Appendix to 'Monetary Policy and Multiple Equilibria'

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A A model with Calvo-Yun-type price staggering

In this appendix, we develop a continuous-time, money-in-the-production function version of Tack Yun's (1996) sticky-price model. Yun's model is a variation of Guillermo A. Calvo's (1983) model in which firms are assumed to set prices so as to maximize the present value of profits, instead of following a rule of thumb as assumed by Calvo. We show that the Calvo-Yun framework implies equilibrium conditions that are qualitatively identical to those obtained under the Rotemberg model. Thus, all the results on local determinacy obtained in section V. carry over to environments with Calvo-Yun price staggering.

Households

The representative household's lifetime utility function is assumed to be of the form

$$\int_0^\infty e^{-rt} u(c, m^{np}) dt, \tag{48}$$

where $u(\cdot,\cdot)$ satisfies Assumption 1. The household's instant budget constraint is

$$\dot{a} = (R - \pi)a - Rm^{np} + x - c - \tau, \tag{49}$$

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where x denotes the household's income measured in units of the composite good, which consists of profits from ownership of shares in firms. The household chooses sequences for c, m^{np}, and a so as to maximize (48) subject to (49) and the no-Ponzi-game borrowing constraint (26), taking as given $a(0)$ and the time paths of τ, R, x, and π. The first-order conditions associated with the household's optimization problem are (28), (29), and (31) and (26) holding with equality. Combining (28), (29), and (11) yields

$$(50) \qquad c = c(\lambda, \pi); \quad c_\lambda < 0,\; c_\pi u_{cm} \leq 0$$

Firms

The production technology and market structure are identical to those assumed in section V.. The difference with the Rotemberg model stems from the source of nominal rigidities. Following Calvo (1983), suppose that a firm can change the nominal price of the good it produces only when it receives a price-change signal. If the firm does not receive a signal, then its price is assumed to increase automatically at the steady-state inflation rate. The probability of receiving a price-change signal between periods t and $s > t$ is assumed to be given by

$$(51) \qquad 1 - e^{-\delta(s-t)}, \quad \delta > 0.$$

Consider the problem faced by firm j that receives a price-change signal at time t. The expected stream of profits associated with a particular price $P^j(t)$ is given by

$$(52) \quad \Pi(P^j) = \int_t^\infty e^{-\delta(s-t)} e^{-r(s-t)} \lambda(s) \left[\frac{P^j(t) e^{\pi^*(s-t)}}{P(s)} Y^d(s) d\left(\frac{P^j(t) e^{\pi^*(s-t)}}{P(s)} \right) - R(s) m^p(s)^j \right] ds.$$

The expression within square brackets represents profits at time s in the event that the firm has not received a price-change signal between times t and s. We ignore the profits corresponding to the events in which the firm receives a price-change signal after time t because they are irrelevant to the firm's current price-setting decision. The present discounted value of the firm's profits are multiplied by $e^{-\delta(s-t)}$, the probability that the price set in t will still be in place at time s. The firm discounts profits accruing at time s using the pricing kernel $e^{-r(s-t)}\lambda(s)$ that results from the representative household's optimization problem. This kernel is deterministic because households are assumed to be able to insure against firm-specific risks by holding a portfolio containing shares from all firms in the economy. The firm chooses $P^j(t)$ so as to maximize $\Pi(P^j)$, subject to the constraint that sales are demand determined:

$$y(m^j) \geq Y^d d\left(\frac{P^j}{P}\right).$$

The first-order condition associated with this optimization problem is

$$0 = \int_t^\infty e^{-(\delta+r)s}\lambda(s)Y^d(s)d\left(\frac{P^j(t)e^{\pi^*(s-t)}}{P(s)}\right)\left[\frac{P^j(t)e^{\pi^*(s-t)}}{P(s)}\frac{1+\eta}{\eta} - \frac{R(s)}{y'(m^p(s)^j)}\right]ds, \tag{53}$$

where, as in section V., $\eta < -1$ denotes the price elasticity of the demand faced by an individual firm and is assumed to be constant. The expression within square brackets is the difference between marginal revenue and marginal cost. Thus, the firm chooses to set today's price so that on average marginal revenue equals marginal cost.

