ON TAYLOR RULES AND MONETARY POLICY[†]

Chaotic Interest-Rate Rules

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In much of the recent literature on monetary economics it is assumed that monetary policy takes the form of an interest-rate feedback rule whereby the central bank sets the nominal interest rate as a function of some measure of inflation and the level of aggregate activity. One justification for this modeling strategy is empirical. Several authors, beginning with John B. Taylor (1993) have documented that the central banks of major industrialized countries implement monetary policy through interest-rate feedback rules of this type. These empirical studies have further shown that since the early 1980's interest-rate feedback rules in developed countries have been active in the sense that the nominal interest rate responds more than one for one to changes in the inflation measure.

In his seminal paper, Taylor (1993) also argues on theoretical grounds that active interest-rate feedback rules (which have become known as Taylor rules) are desirable for aggregate stability. The essence of his argument is that, if in response to an increase in inflation the central bank raises nominal interest rates by more than the increase in inflation, the resulting increase in real interest rates will tend to slow down aggregate demand, thereby curbing inflationary pressures. Following Taylor's influential work, a large body of theoretical research has argued in favor of active interest-rate rules. One argument is that Taylor-type rules guarantee local uniqueness of the rational-expectations equilibrium.

The validity of the view that Taylor rules induce determinacy of the rational-expectations equilibrium has been challenged in two ways. First, it has been shown that local determinacy of equilibrium under active interest-rate rules depends crucially on the assumed preference and technology specification (Bill Dupor, 1999; Charles Carlstrom and Timothy Fuerst, 2000, 2001; Benhabib et al., 2001b), as well as on the nature of the accompanying fiscal regime (Eric Leeper, 1991). Second, even in cases in which active interest-rate rules guarantee uniqueness of the rational-expectations equilibrium locally, they may give rise to liquidity traps (Benhabib et al., 2001a, c; Schmitt-Grohé and Uribe, 2000a, b).

In this paper, we identify a third form of instability that may arise under Taylor-type policy rules. Specifically, we show that active interest-rate rules may open the door to equilibrium cycles of any periodicity and even chaos. These equilibria feature trajectories that converge neither to the intended steady state nor to an unintended liquidity trap. Rather the economy cycles forever around the intended steady state in a periodic or aperiodic fashion. Interestingly, such equilibrium dynamics exist precisely when the target equilibrium is unique from a local point of view, that is, when the inflation target is the only equilibrium level of inflation within a sufficiently small neighborhood around the target itself.

I. The Economic Environment

A. *Households*

Consider an economy populated by a large number of infinitely-lived agents with preferences described by the following utility function:

$$(1)\qquad \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \qquad \sigma > 0 \quad \beta \in (0, 1)$$

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where c_t denotes consumption in period t. Agents have access to two types of financial asset: fiat money (M_t) and government bonds (B_t). Government bonds held between periods t and $t + 1$ pay the gross nominal interest rate R_t. In addition, agents receive a stream of real income y_t and pay real lump-sum taxes τ_t. Letting P_t denote the price level in period t, $a_t \equiv (M_t + B_t)/P_t$ denote real financial wealth in period t, $m_t \equiv M_t/P_t$ denote real money balances, and $\pi_t \equiv P_t/P_{t-1}$ the gross rate of inflation, the budget constraint of the representative household can be written as

$$a_t + c_t + \tau_t = \frac{1 - R_{t-1}}{\pi_t} m_{t-1} + \frac{R_{t-1}}{\pi_t} a_{t-1} + y_t. \tag{2}$$

Households are subject to a no-Ponzi-game constraint of the form

$$\lim_{t\to\infty} \frac{a_t}{\prod_{j=0}^{t-1} (R_j/\pi_{j+1})} \geq 0. \tag{3}$$

We motivate a demand for money by assuming that real balances facilitate firms' transactions as in Stanley Fischer (1974), Taylor (1977), and Guillermo A. Calvo (1979). Specifically, we assume that output is an increasing and concave function of real balances. Formally,

$$y_t = f(m_t). \tag{4}$$

We choose this model because in industrialized economies about two-thirds of M1 is held by firms (where M1 denotes the money supply in terms of currency plus checking accounts).

