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slides

chapter 4

the open economy

real-business-cycle model

Motivation:

In the previous chapter, we built a model of the open economy driven by productivity shocks and argued that it can capture the observed countercyclicality of the trade balance. We also established that two features of the model are important for making this prediction possible. First, productivity shocks must be sufficiently persistent. Second, capital adjustment costs must not be too strong. In this chapter, we ask more questions about the ability of that model to explain observed business cycles. In particular, we ask whether it can explain the sign and magnitude of business-cycle indicators, such as the standard deviation, serial correlation, and correlation with output of output, consumption, investment, the trade balance, and the current account.

The Small Open Economy RBC Model

To make the models studied in chapters 2 and 3 more empirically realistic and to give them a better chance to account for observed business-cycle regularities add:

1. endogenous labor supply and demand

2. uncertainty in the technology shock process

3. capital depreciation.

The resulting theoretical framework is known as the Small Open Economy Real-Business-Cycle model, or, succinctly, the SOE-RBC model.

The Household's Maximization Problem

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, h_t) \tag{4.1}$$

subject to

$$c_t + i_t + \Phi(k_{t+1} - k_t) + (1 + r_{t-1})d_{t-1} = y_t + d_t \tag{4.2}$$

$$y_t = A_t F(k_t, h_t) \tag{4.3}$$

$$k_{t+1} = (1 - \delta)k_t + i_t \tag{4.4}$$

$$\lim_{j \to \infty} E_t \frac{d_{t+j}}{\prod_{s=0}^{j}(1 + r_s)} \leq 0 \tag{4.5}$$

Capital adjustment cost, $\Phi(0) = \Phi'(0) = 0$; $\Phi''(0) > 0$

Additions/differences to the model analyzed in Chapter 3

- endogenous labor supply, $U(c_t, h_t)$
- endogenous labor demand, $F(k_t, h_t)$
- uncertainty, A_t is stochastic
- the interest rate is no longer constant, $r_t \neq r$
- depreciation, δ no longer 0

Household's Optimality Conditions

$$c_t + k_{t+1} - (1-\delta)k_t + \Phi(k_{t+1} - k_t) + (1 + r_{t-1})d_{t-1} = A_t F(k_t, h_t) + d_t \qquad (4.6)$$

$$\lambda_t = \beta(1 + r_t)E_t \lambda_{t+1} \qquad (4.7)$$

$$U_c(c_t, h_t) = \lambda_t \qquad (4.8)$$

$$-U_h(c_t, h_t) = \lambda_t A_t F_h(k_t, h_t) \qquad (4.9)$$

$$1 + \Phi'(k_{t+1} - k_t) = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} \left[A_{t+1} F_k(k_{t+1}, h_{t+1}) + 1 - \delta + \Phi'(k_{t+2} - k_{t+1}) \right] \qquad (4.10)$$

Inducing Stationarity: External debt-Elastic Interest Rate (EDEIR)

$$r_t = r^* + p(\widetilde{d}_t) \tag{4.14}$$

r^* = constant world interest rate
$p(\tilde{d}_t)$ = country interest-rate premium
$\tilde{d}_t$ = cross-sectional average of debt

In equilibrium cross-sectional average of debt must equal individual debt

$$\widetilde{d}_t = d_t \tag{4.15}$$

Evolution of Total Factor Productivity, AR(1) process

$$\ln A_{t+1} = \rho \ln A_t + \widetilde{\eta}\epsilon_{t+1} \tag{4.12}$$

The Trade Balance

$$tb_t = y_t - c_t - i_t - \Phi(k_{t+1} - k_t) \tag{4.20}$$

The Current Account

$$ca_t = tb_t - r_{t-1}d_{t-1} \tag{4.21}$$

Equilibrium Conditions

$$-\frac{U_h(c_t,h_t)}{U_c(c_t,h_t)} = A_t F_h(k_t,h_t) \tag{4.11}$$

$$c_t + k_{t+1} - (1-\delta)k_t + \Phi(k_{t+1}-k_t) + [1+r^* + p(d_{t-1})]d_{t-1} = A_t F(k_t,h_t) + d_t \tag{4.16}$$

$$U_c(c_t,h_t) = \beta(1+r^*+p(d_t))E_t U_c(c_{t+1},h_{t+1}) \tag{4.17}$$

$$1 = \beta E_t \left\{ \frac{U_c(c_{t+1},h_{t+1})}{U_c(c_t,h_t)} \frac{[A_{t+1}F_k(k_{t+1},h_{t+1}) + 1 - \delta + \Phi'(k_{t+2}-k_{t+1})]}{1+\Phi'(k_{t+1}-k_t)} \right\} \tag{4.18}$$

This is a system of non-linear stochastic difference equations. It does not have a closed form solution. We will use numerical techniques to find a first-order accurate approximate solution around the nonstochastic steady state. This is a local approximation. (Later we will also consider a global solution method.)

The system is a second-order difference equation as it features k_t, k_{t+1} and k_{t+2}. By defining auxiliry variables, it can be trnsformed into a first-order system. To this end, introduce the auxiliary variable k_t^f, and equation $k_t^f = k_{t+1}$, and replace k_{t+2} by k_{t+1}^f. Note that k_t^f is in the information set of period t. The transformed system is a set of stochastic first-order difference equations in the 5 unknowns: c_t, h_t, d_{t-1}, k_t, and k_t^f.

First-Order Accurate Approximaiton of the Equilibrium Conditions

Let y_t be a vector containing rthe endogenous nonpredetermined variables of the model, c_t h_t, and k_t^f, and x_t a vector containing the predetermined endogenous variables, k_t and d_{t-1}, and exogenous, A_t, variables of the model Then the solution of the model can be written as

$$y_t = g(x_t, \sigma) \quad \text{and} \quad x_{t+1} = h(x_t, \sigma) + \sigma \eta \epsilon_{t+1},$$

where σ is a scalar such that if $\sigma = 0$, the system becomes deterministic. The method consists in applying a Taylor expansion of this system with respect to y_t, x_t, and σ around the deterministic steady state of y_t and x_t and $\sigma = 1$.

The first-order accurate solution is of the form

$$\widehat{y}_t = g_x \widehat{x}_t \quad \text{and} \quad \widehat{x}_{t+1} = h_x \widehat{x}_t + \eta \epsilon_{t+1},$$

where $\widehat{x}_t$ and $\widehat{y}_t$ are (log) deviations of x_t and y_t from their steady-state values. The appendix shows how to obtain this approximation.

Taking stock:

An important element of the implementation of the first-order accurate solution of the model is finding numerical values for the derivatives of the functions $g(.,.)$ and $h(.,.)$ at the non-stochastic steady state.

Our approach is to use the Symbolic Math Toolbox of Matlab to do most of the work. This has several advantages. One is that the room for error is much smaller and the other is that it eliminates any tedious linearization by hand.

To allow a Symbolic Math toolbox to implement the linearization it is convenient to specify functional forms for the utility, production, country premium, and adjustment cost functions. What will matter, given that we perform a first-order approximation to the equilibrium conditions, is at most the first and second derivatives of those functions.

Functional Forms

Period utility function (CRRA(GHH))

$$U(c,h) = \frac{G(c,h)^{1-\sigma} - 1}{1-\sigma} \quad \text{with} \quad G(c,h) = c - \frac{h^\omega}{\omega}; \quad \omega > 1, \sigma > 0$$

Debt-elastic interest rate

$$p(d) = \psi\left(e^{d-\bar{d}} - 1\right); \quad \psi > 0$$

Production function

$$F(k,h) = k^\alpha h^{1-\alpha}; \quad \alpha \in (0,1)$$

Adjustment cost function

$$\Phi(x) = \frac{\phi}{2}x^2; \quad \phi > 0$$

6 structural parameters: σ, ω, ψ, $\bar{d}$, α, ϕ

What's Special about GHH Preferences?