Equilibrium

In a symmetric equilibrium all firms that receive a price-change signal will choose the same price. Let this price be denoted by $\mathcal{P}(t)$. Let $p(t) \equiv \mathcal{P}(t)/P(t)$. Then equation (53) can be written as

$$0 = \int_t^\infty e^{-(\delta+r)s}\lambda(s)Y^d(s)\, d\Big(p(t)e^{-\int_t^s[\pi(r)-\pi^*]dr}\Big)\left[\frac{1+\eta}{\eta}p(t)e^{-\int_t^s[\pi(r)-\pi^*]dr} - \frac{R(s)}{y'(m^p(s)^j}\right]ds.$$

Linearizing this expression around the steady state yields

$$\begin{aligned} 0 \quad = \quad & \int_t^\infty e^{-(\delta+r)s}\left[\frac{1+\eta}{\eta}\left([p(t)-p^*] - \int_t^s [\pi(r)-\pi^*]dr\right)\right. \\ & \left. -\frac{\rho'}{y'}[\pi(s)-\pi^*] + \frac{R^*y''}{y'^2}[m^p(s)^j - m^{p*}]\right]ds. \end{aligned} \tag{54}$$

Assume that the consumption good is a composite of the goods produced by each firm. Let the aggregator function be of the Dixit-Stiglitz form with an elasticity of substitution across goods of

η. Then, the price of the composite good is given by

$$P(t) = \left[\int_{-\infty}^{t} \delta e^{-\delta(t-s)} \mathcal{P}(s)^{1+\eta}\right]^{\frac{1}{1+\eta}} ds,$$

Differentiate this expression to obtain

$$\pi(t) - \pi^* = \frac{\delta}{1+\eta}\left[p(t)^{1+\eta} - 1\right], \tag{55}$$

which after linearizing can be written as

$$[p(t) - p^*] = \frac{1}{\delta}[\pi(t) - \pi^*]$$

Using this equation to eliminate $p(t) - p^*$ from equation (54) and differentiating the result with respect to t yields

$$\dot{\pi} = r[\pi - \pi^*] + (\delta + r)\delta\frac{\eta}{1+\eta}\left[-\frac{\rho'}{y'}[\pi - \pi^*] + \frac{R^* y''}{y'^2}[m^{pj} - m^{p*}]\right]. \tag{56}$$

Using equations (50) and (55) and the fact that in equilibrium $y(m^{pj}) = cd(p)$ one can express m^{pj} as a function of λ and π, whose linearized form is

$$m^{pj} - m^{p*} = \frac{1}{y'}\left[c_\lambda(\lambda - \lambda^*) + (c_\pi + \eta/\delta)(\pi - \pi^*)\right]$$

Finally, use this expression to eliminate $(m^{pj} - m^{p*})$ from (56), to get the following aggregate supply equation, or new Keynesian Phillips curve:

$$\dot{\pi} = \tilde{A}_{21}(\lambda - \lambda^*) + \tilde{A}_{22}(\pi - \pi^*) \tag{57}$$

where

$$\begin{aligned}
\tilde{A}_{21} &= (\delta + r)\delta\frac{y''}{y'^2}c_\lambda > 0 \\
\tilde{A}_{22} &= r - (\delta + r)\delta\left[\frac{\rho'}{y'}\frac{\eta}{1+\eta} - \frac{y''}{y'^2}[c_\pi + \eta/\delta]\right]
\end{aligned}$$

The remaining equilibrium conditions are identical to those of the Rotemberg sticky-price model developed in section V.. Comparing $\tilde{A}_{21}$ with A_{21} in the aggregate supply function of the Rotemberg model (equation (42)), it follows that the determinants of the Jacobian matrices of the Rotemberg and Calvo-Yun models have the same sign. This implies that the results on local indeterminacy under passive monetary policy are identical under both models. Furthermore, by an analysis similar to the one carried out in section V., it is possible to show that, like A_{22}, $\tilde{A}_{22}$ may take either sign. This is important because it implies that, like the Rotemberg model, the Calvo-Yun model can generate local indeterminacy under active monetary policy (regardless of the stance of fiscal policy). This result is entirely due to the assumption that money affects real variables through production. As mentioned earlier, in the Calvo-Yun model without money in the production function, the trace of the Jacobian is always positive and equal to r.