Households choose sequences $\{c_t, m_t, y_t, a_t\}_{t=0}^{\infty}$ so as to maximize the utility function (1) subject to (2)–(4), given a_{-1}. The first-order optimality conditions are constraints (2)–(4) holding with equality and

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \frac{R_t}{\pi_{t+1}} \tag{5}$$

$$f'(m_t) = \frac{R_t - 1}{R_t}. \tag{6}$$

The first optimality condition is a standard Euler equation requiring that in the margin a dollar spent on consumption today provides as much utility as that dollar saved and spent tomorrow. The second condition says that the marginal productivity of money at the optimum is equal to the opportunity cost of holding money, $(R_t - 1)/R_t$.

B. *The Monetary and Fiscal Policy Regime*

We postulate that the government conducts monetary policy in terms of an interest-rate feedback rule of the form

$$R_t = \rho(\pi_{t+1}). \tag{7}$$

Under the feedback rule the central bank sets the current nominal interest rate as a function of the inflation rate between periods t and $t + 1$. We adopt this specification because a number of authors have argued that in the post-Volker era, U.S. monetary policy is better described as incorporating a forward-looking component (see Athanasios Orphanides, 1997; Richard Clarida et al., 1998).[1] We impose four conditions on the functional form of the interest-rate feedback rule. First, in the spirit of Taylor (1993) we assume that monetary policy is active around a target rate of inflation $\pi^* > \beta$; that is, the interest elasticity of the feedback rule at π^* is greater than unity, or $\rho'(\pi^*)\pi^*/\rho(\pi^*) > 1$. Second, we impose the restriction $\rho(\pi^*) = \pi^*/\beta$, which ensures the existence of a steady-state consistent with the target rate of inflation. Third, we assume that the feedback rule satisfies the zero bound on nominal interest rates, $\rho(\pi) > 1$ for all π. Finally, we assume that the feedback rule is nondecreasing, $\rho'(\pi) \geq 0$ for all π.

Government consumption is assumed to be zero. Each period, the government faces the following budget constraint:

$$a_t = \frac{R_{t-1}}{\pi_t} a_{t-1} - \left[\frac{R_{t-1} - 1}{\pi_t} m_{t-1} + \tau_t\right]. \tag{8}$$

[1] In the working-paper version of this paper (Benhabib et al., 2001d), we also consider the case of contemporaneous rules, $R_t = \rho(\pi_t)$.

We assume that the fiscal regime consists of setting consolidated government revenues equal to a constant fraction of total government liabilities each period. Formally,

$$\tau_t + \frac{R_{t-1} - 1}{\pi_t} m_{t-1} = \omega a_{t-1} \qquad \omega > 0.$$

The above two expressions imply that

$$\lim_{t \to \infty} \frac{a_t}{\prod_{j=0}^{t-1} (R_j / \pi_{j+1})} = 0.$$

Therefore, the assumed fiscal policy ensures that the household's borrowing limit holds with equality under all circumstances.

C. *Equilibrium*

Combining equations (2) and (8) implies that the goods market clears at all times:

$$y_t = c_t. \tag{9}$$

An equilibrium real allocation is a set of sequences $\{m_t, R_t, c_t, \pi_t, y_t\}_{t=0}^{\infty}$ satisfying $R_t > 1$, equations (4)–(7), and equation (9).

We focus on the following specific parameterizations of the policy rule and the production function:

$$\rho(\pi_{t+1}) = 1 + (R^* - 1)\left(\frac{\pi_{t+1}}{\pi^*}\right)^{A/R^* - 1} \tag{10}$$

$$R^* = \pi^*/\beta$$

and

$$f(m_t) = [a m_t^{\mu} + (1 - a)\bar{y}^{\mu}]^{1/\mu} \tag{11}$$

$$\mu < 1 \quad a \in (0, 1].$$

We assume that $A/R^* > 1$, so that at the target rate of inflation the feedback rule satisfies the Taylor criterion, $\rho'(\pi^*)\pi^*/\rho(\pi^*) > 1$. In other words, at the target rate of inflation, the interest-rate feedback rule is active. The parameter $\bar{y} > 0$ is meant to reflect the presence of a fixed factor of production. Under this production technology, one may view real balances either as directly productive or as decreasing the transaction costs of exchange.[2] With these particular functional forms, an equilibrium real allocation is defined as a set of sequences $\{m_t, R_t, c_t, \pi_t, y_t\}_{t=0}^{\infty}$ satisfying $R_t > 1$ and equations (5), (6), and (9)–(11).

