as a discipline devise. Let’s get back, then, to the characterization of the optimal incentive-compatible debt contract.

Because the debt contracting problem is stationary, in the sense that the contract must be time
independent, it suffices to maximize the period utility index,

$$
\int_{\epsilon_L}^{\epsilon_H} u(\bar{y} + \epsilon - d(\epsilon)) \pi(\epsilon) d\epsilon,
$$

subject to the the participation constraint (12.1) and the incentive-compatibility constraint (12.20). The Lagrangian associated with this problem is

$$
\mathcal{L} = \int_{\epsilon_L}^{\epsilon_H} u(\bar{y} + \epsilon - d(\epsilon)) \pi(\epsilon) d\epsilon + \lambda \int_{\epsilon_L}^{\epsilon_H} d(\epsilon) \pi(\epsilon) d\epsilon + \int_{\epsilon_L}^{\epsilon_H} \gamma(\epsilon) \left[ u(\bar{y} + \epsilon - d(\epsilon)) + \frac{\beta}{1 - \beta} \int_{\epsilon_L}^{\epsilon_H} u(\bar{y} + \epsilon' - d(\epsilon')) \pi(\epsilon') d\epsilon' \right] \pi(\epsilon) d\epsilon,
$$

where $\lambda$ and $\pi(\epsilon) \gamma(\epsilon)$ denote the Lagrange multipliers on (12.1) and (12.20), respectively. The first-order conditions associated with the problem of choosing the transfer schedule $d(\epsilon)$ are (12.1), (12.20),

$$
u'(\bar{y} + \epsilon - d(\epsilon)) = \frac{\lambda}{1 + \gamma(\epsilon) + \frac{\beta}{1 - \beta} \int_{\epsilon_L}^{\epsilon_H} \gamma(\epsilon') \pi(\epsilon') d\epsilon'},
$$

(12.21)

$$
\gamma(\epsilon) \geq 0 \quad (12.22)
$$

and the slackness condition

$$
\gamma(\epsilon) \left[ u(\bar{y} + \epsilon - d(\epsilon)) + \frac{\beta}{1 - \beta} \int_{\epsilon_L}^{\epsilon_H} u(\bar{y} + \epsilon' - d(\epsilon')) \pi(\epsilon') d\epsilon' - u(\bar{y} + \epsilon) - \frac{\beta}{1 - \beta} \int_{\epsilon_L}^{\epsilon_H} u(\bar{y} + \epsilon') \pi(\epsilon') d\epsilon' \right] = 0.
$$

(12.23)

In states in which the incentive-compatibility constraint (12.20) is not binding, the slackness condition (12.23) stipulates that the Lagrange multiplier $\gamma(\epsilon)$ must vanish. It follows that in these
states, the optimality condition (12.21) becomes

\[ u'(\bar{y} + \epsilon - d(\epsilon)) = \frac{\lambda}{1 + \frac{\beta}{1-\beta} \int_{\epsilon}^{\epsilon'} \gamma(e') \pi(e') de'} \]  \hspace{1cm} (12.24)

Because the right-hand side of this expression is independent of the current value of \( \epsilon \), we have that the marginal utility of consumption must be constant across states in which the incentive compatibility constraint does not bind. This, in turn, implies that consumption is constant across these states, and that transfers are of the form \( d(\epsilon) = \bar{d} + \epsilon \), where \( \bar{d} \) is a constant. Over these states, consumption and payments to or from the rest of the world behave exactly as in the case with direct sanctions: domestic risk-averse agents transfer their endowment shock plus a constant to risk-neutral foreign lenders.

In states in which the incentive-compatibility constraint (12.20) is binding, consumption is greater than or equal to consumption in states in which the incentive-compatibility constraint is not binding. To see this, notice that because \( \gamma(\epsilon) \geq 0 \) for all \( \epsilon \), the right-hand side of (12.21) is smaller than or equal to the right-hand side of (12.24). It then follows from the concavity of the period utility function that consumption must be higher in states in which the incentive-compatibility constraint is binding.

It is again natural to expect that the incentive-compatibility constraint will bind in high-endowment states and that it will not bind in low-endowment states. The intuition is, again, that the debt contract should stipulate payments to the rest of the world in high-endowment states and transfers from the rest of the world to the domestic households in low endowment states, creating the largest incentives to default in high endowment states. To see that this intuition is correct, consider an \( \epsilon_1 \) for which the incentive-compatibility constraint is not binding, that is, \( \gamma(\epsilon_1) = 0 \). Consider now any endowment \( \epsilon_2 < \epsilon_1 \). We wish to show that \( \gamma(\epsilon_2) = 0 \). The proof is by contradiction. Suppose that \( \gamma(\epsilon_2) > 0 \). The analysis of the previous paragraph implies that \( c(\epsilon_2) > c(\epsilon_1) \).
and hence that $\epsilon_2 - d(\epsilon_2) > \epsilon_1 - d(\epsilon_1)$. This, in turn, implies that as the endowment shock falls from $\epsilon_1$ to $\epsilon_2$, the left-hand side of the incentive-compatibility constraint (12.20) increases and its right-hand side decreases. This means that (12.20) must hold with strict inequality at $\epsilon_2$. It follows that the slackness condition (12.23) is violated, which shows that $\gamma(\epsilon_2)$ cannot be positive. It follows from this analysis that there exists a threshold level of the endowment shock $\bar{\epsilon} \leq \epsilon_H$ such that the incentive-compatibility constraint binds for all $\epsilon > \bar{\epsilon}$ and does not bind for all $\epsilon < \bar{\epsilon}$. That is,

$$\gamma(\epsilon) \begin{cases} 0 & \text{for } \epsilon < \bar{\epsilon} \\ > 0 & \text{for } \epsilon > \bar{\epsilon} \end{cases}.$$

Consider the question of how the optimal transfer $d(\epsilon)$ varies across states in which the collateral constraint is binding. Does it increase as one moves from low- to high-endowment states, and by how much? To address this question, let us examine the incentive-compatibility constraint (12.20) holding with equality:

$$u(\bar{y} + \epsilon - d(\epsilon)) + \frac{\beta}{1 - \beta} \int_{\epsilon_L}^{\epsilon_H} u(\bar{y} + \epsilon' - d(\epsilon')) \pi(\epsilon') d\epsilon' = u(\bar{y} + \epsilon) + \frac{\beta}{1 - \beta} \int_{\epsilon_L}^{\epsilon_H} u(\bar{y} + \epsilon') \pi(\epsilon') d\epsilon'. \quad (12.25)$$

Notice that in this expression the terms $\frac{\beta}{1 - \beta} \int_{\epsilon_L}^{\epsilon_H} u(\bar{y} + \epsilon' - d(\epsilon')) \pi(\epsilon') d\epsilon'$ and $\frac{\beta}{1 - \beta} \int_{\epsilon_L}^{\epsilon_H} u(\bar{y} + \epsilon') \pi(\epsilon') d\epsilon'$ are both independent of the current endowment shock $\epsilon$. Only the first terms on the right- and left-hand sides of (12.25) change with the current level of endowment. Differentiating (12.25) with respect to the current endowment, $\epsilon$, yields

$$d'(\epsilon) = \frac{u'(\bar{y} + \epsilon - d(\epsilon)) - u'(\bar{y} + \epsilon)}{u'(\bar{y} + \epsilon - d(\epsilon))}.$$
foreign agent—and because the utility index is strictly concave, it follows that $u'(\bar{y} + \epsilon - d(\epsilon)) > u'(\bar{y} + \epsilon)$ in all states in which the incentive-compatibility constraint binds. This implies that when the incentive-compatibility constraint binds we have

$$0 < d'(\epsilon) < 1.$$  

That is, payments to the foreign lender increase with the level of income, but less than one for one.

It might seem counterintuitive that as the current endowment increases the payment to creditors that can be supported without default also increases. After all, the higher is the current level of endowment, the higher is the level of current consumption that can be achieved upon default. The intuition behind the direct relation between income and payments is that given a positive level of current payments, $d(\epsilon) > 0$, a small increase in current endowment, $\epsilon$, raises the current-period utility associated with not defaulting, $u(\bar{y} + \epsilon - d(\epsilon))$, by more than it raises the utility associated with the alternative of defaulting, $u(\bar{y} + \epsilon)$. (This is because the period utility function is assumed to be strictly concave.) It follows that in states in which $d(\epsilon) > 0$, the higher is the current endowment, the higher is the level of payments to foreign lenders that can be supported without inducing default. This does not mean that default incentives are weaker the higher is the level of the endowment. Recall that the analysis in this paragraph is restricted to states in which the incentive-compatibility constraint is binding. The incentive-compatibility constraint tends to bind in relatively high-endowment states.

The positive slope of the payment schedule with respect to the endowment (when the incentive-compatibility constraint is binding) presents a contrast with the pattern that emerges in the case of direct sanctions. In the direct-sanction economy, when the incentive-compatibility constraint binds, payments equal the maximum punishment $k$, which implies that the slope of the payment schedule equals zero.
Figure 12.7: Consumption Profiles Under Full Commitment and No Commitment in a Reputational Model of Debt

\[ y(\epsilon) = \bar{y} + \epsilon \]

\[ c^c(\epsilon) = \bar{y} \]

\[ c^R(\epsilon) \]

\[ d \]

\[ 0 \]

\[ \epsilon \]

Note: \( c^c(\epsilon) \) and \( c^R(\epsilon) \) denote the levels of consumption in state \( \epsilon \) under commitment and no commitment, respectively, \( y(\epsilon) \equiv \bar{y} + \epsilon \) denotes output, and \( \epsilon \) denotes the endowment shock.

Finally, the fact that the present reputational model the slope of the payment schedule is positive but less than one implies that when the incentive-compatibility constraint is binding consumption is strictly increasing in the endowment. Figure 12.7 plots the consumption schedule as a function of the endowment shock.

### 12.4 Default Incentives With Non-State-Contingent Contracts

In a world with complete financial markets, optimal risk-sharing arrangements stipulate positive payoffs in low-income states and negative payoffs in high-income states. In this way, the optimal
financial contract facilitates a smooth level of consumption across states of nature. An implication of this result is that default incentives are stronger in high-income states and weaker in low-income states. In the real world, however, as documented earlier in this chapter, countries tend to default during economic contractions. One goal of this section is to explain this empirical regularity. To this end, we remove the assumption that financial markets are complete. Indeed, we focus on the polar case of a single non-state-contingent asset. In this environment, debts assumed in the current period impose financial obligations in the next period that are independent of whether income in that period is high or low. The debtor is no longer able to design debt contracts that carry a high interest rate in good states and a low interest rate in bad states. As a result, debtors facing high debt obligations and low endowments will have strong incentives to default. The pioneer model of Eaton and Gersovitz (1981), which we study in this section, represents the first formalization of this idea. Our version of the Eaton-Gersovitz model follows Arellano (2008).

12.4.1 The Eaton-Gersovitz Model

Consider a small open economy populated by a large number of identical individuals. Preferences are described by the utility function

\[ E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \]

where \( c_t \) denotes consumption in period \( t \), \( u \) is a period utility function assumed to be strictly increasing and strictly concave, and \( \beta \in (0,1) \) is a parameter denoting the subjective discount factor. Throughout our analysis, we will use the terms household, country, and government indiscriminably. We have in mind an arrangement in which the government makes all decisions concerning international borrowing and default in a benevolent fashion. Each period \( t \geq 0 \), the representative country is endowed with \( y_t \) units of consumption goods. This endowment is assumed to be exogenous, stochastic, and i.i.d., with a distribution featuring a bounded support \( Y = [\underline{y}, \bar{y}] \).
At the beginning of each period, the country can be either in good financial standing or in bad financial standing. If the country is in bad financial standing, then it is prevented from borrowing or lending in financial markets. As a result, the country is forced to consume its endowment. Formally, consumption under bad financial standing is given by

\[ c = y. \]

We drop the time subscript in expressions where all variables are dated in the current period.

If the country is in good financial standing, it can choose to default on its debt obligations or to honor its debt. If it chooses to default, then it immediately acquires a bad financial status. If it chooses to honor its debt, then it maintains its good financial standing until the beginning of the next period. If the country is in good standing and chooses not to default, its budget constraint is given by

\[ c + d = y + q(d')d', \]

(12.26)

where \( d \) denotes the country’s debt due in the current period, \( d' \) denotes the debt acquired in the current period and due in the next period, and \( q(d') \) denotes the market price of the country’s debt. Note that the price of debt depends on the amount of debt acquired in the current period and due next period, \( d' \), but not on the level of debt acquired in the previous period and due in the current period, \( d \). This is because the default decision in the next period depends on the amount of debt due then. Notice also that \( q(\cdot) \) is independent of the current level of output. This is because of the assumed i.i.d. nature of the endowment, which implies that its current value conveys no information about future expected endowment levels. If instead we had assumed that \( y \) was serially correlated, then bond prices would depend on the current level of the endowment, since it would be informative of the state of the business cycle—and hence of the probability of default—next period.

We assume that ‘bad financial standing’ is an absorbent state. This means that once the country
falls into bad standing, it remains in that status forever. The country acquires a bad standing when it defaults on its financial obligations. The value function associated with bad financial standing is denoted \( v^b(y) \) and is given by

\[
v^b(y) = u(y) + \beta E v^b(y').
\]

Here, \( y' \) denotes next period’s endowment, and \( E \) denotes the \( l \) expectations operator.

If the country is in good standing, the value function associated with continuing to participate in capital markets by honoring its current debts is denoted \( v^c(d, y) \) and is given by

\[
v^c(d, y) = \max_{d'} \left\{ u(y + q(d')d' - d) + \beta E v^g(d', y') \right\},
\]

subject to

\[
d' \leq \bar{d},
\]

where \( v^g(d, y) \) denotes the value function associated with being in good financial standing, and is given by

\[
v^g(d, y) = \max \{ v^b(y), v^c(d, y) \}.
\]

The parameter \( \bar{d} > 0 \) is a debt limit that prevents agents from engaging in Ponzi games. In this economy, the country chooses to default when servicing the debt entails a cost in terms of forgone current consumption that is larger than the inconvenience of living in financial autarky forever. It is then reasonable to conjecture that default is more likely the larger the level of debt and the lower the current endowment. In what follows, we demonstrate that this intuition is in fact correct. We do so in steps.
12.4.2 The Default Set

The default set contains all endowment levels at which a country chooses to default given a particular level of debt. We denote the default set by $D(d)$. Formally, the default set is defined by

$$D(d) = \{ y \in Y : v^b(y) > v^c(d, y) \}.$$

Because it is never in the agent’s interest to default when its asset position is nonnegative (or $d \leq 0$), it follows that $D(d)$ is empty for all $d \leq 0$.

The trade balance is given by $tb \equiv y - c$. The budget constraint (12.26) then implies that

$$tb = d - q(d')d'.$$

The following proposition shows that at debt levels for which the default set is not empty, an economy that chooses not to default will run a trade surplus.

**Proposition 12.1** If $D(d) \neq \emptyset$, then $tb = d - q(d')d' > 0$ for all $d' \leq \bar{d}$.

**Proof:** The proof is by contradiction. Suppose that $D(d) \neq \emptyset$ and that $q(\hat{d}) \hat{d} - d \geq 0$ for some $\hat{d} \leq \bar{d}$. Then,

$$v^c(d, y) \equiv \max_{d' < \hat{d}} \left\{ u(y + q(d')d' - d) + \beta Ev^g(d', y') \right\}$$

$$\geq u(y + q(\hat{d}) \hat{d} - d) + \beta Ev^g(\hat{d}, y')$$

$$\geq u(y) + \beta Ev^b(y')$$

$$\equiv v^b(y),$$

for all $y \in Y$. But if $v^c(d, y) \geq v^b(y)$ for all possible endowments, then the default set must be empty ($D(d) = \emptyset$), which contradicts the assumptions of the proposition. In the above expression,
the first inequality follows from the definition of a maximum. The second inequality holds because, by assumption, \( q(\hat{d})\hat{d} - d \geq 0 \) and because, by definition, \( v^g(\hat{d}, y') \geq v^b(y') \).

The object \( q(d')d' - d \) represents the trade balance deficit (i.e., \( q(d')d' - d = c - y \)) for a country that chooses to continue to pay its debt. Thus, the proposition states that a country that has a level of debt that puts it at risk of default (\( D(d) \neq \emptyset \)) and that chooses to continue to participate in the financial market will devote part of its current endowment to servicing the debt, by running a trade balance surplus. Put differently, the proposition states that the economy runs trade deficits only when its debt position is such that the probability of default is nil (\( D(d) = \emptyset \)). This result is robust to assuming that the endowment is serially correlated (see exercise 12.9).

We now establish that in this economy the country tends to default in bad times. Specifically, we show that if a country with a certain level of debt and income chooses to default then it will also choose to default at the same level of debt and a lower level of income. In other worlds, if the default set is not empty then it is indeed an interval with lower bound given by the lowest endowment level \( y \).

**Proposition 12.2** If \( y_1 \in D(d) \) and \( y \leq y_2 < y_1 \), then \( y_2 \in D(d) \).

**Proof:** Suppose \( D(d) \neq \emptyset \). Consider any \( y \in Y \) such that \( y \in D(d) \). Let \( v_y^b(y) = \partial v^b(y)/\partial y \) and \( v_y^c(d, y) = \partial v^c(d, y)/\partial y \). By the envelope theorem, \( v_y^b(y) = u'(y) \) and \( v_y^c(d, y) = u'(y + q(d')d' - d) \).

\[ v^b(y_2) - v^c(d, y_2) = u(y_1) - u(y_1) + u(y_2) + \beta Ev^b(y') - \max_{d'} \{u(y_2 + q(d')d' - d) + \beta Ev^c(d', y')\} \]

\[ u(y_2) - u(y_1) + v^c(y_1) - \max_{d'} \{u(y_2 + q(d')d' - d) + \beta Ev^c(d', y')\} \]

\[ = u(y_2) - u(y_1) + v^b(y_1) - \max_{d'} \{u(y_2 + q(d')d' - d) - u(y_1 + q(d')d' - d) + u(y_1 + q(d')d' - d) + \beta Ev^c(d', y')\} \]

\[ \geq u(y_2) - u(y_1) + v^b(y_1) - \max_{d'} \{u(y_2 + q(d')d' - d) - u(y_1 + q(d')d' - d) \} - \max_{d'} \{u(y_1 + q(d')d' - d) + \beta Ev^c(d', y')\} \]

\[ = u(y_2) - u(y_1) + v^b(y_1) - u(y_1 + q(d)d - d) + u(y_1 + q(d)d - d) - \max_{d'} \{u(y_1 + q(d')d' - d) + \beta Ev^c(d', y')\} \]

\[ = [u(y_1 + q(d)d - d) - u(y_1 + q(d)d - d)] - [u(y_1) - u(y_2)] + v^b(y_1) - v^c(d, y_1) \]

\[ \hat{\alpha} \]

An alternative proof that does not rely on the differentiability of the value function is as follows.
By proposition 12.1, we have that $q(d')d' - d < 0$ for all $d' \leq \bar{d}$. This implies, by strict concavity of $u$, that $u'(y + q(d')d' - d) > u'(y)$. It follows that $v^b_y(y) - v^c_y(d, y) < 0$, for all $y \in D(d)$. That is, $v^b_y(y) - v^c_y(d, y)$ is a decreasing function of $y$ for all $y \in D(d)$. This means that if $v^b(y_1) > v^c(d, y_1)$, then $v^b(y_2) > v^c(d, y_2)$ for $y_1 \leq y_2 < y_1$. Equivalently, if $y_1 \in D(d)$, then $y_2 \in D(d)$ for any $y_1 \leq y_2 < y_1$. We have shown that the default set is an interval with a lower bound given by the lowest endowment $y$. We now show that the default set $D(d)$ is a larger interval the larger the stock of debt. Put differently, the higher the debt, the larger the probability of default.

**Proposition 12.3** If $D(d) \neq \emptyset$, then $D(d)$ is an interval, $[y, y^*(d)]$, where $y^*(d)$ is increasing in $d$ if $y^*(d) < \bar{y}$.

**Proof:** We already proved that the default set $D(d)$ is an interval. By definition, every $y \in D(d)$ satisfies $v^b_y(y) - v^c_y(d, y) > 0$. At the same time, we showed that $v^b_y(y) - v^c_y(d, y) < 0$ for all $y \in D(d)$. It follows that $y^*(d)$ is given either by $\bar{y}$ or (implicitly) by $v^b_y(y^*(d)) = v^c_y(d, y^*(d))$. Differentiating this expression yields

$$\frac{dy^*(d)}{dd} = \frac{v^c_y(d, y^*(d))}{v^b_y(y^*(d)) - v^c_y(d, y^*(d))},$$

where $v^c_y(d, y) \equiv \partial v^c_y(d, y)/\partial d$. We have shown that $v^b_y(y^*(d)) - v^c_y(d, y^*(d)) < 0$. Using the definition of $v^c_y(d, y)$ and applying the envelope theorem, it follows that $v^c_y(d, y^*(d)) = -u'(y^*(d) + q(d')d' - d) < 0$. We then conclude that

$$\frac{dy^*(d)}{dd} > 0,$$

as stated in the proposition.

$$> v^b(y_1) - v^c(d, y_1)$$

where $\bar{d}$ is the value of $d'$ that maximizes $\max_{d'} \{u(y_2 + q(d')d' - d) - u(y_1 + q(d')d' - d)\}$. The first inequality holds because we are distributing the max operator. The second inequality holds because $u$ is concave. And the third inequality follows because, by assumption, $y_1 \in D(d)$.
Summarizing, we have obtained two important results: First, given the stock of debt, default is more likely the lower the level of output. Second, the larger the stock of debt, the higher the probability of default. These two results are in line with the stylized facts presented earlier in this chapter, indicating that at the time of default countries tend to display above-average debt-to-GNP ratios (see table 12.3).

12.4.3 Default Risk and the Country Premium

We now characterize the behavior of the country interest-rate premium in this economy. Let the world interest rate be constant and equal to \( r^* > 0 \). We assume that foreign lenders are risk neutral and perfectly competitive. It follows that the expected rate of return on the country’s debt must equal \( r^* \). If the country does not default, foreign lenders receive \( \frac{1}{q(d')}(d') \) units of goods per unit lent. If the country does default, foreign lenders receive nothing. Therefore, equating the expected rate of return on the domestic debt to the risk-free world interest rate, one obtains

\[
1 + r^* = \frac{\text{Prob}\{y' \geq y^*(d')\}}{q(d')}.
\]

The numerator on the right side of this expression is the probability that the country will not default next period. Letting \( F(y) \) denote the cumulative density function of the endowment shock, we can write

\[
q(d') = \frac{1 - F(y^*(d'))}{1 + r^*}.
\]

This expression states that the gross country premium, \( 1/[q(d')(1 + r^*)] \), equals the inverse of the probability of repayment, or approximately one plus the probability of default. Hence, the net country premium is approximately equal to the probability of default. The above expression
implies that the derivative of the price of debt with respect to next period’s debt is given by

\[
\frac{dq(d')}{dd'} = -\frac{F'(y^*(d'))y^*(d')}{1 + r^*} \leq 0.
\]

The inequality follows because by definition \(F' \geq 0\) and because, by proposition 12.3, \(y^*(d') \geq 0\). It follows that the country spread, given by the difference between \(1/q(d')\) and \(1 + r^*\), is nondecreasing in the stock of debt. We summarize this result in the following proposition:

**Proposition 12.4** The country spread, given by \(1/q(d') - 1 - r^*\) is nondecreasing in the stock of debt.

### 12.5 Saving and the Breakdown of Reputational Lending

A key assumption of the reputational model of sovereign debt is that when a country defaults foreign lenders coordinate to exclude it from the possibility to borrow or lend in international financial markets. At a first glance, it might seem that what is important is that defaulters be precluded from borrowing in international financial markets. Why should defaulting countries not be allowed to save? Bulow and Rogoff (1989) have shown that prohibiting defaulters to lend to foreign agents, or to hold a positive net foreign asset position, is crucial for the reputational model to work. If delinquent countries were not allowed to borrow but could run current account surpluses, no lending at all could be supported on reputational grounds alone.