Recall the period utility function is of the form

$$U(c,h) = \frac{G(c,h)^{1-\sigma} - 1}{1-\sigma} \quad \text{with} \quad G(c,h) = c - \frac{h^\omega}{\omega}$$

The marginal rate of substitution between consumption and labor is independent of consumption:

$$-\frac{U_h(c,h)}{U_c(c,h)} = \frac{G(c,h)^{-\sigma} G_h(c,h)}{G(c,h)^{-\sigma} G_c(c,h)} = h^{\omega-1}.$$

Implicaiton: The labor supply, $-Uh/Uc = w$ is independent of c_t, i.e., depends on the wage rate only. Put differently, GHH preferences kill the wealth effect on labor supply.

Deterministic Steady State

The steady state is the quadruple (d, k, c, h) satisfying

$$-\frac{U_h(c,h)}{U_c(c,h)} = AF_h(k,h) \tag{4.11'}$$

$$c + \delta k + (r^* + p(d))d = AF(k,h) \tag{4.16'}$$

$$1 = \beta(1 + r^* + p(d)) \tag{4.17'}$$

$$1 = \beta\left[AF_k(k,h) + 1 - \delta\right] \tag{4.18'}$$

$$A = 1.$$

Using the assumed functional forms the steady state becomes

$$h^{\omega-1} = (1-\alpha)(k/h)^{\alpha} \tag{4.11''}$$

$$c + \delta k + (r^* + \psi(e^{d-\bar{d}} - 1))d = (k/h)^{\alpha} h \tag{4.16''}$$

$$1 = \beta(1 + r^* + \psi(e^{d-\bar{d}} - 1)) \tag{4.17''}$$

$$1 = \beta\left[\alpha(k/h)^{\alpha-1} + 1 - \delta\right] \tag{4.18''}$$

This is a system of 4 equations in 4 unknown endogenous variables, (c, d, h, k) and 7 unknown parameters, $\omega, \alpha, \delta, r^*, \psi, \bar{d}, \beta$. The model has 4 additional structural parameters, σ, ϕ, ρ, $\tilde{\eta}$, which do not enter the steady state but which also need to be assigned values to. In sum, there are 11 structural parameters to be calibrated. They are:

$$\left[\begin{array}{ccccccccccc} \omega & \alpha & \delta & r^* & \beta & \sigma & \phi & \rho & \tilde{\eta} & \bar{d} & \psi \end{array}\right]$$

Calibration

Assume the time unit is one year and calibrate the model to the Canadian economy. We adopt (almost) the same calibration as Mendoza (1991).

σ	r^*	δ	α	ω	ϕ	ρ	σ_ϵ	$\bar{d}$
2	0.04	0.1	0.32	1.455	0.028	0.42	0.0129	0.7442

Comment: Mendoza's model uses a different stationarity inducing device (an internal discount factor (IDF) model, which we discuss in detail in section 4.10.4) and hence his calibration does not assign a value to ψ. As in Schmitt-Grohé and Uribe (2003), we set ψ to ensure that the EDEIR model predicts the same volatility of the trade-balance-to-output ratio as the IDF model. The value that achieves that is

$$\psi = 0.000742$$

Given values for the structural parameters, a numerically exact solution of the steady state can be radily obtained (see, for example, the Matlab program `edeir_ss.m` available on the book's Website.) This yields

c	d	h	k
1.1170	0.7442	1.0074	3.3977

The Calibration Strategy

To obtain the values of the structural parameters shown in the previous slide, three types of restrictions were imposed:
Category a: restrictions using sources unrelated to the data that the model aims to explain, **4** parameters: $\sigma = 2$, $\delta = 0.1$, $r^* = 0.04$, $\beta = 1/(1 + r^*)$ (or $p(d) = 0$).
Category b: restrictions to match first moments of the data that the model aims to explain (geared toward pinning down α and $\bar{d}$):

$$\text{labor share} = 0.68$$
$$\text{trade-balance-to-output ratio} = 0.02$$

Category c: restrictions to match second moments of the data that the model aims to explain (geared toward pinning down 5 parameters, ω, ϕ, ψ, ρ, $\tilde{\eta}$). The second moments to be matched are:
σ_y, σ_h, σ_i, $\sigma_{tb/y}$, $\text{corr}(\ln y_t, \ln y_{t-1})$

How to implement this calibration strategy? The restrictions in category a translate immediately into values for structural parameters. To go from the restrictions in categories b and c to the values of the structural parameters shown in Table 4.1, one proceeds as follows:

The labor share, s_h, is defined as

$$s_h = \frac{wh}{y}$$

In the decentralized economy we have

$$A_t F_2(k_t, h_t) = w_t$$

Thus, in the steady state:

$$s_h = \frac{AF_2(k, h)h}{AF(k, h)}$$

Using the assumed functional form $Fk, h) = k^\alpha h^{1-\alpha}$ yields

$$s_h = (1 - \alpha)$$

Hence we have that $\alpha = 1 - s_h = 1 - 0.68$, that is,

$$\alpha = 0.32$$

At this stage the calibration strategy proceeds in the following steps:
Step 1: Let θ denote the vector of structural parameters we still need to assign numerical values to, that is, let

$$\theta \equiv \left[\begin{array}{cccccc} \omega & \bar{d} & \phi & \psi & \rho & \tilde{\eta} \end{array} \right]$$

Guess a value for each element of θ except $\bar{d}$.
Step 2: Given the guess for ω find h get k/h by solving the capital Euler equation $1 = \beta \left[\alpha(k/h)^{\alpha-1} + 1 - \delta\right]$:

$$\frac{k}{h} = \left(\frac{r^* + \delta}{A\alpha}\right)^{\frac{1}{\alpha-1}}$$

Plug k/h into the labor market equation $h^{\omega-1} = (1-\alpha)(k/h)^{\alpha}$ and solve for h

$$h = ((1-\alpha)A(k/h)^{\alpha})^{1/(\omega-1)}$$

With h in hand, find k and y, as $k = (k/h)h$ and $y = A(k/h)^{\alpha}h$, respectively.

Step 3: Let s_{tb} denote the average trade-balance-to-output ratio (a targeted first moment for which we have a number). In the steady state,

$$s_{tb} = \frac{r^* \bar{d}}{y}$$

which uses the fact that, by the restriction $\beta(1 + r^*) = 1$ (or $p(d) = 0$), $d = \bar{d}$. Solve this expression for $\bar{d}$

$$\bar{d} = \frac{s_{tb} y}{r^*}$$

Step 4: Find c from the resource constraint $c + \delta k + (r^* + \psi(e^{d-\bar{d}} - 1))d = (k/h)^\alpha h$

$$c = y - \delta k - r^* d$$

Step 5: With the steady state values of (c, k, d, h) and values for all structural parameters in hand, compute the model's predictions for the targeted second moments

$$x(\theta) \equiv \left[\begin{array}{ccccc} \sigma_y & \sigma_h & \sigma_i & \sigma_{tb/y} & corr(\ln y_t, \ln y_{t-1}) \end{array} \right]$$

Step 6: Find the distance

$$D = |x(\theta) - x^*|$$

where x^* denotes the vector of targeted moments observed in Canadian data

Step 7: Keep adjusting θ until D is less than some threshold D^*.

Comment: In general there does not exist a θ that makes the distance D exactly equal to zero. Hence one has to pick some threshold for the distance, D^*.

