Inflation Stabilization and Welfare: The Case of a Distorted Steady State*

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Abstract

This paper considers the appropriate stabilization objectives for monetary policy in a microfounded model with staggered price-setting. Rotemberg and Woodford (1997) and Woodford (2002) have shown that under certain conditions, a local approximation to the expected utility of the representative household in a model of this kind is related inversely to the expected discounted value of a conventional quadratic loss function, in which each period's loss is a weighted average of squared deviations of inflation and an output gap measure from their optimal values (zero). However, those derivations rely on an assumption of the existence of an output or employment subsidy that offsets the distortion due to the market power of monopolistically-competitive price-setters, so that the steady state under a zero-inflation policy involves an efficient level of output. Here we show how to dispense with this unappealing assumption, so that a valid linear-quadratic approximation to the optimal policy problem is possible even when the steady state is distorted to an arbitrary extent (allowing for tax distortions as well as market power), and when, as a consequence, it is necessary to take account of the effects of stabilization policy on the average level of output.

We again obtain a welfare-theoretic loss function that involves both inflation and an appropriately defined output gap, though the degree of distortion of the steady state affects both the weights on the two stabilization objectives and the definition of the welfare-relevant output gap. In the light of these results, we reconsider the conditions under which complete price stability is optimal, and find that they are more restrictive in the case of a distorted steady state. We also consider the conditions under which pure randomization of monetary policy can be welfare-improving, and find that this is possible in the case of a sufficiently distorted steady state, though the parameter values required are probably not empirically realistic.

According to a common conception of the goals of monetary stabilization policy, it is appropriate for the monetary authority to aim to stabilize both some measure of inflation and some measure of real activity relative to potential. This is often represented by supposing that the authority should seek to minimize the expected discounted value of a quadratic loss function, in which each period's loss consists of a weighted average of the square of the inflation rate and the square of the "output gap." It is furthermore typically argued that the two stabilization goals are not fully compatible with one another, owing to the occurrence of "cost-push shocks," which prevent a zero output gap from being consistent with zero inflation. The problem of finding an optimal tradeoff between the two goals is then non-trivial.[1]

This familiar framework raises a number of questions, however. Most obvious is the question of how to define the "output gap" that policy should seek to stabilize. Should this be understood to mean output relative to some smooth trend, or should the target output level vary in response to real disturbances of various sorts? A closely related question is the definition of the "cost-push shocks": how should these be identified in practice, and how often do disturbances of this kind actually occur? And even supposing that we know how to identify the output gap and the cost-push disturbances, what relative weight should be placed on output-gap stabilization as opposed to inflation stabilization?

Here we propose to answer such questions on welfare-theoretic grounds. The ultimate aim of monetary policy, in our view, should be the maximization of the expected utility of households. We show, however (following a method introduced by Rotemberg and Woodford, 1997, and further expounded in Woodford, 2002; 2003b, chap. 6), that it is possible to derive a quadratic approximation to the expected utility of the representative household that takes the form of a discounted quadratic loss function of the kind assumed in the traditional literature on monetary policy evaluation. In the case that the exogenous disturbances are sufficiently small in amplitude, the best policy (in terms of expected utility) will also be the one that minimizes the discounted quadratic loss function. We thus obtain precise answers to the question of what terms should appear in a quadratic loss function, and with which relative weights, that depend on the specification of one's model of the monetary transmission mechanism.[2]

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[1]See, *e.g.*, Clarida *et al.* (1999) and Walsh (2003, chaps. 8, 11) for a number of analyses in this vein.

[2]For examples of the way in which alternative model specifications lead to alternative welfare-theoretic loss functions, see Woodford (2003b, chap. 6) and Giannoni and Woodford (2005).

