Multiple Equilibria in Open Economy Models with Collateral Constraints: Overborrowing Revisited

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Starting Point

• Open-economy models with collateral constraints have been used to explain:
  — sudden stops in response to fundamental shocks
  — amplification of business cycles
  — overborrowing
  — the desirability of capital control taxes

• Open-economy models with collateral constraints are prone to multiple equilibria. This issue has been little explored (indeed avoided) in the related literature.

• Exceptions are Mendoza (2005) and Jeanne and Korinek (2010) who present heuristic analysis of the multiplicity problem.
This Paper

• characterizes (analytically and numerically) equilibrium multiplicity in open economy models with flow collateral constraints.

• shows that multiple equilibria exist when the value of collateral is elastic with respect to debt.

• shows that such an economy displays underborrowing (as opposed to overborrowing).

• shows that self-fulfilling financial crises exist in which the economy deleverages, there is a fire sale, a Fisherian deflation, and a sudden stop (as opposed to crises triggered by fundamental shocks).
An Open Economy Model with a Flow Collateral Constraint

Households maximize

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1 - \sigma}$$

subject to

$$c_t = \left[ ac_t^{T1-\frac{1}{\xi}} + (1 - a)c_t^{N1-\frac{1}{\xi}} \right]^{\frac{1}{1-\frac{1}{\xi}}}$$

$$c_t^T + p_t c_t^N + d_t = y^T + p_t y^N + \frac{d_{t+1}}{1 + r}$$

$$d_{t+1} \leq \kappa(y^T + p_t y^N)$$

where $c_t =$consumption; $c_t^T, c_t^N =$consumption of tradables, nontradables; $\xi > 0, a \in (0, 1)$, $d_t =$debt due in $t$; $d_{t+1} =$debt assumed in $t$ and due in $t + 1$; $y^T, y^N =$endowments of tradables, nontradables; $p_t =$relative price of nontradables; $r =$interest rate.
Observations

(1) The last equation is the flow collateral constraint (CC). It says that the amount of debt issued in period $t$, $d_{t+1}$, cannot exceed a fraction $\kappa$ of income, $y^T + p_t y^N$.

(2) From the individual agent’s point of view, the CC is well behaved: the larger $d_{t+1}$ is, the smaller the slack in the collateral constraint will be. This is because agents take as exogenous all of the objects on the RHS of the collateral constraint (in particular $p_t$).

(3) Also, from the perspective of the individual agent, the collateral constraint defines a convex set of feasible debt levels: if $d'$ and $d''$ satisfy the budget constraint and the collateral constraint, then so does the debt level $\alpha d' + (1 - \alpha) d''$, for any $\alpha \in [0, 1]$.

(4) As we will see shortly, (2) and possibly (3) do not hold from an aggregate perspective.
Three Equilibrium Conditions of Interest

\[ d_{t+1} \leq \kappa (y^T + p_t y^N) \]

\[ p_t = \frac{1 - a}{a} \left( \frac{c_t^T}{y^N} \right)^{1/\xi} \]

\[ c_t^T + d_t = y^T + \frac{d_{t+1}}{1 + r} \]

These three conditions give rise to the following equilibrium collateral constraint

\[ d_{t+1} \leq \kappa \left[ y^T + \left( \frac{1 - a}{a} \right) \left( y^T + \frac{d_{t+1}}{1 + r} - d_t \right)^{\frac{1}{\xi}} y^N^{1-\frac{1}{\xi}} \right] \]
Observations

(1) $d_{t+1}$ appears on both sides of the equilibrium collateral constraint.

(2) Because $\xi > 0$, the equilibrium value of collateral increases with the level of debt, giving rise to the possibility that the higher $d_{t+1}$ is, the more slack the collateral constraint will be.

(3) Collateral (i.e., the RHS of the equilibrium collateral constraint) is in general nonlinear in $d_{t+1}$. If $\xi < 1$, then the set of debt levels that satisfy the equilibrium collateral constraint may be nonconvex, that is, if $d'$ and $d''$ satisfy the equilibrium collateral constraint, then $\alpha d' + (1 - \alpha)d''$ may not for some $\alpha \in (0, 1)$. 
Three Simplifying Assumptions

(1) $\sigma = 1/\xi$. (2) $y^N = 1$. (3) $\beta(1 + r) = 1$.