The Role of Persistence and Capital Adjustment Costs

In chapter 3, we showed that

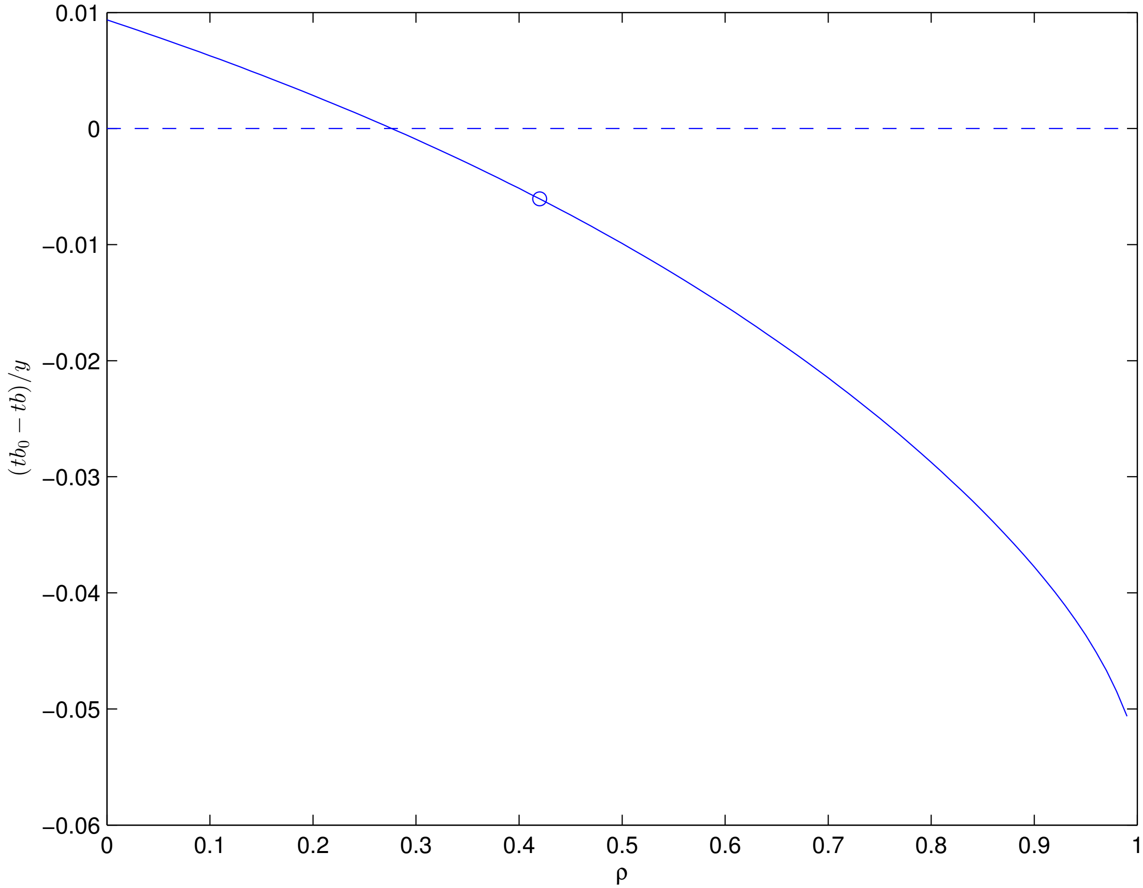
- the more persistent productivity shocks are, the more likely it will be that a positive productivity shock will cause a deterioration of the trade balance.

- the more pronounced are capital adjustment costs, the smaller will be the initial trade balance deterioration in response to a positive productivity shock.

- the more persistent the technology shock is, the higher the volatility of consumption relative to output will be.

The next three figures show that these analytical results do indeed hold in the fully-fledged stochastic dynamic open economy real-business-cycle model.

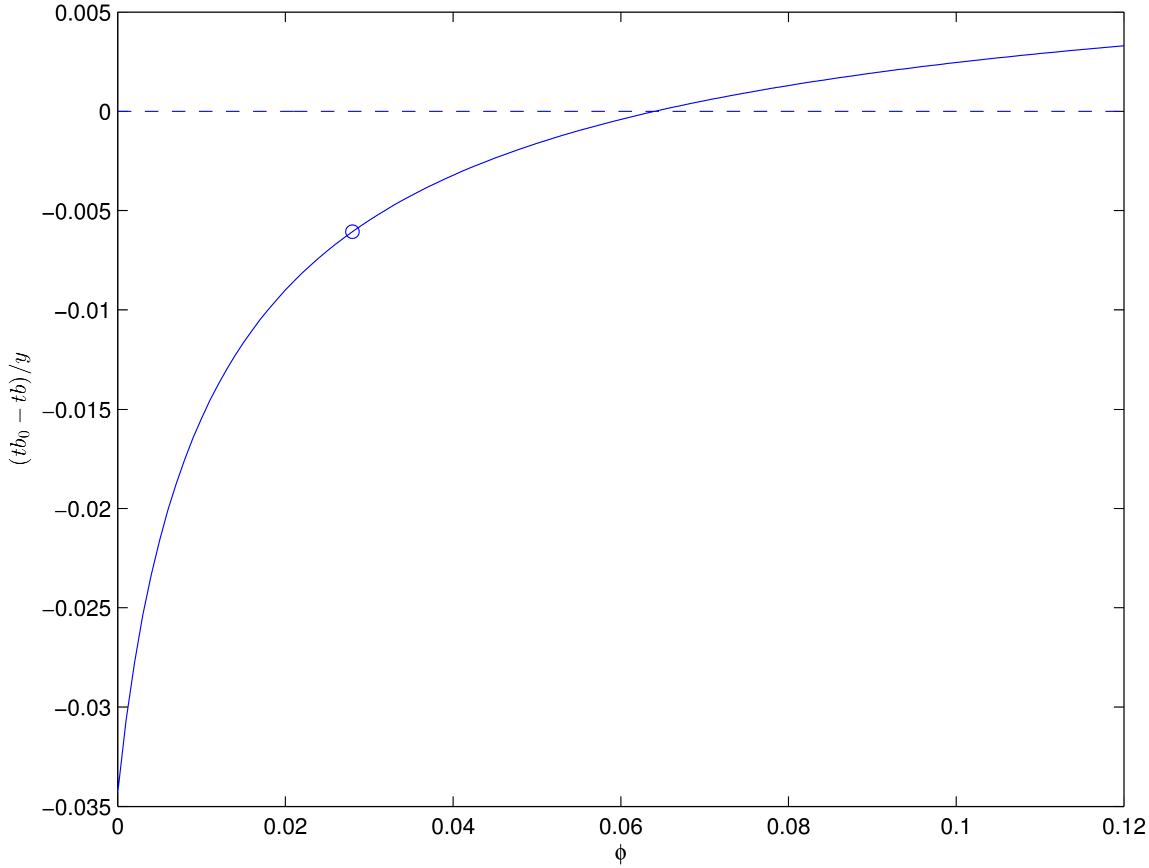
Impact response of the trade balance as a function of the persistence of the technology shock



The figure shows the impact response of the trade balance to a one percent positive innovation in productivity predicted by the EDEIR model presented in Chapter 4. The response of the trade balance is measured in units of steady-state output. All parameters other than ρ take the values shown in Table 4.1. The open circle indicates the baseline value of ρ.

Comments: The figure shows that the more persistent the productivity shock is the smaller the impact response of the trade balance will be. For $\rho > 0.3$, the response of the trade balance is negative, confirming the analytical results of chapters 2 and 3.