B Proof of proposition 7

In the economy under analysis, the equilibrium conditions (37) and (38) take the form

$$\frac{\dot{\lambda}}{\lambda} = (1-a)(\pi - \pi^*) \tag{58}$$

$$\dot{\pi} = r\,(\pi - \pi^*) - \gamma^{-1}\lambda^{((s-1)/s)}\left[1 + \eta\left(1 - \frac{R^* + a(\pi - \pi^*)}{\alpha\lambda^{((1-\alpha)/\alpha s)}}\right)\right] \tag{59}$$

To prove orbital stability, we use the formula provided by J. Guckenheimer and P. Holmes (1983, p. 152), which requires a change of variables and expressing the above system at the Hopf bifurcation as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}$$

where ω is a function of the parameters of the model and $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ satisfy $f(0,0) = g(0,0) = 0$ and $f_i(0,0) = g_i(0,0) = 0$ for $i = 1,2$; that is, $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ have no constant or linear terms.

The Hopf bifurcation is supercritical (and thus stable cycles exist) if

$$\begin{aligned}\kappa &\equiv (f_{uuu}+f_{uvv}+g_{uuv}+g_{vvv})\\ &+\frac{1}{\omega}[f_{uv}(f_{uu}+f_{vv})-g_{uv}\left(g_{uu}+g_{vv}\right)-f_{uu}g_{uu}+f_{vv}g_{vv}]<0\end{aligned}$$

at $u=v=0$. We obtain this condition by steps.

STEP 1: Let $p=\pi-\pi^{*}$ and $z=\ln(\lambda/\lambda^{*})$, where λ^{*} denotes the steady-state value of λ. Then equations (58) and (59) can be written as

$$\begin{aligned}\dot{z} &= (1-a)\,p\\ \dot{p} &= rp+M\left(e^{\beta z}-e^{\sigma z}\right)+Npe^{\beta z}\end{aligned}$$

where $\sigma=\frac{s-1}{s}$, $\beta=\sigma+\frac{\alpha-1}{\alpha s}$, $M=\gamma^{-1}\left(1+\eta\right)\lambda^{*\sigma}<0$, and $N=\gamma^{-1}\eta\alpha^{-1}a\lambda^{*\beta}<0$.

STEP 2: Write the system of differential equations as:

$$\begin{bmatrix}\dot{z}\\ \dot{p}\end{bmatrix}=\begin{bmatrix}0 & 1-a\\ M\left(\beta-\sigma\right) & 0\end{bmatrix}\begin{bmatrix}z\\ p\end{bmatrix}+\begin{pmatrix}0\\ G\left(z,p\right)\end{pmatrix}$$

where

$$G\left(z,p\right)=M[e^{\beta z}-e^{\sigma z}-(\beta-\sigma)z]+p(Ne^{\beta z}+r)$$

Note that the matrix in the linear part satisfies:

$$DET=-M(\beta-\sigma)(1-a)$$

$$TRACE=0$$

Assume that $a>1$ and $N=-r$. That is, the parameter configuration corresponds to a Hopf bifurcation. Then letting $\omega=\sqrt{DET}$, $\mu=\sqrt{\frac{M(\beta-\sigma)}{a-1}}$, $v=-z$, and $u=\mu^{-1}p$, the two differential equations become:

$$\begin{pmatrix} \dot{u} \\ \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & -\omega & \\ & & 0 \\ \omega & 0 & \end{bmatrix} \begin{pmatrix} u \\ \\ v \end{pmatrix} + \begin{pmatrix} f(u,v) \\ \\ 0 \end{pmatrix}$$

where

$$f(u,v) = \frac{M}{\mu}[e^{-\beta v} - e^{-\sigma v} + (\beta - \sigma)v] + u(Ne^{-\beta v} + r)$$

Note that $f(0,0) = f_u(0,0) = f_v(0,0) = 0$. Also, in our formulation $g(\cdot,\cdot) = 0$.

