Optimal Target Criterion in Chap. 8 of Woodford (2003)

Marc P. Giannoni Michael Woodford

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Contents

A Optimal Target Criterion in the Small Quantitative Model

This section characterizes the optimal target criterion for the estimated structural model of Giannoni and Woodford ("Optimal Inflation Targeting Rules, Section 2, 2003). The optimal target criterion derived here is the one presented in the chapter 8 of Woodford (2003). (The long-term target criterion presented below differs from the one presented in Giannoni and Woodford (2003).) The equations numbers followed by XXX below refer to the equation numbers in Giannoni and Woodford (2003).

A.1 Analytical derivation

The constraints relevant for optimal monetary policy are the aggregate supply equation (2.14 XXX)

$$\pi_t - \gamma_p \pi_{t-1} = \xi_p \omega_p E_{t-1} x_t + \xi_p E_{t-1}\left(w_t - w_t^n\right) + \beta E_{t-1}\left(\pi_{t+1} - \gamma_p \pi_t\right) \tag{1}$$

and the wage inflation equation (2.11 XXX)

$$\pi_t^w - \gamma_w \pi_{t-1} = \xi_w E_{t-1}\left(\omega_w x_t + \varphi \tilde{x}_t\right) - \xi_w E_{t-1}\mu_t + \xi_w E_{t-1}\left(w_t^n - w_t\right) + \beta E_{t-1}\left(\pi_{t+1}^w - \gamma_w \pi_t\right),$$

where $\tilde{x}_t \equiv (x_t - \eta x_{t-1}) - \beta\eta\left(E_t x_{t+1} - \eta x_t\right)$. However, because there is no constraint on what the surprise component $E_{t-1}\mu_t$ may be (except that it must be unforecastable at date $t-2$), the only constraint implied by the wage inflation equation is

$$E_{t-2}\left(\pi_t^w - \gamma_w \pi_{t-1}\right) = \xi_w E_{t-2}\left(\omega_w x_t + \varphi \tilde{x}_t\right) + \xi_w E_{t-2}\left(w_t^n - w_t\right) + \beta E_{t-2}\left(\pi_{t+1}^w - \gamma_w \pi_t\right). \tag{2}$$

In addition, the identity

$$w_t = w_{t-1} + \pi_t^w - \pi_t \tag{3}$$

must be satisfied at all dates. The constraints (1) – (3) generalize the constraints (1.29 XXX) – (1.31 XXX) of section 1.4.

Because of the delays assumed in the underlying model, the variables $\pi_t, \pi_t^w, w_t,$ and x_t are all determined at date $t-1$. It will thus be convenient to define the following variables $\bar{\pi}_t \equiv E_t \pi_{t+1} = \pi_{t+1}$ and $\bar{\pi}_t^w \equiv E_t \pi_{t+1}^w = \pi_{t+1}^w$, and $\bar{w}_t \equiv E_t w_{t+1} = w_{t+1}$, all determined at date t. Furthermore, because consumption at date t is determined at date $t-2$, the output gap satisfies

$$x_t = v_{t-2} + s_{t-1}$$

where v_{t-2} is an endogenous variable determined at date $t-2$ and s_{t-1} is an exogenous variable determined at date $t-1$ and unforecastable at date $t-2$.

The objective function (3.1 XXX) can then be rewritten as

$$\begin{aligned} & E_0 \sum_{t=0}^{\infty} \beta^t \left[\lambda_p \left(\bar{\pi}_{t-1} - \gamma_p \bar{\pi}_{t-2}\right)^2 + \lambda_w \left(\bar{\pi}_{t-1}^w - \gamma_w \bar{\pi}_{t-2}\right)^2 + \lambda_x \left(v_{t-2} + s_{t-1} - \delta v_{t-3} - \delta s_{t-2} - \hat{x}^*\right)^2\right] \\ = \quad & \beta E_0 \sum_{t=0}^{\infty} \beta^t \left[\lambda_p \left(\bar{\pi}_t - \gamma_p \bar{\pi}_{t-1}\right)^2 + \lambda_w \left(\bar{\pi}_t^w - \gamma_w \bar{\pi}_{t-1}\right)^2 + \beta\lambda_x \left(v_t - \delta v_{t-1} - \delta s_t - \hat{x}^*\right)^2\right] + tip \end{aligned}$$

where *tip* represents again terms independent of policy adopted at date 0, such as endogenous variables determined before date 0. Note that to get the second line we also used the fact that s_t is unforecastable, so that $E_0 z_t s_{t+1} = E_0 \left[z_t \left(E_t s_{t+1}\right)\right] = 0$ for any date $t \geq 0$ and any variable z_t determined at date t or earlier.

Combining the objective function with the constraints (1) – (3), we may write the following Lagrangian

$$\begin{aligned}
\mathcal{L}_0 \quad = \quad & E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2} \left[\lambda_p \left(\bar{\pi}_t - \gamma_p \bar{\pi}_{t-1}\right)^2 + \lambda_w \left(\bar{\pi}_t^w - \gamma_w \bar{\pi}_{t-1}\right)^2 + \beta \lambda_x \left(v_t - \delta v_{t-1} - \delta s_t - \hat{x}^*\right)^2 \right] \right. \\
& + \varphi_{1,t} \left[\bar{\pi}_t - \gamma_p \bar{\pi}_{t-1} - \xi_p \omega_p v_{t-1} - \xi_p \bar{w}_t - \beta \bar{\pi}_{t+1} + \beta \gamma_p \bar{\pi}_t \right] \\
& + \varphi_{2,t-1} \left[\bar{\pi}_t^w - \gamma_w \bar{\pi}_{t-1} - \xi_w \left(\omega_w v_{t-1} + \varphi \left[\left(1 + \beta \eta^2\right) v_{t-1} - \eta v_{t-2} - \beta \eta v_t \right] \right) + \xi_w \bar{w}_t - \beta \bar{\pi}_{t+1}^w + \beta \gamma_w \bar{\pi}_t \right] \\
& \left. + \varphi_{3,t} \left[\bar{w}_t - \bar{w}_{t-1} - \bar{\pi}_t^w + \bar{\pi}_t \right] \right\}.
\end{aligned}$$

The associated system of first-order conditions is given by

$$\begin{aligned}
0 \quad &= \quad \lambda_p \left[\left(\bar{\pi}_t - \gamma_p \bar{\pi}_{t-1}\right) - \beta \gamma_p \left(E_t \bar{\pi}_{t+1} - \gamma_p \bar{\pi}_t\right) \right] - \beta \lambda_w \gamma_w \left(E_t \bar{\pi}_{t+1}^w - \gamma_w \bar{\pi}_t\right) \\
& \quad\quad + \varphi_{1,t} - \beta \gamma_p E_t \varphi_{1,t+1} - \left(\varphi_{1,t-1} - \beta \gamma_p \varphi_{1,t}\right) - \beta \gamma_w \left(\varphi_{2,t} - \varphi_{2,t-1}\right) + \varphi_{3,t} \qquad (4) \\
0 \quad &= \quad \lambda_w \left(\bar{\pi}_t^w - \gamma_w \bar{\pi}_{t-1}\right) + \left(\varphi_{2,t-1} - \varphi_{2,t-2}\right) - \varphi_{3,t} \qquad (5) \\
0 \quad &= \quad -\xi_p \varphi_{1,t} + \xi_w \varphi_{2,t-1} + \varphi_{3,t} - \beta E_t \varphi_{3,t+1} \qquad (6) \\
0 \quad &= \quad \beta \lambda_x \left[\left(v_t - \delta v_{t-1} - \delta s_t - \hat{x}^*\right) - \beta \delta E_t \left(v_{t+1} - \delta v_t - \delta s_{t+1} - \hat{x}^*\right) \right] \\
& \quad\quad - \beta \xi_p \omega_p E_t \varphi_{1,t+1} + \beta \xi_w E_t \left[B(L) \varphi_{2,t+1} \right] \qquad (7)
\end{aligned}$$

for each $t \geq t_0$, where

$$\begin{aligned}
B(L) \quad &\equiv \quad \varphi \left(\eta \beta - L\right) \left(1 - \eta L\right) - \omega_w L \\
&\equiv \quad B_0 + B_1 L + B_2 L^2.
\end{aligned}$$

The optimal plan must in addition satisfy a transversality condition. The latter is however necessarily satisfied as we restrict our attention to bounded solutions.

