Implementing Iskrev's Identifiability Test

Stephanie Schmitt Grohé Martín Uribe*

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This document describes Iskrev's (2010) test and the suite of MATLAB code we created to implement it.

Iskrev's test consists in checking whether the derivative of the predicted autocovariogram of the vector of observables with respect to the vector of estimated parameters has a rank equal to the length of the vector of estimated parameters. Formally, let

$$m(t) \equiv \frac{\partial \text{vec} E(d_t d_0')}{\partial \theta},$$

for $t = 0, \ldots T-1$, where d_t is the theoretical counterpart of the vector of observables used to estimate the model, θ is a vector of model parameters whose identifiability the test establishes, and T is the sample size. Let

$$M \equiv \begin{bmatrix} m(0) \\ \vdots \\ m(T-1) \end{bmatrix}.$$

Then the estimated parameter θ is identifiable if M has full column rank. The test is performed by our matlab code `iskrev_test.m`.

Using the notation in Schmitt-Grohé and Uribe (2004), we can write the solution of the DSGE model up to first order as

$$y_t = g_x x_t$$

and

$$x_{t+1} = h_x x_t + \eta \epsilon_{t+1}, \tag{1}$$

where y_t is a vector of endogenous controls, x_t is a vector of endogenous and exogenous states, and ϵ_{t+1} is a white noise vector with identity variance/covariance matrix. The vectors y_t and x_t are deviations of the control and state variables of the model from their respective deterministic steady-state values $\bar{y}$ and $\bar{x}$. The elements of d_t are linear combinations of the elements of y_t. The two vectors are related by an expression of the form

$$d_t = D y_t,$$

where D is a matrix of known coefficients. This relation implies that

$$\text{vec}(E d_t d_0') = (D \otimes D) \text{vec}(E y_t y_0'),$$

*Columbia University and NBER. E-mail: martin.uribe@columbia.edu.

and therefore

$$m(t) = (D \otimes D)\frac{\partial \text{vec} E(y_t y_0')}{\partial \theta}.$$

Given the structure of the solution of the linearized DSGE model, we can write

$$E(y_t y_0') = g_x h_x^t \Sigma_x g_x',$$

where $\Sigma_x \equiv E x_t x_t'$. Taking the derivative of $\text{vec}E(y_t y_0')$ with respect to θ, we obtain

$$\frac{\partial \text{vec}(g_x h_x^t \Sigma_x g_x')}{\partial \theta} = (I_y \otimes g_x h_x^t \Sigma_x) dg_x' + (g_x \otimes g_x h_x^t) d\Sigma_x + (g_x \Sigma_x \otimes g_x) d(h_x^t) + (g_x \Sigma_x {h_x'}^t \otimes I_y) dg_x.$$

In this expression, the object dg_x denotes $\partial \text{vec}(g_x)/\partial\theta$, and is a matrix of order $n_y n_x \times n_\theta$, where n_y, n_x, and n_θ are the lengths of y_t, x_t, and θ, respectively. Similar notation applies to other objects.

Deriving dg_x and dh_x

The equilibrium conditions of the class of DSGE models considered here take the form $E_t f(y_{t+1} + \bar{y}, y_t + \bar{y}, x_{t+1} + \bar{x}, x_t + \bar{x}; \theta) = 0$. Up to first order, this expression can be written as

$$\begin{bmatrix} f_{y'} & f_{x'} \end{bmatrix} E_t \begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix} = -\begin{bmatrix} f_y & f_x \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix},$$

where $f_{y'}$, f_y, $f_{x'}$, f_x, denote the partial derivatives of f with respect to y_{t+1}, y_t, x_{t+1}, and x_t, respectively, evaluated at the steady stead point $(\bar{y}, \bar{y}, \bar{x}, \bar{x})$. Using the solution to the linearized model in the linearized equilibrium conditions, we obtain

