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An OLS approach to computing Ramsey equilibria in medium-scale macroeconomic models

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ABSTRACT

This note describes a general procedure for solving for the steady state and the dynamics implied by the Ramsey equilibrium of medium-scale macroeconomic models. The procedure yields an exact numerical solution for the steady state and second-order accurate dynamics. It introduces a novel projection-based approach to calculating exact solutions to the steady state of Ramsey equilibria.

1. The model

Given a policy regime, described by the process $\{\tau_t\}$, the competitive equilibrium conditions of a medium-scale macroeconomic model, such as the ones used in Schmitt-Grohé et al. (2006, 2007), can be written as

$$E_t\mathbf{C}(x_t, y_t, \tau_t, s_t, x_{t+1}, y_{t+1}, \tau_{t+1}, s_{t+1}) = 0, \tag{1}$$

$$s_{t+1} - s = \rho(s_t - s) + \eta\sigma\epsilon_{t+1}, \tag{2}$$

where x_t is an $n_x \times 1$ vector of endogenous predetermined variables, y_t is an $n_y \times 1$ vector of endogenous nonpredetermined variables, τ_t is an $n_\tau \times 1$ vector of policy instruments chosen by the government, s_t is an $n_s \times 1$ vector of exogenous predetermined variables, and ϵ_t is an $n_\epsilon \times 1$ vector of exogenous i.i.d. innovations with mean zero and unit standard deviations. The matrix of parameters ρ is of order $n_s \times n_s$, the vector of parameters s is of order $n_s \times 1$, the matrix of parameters η is of order $n_s \times n_\epsilon$, and $\sigma > 0$ is a scalar scaling the amount of uncertainty in the economy. The function $\mathbf{C}$ maps $\mathbb{R}^{2(n_x+n_y+n_\tau+n_s)}$ into $\mathbb{R}^{n_y+n_x}$.

The period-t objective function of the Ramsey planner (often identical to the period utility function of the representative agent) can be written as

$$U(x_t, y_t, \tau_t, s_t),$$

for all t. The vector y_t enters into the function U because it includes variables such as consumption and hours in period t. The vector x_t may enter into the function U because it includes variables such as lagged consumption, which are relevant, for instance, in models with habit formation. The vector τ_t may enter into the function U because it includes variables such as government purchases, which is relevant in models in which the private sector values public spending. The vector s_t may enter into the function U because it includes exogenous shocks such as preference shocks.

The Ramsey planner discounts time at the rate $\beta \in (0, 1)$. The portion of the Lagrangian associated with the Ramsey planner's optimization problem that is relevant for the purpose of computing optimal policy from the timeless perspective is given by

$$\begin{aligned}\mathcal{L} = \cdots &+ U(x_t, y_t, \tau_t, s_t) + \beta E_t U(x_{t+1}, y_{t+1}, \tau_{t+1}, s_{t+1}) \\ &+ \beta^{-1}\Lambda'_{t-1}\mathbf{C}(x_{t-1}, y_{t-1}, \tau_{t-1}, s_{t-1}, x_t, y_t, \tau_t, s_t) \\ &+ \Lambda'_t E_t\mathbf{C}(x_t, y_t, \tau_t, s_t, x_{t+1}, y_{t+1}, \tau_{t+1}, s_{t+1}) \\ &+ \beta E_t\Lambda'_{t+1}\mathbf{C}(x_{t+1}, y_{t+1}, \tau_{t+1}, s_{t+1}, x_{t+2}, y_{t+2}, \tau_{t+2}, s_{t+2}) \\ &+ \cdots.\end{aligned}$$

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Let w_t denote the vector of variables that the Ramsey planner chooses in period t. The vector w_t is given by

$$w_t = \begin{bmatrix} x_{t+1} \\ y_t \\ \tau_t \end{bmatrix}.$$

Notice that the vector x_{t+1} is in the information set of period t for it consists of endogenous predetermined variables. The first-order condition of the Ramsey planner with respect to w_t is given by $\frac{\partial \mathcal{L}}{\partial w_t} = 0$, or

$$\frac{\partial U(t)}{\partial w_t} + \beta E_t \frac{\partial U(t+1)}{\partial w_t} + \beta^{-1} \Lambda'_{t-1} \frac{\partial \mathbf{C}(t-1)}{\partial w_t} + \Lambda'_t E_t \frac{\partial \mathbf{C}(t)}{\partial w_t} + \beta E_t \Lambda'_{t+1} \frac{\partial \mathbf{C}(t+1)}{\partial w_t} = 0, \quad (3)$$

where $U(t) \equiv U(x_t, y_t, \tau_t, s_t)$ and $\mathbf{C}(t) \equiv \mathbf{C}(x_t, y_t, \tau_t, s_t, x_{t+1}, y_{t+1}, \tau_{t+1}, s_{t+1})$. The first-order condition with respect to the vector of Lagrange multipliers, Λ_t, is $\frac{\partial \mathcal{L}}{\partial \Lambda_t} = 0$, or Eq. (1).

Obtaining the set of Ramsey equilibrium conditions given in Eq. (3) in a medium-scale macroeconomic model can be extremely tedious if done manually. Therefore, our methodology obtains those equilibrium conditions analytically using Matlab's symbolic math toolbox. At no point does our method employ numerical derivatives. Avoiding numerical derivatives maximizes accuracy and minimizes computational time.