To illustrate this insight in a simple setting, consider a deterministic economy. Suppose that a reputational equilibrium supports a path for external debt given by \(\{d_t\}_{t=0}^{\infty}\), where \(d_t\) denotes the level of external debt assumed in period \(t\) and due in period \(t+1\).\(^9\) Assume that default is punished with perpetual exclusion from borrowing in international financial markets, but that saving in these

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\(^9\)For an example of a deterministic model with sovereign debt supported by reputation, see Eaton and Fernández (1995).
markets is allowed after default. This assumption and the fact that the economy operates under perfect foresight imply that any reputational equilibrium featuring positive debt in at least one date must be characterized by no default. To see this, notice that if the country defaults at some date \( T > 0 \), then no foreign investor would want to lend to this country in period \( T - 1 \), since default would occur for sure one period later. Thus, \( d_{T-1} \leq 0 \). In turn, if the country is excluded from borrowing starting in period \( T - 1 \), then it will have no incentives to honor any debts outstanding in that period. As a result, no foreign investor will be willing to lend to the country in period \( T - 2 \). That is, \( d_{T-2} \leq 0 \). Continuing with this logic, we arrive at the conclusion that default in period \( T \) implies no debt at any time. That is, \( d_t \leq 0 \) for all \( t \geq 0 \).

It follows from this result that in an equilibrium with positive external debt the interest rate must equal the world interest rate \( r^* > 0 \), because the probability of default is nil. The country premium is therefore also nil. The evolution of the equilibrium level of debt is then given by

\[
d_t = (1 + r^*)d_{t-1} - tb_t,
\]

for \( t \geq 0 \), where \( tb_t \equiv y_t - c_t \) denotes the trade balance in period \( t \). Assume that the endowment path \( \{y_t\}_{t=0}^{\infty} \) is bounded. Let \( d_T > 0 \) be the maximum level of external debt in this equilibrium sequence.\(^{10}\)

That is, \( d_T \geq d_t \) for all \( t \geq -1 \). Does it pay for the country to honor this debt? The answer is no. The reason is that the country could default in period \( T + 1 \)—and therefore be excluded from borrowing internationally forever thereafter—and still be able to maintain a level of consumption no lower than the one that would have obtained in the absence of default. To see this, let \( \tilde{d}_t \) for \( t > T \) denote the post-default path of external debt, or external assets if negative. Let the debt

\(^{10}\)Problem 12.10 asks you to derive the main result of this section when a maximal debt level does not exist.
position acquired in the period of default be

\[ \tilde{d}_{T+1} = -tb_{T+1}, \]

where \( tb_{T+1} \) is the trade balance prevailing in period \( T + 1 \) under the original debt sequence \( \{d_t\} \).

By (12.27) we have that 

\[ -tb_{T+1} = d_{T+1} - (1 + r^*)d_T, \]

which implies that

\[ \tilde{d}_{T+1} = d_{T+1} - (1 + r^*)d_T. \]  \hspace{1cm} (12.28)

Because by assumption \( d_T \geq d_{T+1} \) and \( r^* > 0 \), we have that

\[ \tilde{d}_{T+1} < 0. \]

That is, in period \( T + 1 \) the country can achieve the same level of trade balance (and hence consumption) under the default strategy as under the no-default strategy, without having to borrow internationally. Let the external debt position in period \( T + 2 \) in the default strategy be

\[ \tilde{d}_{T+2} = (1 + r^*)\tilde{d}_{T+1} - tb_{T+2}, \]

where, again, \( tb_{T+2} \) is the trade balance prevailing in period \( T + 2 \) under the original (no default) debt sequence \( \{d_t\} \). Using (12.27) and (12.28) we can rewrite the above expression as

\[ \tilde{d}_{T+2} = d_{T+2} - (1 + r^*)^2d_T < 0. \]

The inequality follows because by assumption \( d_{T+2} \leq d_T \) and \( r^* > 0 \). We have shown that the defaulting strategy can achieve the no-default level of trade balance in period \( t+2 \) without requiring any international borrowing. Continuing in this way, one obtains that the no-default sequence of
trade balances, \( tb_t \) for \( t \geq T + 1 \), can be supported by the debt path \( \tilde{d}_t \) satisfying

\[
\tilde{d}_t = d_t - (1 + r^*)^{t-T}d_T,
\]

which is strictly negative for all \( t \geq T + 1 \). The fact that the entire post-default debt path is negative implies that the country could also implement a post default path of trade balances \( \tilde{tb}_t \) satisfying \( \tilde{tb}_t \leq tb_t \) for \( t \geq T + 1 \) and \( \tilde{tb}_{t'} < tb_{t'} \) for at least one \( t' \geq T + 1 \) and still generate no positive debt at any date \( t \geq T + 1 \). This new path for the trade balance would be strictly preferred to the no-default path because it would allow consumption to be strictly higher than under the no-default strategy in at least one period and to be at least as high as under the no-default strategy in all other periods (recall that \( tb_t = y_t - c_t \)). It follows that it pays for the country to default immediately after reaching the largest debt level \( d_T \).

But we showed that default in this perfect foresight economy implies zero debt at all times. Therefore, no external debt can be supported in equilibrium. In other words, allowing the country to save in international markets after default implies that no equilibrium featuring strictly positive levels of debt can be supported on reputational grounds alone. For simplicity, we derived this breakdown result using a model without uncertainty. But the result also holds in a stochastic environment (see Bulow and Rogoff, 1989).

### 12.6 Quantitative Analysis Of The Eaton-Gersovitz Model

The reputational model of default analyzed in section 12.4 has been subject to intense quantitative scrutiny. However, as formulated there, the model is too stylized to capture salient features of actual defaults. To give the model a chance to match the data, researchers have enriched it along a number of dimensions. Three basic features that can be found in virtually all quantitative
models are serial correlation of the endowment process, a finite exclusion period from international
credit markets after default, and an output cost of default. These features render the model less
analytically tractable, but a full characterization of the equilibrium dynamics is possible using
numerical methods.

12.6.1 Serially Correlated Endowment Shocks

We begin by introducing the assumption that the endowment process has an autoregressive com-
ponent. Specifically, assume that the variable $y_t$ has the AR(1) law of motion

$$\ln y_t = \rho \ln y_{t-1} + \sigma \epsilon_t,$$

where $\ln$ denotes the natural logarithm, $\rho \in [0, 1)$ is a parameter denoting the serial correlation of
the endowment process, $\sigma > 0$ is a parameter denoting the standard deviation of the innovations to
the endowment process, and $\epsilon_t$ is an i.i.d. random variable following a standard normal distribution,
$\epsilon_t \sim N(0, 1)$. When $\rho = 0$, this process nests as a special case the white noise specification
assumed in sections 12.3 and 12.4. On the other extreme, Aguiar and Gopinath (2006) consider an
endowment process that is nonstationary in levels but follows an AR(1) law of motion in growth
rates, as in the small open endowment economy studied in chapter 2, section 2.4.

A theoretical implication of assuming a serially correlated output process is that now the period
$t$ price of debt assumed in period $t$ and due in period $t+1$ is no longer only a function of the amount
of debt assumed in $t$, $d_{t+1}$, but also of the current endowment, $y_t$. The reason is that, as we have
seen, the price of debt in $t$ depends on the expected value of default in $t + 1$. In turn, the decision
to default in $t + 1$ depends on that period’s output, $y_{t+1}$. When the output process is serially
correlated, $y_t$ provides information on $y_{t+1}$, and therefore affects the current price of debt. So we
can write the price of debt as $q(y_t, d_{t+1})$. 
12.6.2 Reentry

A second ubiquitous generalization of the default model is to assume that upon default the country is not perpetually excluded from international credit markets. This assumption makes the model more realistic, as defaulting countries are not excluded from international financial markets indefinitely. As discussed in section 12.2.1, depending on the empirical strategy, the typical exclusion period is estimated to last between 4.7 and 13.7 years (see table 12.6).

The assumption of permanent exclusion upon default is typically replaced with the assumption that after default the country regains access to financial markets with constant probability $\theta \in [0, 1)$ each period. This assumption implies that the average exclusion period is $1/\theta$ periods. To see this, assume that the first period of exclusion is the period of default. Then, the probability that the country will be excluded for exactly 1 period is $\theta$. The probability that the country will be excluded for exactly 2 periods is $(1 - \theta)\theta$. In general, the probability of being excluded for exactly $j$ periods is given by $(1 - \theta)^{j-1}\theta$. Thus we have

$$\text{average exclusion period} = 1 \times \theta + 2 \times (1 - \theta)\theta + 3 \times (1 - \theta)^2\theta + \ldots$$

$$= \theta \sum_{j=1}^{\infty} j(1 - \theta)^{j-1}$$

$$= \frac{1}{\theta}.$$  \hspace{1cm} (12.30)

Most calibrations of the default model use an estimate of the left-hand side of this expression (the average length of the exclusion period) to identify the value of $\theta$ (see, for instance, section 12.6.5).

The assumption of a constant probability of reentry regardless of how long the country has been excluded from credit markets has some empirical support. As shown in figure 12.1, the empirical distribution of the length of time defaulters are in default status resembles that of an exponential distribution.
The larger is \( \theta \), the quicker the country regains credit access after default. As a result, \( \theta \) affects the model’s predictions regarding default frequency, average risk premium, and the amount of debt that can be sustained in equilibrium. The assumed specification of reentry nests as a special case \( (\theta = 0) \), the setup studied earlier, in which financial autarky is an absorbent state.

It is common to assume that when the country regains good financial standing, it starts with no external obligations. As we saw in section 12.1.2, this assumption is unrealistic. There we documented that available estimates indicate that the typical haircut is about 40 percent of the external debt. Later, we will discuss theoretical studies that attempt to make the default model more realistic in this regard.

12.6.3 Output Costs

As it turns out, exclusion by itself is not enough punishment for defaulting countries, in the sense that it is unable to support empirically realistic levels of external debt.\(^{11}\) For this reason, a third generalization of the standard Eaton-Gersovitz model that has become commonplace in quantitative applications is the introduction of direct output costs of default. Typically the output cost of default is assumed to be exogenously given. Thus, upon default, countries are assumed to be punished not only by being excluded from international credit markets, but also by losing part of their endowment for the duration of the bad standing status. The empirical work surveyed in section 12.2.2 provides strong evidence that defaults are associated with sizable and protracted contractions in output. However, as discussed there, the direction of causality has not yet been established.

Assume that the endowment received by the country is not \( y_t \), but \( \tilde{y}_t \leq y_t \), where \( \tilde{y}_t \) is defined

\(^{11}\)We demonstrate this point in section 12.6.11 below, where we show that with exclusion as the only punishment the maximum predicted level of debt is zero.
as
\[ \tilde{y}_t = \begin{cases} 
    y_t & \text{if the country is in good standing} \\
    y_t - L(y_t) & \text{if the country is in bad standing}
\end{cases}, \]
where \( L(y_t) \) is an output loss function assumed to be positive and nondecreasing. The introduction of this type of direct costs affects the model’s predictions along two dimensions. First, it discourages default, and therefore tends to increase the amount of debt sustainable in equilibrium and to reduce the risk premium. Second, it discourages default in good states of nature, i.e., when \( y_t \) is high. This is because the higher is \( y_t \), the higher is the output loss in case of default, as \( L(y_t) \) is positive and nondecreasing.

This way of modeling output losses caused by default is, however, ad-hoc, as it is not based on micro-foundations. There have been some attempts at endogenizing this feature of the model. Mendoza and Yue (2012), for example, assume that imported inputs require working-capital financing. Default causes an increase in the cost of working capital, inducing an endogenous inefficient substitution toward domestic inputs. Na et al. (2014) study a model with downward nominal wage rigidity. Under suboptimal exchange-rate policy, such as a currency peg, default causes an endogenous increase in unemployment, which results in an inefficiently low level of output.

We assume the following specification for the loss function \( L(y_t) \)
\[ L(y_t) = \max\{0, a_0 + a_1 y_t + a_2 y_t^2\}. \] (12.31)
This specification encompasses a number of cases of interest. For example, Arellano (2008) assumes that when the country is in bad standing, it loses any endowment above a certain threshold, \( \bar{y} \), that is,
\[ y_t - L(y_t) = \begin{cases} 
    y_t & \text{if } y_t < \bar{y} \\
    \bar{y} & \text{if } y_t \geq \bar{y}
\end{cases}. \] (12.32)
Figure 12.8: Output Cost of Default

Figure 12.8 displays with a dashed line the endowment net of the output cost as a function of the endowment itself for the Arellano specification. This specification obtains as a special case of (12.31) when one sets $a_0 = -\bar{y}$, $a_1 = 1$, and $a_2 = 0$.

Chatterjee and Eyigungor (2012) adopt a two-parameter specification of the output loss function under which the output loss is quadratic at sufficiently large values of $y_t$. This case obtains by setting $a_0 = 0$, $a_1 < 0$, and $a_2 > 0$ in (12.31). Figure 12.8 displays with a dash-dotted line the endowment net of output cost under the Chatterjee-Eyigungor specification.

12.6.4 The Model

With these three modifications, the Eaton-Gersovitz model of section 12.4 changes as follows. The value of continuing to participate in financial markets, $\nu^c(d, y)$, is the solution to the Bellman
equation

\[ v^c(d, y) = \max_{d'} \left\{ u(y + q(d', y)d' - d) + \beta E_y v^g(d', y') \right\}, \]

where \( E_y \) denotes the expectations operator conditional on \( y \), and \( v^g(d, y) \) is the value of being in good financial standing. As before, we drop time subscripts and denote variables dated next period with a prime.

The value of being in bad financial standing is given by

\[ v^b(y) = u(y - L(y)) + \beta \theta E_y v^g(0, y') + \beta (1 - \theta) E_y v^b(y'). \]

The value of being in good financial standing, \( v^g(d, y) \), is given by

\[ v^g(d, y) = \max\{v^c(d, y), v^b(y)\}. \]

Finally, given the assumption that foreign lenders are risk neutral, the price of debt must satisfy

\[ q(d', y) = \frac{\text{Prob}_y\{v^c(d', y') \geq v^b(y')\}}{1 + r^*}, \tag{12.33} \]

where \( \text{Prob}_y \) denotes probability conditional on \( y \), and \( r^* \) is the risk free real interest rate, assumed to be constant.

The country interest rate, denoted \( r \) is given by the inverse of the price of debt minus one, \( 1/q(d', y) - 1 \),

\[ r \equiv \frac{1}{q(d', y)} - 1. \]

In turn, the country premium, or country spread, is defined as the difference between the country
interest rate, $r$, and the world interest rate $r^*$, that is

$$\text{country premium} = r - r^*.$$  

### 12.6.5 Calibration and Functional Forms

Obtaining quantitative predictions of the model requires assigning functional forms to preferences and technologies and numerical values to the structural parameters. We begin by assuming that one period in the model corresponds to one quarter (of a year). We adopt a CRRA form for the period utility function,

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma},$$

and set the intertemporal elasticity of substitution, $\sigma$, at 2, as in much of the related literature. We set the world interest rate, $r^*$, to 1 percent per quarter.

The remaining parameters are calibrated to match characteristics of the Argentine economy. To calibrate the probability of reentry, $\theta$, we revisit the evidence on the average exclusion period presented in section 12.2.1. Consider first measuring the exclusion period by the number of years a country is in default status. According to table 12.19, after the 1982 default Argentina was in default status until 1993, or 11 years. And after the 2001 default, Argentina was in default status until 2005, or 4 years. Thus, on average, Argentina was in default status for 7.5 years. Consider now measuring the end of the exclusion period by the first year of issuance of new debt. Cruces and Trebesch (2013, table A.2) find that for both Argentine defaults, the first period of issuance of new debt coincided with the end of default status, also suggesting an average exclusion period of 7.5 years. However, Gelos et al. (2011, table A.7) estimate that after the default of 1982, Argentina was able to issue new debt already in 1986, resulting in an exclusion period of 4 years. Their sample does not include the exit from the 2001 default. A simple average of the above estimates,
(7.5+7.5+4)/3, yields 6.33 years. We round this number to 6.5 years. Applying the formula in equation (12.30), this estimate yields a value of $\theta$ of 0.0385 at a quarterly frequency. This value is the same as the one used in Chatterjee and Eyigungor (2012).

We use data from Argentina to estimate the output process. Choosing a proxy for $y_t$ is complicated by the fact that in the model output is fully traded internationally. In reality, a large fraction of the goods and services produced by any country is nontraded. We choose to proxy $y_t$ by a measure of tradable output. In turn, as in chapter 9, we measure tradable output as the sum of GDP in agriculture, forestry, fishing, mining, and manufacturing in Argentina over the period 1983:Q1 to 2001:Q4. We obtain the cyclical component of output by removing a quadratic trend. The OLS estimate of (12.29) is then

$$\ln y_t = 0.9317 \ln y_{t-1} + 0.037 \epsilon_t. \quad (12.34)$$

The data used in the estimation is in the matlab file lgdp_traded.mat. The estimated process is quite persistent, with a serial correlation of 0.93. It is also quite volatile. The implied unconditional standard deviation of output is 10%. Such a volatile process gives risk averse domestic agents a strong incentive to use the current account to smooth consumption over time.

Following Chatterjee and Eyigungor (2012), we consider a two-parameter specification of the output loss function by setting $a_0 = 0$ in equation (12.31). We set $a_1 = -0.35$ and $a_2 = 0.4403$, which are the values assigned in Na, Schmitt-Grohé, Uribe, and Yue (2014). Also following these authors, we calibrate $\beta$, the subjective discount factor, at 0.85. Together with the rest of the parameter values, the chosen value for the triplet $(a_1, a_2, \beta)$ produces the following three equilibrium implications: (a) the average debt to GDP ratio in periods of good financial standing is about 60 percent per quarter; (b) the frequency of default is 2.6 times per century; and (c) the average output loss is 7 percent per year conditional on being in financial autarky. These three targets are
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Table 12.7: Calibration of the Default Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>2</td>
<td>Inverse of intertemporal elasticity of consumption</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.85</td>
<td>Quarterly subjective discount factor</td>
</tr>
<tr>
<td>$r^*$</td>
<td>0.01</td>
<td>World interest rate</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.0385</td>
<td>Probability of reentry</td>
</tr>
<tr>
<td>$a_0$</td>
<td>0</td>
<td>Parameter of output loss function</td>
</tr>
<tr>
<td>$a_1$</td>
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<td>Parameter of output loss function</td>
</tr>
<tr>
<td>$a_2$</td>
<td>0.4403</td>
<td>Parameter of output loss function</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.9317</td>
<td>Serial correlation of $\ln y_t$</td>
</tr>
<tr>
<td>$\sigma_\epsilon$</td>
<td>0.037</td>
<td>Std. dev. of innovation $\epsilon_t$</td>
</tr>
</tbody>
</table>

Discretization of State Space

| $n_y$ | 200 | Number of output grid points (equally spaced in logs) |
| $n_d$ | 200 | Number of debt grid points (equally spaced)           |
| $[y, \bar{y}]$ | [0.6523, 1.5330] | Output range                                           |
| $[d, \bar{d}]$ | [0, 1.5] | Debt range                                             |

Note. The time unit is one quarter.

empirically justified as follows:

(a) The target of a 60 percent quarterly debt-to-output ratio is motivated as follows. The net external debt in Argentina over the inter-default period 1994 to 2001 fluctuated around 30 percent of GDP (Lane and Milesi-Ferretti, 2007). At the same time, the haircuts in the 1982 and 2001 defaults were on average about 50 percent (Cruces and Trebesch, 2013).\footnote{Since 1975, Argentina restructured its debt 4 times, in August 1985, in August 1987, in April 1993, and in April 2005. The corresponding haircuts were, respectively, 0.303, 0.217, 0.325, and 0.768, and the amount of debt upon which these haircuts were applied were, respectively, 9.9, 29.5, 28.5, and 60.6 billion dollars. Therefore, the debt-weighted average haircut is 50.7 percent.} Since the Eaton-Gersovitz model assumes that the country defaults on 100 percent of the debt, we assume that only 50 percent of the country’s external debt is unsecured, and thus target an annual debt-to-output ratio of 15 percent or a quarterly debt-to-output ratio of 60 percent.

(b) The predicted average frequency of default of 2.6 times per century is in line with the
Argentine experience since the late 19th century. Table 12.1 implies that Argentina defaulted 5 times between 1824 and 2014. That is, 5 defaults over 191 years, or 2.6 times per century.

(c) The implied average output loss of 7 percent per year for the duration of the default status is in the lower range of the our estimates based on Zarazaga’s methodology for calculating output losses in the Argentine defaults of 1982 and 2001 (see the discussion in section 12.2.2 of this chapter).

The assumed value of $\beta$ is low compared the values used in models without default, but not uncommon in models à la Eaton-Gersovitz (see, for example, Mendoza and Yue, 2012).

Table 12.7 summarizes the calibration of the model.

12.6.6 Computation

We approximate the equilibrium dynamics by value-function iteration over a discretized state space. The AR(1) process for $\ln y_t$ given in equation (12.29) takes on continuous values, because the innovation $\epsilon_t$ is assumed to be normally distributed. We discretize this process using 200 equally spaced points for $\ln y_t$. The first and last values of the grid for $\ln y_t$ are set to $\pm 4.2$ times the unconditional standard deviation of the estimated process. Given the assumed normality of the process, the probability that (the log of) an output observation falls outside of this range is less than $10^{-4}$. The values of $\sigma_\epsilon$ and $\rho$ given in equation (12.34) imply an unconditional standard deviation of $\ln y_t$ of $0.037/\sqrt{1-0.9317^2}$ or 0.102. Thus, the first and last points of the grid for $\ln y_t$ are $\pm 0.427$.

To construct the transition probability matrix of the process $\ln y_t^T$, we apply the iterative procedure proposed by Schmitt-Grohé and Uribe (2013). Specifically, we simulate a time series of length 10 million drawn from the process (12.34). We associate each observation in the time series with one of the 200 possible discrete values of $\ln y_t$ by distance minimization. The resulting discrete-valued time series is used to compute the probability of transitioning from a particular discrete state in one period to a particular discrete state in the next period. Given the discretized
series of draws, the algorithm proceeds as follows. Start with a $200 \times 200$ matrix of zeros. Suppose
the first draw is element $i$ of the grid and the second draw is element $j$ of the grid. Then, add 1 to
element $(i, j)$ of the $200 \times 200$ matrix. Now suppose that the third draw is element $k$ of the grid.
Then add 1 to element $(j, k)$ of the $200 \times 200$ matrix. Continue in this way until draw $10^7$. Then
divide each row of the $200 \times 200$ matrix by the sum of its 200 elements. The resulting matrix is
the transition probability matrix we wished to estimate. It captures well the covariance matrices
of order 0 and 1. It is available in the file tpm.mat and the Matlab code used to compute it is
available in the file tpm.m.$^{13}$

Finally, the stock of net external debt, $d_t$, is also a continuous state of the model. We discretize
this variable with a grid of 200 equally spaced points starting at 0 and ending at 1.5. These two
values were chosen by a try-and-error procedure. Widening the grid did not produce significant
changes in the shape and position of the debt distribution. The matlab code eg.m computes the
equilibrium policy functions.