$$\begin{bmatrix} f_{y'} g_x h_x & f_{x'} h_x \end{bmatrix} \begin{bmatrix} x_t \\ x_t \end{bmatrix} = -\begin{bmatrix} f_y g_x & f_x \end{bmatrix} \begin{bmatrix} x_t \\ x_t \end{bmatrix},$$

which implies that

$$f_{y'} g_x h_x + f_{x'} h_x = -f_y g_x - f_x.$$

Taking derivative with respect to θ, we obtain

$$\begin{aligned} (I_x \otimes f_{y'} g_x) dh_x + (h_x' \otimes f_{y'}) dg_x + (h_x' g_x' \otimes I_n) df_{y'} + (I_x \otimes f_{x'}) dh_x + (h_x' \otimes I_n) df_{x'} \\ = -(I_x \otimes f_y) dg_x - (g_x' \otimes I_n) df_y - df_x \end{aligned} .$$

Let

$$A \equiv (h_x' \otimes f_{y'}) + (I_x \otimes f_y),$$

$$B \equiv (I_x \otimes f_{y'} g_x) + (I_x \otimes f_{x'}),$$

and

$$C \equiv -(h_x' g_x' \otimes I_n) df_{y'} - (h_x' \otimes I_n) df_{x'} - (g_x' \otimes I_n) df_y - df_x.$$

Then, we can write

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} dg_x \\ dh_x \end{bmatrix} = C,$$

which can be solved to obtain

$$\left[\begin{array}{c} dg_x \\ dh_x \end{array}\right] = \left[\begin{array}{cc} A & B \end{array}\right]^{-1} C.$$

We now explain how to obtain the objects df_x, $df_{x'}$, df_y, and $df_{y'}$. We explain in detail how to obtain df_x, the other derivations follow similar steps. We view f_x as a function of the parameter vector θ and of the vector $z(\theta) \equiv [\bar{y}(\theta)' \quad \bar{x}(\theta)']'$, which is the vector of steady-sate values of controls and states, respectively. Thus, we write $f_x(z(\theta);\theta)$. Then, we have

$$df_x = \frac{\partial f_x}{\partial \theta} + \frac{\partial f_x}{\partial z}\frac{\partial z(\theta)}{\partial \theta}$$

The objects$\frac{\partial f_x}{\partial \theta}$ and $\frac{\partial f_x}{\partial z}$ are produced analytically by our Matlab code `iskrev_anal_deriv.m`. To facilitate the numerical evaluation of these symbolic expressions, the code writes these derivatives to a Matlab script file called `filename_iskrev_anal_deriv.m`, where the prefix filename is an input of iskrev_anal_deriv.m chosen by the user.

To obtain $\frac{\partial z(\theta)}{\partial \theta}$ note that one can write the steady state of the model as $f(z;\theta) = 0$, which implicitly defines $z(\theta)$. Differentiating we get

$$\frac{\partial f(z(\theta);\theta)}{\partial \theta} + \frac{\partial f(z(\theta);\theta)}{\partial z}\frac{\partial z(\theta)}{\partial \theta} = 0,$$

which can be solved to obtain

$$\frac{\partial z(\theta)}{\partial \theta} = -\left[\frac{\partial f(z(\theta);\theta)}{\partial z}\right]^{-1}\frac{\partial f(z(\theta);\theta)}{\partial \theta}.$$

The Matlab code `iskrev_anal_deriv.m` writes this formula into the Matlab script `filename_iskrev_anal_deriv.m`.

Deriving dg_x' and dh_x'

Let R_h be a matrix such that

$$\text{vec}(h_x') = R_h \text{vec}(h_x)$$

The matrix R_h is a permutation matrix of order n_x^2. Its unitary elements are located in row i column fix$((i-1)/n_x) + 1$+rem$(i-1, n_x)n_x$, for $i = 1, \ldots, n_x^2$. Then we have that

$$dh_x' = R_h dh_x$$

Similarly, we can deduce that

$$dg_x' = R_g dg_x,$$

where the matrix R_g is a permutation matrix (i.e., a square matrix with only one element equal to unity per row and per column and all remaining elements equal to zero) of order $n_x n_y$. Its unitary elements are located in row i column fix$((i-1)/n_x) + 1$+rem$(i-1, n_x)n_y$, for $i = 1, \ldots, n_x n_y$.