2. An OLS approach to computing the Ramsey steady state

In a deterministic steady state we have that $\sigma = 0$ and that all endogenous and exogenous variables are constant, so we drop the time subscripts. The Ramsey equilibrium conditions, then simplify to:

$$A(x, y, \tau; s) + B(x, y, \tau; s)\Lambda = 0 \quad (4)$$

and

$$C(x, y, \tau; s) = 0, \quad (5)$$

where $A(x, y, \tau; s)'$ is the steady state of $\frac{\partial U(t)}{\partial w_t} + \beta \frac{\partial U(t+1)}{\partial w_t}$, $B(x, y, \tau; s)'$ is the steady state of $\beta^{-1} \frac{\partial \mathbf{C}(t-1)}{\partial w_t} + \frac{\partial \mathbf{C}(t)}{\partial w_t} + \beta \frac{\partial \mathbf{C}(t+1)}{\partial w_t}$, and $C(x, y, \tau; s)$ is the steady state of $\mathbf{C}(t)$. Note that $A(x, y, \tau; s)$, $B(x, y, \tau; s)$, and $C(x, y, \tau; s)$ are analytical expressions. Furthermore, in many applications, condition (5) can be solved for $x(\tau; s)$ and $y(\tau; s)$ analytically. This step amounts to solving for the steady state of the competitive equilibrium given the steady-state of the policy vector τ_t.

The steady-state value of s_t, given by s, is known. The goal is to obtain the Ramsey steady-state values of x_t, y_t, and τ_t. The algorithm we propose below consists in first constructing a non-negative function $\delta(\tau)$ mapping $\mathbb{R}^{n_\tau}$ into $\mathbb{R}^+$. Once the function $\delta(\tau)$ has been constructed, a numerical minimization package can be used to find that value of τ that minimizes $\delta(\tau)$.

The function $\delta(\tau)$ measures the distance between the steady state value of $\frac{\partial \mathcal{L}}{\partial w_t}$ and zero, given τ and imposing that x and y solve Eq. (5). The numerical procedure for evaluating the function $\delta(\tau)$ exploits the insight that the Ramsey equilibrium conditions are linear in the steady-state value of the vector of Lagrange multipliers, Λ. The specific computational steps for constructing $\delta(\tau)$ are as follows.

1. Guess a value for the vector τ.
2. Solve the system of Eq. (5) for the vectors $x(\tau; s)$ and $y(\tau; s)$. In most applications, the functions $x(\tau; s)$ and $y(\tau; s)$ are available analytically. In this case, this step amounts to simply evaluating these functions numerically at the candidate value of τ.
3. With numerical values for $x(\tau; s)$, $y(\tau; s)$, and τ in hand, evaluate $A(x(\tau; s), y(\tau; s), \tau; s)$ and $B(x(\tau; s), y(\tau; s), \tau; s)$.
4. To solve for Λ, notice that (4) is a set of $n_x + n_y + n_\tau$ equations in $n_x + n_y$ unknowns, where $n_x + n_y$ is the size of the vector Λ. Thus, there are more equations than unknowns. Construct the projection of $A(x(\tau; s), y(\tau; s), \tau; s)$ onto $-B(x(\tau; s), y(\tau; s), \tau; s)$ as

   $$\hat{\Lambda}(\tau) = -B(x(\tau; s), y(\tau; s), \tau; s) \setminus A(x(\tau; s), y(\tau; s), \tau; s).$$

   Here the operator denotes the Matlab operator for OLS.
5. Construct the regression residual $\hat{e}(\tau)$:

   $$\hat{e}(\tau) = A(x(\tau; s), y(\tau; s), \tau; s) + B(x(\tau; s), y(\tau; s), \tau; s)\hat{\Lambda}(\tau).$$
6. Define $\delta(\tau)$ as

   $$\delta(\tau) = \hat{e}(\tau)'\hat{e}(\tau).$$

A numerical package, such as the Matlab function `fminsearch.m`, can be used to find the value of τ that minimizes $\delta(\tau)$. A Ramsey steady state is a vector τ^* such that $\delta(\tau^*) = 0$.

3. Computing Ramsey dynamics

The complete set of Ramsey equilibrium conditions is given by (1)–(3), which is a system of $2n_y + 2n_x + n_\tau + n_s$ stochastic difference equations in the $2n_y + 2n_x + n_\tau + n_s$ variables $x_{t+1}, y_t, \tau_t, s_{t+1}$, and Λ_t. To compute first- and second-order accurate approximations to the set of Ramsey equilibrium conditions, use a perturbation package, such as the one described in Schmitt-Grohé et al. (2004) and available on our websites. The equilibrium dynamics are approximated around the Ramsey steady state obtained in the previous section. The procedure involves expanding the system given in Eqs. (1) and (3). This task becomes close to impossible to perform by hand in medium-scale models of the macroeconomy. For this reason, in practice one must perform this step analytically by applying, for example, the symbolic math toolbox of Matlab (see the Matlab code accompanying Schmitt-Grohé et al., 2006, 2007).

4. Approximating welfare

A second-order accurate approximation of the value function v_t, defined as

$$v_t = U(x_t, y_t, \tau_t, s_t) + \beta E_t v_{t+1}$$

can be obtained by appending this expression to the complete set of first-order conditions of the Ramsey problem, given by (1)–(3). The resulting expanded dynamic system, has one more equation and one more endogenous nonpredetermined variable, v_t. The steady-state value of v_t can be readily obtained from the solution to the steady state of the Ramsey problem. Then, the expanded dynamic system can be approximated up to second order using standard perturbation packages, such as the one associated with Schmitt-Grohé et al. (2004), which is publicly available. This procedure guarantees a correct second-order accurate approximation of welfare and is computationally advantageous.

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