The equilibrium collateral constraint then becomes

$$d_{t+1} \leq \kappa \left[ y^T + \left( \frac{1 - a}{a} \right) \left( y^T + \frac{d_{t+1}}{1 + r} - d_t \right)^{\frac{1}{\xi}} \right]$$
The Steady-State Collateral Constraint

\[ d \leq \kappa \left[ y^T + \left( \frac{1-a}{a} \right) \left( y^T - \frac{r}{1+r} d \right)^{\frac{1}{\xi}} \right] \]

Assumption: \( \frac{1+r}{r} > \kappa \)
Observations

(1) The expression under the power of $1/\xi$ is steady-state consumption of tradables, $y^T - r/(1 + r)d$.

(2) Steady-state consumption is zero at the natural debt limit, that is, when $d = \overline{d} \equiv (1 + r)y^T/r$ and positive for $d < \overline{d}$.

(3) The steady-state collateral constraint is well behaved, that is, the larger is debt, the tighter it will be.

(4) The maximum level of debt sustainable in the steady state, $\tilde{d}$, is

$$\tilde{d} = \kappa \left[ y^T + \left( \frac{1-a}{a} \right) \left( y^T - \frac{r}{1+r}\tilde{d} \right)^{\frac{1}{\xi}} \right].$$

As long as $(1 + r)/r > \kappa$, $\tilde{d} < \overline{d}$.

(5) The steady-state collateral constraint binds at $\tilde{d}$, and is violated at any level of debt larger than $\tilde{d}$. ⇒ No steady-state equilibrium is possible to the right of $\tilde{d}$.

(6) For any initial debt $d_0 < \tilde{d}$, the constant debt path $d_t = d_0$ for all $t$ satisfies the resource constraint with a positive and constant level of consumption, $c_t^T = c_0^T > 0$ for all $t$, and ensures that the collateral constraint never binds. Is this an equilibrium? Yes, if the Euler equations holds.
The Euler Equation in a Steady-State Equilibrium

When the collateral constraint is slack, the Euler equation is

$$\left( \frac{c_t^{T+1}}{c_t^T} \right)^\sigma = \beta(1 + r).$$

By assumption (3) \( \beta(1 + r) = 1 \). It follows that when the collateral constraint is slack, consumption must be constant over time. Therefore the Euler equation is satisfied with a slack collateral constraint in any steady state equilibrium.
Taking stock:

We have established that if

\[ d_0 \leq \tilde{d}, \]

then

\[ d_{t+1} = d_0 \quad \text{and} \quad c_t^T = y^T - \frac{r}{1 + r}d_0 \]

for all \( t \geq 0 \) is an equilibrium.

And this equilibrium is first best.

Are there other equilibria?
Definition 1 (Self-Fulfilling Financial-Crisis Equilibrium) For any initial level of debt $d_0 < \tilde{d}$, a self-fulfilling financial-crisis equilibrium is a set of deterministic paths $\{c_t^T, d_{t+1}, \mu_t\}_{t=0}^\infty$ satisfying $d_1 < d_0$ (deleveraging), $d_{t+1} = d_1$ (steady state after period 0), $\mu_t \geq 0$, $c_t^T > 0$, and

\[ \Lambda(c_t^T)[1 - (1 + r)\mu_t] = \Lambda(c_{t+1}^T), \text{ where } \Lambda(c_t^T) \equiv U'(A(c_t^T, 1))A_1(c_t^T, 1) \]

\[ c_t^T + d_t = y^T + \frac{d_{t+1}}{1 + r} \]

\[ d_{t+1} \leq \kappa \left[ y^T + \frac{(1 - a)}{a} \left( y^T + \frac{d_{t+1}}{(1 + r) - d_t} \right)^{\frac{1}{\xi}} \right] \]

\[ \mu_t \left\{ \kappa \left[ y^T + \frac{(1 - a)}{a} \left( y^T + \frac{d_{t+1}}{(1 + r) - d_t} \right)^{\frac{1}{\xi}} \right] - d_{t+1} \right\} = 0 \]
Are there other equilibria?