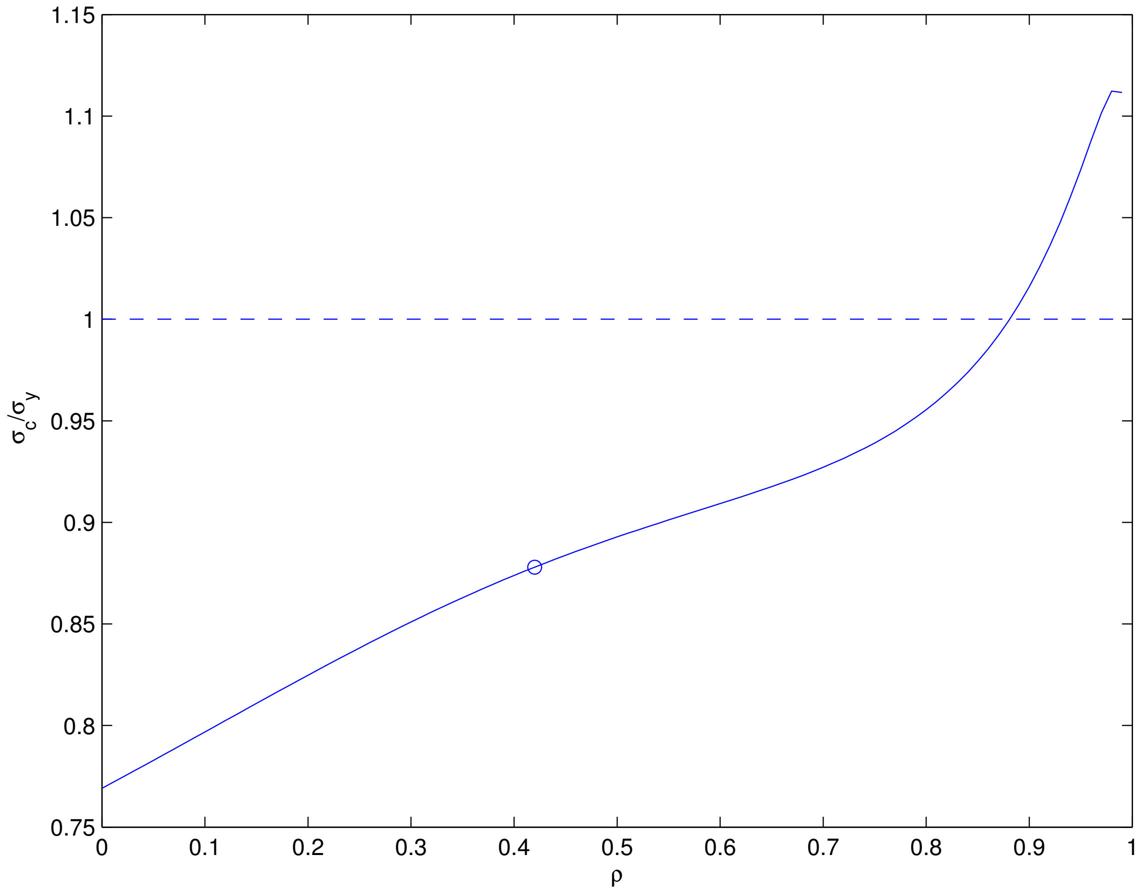
Impact response of the trade balance as a function of capital adjustment costs



Notes. The figure shows the impact response of the trade balance in response to a one percent positive innovation in productivity as a function of the size of capital adjustment costs, ϕ, predicted by the EDEIR model presented in Chapter 4. The response of the trade balance is measured in units of steady-state output. All parameters other than ρ take the values shown in Table 4.1. The open circle indicates the baseline ϕ value.

Comments: The figure shows that the higher capital adjustment costs are the larger the impact response of the trade balance will be. For $\phi > 0.06$, the response of the trade balance turns positive, confirming the analytical results of chapters 2 and 3.

Relative volatility of consumption as a function of the persistence of the stationary technology shock



Notes. The relative standard deviation shown is that implied by the EDEIR model presented in Chapter 4. All parameters other than ρ take the values shown in Table 4.1. The open circle indicates the baseline value of ρ.

Comments: The figure shows that the more persistent stationary productivity shocks are, the higher the standard deviation of consumption relative to the standard deviation of output will be, just as derived analytically in the permanent income model of Chapter 2.

We now turn an analysis of second moments predicted by the SOE-RBC model and compare them to the Canadian data.

Some Empirical Regularities of the Canadian Economy

Why Canada? Because it is the (open) economy on which we based the calibration of the model, following Mendoza (1991).

Variable	Canadian Data		
	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}
y	2.8	0.61	1
c	2.5	0.7	0.59
i	9.8	0.31	0.64
h	2	0.54	0.8
$\frac{tb}{y}$	1.9	0.66	-0.13

Source: Mendoza AER, 1991. Annual data. Log-quadratically detrended.

Comments

- Volatility ranking: $\sigma_{tb/y} < \sigma_c < \sigma_y < \sigma_i$.
- Consumption, investment, and hours are procyclical.
- The trade-balance-to-output ratios is countercyclical.
- All variables considered are positively serially correlated.
- Similar stylized facts emerge from other small developed countries (see, e.g., chapter 1).

Empirical and Theoretical Second Moments

	Canadian Data 1946 to 1985			Canadian Data 1960 to 2011			Model		
	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,y_t}	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,y_t}	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,y_t}
y	2.8	0.6	1	3.7	0.9	1	3.1	0.6	1
c	2.5	0.7	0.6	2.2	0.7	0.6	2.7	0.8	0.8
i	9.8	0.3	0.6	10.3	0.7	0.8	9.0	0.1	0.7
h	2.0	0.5	0.8	3.6	0.7	0.8	2.1	0.6	1
$\frac{tb}{y}$	1.9	0.7	-0.1	1.7	0.8	0.1	1.8	0.5	-0.04
$\frac{ca}{y}$							1.4	0.3	0.05

Comments:

- σ_h, σ_i, σ_y, $\sigma_{tb/y}$, and $\rho_{y_t,y_{t-1}}$ were targeted by calibration, so no real test here.
- model correctly places σ_c below σ_y and σ_i and above σ_h and $\sigma_{tb/y}$.
- model correctly makes tb/y countercyclical.
- model overestimates the correlations of hours and consumption with output.

Why is corr($\mathrm{h_t, y_t}$) Exactly Equal to One?

Recall the equilibrium condition in the labor market:

$$-\frac{U_h(c_t, h_t)}{U_c(c_t, h_t)} = A_t F_h(k_t, h_t)$$

with GHH preferences and Cobb-Douglas technology, this becomes

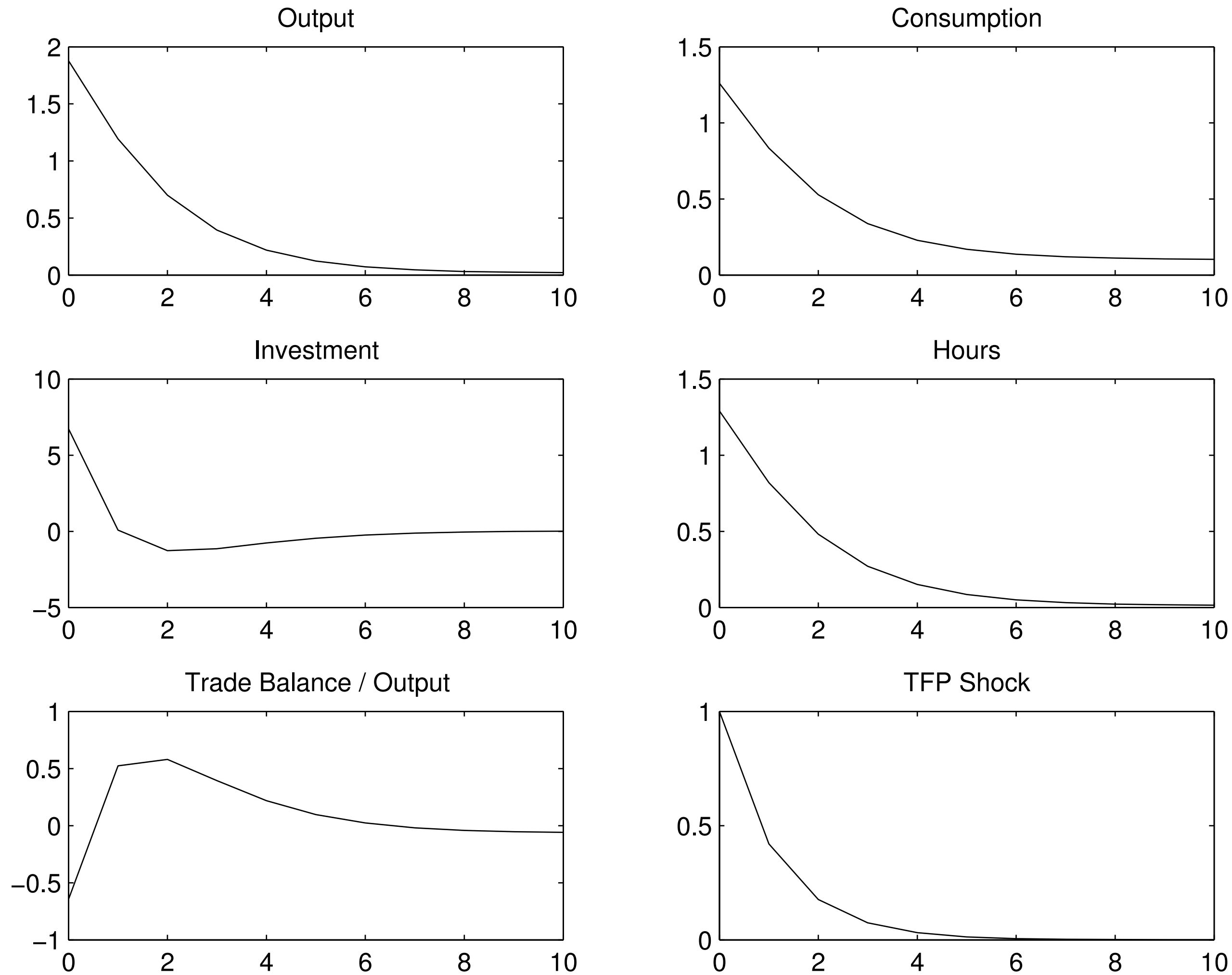
$$\begin{aligned} h_t^{\omega-1} &= A_t(1-\alpha)k_t^{\alpha-1}h_t^{\alpha} \\ &= (1-\alpha)\frac{y_t}{h_t} \end{aligned}$$

Log-linearizing,

$$\omega \widehat{h}_t = \widehat{y}_t,$$

which says taht up to first order, hours and output are perfectly correlated.

Response to a Positive Technology Shock



Source: Schmitt-Grohé and Uribe (JIE, 2003)

Comments:

- Output, consumption, investment, and hours expand.

- The trade balance deteriorates.

4.9 The Complete Asset Markets (CAM) Model

$$E_t r_{t+1} b_{t+1} = b_t + y_t - c_t - i_t - \Phi(k_{t+1} - k_t),$$

$$\lim_{j\to\infty} E_t q_{t+j} b_{t+j} \geq 0,$$

$$q_t = r_1 r_2 \ldots r_t,$$

$$\lambda_t r_{t+1} = \beta \lambda_{t+1}.$$

$$\lambda_t^* r_{t+1} = \beta \lambda_{t+1}^*.$$

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{\lambda_{t+1}^*}{\lambda_t^*}.$$

$$\lambda_t = \xi \lambda_t^*,$$

$$\lambda_t = \psi_4,$$

Calibration: Set ψ_4 so that steady-state consumption equals steady-state consumption in the model with Uzawa preferences.

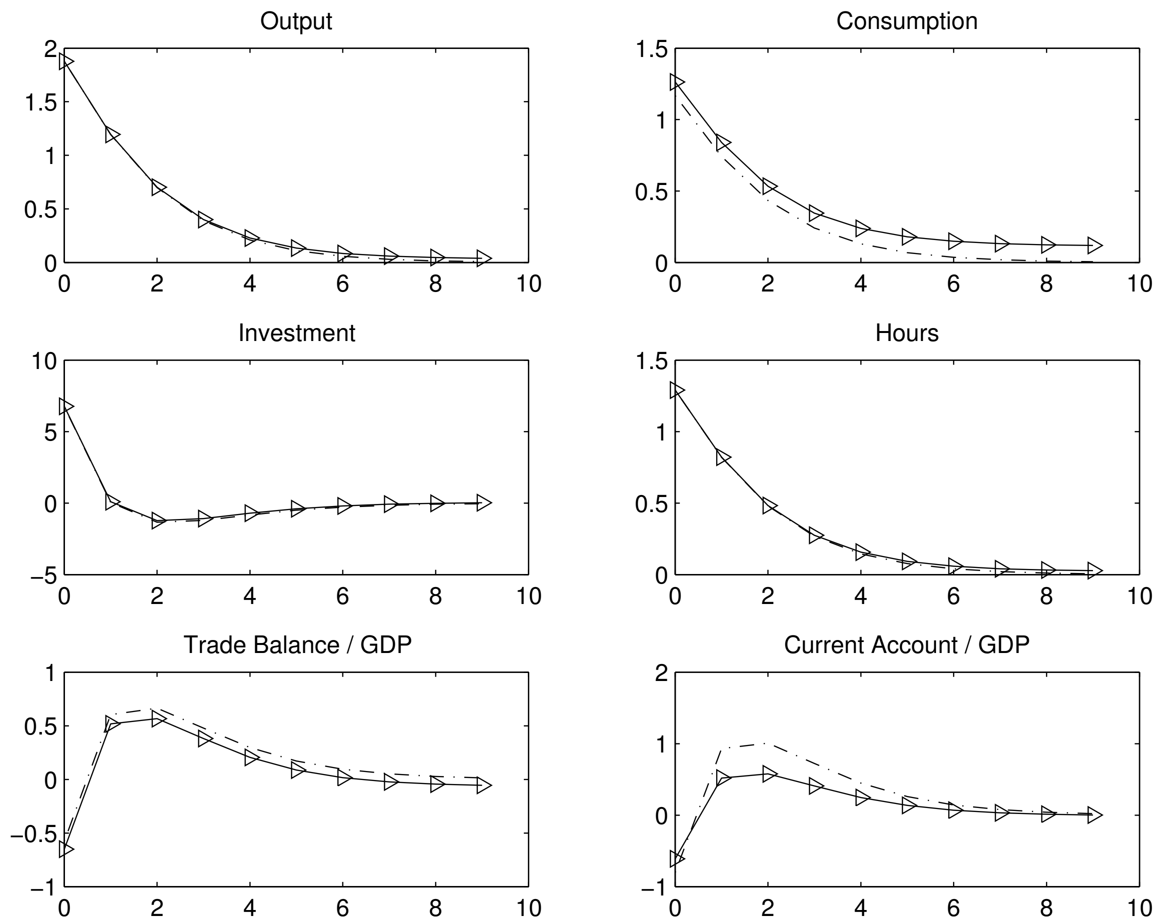
The SOE-RBC Model With Complete Asset Markets: Predicted Second Moments

	σ_{x_t}		$\rho_{x_t,x_{t-1}}$		ρ_{x_t,GDP_t}	
variable	CAM	EDEIR	CAM	EDEIR	CAM	EDEIR
y	3.1	3.1	0.61	0.62	1.00	1.00
c	1.9	2.71	0.61	0.78	1.00	0.84
i	9.1	9.0	0.07	0.07	0.66	0.67
h	2.1	2.1	0.61	0.62	1.00	1.00
$\frac{tb}{y}$	1.6	1.78	0.39	0.51	0.13	-0.04
$\frac{ca}{y}$	3.1	1.45	-0.07	0.32	-0.49	0.05

Note. Standard deviations are measured in percentage points. The columns labeled CAM are produced with the Matlab program `cam_run.m` available at

`http://www.columbia.edu/~mu2166/closing.htm`.