An important limitation of the method introduced by Rotemberg and Woodford (1997) is that it requires that the zero-inflation steady state of one's model involve an efficient level of output.[3] (They imagine a model in which this is true by assuming the existence of an output subsidy that offsets the distortion resulting from the market power of monopolistically competitive suppliers, though this is obviously not literally true in actual economies.) For if one were instead to consider the more realistic case of an economy in which steady-state output is inefficiently low, one would find that expected utility would depend on the expected level of output. An estimate of expected utility that is accurate to second order would then require a solution for output (or at any rate, for the expected discounted level of output) that is accurate to second order in the amplitude of the exogenous disturbances. A log-linear approximation to the structural equations of one's model will then not suffice to allow one to determine the evolution of output under one policy or another to a sufficient degree of accuracy. As a consequence, a linear-quadratic methodology — in which a linear policy rule is derived so as to minimize a quadratic approximation to the true welfare objective subject to linear constraints that are first-order approximations to the true structural equations — will not generally yield a correct linear approximation to the optimal policy rule.[4]

Here we show how the method of Rotemberg and Woodford can be extended to deal with the case in which the steady-state level of output is inefficient (owing to the existence of distorting taxes on sales revenues or labor income, in addition to the distortions created by market power). Our approach involves computation of a second-order approximation to the model structural relations (specifically, to the aggregate-supply relation in the present application), and using this to solve for the expected discounted value of output as a function of purely quadratic terms. This solution can then be used to substitute for the terms proportional to expected discounted output in the quadratic approximation to expected utility. In this way, we obtain an approximation to expected utility — that holds regardless of the policy contemplated (as long as it involves inflation that is not too extreme) — and that is

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[3]Strictly speaking, it is not essential to the method that zero be the inflation rate that leads to the efficient level of output; it is only necessary that there be *some* such steady state, and that the policies that one intends to compare all be close enough to being consistent with that steady state.

[4]See Woodford (2003b, chap. 6) and Benigno and Woodford (2005) for discussion of the conditions required for validity of an LQ approach.

purely quadratic, in the sense of lacking any linear terms. This alternative quadratic loss function can then be evaluated to second order using an approximate solution for the endogenous variables of one's model that is accurate only to first order. One is then able to compute a linear approximation to optimal policy using a simple linear-quadratic methodology.

Our proposal to substitute purely quadratic terms for the discounted linear terms in the Taylor approximation to expected utility builds upon an idea of Sutherland (2002), who showed how it was possible to take account of the effects of macroeconomic volatility on the average levels of variables in welfare calculations for a model with Calvo pricing like the baseline model considered here. Sutherland's crucial insight was that it is not necessary to compute a complete second-order solution for the evolution of the endogenous variables under each of the policies that one wishes to consider in order to evaluate the discounted linear terms needed for the welfare calculation. Sutherland's approach, however, still requires that one restrict attention to a particular parametric family of policy rules before computing the second-order approximations that are used to substitute for the discounted linear terms in the welfare criterion. Instead, we show that one can substitute out the linear terms using *only* a second-order approximation to the structural equations; one thus obtains a welfare criterion that applies to arbitrary policies.[5]

An alternative way of attaining a welfare measure that is accurate to second order even in the case of a distorted steady state, that has recently become popular, is to solve for a second-order approximation to the complete evolution of the endogenous variables under any given policy rule, and then use this solution to evaluate a quadratic approximation to expected utility (*e.g.*, Kim *et al.*, 2002). However, the requirement that a system of quadratic expectational difference equations be solved for each policy rule that is contemplated is much more computationally demanding than the implementation of our LQ methodology. For we are required to consider the second-order approximation to our structural equations only *once* — when deriving the appropriate quadratic loss function, a calculation undertaken in this paper — after which the evaluation of individual policies requires only that one solve a system of linear equations. In addition, the method illustrated by Kim *et al.* requires that

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[5]It might seem fortuitous that we are able to do this in the present case, but Benigno and Woodford (2005) shows that substitutions of this kind can be used quite generally to obtain a purely quadratic loss function.

one restrict one's attention to a particular parametric family of policy rules, since the system of equations that is solved to second order must include a specification of the policy rule. Our method, by contrast, allows us to determine what variables it is desirable for policy to depend on without having to prejudge that issue.