STEP 3: The relevant derivatives of f are

$$\begin{aligned} f_{vv} &= \frac{M}{\mu}[\beta^2 e^{-\beta v} - \sigma^2 e^{-\sigma v}] + \mu u N \beta^2 e^{-\beta v} \\ f_{uu} &= 0 \\ f_{uuu} &= 0 \\ f_{uv} &= -\beta N e^{-\beta v} \\ f_{uvv} &= \beta^2 N e^{-\beta v} \end{aligned}$$

Setting $u = v = 0$, it follows that

$$\kappa = (\beta N)[\beta - \mu^{-1}\omega^{-1}M(\beta^2 - \sigma^2)]$$

Noting that $\mu^{-1}\omega^{-1}M = (\beta - \sigma)^{-1}$ and recalling that $N = -r$, κ reduces to

$$\kappa = \beta r \sigma$$

Thus κ will be less than zero if and only if

$$1 < s < \frac{1}{\alpha}.$$

C Backward- and forward-looking interest-rate feedback rules

C.1 Flexible-price model

Backward-looking feedback rules

Differentiating (47) with respect to time yields

$$\dot{\pi}^p = b(\pi - \pi^p) \tag{60}$$

In equilibrium, the evolution of the nominal interest rate is given by

$$\lambda'(R)\dot{R} = \lambda(R)[r + \pi - R] \tag{61}$$

where

$$\lambda'(R) = \left[u_{cc} y' m^{p\prime} + u_{cm}(m_c^{np} y' m^{p\prime} + m_R^{np})\right] \tag{62}$$

Using equation (46) to eliminate π from (60) and (61) and linearizing around the steady state results in the following system of linear differential equations, valid for $0 < q \leq 1$:

$$\begin{bmatrix} \dot{R} \\ \dot{\pi}^p \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{\lambda'}\left(\frac{1}{\rho' q} - 1\right) & -\frac{\lambda}{\lambda'}\frac{(1-q)}{q} \\ b\frac{1}{\rho' q} & -\frac{b}{q} \end{bmatrix} \begin{bmatrix} R - R^* \\ \pi^p - \pi^* \end{bmatrix}; \quad q \in (0, 1].$$

Let J denote the Jacobian matrix of this system. Because R is a jump variable and π^p is predetermined, the real allocation is locally unique if the real parts of the eigenvalues of J have opposite signs, or, equivalently, if the determinant of J is negative. On the other hand, the real allocation is locally indeterminate if both eigenvalues have negative real parts, that is, if the determinant of J is positive and its trace is negative. The determinant and trace of J are given by

$$det(J) = \frac{\lambda}{\lambda'}\frac{b}{\rho' q}(\rho' - 1)$$

$$trace(J) = \frac{\lambda}{\lambda'}\left(\frac{1}{\rho' q} - 1\right) - \frac{b}{q}$$

If $u_{cm} > 0$ and $y' = 0$, then equation (62) implies that λ' is negative. It follows that the equilibrium is unique if $\rho' > 1$ and is indeterminate if $\rho' < 1$. If $u_{cm} < 0$ and $y' = 0$ or if $u_{cm} = 0$ and $y' > 0$, then $\lambda' > 0$. Thus, the equilibrium is always locally determinate when $\rho' < 1$. If $\rho' > 1$, the determinant of J is positive, so that the real parts of the roots of J have the same sign as the trace of J. However, the trace of J can have either sign. If the trace is positive, then no equilibrium converging to the steady state exists. If it is negative, the equilibrium is indeterminate. For large enough values of ρ' the trace of J becomes negative and the equilibrium is indeterminate. Furthermore, the larger q and/or the larger b, the smaller is the minimum value of ρ' above which the equilibrium becomes indeterminate. As $q \to 1$ and/or $b \to \infty$, the equilibrium becomes indeterminate for any $\rho' > 1$.