As in Giannoni and Woodford (2002a, 2002b), we combine these first-order conditions to obtain a single equation that involves only target variables, i.e., inflation, wage inflation, and the output gap. To simplify the algebra, we now specialize the analysis to the case $\gamma_p = \gamma_w = 1$, in accordance with our estimates (as well as the model of Christiano et al., 2001). In this case, adding (4) to (5) yields

$$E_t \left\{ \left(1 - \beta L^{-1}\right) \left[a_t + (1 - L) \xi_{1,t} \right] \right\} = 0 \qquad (8)$$

for all $t \geq t_0$, where the variable a_t and the new multiplier $\xi_{1,t}$ are defined as

$$\begin{aligned}
a_t \quad &\equiv \quad \beta \left[\lambda_p \left(\bar{\pi}_t - \bar{\pi}_{t-1}\right) + \lambda_w \left(\bar{\pi}_t^w - \bar{\pi}_{t-1}\right) \right] \\
\xi_{1,t} \quad &\equiv \quad \beta \left(\varphi_{1,t} + \varphi_{2,t-1}\right).
\end{aligned}$$

As a_t and $\xi_{1,t}$ are bounded, (8) is equivalent to

$$a_t + (1 - L)\,\xi_{1,t} = 0 \tag{9}$$

in the sense that (8) holds for all $t \geq t_0$ if and only if (9) holds for all $t \geq t_0$.

Applying the linear operator $\beta E_t\left[\left(1 - \beta L^{-1}\right)(\cdot)\right]$ to (5), we obtain

$$E_t\left\{\left(1 - \beta L^{-1}\right)\left[b_t + (1 - L)\,\beta\varphi_{2,t-1}\right]\right\} = \xi_{2,t} \tag{10}$$

for all dates $t \geq t_0$, where the variable b_t and the new multiplier $\xi_{2,t}$ are defined as

$$\begin{aligned} b_t &\equiv \beta\lambda_w\left(\bar{\pi}_t^w - \bar{\pi}_{t-1}\right) \\ \xi_{2,t} &\equiv \beta\left(\xi_p\varphi_{1,t} - \xi_w\varphi_{2,t-1}\right). \end{aligned}$$

Note that the variable $\left(\xi_{2,t} - \xi_p\xi_{1,t}\right)$ satisfies

$$\xi_{2,t} - \xi_p\xi_{1,t} = E_{t-1}\left(\xi_{2,t} - \xi_p\xi_{1,t}\right). \tag{11}$$

Subtracting (10) from (8) yields

$$E_t\left\{\left(1 - \beta L^{-1}\right)\left[c_t + (1 - L)\,\beta\varphi_{1,t}\right]\right\} = -\xi_{2,t}, \tag{12}$$

where

$$c_t \equiv \beta\lambda_p\left(\bar{\pi}_t - \bar{\pi}_{t-1}\right).$$

Then multiplying (12) by ξ_p and subtracting from it ξ_w times (10), one obtains

$$E_t\left\{A(L)\,\xi_{2,t+1}\right\} = d_t \tag{13}$$

for all $t \geq t_0$, where

$$\begin{aligned} A(L) &\equiv (L - \beta)(1 - L) + \left(\xi_p + \xi_w\right)L \qquad (14)\\ d_t &\equiv E_t\left[\left(1 - \beta L^{-1}\right)\left(\xi_w b_t - \xi_p c_t\right)\right]. \end{aligned}$$

Because the quadratic polynomial $A(L)$ satisfies $A(0) = -\beta < 0$, $A(1) = \xi_p + \xi_w > 0$ and $A(+\infty) = -\infty$, it must have two positive real roots, one smaller than 1 and one larger than 1. Factoring $A(L) = -\beta(1 - \mu_1 L)(1 - \mu_2 L)$, where $0 < \mu_1 < 1 < \mu_2$ and $\mu_2 = (\beta\mu_1)^{-1}$, we can rewrite equation (13) equivalently as

$$E_t\left\{(1 - \mu_1 L)\left(1 - \mu_2^{-1}L^{-1}\right)\beta\mu_2\xi_{2,t}\right\} = d_t.$$

Given that $\xi_{2,t}$ and d_t are both bounded variables, and that $\left|\mu_2^{-1}\right| < 1$, the previous equation is equivalent to

$$(1 - \mu_1 L)\,\xi_{2t} = \mu_1 E_t\left[\left(1 - \mu_2^{-1}L^{-1}\right)^{-1} d_t\right].$$

Thus (9) and the previous equation can be written as

$$\begin{aligned} (1-L)\,\xi_{1,t} &= -f_t && (15)\\ (1-\mu_1 L)\,\xi_{2,t} &= g_t && (16) \end{aligned}$$

for all dates $t \geq t_0$, where

$$\begin{aligned} f_t &\equiv \beta\left[\lambda_p\left(\bar{\pi}_t - \bar{\pi}_{t-1}\right) + \lambda_w\left(\bar{\pi}^w_t - \bar{\pi}_{t-1}\right)\right]\\ g_t &\equiv \beta\mu_1 E_t\left\{\left(1-\mu_2^{-1}L^{-1}\right)^{-1}\left(1-\beta L^{-1}\right)\left[\xi_w\lambda_w\left(\bar{\pi}^w_t - \bar{\pi}_{t-1}\right) - \xi_p\lambda_p\left(\bar{\pi}_t - \bar{\pi}_{t-1}\right)\right]\right\} \end{aligned}$$

are functions of target variables.

Note that

$$g_t + \xi_p f_t = E_{t-1}\left(g_t + \xi_p f_t\right), \tag{17}$$

as a consequence of (11), (15), and (16). This is a restriction on the path of target variables at all dates $t \geq t_0$.