Yet another approach that allows a correct calculation of a linear approximation to the optimal policy rule even in the case of a distorted steady state is to compute first-order conditions that characterize optimal policy in the exact model (*i.e.,* without approximating either the welfare measure or the structural equations), and then log-linearize these optimality conditions in order to obtain an approximate characterization of optimal policy (*e.g.,* King and Wolman, 1999; Khan *et al.,* 2003). A disadvantage of this approach is that it is *only* suitable for computing the optimal policy; as we discuss in section 4, our quadratic approximate welfare measure also yields a correct ranking of alternative sub-optimal policy rules, as long as disturbances are small enough, and the policies under comparison all involve low inflation. Furthermore, our LQ approach makes it straightforward to consider whether the *second-order conditions* for a policy to be a local optimum are satisfied, and not just the first-order conditions that are typically considered in the literature on "Ramsey policy", as we show in section 3.1 below. Under conditions where the second-order conditions are satisfied, our approach and the one used by Khan *et al.* yield identical approximate linear characterizations of optimal policy; but we believe that the LQ approach provides useful insight into the aspects of the policy problem that are responsible for the conclusions obtained. We illustrate this in sections 3.2 and 3.3 by providing an analytical derivation of results with the same qualitative features as the numerical results reported by Khan *et al.* for a related model.

1   Monetary Stabilization Policy: Welfare-Theoretic Foundations

Here we describe our assumptions about the economic environment and pose the optimization problem that a monetary stabilization policy is intended to solve. The approximation method that we use to characterize the solution to this problem is then presented in the following section. Further details of the derivation of the structural equations of our model of nominal price rigidity can be found in Woodford (2003b,

chapter 3).

1.1 Objective and Constraints

The goal of policy is assumed to be the maximization of the level of expected utility of a representative household. In our model, each household seeks to maximize

$$U_{t_0} \equiv E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[\tilde{u}(C_t; \xi_t) - \int_0^1 \tilde{v}(H_t(j); \xi_t) dj \right], \tag{1.1}$$

where C_t is a Dixit-Stiglitz aggregate of consumption of each of a continuum of differentiated goods,

$$C_t \equiv \left[\int_0^1 c_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}, \tag{1.2}$$

with an elasticity of substitution equal to $\theta > 1$, and $H_t(j)$ is the quantity supplied of labor of type j. Each differentiated good is supplied by a single monopolistically competitive producer. There are assumed to be many goods in each of an infinite number of "industries"; the goods in each industry j are produced using a type of labor that is specific to that industry, and suppliers in the same industry also change their prices at the same time. The representative household supplies all types of labor as well as consuming all types of goods.[6] To simplify the algebraic form of our results, in our main exposition we shall restrict attention to the case of isoelastic functional forms,

$$\tilde{u}(C_t; \xi_t) \equiv \frac{C_t^{1-\tilde{\sigma}^{-1}} \bar{C}_t^{\tilde{\sigma}^{-1}}}{1 - \tilde{\sigma}^{-1}}, \tag{1.3}$$

$$\tilde{v}(H_t; \xi_t) \equiv \frac{\lambda}{1+\nu} H_t^{1+\nu} \bar{H}_t^{-\nu}, \tag{1.4}$$

where $\tilde{\sigma}, \nu > 0$, and $\{\bar{C}_t, \bar{H}_t\}$ are bounded exogenous disturbance processes. (We use the notation ξ_t to refer to the complete vector of exogenous disturbances, including $\bar{C}_t$ and $\bar{H}_t$.)[7]

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[6]We might alternatively assume specialization across households in the type of labor supplied; in the presence of perfect sharing of labor income risk across households, household decisions regarding consumption and labor supply would all be as assumed here.

[7]The extension of our results to the case of more general preferences is taken up in a longer version of this paper (Benigno and Woodford, 2004a).