Impulse Response to a Unit Technology Shock
One-Bond Versus Complete Asset Market Models



Dash-diamond, EDEIR model. Dash-dotted, complete-asset-market model.

4.10 Alternative Ways to Induce Stationarity

4.10.1 The Internal Debt-Elastic Interest Rate (IDEIR) Model

$$r_t = r + p(d_t),$$

The Euler equation becomes

$$\lambda_t = \beta[1 + r + p(d_t) + p'(d_t)d_t]E_t\lambda_{t+1}$$

$$p(d) = \psi_2\left(e^{d-\bar{d}} - 1\right),$$

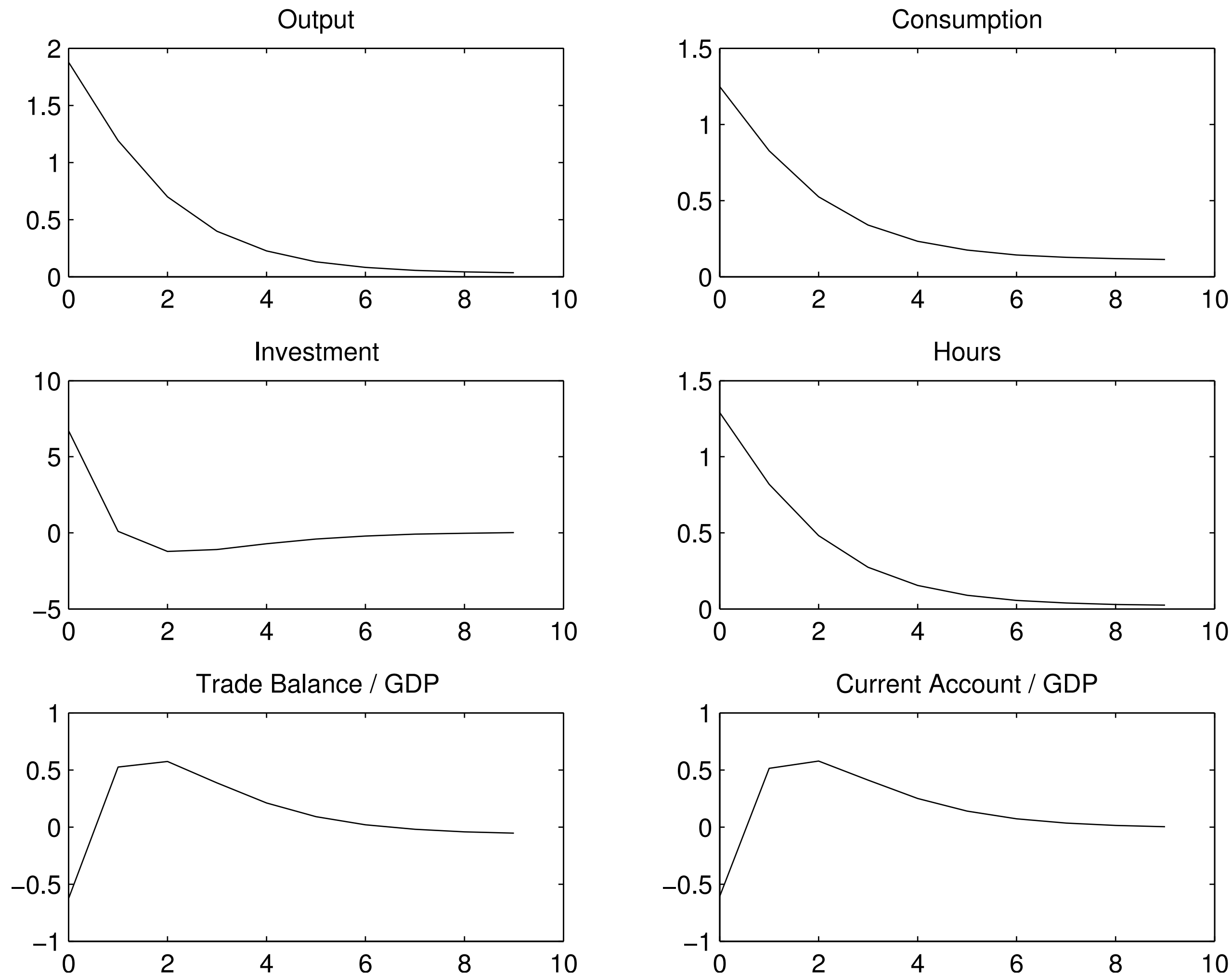
Calibration: Same as in the external case. Note that the steady-state value of debt is no longer equal to $\bar{d}$. Instead, d solves

$$(1 + d)e^{d-\bar{d}} = 1 \Rightarrow d = 0.4045212.$$

Internal Debt-Elastic Interest-Rate

Variable	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}
y	3.1	0.62	1
c	2.5	0.76	0.89
i	9	0.068	0.68
h	2.1	0.62	1
tb/y	1.6	0.43	-0.036
ca/y	1.4	0.31	0.041

Internal Debt-Elastic Interest Rate Premium Response to a Positive Technology Shock



Comment: The economy with internal debt-elastic interest rate premium behaves very simularly to the economies featuring other stationarity inducing devices.

4.10.2 The Portfolio Adjustment Cost (PAC) Model

$$d_t = (1 + r_{t-1})d_{t-1} - y_t + c_t + i_t + \Phi(k_{t+1} - k_t) + \frac{\psi_3}{2}(d_t - \bar{d})^2$$

$$\lambda_t[1 - \psi_3(d_t - \bar{d})] = \beta(1 + r_t)E_t\lambda_{t+1}$$

Calibration

β	$\bar{d}$	ψ_3	r
0.96	0.7442	0.00074	$\beta^{-1} - 1$

4.10.3 The External Discount Factor (EDF) Model

$$\theta_{t+1} = \beta(\tilde{c}_t, \tilde{h}_t)\theta_t \qquad t \geq 0,$$

$$\theta_0 = 1,$$

where $\tilde{c}_t$ and $\tilde{h}_t$ denote per capita consumption and hours worked.

$$\lambda_t = \beta(\tilde{c}_t, \tilde{h}_t)(1 + r_t)E_t\lambda_{t+1}$$

$$\lambda_t = U_c(c_t, h_t)$$

$$-U_h(c_t, h_t) = \lambda_t A_t F_h(k_t, h_t)$$

$$\begin{aligned} \lambda_t[1 + \Phi'_t] &= \beta(\tilde{c}_t, \tilde{h}_t)E_t\lambda_{t+1}[A_{t+1}F_k(k_{t+1}, h_{t+1}) \\ &+ 1 - \delta + \Phi'_{t+1}] \end{aligned}$$

In Equilibrium

$$c_t = \tilde{c}_t \text{ and } h_t = \tilde{h}_t$$

To be added:

4.10.4 The Internal Discount Factor (IDF) Model

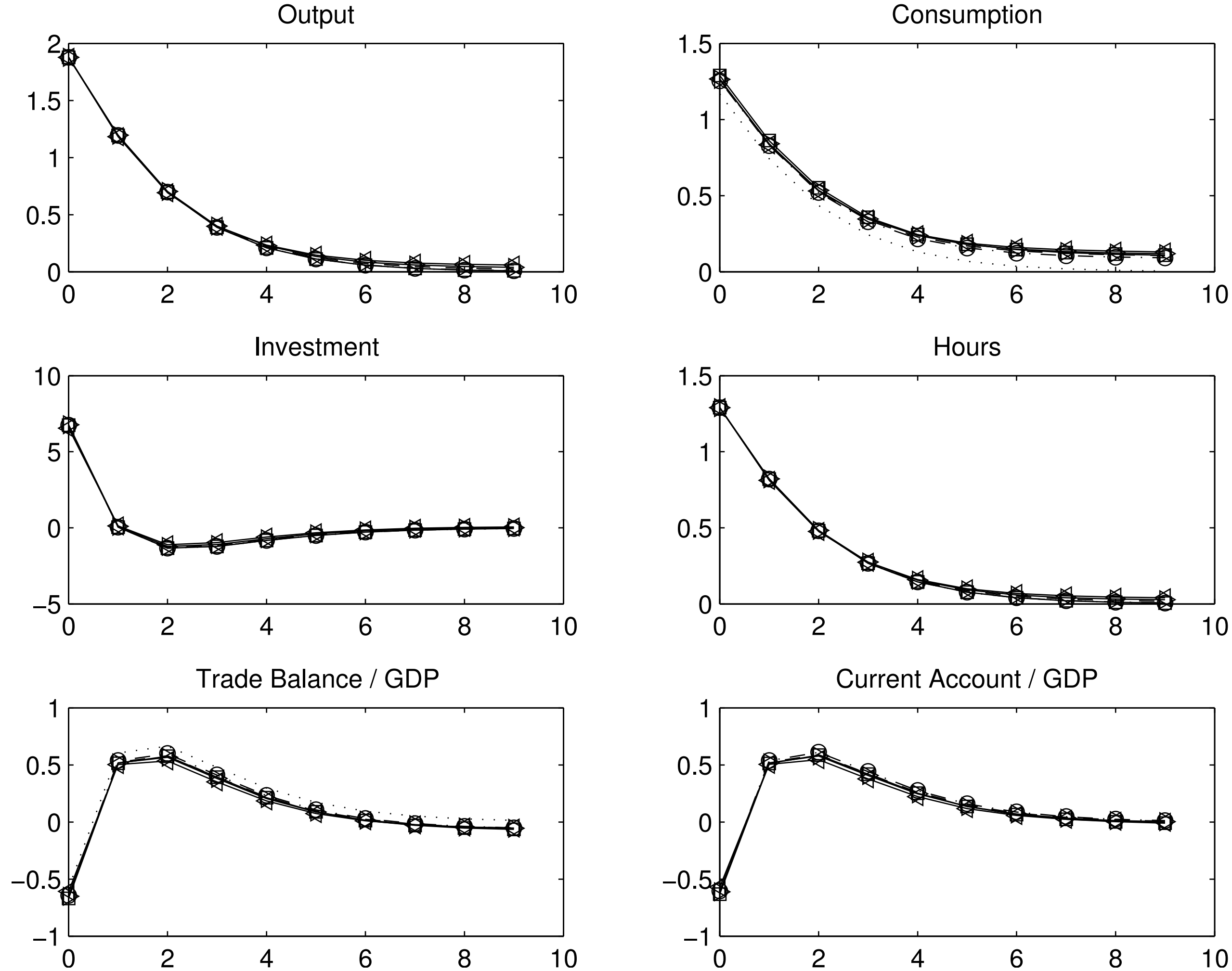
4.10.5 The Model with No Stationarity-Inducing Features (NSIF)

4.11 The Perpetual-Youth (PY) Model

to be added

4.12 Inducing Stationarity: Quantitative Comparison of Alternative Methods

Impulse Response to a Unit Technology Shock in Models 1 Through 5



Source: Schmitt-Grohé and Uribe (JIE, 2003). Note. Solid line, endogenous discount factor. Squares, endogenous discount factor without internalization. Dashed line, Debt-elastic interest rate. Dash-dotted line, Portfolio adjustment cost. Dotted line, complete asset markets. Circles, No stationarity inducing elements.

Observed and Implied Second Moments

	Data	Model 1	Model 1a	Model 2	Model 3	Model 4
Standard Deviations						
y	2.8	3.1	3.1	3.1	3.1	3.1
c	2.5	2.3	2.3	2.7	2.7	1.9
i	9.8	9.1	9.1	9	9	9.1
h	2	2.1	2.1	2.1	2.1	2.1
tb/y	1.9	1.5	1.5	1.8	1.8	1.6
ca/y		1.5	1.5	1.5	1.5	
Serial Correlations						
y	0.61	0.61	0.61	0.62	0.62	0.61
c	0.7	0.7	0.7	0.78	0.78	0.61
i	0.31	0.07	0.07	0.069	0.069	0.07
h	0.54	0.61	0.61	0.62	0.62	0.61
tb/y	0.66	0.33	0.32	0.51	0.5	0.39
ca/y		0.3	0.3	0.32	0.32	
Correlations with Output						
c	0.59	0.94	0.94	0.84	0.85	1
i	0.64	0.66	0.66	0.67	0.67	0.66
h	0.8	1	1	1	1	1
tb/y	-0.13	-0.012	-0.013	-0.044	-0.043	0.13
ca/y		0.026	0.025	0.05	0.051	

Source: Schmitt-Grohé and Uribe (JIE, 2003)

Note. Standard deviations are measured in percent per year.

4.13 Global Solution Methods

To be added

Appendix

First-Order Accurate Approximaiton of Equilibrium Conditions

Derivation of a first-order accurate approximation

Based on Schmitt-Grohé and Uribe JEDC 2004.

The EDEIR model developed above gives rise to equilibrium conditions of the form

$$E_t f(y_{t+1}, y_t, x_{t+1}, x_t) = 0 \tag{1}$$

where
$x_t = n_x \times 1$ vector of predetermined (or state) variables
$y_t = n_y \times 1$ vector of nonpredetermined (or control) variables
x_0 is an $n_x \times 1$ vector of initial conditions
Terminal condition: $\lim_{j\to\infty} E_t \begin{bmatrix} x_{t+j} & y_{t+j} \end{bmatrix}' \to \begin{bmatrix} \bar{x} & \bar{y} \end{bmatrix}'$
Let

$$n = n_x + n_y$$

Then we have that

$$f : R^{n_y} \times R^{n_y} \times R^{n_x} \times R^{n_x} \to R^n$$

A large class of dynamic stochastic general equilibrium models can be written in the form given in (1). And most studies in real and monetary business cycle analysis use models belonging to this class. Of course, there are also many types of models that do not fit into that class. For example, models with occasionally binding constraints.

Partition state vector x_t

$$x_t = \begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix}$$

$x_t^1 =$ vector of endogenous predetermined state variables
$x_t^2 =$ vector of exogenous state variables

We assume that the exogenous state evolves as:

$$x_{t+1}^2 = \Lambda + \sigma\tilde{\eta}\epsilon_{t+1}, \tag{2}$$

where Λ is a square matrix of coefficients, typically diagonal (with diagonal elements representing the serial correlations of the exogenous shocks), and $\sigma =$ parameter scaling the amount of uncertainty. ($\sigma = 0$ is perfect foresight.)

Solution to models that are described by (1) and (2) can be expressed as:

$$y_t = g(x_t, \sigma) \tag{3}$$
$$x_{t+1} = h(x_t, \sigma) + \sigma\eta\epsilon_{t+1} \tag{4}$$

where

$$\eta = \begin{bmatrix} \emptyset \\ \tilde{\eta} \end{bmatrix}$$

Perturbation methods perform a *local* approximation of $g(x, \sigma)$ and $h(x, \sigma)$ around a particular point $(\bar{x}, \bar{\sigma})$

First-order Taylor series expansion of g and h around $(x,\sigma)=(\bar{x},\bar{\sigma})$

$$\begin{aligned} g(x,\sigma) &= g(\bar{x},\bar{\sigma})+g_x(\bar{x},\bar{\sigma})(x-\bar{x})+g_\sigma(\bar{x},\bar{\sigma})(\sigma-\bar{\sigma})+h.o.t. \\ h(x,\sigma) &= h(\bar{x},\bar{\sigma})+h_x(\bar{x},\bar{\sigma})(x-\bar{x})+h_\sigma(\bar{x},\bar{\sigma})(\sigma-\bar{\sigma})+h.o.t. \end{aligned}$$

$h.o.t.=$ higher order terms
Unknowns: $g(\bar{x},\bar{\sigma}), g_x(\bar{x},\bar{\sigma}), g_\sigma(\bar{x},\bar{\sigma}), h(\bar{x},\bar{\sigma}), h_x(\bar{x},\bar{\sigma}), h_\sigma(\bar{x},\bar{\sigma})$
To identify these terms, substitute the proposed solution given by equations (3) and (4) into equation (1), and define

$$\begin{aligned} F(x,\sigma) &\equiv E_t f(g(h(x,\sigma)+\eta\sigma\epsilon',\sigma), g(x,\sigma), h(x,\sigma)+\eta\sigma\epsilon', x) \\ &= 0. \end{aligned} \tag{5}$$

Here we are dropping time subscripts. We use a prime to indicate variables dated in period $t+1$.