Furthermore, using the identities

$$\begin{aligned} \varphi_{1,t} &= \beta^{-1}\left(\xi_p + \xi_w\right)^{-1}\left(\xi_{2,t} + \xi_w\xi_{1,t}\right)\\ \varphi_{2,t-1} &= \beta^{-1}\left(\xi_p + \xi_w\right)^{-1}\left(\xi_p\xi_{1,t} - \xi_{2,t}\right), \end{aligned}$$

we can eliminate $\varphi_{1,t}$ and $\varphi_{2,t-1}$ from (7) and obtain

$$e_t = \left(\xi_p + \xi_w\right)^{-1} E_t\left[\xi_p\omega_p\left(\xi_{2,t+1} + \xi_w\xi_{1,t+1}\right) - \xi_w B\left(L\right)\left(\xi_p\xi_{1,t+2} - \xi_{2,t+2}\right)\right] \tag{18}$$

where

$$\begin{aligned} e_t &\equiv \beta\lambda_x E_t\left[\left(1-\beta\delta L^{-1}\right)\left(v_t - \delta v_{t-1} - \delta s_t - \hat{x}^*\right)\right]\\ &= \beta\lambda_x E_t\left[\left(1-\beta\delta L^{-1}\right)\left(x_{t+2} - \delta x_{t+1} - \hat{x}^*\right)\right]. && (19) \end{aligned}$$

Using (15) and (16) to substitute for $E_t\xi_{i,t+j}$ terms in (18), we obtain

$$\begin{aligned} e_t = {} & \left(\xi_p+\xi_w\right)^{-1} E_t\left\{\xi_p\omega_p\left(g_{t+1} - \xi_w f_{t+1}\right) + \xi_p\omega_p\left(\mu_1\xi_{2,t} + \xi_w\xi_{1,t}\right) + \xi_w B_0\left(\xi_p f_{t+2} + g_{t+2}\right)\right\}\\ & - \left(\xi_p+\xi_w\right)^{-1}\xi_w E_t\left\{B_0\left(\xi_p\xi_{1,t+1} - \mu_1\xi_{2,t+1}\right) + B_1\left(\xi_p\xi_{1,t+1} - \xi_{2,t+1}\right) + B_2\left(\xi_p\xi_{1,t} - \xi_{2,t}\right)\right\}\\ = {} & \left(\xi_p+\xi_w\right)^{-1} E_t\left\{\xi_p\omega_p\left(g_{t+1} - \xi_w f_{t+1}\right) + \xi_p\omega_p\left(\mu_1\xi_{2,t} + \xi_w\xi_{1,t}\right) + \xi_w B_0\left(\xi_p f_{t+2} + g_{t+2}\right)\right\}\\ & - \left(\xi_p+\xi_w\right)^{-1}\xi_w E_t\left\{\left(B_0+B_1\right)\xi_p\left(-f_{t+1}\right) - \left(B_0\mu_1 + B_1\right)g_{t+1} + B_0\left(\xi_p\xi_{1,t} - \mu_1^2\xi_{2,t}\right)\right.\\ & \left. + B_1\left(\xi_p\xi_{1,t} - \mu_1\xi_{2,t}\right) + B_2\left(\xi_p\xi_{1,t} - \xi_{2,t}\right)\right\}\\ = {} & \left(\xi_p+\xi_w\right)^{-1} E_t\left\{\xi_p\omega_p\left(g_{t+1} - \xi_w f_{t+1}\right) + \xi_p\omega_p\left(\mu_1\xi_{2,t} + \xi_w\xi_{1,t}\right) + \xi_w\xi_p\left[B_0 + \left(B_0+B_1\right)L\right]f_{t+2}\right\}\\ & + \left(\xi_p+\xi_w\right)^{-1}\xi_w E_t\left\{\left[B_0 + \left(B_0\mu_1 + B_1\right)L\right]g_{t+2} - \xi_p B\left(1\right)\xi_{1,t} + \mu_1^2 B\left(\mu_1^{-1}\right)\xi_{2,t}\right\}. \end{aligned}$$

We can thus rewrite the previous equation as

$$h_t = -\alpha_1\xi_{1,t} + \alpha_2\xi_{2,t} \tag{20}$$

where

$$\begin{aligned}
h_t &\equiv e_t - E_t\left\{F(L)\, f_{t+2}\right\} - E_t\left\{G(L)\, g_{t+2}\right\} \\
F(L) &\equiv \xi_p \xi_w \left(\xi_p + \xi_w\right)^{-1} \left[B_0 + (B_0 + B_1) L - \omega_p L\right] \equiv F_0 + F_1 L \\
G(L) &\equiv \left(\xi_p + \xi_w\right)^{-1} \left[\xi_w B_0 + \xi_w (B_0 \mu_1 + B_1) L + \xi_p \omega_p L\right] \equiv G_0 + G_1 L
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1 &\equiv \xi_p \xi_w \left(\xi_p + \xi_w\right)^{-1} \left(B(1) - \omega_p\right) \\
\alpha_2 &\equiv \left(\xi_p + \xi_w\right)^{-1} \left(\xi_w \mu_1^2 B\left(\mu_1^{-1}\right) + \xi_p \omega_p \mu_1\right).
\end{aligned}$$

Equation (20) is a restriction that must be satisfied by the projected paths of the target variables at all dates $t \geq t_0$, and that depends only on the multipliers $\xi_{1,t}$ and $\xi_{2,t}$. Quasi-differentiating (20), and using (15), (16) to substitute for the multipliers, we obtain finally

$$(1 - L)(1 - \mu_1 L)\, h_t = \alpha_1 (1 - \mu_1 L)\, f_t + \alpha_2 (1 - L)\, g_t. \tag{21}$$

for every $t \geq t_0 + 2$.

Note that both (17) and (21) must be satisfied by the processes $\{e_t, f_t, g_t\}$ at all dates. Thus (17) and (21) are two criteria that represent optimal policy and that involve only the projected paths of the target variables.

A.1.1 Special case: Flexible wages

To give some intuition about the two target criteria (17) and (21) it may be useful to consider the special case in which wages are flexible ($\xi_w \to +\infty$), as the optimal target criteria are simple to characterize analytically. In this case, we have $\lambda_w = 0$, $\lambda_p = 1$, and the roots of (14) satisfy $\mu_1 \to 0$ and $\mu_2 \to +\infty$. It follows that

$$\begin{aligned}
f_t &= \beta\left(\pi_{t+1} - \pi_t\right) \\
g_t &= 0.
\end{aligned}$$

The short-run optimal target criterion (17) reduces thus to

$$f_t = E_{t-1} f_t$$

or equivalently to

$$\pi_{t+1} = E_{t-1} \pi_{t+1}.$$

This indicates that under optimal policy, the central bank has to make inflation totally predictable two periods in advance. The long-run optimal target criterion (21) reduces in turn to

$$\begin{aligned}
0 &= (1 - L)\, h_t - \alpha_1 f_t \\
&= (1 - L)\left[e_t - E_t\left\{F(L)\, f_{t+2}\right\} - \alpha_1 \beta \pi_{t+1}\right] \\
&= (1 - L)\left[e_t - \beta \xi_p E_t\left\{\left[\varphi \eta \beta L^{-1} - \left(\varphi\left(1 + \beta \eta^2\right) + \omega\right) + \eta \varphi L\right] \pi_{t+2}\right\}\right] \\
&= (1 - L)\left[e_t + \beta \xi_p \varphi \vartheta E_t\left\{\left[\left(1 + \beta \delta^2\right) - \delta L - \beta \delta L^{-1}\right] \pi_{t+2}\right\}\right]
\end{aligned}$$

where we use (**??**). Using (19), we then can write

$$0 = (1 - L)\, E_t \left\{ \left(1 - \beta\delta L^{-1}\right)(1 - \delta L) \left[\beta\xi_p\varphi\vartheta\pi_{t+2} + \beta\lambda_x x_{t+2}\right] \right\}$$

or

$$0 = (1 - L)\, E_t \left\{ \left(1 - \beta\delta L^{-1}\right)(1 - \delta L) \left[\pi_{t+2} + \phi x_{t+2}\right] \right\}$$

where

$$\phi = \frac{\lambda_x}{\xi_p\varphi\vartheta} = \theta_p^{-1},$$

when we use the definition of the weight λ_x. One can simplify this criterion even further by noting that an equivalent target criterion is given by

$$E_t \left\{ \left(1 - \beta\delta L^{-1}\right)(1 - \delta L) \left[\pi_{t+2} + \phi x_{t+2}\right] \right\} = \tilde{\pi}^*, \tag{22}$$

where $\tilde{\pi}^*$ is a constant. As $|\beta\delta| < 1$, a commitment to (22) at all dates $t \geq 0$ is then equivalent to a commitment to

$$E_t \left\{ (1 - \delta L) \left[\pi_{t+2} + \phi x_{t+2}\right] \right\} = \pi^*, \tag{23}$$

at all dates $t \geq 0$, where $\pi^* \equiv (1 - \beta\delta)^{-1}\, \tilde{\pi}^*$. The long-run optimal target criterion can finally be expressed as

$$E_t \left[\pi_{t+2} + \phi x_{t+2}\right] = \pi^* + \delta \left[\pi_{t+1} + \phi x_{t+1}\right].$$