12.6.7 Quantitative Predictions of the Eaton-Gersovitz Model

Table 12.8 displays selected empirical and theoretical first and second moments. By design, the
model fits the average default frequency of 2.6 times per century and the average quarterly debt-to-
GDP ratio of about 60 percent. But the model also does quite well at replicating moments that were
not targeted in the calibration. Specifically, it explains the observed volatility and countercyclicality
of the country premium, as well as its positive correlation with the trade-balance-to-GDP ratio.
The countercyclicality of default risk is intuitive. The lower is output, the harder it is for the
country to give up goods to service the debt. This result is also in line with the theoretical analysis
of section 12.4. There, we proved that the default set is an interval, implying that, all other things

$^{13}$An alternative method for computing the transition probability matrix of the exogenous state is the quadrature-
Table 12.8: Selected First and Second Moments: Data and Model Predictions

<table>
<thead>
<tr>
<th></th>
<th>Default frequency</th>
<th>$E(d/y)$</th>
<th>$E(r - r^*)$</th>
<th>$\sigma(r - r^*)$</th>
<th>corr($r - r^*, y$)</th>
<th>corr($r - r^*, tb/y$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>2.6</td>
<td>58.0</td>
<td>7.4</td>
<td>2.9</td>
<td>-0.64</td>
<td>0.72</td>
</tr>
<tr>
<td>Model</td>
<td>2.6</td>
<td>59.0</td>
<td>3.5</td>
<td>3.2</td>
<td>-0.54</td>
<td>0.81</td>
</tr>
</tbody>
</table>

Note. Data moments are from Argentina over the inter-default period 1994:1 to 2001:3, except for the default frequency, which is calculated over the period 1824 to 2014. The variable $d/y$ denotes the quarterly debt-to-GDP ratio in percent, $r - r^*$ denotes the country premium, in percent per year, $y$ denotes (quarterly detrended) output, and $tb/y$ denotes the trade-balance-to-GDP ratio. The symbols $E$, $\sigma$, and corr denote, respectively, the mean, the standard deviation, and the correlation. In the theoretical model, all moments are conditional on the country being in good financial standing. Theoretical moments were computed by running the Matlab script `statistics_model.m`.

equal, if the country defaults at a certain level of the endowment, it also defaults at all other lower levels. The quantitative model studied here features an additional incentive not to default when output is high that is absent in the canonical model of section 12.4, namely an output cost of default, $L(y)$, that increases more than proportionally with the level of $y$.

The positive correlation between the trade balance and the country premium is also in line with the analytical results of section 12.4. Proposition 12.1 shows that in any period in which the default risk is positive, the trade balance must also be positive. Intuitively, when the country is at risk of default, foreign lenders demand that the economy makes an effort to improve its financial situation by at least paying part of the interest due.

The model, however, explains only half of the observed average country premium in Argentina (3.5 versus 7.4 percent per year). Indeed, in the context of the present model, it is impossible to explain both the observed average frequency of default and the observed average country premium. This is because in the model the average country premium is approximately equal to the average
frequency of default. To see this, note that the country premium, \( r - r^* \equiv 1/q(d', y) - (1 + r^*) \), is approximately equal to \( \ln \left( \frac{1}{q(d', y)(1+r^*)} \right) \). Then, from equation (12.33) we have that

\[
r - r^* \approx \ln \left( \frac{1}{q(d', y)(1+r^*)} \right) = \ln \left( \frac{1}{\text{Prob}\{\text{repayment in } t+1 \text{ given information in } t\}} \right) = \ln \left( \frac{1}{1 - \text{Prob}\{\text{default in } t+1 \text{ given information in } t\}} \right) \approx \text{Prob}\{\text{default in } t+1 \text{ given information in } t\}.
\] (12.35)

According to this expression, the model can explain either the average country premium or the average frequency of default, but not both at the same time, unless both moments are the same in the data.\(^{14}\)

A natural question is why the frequency of default and the average country premium are so different in the data. One reason for the discrepancy, discussed in section 12.1.5, is the sample mismatch problem. The average country premium is based on relatively few observations, the 31 quarters covering the period 1994:Q1 to 2001:Q3. By contrast, the default frequency is computed for a much longer sample, spanning the period 1824 to 2014. The sample mismatch problem could introduce a bias in the measured spread-default-frequency differential because the structure of the economy could have changed substantially over time. For it could be possible that the default frequency has increased in the past few decades rendering inappropriate the use of a long historical sample. Consider, for instance, the default history of Argentina over the past four decades. Between 1975 and 2014, Argentina defaulted twice, in 1982 and in 2001, which implies a default frequency of 5 defaults per one hundred years. This number is almost twice as large as the one based on the

\(^{14}\)The frequency of default reported in table 12.8 is defined as the number of defaults per one hundred years, whereas the probability of default that appears on the right-hand side of the above expression (\( \text{Prob}\{\text{default in } t+1 \text{ given information in } t\} \)), indicates the average number of defaults per one hundred years of good financial standing. In the model, these two moments are quite similar, 2.6 and 3.2, respectively.
There are also theoretical reasons, not captured in the present model, for a discrepancy between the average country premium and the frequency of default. We derived the result that the default frequency is approximately equal to the country premium under the assumptions that when the government defaults it does so on the totality of debt (no partial default) and that foreign lenders are risk neutral. Consider the case of partial default. Specifically, assume that if the country decides to default, it still honors a fraction $\lambda \in (0, 1)$ of its obligations. In this case, the price of debt must satisfy

$$q(d', y) = \frac{\lambda \text{Prob}(v^c(d', y') < v^b(y')) + \text{Prob}(v^c(d', y') \geq v^b(y'))}{1 + r^*}. \quad (12.36)$$

A derivation similar to the one presented in equation (12.35) yields that the country premium, $r - r^*$, satisfies

$$r - r^* \approx (1 - \lambda) \text{Prob}\{\text{default in } t + 1 \text{ given information in } t\}. \quad (12.37)$$

That is, the size of the haircut, $1 - \lambda$, is a wedge between the country premium and the probability of default. Note, however, that this wedge makes the country premium smaller than the probability of default—which is intuitive because the fact that the country honors a fraction of the debt no matter what, implies that lenders require less compensation than in the case of full default. But this makes the puzzle even worse, since in the data the premium exceeds the default risk.

Finally, altering the assumption of risk-neutral lenders may also break the equality result. Lizarazo (2013) shows that this is indeed the case by augmenting an otherwise standard Eaton-Gersovitz model with the assumption of risk averse lenders. She shows that for plausible calibrations this assumption increases the predicted average country spread substantially without significantly affecting the average probability of default. The reason for the predicted increase in the country spread is that in this type of environment, the country spread is the sum of two compensations,
one for the possibility of default and a second one necessary to induce risk sensitive creditors to accept the default risk. Lizarazo’s result relies on the assumption that default has a sizable negative wealth effect on the creditor. In section 12.9, we show that under the realistic assumption that the emerging country is too small to affect the wealth of the creditor country (and hence the world interest rate), the model implies a near zero spread-default-frequency differential, even under high degrees of risk aversion on the part of foreign lenders.

The Eaton-Gersovitz model captures some of the standard regularities emphasized in the business-cycle literature, as shown in table 12.9. One is the excess volatility of consumption. In chapter 1, we documented that in emerging countries consumption is at least as volatile as output. The model predicts that consumption is 1.22 times as volatile as output, which is in line with the values of 1.23 and 1.11 observed in emerging countries and Argentina, respectively. The reason for the predicted excess volatility of consumption is that during periods of good financial standing, the country is buffeted by large movements in the interest rate due to cyclical variations in default risk, which causes consumption to move significantly over the business cycle. During periods of bad financial standing, movements in consumption mimic exactly the movements in the endowment net of the output loss, which makes these two variables equally volatile.

Further, the model can explain well the observed relative volatility of the trade-balance-to-output ratio relative to the volatility of output (0.57 in the model versus 0.69 in emerging countries and 0.48 in Argentina). The model is equally successful in matching the procyclicality of consumption, as measured by its correlation with output (0.88 in the model versus 0.72 and 0.75 in emerging countries and Argentina, respectively.) Finally, although the model captures the countercyclicality of the trade balance qualitatively, as it correctly predicts a negative correlation of the trade-balance-to-output ratio with output, it significantly underpredicts its magnitude (the correlation of \( tb/y \) with \( y \) is -0.14 in the model versus -0.51 and -0.87 in emerging countries and Argentina, respectively.) The predicted countercyclicality of the trade balance is a step forward relative to an
identical model without default risk. We saw in chapter 2, that in the absence of default, positive temporary output shocks induce an increase in savings and therefore an improvement in the trade balance. Thus, the model without default risk predicts a procyclical trade balance. The presence of default risk makes the interest rate countercyclical (see table 12.8), because in the Eaton-Gersovitz model default incentives are weak when output is high and strong when output is low. In turn, the countercyclicality of the interest rate makes savings countercyclical, allowing for the possibility that the trade balance itself become countercyclical.

12.6.8 Dynamics Around A Typical Default Episode

What happens around default episodes? To answer this question, we simulate 1.1 million time periods from the theoretical model. After discarding the first 0.1 million periods, we identify all periods in which a default occurs and extract a window of 12 quarters prior to default and 12 quarters after default. Finally, we compute the median period by period across all windows and normalize the period of default to 0. Figure 12.9 presents the behavior of the economy around the typical default episode. Defaults occur after a sudden contraction in output. As shown in the upper

Table 12.9: Data and Model Predictions: Additional Business-Cycle Statistics

<table>
<thead>
<tr>
<th></th>
<th>$\sigma(c)/\sigma(y)$</th>
<th>$\sigma(tb/y)/\sigma(y)$</th>
<th>$\text{corr}(c, y)$</th>
<th>$\text{corr}(tb/y, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Emerging Countries</td>
<td>1.23</td>
<td>0.69</td>
<td>0.72</td>
<td>-0.51</td>
</tr>
<tr>
<td>Argentina</td>
<td>1.11</td>
<td>0.48</td>
<td>0.75</td>
<td>-0.87</td>
</tr>
<tr>
<td>Model</td>
<td>1.22</td>
<td>0.57</td>
<td>0.88</td>
<td>-0.14</td>
</tr>
</tbody>
</table>

Note. Data moments for emerging countries and Argentina are taken from chapter 1, tables 1.7 and 1.9, respectively. The symbols $c$ and $y$ denote the log deviation from trend, $tb/y$ denotes the trade-balance-to-output ratio, and $\sigma$ and corr denote, respectively, standard deviation and correlation.
Figure 12.9: Typical Default Episode

Note. Solid lines display medians of 25-quarter windows centered around default episodes occurred in an artificial time series of 1 million quarters. The default date is normalized to 0. Dotted lines display medians conditional on continuing to participate in financial markets. The figure is produced by running the matlab script typical_default_episode.m
left panel, \( y \) is at its mean level of unity until three quarters prior to default. Then, three consecutive negative shocks push \( y \) 1.3 standard deviations below normal. One may wonder whether a fall in traded output of slightly more than one standard deviation squares with a predicted average default frequency of only 2.6 per century (see table 12.8). The reason why it does is that it is the sequence of output shocks that matters. The probability of traded output falling from its mean value to 1.3 standard deviations below mean in only three quarters is much lower than the unconditional probability of traded output being 1.3 standard deviations below mean.

In period 0, the government defaults, triggering a loss of output \( L(y_t) \), as shown by the difference between the solid and the broken lines in the upper left panel. After the default, output begins to recover. Thus, the period of default coincides with the trough of the contraction in output, \( y_t \). Therefore, the model captures the empirical regularity regarding the cyclical behavior of output around default episodes documented in figure 12.2 and first identified by Levy-Yeyati and Panizza (2011), according to which default marks the end of a contraction and the beginning of a recovery.

As can be seen in figure 12.9, the model predicts that the country does not smooth out the temporary decline in the endowment. Instead, the country sharply reduces consumption, by 14 percent. The contraction in consumption is actually larger than the contraction in the endowment so that the trade balance improves. In fact, the trade balance surplus is large enough to generate a slight decline in the level of external debt. These dynamics seem at odds with the quintessential dictum of the intertemporal approach to the balance of payments according to which countries should finance temporary declines in income by external borrowing. The country deviates from this prescription because foreign lenders raise the interest rate premium prior to default. The bottom right panel of figure 12.9 shows that the country spread doubles from 3 to 6 percent per year in the run up to default. This increase in the cost of credit discourages borrowing and induces agents to postpone consumption.
12.6.9 Goodness of Approximation of the Eaton-Gersovitz Model

We used the quantitative predictions of the Eaton-Gersovitz model to gauge its ability to explain observed patterns of default and country spread dynamics and their comovement with other macroeconomic indicators. Because the model does not have a closed form solution, its quantitative predictions are based on an approximation. As a result, the validity of the model evaluation requires trust in the accuracy of the approximate solution. The question of how close the approximation is to the true solution is impossible to answer with certainty because the latter is unknown. One way to address this issue is based on the reasonable assumption that as the number of grid points is increased, the approximate solution gets closer to the true solution. This suggests an accuracy test consisting in examining how stable the quantitative predictions of the model are to varying the number of grid points.

Hatchondo, Martínez, and Sapriza (2010) find that the numerical solution of the Eaton-Gersovitz model deteriorates significantly when the endowment grid is coarsely specified. The deterioration affects primarily the volatility and comovement of the country premium. To check the validity of their result in the context of the present parameterization of the Eaton-Gersovitz model, in Table 12.10 we present the predictions of the model under a number of alternative grid specifications. Consider an approximation based on an endowment grid containing 25 equally spaced points, 8 times coarser than the baseline grid specification, which contains 200 equally spaced endowment points. A specification with 25 grid points is of interest because it is representative of the one used in most early quantitative default studies (e.g., Arellano, 2008; and Aguiar and Gopinath, 2006). The table shows that the coarser approximation affects mostly the correlation of the country premium with output and with the trade-balance-to-output ratio. Although the sign is preserved, the magnitude of both correlations falls to about half. Also affected are the standard deviation of the country premium and the average debt-to-output ratio. The former is an entire percentage point
### Table 12.10: Approximating the Eaton-Gersovitz Model: Accuracy Tests

<table>
<thead>
<tr>
<th>Grid Points</th>
<th>Default frequency</th>
<th>E(d/y)</th>
<th>$E(r - r^*)$</th>
<th>$\sigma(r - r^*)$</th>
<th>$(r - r^*, y)$</th>
<th>$(r - r^*, tb/y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>2.6</td>
<td>58.0</td>
<td>7.4</td>
<td>2.9</td>
<td>-0.64</td>
<td>0.72</td>
</tr>
<tr>
<td>Model*</td>
<td>200 200</td>
<td>2.65</td>
<td>59.05</td>
<td>3.47</td>
<td>3.21</td>
<td>-0.54 0.81</td>
</tr>
<tr>
<td>Model</td>
<td>25 200</td>
<td>2.30</td>
<td>69.43</td>
<td>3.01</td>
<td>4.20</td>
<td>-0.28 0.44</td>
</tr>
<tr>
<td>Model</td>
<td>400 200</td>
<td>2.63</td>
<td>58.64</td>
<td>3.43</td>
<td>3.12</td>
<td>-0.55 0.82</td>
</tr>
<tr>
<td>Model</td>
<td>200 400</td>
<td>2.65</td>
<td>59.46</td>
<td>3.44</td>
<td>3.13</td>
<td>-0.55 0.83</td>
</tr>
<tr>
<td>Model</td>
<td>400 400</td>
<td>2.65</td>
<td>59.46</td>
<td>3.44</td>
<td>3.13</td>
<td>-0.55 0.83</td>
</tr>
</tbody>
</table>

Note. Data moments are from Argentina over the inter-default period 1994:1 to 2001:3, except for the default frequency, which is calculated over the period 1824 to 2014. The variable $d/y$ denotes the quarterly debt-to-GDP ratio in percent, $r - r^*$ denotes the country premium, in percent per year, $y$ denotes (quarterly detrended) output, and $tb/y$ denotes the trade-balance-to-GDP ratio. The symbols $E$ and $\sigma$ denote, respectively, the mean and the standard deviation. The symbols $n_y$ and $n_d$ denote the number of grid points for the endowment and debt, respectively. In the theoretical model, all moments are conditional on the country being in good financial standing. Theoretical moments were computed by running the Matlab script `statistics.m` after appropriately adjusting the number of grid points in `eg.m`.  

*Baseline grid specification.
higher than under the baseline grid specification and the latter is 10 percentage points higher.

A natural question is whether the predictions of the model also change as one increases the number of endowment points above the baseline value of 200. Table 12.10 shows that this not the case. All first and second moments displayed are quite stable as the number of endowment grid points is doubled from 200 to 400. Furthermore, the predictions under the baseline grid specification do not change substantially as one doubles the number of debt grid points from 200 to 400 either holding constant or doubling the number of endowment grid points. We therefore conclude that the baseline grid specification (with 200 points for the endowment and 200 points for debt) yields a reasonable numerical approximation to the equilibrium dynamics of the Eaton-Gersovitz model studied here.

12.6.10 Alternative Output Cost Specification

Thus far, we have assumed a two-parameter specification of the output cost function \( L(y_t) \) that is quadratic above some level of output. Another form that is often used in the default literature is one in which during periods of bad financial standing all output beyond a certain threshold is lost. This specification is given in equation (12.32) and illustrated with a dashed line in figure 12.8. As mentioned earlier, it is a special case of the three-parameter quadratic form given in (12.31) that arises when the coefficient of the constant term, \( a_0 \), takes a negative value, the coefficient of the linear term, \( a_1 \), is unity, and the coefficient of the quadratic term, \( a_2 \), is nil.

To make the calibration of the model under this cost function comparable to the one associated with the baseline (quadratic) specification, we set \( \beta \) and \( a_0 \) to match the average debt-to-output ratio and the average default frequency observed in Argentina. This yields \( \beta = 0.875 \) and \( a_0 = -0.88 \). The fact that the present specification features one parameter less than the quadratic specification means that one targeted empirical statistic must be left out. In this case, it is the average output cost during periods of bad financial standing. The present parameterization delivers
Table 12.11: The Eaton-Gersovitz Model: Alternative Output Cost Specification

<table>
<thead>
<tr>
<th></th>
<th>Default frequency</th>
<th>E(d/y)</th>
<th>E(r – r*)</th>
<th>σ(r – r*)</th>
<th>corr(r – r*, y)</th>
<th>corr(r – r*, tb/y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>2.6</td>
<td>58.0</td>
<td>7.4</td>
<td>2.9</td>
<td>-0.64</td>
<td>0.72</td>
</tr>
<tr>
<td>Model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quadratic</td>
<td>2.6</td>
<td>59.0</td>
<td>3.5</td>
<td>3.2</td>
<td>-0.54</td>
<td>0.81</td>
</tr>
<tr>
<td>Flat</td>
<td>2.8</td>
<td>59.9</td>
<td>3.5</td>
<td>4.2</td>
<td>-0.43</td>
<td>0.74</td>
</tr>
<tr>
<td>Flat and n_y = 25</td>
<td>2.4</td>
<td>71.4</td>
<td>3.1</td>
<td>5.7</td>
<td>-0.26</td>
<td>0.48</td>
</tr>
</tbody>
</table>

Note. See note to table 12.8. Quadratic refers to the baseline specification, \( L(y) = \max(0, -0.35y + 0.4403y^2) \). Flat refers to the specification \( L(y) = \max(0, -0.88 + y) \).

an average output cost of default of 8.3 percent of the endowment per period for the duration of the bad financial status. This number is larger than the one corresponding to the baseline calibration (7 percent) but still within the range estimated for Argentina by Zarazaga (2012), which we discuss earlier in section 12.2.2.

The predictions of the model are displayed in table 12.11. For comparison, the table reproduces from table 12.8 the empirical moments and the moments predicted by the baseline model. Overall, the model with a flat post-default endowment performs as well as the model with quadratic post default output. Moreover, as one makes the output grid coarser (in the table \( n_y \) is reduced from 200 to 25), the model deteriorates along the same dimensions as it does under the quadratic specification, namely, by overpredicting the volatility of the country spread and by underpredicting the correlation of the spread with output and the correlation of the spread with the trade-balance-to-output ratio.

A third specification of the output loss function, employed in a number of existing studies, is of the form

\[ L(y) = a_1 y, \]
with $a_1 \in (0, 1)$. Under this formulation, the output cost of default is proportional to the endowment level. Unlike the two specifications considered thus far, the present one does not punish default relatively more when the endowment is high than when it is low. Exercise 12.11 explores the quantitative implications of this formulation and in particular the way it affects the model’s ability to explain why countries default in bad times.

### 12.6.11 The Quantitative Importance of Output Costs of Default

The original formulation of the Eaton-Gersovitz model (Eaton and Gersovitz, 1981) contemplates a single cost of default, namely, exclusion from international financial markets. As explained earlier, quantitative analyses of this model invariably include an output cost of default, as embodied in the function $L(y)$. The main purpose of this source of cost of default is to support realistic levels of debt in equilibrium.

To highlight the role of the output cost of default, we now consider the case $L(y) = 0$ for all $y$, by setting $a_0 = a_1 = a_2 = 0$ in equation (12.31). The debt grid is set to range from -0.001 to 0.001 with 200 equally spaced points. All other parameters of the model are set at their baseline values shown in table 12.7. Under this parameterization, the model is unable to support any debt in equilibrium. The intuition for this result is that in the absence of an output cost of default, the only cost of debt repudiation is financial autarky, which makes it impossible for the country to use the current account to smooth output shocks. However, we know from the work of Lucas (1987) that the welfare cost of consumption volatility is quite small, which suggests that the cost of financial autarky is small. On the other hand, the benefit of default is a permanent increase in consumption as the obliteration of the external debt frees up resources that would have otherwise been allocated to service external obligations. Under the current calibration, this benefit outweighs the cost for all values of debt and income.

Figure 12.10 displays the default decision in the vicinity of zero debt. It plots $v^c(d, y) - v^b(y)$,
as a function of output for three different values of debt. The solid line corresponds to the case of zero debt. In this case, the country is indifferent between defaulting and not defaulting at any level of output. This can only be the case if the country assigns no value to the possibility of accessing financial markets. The broken line displays the case of debt positive and equal to $10^{-5}$, which is the smallest possible debt value in the grid. For this value of debt, the value of continuing to participate in credit markets is below the value of being in bad financial standing for all levels of output, meaning that the country defaults with probability one. Finally, the dashed-dotted line shows the case of a level of debt equal to $-10^{-5}$, the smallest possible level of assets in the grid. Obviously, in this case, the value of not defaulting exceeds the value of defaulting at all levels of output.

We conclude that incorporating some sort of output loss due to default is essential for the empirical performance of the Eaton-Gersovitz model.
12.6.12 The Quantitative Irrelevance of Exclusion

The quintessential element of the Eaton-Gersovitz model is that default is punished by exclusion from international credit markets. In the original formulation of the model, this feature allows for the existence of debt in equilibrium in an environment in which the country cannot commit to repay. Quantitative implementations of the Eaton-Gersovitz model have introduced an additional cost of default in the form of an output loss. Here, we gauge the quantitative relevance of the exclusion assumption. To this end, we remove exclusion from the quantitative model analyzed thus far, keeping all other features unchanged.

Specifically, suppose now that even if a country is in bad financial standing because it defaulted, it can still borrow and save internationally. Continue to assume, however, that if a country defaults it suffers the output cost of default $L(y)$ until it exogenously exits default status with probability $\theta$ each period. Under this modification, the model changes as follows. The value of being in good financial standing and continuing to honor financial obligations, $v^{gc}(d, y)$, is the solution to the Bellman equation

$$v^{gc}(d, y) = \max_{d'} \left\{ u(y + q^g(d', y)d' - d) + \beta E_y v^{gc}(d', y') \right\},$$

where, as before, $E_y$ denotes the expectations operator conditional on $y$, $v^g(d, y)$ is the value of being in good financial standing, and $q^g(d', y)$ denotes the price of debt if the country ends the period in good financial standing.

The value of defaulting, $v^d(y)$, is given by

$$v^d(y) = \max_{d'} \left\{ u(y - L(y) + q^b(d', y)d') + \beta \theta E_y v^g(d', y') + \beta (1 - \theta) E_y v^b(d', y') \right\},$$

where $v^b(d, y)$ denotes the value of being in bad financial standing. Note that the value of defaulting
is independent of whether the country started the period in good or bad financial standing.

The value of being in good financial standing is then given by

$$v^g(d, y) = \max\{v^{gc}(d, y), v^d(y)\}.$$ 

When the country ends the period in good financial standing, the price of debt satisfies

$$q^g(d', y) = \frac{\text{Prob}_y\{v^{gc}(d', y') \geq v^d(y')\}}{1 + r^*},$$

where, as before, $\text{Prob}_y$ denotes probability conditional on $y$, and $r^*$ is the constant risk free real interest rate.