We assume a common technology for the production of all goods, in which (industry-specific) labor is the only variable input,

$$y_t(i) = A_t f(h_t(i)) = A_t h_t(i)^{1/\phi},$$

where A_t is an exogenously varying technology factor, and $\phi > 1$.[8] Inverting the production function to write the demand for each type of labor as a function of the quantities produced of the various differentiated goods, and using the identity

$$Y_t = C_t + G_t$$

to substitute for C_t, where G_t is exogenous government demand for the composite good, we can write the utility of the representative household as a function of the expected production plan $\{y_t(i)\}$.[9]

The utility of the representative household (our welfare measure) can be expressed as a function of equilibrium production,

$$U_{t_0} \equiv E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[u(Y_t; \xi_t) - \int_0^1 v(y_t^j; \xi_t) dj \right], \tag{1.5}$$

where

$$u(Y_t; \xi_t) \equiv \tilde{u}(Y_t - G_t; \xi_t),$$
$$v(y_t^j; \xi_t) \equiv \tilde{v}(f^{-1}(y_t^j/A_t); \xi_t).$$

In this last expression we make use of the fact that the quantity produced of each good in industry j will be the same, and hence can be denoted y_t^j; and that the quantity of labor hired by each of these firms will also be the same, so that the total demand for labor of type j is proportional to the demand of any one of these firms.

We can furthermore express the relative quantities demanded of the differentiated goods each period as a function of their relative prices. This allows us to write the utility flow to the representative household in the form $U(Y_t, \Delta_t; \xi_t)$, where

$$\Delta_t \equiv \int_0^1 \left(\frac{p_t(i)}{P_t} \right)^{-\theta(1+\omega)} di \geq 1 \tag{1.6}$$

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[8] Again, more general production functions are considered in Benigno and Woodford (2004a).

[9] The government is assumed to need to obtain an exogenously given quantity of the Dixit-Stiglitz aggregate each period, and to obtain this in a cost-minimizing fashion. Hence the government allocates its purchases across the suppliers of differentiated goods in the same proportion as do households, and the index of aggregate demand Y_t is the same function of the individual quantities $\{y_t(i)\}$ as C_t is of the individual quantities consumed $\{c_t(i)\}$, defined in (1.2).

is a measure of price dispersion at date t, in which $\omega \equiv \phi(1+\nu) - 1 > 0$,[10] P_t is the Dixit-Stiglitz price index

$$P_t \equiv \left[\int_0^1 p_t(i)^{1-\theta} di \right]^{\frac{1}{1-\theta}},\qquad (1.7)$$

and the vector ξ_t now includes the exogenous disturbances G_t and A_t as well as the preference shocks. Hence we can write our objective (1.5) as

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} U(Y_t, \Delta_t; \xi_t).\qquad (1.8)$$

The producers in each industry fix the prices of their goods in monetary units for a random interval of time, as in the model of staggered pricing introduced by Calvo (1983). We let $0 \leq \alpha < 1$ be the fraction of prices that remain unchanged in any period. A supplier that changes its price in period t chooses its new price $p_t(i)$ to maximize

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \Pi(p_t(i), p_T^j, P_T; Y_T, \xi_T) \right\},\qquad (1.9)$$

where $Q_{t,T}$ is the stochastic discount factor by which financial markets discount random nominal income in period T to determine the nominal value of a claim to such income in period t, and α^{T-t} is the probability that a price chosen in period t will not have been revised by period T. In equilibrium, this discount factor is given by

$$Q_{t,T} = \beta^{T-t} \frac{\tilde{u}_c(C_T; \xi_T)}{\tilde{u}_c(C_t; \xi_t)} \frac{P_t}{P_T}.\qquad (1.10)$$

The function

$$\Pi(p, p^j, P; Y, \xi) \equiv (1-\tau)pY(p/P)^{-\theta} - \mu^w \frac{\tilde{v}_h(f^{-1}(Y(p^j/P)^{-\theta}/A); \xi)}{\tilde{u}_c(Y-G; \xi)} P \cdot f^{-1}(Y(p/P)^{-\theta}/A)\qquad (1.11)$$

indicates the after-tax nominal profits of a supplier with price p, in an industry with common price p^j, when the aggregate price index is equal to P and aggregate demand is equal to Y. Here τ_t is the proportional tax on sales revenues in period t; we treat $\{\tau_t\}$ as an exogenous disturbance process, taken as given by the monetary