We will look for “self-fulfilling financial crisis equilibria”

The period-0 collateral constraint

\[
d_1 \leq \kappa y^T + \kappa \frac{1-a}{a} \left( y^T + \frac{d_1}{1+r} - d_0 \right)^\frac{1}{\xi}
\]

The slope of the period-0 collateral constraint

\[
S(d_1; d_0) = \kappa \frac{1-a}{a} \frac{1}{\xi} \frac{1}{1+r} \left( y^T + \frac{d_1}{1+r} - d_0 \right)^{\frac{1}{\xi}-1}
\]
Properties of the RHS of the period-0 collateral constraint:

(1) increasing in $d_1$

(2) convex in $d_1$ if $\xi < 1$ (concave in $d_1$ if $\xi > 1$)

(3) well defined for $c_0^T > 0 \iff d_1 > (1 + r)(d_0 - y^T)$

(4) slope is zero at $c_0^T = 0 \iff d_1 = (1 + r)(d_0 - y^T)$

(5) equal to $\kappa y^T$ at $c_0^T = 0$

(6) intersect RHS of steady-state collateral constraint at $d_1 = d_0$
Low intratemporal elasticity of substitution: $0 < \xi < 1$

⇒ self-fulfilling financial crises equilibria may exist
Observations

(1) In the figure, $d_0 < \bar{d}$. From the previous analysis we have that point $A$ is an equilibrium featuring $d_t = d_0$ for all $t \geq 0$.

(2) In the figure $\xi < 1 \Rightarrow$ the period-0 CC is convex in $d$.

(3) $c_0^T$ must be positive, $\Rightarrow$ feasible choices of $d$ are $d > (1 + r)(d_0 - y^T)$.

(4) in the figure $\kappa y^T < (1 + r)(d_0 - y^T)$

(5) Consider point $B$. At $B$ the period-0 CC holds with equality (binding), since it is on the 45-degree line. Must still check
(a) that at $B$ the Euler equation holds in period 0, and
(b) that all equilibrium conditions are satisfied after period 1.

We wish to consider an equilibrium in which the economy jumps from $A$ to $B$ in period 0 and then stays at $B$ forever thereafter.

Note first that deleveraging from $A$ to $B$ and then staying at $B$ forever requires that $c_0^T < c_1^T$. Knowing this makes it easy to see that at $B$ the Euler equation holds in period 0, as we show next.
The Euler Equation at Point B

When the collateral constraint binds (ex: at point $B$ in period 0) the Euler equation becomes

$$
\left( \frac{c_1^T}{c_0^T} \right)^\sigma = \frac{(1 + r)\beta}{1 - \mu_0(1 + r)},
$$

where $\mu_0$ is the Lagrange multiplier on the collateral constraint in period 0.
Observations on the Euler equation

(1) Is $\mu_0 \geq 0$? Yes, because $(1 + r)\beta = 1$ and $c_1^T > c_0^T$.

(2) From period 1 on, we can use the previous analysis to state that point $B$ represents an equilibrium with $d_t = d_1$ for all $t \geq 1$. We have therefore shown that $B$ is indeed an equilibrium.

(3) At point $B$, the economy experiences a self-fulfilling deleveraging crisis, caused by an arbitrary, generalized belief that the value of collateral is low.

(4) The equilibrium at point $B$ is welfare inferior to equilibrium at point $A$, because the former features a drop in consumption in period 0 and a recovery in period 1, whereas the latter ensures perfect consumption smoothing.
For $0 < \xi < 1$, the necessary and sufficient condition for the existence of multiple equilibria is

$$S(\bar{d}; \bar{d}) > 1$$

Notes. The downward-sloping solid line is the right-hand side of the steady-state collateral constraint. The upward-sloping dashed and dashed-dotted lines are the right-hand sides of the period-0 collateral constraint for $d_0 = \bar{d}$ and $d_0 < \bar{d}$, respectively. The figure is drawn under the assumptions that $S(\bar{d}; \bar{d}) > 1$ and $0 < \xi < 1$. 
What if $S(\tilde{d}; \tilde{d}) < 1$?