Suppose now that the country is in bad standing. Then it must pay the output cost, so that its income is $y - L(y)$. Contrary to the standard model, we assume that the country can access international financial markets even when it is in bad financial standing. The value of being in bad financial standing and continuing to service the debt, $v^{bc}$, is

$$v^{bc}(d, y) = \max_{d'} \left\{ u(y - L(y) + q^b(d', y)d' - d) + \beta \theta E_y v^g(d', y') + \beta(1 - \theta) E_y v^b(d', y') \right\}.$$

The value of being in bad financial standing is then given by

$$v^b(d, y) = \max\{v^{bc}(d, y), v^d(y)\}.$$

The reason why $v^b(d, y)$ may not always be equal to $v^d(y)$ is that a country in bad standing may have assets $d < 0$, in which case it will never default. The price of debt in periods of bad financial
Table 12.12: The Quantitative Irrelevance of Exclusion: Selected First and Second Moments

<table>
<thead>
<tr>
<th></th>
<th>Default frequency</th>
<th>$E(d/y)$</th>
<th>$E(r - r^*)$</th>
<th>$\sigma(r - r^*)$</th>
<th>$\text{corr}(r - r^*, y)$</th>
<th>$\text{corr}(r - r^*, tb/y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Data</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Baseline</strong></td>
<td>2.6</td>
<td>59.0</td>
<td>7.4</td>
<td>2.9</td>
<td>-0.64</td>
<td>0.72</td>
</tr>
<tr>
<td><strong>No Exclusion</strong></td>
<td>2.6</td>
<td>59.0</td>
<td>3.5</td>
<td>3.2</td>
<td>-0.54</td>
<td>0.81</td>
</tr>
</tbody>
</table>

Note. See notes to table 12.8. The predictions of the theoretical model under no exclusion are produced by running the matlab script statistics_model_no_exclusion.m.

As in the baseline case, we solve this version of the model numerically using state space methods and employ the baseline parameter values shown in table 12.7. The approximation is performed using the matlab script eg_no_exclusion.m.

Table 12.12 presents the predictions of this version of the model. For comparison it reproduces the data moments and the theoretical moments predicted by the baseline model. The model in which default is not punished by exclusion from financial markets behaves remarkably similar to the baseline model. Specifically, the no-exclusion model can support about the same amount of debt as the model with exclusion. The mean debt to output ratio in times of good financial standing is 53 percent per quarter in the no-exclusion model, slightly below the value of 59 percent predicted by the baseline model. This means that exclusion can support a level of debt of 6 percent of quarterly output, whereas the direct output cost $L(y)$ explains a level of debt of 53 percent of quarterly output. The average country premium in times of good standing is predicted to be 4.1
percent in the no-exclusion model compared to 3.5 percent in the baseline case. The volatility of the
country premium and the correlation of the country premium with output and the trade balance
to output ratio are also little changed. The model predicts that on average the country defaults
three times per century compared to a default frequency of 2.6 times per century predicted by the
baseline model.

We conclude that exclusion from credit markets plays a negligible role for the quantitative
performance of the Eaton-Gersovitz model. The main mechanism supporting debt in equilibrium
is the output loss associated with default.

12.6.13 The Role Of Discounting

The predictions of the Eaton-Gersovitz model are particularly sensitive to the assumed value for the
subjective discount factor $\beta$. One reason is that the lower is $\beta$, the higher is the household’s appetite
for present consumption and therefore the higher the demand for debt. Under commitment, the
equilibrium level of debt is higher the lower is $\beta$. With default risk, this relationship becomes more
complex because foreign lenders take into account the household’s impatience in determining the
supply of funds to the borrowing country. Thus, $\beta$ affects not only the demand for debt but also
the supply of funds to the country.

Another channel through which $\beta$ affects the predictions of the Eaton-Gersovitz model is through
its effect on the costs of default. The reason is that default condemns agents to financial autarky
and to an output loss for multiple periods. The more agents care about the future (i.e., the higher
is $\beta$) the larger is the present value of these two types of cost. We should therefore expect that
the frequency of default falls as $\beta$ is increased. In turn, a lower frequency of default should be
associated with a lower country premium. This incentivizes households to borrow more, which
means that we should expect a higher equilibrium level of debt during periods of good standing.

Table 12.13 displays the predictions of the model for values of $\beta$ of 0.85, 0.90 and 0.95. Recall
that the baseline value is 0.85. The table confirms the intuition given above. The default frequency falls from 2.6 defaults per century to 0.4 defaults as $\beta$ rises from 0.85 to 0.95. The mirror image of this fall in the frequency of default is a drop in the average country premium from 3.5 percent per year to 0.5 percent. Concurrently, the average debt to output ratio during periods of good standing increases from 59 percent per quarter to 87.8 percent. This results suggest that the equilibrium level of debt is increasing in $\beta$, which is the opposite of what happens under commitment.

### 12.6.14 Changing the Volatility Of The Endowment Process

An increase in the volatility of the income process has two effects on the predictions of the model. One is that agents are more frequently exposed to large negative income shocks and as a result have a higher incentive to default. This should drive up the default frequency and the country premium. The second effect is that increased uncertainty induces a rise in precautionary savings and consequently a fall in the desired level of external debt. Table 12.14 displays the predictions of the model for three values of $\sigma_\epsilon$, the standard deviation of the innovation to the (log of the) endowment process, the baseline value of 0.037, a lower value of 0.03, and a higher value of 0.045. The latter two values represent a change in the standard deviation of the log of output of $\pm 20$
percent. The table shows that, in line with the intuition provided above, the frequency of default and the country premium rise from 2.1 times per century and 2.6 percent per annum to 3.0 and 4.2, respectively, as $\sigma_\epsilon$ increases from 0.03 to 0.045. At the same time, the average net external debt in periods of good financial standing falls from 73 percent of quarterly output to 50 percent.

**Time-Varying Volatility, Country Spreads, And Default**

The preceding analysis concerns the effect of permanent changes in output volatility. But changes in volatility can also be temporary. Indeed, it has been documented that emerging countries experience important changes in volatility over the business cycle (Fernández-Villaverde, et al., 2011). Seoane (2014) studies the effect of time-varying output volatility on the equilibrium behavior country spreads and default in the context of the Eaton-Gersovitz model. He begins by documenting a positive correlation between output volatility and country spreads for four European peripheral countries (Greece, Italy, Portugal, and Spain). Then Seoane augments the Eaton-Gersovitz model to allow for time-varying volatility in the endowment process. In particular, he formulates the

<table>
<thead>
<tr>
<th>$\sigma_\epsilon$</th>
<th>Default frequency</th>
<th>$E(d/y)$</th>
<th>$E(r - r^*)$</th>
<th>$\sigma(r - r^*)$</th>
<th>$\text{corr}(r - r^*, y)$</th>
<th>$\text{corr}(r - r^*, tb/y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>2.6</td>
<td>58.0</td>
<td>7.4</td>
<td>2.9</td>
<td>-0.64</td>
<td>0.72</td>
</tr>
<tr>
<td>Model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_\epsilon = 0.030$</td>
<td>2.1</td>
<td>72.6</td>
<td>2.6</td>
<td>2.3</td>
<td>-0.56</td>
<td>0.85</td>
</tr>
<tr>
<td>$\sigma_\epsilon = 0.037^*$</td>
<td>2.6</td>
<td>59.0</td>
<td>3.5</td>
<td>3.2</td>
<td>-0.54</td>
<td>0.81</td>
</tr>
<tr>
<td>$\sigma_\epsilon = 0.045$</td>
<td>3.0</td>
<td>49.7</td>
<td>4.2</td>
<td>4.0</td>
<td>-0.49</td>
<td>0.74</td>
</tr>
</tbody>
</table>

Note. See notes to table 12.8. $^*$ = baseline value.
following law of motion for the natural logarithm of the endowment

$$\ln y' = \rho \ln y + e^{\sigma'} \epsilon',$$

with

$$\sigma' - \bar{\sigma} = \rho_\sigma (\sigma - \bar{\sigma}) + \sigma_\mu \mu',$$

where $\epsilon$ and $\mu$ are normally distributed i.i.d. innovations with mean 0 and unit variance. Here, $e^{\sigma}$ is a time-varying volatility with a log-normal distribution with serial correlation $\rho_\sigma$. Seoane estimates the above law of motion separately for Greece, Italy, Portugal, and Spain using quarterly data on output over the period 1980 to 2011. Estimates are quite homogeneous across countries. The average estimated parameter values are $\bar{\sigma} = -4.55$, $\rho = 0.989$, $\rho_\sigma = 0.95$, and $\sigma_\mu = 0.16$. Seoane finds that when fed with a process of this type, the Eaton-Gersovitz model predicts a positive correlation between output volatility and the country spread, which is in line with the data. He also reports that this correlation is larger in the data (around 0.6 for the four European countries mentioned above) than in the model (below 0.2). Taken together this result and those presented in table 12.14 suggest that the positive link between output volatility and spreads predicted by the Eaton and Gersovitz holds for both permanent and transitory changes in output volatility.

12.6.15 Varying The Persistence Of The Output Process

Table 12.15 displays the predictions of the model for different values of $\rho$, the parameter measuring the first-order autocorrelation of the endowment process. The changes in $\rho$ are variance preserving. That is, changes in $\rho$ are accompanied by adjustments in $\sigma_\epsilon$ (the standard deviation of the innovation to the endowment process) to ensure that the unconditional variance of (the log of) the endowment is constant at its baseline value. The table displays predicted moments for values of $\rho$ ranging from 0 to 0.97. For low values of $\rho$ the frequency of default and the country premium are small. This is
because when the endowment process is not highly serially correlated, following negative shocks, the endowment is expected to recover quickly to values at which the output cost of default is nonzero (recall that $L(y)$ is increasing in $y$). As a result the economy defaults infrequently. By contrast, when the endowment process is highly serially correlated, negative income shocks are expected to persist over time causing the economy to default more frequently in bad states. Because the frequency of default is low for low values of $\rho$, the economy is able to borrow more, resulting in relatively high equilibrium levels of debt. This negative relationship between $\rho$ and the level of debt is broken at very high values of $\rho$ (above 0.94). When the endowment is very persistent, positive shocks are expected to last for a very long period of time. Because the output loss associated with default is high conditional on the endowment being high, the foreign lenders charge low interest rate premia, inducing the economy to take on relatively high levels of debt. At the same time, conditional on the endowment being low, the economy defaults more frequently, because bad states are expected to persist. Thus, at high values of $\rho$ the model predicts a paradoxical mix of high

<table>
<thead>
<tr>
<th>Data</th>
<th>Model</th>
<th>Default frequency</th>
<th>$E(d/y)$</th>
<th>$E(r - r^*)$</th>
<th>$\sigma(r - r^*)$</th>
<th>corr($r - r^*, y$)</th>
<th>corr($r - r^*, tb/y$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.6</td>
<td>0</td>
<td>0.1</td>
<td>274.1</td>
<td>0.1</td>
<td>0.1</td>
<td>-0.68</td>
<td>0.65</td>
</tr>
<tr>
<td>58.0</td>
<td>0.1</td>
<td>176.5</td>
<td>0.2</td>
<td>0.2</td>
<td>-0.68</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>7.4</td>
<td>0.2</td>
<td>104.3</td>
<td>0.8</td>
<td>0.6</td>
<td>-0.57</td>
<td>0.66</td>
<td></td>
</tr>
<tr>
<td>2.9</td>
<td>0.7</td>
<td>1.48</td>
<td>74.0</td>
<td>1.7</td>
<td>-0.52</td>
<td>0.73</td>
<td></td>
</tr>
<tr>
<td>-0.64</td>
<td>0.85</td>
<td>2.6</td>
<td>59.0</td>
<td>3.5</td>
<td>-0.54</td>
<td>0.81</td>
<td></td>
</tr>
<tr>
<td>0.72</td>
<td>0.9317*</td>
<td>2.6</td>
<td>59.7</td>
<td>3.8</td>
<td>-0.58</td>
<td>0.85</td>
<td></td>
</tr>
<tr>
<td>0.85</td>
<td>2.8</td>
<td>67.3</td>
<td>3.7</td>
<td>3.7</td>
<td>-0.62</td>
<td>0.85</td>
<td></td>
</tr>
</tbody>
</table>

Note. See notes to table 12.8. * = baseline value.
12.7 The Welfare Cost of Lack of Commitment

Lacking commitment to repay debt results in an equilibrium in which the country holds less debt than it would choose to hold under commitment. Thus, in the absence of other distortions, lack of commitment is welfare decreasing. In the calibrated version of the Eaton-Gersovitz model of section 12.6, this cost turns out to be extremely large. The reason is that under that calibration consumers are highly impatient relative to the risk-free interest rate. The subjective discount rate, $1/\beta - 1$, is 17.7 percent per quarter, compared with a risk-free interest rate of 1 percent per quarter. Private households do not place a high value on future consumption and prefer to spend much of their lifetime wealth on current consumption. As a result, with full commitment, the equilibrium displays a high level of debt of 65.88, which is about a hundred times larger than the average debt under lack of commitment. In this regard, imperfect enforcement of international debt contracts manifests itself in equilibrium as a borrowing constraint. In the present calibration, this endogenous borrowing constraint is quite severe.

The model economy under commitment is simple. Welfare, denoted $v^{\text{com}}(d, y)$, is given by

$$v^{\text{com}}(d, y) = \max_{d'} \{ u(y + d'(1 + r^*)^{-1} - d) + \beta E_y v^{\text{com}}(d', y') \}$$

subject to a no-Ponzi-game constraint of the form $d' < \bar{d}$, where the limit $\bar{d}$ is finite, but can be set arbitrarily large. The solution to this Bellman equation delivers the equilibrium processes for debt and welfare. Equilibrium consumption, denoted $c^{\text{com}}$, is then derived residually from the resource constraint as $c^{\text{com}} = y + d'/(1 + r^*) - d$.

Given the state of the economy, $(d, y)$, we define the welfare cost of lack of commitment as the
proportional increase in the level of consumption under lack of commitment necessary to make the
representative agent as well off as under commitment. Letting \( c_t^{\text{com}} \) denote consumption under
lack of commitment, the welfare cost of no commitment, denoted \( \Lambda(d_t, y_t) \) is given by

\[
E_0 \sum_{t=0}^{\infty} \beta^t \left( \frac{c_t^{\text{com}}}{1 - \sigma} - 1 \right) = E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{(1 + \Lambda(d_0, y_0)) c_t^{\text{nocom}}}{1 - \sigma} - 1 \right].
\]

Solving for \( \Lambda(d, y) \) yields

\[
\Lambda(d, y) = \left[ \frac{v^{\text{com}}(d, y)(1 - \sigma)(1 - \beta) + 1}{v^{\text{nocom}}(d, y)(1 - \sigma)(1 - \beta) + 1} \right]^{\frac{1}{1 - \sigma}} - 1,
\]

where \( v^{\text{nocom}}(d, y) \) denotes the welfare level under no commitment to repay, which corresponds to
\( v^g(d, y) \) if the country is in good financial standing or to \( v^b(y) \) if the country is in bad standing.
Notice that because both \( d \) and \( y \) are random variables, so is the welfare cost of lack of commitment,
\( \Lambda(d, y) \).

We are interested in the distribution of \( \Lambda(d, y) \) when the pair \( (d, y) \) is drawn from the distribution
induced by the economy with lack of commitment. The resulting distribution allows one to compute
moments of the welfare gains of migrating from the economy with no commitment to repay debt
to the economy with commitment. These moments take into account the effect of the transitional
dynamics involved in the migration from one economy to the other.

We find that the unconditional mean of the welfare cost of lack of commitment is 273 percent,
that is, \( 100 \times E \Lambda(d, y) = 273 \). This is an enormous value. It means that the consumption stream
of an individual living in the economy without commitment must almost quadruplicate in order for
him to be as well off as living in the economy with commitment to repay debts. The totality of this
welfare cost is due to the transitional dynamics of switching from no commitment to commitment.
Along this transition, debt increases from a mean of 0.59 to a mean of 65.88, and consumption
declines from a mean of 0.98 to a mean of 0.36. Of course, along this transition consumption is temporarily much higher than 0.98. Figure 12.11 shows the typical transition paths for consumption and the stock of debt from the equilibrium with lack of commitment to the equilibrium with commitment. The typical transition path is the mean of 10,000 transition paths each starting at a pair \((d, y)\) drawn from the ergodic distribution under lack of commitment. As shown in the left panel of the figure, at the beginning of the typical transition, consumption rises to about 6 and stays above its no-commitment mean for 25 quarters. The high consumption in the early transition is financed with external debt, which increases 100 fold in 25 quarters.

It is clear from this analysis that the transitional dynamics are the key determinant of the welfare gains of commitment. Consider a naive approach to welfare evaluation consisting in computing the unconditional welfare in each economy separately. Because in the stationary state average consumption in the commitment economy is one third as high as consumption in the no-commitment economy, one would erroneously conclude that lack of commitment is welfare improving.
12.8 Decentralization Of The Eaton-Gersovitz Model

The Eaton-Gersovitz model is cast in terms of a social planner’s problem. A benevolent government aims to maximize the lifetime welfare of households. In doing so, it chooses how much to borrow, when to default, and how much the household should consume each period. The household itself makes no relevant decision. It does not participate in financial markets, nor in goods markets, but passively consumes the goods it receives from the government each period.

In this section, we wish to think about optimal sovereign default (i.e., about the government’s decision to repudiate the country’s external obligations) in an environment in which private households participate in credit markets and choose optimally how much to consume each period. In this new environment, the government retains only the decisions to default and to conduct fiscal policy. A central question that we will address is whether there exist fiscal instruments that the government can use to induce households to undertake borrowing and consumption decisions that mimic the social planner’s allocation. This exercise is known as the decentralization of the social planner’s equilibrium.

The reason why the government may need fiscal instruments to alter the behavior of private households is that while the former internalizes that the interest rate faced by the country in international financial markets depends on its net external debt position, the latter does not. Individual households are too small to affect with their borrowing the country’s credit conditions. By applying fiscal distortions, the government makes its borrowing decisions and the private sector’s coincide. We will show that the social planner’s equilibrium can be decentralized via capital controls. An interesting byproduct of characterizing the decentralized equilibrium is that it will allow us to back-out the optimal fiscal policy. That is, the capital-control policy that induces households to carry out the planner’s desired debt and consumption decisions.

Clearly, under the present approach, we must build the economy from the private sector upward.
Accordingly, we begin by studying the household’s problem. The exposition that follows draws from Na, Schmitt-Grohé, Uribe, and Yue (2014).

12.8.1 Households

Consider an economy populated by a large number of identical households with preferences described by the utility function

\[ E_0 \sum_{t=0}^{\infty} \beta^t u(c_t). \]

Households can borrow and lend in credit markets in a non-state-contingent bond with price \( q^d_t \). Households can commit to repay their debts. Their sequential budget constraint is of the form

\[ c_t + d_t = (1 + \tau^y_t) \tilde{y}_t + (1 - \tau^d_t) q^d_t d_{t+1}, \]

where \( \tau^y_t \) is an income subsidy (tax if negative) and \( \tau^d_t \) is a tax (subsidy if negative) on debt. Households choose processes for consumption and debt to maximize their utility function subject to the above sequential budget constraint and the natural debt limit, which prevents them from engaging in Ponzi schemes. The optimality conditions associated with this problem are (12.40), the no-Ponzi-game constraint, and

\[ u'(c_t)(1 - \tau^d_t) q^d_t = \beta E_t u'(c_{t+1}). \]

12.8.2 The Government

Default decisions are assumed to be made by the government. Each period the country can be either in good financial standing or in bad financial standing. If it is in good financial standing, it can choose to honor its international debts or default. Let \( I_t \) be a binary variable taking the
value 1 if the country is in good standing in period $t$ and chooses to honor its debt and 0 if it is in bad standing. If the country defaults in period $t$, it immediately acquires bad financial standing and $I_t$ takes the value 0. If the country is in bad standing in period $t$, it regains good standing in period $t + 1$ with constant and exogenous probability $\theta$, and maintains its bad financial standing with probability $1 - \theta$.

When the country is in bad financial standing, it is excluded from international financial credit markets and is therefore unable to borrow or lend internationally. We then have that

$$(1 - I_t)d_{t+1} = 0. \quad (12.41)$$

In periods in which the country is in bad standing ($I_t = 0$), the government confiscates any payments of households to foreign lenders and returns the proceeds to households via income subsidies. The government also uses the income subsidy to rebate the proceeds from the debt tax. The resulting sequential budget constraint of the government is then given by

$$\tau^y l_t y_t = \tau^d q_i d_{t+1} + (1 - I_t)d_t. \quad (12.42)$$

Let $q_t$ denote the price of debt charged by foreign lenders to domestic borrowers during periods in which the government maintains good financial standing. As before, the price of debt, $q_t$, must satisfy the condition that the expected return of lending to the domestic country equal the opportunity cost of funds. Formally,

$$q_t = \frac{\text{Prob}\{I_{t+1} = 1|I_t = 1\}}{1 + r^*}. \quad (12.43)$$
This expression can be equivalently written as

\[ I_t \left[ q_t - \frac{E_t I_{t+1}}{1 + r^*} \right] = 0. \]

### 12.8.3 Competitive Equilibrium

Because all domestic households are identical, there is no borrowing or lending among them. This means that in equilibrium the household’s net asset position equals the country’s net foreign asset position. This in turn implies that the debt tax, \( \tau^d_t \), can be interpreted as a capital control tax. Because when the country is in bad standing external debt is nil, the value of \( \tau^d_t \) in periods of bad standing is immaterial. Without loss of generality, we set \( \tau^d_t = 0 \) when \( I_t = 0 \), that is,

\[ (1 - I_t)\tau^d_t = 0. \]  \hspace{1cm} (12.44)

As before, the endowment received by the household, \( \tilde{y}_t \), is given by

\[ \tilde{y}_t = \begin{cases} y_t & \text{if } I_t = 1 \\ y_t - L(y_t) & \text{otherwise} \end{cases}. \]  \hspace{1cm} (12.45)

In any period \( t \) in which the country is in good financial standing, the domestic price of debt, \( q^d_t \), must equal the price of debt offered by foreign lenders, \( q_t \), that is

\[ I_t(q^d_t - q_t) = 0. \]  \hspace{1cm} (12.46)

Combining (12.40)-(12.42), (12.45), and (12.46) yields the following market-clearing condition

\[ c_t = y_t - (1 - I_t)L(y_t) + I_t[q_t d_{t+1} - d_t]. \]
A competitive equilibrium is a set of stochastic processes \( \{c_t, d_{t+1}, q_t, q^d_t\} \) satisfying

\[
c_t = y_t - (1 - I_t)L(y_t) + I_t[q_t d_{t+1} - d_t],
\]

\[(1 - I_t)d_{t+1} = 0,\]

\[(1 - \tau_t^d) q^d_t u'(c_t) = \beta E_t u'(c_{t+1}),\]

\[I_t(q_t^d - q_t) = 0,\]

\[I_t \left[ q_t - \frac{E_t I_{t+1}}{1 + r^*} \right] = 0.\]

given processes \( \{y_t, \tau_t^d, I_t\} \) and the initial condition \( d_0 \).

### 12.8.4 Equilibrium Under Optimal Capital Control Policy

Here, we characterize the optimal default and capital-control policies. When the government can choose freely the capital-control tax, \( \tau_t^d \), the competitive equilibrium can be written in the following more compact form.

**Proposition 12.5 (Competitive Equilibrium When \( \tau_t^d \) Is Unrestricted)** When the government can choose \( \tau_t^d \) freely, stochastic processes \( \{c_t, d_{t+1}, q_t\} \) can be supported as a competitive equilibrium if and only if they satisfy

\[
c_t = y_t - (1 - I_t)L(y_t) + I_t[q_t d_{t+1} - d_t],
\]

\[(1 - I_t)d_{t+1} = 0,\]

\[I_t \left[ q_t - \frac{E_t I_{t+1}}{1 + r^*} \right] = 0,\]
given processes \( \{y_t, I_t\} \) and the initial condition \( d_0 \).