A.2 Alternative Representation of the Optimal Target Criterion

We now describe how the optimal target criteria (17) and (21) derived above can be rewritten as (3.3 XXX) – (3.7 XXX) in the text. As described in the text, we find more convenient to express the target criteria in terms of the real wage rather than wage inflation. Using the wage identity (3) and noting that w_t is determined at date $t-1$, since both π_t and π_t^w are determined at date $t-1$, we have

$$\bar{\pi}_t^w = \bar{\pi}_t + \bar{w}_t - \bar{w}_{t-1}$$

where $\bar{w}_t \equiv E_t w_{t+1} = w_{t+1}$. Using this to substitute for wage inflation, we may rewrite the variables f_t and g_t as

$$\begin{aligned} f_t &= \beta \left[\lambda_p \left(\bar{\pi}_t - \bar{\pi}_{t-1}\right) + \lambda_w \left(\bar{\pi}_t - \bar{\pi}_{t-1}\right) + \lambda_w \left(\bar{w}_t - \bar{w}_{t-1}\right)\right] \\ &= (1 - L)\, \beta \left[\left(\lambda_p + \lambda_w\right) \bar{\pi}_t + \lambda_w \bar{w}_t\right] \end{aligned}$$

and

$$\begin{aligned} g_t &= \beta\mu_1 E_t \left\{ \left(1 - \mu_2^{-1} L^{-1}\right)^{-1} \left(1 - \beta L^{-1}\right) \left[\left(\xi_w\lambda_w - \xi_p\lambda_p\right)\left(\bar{\pi}_t - \bar{\pi}_{t-1}\right) + \xi_w\lambda_w \left(\bar{w}_t - \bar{w}_{t-1}\right)\right] \right\} \\ &= \beta\mu_1 E_t \left\{ \left(1 - \mu_2^{-1} L^{-1}\right)^{-1} \left(1 - \beta L^{-1}\right)(1 - L) \left[\left(\xi_w\lambda_w - \xi_p\lambda_p\right)\bar{\pi}_t + \xi_w\lambda_w \bar{w}_t\right] \right\} \\ &= \beta\mu_1 E_t \left\{ \left[-L + \left(1 + \beta - \mu_2^{-1}\right) + \left(\mu_2^{-1} - \beta\right)\left(1 - \mu_2^{-1}\right) L^{-1} \left(1 - \mu_2^{-1} L^{-1}\right)^{-1}\right] \left[\left(\xi_w\lambda_w - \xi_p\lambda_p\right)\bar{\pi}_t + \xi_w\lambda_w \bar{w}_t\right] \right\} \end{aligned}$$

A.2.1 Short-run target criterion

The short-run target criterion (17) can be rewritten as

$$m_t = E_{t-1} m_t \tag{24}$$

where

$$\begin{aligned} m_t \equiv\ & \beta\mu_1 E_t \left\{ \left[\left(1+\beta-\mu_2^{-1}\right) + \left(\mu_2^{-1}-\beta\right)\left(1-\mu_2^{-1}\right) L^{-1}\left(1-\mu_2^{-1}L^{-1}\right)^{-1} \right] \left[\left(\xi_w\lambda_w - \xi_p\lambda_p\right)\bar{\pi}_t + \xi_w\lambda_w\bar{w}_t \right] \right\} \\ & + \xi_p\beta\left[\left(\lambda_p+\lambda_w\right)\bar{\pi}_t + \lambda_w\bar{w}_t\right]. \end{aligned}$$

Here we note that the terms at date $t-1$ cancel out on both sides of the equation. We can then rewrite m_t as

$$\begin{aligned} m_t =\ & \left[\xi_p\beta\left(\lambda_p+\lambda_w\right) + \beta\mu_1\left(1+\beta-\mu_2^{-1}\right)\left(\xi_w\lambda_w-\xi_p\lambda_p\right)\right]\bar{\pi}_t \\ & + \beta\mu_1\left(\mu_2^{-1}-\beta\right)\left(1-\mu_2^{-1}\right)\left(\xi_w\lambda_w-\xi_p\lambda_p\right)\sum_{k=0}^{\infty}\mu_2^{-k}E_t\bar{\pi}_{t+k+1} \\ & + \left[\xi_p\beta\lambda_w + \beta\mu_1\left(1+\beta-\mu_2^{-1}\right)\xi_w\lambda_w\right]\bar{w}_t + \beta\mu_1\left(\mu_2^{-1}-\beta\right)\left(1-\mu_2^{-1}\right)\xi_w\lambda_w\sum_{k=0}^{\infty}\mu_2^{-k}E_t\bar{w}_{t+k+1} \end{aligned}$$

or as

$$m_t = S_\pi \sum_{k=1}^{\infty}\alpha_k^\pi E_t\pi_{t+k} + S_w\sum_{k=1}^{\infty}\alpha_k^w E_t w_{t+k} \tag{25}$$

where S_π and S_w are the sums of coefficients and α_k^π, α_k^w are the weights defined by

$$\begin{aligned} S_\pi &= \xi_p\beta\left(\lambda_p+\lambda_w\right) + \beta\mu_1\left(1+\beta-\mu_2^{-1}\right)\left(\xi_w\lambda_w-\xi_p\lambda_p\right) + \beta\mu_1\left(\mu_2^{-1}-\beta\right)\left(1-\mu_2^{-1}\right)\left(\xi_w\lambda_w-\xi_p\lambda_p\right)\sum_{k=0}^{\infty}\mu_2^{-k} \\ &= \xi_p\beta\left(\lambda_p+\lambda_w\right) + \beta\mu_1\left(\xi_w\lambda_w-\xi_p\lambda_p\right) \\ \alpha_1^\pi &= \left[\xi_p\beta\left(\lambda_p+\lambda_w\right) + \beta\mu_1\left(1+\beta-\mu_2^{-1}\right)\left(\xi_w\lambda_w-\xi_p\lambda_p\right)\right]/S_\pi \\ \alpha_k^\pi &= \beta\mu_1\left(\mu_2^{-1}-\beta\right)\left(1-\mu_2^{-1}\right)\left(\xi_w\lambda_w-\xi_p\lambda_p\right)\mu_2^{-k+1}/S_\pi, \qquad \text{for } k\geq 2 \end{aligned}$$

where we used $\beta\mu_1 = \mu_2^{-1}$, and

$$\begin{aligned} S_w &= \xi_p\beta\lambda_w + \beta\mu_1\left(1+\beta-\mu_2^{-1}\right)\xi_w\lambda_w + \beta\mu_1\left(\mu_2^{-1}-\beta\right)\left(1-\mu_2^{-1}\right)\xi_w\lambda_w\sum_{k=0}^{\infty}\mu_2^{-k} \\ &= \lambda_w\beta\left(\xi_p+\xi_w\mu_1\right) \\ \alpha_1^w &= \lambda_w\beta\left[\xi_p+\xi_w\mu_1\left(1+\beta-\mu_2^{-1}\right)\right]/S_w \\ \alpha_k^w &= \lambda_w\xi_w\left(\mu_2^{-1}-\beta\right)\left(1-\mu_2^{-1}\right)\mu_2^{-k+1}/S_w, \qquad \text{for } k\geq 2. \end{aligned}$$