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[10]Under this definition, ω corresponds to the elasticity of real marginal cost with respect to a firm's own output.

policymaker.[11] We assume that τ_t fluctuates over a small interval around a non-zero steady-state level $\bar\tau$; this is another of the possible reasons for inefficiency of the steady-state level of output that we consider.[12] Profits are equal to after-tax sales revenues net of the wage bill, and the real wage demanded for labor of type j is assumed to be given by

$$w_t(j) = \mu_t^w \frac{\tilde v_h(H_t(j); \xi)}{\tilde u_c(C_t; \xi_t)}, \tag{1.12}$$

where $\mu_t^w \geq 1$ is an exogenous markup factor in the labor market (allowed to vary over time, but assumed common to all labor markets),[13] and firms are assumed to be wage-takers. We allow for exogenous variations in both the tax rate and the wage markup in order to include the possibility of "pure cost-push shocks" that affect equilibrium pricing behavior while implying no change in the efficient allocation of resources.[14] The disturbances τ_t and μ_t^w are also included as elements of the vector ξ_t.

Each of the suppliers that revise their prices in period t choose the same new price p_t^*, that maximizes (1.9). Note that supplier i's profits are a concave function of the quantity sold $y_t(i)$, since revenues are proportional to $y_t(i)^{\frac{\theta-1}{\theta}}$ and hence concave in $y_t(i)$, while costs are convex in $y_t(i)$. Moreover, since $y_t(i)$ is proportional to $p_t(i)^{-\theta}$, the profit function is also concave in $p_t(i)^{-\theta}$. The first-order condition for the optimal choice of the price $p_t(i)$ is the same as the one with respect to $p_t(i)^{-\theta}$; hence the first-order condition with respect to $p_t(i)$,

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \Pi_1(p_t(i), p_T^j, P_T; Y_T, \xi_T) \right\} = 0,$$

is both necessary and sufficient for an optimum. The equilibrium choice p_t^* (which is the same for each firm in industry j) is the solution to the equation obtained by

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[11] The extension to the case in which the tax rate is also chosen optimally in response to other shocks is treated in Benigno and Woodford (2003).

[12] Other types of distorting taxes would have similar consequences, since it is the overall size of the steady-state inefficiency wedge that is of greatest importance for our analysis, as we show below. To economize on notation, we assume that the only distorting tax is of this particular kind.

[13] In the case that we assume that $\mu_t^w = 1$ at all times, our model is one in which both households and firms are wage-takers, or there is efficient contracting between them.

[14] We show below, however, that these two disturbances are not, in general, the only reasons for the existence of a "cost-push" term in our aggregate-supply relation, in the sense of a term that creates a tension between the goals of inflation stabilization and output-gap stabilization.

substituting $p_t(i) = p_t^j = p_t^*$ into the above.

Under our assumed isoelastic functional forms, the optimal choice has a closed-form solution

$$\frac{p_t^*}{P_t} = \left(\frac{K_t}{F_t}\right)^{\frac{1}{1+\omega\theta}}, \tag{1.13}$$

where F_t and K_t are functions of current aggregate output Y_t, the current exogenous state ξ_t, and the expected future evolution of inflation, output, and disturbances, defined by

$$F_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} (1 - \tau_T) f(Y_T; \xi_T) \left(\frac{P_T}{P_t}\right)^{\theta-1}, \tag{1.14}$$

$$K_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} k(Y_T; \xi_T) \left(\frac{P_T}{P_t}\right)^{\theta(1+\omega)}, \tag{1.15}$$

in which expressions

$$f(Y; \xi) \equiv u_y(Y; \xi)Y, \tag{1.16}$$

$$k(Y; \xi) \equiv \frac{\theta}{\theta - 1} \mu^w v_y(Y; \xi)Y. \tag{1.17}$$

The price index then evolves according to a law of motion

$$P_t = \left[(1 - \alpha)p_t^{*1-\theta} + \alpha P_{t-1}^{1-\theta}\right]^{\frac{1}{1-\theta}}, \tag{1.18}$$