The only nontrivial step involved in establishing this proposition is to show that if processes \( \{c_t, d_{t+1}, q_t\} \) satisfy conditions (12.47), (12.48), and (12.51) then they also satisfy the remaining conditions defining a competitive equilibrium, namely, conditions (12.49) and (12.50). To see this, proceed as follows. When \( I_t \) equals 1, set \( q_t^d \) to satisfy (12.50) and set \( \tau_t^d \) to satisfy (12.49). When \( I_t \) equals 0, set \( \tau_t^d = 0 \) (recall convention (12.44)) and set \( q_t^d \) to satisfy (12.49).

The government is assumed to be benevolent. It chooses a default policy \( I_t \) to maximize the welfare of the representative household subject to the constraint that the resulting allocation can be supported as a competitive equilibrium. The Eaton-Gersovitz model imposes an additional restriction on the default policy. Namely, that the government has no commitment to honor past promises regarding debt payments or defaults. The lack of commitment opens the door to time inconsistency. For this reason the Eaton-Gersovitz model assumes that the government has the ability to commit to a default policy that makes the default decision in period \( t \) an invariant function of the minimum set of aggregate states of the competitive equilibrium of the economy in period \( t \). The states appearing in the conditions of the competitive equilibrium listed in proposition 12.5 are the endowment, \( y_t \), and the stock of net external debt, \( d_t \). Thus, we impose that the default decision in period \( t \) is a time invariant function of \( y_t \) and \( d_t \). We can then define the Eaton-Gersovitz problem as follows.

**Definition 1 (Equilibrium in the Eaton-Gersovitz Model)** An equilibrium in the Eaton-Gersovitz model is a set of processes \( \{c_t, d_{t+1}, q_t, I_t\} \) that maximizes

\[
E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{12.39}
\]

subject to

\[
c_t = y_t - (1 - I_t)L(y_t) + I_t[q_{t}d_{t+1} - d_t], \tag{12.47}
\]
\[(1 - I_t)d_{t+1} = 0, \quad (12.48)\]

\[I_t \left[ q_t - \frac{E_t I_{t+1}}{1 + r^*} \right] = 0 \quad (12.51)\]

and to the constraint that if \(I_{t-1} = 1\), then \(I_t\) is an invariant function of \(y_t\) and \(d_t\) and if \(I_{t-1} = 0\), then \(I_t = 0\) except when reentry to credit markets occurs exogenously, and the natural debt limit, given the initial conditions \(d_0\) and \(I_{-1}\).

The equilibrium in the Eaton-Gersovitz model is generically unique. To see this note that the equilibrium is the solution to an optimization problem. This means that if there were more than one equilibrium, all of them should deliver the same level of welfare. To understand this result, it is important to note that \(I_t\) and \(q_t\) are objects that are chosen as part of the optimization problem that defines the equilibrium and that the participation constraint (12.51) is part of the constraints of that optimization problem.

Moreover, the Eaton-Gersovitz problem is time consistent because none of the constraints contains a conditional expectation of a future nonpredetermined endogenous variable. To see that this is true for constraint (12.51), notice that by the restrictions imposed on the default decision, \(I_{t+1}\) depends only upon \(y_{t+1}\) and \(d_{t+1}\), and that \(d_{t+1}\) is chosen in period \(t\).

**Proposition 12.6 (Uniqueness and Time Consistency of the Eaton-Gersovitz Equilibrium)**

*The equilibrium in the Eaton-Gersovitz model is generically unique and time consistent.*

A further implication of the restrictions imposed on the default decision \(I_t\) and of the assumption that output follows an autoregressive process of order one is that, by equation (12.51), the price of debt depends only upon \(y_t\) and \(d_{t+1}\), hence we can write equation (12.51) as

\[I_t \left[ q_t - q(y_t, d_{t+1}) \right] = 0. \quad (12.52)\]
12.8.5 The Optimal-Policy Equilibrium As A Decentralization Of The Eaton-Gersovitz Model

We now show that the optimal default policy problem given in definition 1 is identical to the Eaton-Gersovitz model presented in subsection 12.6.4. To this end, we express the optimal policy problem in recursive form as follows. If the country is in good financial standing in period $t$, $I_{t-1} = 1$, the value of continuing to service the external debt, denoted $v^c(d_t, y_t)$, i.e., the value of setting $I_t = 1$, is given by

$$v^c(d_t, y_t) = \max \{ c_t, d_t + 1 \} \{ u(c_t) + \beta E_t v^g(d_{t+1}, y_{t+1}) \}$$

subject to

$$c_t + d_t = y_t + q(y_t, d_{t+1})d_{t+1},$$

where $v^g(d_t, y_t)$ denotes the value of being in good financial standing at the beginning of period $t$. The constraint (12.54) results from evaluating (12.47) and (12.52) at $I_t = 1$ and using the latter to eliminate $q_t$ from the former.

The value of being in bad financial standing in period $t$, denoted $v^b(y_t)$, is given by

$$v^b(y_t) = \left\{ u(y_t - L(y_t)) + \beta E_t \left[ \theta v^g(0, y_{t+1}) + (1 - \theta) v^b(y_{t+1}) \right] \right\}.$$  

This equation results from expressing the utility function (12.39) in recursive form, evaluating (12.47) at $I_t = 0$ to eliminate $c_t$ from the period utility function, and taking into account that the expected value of future lifetime utility is conditional on $I_t = 0$.

In any period $t$ in which the economy is in good financial standing, it has the option to either continue to service the debt obligations or to default. It follows that the value of being in good
standing in period $t$ is given by

$$v^g(d_t, y_t) = \max \left\{ v^c(d_t, y_t), v^b(y_t) \right\}. \quad (12.56)$$

In a period into which the government enters with good standing, $I_{t-1} = 1$, it chooses to default whenever the value of continuing to participate in financial markets is smaller than the value of being in bad financial standing, $v^c(d_t, y_t) < v^b(y_t)$. Therefore, in periods in which $I_t = 1$, we have that $E_t I_{t+1} = \text{Prob}\{v^c(y_{t+1}, d_{t+1}) \geq v^b(y_{t+1})|y_t\}$. It follows that when $I_t = 1$, equilibrium condition (12.51) can be written as

$$q_t = \frac{\text{Prob}_y\{v^c(y_{t+1}, d_{t+1}) \geq v^b(y_{t+1})\}}{1 + r^*} \quad (12.57)$$

Combining this expression with (12.52) we obtain

$$q(y_t, d_{t+1}) = \frac{\text{Prob}_y\{v^c(y_{t+1}, d_{t+1}) \geq v^b(y_{t+1})\}}{1 + r^*},$$

which is condition (12.33) of section 12.6.4. We have therefore demonstrated the equivalence between the optimal default policy problem in the decentralized equilibrium as stated in definition 1 and the optimal default policy problem in the Eaton and Gersovitz model as stated in section 12.6.4. We highlight this result in the following proposition:

**Proposition 12.7 (Decentralization)** Sovereign default models in the tradition of Eaton and Gersovitz (1981) can be interpreted as the centralized version of competitive economies with default risk and optimal capital control policy.

The need for capital controls in the decentralization of Eaton-Gersovitz-style models arises from the fact that the government internalizes the effect of aggregate external debt on the country premium,
Figure 12.12 displays the dynamics of optimal capital controls around the typical default episode under the baseline calibration (table 12.7). Since the optimal capital control policy is the one that supports the social planner’s equilibrium, the capital controls shown in the figure are associated with the dynamics for the other variables of the model displayed in figure 12.9. The government increases capital controls sharply in the three quarters prior to the default from 9 to 17 percent. This increase in capital control taxes increases the effective interest rate faced by households. In this way, the government makes private agents internalize the increased sensitivity of the interest rate premium with respect to debt as the debt crisis nears. The debt elasticity of the country premium is larger in the run up to the default because foreign lenders understand that the lower
is output the higher is the incentive to default, as the output loss that occurs upon default, \( L(y_t) \), decreases in absolute and relative terms as \( y_t \) falls. This capital control tax is implicitly present in every default model à la Eaton-Gersovitz. Analyzing the decentralized version of the model, as we did in this section, makes its presence explicit.

12.8.7 Optimal Default Policy Without Capital Controls

The decentralization of the Eaton-Gersovitz model studied thus far relies on the government having access to tax instruments (e.g., capital controls) to induce households to make consumption and borrowing decisions that mimic the ones that are optimal from the social planners’ perspective. Suppose instead, that the government makes optimal default decision as before, but does not have the ability to set fiscal instruments optimally. In this case, the borrowing decisions of the households may differ from the socially optimal ones. The reason is that in this environment, the decision to default resides with the government and therefore depends on the aggregate level of debt of the economy. As a result the interest foreign lenders charge to the country also depends on the aggregate stock of debt. Individual households are too small to affect the aggregate level of debt, and as a result take the interest rate as independent of their own borrowing decisions. Collectively, however, the borrowing decisions of the households do affect the aggregate level of debt and hence the interest rate. The model without capital controls therefore features a pecuniary externality. Kim and Zhang (2012) quantitatively characterize debt and default dynamics in this environment.

The Eaton-Gersovitz model with the pecuniary externality is of interest because in practice the interest rate charged to private borrowers in emerging markets depends on aggregate macroeconomic indicators, rather than on individual agents’ ability to repay. For instance, it has been documented that foreign lenders rely heavily on the assessments of credit rating agencies of countries’ ability to repay. And, in turn, such assessments are based on macroeconomic indicators. See, for instance, Ferri and Liu (2003), Agca and Celasun (2012), and Borensztein, Cowan, and Valenzuela (2013).
In the economy studied here, in which private households make borrowing decision and the government makes default decisions and cannot apply fiscal instruments to support the socially optimal level of borrowing, a competitive equilibrium is a set of stochastic processes \( \{c_t, d_{t+1}, q_t, q^d_t\} \) satisfying (12.47)-(12.51) given \( \tau^d_t = 0 \), processes \( \{y_t, I_t\} \) and the initial condition \( d_0 \). Taking these equilibrium conditions as constraints, the benevolent government chooses the default decision, \( I_t \), to maximize the lifetime utility of the representative household.

To state the problem in recursive form, we follow Kim and Zhang (2012) and distinguish the individual household debt position, \( d \), from the aggregate level of debt, denoted, \( D \). At the household level, the welfare function in periods in which the government continues to service the debt is given by

\[
v^c(d, y, D') = \max_{d'} u(y + q(D', y)d' - d) + \beta E_y \left[ \delta(D', y') v^b(y') + (1 - \delta(D', y')) v^c(d', y', D'') \right],
\]

subject to

\[
D'' = \Gamma(D', y'),
\]

where \( D'' \) denotes the aggregate level of debt assumed in the next period and due in two periods after the current period. The function \( \delta(D', y') \) is an indicator that takes the value one if the government defaults given \( D' \) and \( y' \) and zero otherwise. Notice that now the value function of the individual household features a third argument, the aggregate level of debt, \( D' \), due in the next period. The reason why this variable affects welfare is that it affects the price of debt in the current period, the probability that the government will default in the next period, and the aggregate level of debt in the next period. Note that although households take the aggregate level of debt as given, they understand its law of motion, as given in the constraint of this problem. The function \( \Gamma(D', y') \) is taken as given by the households, but is endogenously determined in equilibrium.

The welfare of the individual households in periods in which the country is in bad financial
standing is given by

\[ v^b(y) = u(y - L(y)) + \beta E_y \left[ \theta v^c(0, y', D'') + (1 - \theta) v^h(y') \right]. \]

The solution to this problem delivers value functions \( v^c(d, y, D') \) and \( v^h(y) \) and a decision rule for debt \( d' = \gamma(d, y, D') \).

Because all agents are identical and assuming that there is a continuum of households with measure one, we have that in any competitive equilibrium the aggregate and individual levels of debt must equal each other

\[ d = D. \]

The decision to default is determined by the benevolent government to maximize the lifetime utility of the representative household. That is,

\[ \delta(D, y) = \begin{cases} 1 & \text{if } v^b(y) > v^c(D, y, \Gamma(D, y)) \\ 0 & \text{otherwise} \end{cases}. \]

As before, with risk-neutral lenders the price of debt must satisfy

\[ q(D', y) = \frac{1 - E_y \delta(D', y')}{1 + r^*}. \]

This completes the presentation of the model. The emergence of the third endogenous state variable, \( D \), makes the computation of equilibrium more demanding both in terms of computer memory and time. Kim and Zhang (2012) calibrate the model to Argentina using similar values to those that appear in table 12.7, except that they assume a flat specification for the post-default output, \( y - L(y) \), by setting \( a_0 = -0.9, a_1 = 1, \) and \( a_2 = 0. \)

What differences should we expect between the equilibrium allocations with and without the
pecuniary externality (i.e., with and without optimal capital controls). Because households do not internalize that the interest rate is an increasing function of their own demand for debt, one would expect an increase in the demand schedule for debt. All other things equal, such a shift in the demand for debt should imply higher levels of debt and higher interest rates in equilibrium. However, a striking finding of Kim and Zhang (2012) is that although the equilibrium interest rate does increase significantly as expected, the equilibrium quantity of debt is not significantly higher in the equilibrium with the externality than in the equilibrium without the externality. Thus, the pecuniary externality does not necessarily induce overborrowing. The reason for the lack of overborrowing is that the externality shifts the supply schedule of credit, \( q(D', y) \), up leading to a reduction in the supply of funds to the country for each level of the country interest rate. In other words, for given levels of debt, \( D' \), and current output, \( y \), the probability of default next period is higher in the economy with the externality than in the economy without the externality. This shift in the supply schedule of funds offsets the increase in the demand schedule for funds, making it possible that the externality may or may not lead to overborrowing. However, the country interest rate and the equilibrium frequency of default increase unambiguously with the introduction of the pecuniary externality.

### 12.9 Risk Averse Lenders

Observed country spreads tend to be larger than observed default probabilities. Tables 12.1 and 12.4 document, using data from nine emerging countries, an average country premium of 4.5 percent per year and a default frequency of 3.9 percent per year, implying an average spread-default-frequency differential of 60 basis points. The spread-default-frequency differential is more pronounced when one corrects for the sample mismatch problem identified in section 12.1.5. As table 12.5 shows, after applying this correction the differential increases to 230 basis points.
By contrast, as established earlier in this chapter, the Eaton-Gersovitz model with risk-neutral lenders predicts that the average country premium must equal the probability of default, or that the spread-default-frequency differential must be zero (see equation (12.35)). In fact, we established that under the empirically relevant assumption of partial default, the model predicts that the country premium is smaller than the probability of default, or that the spread-default-frequency differential is negative (see equation (12.37)).

One possible way to drive a wedge between the country spread and the probability of default is to introduce risk aversion on the part of foreign lenders. In this case, foreign investors will not just require that the expected return on emerging country debt be the same as the risk free rate. In addition, they will need to be compensated for the elevated risk involved in investing in emerging markets.

Consider the Eaton-Gersovitz model of section 12.6.4. Suppose now, however, that foreign lenders are risk averse. In particular, assume that their utility function is given by

$$E_0 \sum_{t=0}^{\infty} \tilde{\beta}^t u(\tilde{c}_t),$$

where $\tilde{\beta} \in (0, 1)$ is the foreign lender’s subjective discount factor, $\tilde{c}_t$ denotes consumption of foreign lenders, and $u$ is the foreign lenders period utility function, which takes the CRRA form

$$u(\tilde{c}) = \frac{\tilde{c}^{1-\tilde{\sigma}} - 1}{1-\tilde{\sigma}},$$

where $\tilde{\sigma} > 0$ is a parameter representing the foreign lender’s coefficient of relative risk aversion. Let

$$m' \equiv \tilde{\beta} \left( \frac{\tilde{c}'}{\tilde{c}} \right)^{-\tilde{\sigma}}$$

denote the foreign lender’s intertemporal marginal rate of consumption substitution. (As usual, we
are dropping the time subscript and using a prime to denote next-period values.) The variable \( m' \) can be interpreted as a pricing kernel, as it indicates the value of one unit of consumption delivered in a particular state in the next period in terms of current consumption. We assume that the emerging country is too small to affect the pricing kernel in the rest of the world. In other words, we assume that changes in the return of the emerging country’s debt have no wealth effect upon foreign lenders. As a result of this assumption, the pricing kernel \( m' \) is independent of economic conditions in the emerging economy.\(^{15}\)

Let \( g' \) denote the gross growth rate of foreign consumption between the current period and the next, that is,

\[
g' \equiv \frac{\tilde{c}'}{\tilde{c}}.
\]

We assume that \( g' \) follows a univariate AR(1) process of the form

\[
\ln \left( \frac{g'}{\bar{g}} \right) = \rho_g \ln \left( \frac{g}{\bar{g}} \right) + \mu',
\]

where \( \bar{g} > 0 \) and \( |\rho_g| < 1 \) are constant parameters and \( \mu' \) is a normally distributed disturbance with mean 0 and variance \( \sigma_{\mu'}^2 \). The parameter \( \bar{g} \) denotes the average gross growth rate of foreign consumption. Note that \( g \), denoting the gross growth rate of foreign consumption between the previous and the current period, is in the current information set, whereas \( g' \) and \( \mu' \) are in next period’s information set.

The foreign-lender’s optimality condition associated with investing in emerging-country debt is

\[
q = E\{m'I' | I = 1, y, g, d'\},
\]

where, as before, \( q \) denotes the price of the emerging country’s debt, \( y \) denotes the endowment, \( d' \)

\(^{15}\)Alternatively, one could assume that the emerging country is large enough to affect the world’s pricing kernel. Lizarazo (2012) pursues this alternative.
denotes the emerging country’s debt, and $I$ is an indicator function that takes the value 1 if the emerging country is in good standing in the current period and chooses to honor its debt and 0 otherwise.

The world pricing kernel $m$ can be used to price other financial assets. Of particular interest is the price of a risk-free asset. Specifically, the risk-free interest rate, $r^*$, which in the standard Eaton-Gersovitz model is taken to be constant, is now time varying and satisfies

$$1 = (1 + r^*)E\{m'|g\}.$$  

(12.60)

For $\rho_g \neq 0$, $E\{m'|g\}$ depends on $g$, which is a random variable. As a result, $r^*$ is also a random variable. When $\tilde{\sigma}$ equals 0, foreign lenders are risk neutral and $m = \tilde{\beta}$. In this case, the risk-free interest rate $r^*$ is constant over time and equal to $\tilde{\beta}^{-1} - 1$. Thus, the present formulation nests the baseline Eaton-Gersovitz model with risk-neutral lenders studied in section 12.6 as a special case.

To derive an expression for the country premium, define the gross country interest rate, $1 + r$, as

$$1 + r = \frac{1}{q}.$$

Then, the gross country premium is given by $(1+r)/(1+r^*)$. Combining the Euler equations (12.59) and (12.60), we can express the gross country premium in periods in which the emerging country is in good financial standing ($I = 1$) as

$$\frac{1 + r}{1 + r^*} = \frac{E\{m'|g\}}{E\{m'I'|y,g,d'|y,g,d'\}}.$$

Now use the fact that $E\{m'I'|y,g,d'\} = \text{Cov}(m', I'|y,g,d') + E\{m'|g\}E\{I'|y,g,d'\}$ to obtain

$$\frac{1 + r}{1 + r^*} = \frac{E\{m'|g\}}{\text{Cov}(m', I'|y,g,d') + E\{m'|g\}E\{I'|y,g,d'\}}.$$
Recalling that \( E\{m'|g\} = 1/(1+r^*) > 0 \) and that \( E\{I'|y, g, d'\} = 1 - \text{Prob}\{\text{default in } t+1|y, g, d'\} \), we can write the gross country premium as

\[
\frac{1 + r}{1 + r^*} = \frac{1}{(1 + r^*)\text{Cov}(m', I'|y, g, d') + 1 - \text{Prob}\{\text{default in } t+1|y, g, d'\}}.
\]

Taking logs on both sides of this expression yields

\[
r - r^* \approx \text{Prob}\{\text{default in } t + 1|y, g, d'\} - (1 + r^*)\text{Cov}(m', I'|y, g, d'),
\]

which implies that the country premium will exceed the probability of default if and only if the conditional covariance between the pricing kernel and the decision to repay is negative. Finally, averaging over all states in which the emerging country is in good financial standing and chooses to repay \((I = 1)\), we obtain

\[
E(r - r^*) > \text{Prob}\{\text{default in } t + 1\} \text{ if and only if } \text{Cov}(m', I') < 0.
\]

According to this expression, the more negative the covariance between the world pricing kernel and the decision to repay, the larger is the average spread-default-frequency differential. Although the world pricing kernel \( m \) is an exogenous stochastic process, its covariance with the decision to repay, \( I \), need not be nil. The reason is that \( m \) determines the world interest rate, and therefore it affects the emerging country’s cost of external funds and its decision to default or repay.

How large is the spread-default-frequency differential induced by the assumption of risk averse foreign lenders in the context of the Eaton-Gersovitz model? To answer this question, we next characterize the predictions of a calibrated version of the present model.
12.9.1 Quantitative Predictions of the Eaton-Gersovitz Model with Risk Averse Lenders

With risk-averse foreign lenders, the model contains one additional exogenous state variable, namely $g$, which makes the computation of equilibrium more time consuming. Before plunging into computations, we present the equations describing the model in recursive form.

The value of continuing to honor the debt for the emerging country is now given by

$$v^c(y, g, d) = \max_d \left\{ u(y + qd' - d) + \beta E_{y,g} v^g(y', g', d') \right\}.$$  

The value of being in bad financial standing is given by

$$v^b(y, g) = u(y - L(y)) + \beta \left[ \theta E_{y,g} v^g(y', g', 0) + (1 - \theta) E_{y,g} v^b(y', g') \right].$$

The value of being in good financial standing is given by

$$v^g(y, g, d) = \max \{ v^c(y, g, d), v^b(y, g) \}.$$  

The price of debt satisfies the participation constraint

$$q = E_{y,g} \left\{ m' I(v^c(y', g', d') \geq v^b(y', g')) \right\},$$

where $I(v^c(y', g', d') \geq v^b(y', g'))$ is an indicator function taking the value 1 if $v^c(y', g', d') \geq v^b(y', g')$ and 0 otherwise, and $m'$ is the pricing kernel defined as

$$m' = \tilde{\beta} (g')^{-\tilde{\sigma}}.$$
The model closes with the law of motion of the gross growth rate of foreign consumption,

\[
\ln\left(\frac{g'}{\bar{g}}\right) = \rho_g \ln\left(\frac{g}{\bar{g}}\right) + \mu'.
\]  

(12.58)

We calibrate all parameters common to the baseline specification (the one with risk-neutral foreign lenders) using the values displayed in table 12.7. We estimate the process for the growth rate of world consumption, given in equation (12.58), using data on nominal U.S. per capita consumption expenditures on nondurables and services over the period 1948:Q1 to 2013:Q4, deflated using the U.S. GDP deflator. The estimated values of \( \rho_g \) and \( \sigma_\mu \) are, respectively, 0.2553 and 0.0055. We discretize this process using 100 equally spaced grid points in \( \ln(g/\bar{g}) \) with upper/lower bounds equal to \( \pm 0.0237 \). The stock of external debt, \( d_t \), is discretized with a grid of 100 equally spaced points in the interval \([0, 1.5]\). We set \( \tilde{\sigma} \) equal to 2, which implies the same coefficient of relative risk aversion in the emerging country as in the rest of the world. We also consider a case with highly risk-averse foreign lenders by setting \( \tilde{\sigma} \) to 5. We calibrate the growth-adjusted discount factor \( \tilde{\beta}\bar{g}^{-\tilde{\sigma}} \) to 0.990041 per quarter, to ensure that the unconditional mean of the risk-free interest rate, \( E r^* = E \left\{ \frac{1}{\tilde{\beta}\bar{g}^{-\tilde{\sigma}} E \left\{(g'/\bar{g})^{-\tilde{\sigma}}|g\right\}} \right\} - 1 \), is 1 percent per quarter, as in the baseline economy. Note that the calibration of the model does not require information on \( \tilde{\beta} \) and \( \bar{g} \) individually, but only on the growth-adjusted discount factor \( \tilde{\beta}\bar{g}^{-\tilde{\sigma}} \).