Finally, we may rewrite (24) – (25) more compactly as

$$F_t(\pi) + \phi_w\left[F_t(w) - w_t\right] = E_{t-1}\left\{F_t(\pi) + \phi_w\left[F_t(w)-w_t\right]\right\} \tag{26}$$

which corresponds to the target criterion given by (3.3 XXX), (3.5 XXX). The expression $F_t\left(z\right)$ refers to the weighted average of forecasts of the variable z given by

$$F_t\left(z\right) \equiv \sum_{k=1}^{\infty} \alpha_k^z E_t z_{t+k} \tag{27}$$

where the sums $\sum_{k=1}^{\infty} \alpha_k^{\pi} = \sum_{k=1}^{\infty} \alpha_k^{w} = 1$, and where

$$\phi_x = \frac{S_w}{S_\pi} = \frac{\lambda_w \beta\left(\xi_p + \xi_w \mu_1\right)}{\lambda_p \beta \xi_p\left(1-\mu_1\right) + \lambda_w \beta\left(\xi_p + \xi_w \mu_1\right)}$$

lies between 0 and 1.

A.2.2 Long-run target criterion

To express the long-run target criterion (21) as in (3.6 XXX) – (3.7 XXX), we rewrite f_t, g_t, and e_t as follows

$$f_t = E_t\left\{\sum_{k=0}^{\infty} \alpha_k^f q_{t+k-1}\right\}$$

where

$$q_t \equiv \left[\begin{array}{c} \bar{\pi}_t \\ \bar{w}_t \end{array}\right] = E_t\left[\begin{array}{c} \pi_{t+1} \\ w_{t+1} \end{array}\right]$$

and $\alpha_0^f = -\alpha_1^f = -\beta\left[\left(\lambda_p + \lambda_w\right),\ \ \lambda_w\right]$, $\alpha_k^f = 0$ for all $k \geq 2$. Similarly,

$$g_t = E_t\left\{\sum_{k=0}^{\infty} \alpha_k^g q_{t+k-1}\right\}$$

where

$$\begin{aligned} \alpha_0^g &= -\beta\mu_1\left[\left(\xi_w\lambda_w - \xi_p\lambda_p\right),\ \ \xi_w\lambda_w\right] \\ \alpha_1^g &= -\left(1+\beta-\mu_2^{-1}\right)\alpha_0^g \\ \alpha_k^g &= \left(\beta-\mu_2^{-1}\right)\left(1-\mu_2^{-1}\right)\mu_2^{-(k-2)}\alpha_0^g, \qquad \text{for all } k \geq 2. \end{aligned}$$

Next, it is convenient to write

$$E_t\left\{F\left(L\right)f_{t+2} + G\left(L\right)g_{t+2}\right\} = E_t\left\{\sum_{k=0}^{\infty} \alpha_k^h q_{t+k}\right\}$$

where

$$\begin{aligned} \alpha_0^h &= F_1\alpha_0^f + G_1\alpha_0^g \\ \alpha_k^h &= F_0\alpha_{k-1}^f + F_1\alpha_k^f + G_0\alpha_{k-1}^g + G_1\alpha_k^g, \qquad \text{for all } k \geq 1. \end{aligned}$$

In addition, the variable e_t defined in (19) may be expressed as

$$e_t = \beta\lambda_x\left[S_x F_t\left(x\right) - \left(1-\beta\delta\right)\hat{x}^*\right] \tag{28}$$

where $F_t(x)$ is again of the form (27) and the weights are given by

$$\begin{aligned} S_x &= 1+\beta\delta^2-\delta-\beta\delta \\ \alpha_1^x &= -\delta/S_x \\ \alpha_2^x &= \left(1+\beta\delta^2\right)/S_x \\ \alpha_3^x &= -\beta\delta/S_x \\ \alpha_k^x &= 0, \qquad \text{for all } k\geq 4. \end{aligned}$$

Using this, we can rewrite the target criterion (21) as

$$\begin{aligned} E_t\left\{\sum_{k=0}^{\infty}\alpha_k^0 q_{t+k}\right\}-\beta\lambda_x S_x F_t(x) &= E_{t-1}\left\{\sum_{k=0}^{\infty}\alpha_k^1 q_{t+k-1}\right\}-\beta\lambda_x S_x\left(1+\mu_1\right)F_{t-1}(x) \\ &\quad +E_{t-2}\left\{\sum_{k=0}^{\infty}\alpha_k^2 q_{t+k-2}\right\}+\beta\lambda_x S_x\mu_1 F_{t-2}(x) \end{aligned}$$

where

$$\begin{aligned} \alpha_k^0 &= \alpha_k^h+\alpha_1\alpha_{k+1}^f+\alpha_2\alpha_{k+1}^g, \qquad \text{for all } k\geq 0 \\ \alpha_0^1 &= -\left(\alpha_1\alpha_0^f+\alpha_2\alpha_0^g\right)+\left(1+\mu_1\right)\alpha_0^h+\alpha_1\mu_1\alpha_1^f+\alpha_2\alpha_1^g, \\ \alpha_k^1 &= \left(1+\mu_1\right)\alpha_k^h+\alpha_1\mu_1\alpha_{k+1}^f+\alpha_2\alpha_{k+1}^g, \qquad \text{for all } k\geq 1 \\ \alpha_0^2 &= \alpha_1\mu_1\alpha_0^f+\alpha_2\alpha_0^g-\mu_1\alpha_0^h \\ \alpha_k^2 &= -\mu_1\alpha_k^h, \qquad \text{for all } k\geq 1. \end{aligned}$$

Premultiplying each of the infinite sums by the sum of coefficients

$$\left[S_{\pi j},S_{wj}\right]=E_t\left\{\sum_{k=0}^{\infty}\alpha_k^j\right\}$$

for $j=0,1,2$, we can equivalently the rewrite this as

$$\begin{aligned} S_{\pi 0}F_t^*(\pi)+S_{w0}F_t^*(w)-\beta\lambda_x S_x F_t(x) &= S_{\pi 1}F_{t-1}^1(\pi)+S_{w1}F_{t-1}^1(w)-\beta\lambda_x S_x\left(1+\mu_1\right)F_{t-1}(x) \\ &\quad +S_{\pi 2}F_{t-2}^2(\pi)+S_{w2}F_{t-2}^2(w)+\beta\lambda_x S_x\mu_1 F_{t-2}(x) \end{aligned}$$

where the $F_t(z)$ are again weighted average of forecasts are different horizons with relative weights that sum to one. Noting that $S_{\pi 1}=S_{\pi 0}-S_{\pi 2}$ and $S_{w1}=S_{w0}-S_{w2}$, we can rewrite the target criterion as

$$\begin{aligned} F_t^*(\pi)+\phi_w^*F_t^*(w)+\phi_x^*F_t(x) &= F_{t-1}^1(\pi)+\phi_w^*F_{t-1}^1(w)+\phi_x^*F_{t-1}(x)+\theta_\pi^*\left[F_{t-1}^1(\pi)-F_{t-2}^2(\pi)\right] \\ &\quad +\theta_w^*\left[F_{t-1}^1(w)-F_{t-2}^2(w)\right]+\theta_x^*\left[F_{t-1}(x)-F_{t-2}^2(x)\right] \qquad (29) \end{aligned}$$

where

$$\begin{aligned} \phi_w^* &= \frac{S_{w0}}{S_{\pi 0}}, \qquad \phi_x^*=-\frac{\beta\lambda_x S_x}{S_{\pi 0}} \\ \theta_\pi^* &= -\frac{S_{\pi 2}}{S_{\pi 0}}, \qquad \theta_w^*=-\frac{S_{w2}}{S_{\pi 0}}, \qquad \theta_x^*=\phi_x^*\mu_1. \end{aligned}$$

Subtracting $\phi_w^*w_t+\phi_x^*x_t$ on both sides yields finally (3.6 XXX) – (3.7 XXX).