If $q = 0$, the equilibrium conditions reduce to a single differential equation:

$$\dot{\pi}^p = A[r + \pi^p - \rho(\pi^p)], \tag{63}$$

where

$$A = \frac{b}{b\rho'\lambda'/\lambda - 1}. \tag{64}$$

Because π^p is a predetermined variable, the equilibrium cannot be locally indeterminate. The equilibrium exists locally if $A(1-\rho') < 0$ and fails to exist locally if $A(1-\rho') > 0$. If $u_{cm} > 0$ and $y' = 0$, then $\lambda' < 0$. It follows that the equilibrium is determinate if $\rho' < 1$ and does not exist locally if $\rho' > 1$. If $u_{cm} < 0$ and $y' = 0$ or $u_{cm} = 0$ and $y' > 0$, then $\lambda' > 0$. Thus, for large enough values of b ($b > \lambda/\lambda'$), the equilibrium exists for $\rho' > 1$ as well as for ρ' close enough to 0. For low enough values of b ($b < \lambda/\lambda'$), the equilibrium exists for $\rho' < 1$ as well as for $\rho' > 1$ and large enough, but does not exist for moderately active feedback rules.

Forward-looking feedback rules

It is straightforward to show that the system describing the equilibrium dynamics is identical to the one obtained under backward-looking feedback rules with b replaced by $-d$ and π^p replaced by

π^f. Thus, in this case, provided $q > 0$, the determinant and trace of the Jacobian become

$$det(J) = -\frac{\lambda}{\lambda'}\frac{d}{\rho' q}(\rho' - 1)$$

$$trace(J) = \frac{\lambda}{\lambda'}\left(\frac{1}{\rho' q} - 1\right) + \frac{d}{q}$$

Because neither R nor π^f are predetermined variables, the real allocation is locally unique if both eigenvalues of J have positive real parts (det(J)> 0 and trace(J)> 0) and is locally indeterminate if at least one of the roots of J has a negative real part (det(J)< 0). If $u_{cm} < 0$ and $y' = 0$ or $u_{cm} = 0$ and $y' > 0$, then $\lambda' > 0$. If $\rho' > 1$, then det(J)< 0. If $\rho' < 1$, then both det(J)> 0 and trace(J)> 0. On the other hand, if $y' = 0$ and $u_{cm} > 0$, then $\lambda' < 0$. If $\rho' < 1$, then det(J)< 0. When $\rho' > 1$, we have that det(J)> 0 but trace(J) can have either sign. For q or d large, trace(J) is positive. For q and d/q small enough, trace(J) becomes negative.

If $q = 0$, the equilibrium conditions reduce to (63) and (64) with π^p replaced by π^f and b by $-d$. Because π^f is nonpredetermined, at least one equilibrium always exists locally. The equilibrium is locally indeterminate if $A(1 - \rho') < 0$ and is locally determinate if $A(1 - \rho') > 0$. If $u_{cm} < 0$ and $y' = 0$ or if $u_{cm} = 0$ and $y' > 0$, then $\lambda' > 0$. It follows that the equilibrium is determinate if $\rho' > 1$ and is indeterminate if $\rho' < 1$. If $u_{cm} > 0$ and $y' = 0$, then $\lambda' < 0$. Thus, for large enough values of d ($d > -\lambda/\lambda'$), the equilibrium is determinate for $\rho' > 1$ as well as for ρ' close enough to 0. For low enough values of d ($d < -\lambda/\lambda'$), the equilibrium is determinate for $\rho' < 1$ as well as for $\rho' > 1$ and large enough, but is indeterminate for moderately active feedback rules.