Table 12.16 presents the predictions of the Eaton-Gersovitz model with risk-averse foreign lenders (\( \tilde{\sigma} = 2 \) and 5). For comparison, the table also presents the predictions of the baseline model, with risk-neutral lenders (\( \tilde{\sigma} = 0 \)). The key message of the table is that the assumption of risk-averse foreign lenders has quantitatively negligible effects on the predictions of the Eaton-Gersovitz model. In particular, the assumption of risk-averse foreign lenders does not change the prediction of a near-zero spread-default-frequency differential.

The insensitivity of the endogenous variables of the model to changes in foreign risk aversion is
remarkable because the volatility of the world interest rate (not shown in table 12.16) does increase significantly with $\bar{\sigma}$. Specifically, the standard deviation of $r^*$ is 0, 1.2, and 3.0 percent per year for $\bar{\sigma}$ equal to 0, 2, and 5, respectively. Why is it then that this sizable increase in the volatility of the world interest rate does not affect the domestic economy? The reason is that this is an economy with highly impatient agents, who can borrow much less than what they would like to borrow under commitment. As a result, the present model behaves quite similarly to one in which the agent is up against a borrowing constraint most of the time. In such a setting, the price of credit is little allocative, and hence variation therein do not affect much consumption or borrowing decisions. This result is likely to change in a setting with default and more patient consumers.

### 12.10 Long-Term Debt And Default

In the preceding analysis, we assumed that external debt has a maturity of one period. In addition, in the calibration of the model, a period is meant to represent a quarter of a year. This means that
the totality of the debt has a maturity of one quarter. This assumption is clearly unrealistic. For example, Broner, Lorenzoni, and Schmukler (2013) report that of the foreign-currency denominated bonds issued by Argentina between 1993 and 2003, 91 percent had a maturity longer than 3 years, and 51 percent had a maturity longer than 9 years. A similar pattern holds for other emerging countries. In this section, we introduce long-term debt into the Eaton and Gersovitz model and then ask whether the so amended model can still account quantitatively for observed debt and spread dynamics. We begin by presenting two tractable models of long-term debt and default that preserve the number of states of the Eaton-Gersovitz model. Later, we extend the model to allow for an endogenous choice of maturity structure.

12.10.1 A Random-Maturity Model

Here, we present the long-term debt specification introduced by Chatterjee and Eyigungor (2012). Consider a sovereign bond with the following characteristics. With probability \( \lambda \in [0, 1] \) the bond matures next period and pays out one unit of the consumption good. With probability \( 1 - \lambda \) the bond does not mature, but pays a coupon equal to \( z > 0 \) units of consumption. The country is assumed to hold a portfolio with a continuum of this type of bond. The realization of maturity is independent across bonds. Hence, if the country has \( d \) units of debt outstanding, a share \( \lambda \) will mature each period with certainty and the remaining share \( 1 - \lambda \) will not. The non-maturing bonds pay the coupon \( z \) and trade at the price \( q \) per unit. Because a newly-issued bond is indistinguishable from an existing bond that did not mature, the ex-coupon price of old bonds and new bonds must be equal. Provided the debtor does not default, \( d \) units of debt pay \( [\lambda + (1 - \lambda)z + (1 - \lambda)q]d \) units of consumption. If the debtor defaults the bond pays zero.

The main difference between the models with long-term and one-period debt is that long-term debt results in a state contingent payoff, which may provide hedging against income risk to the borrower. Specifically, the payoff on the long-term bond, \( \lambda + (1 - \lambda)(z + q) \) depends on \( q \), which
is state dependent. In particular, in periods of low income, \( q \) is likely to be low, resulting in an ex-post low interest rate paid by the borrower. Because periods of low income are associated with low consumption, the long-term bond provides insurance against income risk. By contrast, the payoff on a one-period bond is unity and hence non-state contingent, providing no insurance against income risk. Therefore, we should expect that all other things equal, the borrower will hold more debt if debt is long term rather than short term.

Introducing this new asset structure into the Eaton and Gersovitz model of section 12.6.4 is straightforward. It only requires modifying the Bellman equation for the value of continuing to service the debt and the participation constraint for foreign lenders. As before let \( v^c(d, y) \) denote the value of continuing to service the debt. With long-term debt the value of continuation is given by

\[
v^c(d, y) = \max_{d', c} \left\{ u(c) + \beta E_y v^g(d', y') \right\},
\]

subject to

\[
c = y - \lambda d - (1 - \lambda) zd + q(d', y)(d' - (1 - \lambda)d).
\]

The value of being in bad financial standing is unchanged. It continues to be given by

\[
v^b(y) = u(y - L(y)) + \beta \theta E_y v^g(0, y') + \beta (1 - \theta) E_y v^b(y').
\]

The value of being in good financial standing, \( v^g(d, y) \), then is

\[
v^g(d, y) = \max\{v^c(d, y), v^b(y)\}.
\]

Finally, we continue to assume that foreign lenders are risk neutral. Therefore, the expected one-
period return on long-term debt must equal the one-period risk free rate, that is,

\[ 1 + r^* = \frac{E_y \left\{ I(v^c(d', y') \geq v^b(y')) [\lambda + (1 - \lambda)(z + q(d'', y'))]\right\}}{q(d', y)}, \tag{12.62} \]

where \( I(v^c(d', y') \geq v^b(y')) \) is an indicator function that takes the value 1 if \( v^c(d', y') \geq v^b(y') \) and 0 otherwise, and \( d'' \) denotes the debt chosen in the next period, which is a function of \( d' \) and \( y' \).

An appealing feature of the random-maturity model is that it allows for long-term debt without expanding the state space. Specifically, as in the case of one-period debt, here the states variables are the current endowment, \( y \), and the stock of debt, \( d \). Further, the random-maturity model nests the one-period debt model as a special case when \( \lambda \) equals unity, that is, when one hundred percent of the debt matures each period.

Chatterjee and Eyigungor (2012) calibrate this model to Argentina. They set \( \lambda = 0.05 \) to capture an average maturity of debt of 20 quarters. To see how the parameter \( \lambda \) relates to the average maturity of debt, notice that the fraction of the current stock of debt that will mature after exactly one period is \( \lambda \). The fraction of the current debt that will mature in exactly 2 periods is \((1 - \lambda)\lambda\). In general, the fraction of the current debt that will mature in exactly \( j \) periods is \((1 - \lambda)^{j-1}\lambda\). This means that the average maturity of the current debt is \(1 \times \lambda + 2 \times (1 - \lambda)\lambda + 3 \times (1 - \lambda)^2\lambda + \ldots\), or

\[
\text{average maturity} = \frac{1}{\lambda}.
\]

Thus, a value of \( \lambda \) of 0.05 implies an average maturity of 20 quarters. The parameter \( z \) is set at 0.03 to match an annual coupon rate of 12 percent.

The remaining parameters are similar to those appearing in table 12.7, with the exception of \( \beta, a_1, \) and \( a_2 \). Chatterjee and Eyigungor calibrate these parameters to match an external debt-to-output ratio of 70 percent per quarter, an average interest rate spread of 8.15 per year, and a standard deviation of the interest-rate spread of 0.0443. This strategy yields \( \beta = 0.954, \)
\[ a_1 = -0.188, \text{ and } a_2 = 0.246. \] The performance of the so calibrated long-term debt model is at least as good as the performance of the baseline model presented in section 12.6. An appealing property of the long-term-debt model is that in order to match key statistics associated with debt and default, it does not require making households very impatient (compare the value of \( \beta \) of 0.85 in the baseline model versus 0.954 in the long-run debt model). The low value of \( \beta \) required by the baseline model is not necessarily unrealistic, since no direct evidence on this parameter is available. However, the value of 0.954 is more in line with values used in business-cycle studies outside the default literature. The reason why the model with long-term debt can support the same amount of debt as the model with one-period debt and at the same time feature a higher value of the discount factor is related to the intuition given above, namely that long-term debt provides insurance against endowment shocks. On the downside, the model with long-term debt shares with the baseline model a tight relationship between the average country premium and the default frequency, and therefore has a hard time explaining the fact that these two moments are different in the data.

### 12.10.2 A Perpetuity Model

Following Hatchondo and Martínez (2009), we now introduce long-duration debt into the Eaton-Gersovitz model in the form of a bond that pays a coupon with decaying value at perpetuity. Specifically, consider a bond that sells at the price \( q \) measured in terms of consumption goods and promises a stream of payments of \( \delta^{j-1} \) units of goods, where \( j = 1, 2, \ldots \) denotes the number of periods after issuance. The parameter \( \delta \) lies in the interval \([0, 1 + \rho^\ast)\). Let \( s \) denote the number of bonds issued in the current period and \( s_j \), for \( j = 1, 2, \ldots \) denote the number of bonds issued \( j \) periods prior to the current period. The debt service due in the current period, denoted \( d \), is given by the sum of all coupon payments associated with bonds issued in the past. That is,

\[
d = s_1 + \delta s_2 + \delta^2 s_3 + \ldots
\]
Similarly, the debt service in the next period, denoted $d'$, is given by

$$d' = s + \delta s_1 + \delta^2 s_2 + \ldots$$

Combining these two expressions, we obtain the following law of motion of debt services

$$d' = \delta d + s.$$  

Then, the budget constraint of the country conditional on continuing to service the debt is given by

$$c + d = y + q(d', y)[d' - \delta d].$$

The maturity of debt in the present model is infinity, because a perpetuity never matures. However, a concept related to maturity is duration. The duration of a bond is given by the average time to maturity of payments. In the present formulation bonds pay $\delta^{j-1}$ units of goods in period $j = 1, 2, \ldots$ after issuance. The present value of all payments discounted at the risk-free rate $r^*$ is given by $\sum_{j=1}^{\infty} (1 + r^*)^{-j} \delta^{j-1} = 1/(1 + r^* - \delta)$. This means that the bond pays a fraction $(1 + r^*)^{-j} \delta^{j-1} (1 + r^* - \delta)$ of total (discounted) payments in period $j$. Therefore, the average time to maturity of payments is given by $(1 + r^* - \delta) \sum_{j=1}^{\infty} j(1 + r^*)^{-j} \delta^{j-1} = (1 + r^*)/(1 + r^* - \delta)$. In other words,

$$\text{Duration} = \frac{1 + r^*}{1 + r^* - \delta}.$$

This expression is quite intuitive. Suppose, for example, that $\delta$ is zero. In this case, the bond makes a single payment one period after issuance, and the duration is one period. In the polar case that $\delta \to 1 + r^*$, the bond makes a payment with a present value of 1 every period ad infinitum, and its duration becomes infinitely large. In general, the larger is $\delta$, the larger is the duration of
the bond.

The remaining features of the model are familiar. The value of continuing to service the debt, \( v^c(d, y) \) is

\[
v^c(d, y) = \max \left\{ u(c) + \beta E_y v^g(d', y') \right\},
\]

subject to

\[
c + d = y + q(d', y)(d' - \delta d).
\]

Assume, as before, that if the government defaults it does so on its entire debt obligations. Upon default, the country loses access to financial markets and suffers the output loss, \( L(y) \). Access to credit markets is regained with constant probability \( \theta \) each period. Then, the value of being in bad financial standing is,

\[
v^b(y) = u(y - L(y)) + \beta \theta E_y v^g(0, y') + \beta(1 - \theta) E_y v^b(y').
\]

Also unchanged is the value of being in good financial standing, \( v^g(d, y) \),

\[
v^g(d, y) = \max \{ v^c(d, y), v^b(y) \}.
\]

Continuing to assume that foreign lenders are risk neutral, the expected one-period return on long-term debt must equal the one-period risk free rate. Consider, first the case of long-term debt issued in the current period. In this case, the participation constraint is

\[
1 + r^* = \frac{E_y \left\{ I(v^c(d', y') \geq v^b(y')) \left[ 1 + \delta q(d'', y') \right] \right\}}{q(d', y)}.
\]

The numerator of the fraction appearing on the right-hand side of this expression, \( 1 + \delta q(d'', y') \) is next period’s payoff of a perpetuity issued in the current period conditional on no default. The
payoff consists of two terms. The first term is unity, which is the first coupon payment. The second term, \( \delta q(d''', y') \), is the price next period of a perpetuity issued in the current period. Notice that from the next period onward, the stream of coupon payments of a perpetuity issued in the current period is \( \delta \) times the stream of coupon payments of a perpetuity issued in the next period. Therefore, the price in the next period of a perpetuity issued in the current period must be \( \delta \) times the price of a perpetuity issued next period, or \( \delta q(d''', y') \), which is the second term in the numerator.

The participation constraint (12.64) also holds for perpetuities issued \( k \) periods prior to the current period, for \( k = 1, 2, \ldots \). The current period price of such a perpetuity is \( \delta^k q(d', y) \), and the payoff tomorrow conditional on no default is \( \delta^k + \delta^{k+1} q(d''', y') \). Therefore, the participation constraint is

\[
1 + r^* = \frac{\mathbb{E}_y \left\{ I(u'(d', y') \geq v^b(y')) \left[ \delta^k + \delta^{k+1} q(d''', y') \right] \right\}}{\delta^k q(d', y)},
\]

which is the same as equation (12.64).

This completes the Eaton-Gersovitz model expanded to allow for long-duration debt in the form of a perpetuity.

The Perpetuity Model As A Special Case Of The Random-Maturity Model

The structure of the perpetuity model is similar to that of the random-maturity model presented in the previous section. Indeed, the random-maturity model nests the perpetuity model as a special case when \( \delta < 1 \). Specifically, setting \( \lambda = 1 - \delta \) and \( z = 1 \) renders the random-maturity model identical to the perpetuity model. This is straightforward to show. Under this parameterization the budget constraint and the lender participation constraint of the random-maturity model (equations (12.61) and (12.62)) become identical to their respective counterparts in the perpetuity model (equations (12.63) and (12.64)). All other equations are common to both models. We summarize this result in the following statement

Random-Maturity Model with \( \lambda = 1 - \delta \) and \( z = 1 \) \( \Rightarrow \) Perpetuity Model
In turn, it can be readily verified that the perpetuity model nests the one-period-debt model in the special case that \( \delta = 0 \), that is, in the special case in which the perpetuity pays only one coupon of one unit of consumption goods in the first period after issuance.

### 12.10.3 Endogenous Choice of Maturity

A characteristic common to the random-maturity and the perpetuity models of long-term debt is that in both there is a single maturity or duration of debt. In reality, sovereigns issue debt with various maturities. It is therefore of interest to consider environments with more than one maturity and to compare their predictions to observed movements in the maturity structure of debt and in the term structure of country spreads.

Broner, Lorenzoni, and Schmukler (2013), using weekly observations on foreign-currency sovereign bond prices and bond issuance for 11 emerging countries over the period 1990 to 2009, identify the following three stylized facts:

1. During crises, issuance of new debt shifts toward relatively shorter maturities. In particular, issuance of short-term bonds (less than 3 years in maturity) is not significantly affected during crises. But medium- and long-term issuance (longer than 3-year bonds) decreases sharply.\(^\text{16}\)

2. On average, the risk premium on long-term bonds exceeds the risk premium on short-term bonds (the term structure of country spreads is upward sloping). For example the average difference between the risk premia of 12-year and 3-year maturities is around 3 percent.

3. During crises the cost of long-term borrowing increases relative to the cost of short-term borrowing. The difference between the risk premium on long-term bonds and the risk premium on short term bonds is 30 percent during crisis, but less than 1 percent outside of crises.

\(^{16}\)Broner, Lorenzoni, and Schmukler date the beginning of a crisis when the 9-year spread is 300 basis points above its average level over the past 6 months. They date the end of a crisis when the 9-year spread falls below the threshold of 300 basis points above average for more than 4 weeks. Spreads are defined as the difference between the return on emerging-market bonds and the return on comparable U.S. or German bonds.
Why do we observe a shortening of maturities during crises? One line of research (e.g., Rodrik and Velasco, 2000; and Cole and Kehoe, 1996) argues that the accumulation of short-term debt may cause rollover problems that can lead the country to a debt crisis. Under this hypothesis, the causality direction is from the shortening of maturities to crises. Often, this type of analysis is taken to suggest policies conducive to maintaining a relatively long maturity structure of sovereign debt. An alternative explanation for the observed shortening of maturities during crises is that it is optimal for countries to shift new issuance of debt to shorter maturities in response to adverse economic conditions. In this case, the causality runs from crises to maturity, and a policy of lengthening the maturity structure could be counterproductive. For example, Bi (2006) and Arellano and Ramanarayanan (2012) find that the Eaton-Gersovitz model augmented with multiple maturities predicts an optimal shortening of maturities during crises. In what follows, we study this explanation in more detail.

The intuition for why the maturity structure shortens in times of crisis in the Eaton-Gersovitz default model is that short-term debt enhances the country’s incentive to repay. The following simple example, adapted from Arellano and Ramanarayanan (2012) illustrates this incentive-enhancing role of short-term debt. Consider a three-period economy with linear preferences given by

\[ U(c_0, c_1, c_2) = c_0 + \beta c_1 + \beta^2 c_2, \]

where \( c_i \) denotes consumption in period \( i = 0, 1, 2 \), and \( \beta \in (0, 1) \) denotes the subjective discount factor. Suppose that the endowment is 0 in period 0 and \( y > 0 \) in periods 1 and 2. Suppose that the world interest rate, \( r^* \), is zero and that the economy starts period 0 with no debt. Because preferences are linear in consumption and because agents discount future utility at a rate higher than the world interest rate, \( \beta(1+r^*) < 1 \), the first-best allocation is to concentrate all consumption

\[^{17}\text{Jeanne (2009) and Rodrik and Velasco (2000) present models with a similar flavor.}\]
in period 0, that is, to set \( c_0 = 2y \) and \( c_1 = c_2 = 0 \). Suppose that the country can borrow and lend in international markets using a two-period, zero coupon bond or a one-period bond. Suppose further that default results in a complete loss of current and future endowments.

There are two independent maturity structures that implement the first-best allocation. One is to borrow in period 0 \( y \) units long-term and \( y \) units short term. Under this strategy, in period 1, the endowment is used to retire the short-term bond and in period 2 the endowment is used to retire the long-term bond. The second strategy is to borrow \( 2y \) units short term in period 0, use the endowment in period 1 to pay off half of the debt and roll over the remaining debt by borrowing short term \( y \) units. In period 2, the endowment is used to retire the outstanding short-term debt. Both of these asset portfolios can be supported in equilibrium even though the government lacks commitment to repay its debts. Notice that both strategies contain short-term debt issued in period 0 and maturing in period 1. By contrast, any strategy that achieves the first-best allocation under commitment, but involves no issuance of short-term debt in period 0 is unsupportable in equilibrium in the absence of commitment. Consider, for example, a case in which in period 0 the country issues \( 2y \) units of long-term debt and 0 units of short-term debt. In period 1, the government saves the endowment in short-term bonds. In period 2, the government uses its savings and its endowment to retire all of the long-term debt. This asset allocation would, however, result in default in period 2. This is because in period 1, the government could consume the endowment rather than saving it. This would increase utility by \( \beta y \) relative to the first-best allocation. In period 2, the government fails to meet its obligations and defaults, losing its endowment. Period-2 consumption is zero as in the first-best allocation. Hence, lifetime utility under default is higher than under repayment. Foreign lenders understand this and therefore would refuse to lend long term in period 0. International lending breaks down. It follows that the issuance of short-term debt in period 0 is necessary to create incentives for the country not to default.

The simple example presented above suggests that short-term debt may allow for international
lending in situations in which otherwise it would breakdown. To the extent that during crises the value of commitment increases, the desirability of shortening the maturity of debt should also increase. To see how a crisis can lead to a shift in maturity toward the short run, consider modifying the previous example to allow for curvature in the period utility index. Specifically suppose that now the lifetime utility function is of the form

\[ U(c_0, c_1, c_2) = \ln(c_0) + \beta \ln(c_1) + \beta^2 \ln(c_2). \]

Suppose further that the endowment equals \( y_0 > 0 \) in period 0 and \( y > 0 \) in periods 1 and 2. Suppose that initially \( y_0 = y \). Assume that all other features of the economy are as before. The first-best allocation now features a declining path of consumption, \( c_1 = \beta c_0, c_2 = \beta^2 c_0, \) and \( c_0 = (y_0 + 2y)/(1 + \beta + \beta^2) \). Consider implementing this allocation with a debt issuance in period 0 of \( s = y - c_1 \) units of short-term debt and \( \ell = y - c_2 \) units of long-term debt. Suppose now that the country suffers a crisis that lowers \( y_0 \) and leaves the endowments in periods 1 and 2 unchanged. Clearly, the crisis causes \( c_0 \) to fall, \( \Delta c_0 < 0 \). In turn, the change in short-term borrowing, \( \Delta s \), is positive and equal to \( \Delta s = -\beta \Delta c_0 \) and the change in long-term borrowing, \( \Delta \ell = -\beta^2 \Delta c_0 < \Delta s \), is also positive but less than the change in short-term borrowing. Thus, the crisis leads to a shortening of the maturity structure.

The above two examples illustrate the importance of short-term debt as a vehicle to create incentives to repay debt especially during crises. However, they do not make the case for the desirability of long-term debt. For, in both examples, the first-best allocation can be supported even in the absence of long-term debt, by borrowing short term in both periods. The reason why long-term debt has no advantage over short-term debt in the above two examples is the lack of uncertainty. The benefit of long-run debt is that it provides a hedge against shocks to the short-term interest rate. This hedging property manifests itself in the fact that an increase in
short rates makes borrowing more expensive, but at the same time lowers the value of outstanding long-term debt, which compensates the borrower for the elevation in the cost of funds. Arellano and Ramanarayanan (2012) present a generalization of the log-preference economy of the previous paragraph with uncertainty. In their environment, the optimal allocation under lack of commitment to repay features consumption smoothing across states in periods 1 and across non-default states in period 2. Implementing this allocation requires the use of long-run debt, in the sense that in the absence of long-run debt, the optimal allocation is no longer implementable.

What does a calibrated version of the Eaton-Gersovitz model with short- and long-run debt predict for the behavior of debt maturities and interest-rate spreads over the business cycle? Arellano and Ramanarayanan (2012) combine the model with one-period debt of section 12.6.4 and the model with a perpetuity of section 12.10.2. In the resulting economy, the duration of debt is endogenous. It lies between 1 period (the duration of one-period debt) and \((1 + r^*)/(1 + r^* - \delta)\) periods (the duration of the perpetuity). The government determines optimally the average duration of outstanding debt by setting the quantities of one-period debt and perpetuities each period. The model is calibrated at an annual frequency using Brazilian data. The riskfree interest rate, \(r^*\), is set to 3.2 percent per year and rate of decay of the coupon payment, \(\delta\), to 0.936. The resulting duration of the perpetuity is 10.75 years.

The calibrated model captures the fact that the average duration of new issuance shortens during crisis. Specifically, the model predicts that when the one-year spread is below its median, the average duration of newly issued debt is 4.9 years, whereas when the spread is above its median, the average duration falls to 3.7 years. Arellano and Ramanarayanan report that in the data, the corresponding numbers are 8.2 and 5.8 years, respectively.

In addition, the model predicts that long-term borrowing becomes relatively cheaper than short-term borrowing during crises. Specifically, it implies that when the short-term spread is below its median value, the average long-term spread is 1.5 percentage points above the average short-term
spread. That is, when the short-term spread is low, the term structure of spreads is upward sloping. At the same time, the model predicts that when the short-term spread is above its median value, then the average long-term spread is 1.6 percentage points below the average short-term spread. That is, according to the model, during crises the term structure of country spreads not only becomes less steep, it actually becomes downward sloping.