A.3 Historical time series for the target criterion

This section describes the calculations underlying section 3.3 of the text in which we assess to what extent, under actual policy, the evolution of projections of inflation, the real wage and the output gap have satisfied the optimal target criteria. To perform the projections of future variables, we use the structural VAR (2.2 XXX) which we can rewrite in terms of deviations from a long-run steady-state as

$$\hat{Z}_t = B\hat{Z}_{t-1} + U\bar{e}_t$$

where $\hat{Z}_t \equiv \bar{Z}_t - Z^{lr}$ and $B = T^{-1}A$, $U = T^{-1}$. The vector $\bar{Z}_t$ is given by

$$\bar{Z}_t = \left[i_t, \hat{w}_{t+1}, \pi_{t+1}, \hat{Y}_{t+1}, i_{t-1}, \hat{w}_t, \pi_t, \hat{Y}_t, i_{t-2}, \hat{w}_{t-1}, \pi_{t-1}, \hat{Y}_{t-1}\right]'.$$

and its long-run value satisfies $Z^{lr} = (I - B)^{-1} T^{-1} a$. Because we assume that the errors $\bar{e}_t$ are unforecastable, the VAR has the property that $E_t\hat{Z}_{t+k} = B^k\hat{Z}_t$ for all $k > 0$.

Using this, we can compute for each date t the weighted average of future inflation forecasts as follows

$$\begin{aligned} F_t(\pi) &= \sum_{k=1}^{\infty} \alpha_k^{\pi} E_t \pi_{t+k} = \sum_{k=1}^{\infty} \alpha_k^{\pi} \tilde{P} E_t \bar{Z}_{t+k-1} \\ &= \pi^{lr} + P\hat{Z}_t, \end{aligned}$$

where $\tilde{P}$ is a (1×12) vector with a 1 in the third element and zeros elsewhere, $\pi^{lr} \equiv \tilde{P}Z^{lr}$, and $P \equiv \tilde{P}\sum_{k=1}^{\infty} \alpha_k^{\pi} B^{k-1}$. Similarly, we can compute for each date t the weighted average of real wage forecasts

$$F_t(w) = \sum_{k=1}^{\infty} \alpha_k^{w} E_t w_{t+k} = W\hat{Z}_t,$$

where $W \equiv \tilde{W}\sum_{k=1}^{\infty} \alpha_k^{w} B^{k-1}$ and $\tilde{W}$ is a (1×12) vector with a 1 in the second element and zeros elsewhere. (Note that the long-run value of the variable $\hat{w}$, i.e., the percent deviation in the real wage from its trend is zero).

A.3.1 Short-run target criterion

A historical time series for the adjusted inflation projection (3.3 XXX) is obtained by computing for each date t:

$$F_t(\pi) + \phi_w \left[F_t(w) - w_t\right] = \left(\pi^{lr} + P\hat{Z}_t\right) + \phi_w\left(W\hat{Z}_t - \tilde{W}\hat{Z}_{t-1}\right)$$

A historical time series for the optimal target (3.5 XXX) is then obtained by computing for each date t:

$$E_{t-1}\left\{F_t(\pi) + \phi_w\left[F_t(w) - w_t\right]\right\} = \left(\pi^{lr} + PB\hat{Z}_{t-1}\right) + \phi_w\left(WB - \tilde{W}\right)\hat{Z}_{t-1}.$$

A.3.2 Output gap projections

In addition to inflation projections and real wage projections described above, the long-run target criterion (3.6 XXX) – (3.7 XXX) involves also projections of the output gap. This raises some difficulties that we address in this and the next subsections.

Let us first consider the simple case in which the natural rate of output displays only negligible fluctuations. In this case, the output gap considered in the target criterion (3.6 XXX) – (3.7 XXX) corresponds to the deviation of (log) real output from a linear trend (as is the case in Figures 12 and 13 of the text), i.e., to the time series $\hat{Y}_t$ used in the VAR. The weighted average of future output gap forecasts with the weights used in (28) is then simply obtained by computing

$$F_t\left(\hat{Y}\right) = E_t \sum_{k=1}^{\infty} \alpha_k^x \hat{Y}_{t+k} = E_t R \bar{Z}_{t+2} = RB^2 \hat{Z}_t$$

where

$$R = \left[0, 0, 0, \alpha_3^x, 0, 0, 0, \alpha_2^x, 0, 0, 0, \alpha_1^x\right].$$

Again, we note that the long-run value of the variable $\hat{Y}_t$ is zero and that $\alpha_k^x = 0$ for all $k \geq 4$.

We now turn to the alternative case in which fluctuations in the natural rate of output are recovered from the residuals to the estimated equations of the model. First, we note that the weighted average of projection of future output gaps relevant for the target criterion (3.6 XXX) – (3.7 XXX), i.e., with the weights used in (28) satisfies

$$\begin{aligned} F_t\left(x\right) &= E_t \sum_{k=1}^{\infty} \alpha_k^x x_{t+k} = S_x^{-1} E_t \left[-\delta x_{+1} + \left(1 + \beta\delta^2\right) x_{t+2} - \beta\delta x_{t+3}\right] \\ &= S_x^{-1} E_t \left\{\left(1 - \beta\delta L^{-1}\right)\left(x_{t+2} - \delta x_{t+1}\right)\right\}. \end{aligned} \tag{30}$$

Second, we multiply the price inflation equation (2.14 XXX) by ξ_w and add it to the wage inflation equation (2.11 XXX) multiplied by ξ_p to obtain

$$\begin{aligned} & \xi_w \xi_p E_{t-1}\left\{\left[\omega_p + \omega_w + \varphi\left(1 + \beta\eta^2\right) - \varphi\eta L - \varphi\beta\eta L^{-1}\right] x_t\right\} \\ = \; & \left[\xi_w + \beta\left(\xi_w + \xi_p\right)\gamma_p\right]\pi_t - \left(\xi_w\gamma_p + \xi_p\gamma_w\right)\pi_{t-1} - \beta\xi_w E_{t-1}\pi_{t+1} + \xi_p\pi_t^w - \beta\xi_p E_{t-1}\pi_{t+1}^w + \xi_w\xi_p E_{t-1}\mu_t \end{aligned}$$