C.2 Sticky-price model

To facilitate the analysis, we reproduce here the equilibrium conditions for the sticky-price model.

$$
\begin{aligned}
&(65) \qquad \dot{\lambda} &=& \lambda\left[r+\pi-R\right] \\
&(66) \qquad \gamma\dot{\pi} &=& \gamma r(\pi-\pi^*) - y(m^p(\lambda,R))\lambda\left[1+\eta\left(1-\frac{R}{y'(m^p(\lambda,R))}\right)\right] \\
& \qquad \dot{a} &=& [R-\pi]a - R\left[m^{np}(y(m^p(\lambda,R)),R)+m^p(\lambda,R)\right]-\tau \\
& \qquad \tau &=& \begin{cases} \overline{\tau} & \text{non-Ricardian fiscal policy} \\ \alpha a - R\left[m^{np}(y(m^p(\lambda,R)),R)+m^p(\lambda,R)\right] & \text{Ricardian fiscal policy} \end{cases} \\
& \qquad 0 &=& \lim_{t\to\infty} e^{-\int_0^t [R-\pi]ds} a(t)
\end{aligned}
$$

where $m^p(\lambda, R)$ results from replacing $\rho(\pi)$ by R in equation (35) and is decreasing in λ and increasing (decreasing) in R if u_{cm} is negative (positive).

Backward-looking feedback rules

Combining (46) and (47) to eliminate π^p and linearizing around the steady state yields

$$\dot{R} = q\rho'\dot{\pi} + \rho'\left(\pi-\pi^*\right) - b(R-R^*).$$

Using this expression and linearizing equations (65) and (66), the evolution of λ, π, and R is described by the following system of differential equations:

$$\begin{pmatrix} \dot{\lambda} \\ \dot{\pi} \\ \dot{R} \end{pmatrix} = A \begin{pmatrix} \lambda-\lambda^* \\ \pi-\pi^* \\ R-R^* \end{pmatrix}$$

where

$$A = \begin{bmatrix} 0 & u_c & -u_c \\ A_{21} & r & A_{23} \\ \rho' q A_{21} & \rho'(b+qr) & -b+\rho' q A_{23} \end{bmatrix}$$

and

$$
\begin{aligned}
A_{21} &= -\frac{u_c c^* \eta R^* y'' m^p_\lambda}{\gamma y'^2} > 0 \\
A_{23} &= \left(\frac{u_c c^* \eta}{\gamma y'}\right)\left(1 - \frac{R^*}{y'} y'' m^p_R\right).
\end{aligned}
$$

Because π^p is predetermined and R is a function of π and π^p, it follows that a linear combination of $\pi - \pi^*$ and $R - R^*$ is predetermined. In addition, λ is a jump variable. Assume first that fiscal policy is Ricardian. Then the local determinacy of the perfect-foresight equilibrium is governed by the eigenvalues of A. Specifically, the equilibrium is indeterminate if the real part of at least two roots of A are negative.

Assume that monetary policy is active ($\rho' > 1$). We found in section V. that under a purely contemporaneous feedback rule, the combination of Ricardian fiscal policy and active monetary policy can render the real allocation either locally determinate or indeterminate, depending on parameter values. By contrast, if the feedback rule is sufficiently backward looking ($q, b \to 0$), the equilibrium is always unique. To see this, note that when $\rho' > 1$, the determinant of A, which is given by

$$
Det(A) = b u_c A_{21}\left(1 - \rho'\right),
$$

is negative. Thus, the number of roots of A with a negative real part is either one or three. If at the same time the trace of A is positive, then the number of roots of A with a negative real part is exactly equal to one. The trace of A is given by

$$
Trace(A) = r - b + \rho' q A_{23}.
$$

Clearly, as q and b approach zero, the trace of A becomes positive.