The predicted inversion of the term structure of spreads during crises is quite intuitive within the logic of the Eaton-Gersovitz model. The increase in the short-term spread during a crisis is the consequence of an elevated risk of default. Because the model displays reversion to the mean, future expected short-term spreads must display a declining path. Since the long-term spread is roughly an average of current and future expected spreads, it must be lower than the current short-term spread during a crisis.

Is the predicted inversion of the term structure of country spreads during crises borne out in the data? This question turns out to be difficult to answer, because different empirical studies come to different conclusions about the behavior of the slope of the term structure of country spreads during crises. For instance, as discussed above (see stylized fact (3)), Broner, Lorenzoni, and Schmukler (2012) find that the term structure of spreads steepens during periods of high spreads. That is, long-term debt becomes relatively more expensive during crises. Arellano and Ramanarayanan (2012) find that the average slope of the term structure of country spreads is the same whether one conditions on the short-term spread being above or below its median value. These two pieces of empirical evidence are at odds with the predictions of the model. On the other hand, Arellano and Ramanarayanan also report that conditional on the short-term spread being, extremely high, in the top 90th percentile, the average slope of the term structure of spreads is slightly negative at -0.7 percentage points, which is more in line with the inversion predicted by the model.

The Eaton-Gersovitz model with endogenous maturity discussed here also predicts that the slope of the term structure of country spreads must be about zero on average. This is a direct
consequence of the assumption that foreign lenders are risk neutral. By contrast, the observed
country-spread curve is on average upward sloping. As mentioned above, Broner, Lorenzoni, and
Schmukler estimate an average difference between spreads on 3-year and 12-year maturities of 3
percentage points and Arellano and Ramanarayanan estimate an average difference between 1-year
and 10-year maturities of 1.8 percentage points.

The fact that during crises the maturity structure shortens and the term structure of country
spreads steepen has lead some authors to propose models in which the shortening of maturity is
driven by credit supply factors rather than by credit demand factors, as is the case in the Eaton
and Gersovitz model. In particular, Broner, Lorenzoni, and Schmukler present a model in which
foreign creditors are risk averse as in Lizarazo (2013). In their model, an exogenous increase in
the degree of risk aversion induces foreign creditors to expand the supply of more liquid short-term
securities resulting in a steeper term structure of country spreads and a shortening of the maturity
structure.

12.11 Debt Renegotiation

The standard Eaton and Gersovitz model assumes that upon default the debtor country repudiates
all of its outstanding external debt. That is, it assumes that the haircut is always 100 percent. In
reality, however, haircuts are rarely that large. Section 12.1.2 documents that the average observed
haircut is around 40 percent. One way to bring the predictions of the Eaton and Gersovitz model
closer to the data in this regard is to assume that upon default the debtor country engages in a
renegotiation process with foreign lenders. The haircut is then the outcome of such renegotiation,
and its magnitude will depend on the bargaining power of borrowers and lenders as well as on
the fundamentals of the economy. Contributions in this area include Bulow and Rogoff (1989), Bi
In this section, we extend the baseline Eaton and Gersovitz model of section 12.6.4 to allow for debt renegotiation. In the model renegotiation takes the form a Nash bargaining game between the debtor country and foreign lenders. The model has the following distinctive features. First, after default renegotiation determines the amount of debt the country will owe upon re-entry. Second, the renegotiation can feature multiple restructurings depending on the evolution of the state of economic fundamentals over the renegotiation period. Third, after default the length of the exclusion period is not dictated by the requirement that the debtor clears up all of its arrears. This feature of the model is motivated by the fact that in reality countries emerge from default with positive amounts of external debt outstanding. Fourth, the length of the exclusion period has an endogenous and an exogenous component. The exogenous component is as in the baseline Eaton and Gersovitz model of section 12.6.4. The endogenous component arises because the debtor may choose to restructure its debt during the exclusion period. This feature introduces an endogenous positive correlation between the size of the haircut and the length of the exclusion period. Lastly, the model generates a secondary market for distressed debt. This implication captures the empirical fact that observed debt prices during periods of default status are lower than during normal times but not zero as predicted by the standard Eaton and Gersovitz model.

12.11.1 The Eaton and Gersovitz Model with Debt Renegotiation

The problem of a country in good financial standing that chooses to service its debt is as in the standard Eaton and Gersovitz model. Specifically, the value of continuing to participate in credit markets, \( v^c(d, y) \), solves the Bellman equation

\[
v^c(d, y) = \max_{d'} \left\{ u \left( y + q(d', y)d' - d \right) + \beta E_y r^g(d', y') \right\},
\]

where \( u \) is the utility function, \( q(d', y) \) is the growth rate of the economy, \( r^g \) is the interest rate, and \( E_y \) represents the expectation operator.
where, as before, $v^g(d, y)$ denotes the value of being in good financial standing. If the country starts the current period in good financial standing and decides to default and renegotiate, it is assumed to be excluded from borrowing and lending starting in the current period, with constant and exogenous reentry probability $\theta$ beginning in the next period. During exclusion, the country suffers the output loss $L(y)$. Contrary to what we have assumed thus far, upon reentry, the country owes debt in the amount $\tilde{d}$, which is the result of a renegotiation that took place in the period in which the country defaulted. The value of being in bad standing, denoted $v^b(\tilde{d}, y)$, is given by

$$v^b(\tilde{d}, y) = u(y - L(y)) + \theta \beta E_y v^g(\tilde{d}, y') + (1 - \theta) \beta E_y v^b(\tilde{d}, y').$$

We assume that if the country decides to default and not renegotiate, then it is condemned to permanent financial autarky and a permanent output loss of $L(y)$ per period. Thus, the value of defaulting without renegotiation, denoted $v^a(y)$, is given by

$$v^a(y) = u(y - L(y)) + \beta E_y v^a(y').$$

Renegotiation is a mechanism to distribute surpluses between the debtor country and the foreign lenders in case of default. The surplus of the debtor country is the difference between the value of defaulting and renegotiating and the value of defaulting and not renegotiating, $v^b(\tilde{d}, y) - v^a(y)$. This difference is always nonnegative, because a renegotiating debtor can always replicate the autarky allocation by always defaulting upon reentry. We express the debtor surplus in terms of goods by dividing it by the marginal utility of consumption under autarky. This marginal utility is the value of goods in terms of utility when the country defaults regardless of whether it renegotiates
or not. Thus, the surplus of the debtor expressed in terms of consumption goods is given by

\[
\text{surplus of the debtor country} = \frac{v^b(\bar{d}, y) - v^a(y)}{u'(y - L(y))}.
\]

The surplus of foreign lenders is the difference between the value of the restructured debt, \(q^b(\tilde{d}, y)\tilde{d}\), and the payment received by the lender in the absence of renegotiation, which is 0. The variable \(q^b(\tilde{d}, y)\) denotes the price of distressed debt, that is, the price of debt when the country is in bad financial standing. Formally, the surplus of foreign lenders is given by

\[
\text{surplus of foreign lenders} = q^b(\tilde{d}, y)\tilde{d}.
\]

The outcome of the renegotiation is the level of restructured debt, \(\bar{d}\), which is assumed to be determined by generalized Nash bargaining with exogenous parameter \(\alpha \in [0, 1]\). Specifically, \(\bar{d}\) maximizes the following Cobb-Douglas function

\[
\bar{d} = \arg\max_x \left[ \frac{v^b(x, y) - v^a(y)}{u'(y - L(y))} \right]^\alpha \left[ q^b(x, y)x \right]^{1-\alpha}.
\]

The parameter \(\alpha\) measures the bargaining power of the borrower. The higher is \(\alpha\) the stronger the negotiating power of the debtor country. The level of the restructured debt, \(\bar{d}\), is a function of current output, \(y\), but does not depend upon the level of debt the country defaults on, \(d\). Therefore, we have that

\[
\text{restructured debt} = \bar{d}(y).
\]

Notice that nothing prevents \(\bar{d}(y)\) to be bigger than \(d\), the debt defaulted upon. When positive, we interpret the difference \(\bar{d}(y) - d\) as the threat of additional punishment, such as punitive interest, should the country declare cessation of payments. An alternative formulation, not pursued here,
would be to impose the constraint $x \leq d$ in the above bargaining problem.

The value of default with renegotiation is then given by

$$v^d(y) \equiv v^b(\tilde{d}(y), y),$$

which implies that the value of default and restructuring is independent of the size of the default. Note that both, the haircut, $d - \tilde{d}(y)$, and the recovery rate, $\tilde{d}(y)/d$, do depend on the level of debt defaulted upon. In particular, the model predicts that the size of the haircut is increasing in the level of debt, $d$.

A country in good financial standing has the option either to continue servicing the debt or to default. It follows that the value of being in good standing is given by

$$v^g(d, y) = \max\{v^c(d, y), v^d(y)\}.$$ 

As in the model without renegotiation, foreign lenders are assumed to be risk neutral and perfectly competitive. To derive a recursive representation of the price of debt, it is convenient to define the indicator function $I(d, y)$, which takes the value 1 if a country in good standing chooses to default and renegotiate and 0 otherwise,

$$I(d, y) \equiv \begin{cases} 
1 & \text{if } v^c(d, y) \geq v^d(y) \\
0 & \text{otherwise}
\end{cases}.$$ 

Then, the price of debt in periods in which the country is in good standing and chooses to continue to be engaged in credit markets, denoted $q(d', y')$, satisfies the following participation constraint

$$1 + r^* = \frac{E_y[I(d', y') + E_y[1 - I(d', y')](\tilde{d}(y')/d')q^b(\tilde{d}(y'), y')]}{q(d', y)}.$$
The left-hand side is the opportunity cost of funds for lenders, given by the gross risk-free interest rate, $1 + r^*$. The right-hand side is the expected return on the emerging country debt. A key difference with the model without renegotiation is that now in the case of default the return of one unit of debt purchased prior to default is not zero, but takes the value $(\bar{d}(y')/d')q^b(\bar{d}(y'), y')$, where $\bar{d}(y'/d'$ is the recovery rate and $q^b(\bar{d}(y'), y')$ is the price of restructured debt. Because a unit of distressed debt represents a promise to pay one unit of consumption upon reentry, the price of a unit of distressed debt, $q^b(\bar{d}, y)$, must satisfy

$$1 + r^* = \frac{(1 - \theta)E_yq^b(d', y') + \theta E_yI(d', y')\theta E_yI(1 - I(d', y'))(\bar{d}(y')/d')q^b(\bar{d}(y'), y')}{q^b(d', y)}.$$ 

The numerator on the right-hand side has three components. The first component is the expected payoff of restructured debt in the case that the debtor country does not reenter financial markets next period. The second term is the expected payoff in case the debtor gets to reenter and chooses to honor its debt obligations. And the third term corresponds to the expected payoff in the case that the debtor gets to reenter and chooses to default again, generating another renegotiation and debt restructuring.

This expression embodies a key difference with the model without renegotiation. For in the absence of renegotiation, the price of debt when the country is in bad standing is zero, $q^b(d', y) = 0$. Here, this price is in general positive because the creditor expects to recover part of the distressed debt outstanding. A second property of the current model that is evident from the above participation constraint is that although the probability of reentry, $\theta$, is exogenous, the duration of the exclusion period has an endogenous component. This is because, when the country receives the exogenous signal allowing it to reenter, it can choose to remain in bad financial standing, thereby endogenously extending the exclusion period. This option is never exercised in the model without renegotiation, because in that model, by assumption, the country always reenters with zero out-
standing debt, and hence has no incentive to default at that point. The present model nests the standard Eaton-Gersovitz model in the special case in which all the bargaining power lies with the debtor, \( \alpha = 1 \), (see exercise 12.14).

### 12.11.2 Quantitative Predictions of the Debt-Renegotiation Model

To facilitate comparison of the quantitative predictions of the model with debt renegotiation and those of the standard Eaton-Gersovitz model studied in section 12.6.4, we set the values of all parameters common to both models to those given in table 12.7.\(^{18}\) The present model features a single additional parameter relative to the model without renegotiation, namely \( \alpha \), the parameter governing the bargaining power of the borrower. We set \( \alpha \) at 0.55, to match the observed average haircut of 40 percent (see section 12.1.2). Matlab code to solve the present model is available online. Policy functions are produced by the program egr.m. Simulated time series are computed by the program simur.m and predicted moments are produced by the program statistics_modelr.m.

Table 12.17 displays the predictions of the model concerning the behavior of its novel variable,

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\(^{18}\)An alternative strategy is to recalibrate the model to meet the targets imposed in section 12.6.5. This is the subject of exercise 12.15.
namely, the haircut,

\[ h \equiv 1 - \frac{\bar{d}(y)}{d}. \]

The haircut is a random variable whose mean, as indicated above, is pinned down to about 40 percent by appropriate choice of the bargaining weight \( \alpha \). The range of haircuts predicted by the model varies from 27 to 56 percent, with a standard deviation of 5, which is lower than its empirical counterpart of about 22 (see section 12.1.2). The model also displays a positive but modest correlation between the size of the haircut and the length of the exclusion period. The sign of this correlation is in line with the data (see, for example, Benjamin and Wright, 2013). The positive comovement between the size of the haircut and the length of the exclusion period arises because occasionally the country finds it optimal to restructure its debt while in bad financial standing, which causes both the size of the accumulated haircut and the exclusion period to be larger.

Table 12.18 presents the predictions of the debt renegotiation model regarding default, debt, and the country spread. For comparison, the table reproduces from table 12.8 the predictions of the standard Eaton-Gersovitz model, which corresponds to the case \( \alpha = 1 \). The debt renegotiation model supports more debt in equilibrium than the standard Eaton-Gersovitz model (0.87 versus 0.59). The reason is that with debt renegotiation there is a floor to how much lenders can loose in the event of default. This floor is on average 40 percent of the initial investment. As a result, they are more willing to lend to the emerging economy, which results in lower spreads and higher debt. The model predicts a negative spread-default-frequency differential (i.e., that the spread is on average lower than the default frequency). To a large extent, this is a consequence of the fact that the model features partial default in equilibrium. For the intuition behind this connection, see the discussion surrounding equation (12.37). The negative spread-default-frequency differential induced by debt renegotiation is at odds with the data. For, as discussed in section 12.1.5, observed spread-
Table 12.18: The Eaton-Gersovitz Model with Debt Renegotiation, Predicted First and Second Moments

<table>
<thead>
<tr>
<th>Model</th>
<th>Default Frequency</th>
<th>$E(d/y)$</th>
<th>$E(r - r^*)$</th>
<th>$\sigma(r - r^*)$</th>
<th>$\text{corr}(r - r^*, y)$</th>
<th>$\text{corr}(r - r^*, tb/y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>3.2</td>
<td>59.0</td>
<td>3.5</td>
<td>3.2</td>
<td>-0.54</td>
<td>0.81</td>
</tr>
<tr>
<td>Renegotiation</td>
<td>3.8</td>
<td>86.8</td>
<td>2.8</td>
<td>2.7</td>
<td>-0.53</td>
<td>0.81</td>
</tr>
</tbody>
</table>

Note. Baseline refers to the standard Eaton-Gersovitz model of section 12.6.4. In the debt renegotiation model, the parameter $\alpha$ takes the value 0.55. All other parameter values are as in table 12.7. The variable $d/y$ denotes the quarterly debt-to-output ratio in percent, $r - r^*$ denotes the country premium, in percent per year, $y$ denotes output, and $tb/y$ denotes the trade-balance-to-output ratio. The symbols $E$, $\sigma$, and $\text{corr}$ denote, respectively, the mean, the standard deviation, and the correlation. All moments are conditional on the country being in good financial standing. The Matlab script to produce this table is statistics_modelr.m.

default-frequency differentials are typically positive. The predictions of the model regarding the volatility and cyclicality of the country spread are similar to those of the standard Eaton-Gersovitz model.

It is of interest to analyze the effects of varying the parameter $\alpha$ governing the bargaining power of the debtor. Figure 12.13 shows that as $\alpha$ increases from 0 to 1, the size of the average haircut increases reaching 100 percent when the borrower has all the bargaining power. At first glance this result is intuitive because in the short run a higher haircut represents more resources available to the borrower for consumption. However, the shift of bargaining power from the lender to the borrower has additional equilibrium effects, because lenders incorporate the increase in the size of haircuts into the price of debt. As shown in the bottom right panel of the figure, lenders charge a higher country premium the higher is the average haircut, which raises the cost of borrowing for the debtor. This effect should induce the borrower to choose a smaller haircut. In equilibrium however the first effect dominates and the haircut increases with the bargaining power of the borrower. Because the
average haircut increases with bargaining power, lenders reduce the supply of funds as $\alpha$ increases causing less lending in equilibrium as shown in the top right panel of the figure. Finally, as the bargaining power of borrowers increases they find it optimal to default less frequently (bottom left panel of the figure). This is because, over a period of time, a given desired reduction in the level of debt can be achieved either by defaulting often and imposing small haircuts or by defaulting less frequently and imposing larger haircuts.

In summary, the main payoff of enriching the Eaton and Gersovitz model with debt renegotiation is the emergence of an endogenous and empirically realistic haircut. In addition, the model with renegotiation is able to support more debt in equilibrium. On the downside, the model with
renegotiation exacerbates the difficulties of the standard Eaton and Gersovitz model to generate positive spread-default-frequency differentials.

12.12 Default and Monetary Policy

All of the models of default we have studied thus far are cast in real terms. In those frameworks, there is no room for monetary or exchange-rate policy to matter. However, as we will see shortly, the data suggest that there is a significant relationship between default and exchange rates. To make room for monetary policy, in this section we introduce nominal rigidities into the Eaton-Gersovitz default model. We use this augmented model to characterize the equilibrium dynamics of default, devaluation, and employment under optimal and suboptimal exchange-rate policies. The analysis that follows draws from Na et al. (2014).

12.12.1 The Twin Ds

There exists a strong link between sovereign default and devaluation in emerging countries. Reinhart (2002), using data for 58 countries over the period 1970 to 1999, estimates that the unconditional probability of a large devaluation (25 percent or higher) in any 24-month period is 17 percent. At the same time, she estimates that conditional on the 24-month period containing a default event, the probability of a large devaluation increases to 84 percent. Reinhart refers to this phenomenon as the Twin Ds.

Figure 12.14 provides further evidence of the Twin Ds phenomenon. It displays the median excess depreciation of the nominal exchange rate over 116 sovereign defaults that occurred in 70 countries over the period 1975 to 2013. The nominal exchange rate is expressed as the price of one U.S. dollar in units of local currency. The exchange rate devalues when the price of a dollar in terms of domestic currency increases. The figure shows that in a seven-year window the exchange rate
depreciates 45 percent more if the window contains a default event than if it does not. Figure 12.15 displays the behavior of the nominal exchange rate around six well-known default episodes. In all cases, sovereign default is accompanied by large devaluations of the domestic currency.

In this section, we build a model in which the Twin Ds emerge endogenously as an optimal policy outcome. That is, we will ask the question of whether it is optimal for a benevolent government to couple defaults with large devaluations.

12.12.2 A Model of the Twin Ds

The Twin Ds phenomenon suggests some connection between the decision to default and the decision to devalue. Here, we build a model in which this connection results from combining lack of commitment to repay sovereign debt, as in the Eaton-Gersovitz model studied earlier in this chapter, and downward nominal wage rigidity, as in the model of chapter 9. A prediction of the resulting model is that the country chooses to default when aggregate demand is depressed. In turn, when aggregate demand is low, so is the demand for labor. Full employment then requires real wages to fall. However, given the downward rigidity of nominal wages, a fall in the real wage can occur only if the government erodes the purchasing power of the nominal wage via a devaluation. Hence, in these circumstances, it is optimal for the government to devalue. The model therefore predicts that it is optimal for the government to combine default with currency devaluation. In this way, the Twin Ds phenomenon emerges endogenously as an optimal outcome in this environment.

Consider an economy populated by a large number of identical households with preferences described by the utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t).$$

The consumption good is a composite of tradable consumption, $c_t^T$, and nontradable consumption,
Figure 12.14: Excess Devaluation Around Default, 1975-2013

Note. The solid line displays the median of the cumulative devaluation rate between years -3 and +3 conditional on default in year 0 minus the unconditional median of the cumulative devaluation rate between years -3 and +3. Conditional medians are taken across 7-year windows centered at a default episode. The sample contains 116 default episodes between 1975 and 2013 in 70 countries. Unconditional medians are taken over all 7-year windows in the sample. Data Source: Default dates are taken from table 12.19. Exchange rates, World Development Indicators, code: PA.NUS.FCRF.
Figure 12.15: The Twin Ds: Six Examples

- **Argentina**
  - Exchange Rate

- **Uruguay**
  - Exchange Rate

- **Ukraine**
  - Exchange Rate

- **Russia**
  - Exchange Rate

- **Paraguay**
  - Exchange Rate

- **Ecuador**
  - Exchange Rate

Legend:
- **Nominal Exchange Rate**
- **Default Date**
$c_t^N$. The aggregation technology is of the form

$$c_t = A(c_t^T, c_t^N),$$

(12.66)

where $A$ is an increasing, concave, and linearly homogeneous function. The presence of nontradable goods is one of the differences between the present setup and the default model analyzed in previous sections.

The sequential budget constraint of the household is given by

$$P_t^T c_t^T + P_t^N c_t^N + \varepsilon_t d_t = P_t^T \tilde{y}_t^T + W_t h_t + (1 - \tau^d_t)q^d_t \varepsilon_t d_{t+1} + F_t + \Phi_t,$$

where $P_t^T$ denotes the nominal price of tradable goods, $P_t^N$ the nominal price of nontradable goods, $\varepsilon_t$ the nominal exchange rate defined as the domestic-currency price of one unit of foreign currency, $\tilde{y}_t^T$ the household’s endowment of traded goods, $W_t$ the nominal wage rate, $h_t$ hours worked, $\tau^d_t$ a tax on debt, $F_t$ a lump-sum transfer received from the government, and $\Phi_t$ nominal profits from the ownership of firms. Households are assumed to be subject to the natural debt limit, which prevents them from engaging in Ponzi schemes. The variable $\tilde{y}_t^T$ is stochastic and is taken as given by the household.

Assume that the law of one price holds for tradables. Specifically, letting $P_t^{T*}$ denote the foreign currency price of tradables, the law of one price implies that

$$P_t^T = P_t^{T*} \varepsilon_t.$$

Further, assume that the foreign-currency price of tradables is constant and normalized to unity,
Thus, we have that the nominal price of tradables equals the nominal exchange rate,

$$P^*_t = 1.$$ 

$$P^*_t = \mathcal{E}_t.$$ 

Let $p_t = P^N_t / \mathcal{E}_t$, $w_t = W_t / \mathcal{E}_t$, $f_t \equiv F_t / \mathcal{E}_t$, and $\phi_t \equiv \Phi_t / \mathcal{E}_t$ denote, respectively, the relative price of nontradables, the real wage rate, real lump-sum transfers, and real profits, all expressed in terms of tradables. Then, the sequential budget constraint of the household can be written as

$$c^T_t + p_t c^N_t + d_t = \tilde{y}^T_t + w_t h_t + (1 - \tau^d_t) q_t^d d_{t+1} + f_t + \phi_t.$$  \hspace{1cm} (12.67)

Households supply inelastically $\bar{h}$ hours to the labor market each period. However, sometimes, they are unable to sell all of the hours they supply. Hence, hours worked, $h_t$, satisfy

$$h_t \leq \bar{h}.$$  \hspace{1cm} (12.68)

Households take $h_t$ as given.