Deriving $d\Sigma_x$

From (1), we have that the matrix $\Sigma_x \equiv E x_t x_t'$ satisfies

$$\Sigma_x = h_x \Sigma_x h_x' + \eta\eta'.$$

The derivative of Σ_x with respect to θ must then satisfy

$$d\Sigma_x = (h_x \otimes h_x) d\Sigma_x + (h_x \Sigma_x \otimes I_x) dh_x + (I_x \otimes h_x \Sigma_x) dh_x' + d(\eta\eta')$$

Solving for $d\Sigma_x$, we obtain

$$d\Sigma_x = [I_{n_x^2} - (h_x \otimes h_x)]^{-1} [(h_x \Sigma_x \otimes I_x) dh_x + (I_x \otimes h_x \Sigma_x) dh_x' + d(\eta\eta')]$$

The object $d(\eta\eta')$ is produced symbolically by `iskrev_anal_deriv.m` and then written to the script file `filename_iskrev_anal_deriv.m`.

Deriving dh_x^t

For $t = 1$, it is dh_x, which we already derived. For $t \geq 2$, we proceed iteratively, noticing that $h_x^t = h_x^{t-1} h_x$, whose derivative is given by

$$dh_x^t = (I_x \otimes h_x^{t-1}) dh_x + (h_x' \otimes I_x) dh_x^{t-1}$$

What if M Is Not Full Column Rank

Suppose M is less than full column rank at a parameter value θ_0. Then, we conclude that with the selected observables and sample size, the parameter θ is not identifiable in the vicinity of θ_0. This essentially means that in this case there will be an infinite number of parameter vectors θ that will give rise to the same autocovariogram as θ_0. When θ is not identifiable, we can establish what linear combinations of the elements of θ will deliver the same autocovariogram as θ_0.

Let $V(\theta, T)$ be the vectorized covariogram of the vector of observables, d_t, of order T. That is,

$$V(\theta) = \begin{bmatrix} \text{vech}(E d_0 d_0') \\ \vdots \\ \text{vec}(E d_{T-1} d_0') \end{bmatrix}$$

Then, Taylor-expanding around θ_0 up to first order, we obtain

$$V(\theta) \approx V(\theta_0) + M(\theta_0)(\theta - \theta_0)$$

If $M(\theta_0)$ has full column rank, then $V(\theta) = V(\theta_0)$ if and only if $\theta = \theta_0$ in the neighborhood of θ_0. If, on the other hand, $M(\theta_0)$ is rank deficient, then there exists an infinite number of vectors θ in the vicinity of θ_0 satisfying $V(\theta) = V(\theta_0)$. To obtain these vectors, perform a singular value decomposition of $M(\theta_0)'$. That is, find matrices U, S, and V such that

$$M(\theta)U = VS',$$

where U and V are unitary (i.e., $UU' = I$ and $VV' = I$) and S is diagonal with its diagonal elements nonnegative and decreasing. The matrix S has as many rows as $M(\theta)$ and as many columns as the length of θ. Now partition the matrix U as $[U^1 \, U^2]$, where U^2 has as many columns as S has zero diagonal elements. Then, we have that any vector θ of the form

$$\theta = \theta_0 + u^2 \alpha$$

delivers the same autocovariogram as θ_0 for any (small) scalar α and any vector u^2 taken from the columns of U^2.

References

Iskrev, Nicolay, "Local Identification in DSGE Models," *Journal of Monetary Economics 57,* 2010, 189-210.

Schmitt-Grohé, Stephanie and Martín Uribe, "Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function," *Journal of Economic Dynamics and Control 28,* January 2004, 755-775.