Combining equations (6) and (11) yields the following negative relation between output and the nominal interest rate:

$$R_t = R(y_t) \qquad R' < 0. \tag{12}$$

This expression together with equations (5), (9), and (10), implies a first-order, nonlinear difference equation in output of the form:

$$y_{t+1} = F(y_t) \equiv \beta^{1/\sigma} y_t \left[\frac{R(y_t)}{\rho^{-1}(R(y_t))}\right]^{1/\sigma} \tag{13}$$

where $\rho^{-1}(\cdot)$ denotes the inverse of the function $\rho(\cdot)$. Finding an equilibrium real allocation then reduces to finding a real positive sequence $\{y_t\}_{t=0}^{\infty}$ satisfying equation (13).[3]

II. Local Equilibria

Equation (13) has two steady states, which we denote by y^* and y^{p}. Let y^* denote the steady state associated with $R_t = R^*$. Note that at y^* monetary policy is active. Simple algebra shows that at y^{p} monetary policy is passive. We refer to y^* as the intended steady state.

Consider perfect-foresight equilibrium real allocations in which output remains forever in an arbitrarily small neighborhood around a steady state and converges to it. To this end, we

[2] It is also possible to replace the fixed factor $\bar{y}$ with a function increasing in labor and to add leisure to the utility function. The current formulation then would correspond to the case of an inelastic labor supply.

[3] An additional restriction that solutions to equation (13) must satisfy in order to be able to be supported as equilibrium real allocation is that

$$\left(\frac{1 - a^{1/(1-\mu)}}{1 - a}\right)^{-1/\mu} \bar{y} > y_t > (1 - a)^{1/\mu} \bar{y}$$

when $\mu > 0$ and

$$\left(\frac{1 - a^{1/(1-\mu)}}{1 - a}\right)^{-1/\mu} \bar{y} < y_t < (1 - a)^{1/\mu} \bar{y}$$

when $\mu < 0$. These constraints ensure that $R_t \geq 1$ and that m_t is a positive real number.

log-linearize (13) around a steady state. This yields

$$(14) \qquad \hat{y}_{t+1} = \left[1 + \frac{\varepsilon_R}{\sigma}\left(1 - \frac{1}{\varepsilon_\rho}\right)\right]\hat{y}_t$$

where $\hat{y}_t$ denotes the log-deviation of y_t from its steady-state value. The parameter $\varepsilon_R < 0$ denotes the elasticity of the function $R(\cdot)$, defined by equation (12), with respect to y_t evaluated at the steady-state value of output. The parameter $\varepsilon_\rho > 0$ denotes the elasticity of the interest-rate feedback rule with respect to inflation at the steady state.

Consider first the dynamics around the steady state y^{p}. As discussed above, in this case the feedback-rule is passive, that is, $\varepsilon_\rho < 1$. It follows that the coefficient of the linear difference equation (14) is greater than 1. With y_t being a non-predetermined variable, this implies that the passive steady state is locally the unique perfect-foresight equilibrium. Consider next the dynamics around y^*. At the intended steady state ε_ρ is greater than 1. This implies that the coefficient of the difference equation (14), at least for mildly active policy rules (i.e., ε_ρ greater than but close to 1, lies between 1 and -1). Therefore, for mildly active rules, the coefficient is less than 1 in absolute value, and the rational-expectations equilibrium is indeterminate. It follows from our analysis that the parameter value $\varepsilon_\rho = 1$ is a bifurcation point of the dynamical system (14), because at this value the local stability properties of the system change from determinate to indeterminate.

For sufficiently active policy rules, a second bifurcation point might emerge. In particular, if $\varepsilon_R/\sigma < -2$, then there exists an $\varepsilon_\rho > 1$ at which the coefficient of the linear difference equation (14) equals -1. Above this value of ε_ρ the coefficient of the difference equation is greater than 1 in absolute value, and the equilibrium is again locally unique.

One might conclude from the above characterization of local equilibria that, as long as the policymaker pursues a sufficiently active monetary policy, he can guarantee a unique equilibrium around the inflation target π^*. In this sense, active monetary policy might be viewed as stabilizing. However, this view can be misleading, for the global picture can look very different.

III. Chaos

Consider the case of a sufficiently active monetary policy stance that ensures that the inflation target of the central bank, π^*, is locally the unique equilibrium. Formally, assume that at the active steady state $\varepsilon_\rho > 1/(1 + 2\sigma/\varepsilon_R)$.[4] In what follows, we show that, for such a monetary policy, there may exist equilibria other than the active steady state, with the property that the real allocation fluctuates forever in a bounded region around the target allocation. These equilibria include cycles of any periodicity and even chaos (i.e., nonperiodic deterministic cycles). We first establish theoretically the conditions under which periodic and chaotic dynamics exist. We then demonstrate that these conditions are satisfied under plausible parameterizations of our simple model economy.