Households choose contingent plans $\{c_t, c^T_t, c^N_t, d_{t+1}\}$ to maximize the lifetime utility function (12.65) subject to (12.66)-(12.68) and the natural debt limit, taking as given $p_t$, $w_t$, $h_t$, $\phi_t$, $q^d_t$, $\tau^d_t$, $f_t$, and $\tilde{y}^T_t$. The optimality conditions associated with this problem are (12.66)-(12.68), the natural debt limit, and

$$\frac{A_2(c^T_t, c^N_t)}{A_1(c^T_t, c^N_t)} = p_t,$$

$$\lambda_t = U'(c_t) A_1(c^T_t, c^N_t),$$

$$(1 - \tau^d_t) q^d_t \lambda_t = \beta \mathcal{E}_t \lambda_{t+1},$$

where $\lambda_t$ denotes the Lagrange multiplier associated with (12.67).
Nontraded output, denoted $y^N_t$, is produced by perfectly competitive firms with the production technology

$$ y^N_t = F(h_t). $$

(12.70)

The function $F$ is assumed to be strictly increasing and strictly concave. Firms choose the amount of labor input to maximize profits, which are given by

$$ \Phi_t \equiv P_t^N F(h_t) - W_t h_t. $$

(12.71)

The optimality condition associated with this problem is $P_t^N F'(h_t) = W_t$. Dividing both sides by $E_t$ yields

$$ p_t F'(h_t) = w_t. $$

As in the model of chapter 9, we introduce downward nominal wage rigidity by imposing a lower bound on nominal wage growth of the form

$$ W_t \geq \gamma W_{t-1}, $$

(12.72)

where $\gamma > 0$ is a parameter governing the degree of downward nominal wage rigidity. In what follows, we assume that $\gamma$ is less than unity. We impose that at every point in time wages and employment must satisfy the slackness condition

$$ (\bar{h} - h_t) (W_t - \gamma W_{t-1}) = 0. $$

(12.73)

Default decisions are assumed to be made by the government and are modeled along the lines of the decentralized Eaton-Gersovitz model of section 12.8. Accordingly, the exclusion from international credit markets when the country is in bad financial standing ($I_t = 0$) is formalized by the
expression

\[(1 - I_t)\delta_{t+1} = 0. \]  \hspace{1cm} (12.74)

As in section 12.8, the sequential budget constraint of the government is

\[f_t = \tau^d_t q^d_t \delta_{t+1} + (1 - I_t)\delta_t. \]  \hspace{1cm} (12.75)

The first term on the right-hand side reflects the assumption that the government rebates the proceeds from the debt tax in a lump-sum fashion to households. And the second term captures the assumption that in periods in which the country is in bad standing \((I_t = 0)\), the government confiscates any payments of households to foreign lenders and returns the proceeds to households in a lump-sum fashion.

With risk-neutral foreign lenders the price of debt charged by foreign lenders to the country, \(q_t\), must obey the participation constraint

\[I_t \left[ q_t - \frac{E_t I_{t+1}}{1 + r^*} \right] = 0. \]  \hspace{1cm} (12.76)

As explained in section 12.8, the domestic price of debt, \(q^d_t\), equals \(q_t\) when the country is in good financial standing \((I_t = 1)\), but is in general different from \(q_t\) when the country is in financial autarky \((I_t = 0)\).

**Competitive Equilibrium**

In equilibrium, the market for nontraded goods must clear at all times. That is, the condition

\[c^N_t = y^N_t \]  \hspace{1cm} (12.77)
must hold for all \( t \).

We assume that \( \ln y_t^T \) obeys the law of motion

\[
\ln y_t^T = \rho \ln y_{t-1}^T + \mu_t, \tag{12.78}
\]

where \( \mu_t \) is an i.i.d. innovation with mean 0 and variance \( \sigma^2_\mu \), and \( |\rho| \in [0, 1) \) is a parameter. The variable \( \tilde{y}_t^T \) is defined as the endowment net of the output loss associated with default,

\[
\tilde{y}_t^T = \begin{cases} 
 y_t^T & \text{if } I_t = 1 \\
 y_t^T - L(y_t^T) & \text{otherwise} 
\end{cases} \tag{12.79}
\]

In any period \( t \) in which the country is in good financial standing and chooses to honor its debt \( (I_t = 1) \), the domestic price of debt, \( q_t^d \), must equal the price of debt offered by foreign lenders, \( q_t \), that is,

\[
I_t(q_t^d - q_t) = 0. \tag{12.80}
\]

When the country is in bad financial standing \( (I_t = 0) \), the value of \( \tau_t^d \) is immaterial. This is because in these periods external debt is nil. Therefore, without loss of generality, we set \( \tau_t^d = 0 \) when \( I_t = 0 \), that is,

\[
(1 - I_t)\tau_t^d = 0. \tag{12.81}
\]

Combining (12.67), (12.70), (12.71), (12.74), (12.75), (12.77), (12.79), and (12.80) yields the following market-clearing condition for traded goods:

\[
c_t^T = y_t^T - (1 - I_t)L(y_t^T) + I_t[q_t d_{t+1} - d_t].
\]
Finally, let

\[ \epsilon_t \equiv \frac{E_t}{E_{t-1}} \]

denote the gross devaluation rate of the domestic currency. We are now ready to define a competitive equilibrium.

**Definition 2 (Competitive Equilibrium)** A competitive equilibrium is a set of stochastic processes \( \{c^T_t, h_t, w_t, d_{t+1}, \lambda_t, q_t, q^d_t\} \) satisfying

\[ c^T_t = y^T_t - (1 - I_t)L(y^T_t) + I_t[q_t d_{t+1} - d_t], \]
\[ (1 - I_t) d_{t+1} = 0, \]
\[ \lambda_t = U'(A(c^T_t, F(h_t))) A_1(c^T_t, F(h_t)), \]
\[ (1 - \tau^d_t)q^d_t \lambda_t = \beta E_t \lambda_{t+1}, \]
\[ I_t(q^d_t - q_t) = 0, \]
\[ \frac{A_2(c^T_t, F(h_t))}{A_1(c^T_t, F(h_t))} = \frac{w_t}{F'(h_t)}, \]
\[ w_t \geq \gamma \frac{w_{t-1}}{\epsilon_t}, \]
\[ h_t \leq \bar{h}, \]
\[ (h_t - \bar{h}) \left( w_t - \gamma \frac{w_{t-1}}{\epsilon_t} \right) = 0, \]
\[ I_t \left[ q_t - \frac{E_t I_{t+1}}{1 + r^*} \right] = 0, \]

given processes \( \{y^T_t, \epsilon_t, \tau^d_t, I_t\} \) and initial conditions \( w_{-1} \) and \( d_0 \).
When the government can choose freely $\epsilon_t$ and $\tau^d_t$, the competitive equilibrium can be written in a more compact form, as stated in the following proposition.

**Proposition 12.8 (Competitive Equilibrium When $\epsilon_t$ and $\tau^d_t$ Are Unrestricted)** When the government can choose $\epsilon_t$ and $\tau^d_t$ freely, stochastic processes $\{c^T_t, h_t, d_{t+1}, q_t\}$ can be supported as a competitive equilibrium if and only if they satisfy (12.82), (12.83), (12.89), and (12.91), given processes $\{y^T_t, I_t\}$ and the initial condition $d_0$.

The ‘only if’ part of this proposition is trivial, since processes that satisfy the complete set of equilibrium conditions, that is, conditions (12.82)-(12.91), must obviously satisfy a subset thereof. Establishing the ‘if’ part of the proposition amounts to showing that if processes $\{c^T_t, h_t, d_{t+1}, q_t\}$ satisfy conditions (12.82), (12.83), (12.89), and (12.91), then they also satisfy the remaining conditions defining a competitive equilibrium, namely, conditions (12.84)-(12.88) and (12.90). To see this, pick $\lambda_t$ to satisfy (12.84). When $I_t$ equals 1, set $q^d_t$ to satisfy (12.86) and set $\tau^d_t$ to satisfy (12.85). When $I_t$ equals 0, set $\tau^d_t = 0$ (recall convention (12.81)) and set $q^d_t$ to satisfy (12.85). Set $w_t$ to satisfy (12.87). Set $\epsilon_t$ to satisfy (12.88) with equality. This implies that the slackness condition (12.90) is also satisfied. This establishes proposition 12.8.

### 12.12.3 Optimality of the Full-Employment Devaluation Policy

The benevolent government in this economy then chooses a set of processes $\{c^T_t, h_t, d_{t+1}, q_t, I_t\}$ that maximizes

$$
\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(A(c^T_t, F(h_t)))
$$

subject to compact version of the equilibrium conditions

$$
c^T_t = y^T_t - (1 - I_t)L(y^T_t) + I_t[q_t d_{t+1} - d_t],
$$

(12.82)
\[(1 - I_t)dt_{t+1} = 0, \quad (12.83)\]

\[h_t \leq \bar{h}, \quad (12.89)\]

\[I_t \left[ q_t - \frac{E_t I_{t+1}}{1 + r^*} \right] = 0 \quad (12.91)\]

Notice that \(h_t\) enters only in the objective function (12.92) and the constraint (12.89). Clearly, because \(U, A,\) and \(F\) are all strictly increasing, the solution to the optimal policy problem must feature full employment at all times, \(h_t = \bar{h}\).

Once \(h_t\) is set at \(\bar{h}\) for all \(t\), the above optimization problem becomes identical to the canonical Eaton-Gersovitz model. This becomes evident if one rewrites the above optimization problem in recursive form, as we did in section 12.8. We have therefore established that real models of sovereign default in the tradition of Eaton and Gersovitz (1981) can be interpreted as the centralized version of economies with default risk, downward nominal wage rigidity, optimal capital controls, and optimal devaluation policy. Unlike the family of real Eaton-Gersovitz models, however, the present model delivers precise predictions regarding the optimal behavior of the nominal devaluation rate. In particular, the present formulation allows us to answer the question of whether it is optimal to couple defaults with large devaluations, that is, whether the Twin Ds phenomenon emerges as an optimal outcome.

In chapter 9, we showed that the optimal exchange-rate policy is indeed a family given by the devaluation rates satisfying \(\epsilon_t \geq \frac{\omega_t}{\omega(c_t^T)}\), where \(\omega(c_t^T)\) denotes the full-employment real wage, defined as \(\omega(c_t^T) = \frac{A_2(c_t^T, F(\bar{h}))}{A_1(c_t^T, F(\bar{h}))} F'(\bar{h})\). According to these expressions, under optimal policy, the government must devalue in periods in which consumption of tradables experiences a sufficiently large contraction. To the extent that during contractions of this type the government finds it optimal to default, the current model will predict that devaluations and default happen together. From the family of optimal devaluation policies, we now select the one that stabilizes nominal
wages, which are the source of nominal rigidity in the present model. Specifically, we assume a devaluation rule of the form

$$
\epsilon_t = \frac{w_{t-1}}{\omega(c^T_t)}.
$$

(12.93)

A property of this devaluation rule is that it fully stabilizes the nominal wage rate and guarantees price and exchange rate stability in the long run (exercise 12.13 asks you to show this).

### 12.12.4 Default Dynamics Under Optimal Devaluation Policy and Currency Pegs

We now explore quantitatively the dynamics of the model around defaults under two polar exchange-rate policies, the optimal policy and a currency peg. Where applicable, the parameterization of the model mimics that of the real Eaton-Gersovitz model presented in section 12.6 (see table 12.7). In particular, we set the parameters of the AR(1) process $y^T_t$ equal to the corresponding parameters defining the process $y_t$. The present model, however, has a richer structure than the standard Eaton-Gersovitz model of section 12.6, as it features an additional sector, the nontraded sector, and nominal rigidities. We parameterize these aspects of the model as in the model with nontradables and downward nominal wage rigidity of chapter 9. In particular, we assume that the aggregator function takes the CES form

$$
A(c^T_t, c^N_t) = \left[ a(c^T_t)^{1-\xi} + (1-a)(c^N_t)^{1-\xi} \right]^{1-\frac{1}{\xi}}
$$

and that the production function of nontradables takes the isoelastic form $F(h_t) = h_t^\alpha$, with $\xi = 0.5$, $a = 0.26$, and $\alpha = 0.75$.

Because the intra- and intertemporal elasticity of consumption substitution, $\xi$ and $1/\sigma$, are equal to each other, the equilibrium behavior of $I_t$, $d_t$, $q_t$, $c^T_t$, and $\tau^d_t$ are the same as those implied by the real Eaton-Gersovitz model characterized in sections 12.6 and 12.8. \footnote{One might wonder why the restriction $\xi = 1/\sigma$ is necessary to ensure the exact quantitative behavior of the present model under optimal exchange-rate policy and the model of section 12.6.4. The reason is that only under this restriction is the derivative of $U(A(c^T_t, F(h_t)))$ with respect to $c^T_t$ proportional the corresponding derivative of $U(c^T_t)$.} This is because the optimal exchange-rate policy fully neutralizes the distortions introduced by downward nominal
wage rigidity. Hours worked are always at the full employment level \( \bar{h} \), which we normalize to unity. Given these equilibrium processes, the wage rate, \( w_t \), and the optimal devaluation rate, \( \epsilon_t \), can be readily obtained from equations (12.87) and (12.93), respectively.

Figure 12.16 displays with solid lines the behavior of the model economy around a typical default episode under the optimal exchange-rate policy. The construction of the figure follows the methodology described in section 12.6. As explained above, the dynamics of all variables but the devaluation rate and the real wage are as in the flexible-wage model (see figures 12.9 and 12.12). The typical default event is accompanied by a large devaluation of the local currency of about 40 percent. The model thus captures the Twin Ds phenomenon identified by Reinhart (2002) and documented in figure 12.14. The purpose of this devaluation is to inflate away the real value of wages, thereby facilitating employment in the nontraded sector. Indeed, the figure shows that under the optimal exchange-rate policy the real wage falls sharply by about 40 percent around the default date. In this way, the monetary authority ensures that the external debt crisis does not spill over to the labor market.

It is of interest to compare the default dynamics under optimal devaluation policy with those resulting under a currency peg, for a number of observed debt crises have taken place in the context of fixed exchange rates (e.g., Greece in 2012 and Cyprus in 2013). Figure 12.16 displays with dashed lines the predicted dynamics around the typical default episode under a currency peg. In this case, the monetary authority has its hands tied and is unable to inflate away the real value of wages through a devaluation. As a result, real wages remain high, and the debt crisis spreads to the labor market in the form of massive unemployment, which at the time of the default rises to almost 20 percent. This effect is reminiscent of the high levels of unemployment observed during the debt crisis in the periphery of the eurozone following the global contraction of 2008.

The inability of the pegging country to prevent external crises from spreading to the domestic labor market, generates incentives for the government to boost the domestic absorption of consum-
Figure 12.16: A Typical Default Episode Under Optimal Policy and Under a Peg
tion goods. One way to achieve this is to default and use resources that were previously allocated to servicing the external debt. It follows that in the model peggers have additional incentives to default compared to countries following the optimal devaluation policy. Foreign lenders respond to this elevated incentive to default by reducing the supply of credit. Consequently the model predicts a much lower equilibrium level of debt under a peg. The unconditional mean of debt under a peg is 20 percent of traded output per quarter versus 60 percent under the optimal exchange-rate policy.

Summarizing, the present section establishes that the standard Eaton-Gersovitz model can be viewed as a decentralized economy in which households and firms transact in goods, services, and credit markets and in which prices suffer from nominal rigidity. This way of viewing the Eaton-Gersovitz model allows one to think about the joint properties of optimal default and optimal exchange-rate policy. In this regard, the section presents a model in which both default and large devaluations occur in tandem under optimal policy, which explains the Twin Ds regularity observed in the data. The approach of this section also allows for the analysis of default dynamics when the monetary authority follows sub-optimal policy. In particular, the results of the section establish that under a currency peg optimal defaults are accompanied by large levels of involuntary unemployment, as observed in the periphery of the euro area in the wake of the global contraction of 2008.
Table 12.19: Sovereign Default Episodes 1975-2014

<table>
<thead>
<tr>
<th>Country</th>
<th>Start of Default</th>
<th>End of Default</th>
<th>Data Source</th>
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</thead>
<tbody>
<tr>
<td>Angola</td>
<td>1985</td>
<td>2003</td>
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</tr>
<tr>
<td>Antigua and Barbuda</td>
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<td>Argentina</td>
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</tr>
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<td>2007</td>
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<td>2013</td>
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</tr>
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<td>BC, Table 7; BN.</td>
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<td>1983</td>
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<tr>
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</tr>
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</table>

12.14 Exercises

Exercise 12.1 Sturzenegger and Zettelmeyer (2006) argue that default episodes tend to happen in clusters. Devise an empirical strategy and use the information provided in table 12.19 to ascertain whether the data provided there supports or contradicts their finding.

Exercise 12.2 Figure 12.5 shows that the capital-output ratio in Argentina peaked in the run up to defaults of 1982 and 2001 and fell significantly thereafter. This exercise aims at establishing whether this finding holds more generally. To this end, proceed as follows:

1. Use the World Development Indicators data base to download data on real GDP per capita and real gross capital formation per capita (i.e., investment). The primary series to use here are GDP per capita in constant local currency units (NY.GDP.PCAP.KN) and gross capital formation in percent of GDP (NE.GDI.TOTL.ZS). Let $Y_{it}$ and $I_{it}$ denote, respectively, real per capita output and real per capita investment in country $i$ in year $t$.

2. For each country, compute the average growth rate of real per capita output, denoted $g_i$ (i.e., $g_i = 0.02$ means 2 percent).

3. Assume that the capital stock in country $i$ evolves according to

$$K_{it+1} = (1 - \delta)K_{it} + I_{it}, \quad (12.94)$$

where $\delta$ denotes the depreciation rate, which is assumed to be the same in all countries. Use this expression to construct a time series for capital. Set $\delta = 0.1$ (or 10%). We need an initial value for the capital stock, $K_{i1}$. Assume that $K_{i2} = (1 + g_i)K_{i1}$, that is, assume that between periods 1 and 2, the capital stock grew at the average growth rate of the economy. Use this assumption, equation (12.94), and $I_{i1}$ to obtain $K_{i1}$.
4. Now use $K_{i1}$, the time series $I_{it}$, and iterations on equation (12.94) to derive a time series for $K_{it}$.

5. For each country $i$, use the time series for capital, $K_{it}$, and output, $Y_{it}$, to construct a time series for the capital-output ratio, $K_{it}/Y_{it}$.

6. Combine the data on the capital-output ratio with the data on default dates from table 12.19 to produce a figure (in the spirit of figure 12.2) displaying the typical behavior of the (demeaned) capital output ratio around default episodes.

7. Discuss to which extent the behavior of the capital-output ratio pre and post default in Argentina is representative of what happens around the typical default episode.

Exercise 12.3 ([No Excessive Punishment]) In section 12.3.3 we studied an environment in which creditors seize $k$ units of goods from delinquent debtors. This threat could be viewed as excessive punishment in some states, because it implies that creditors will take away $k$ units of goods even if the size of the defaulted debt is smaller than $k$. A more compelling assumption is that the punishment takes the form $\min\{k, d(\epsilon)\}$, where $k \in (0, \epsilon H)$ and $d(\epsilon)$ denotes the debt obligation in state $\epsilon$. Show that the analysis of section 12.3.3 goes through under this assumption.

Exercise 12.4 (Proportional Sanctions) In the model with direct sanctions of section 12.3.3, replace the assumption of a constant sanction $k$ with the assumption of a proportional sanction $k(\epsilon) \equiv \alpha(\bar{y} + \epsilon)$. Characterize the optimal debt contract. How does it compare with the case of constant sanctions?

Exercise 12.5 (Moral Sanctions) Consider another variant of the model with direct sanctions of section 12.3.3. Suppose that direct sanctions are not possible, that is, $k = 0$. Instead, assume that defaulting countries experience a self-inflicted moral punishment. Specifically, assume that the
utility of the country that defaults in state $\epsilon$ is given by $u(y + \epsilon) - m$, where $m > 0$ is a parameter defining the severity of the moral punishment.

1. Write the incentive-compatibility constraint.

2. Characterize the optimal debt contract $d(\epsilon)$ and compare it to the one corresponding to the case of direct sanctions.

Exercise 12.6 (Non-zero Opportunity Cost of Lending) Consider yet another variant of the model with direct sanctions of section 12.3.3. Suppose that the opportunity cost of funds of foreign lenders is not zero but positive and equal to the constant $r$. Suppose first that the borrowing country does not have the option of not writing a debt contract with foreign lenders.

1. What restrictions on $k$ and $r$ do you need to impose to guarantee the existence of an non-autarkic equilibrium.

2. Write the participation constraint.

3. Characterize the optimal debt contract $d(\epsilon)$ and compare it to the one corresponding to the case of zero opportunity costs.

4. How do the answers to the above two questions change if the borrowing country was assumed to have the option of not writing a contract with foreign lenders.

Exercise 12.7 (Reputation, Complete Asset Markets, And Reentry) Extend the reputational model of section 12.3.4 by allowing for the possibility of regaining access to international capital markets after default. Specifically, assume that with constant probability $\delta \in (0, 1)$ defaulters can reentry capital markets the next period.

1. Derive the value function of a country in bad financial standing, $v^b(\epsilon)$, as a function of current and future expected values of $u(\bar{y} + \epsilon)$ and $u(\bar{y} + \epsilon - d(\epsilon))$ only.
2. Write down the incentive-compatibility constraint.

3. Write down the optimization problem of the country and its associated Lagrangian.

4. Derive the optimality conditions of the country’s problem.

5. Show that all of the results of section 12.3.4 pertaining to the reputation model hold under this extension.

6. It is intuitively obvious that if $\delta = 1$, lending breaks down, since in this case lenders have no way to punish delinquent debtors. Show this result formally.

**Exercise 12.8** Show that in the default model of section 12.4, the current account, denoted $ca$, can be written as $ca = q(d)d - q(d')d'$.

**Exercise 12.9** Show that Proposition 12.1 holds when the endowment process is assumed to be serially correlated.

**Exercise 12.10** Consider a perfect-foresight, endowment economy in which an equilibrium sequence of net external debt $\{d_t\}_{t=-1}^\infty$ is supported on reputational grounds when saving in international markets is not allowed after default. Suppose that this sequence contains no maximal element, but has a positive least upper bound $\bar{d}$. Show that the reputational equilibrium with debt breaks down if the country is allowed to save in international financial markets after default.

**Exercise 12.11 (Proportional Output Loss Function)** Modify the Eaton-Gersovitz model of section 12.6.4 to allow for a proportional output loss function of the form $L(y) = a_1y$. Calibrate $a_1$ to match an average output loss due to default of about 7 percent of output per period conditional on the country being in bad financial standing. Use appropriately modified versions of the matlab scripts eg.m and simu.m, or write your own code, to produce quantitative predictions of the model along the
lines of tables 12.8 and 12.9 and figure 12.9. Discuss your findings, paying particular attention to how the present output loss specification affects the model’s ability to predict that countries default in bad times.

**Exercise 12.12 (Duration in the Random-Maturity Model)** Consider the sovereign debt model with random maturity studied in section 12.10.1.

1. Compute the duration of the current portfolio of debt.
2. Compute the duration of a single bond in the current portfolio of debt.

**Exercise 12.13** Show that the devaluation rule given in equation (12.93) is optimal, that it implies a constant nominal wage rate at all times, and that it induces nontrending paths for the nominal exchange rate, the nominal price of nontradables, and the consumer price level.

**Exercise 12.14 (Eaton-Gersovitz Model As Special Case of Debt-Renegotiation Model)** Consider the debt renegotiation model of section 12.11. Investigate analytically whether for \( \alpha = 1 \), the equilibrium dynamics collapse to those of the standard Eaton-Gersovitz model studied in section 12.6.

**Exercise 12.15 (Recalibration of the Debt Renegotiation Model)** In the model of debt renegotiation studied in section 12.11, calibrate the parameters \( \beta, a_1, a_2, \) and \( \alpha \) to match a quarterly debt-to-output ratio of 60 percent in periods of good financial standing, a default frequency of 2.6 times per century, an average output cost of 7 percent per period conditional on being in bad standing, and an average haircut of 40 percent. Discuss the differences with the calibration used in section 12.11.2, with emphasis on the parameters \( \beta \) and \( \alpha \). Use the matlab scripts egr.m, simur.m and statistics_modelr.m (or your own code) to produce tables and graphs like those presented in section 12.11.2. Discuss your findings.


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