It is convenient to note that the left-hand-side is in fact equal to

$$\xi_w \xi_p \vartheta\varphi E_{t-1}\left\{\left(1 - \beta\delta L^{-1}\right)\left(x_t - \delta x_{t-1}\right)\right\}$$

where $0 \leq \delta < \eta$ is the same value as the one entering the policymaker's objective function, and where $\vartheta \equiv \eta/\delta > 1$. Next, using the wage identity (1.31 XXX) to substitute for π_t^w, we obtain

$$\begin{aligned} \xi_w \xi_p \vartheta\varphi E_{t-1}\left\{\left(1 - \beta\delta L^{-1}\right)\left(x_t - \delta x_{t-1}\right)\right\} &= \left(\xi_w + \xi_p\right)\left(1 + \beta\gamma_p\right)\pi_t - \left(\xi_w\gamma_p + \xi_p\gamma_w\right)\pi_{t-1} - \beta\left(\xi_w + \xi_p\right)E_{t-1}\pi_{t+1} \\ & \quad -\xi_p w_{t-1} + \xi_p\left(1 + \beta\right)w_t - \beta\xi_p E_{t-1}w_{t+1} + \xi_w\xi_p E_{t-1}\mu_t. \end{aligned} \tag{31}$$

Thus, by combining (30), (31) and noting that $E_t\mu_{t+2} = 0$, we obtain a historical time series for projections of future output gaps

$$\begin{aligned} F_t\left(x\right) &= S_x^{-1} Q E_t \bar{Z}_{t+2} \\ &= S_x^{-1} Q B^2 \hat{Z}_t. \end{aligned}$$

where

$$Q=\left[0,-\frac{\beta}{\xi_w\vartheta\varphi},-\frac{\beta\left(\xi_w+\xi_p\right)}{\xi_w\xi_p\vartheta\varphi},0,0,\frac{1+\beta}{\xi_w\vartheta\varphi},\frac{\left(\xi_w+\xi_p\right)\left(1+\beta\gamma_p\right)}{\xi_w\xi_p\vartheta\varphi},0,0,-\frac{1}{\xi_w\vartheta\varphi},-\frac{\left(\xi_w\gamma_p+\xi_p\gamma_w\right)}{\xi_w\xi_p\vartheta\varphi},0\right].$$

Again, it turns out that the constant QB^2Z^{lr} is equal to 0.

A.3.3 A time series for the output gap

The remaining difficulty lies in finding a historical time series for the output gap x_t. For this, we need to express the variable $E_{t-1}\mu_t$ defined in (2.13 XX) in terms of observables. Log-linearizing the optimal condition for consumption (2.4 XXX), replacing $\hat{C}_t \equiv \log\left(C_t/\bar{C}\right)$ with $\bar{Y}/\bar{C}\left(\hat{Y}_t-\hat{G}_t\right)$ where $\hat{Y}_t \equiv \log\left(Y_t/\bar{Y}\right)$ and $\hat{G}_t \equiv \left(G_t-\bar{G}\right)/\bar{Y}$, implies

$$E_{t-2}\hat{\lambda}_t = -\varphi E_{t-2}\left(\tilde{Y}_t-\tilde{g}_t\right) \tag{32}$$

where

$$\begin{aligned}
\tilde{Y}_t &\equiv \left(\hat{Y}_t-\eta\hat{Y}_{t-1}\right)-\beta\eta\left(E_t\hat{Y}_{t+1}-\eta\hat{Y}_t\right)\\
\tilde{g}_t &\equiv \left(g_t-\eta\hat{G}_{t-1}\right)-\beta\eta\left(E_tg_{t+1}-\eta\hat{G}_t\right)\\
g_t &\equiv \hat{G}_t-E_{t-2}\left(\frac{u_{c\xi}}{u_{cc}\bar{Y}}\xi_t\right).
\end{aligned}$$

and $\hat{G}_t$ and g_t are determined at date $t-1$. In addition, integrating forward the Euler equation (2.6 XXX) yields

$$\hat{\lambda}_t = \hat{\lambda}_\infty + \sum_{k=0}^{\infty} E_t\left(\hat{\imath}_{t+k}-\pi_{t+k+1}\right) \tag{33}$$

where $\hat{\lambda}_\infty \equiv \lim_{T\to\infty} E_t\hat{\lambda}_T$. From (32) we know however that $\hat{\lambda}_\infty = 0$, assuming that $\lim_{T\to\infty} E_t\tilde{g}_T = 0$. In addition, the terms $\left(\hat{\imath}_{t+k}-\pi_{t+k+1}\right)$ have by construction a mean equal to zero. A historical time series for $\hat{\lambda}_t$ under actual policy is thus obtained by computing

$$\hat{\lambda}_t = \sum_{k=0}^{\infty} E_tN\hat{Z}_{t+k} = N\left(I-B\right)^{-1}\hat{Z}_t,$$

where

$$N = [1,0,-1,0,0,0,0,0,0,0,0,0].$$

Using (32) and (33) we can then write

$$\begin{aligned}
E_{t-2}\tilde{g}_t &= E_{t-2}\left(\tilde{Y}_t+\varphi^{-1}\hat{\lambda}_t\right)\\
&= E_{t-2}\left[K\hat{Z}_{t-1}+\varphi^{-1}N\left(I-B\right)^{-1}\hat{Z}_t\right]\\
&= H\hat{Z}_{t-2},
\end{aligned}$$

where

$$\begin{aligned} H &= KB + \varphi^{-1} N \left(I - B\right)^{-1} B^2 \\ K &= \left[0, 0, 0, -\beta\eta, 0, 0, 0, \left(1 + \beta\eta^2\right), 0, 0, 0, -\eta\right] B, \end{aligned}$$

so that K satisfies $E_{t-1}\tilde{Y}_t = K\hat{Z}_{t-1}$. Next, it follows from the definition of $\tilde{g}_t$ that

$$\begin{aligned} E_{t-1}\left(g_t - \eta\hat{G}_{t-1}\right) &= \tilde{g}_t + \beta\eta E_{t-1}\left(g_{t+1} - \eta\hat{G}_t\right) \\ &= E_{t-1}\tilde{g}_t + \beta\eta \sum_{k=0}^{\infty} \left(\beta\eta\right)^k E_{t-1}\tilde{g}_{t+k+1} \\ &= E_{t-1}\tilde{g}_t + \beta\eta \sum_{k=0}^{\infty} \left(\beta\eta\right)^k H B^k \hat{Z}_{t-1}. \end{aligned}$$

This implies that

$$E_{t-1}\tilde{g}_t = E_{t-1}\left(g_t - \eta\hat{G}_{t-1}\right) - \beta\eta F \hat{Z}_{t-1}$$

where

$$F = H\left(I - \beta\eta B\right)^{-1},$$

and thus that

$$\begin{aligned} E_{t-1}\tilde{g}_t - E_{t-2}\tilde{g}_t &= E_{t-1}g_t - E_{t-2}g_t - \eta\left(\hat{G}_{t-1} - E_{t-2}\hat{G}_{t-1}\right) - \beta\eta F\left(\hat{Z}_{t-1} - B\hat{Z}_{t-2}\right) \\ &= E_{t-2}\left(\hat{Y}_t - \hat{G}_t\right) - E_{t-1}\left(\hat{Y}_t - \hat{G}_t\right) + E_{t-1}\hat{Y}_t - E_{t-2}\hat{Y}_t - \beta\eta F\left(\hat{Z}_{t-1} - B\hat{Z}_{t-2}\right) \\ &= \left(G - \beta\eta F\right)\left(\hat{Z}_{t-1} - B\hat{Z}_{t-2}\right) \end{aligned}$$

where G is a (1×12) vector with a 1 in the fourth element and zeros elsewhere, so that it satisfies, $\hat{Y}_t = E_{t-1}\hat{Y}_t = G\hat{Z}_{t-1}$. Hence we can write

$$\begin{aligned} E_{t-1}\tilde{g}_t &= E_{t-2}\tilde{g}_t + \left(E_{t-1}\tilde{g}_t - E_{t-2}\tilde{g}_t\right) \\ &= H\hat{Z}_{t-2} + \left(G - \beta\eta F\right)\left(\hat{Z}_{t-1} - B\hat{Z}_{t-2}\right) \end{aligned}$$

and

$$\begin{aligned} E_{t-1}\left(\tilde{Y}_t - \tilde{g}_t\right) &= K\hat{Z}_{t-1} - H\hat{Z}_{t-2} - \left(G - \beta\eta F\right)\left(\hat{Z}_{t-1} - B\hat{Z}_{t-2}\right) \\ E_{t-1}\left(\tilde{Y}_{t+1} - \tilde{g}_{t+1}\right) &= \left(KB - H\right)\hat{Z}_{t-1}. \end{aligned}$$