- - - : Period-0 collateral

___: Steady-state collateral
When one self-fulfilling financial crisis equilibrium exists, then there always also exists a second one.
Self-fulfilling financial crisis equilibria also exist for $\xi > 1$: 
Do self-fulfilling financial-crisis equilibria exist for plausible parameter values?

\[
S(\tilde{d}; \tilde{\delta}) = \kappa \frac{1 - a}{a} \frac{1}{\xi} \frac{1}{1 + r} \left( y^T - \frac{r}{1 + r} \tilde{d} \right)^{\frac{1}{\xi} - 1}
\]

Numerical example: \( \kappa = 0.3, \xi = 0.5, a = 0.25, r = 0.04, y^T = 1 \)

\[ S(\tilde{d}; \tilde{\delta}) > 1.7 \]
$S(\tilde{d}, \tilde{d})$

- $\kappa$: $0$, $0.18$, $0.3$, $0.8$
- $\xi$: $0$, $0.5$, $0.86$, $1$
- $a$: $0$, $0.25$, $0.36$, $0.7$
- $r$: $0$, $0.04$, $0.4$, $0.7$
Sunspots and Persistent Financial Crisis

• under perfect foresight self-fulfilling financial crisis lasts at most one period.

• with extrinsic uncertainty financial crises can be persistent

• sunspot variable $s_t$ is either 0 (slack CC) or 1 (binding CC);

• assume: $s_0 = 1$; $s_t = 0 \ \forall t \geq 2$, and $s_1 = 1$ with probability $\pi$.

• we then have: $c_{1,0}^T > c_0^T > c_{1,1}^T$

• Euler in period 0

$$[1 - (1 + r)\mu_0]\Lambda(c_0^T) = \pi\Lambda(c_{1,1}^T) + (1 - \pi)\Lambda(c_{1,0}^T).$$

• Persistent financial crisis exist if

$$\pi \in (0, \pi^*], \quad \text{where } \pi^* \equiv \frac{\Lambda(c_0^T) - \Lambda(c_{1,0}^T)}{\Lambda(c_{1,1}^T) - \Lambda(c_{1,0}^T)} \in (0, 1).$$
Welfare ranking: $C < B < A$.

The competitive equilibrium at point $A$ is the first-best allocation.

Therefore, if agents coordinate on equilibrium $A$, there is neither overborrowing nor underborrowing.

But if agents coordinate on equilibrium $B$ or $C$, then the economy suffers from underborrowing.
Implementation with Capital Control Taxes

Suppose that the government imposes a proportional tax on debt, $\tau_t$. The budget constraint of the household becomes

$$c^T_t + p_t c^N_t + d_t = y^T_t + p_t y^N_t + \frac{1 - \tau_t}{1 + r} d_{t+1}.$$  

(Capital control taxes can be rebated lump-sum or through income-based transfers.)
Implementation Versus Ramsey-Optimal Taxation

The Ramsey optimal tax rate in this economy is $\tau_t = 0$ at all times. This is because the unregulated competitive equilibrium $A$ delivers the first-best allocation.

However, announcing the policy $\tau_t = 0$ for all $t$ does not guarantee that the Ramsey optimal equilibrium will emerge. Indeed, this tax policy also supports the deleveraging equilibria $B$ or $C$.

What capital control policy can induce the Ramsey-optimal equilibrium? Consider capital-control policy rules of the form

$$\tau_t = \tau(d_{t+1}, d_t)$$

satisfying $\tau(d, d) = 0$ and $\tau_1 < 0$. This policy is consistent with the Ramsey equilibrium, since under the Ramsey equilibrium $d_{t+1} = d_t$ for all $t \geq 0$. 

Off-Equilibrium Threat: The government sets $\tau_t = 0$ if the economy stays at point $A$, but threatens to tax capital outflows if the economy enters a financial panic and deleverages to point $B$ or $C$.