A. *Existence*

We apply a theorem due to Masaya Yamaguti and Hiroshi Matano (1979) on chaotic dynamics in scalar systems. To this end, we introduce the following change of variable: $q_t = \mu \ln(y_t/y^{\mathrm{p}})$. Equation (13) can then be written as

$$(15) \qquad q_{t+1} = H(q_t;\, \alpha) \equiv q_t + \alpha h(q_t)$$

where the parameter α and the function $h(\cdot)$ are defined as $\alpha = (1/\sigma)$ and

$$h(q_t) = (-\mu)\{\ln(\rho^{-1}[R(y^p e^{q_t/\mu})]) - \ln\beta - \ln R(y^p e^{q_t/\mu})\}.$$

We restrict attention to negative values of μ. As we discuss below, this is the case of greatest empirical interest. The function h is continuous and has two zeros, one at $q = 0$ and the other at $q^* \equiv \mu \ln(y^*/y^{\mathrm{p}}) > 0$. Further h is positive for $q_t \in (0, q^*)$ and negative for $q_t \notin [0, q^*]$.

[4] We are implicitly assuming that the second bifurcation point exists, that is, that the condition $\varepsilon_R/\sigma < -2$ is satisfied.

We are now ready to state the Yamaguti and Matano (1979) theorem.

THEOREM 1 (Yamaguti and Matano, 1979): *Consider the difference equation*

$$(16) \qquad q_{t+1} = H(q_t;\, \alpha) \equiv q_t + \alpha h(q_t).$$

Suppose that (*a*) $h(0) = h(q^*) = 0$ *for some* $q^* > 0$; (*b*) $h(q) > 0$ *for* $0 < q < q^*$; *and* (*c*) $h(q) < 0$ *for* $q^* < q < \kappa$, *where the constant* κ *is possibly* $+\infty$. *Then there exists a positive constant* c_1 *such that for any* $\alpha > c_1$ *the difference equation* (16) *is chaotic in the sense of Tien-Yien Li and James A. Yorke* (1975). *Suppose in addition that* $\kappa = +\infty$. *Then there exists another constant* c_2, $0 < c_1 < c_2$, *such that for any* $0 \le \alpha \le c_2$, *the map H has an invariant finite interval* $[0, \gamma(\alpha)]$ (*i.e., H maps* $[0, \gamma(\alpha)]$ *into itself*) *with* $\gamma(\alpha) > q^*$. *Moreover, when* $c_1 < \alpha \le c_2$, *the above-mentioned chaotic phenomenon occurs in this invariant interval.*

The application of this theorem to our model economy is immediate. It follows that there exists some parameterization of the model for which the real allocation cycles perpetually in a chaotic fashion, that is, deterministically and aperiodically. According to the theorem, chaotic dynamics are more likely, the larger is the intertemporal elasticity of substitution, $1/\sigma$. We next study the empirical plausibility of the parameterizations consistent with chaos.

B. *Empirical Plausibility*

Consider the following calibration of the model economy. The time unit is a quarter. Let the intended nominal interest rate be 6 percent per year ($R^* = 1.06^{1/4}$), which corresponds to the average yield on three-month U.S. Treasury bills over the period 1960:1 to 1998:3. We set the target rate of inflation at 4.2 percent per year ($\pi^* = 1.042^{1/4}$). This number matches the average growth rate of the U.S. GDP deflator during the period 1960: 1–1998:3. The assumed values for R^* and π^* imply a subjective discount rate of 1.8 percent per year. Following Taylor (1993), we set the elasticity of the interest-rate feedback rule evaluated at π^* equal to 1.5 (i.e., $A/R^* = 1.5$).

There is a great deal of uncertainty about the value of the intertemporal elasticity of substitution $1/\sigma$. In the real-business-cycle literature, authors have used values as low as ⅓ (e.g., Julio J. Rotemberg and Michael D. Woodford, 1992) and as high as 1 (e.g., Robert G. King et al., 1988). In the baseline calibration, we assign a value of 1.75 to σ. We will also report the sensitivity of the results to variations in the value assumed for this parameter.

Equations (6) and (11) imply a money demand function of the form

$$(17) \qquad m_t = y_t\left(\frac{R_t - 1}{aR_t}\right)^{1/(\mu-1)}.$$

Using U.S. quarterly data from 1960:1 to 1999:3, we estimate the parameters of this money demand function, namely, μ and a. We obtain $\mu = -9$ and $a = 0.000352$. For details, see Benhabib et al. (2001d). Finally, we set the fixed factor $\bar{y}$ at 1.