Combining the above results with (2.13 XXX) we can express $E_{t-1}\mu_t$ is terms of observables as

$$\begin{aligned} E_{t-1}\mu_t &= E_{t-1}\left(\hat{\imath}_t - \pi_{t+1}\right) + \varphi E_{t-1}\left[\left(\tilde{Y}_t - \tilde{g}_t\right) - \left(\tilde{Y}_{t+1} - \tilde{g}_{t+1}\right)\right] \\ &= NB\hat{Z}_{t-1} + \varphi\left[K\hat{Z}_{t-1} - H\hat{Z}_{t-2} - \left(G - \beta\eta F\right)\left(\hat{Z}_{t-1} - B\hat{Z}_{t-2}\right) - \left(KB - H\right)\hat{Z}_{t-1}\right] \\ &= \left[NB + \varphi\left(K - G + \beta\eta F - KB + H\right)\right]\hat{Z}_{t-1} + \varphi\left[\left(G - \beta\eta F\right)B - H\right]\hat{Z}_{t-2}. \end{aligned}$$

Using (31), we can thus write

$$\begin{aligned} E_{t-1}\left\{\left(1-\beta\delta L^{-1}\right)(x_t-\delta x_{t-1})\right\} &= QE_{t-1}\hat{Z}_t+(\vartheta\varphi)^{-1}E_{t-1}\mu_t \\ &= C\hat{Z}_{t-1}+D\hat{Z}_{t-2} \end{aligned}$$

where

$$\begin{aligned} C &= QB+\vartheta^{-1}\left(\varphi^{-1}NB+K-G+\beta\eta F-KB+H\right) \\ D &= \vartheta^{-1}\left[(G-\beta\eta F)B-H\right]. \end{aligned}$$

Integrating forward the above equation, we have

$$\begin{aligned} E_{t-1}(x_t-\delta x_{t-1}) &= C\hat{Z}_{t-1}+D\hat{Z}_{t-2}+\beta\delta E_{t-1}(x_{t+1}-\delta x_t) \\ &= \sum_{k=0}^{\infty}(\beta\delta)^j E_{t-1}\left(C\hat{Z}_{t+k-1}+D\hat{Z}_{t+k-2}\right) \\ &= \tilde{C}\hat{Z}_{t-1}+D\hat{Z}_{t-2} \end{aligned}$$

where

$$\tilde{C}\equiv(C+\beta\delta D)(I-\beta\delta B)^{-1}.$$

We can then obtain a historical series for $x_t=E_{t-1}x_t$ by computing recursively

$$x_t=\tilde{C}\hat{Z}_{t-1}+D\hat{Z}_{t-2}+\delta x_{t-1}, \tag{34}$$

setting the initial output gap x_0 to a particular value which we assume to be 0.

A.3.4 Long-run target criterion

A historical time series for the projections (3.6 XXX) is obtained by computing for each date t:

$$\begin{aligned} F_t^*(\pi)+\phi_w^*\left[F_t^*(w)-w_t\right]+\phi_x^*\left[F_t(x)-x_t\right] &= \left(\pi^{lr}+P^*\hat{Z}_t\right)+\phi_w^*\left(W^*\hat{Z}_t-\tilde{W}\hat{Z}_{t-1}\right) \\ &\quad +\phi_x^*\left(S_x^{-1}QB^2\hat{Z}_t-x_t\right) \end{aligned}$$

where the weights $\alpha_k^{\pi *}$ and α_k^{w*} are those underlying (29), $P^*\equiv\tilde{P}\sum_{k=1}^{\infty}\alpha_k^{\pi *}B^{k-1}$, $W^*\equiv\tilde{W}\sum_{k=1}^{\infty}\alpha_k^{w*}B^{k-1}$, and where the historical series for x_t is determined recursively using (34). Similarly, we can compute a historical time series for the optimal target (3.7 XXX)

$$\begin{aligned} \pi_t^* &= (1+\theta_\pi^*)F_{t-1}^1(\pi)-\theta_\pi^*F_{t-2}^2(\pi)-\phi_w^*w_t+(\phi_w^*+\theta_w^*)F_{t-1}^1(w)-\theta_w^*F_{t-2}^2(w) \\ &\quad -\phi_x^*x_t+(\phi_x^*+\theta_x^*)F_{t-1}(x)-\theta_x^*F_{t-2}(x) \\ &= \pi^{lr}+(1+\theta_\pi^*)P^1\hat{Z}_{t-1}-\theta_\pi^*P^2\hat{Z}_{t-2} \\ &\quad -\phi_w^*\tilde{W}\hat{Z}_{t-1}+(\phi_w^*+\theta_w^*)W^1\hat{Z}_{t-1}-\theta_w^*W^2\hat{Z}_{t-2} \\ &\quad -\phi_x^*x_t+(\phi_x^*+\theta_x^*)S_x^{-1}QB^2\hat{Z}_{t-1}-\theta_x^*S_x^{-1}QB^2\hat{Z}_{t-2}, \end{aligned}$$

where the weights $\alpha_k^{\pi 1}$, $\alpha_k^{\pi 2}$ and α_k^{w1}, α_k^{w2} are those underlying the weighted sums on the right-hand side of (29) and $P^j \equiv \tilde{P} \sum_{k=1}^{\infty} \alpha_k^{\pi j} B^{k-1}$, $W^j \equiv \tilde{W} \sum_{k=1}^{\infty} \alpha_k^{wj} B^{k-1}$ for $j = 1, 2$. Note that the weighted averages $F_{t-j}(x)$ remain unchanged for all j's.

In the case that the natural rate of output displays only negligible fluctuations so that the output gap considered is $\hat{Y}_t$, the contribution to the projections due to output gap fluctuations is given by

$$\phi_x^* \left[F_t \left(\hat{Y} \right) - \hat{Y}_t \right] = \phi_x^* \left[R B^2 \hat{Z}_t - G \hat{Z}_{t-1} \right].$$

The contribution to the optimal target due to output gap fluctuations is then given by

$$-\phi_x^* \hat{Y}_t + \left(\phi_x^* + \theta_x^* \right) F_{t-1} \left(\hat{Y} \right) - \theta_x^* F_{t-2} \left(\hat{Y} \right) = \left[-\phi_x^* G + \left(\phi_x^* + \theta_x^* \right) R B^2 \right] \hat{Z}_{t-1} - \theta_x^* R B^2 \hat{Z}_{t-2}.$$