Result: If the capital outflow tax is strong enough, then points $B$ and $C$ are ruled out, and point $A$ is the unique equilibrium, and capital controls are never imposed.
Implementation (continued)

Under this tax-policy rule, the Euler equation in period 0 becomes:

\[
\left( \frac{c_T^1}{c_T^0} \right)^\sigma = \frac{1}{1 - \tau(d_1, d_0) - (1 + r)\mu_0}
\]

(1) In the intended (Ramsey) equilibrium, \( c_T^1/c_T^0 = 1 \), \( d_1 = d_0 \), and \( \mu_0 = 0 \), so the Euler equation holds and \( \tau(d_1, d_0) = 0 \).

(2) In the unintended equilibrium (point \( B \)), \( c_T^1/c_T^0 > 0 \), and \( d_1 < d_0 \). Make \( \tau(d_1, d_0) \) so large that \( \mu_0 \) has to be negative for the Euler equation to hold. Since \( \mu_0 \) must be nonnegative, this capital-control policy rules out the unintended equilibrium.

Result: To rule out undesired self-fulfilling crises, the Ramsey planner commits to impose sufficiently large capital controls in case of speculative capital outflows.
Quantitative Analysis

Time unit: one quarter.
The economy is driven by endowment and interest-rate shocks. Estimate a bivariate AR(1) process for $(y^T_t, r_t)$ using Argentine data over the period 1983:Q1 to 2001:Q4.

\[ \kappa = 1.2 \quad (\Rightarrow \text{upper limit on debt = 30 percent of annual output}). \]
\[ r = 0.0316 \]
\[ \beta = 0.9635 \]
\[ \sigma = 1/\xi = 2 \]
\[ a = 0.26 \]
\[ y^N = 1. \]

Discretization: 501 points for $d$, 21 for $y^T$, and 11 for $r$. 
Multiple Binding Debt Levels In the Stochastic Economy

Note. The value of collateral is evaluated at the state \((y_t^T, r_t, d_t) = (0.7633, 0.0541, 1.5960)\).
Numerical Algorithm

- The numerical solution must take a stance on how to handle the possibility of indeterminacy.

- Failing to address this issue may result in nonconvergence.

- In searching for an equilibrium the algorithm favors one with a binding constraint, as follows:

  - if for the current state \((y_t^T, r_t, d_t)\) there are one or two values of \(d_{t+1}\) for which the period-\(t\) equilibrium conditions are satisfied and the collateral constraint is binding, pick the larger binding debt level (criterion B); pick the smaller binding debt level (criterion C).
Constrained Optimal Problem

\[ v(y^T, r, d) = \max_{c^T, d'} \left\{ U(A(c^T, y^N)) + \beta \mathbb{E} \left[ v(y^{T'}, r', d') \mid y^T, r \right] \right\} \]

subject to

\[ c^T + d = y^T + \frac{d'}{1 + r} \]

\[ d' \leq \kappa \left[ y^T + \frac{1 - a}{a} \left( \frac{c^T}{y^N} \right)^{\frac{1}{\xi}} y^N \right] \]

Note. Although the constraints of this control problem may not represent a convex set in tradable consumption and debt, the fact that the planner’s allocation is the result of a utility maximization problem, implies that it is generically unique.
The debt distribution under the optimal policy lies to the right of those associated with the unregulated equilibria. It is in this sense that the unregulated economy underborrows.
Notes. $\kappa = 0.8$, $a = 0.5$, $\xi = 0.85$. The debt distribution in the regulated economy lies to the left of that associated with the unregulated equilibria. It is in this sense that the unregulated economy overborrows.
Conclusion

• This paper shows that open economies with flow collateral constraints display multiple equilibria.

• In particular, they are prone to self-fulfilling crises in which deleveraging and Fisherian deflations take place in the absence of changes in fundamentals.

• The competitive equilibrium can display underborrowing, as agents save to protect themselves against expectations-driven financial crises.

• Averting self-fulfilling crises requires the threat of taxing non-fundamental bursts of capital outflows.
EXTRAS
Are Points Between $A$ and $B$ Equilibria? No.

Take another look at the figure.

At any such point: (a) The collateral constraint is satisfied (indeed it is slack); (b) the resource constraint is satisfied with positive consumption.

However, because at any such point $c^T_0 < c^T_1$, the Euler equation is satisfied only if $\mu_t > 0$.

But this violates the slackness condition, which requires $\mu_0 = 0$, because the collateral constraint is slack in period 0.