Because $F(x,\sigma)$ must be equal to zero for any possible values of x and σ, it must be the case that the derivatives of any order of F must also be equal to zero. Formally,

$$F_{x^k\sigma^j}(x,\sigma)=0 \quad \forall x,\sigma,j,k, \tag{6}$$

where $F_{x^k\sigma^j}(x,\sigma)$ denotes the derivative of F with respect to x taken k times and with respect to σ taken j times.

What point to approximate around?

We need to evaluate the derivatives of $F(x,\sigma)$, $F_{x^k\sigma^j}(x,\sigma)$, at the point we are approximating the equilibrium around. In general this is difficult if not impossible. But there are some points for which evaluation of those derivatives is possible.

One such point is the non-stochastic steady state, $(x,\sigma) = (\bar{x}, 0)$, where $\bar{x}$ denotes the non-stochastic steady state value of x_t. For this point we know: $y_t = \bar{y}$, $y_{t+1} = \bar{y}$, and $x_{t+1} = \bar{x}$, where $\bar{y}$ denotes the non-stochastic steady state of y_t.

Another point one can evaluate the derivatives of $F(x,\sigma)$ at is $x_t \neq \bar{x}$ and $\sigma = 0$. This works in cases in which one can find the exact deterministic solution of a model. In that case one can find y_t, y_{t+1} and x_{t+1} for $(x_t,\sigma) = (x_t, 0)$ but needs to resort to approximation techniques to characterize the solution to the stochastic version of the economy.

For the remainder of this chapter we will focus on approximation around the non-stochastic steady state $(x,\sigma) = (\bar{x}, 0)$.

Let's write again the first-order Taylor series expansion of g and h but this time around the non-stochastic steady state, $(x, \sigma) = (\bar{x}, 0)$

$$g(x, \sigma) = g(\bar{x}, 0) + g_x(\bar{x}, 0)(x - \bar{x}) + g_\sigma(\bar{x}, 0)(\sigma - 0)$$

$$h(x, \sigma) = h(\bar{x}, 0) + h_x(\bar{x}, 0)(x - \bar{x}) + h_\sigma(\bar{x}, 0)(\sigma - 0)$$

We wish to find:

$g(\bar{x}, 0)$

$g_x(\bar{x}, 0)$

$g_\sigma(\bar{x}, 0)$

$h(\bar{x}, 0)$

$h_x(\bar{x}, 0)$

$h_\sigma(\bar{x}, 0)$

Find $g(\bar{x}, 0)$ **and** $h(\bar{x}, 0)$

From (3)

$$g(\bar{x}, 0) = \bar{y}$$

From (4)

$$h(\bar{x}, 0) = \bar{x}$$

Find h_σ and g_σ

Recall (5)

$$\begin{aligned} 0 &= F(x,\sigma) \\ &= E_t f(g(h(x,\sigma)+\eta\sigma\epsilon',\sigma), g(x,\sigma), h(x,\sigma)+\eta\sigma\epsilon', x) \end{aligned}$$

The first derivative of $F(x,\sigma)$ with respect to σ evaluated at $(x,\sigma) = (\bar{x}, 0)$

$$\begin{aligned} 0 &= F_\sigma(\bar{x},0) \\ &= f_{y'}(\bar{y},\bar{y},\bar{x},\bar{x})\left[g_x(\bar{x},0)h_\sigma(\bar{x},0)+g_\sigma(\bar{x},0)\right] \\ &\quad + f_y(\bar{y},\bar{y},\bar{x},\bar{x})g_\sigma(\bar{x},0) \\ &\quad + f_{x'}(\bar{y},\bar{y},\bar{x},\bar{x})h_\sigma(\bar{x},0) \end{aligned}$$

Let $f_i \equiv f_i(\bar{y},\bar{y},\bar{x},\bar{x})$ for $i = y',' y, x', x$

Note that we can evaluate f_i because we know the function f and we know the steady state $(\bar{y},\bar{x})$

Rearrange to obtain

$$\begin{bmatrix} f_{y'}g_x + f_{x'} & f_{y'} + f_y \end{bmatrix} \begin{bmatrix} h_\sigma \\ g_\sigma \end{bmatrix} = 0$$

This is a linear homogenous equation in n unknowns. For it to have a unique solution it must be that

$$\begin{bmatrix} h_\sigma \\ g_\sigma \end{bmatrix} = \begin{bmatrix} \emptyset \\ \emptyset \end{bmatrix} \tag{7}$$

This is an important result. It says that up to first-order accuracy one need not correct the constant term or the slope term of the approximation for the presence of uncertainty. The policy function is the same as under perfect foresight but for the additive stochastic error term. (the solution displays the certainty equivalence principle)

Up to first order accuracy the solution is:

$$\begin{aligned} y_t &= \bar{y} + g_x(\bar{x}, 0)(x - \bar{x}) \\ x_{t+1} &= \bar{x} + h_x(\bar{x}, 0)(x - \bar{x}) + \sigma\eta\epsilon_{t+1} \end{aligned}$$

Consider the unconditional expectations of x_t of the first-order accurate approximation:

$$\begin{aligned} E(x_t) &= E\left\{\bar{x} + h_x(\bar{x},0)(x_t - \bar{x}) + h_\sigma(\bar{x},0)(\sigma - 0)\right\} \\ &= \bar{x} + h_x(\bar{x},0)(E(x_t) - \bar{x}) + 0 \end{aligned}$$

It follow that up to first order accuracy:

$$Ex_t = \bar{x} \quad \text{and} \quad Ey_t = \bar{y}$$

or in words the unconditional expectation is the same as the mean. Hence first-order accurate approximations will not be helpful to approximate average risk premia (they would all be zero) or the average welfare associated with different monetary or fiscal policy that all give rise to the same nonstochastic steady state (all policies give the same welfare in the steady state).

Find $h_x(\bar{x},0)$ and $g_x(\bar{x},0)$

Start again from (5)

$$\begin{aligned} 0 &= F(x,\sigma) \\ &= E_t f(g(h(x,\sigma)+\eta\sigma\epsilon',\sigma), g(x,\sigma), h(x,\sigma)+\eta\sigma\epsilon', x) \end{aligned}$$

The first derivative of $F(x,\sigma)$ with respect to x evaluated at $(x,\sigma) = (\bar{x},0)$

$$\begin{aligned} 0 &= F_x(\bar{x},0) \\ &= f_{y'} g_x h_x + f_y g_x + f_{x'} h_x + f_x \end{aligned}$$

Rearrange to

$$\begin{bmatrix} f_{x'} & f_{y'} \end{bmatrix} \begin{bmatrix} I \\ g_x \end{bmatrix} h_x = -\begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} I \\ g_x \end{bmatrix}$$

To solve this expression for h_x and g_x use a Schur decomposition. We describe this in detail in Appendix 4.14 of the Chapter. The Matlab program `gx_hx.m` posted on our website with the materials for Chapter 4 performs this step.