Assume now that monetary policy is passive ($\rho' < 1$). As shown in section V., the combination of Ricardian fiscal policy and passive monetary policy always renders the real allocation locally indeterminate under a purely contemporaneous feedback rule. It is straightforward to show that introducing a backward-looking component in the feedback rule cannot bring about local determinacy. To see this, note that if $\rho' < 1$, the determinant of A is positive, so the number of roots of A with a negative real part can never be exactly equal to one. Unlike the case of purely contemporaneous rules, though, a perfect-foresight equilibrium in which the real allocation converges to its steady state may not exist. This will be the case when all eigenvalues of A have positive real parts. However, if the feedback rule is highly contemporaneous either because q approaches unity or because b approaches infinity, then the equilibrium is always locally indeterminate. To see this, we appeal to the following condition:[1] The number of roots of A with positive real parts is equal to the number of variations of sign in the scheme:

$$(67) \qquad -1 \qquad Trace(A) \qquad -B + \frac{Det(A)}{Trace(A)} \qquad Det(A),$$

where

$$B = \text{Sum of the principal minors of } A = -u_c A_{21}(1 - q\rho') - rb - b\rho' A_{23}.$$

This condition implies that in order for all roots of A to have a positive real part, it is necessary that both B and the trace of A be positive. Consider first the case in which $q \to 1$. Then the trace of A is positive if and only if $r + \rho' A_{23} > 0$. But $r + \rho' A_{23} > 0$ implies that B is negative. To see that the equilibrium is also indeterminate when $b \to \infty$, note that in this case the trace of A becomes negative.[2]

If fiscal policy is non-Ricardian, then the local determinacy of the perfect-foresight equilibrium is governed by the eigenvalues of a four-by-four Jacobian matrix defining the law of motion of λ, π, R, and a. One of the eigenvalues of this matrix is $r > 0$ and the other three are those of the matrix A. Because a and a combination of R and π are predetermined, the equilibrium is locally unique if and only if the Jacobian has exactly two roots with positive real parts. If monetary policy is active, it follows from our previous analysis that the Jacobian matrix has either one or three roots with negative real parts. Thus, local determinacy is impossible. This is the same result as under purely contemporaneous feedback rules. However, if the feedback rule is strongly backward-looking

[1] This is an application to our special case of a more general theorem due to Routh (see F. R. Gantmacher, 1960).

[2] Highly backward-looking policies do not necessarily eliminate the local existence of equilibrium. For example, the equilibrium is indeterminate when the feedback rule places a relatively high weight on inflation rates observed in the distant past ($b \to 0$). This is because in this case B is negative.

$(b, q \to 0)$, then no equilibrium in which the real allocation converges to its steady state exists. If monetary policy is passive, the determinant of the Jacobian is positive, implying that there exist either two or zero roots with negative real parts. Thus, as in the case of purely contemporaneous feedback rules, local indeterminacy is impossible. However, unlike the case of contemporaneous rules, under backward-looking monetary policy an equilibrium may not exist.

Forward-looking feedback rules

Finally, consider the case of a forward-looking feedback rule. We will limit the analysis to the case of Ricardian fiscal policy, leaving the non-Ricardian case to the reader. The law of motion of the vector $(\lambda \ \pi \ R)$ is described by a Jacobian matrix that is identical to A with b replaced by $-d$. In addition, the three variables of the system are nonpredetermined. Therefore, as long as the Jacobian has at least one root with a negative real part, the perfect-foresight equilibrium is locally indeterminate. Local determinacy requires that all three roots have positive real parts.

Suppose first that monetary policy is active. Under contemporaneous feedback rules, the equilibrium can be locally determinate or indeterminate. The same result obtains under forward-looking rules. However, if the rule is strongly forward-looking $(d, q \to 0)$, the equilibrium is necessarily locally indeterminate. To see this, note that in this case the trace of the Jacobian tends to $r > 0$ and that B tends to $-u_c A_{21} < 0$, so that the pattern of signs in the scheme (67) is $- + ++$. If monetary policy is passive, then the determinant is negative, therefore, as in the case of purely contemporaneous rules, the equilibrium is locally indeterminate.

References

Gantmacher, F. R. "The Theory of Matrices." New York: Chelsea, 1960.