as a consequence of (1.7). Substitution of (1.13) into (1.18) implies that equilibrium inflation in any period is given by

$$\frac{1 - \alpha\Pi_t^{\theta-1}}{1 - \alpha} = \left(\frac{F_t}{K_t}\right)^{\frac{\theta-1}{1+\omega\theta}}, \tag{1.19}$$

where $\Pi_t \equiv P_t/P_{t-1}$. This defines a short-run aggregate supply relation between inflation and output, given the current disturbances ξ_t, and expectations regarding future inflation, output, and disturbances.[15] This is the only relevant constraint on

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[15]Note that K_t is an increasing function of Y_t, given expectations and the exogenous state, while F_t is a decreasing function of Y_t, given expectations and the state; so the right-hand side of (1.19) is a decreasing function of Y_t (given expectations and the state), while the left-hand side is a decreasing function of Π_t. Hence the short-run AS relation defined by (1.19) is upward-sloping. As discussed below, a log-linear approximation to (1.19) takes the form of the familiar "new-Keynesian Phillips curve."

the monetary authority's ability to simultaneously stabilize inflation and output in our model.

Because the relative prices of the industries that do not change their prices in period t remain the same, we can also use (1.18) to derive a law of motion of the form

$$\Delta_t = h(\Delta_{t-1}, \Pi_t) \tag{1.20}$$

for the dispersion measure defined in (1.6), where

$$h(\Delta, \Pi) \equiv \alpha \Delta \Pi^{\theta(1+\omega)} + (1-\alpha) \left(\frac{1 - \alpha \Pi^{\theta-1}}{1 - \alpha} \right)^{\frac{\theta(1+\omega)}{\theta-1}}.$$

This is the source in our model of welfare losses from inflation or deflation.

We assume the existence of a lump-sum source of government revenue (in addition to the fixed tax rate τ), and assume that the fiscal authority ensures intertemporal government solvency regardless of what monetary policy may be chosen by the monetary authority.[16] This allows us to abstract from the fiscal consequences of alternative monetary policies in our consideration of optimal monetary stabilization policy, as is common in the literature on monetary policy rules. An extension of our analysis to the case in which only distorting taxes exist is presented in Benigno and Woodford (2003).

Finally, we abstract here from any monetary frictions that would account for a demand for central-bank liabilities that earn a substandard rate of return; we nonetheless assume that the central bank can control the riskless short-term nominal interest rate i_t,[17] which is in turn related to other financial asset prices through the arbitrage relation

$$1 + i_t = [E_t Q_{t,t+1}]^{-1}.$$

We shall assume that the zero lower bound on nominal interest rates never binds under the optimal policies considered below,[18] so that we need not introduce any

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[16]Thus we here assume that fiscal policy is "Ricardian," in the terminology of Woodford (2001). A non-Ricardian fiscal policy would imply the existence of an additional constraint on the set of equilibria that could be achieved through monetary policy. The consequences of such a constraint for the character of optimal monetary policy will be considered elsewhere.

[17]For discussion of how this is possible even in a "cashless" economy of the kind assumed here, see Woodford (2003b, chapter 2).

[18]This can be shown to be true in the case of small enough disturbances, given that the nominal interest rate is equal to $\bar{r} = \beta^{-1} - 1 > 0$ under the optimal policy in the absence of disturbances.

additional constraint on the possible paths of output and prices associated with a need for the chosen evolution of prices to be consistent with a non-negative nominal interest rate. We also note that the ability of the central bank to control i_t in each period gives it one degree of freedom each period (in each possible state of the world) with which to determine equilibrium outcomes. Because of the existence of the aggregate-supply relation (1.19) as a necessary constraint on the joint evolution of inflation and output, there is exactly one degree of freedom to be determined each period, in order to determine particular stochastic processes $\{\Pi_t, Y_t\}$ from among the set of possible rational-expectations equilibria.[19] Hence we shall suppose that the monetary authority can choose from among the possible processes $\{\Pi_t, Y_t\}$ that constitute rational-expectations equilibria, and consider which equilibrium it is optimal to bring about; the detail that policy is implemented through the control of a short-term nominal interest rate will not actually matter to our calculations.