In Figure 1, we show the first three iterates of the difference equation (13), which describes the equilibrium dynamics of output for the baseline calibration. In all three panels, the broken line represents the 45° degree line. The range of values plotted for output starts at a point $\underline{y}$ located below the active steady state, y^*, (i.e., $\underline{y} < y^*$) and ends at the passive steady state, y^p. On the interval $[\underline{y}, y^p]$ the mapping $F(\cdot)$ is invariant in the sense that for any $y \in [\underline{y}, y^p]$ it is the case that $F(y) \in [\underline{y}, y^p]$. The figure shows that the second and third iterates of F have fixed points other than the steady-state values y^* and y^p. This means that there exist two- and three-period cycles. The presence of three-period cycles is of particular importance, for, by A. N. Sarkovskii's (1964) theorem, the existence of three-period cycles implies that the map F has cycles of any periodicity. Moreover, as a consequence of the result of Li and Yorke (1975), the existence of three-period cycles implies chaos. That is, for the baseline calibration there exist perfect-foresight equilibria in which the real allocation fluctuates perpetually in an aperiodic fashion.

Indeed, three-period cycles emerge for any value of σ below 1.75. This finding is in line with Theorem 1, which states that there exists a value for σ below which chaotic dynamics necessarily occur. For values of σ greater than 1.75, three-period cycles disappear. This does not mean, however, that for such values

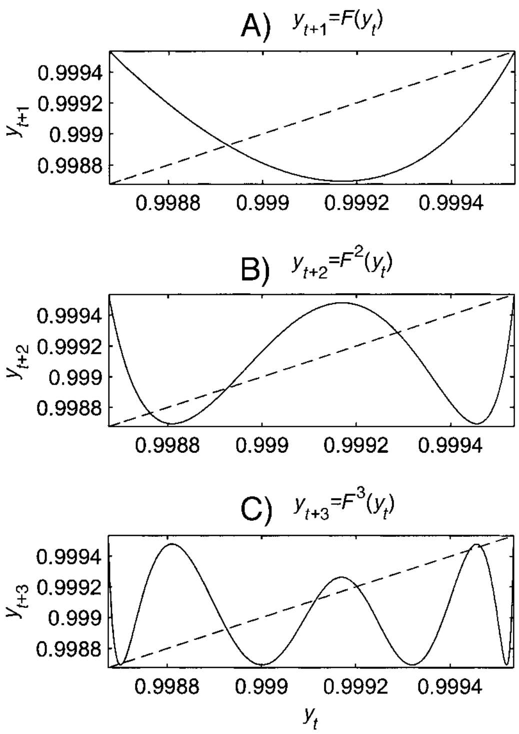

FIGURE 1. FORWARD-LOOKING TAYLOR RULES: THREE-PERIOD CYCLES

of σ the equilibrium dynamics cannot be quite complex. For example, for σ between 1.75 and 1.88, we could detect six-period cycles. Sarkovskii's theorem guarantees that, if six-period cycles exist, then cycles of periodicities $2^n \times 3$ for all $n \geq 1$ also exist. For σ between 1.88 and 2, four-period and two-period cycles exist.

For $\sigma > 1.71$, all aforementioned cycles occur in a feasible invariant interval, that is, in a feasible interval A such that $F(A) \in A$. The interval A contains both steady states. The upper end of the interval coincides with y^{p}, and the lower end is below y^*.[5] Finally, we find that for values of μ less than -9, the economy has three-period cycles when all other parameters take their baseline values. For values of μ greater than -9, three-period cycles cease to exist. Therefore, the more inelastic is the money demand function, the more likely it is that chaotic dynamics emerge.

[5] In terms of the notation of the Yamaguti and Matano (1979) theorem, the values of $1/\sigma$ of 1/1.75 and 1/1.71 correspond to the constants c_1 and c_2, respectively.

C. *Contemporaneous Interest-Rate Rules*

Complex dynamics may also arise under contemporaneous interest-rate feedback rules, that is, when $R_t = \rho(\pi_t)$. In Benhabib et al. (2001d) we show that, in the model under study, contemporaneous rules can give rise to instantaneous indeterminacy, in the sense that for a given equilibrium real allocation in period t there exist two distinct equilibrium real allocations in period $t + 1$. Furthermore, even if one limits attention to instantaneously determinate equilibria, the real allocation is always indeterminate: either it is locally indeterminate, or it is locally determinate but cycles of various periods and even chaos exist.

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