1.2   Optimal Policy from a "Timeless Perspective"

Under the standard (Ramsey) approach to the characterization of an optimal policy commitment, one chooses among state-contingent paths $\{\Pi_t, Y_t, \Delta_t\}$ from some initial date t_0 onward that satisfy (1.19) and (1.20) for each $t \geq t_0$,[20] given initial price dispersion Δ_{t_0-1}, so as to maximize (1.8). Such a t_0-optimal plan requires commitment, insofar as the corresponding $t-$optimal plan for some later date t, given the condition Δ_{t-1} obtaining at that date, will not involve a continuation of the t_0-optimal plan. This failure of time consistency occurs because the constraints on what can be achieved at date t_0, consistent with the existence of a rational-expectations equilibrium, depend on the expected paths of inflation and output at later dates; but in the absence of a prior commitment, a planner would have no motive at those later dates to choose a policy consistent with the anticipations that it was desirable to create at date t_0.

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[19]At least, this is the case if one restricts attention to those equilibrium in which inflation and output remain forever within certain neighborhoods of the steady-state values defined below. We are here concerned solely with the choice of an optimal policy from among those policies consistent with a nearby equilibrium of this kind, as this is the problem to which our approximation technique may be applied.

[20]Here the definitions (1.14) – (1.15) are understood to have been substituted for F_t and K_t in equation (1.19).

However, the degree of advance commitment that is necessary to bring about an optimal equilibrium is of only a limited sort. Let $x_t \equiv (\Pi_t, Y_t, \Delta_t)$, $X_t \equiv (F_t, K_t)$, and let $\mathcal{F}(\xi_t)$ be the set of values for (Δ_{t-1}, X_t) such that there exist paths $\{x_T\}$ for dates $T \geq t$ that satisfy (1.19) and (1.20) for each T, that are consistent with the specified values for the elements of X_t, and that imply a well-defined value for the objective U_t defined in (1.8).[21] Furthermore, for any $(\Delta_{t-1}, X_t) \in \mathcal{F}(\xi_t)$, let $V(\Delta_{t-1}, X_t; \xi_t)$ denote the maximum attainable value of U_t among the state-contingent paths that satisfy the constraints just mentioned. Then the t_0−optimal plan can be obtained as the solution to the following two-stage optimization problem.

In the first stage, values of the endogenous variables x_{t_0} and state-contingent commitments $X_{t_0+1}(\xi_{t_0+1})$ for the following period, are chosen so as to maximize an objective defined below. Then in the second stage, the equilibrium evolution from period t_0+1 onward is chosen to solve the maximization problem that defines the value function $V(\Delta_{t_0}, X_{t_0+1}; \xi_{t_0+1})$, given the state of the world ξ_{t_0+1} and the precommitted values for X_{t_0+1} associated with that state.

In defining the objective for the first stage of this equivalent formulation of the Ramsey problem, it is useful to let $\Pi(F, K)$ denote the value of Π_t that solves (1.19) for given values of F_t and K_t. We also define the functional relationships

$$\hat{J}[x_t, X_{t+1}(\cdot)](\xi_t) \equiv U(Y_t, \Delta_t; \xi_t) + \beta E_t V(\Delta_t, X_{t+1}; \xi_{t+1}),$$

$$\hat{F}[x_t, X_{t+1}(\cdot)](\xi_t) \equiv (1 - \tau_t) f(Y_t; \xi_t) + \alpha \beta E_t \{\Pi(F_{t+1}, K_{t+1})^{\theta-1} F_{t+1}\},$$

$$\hat{K}[x_t, X_{t+1}(\cdot)](\xi_t) \equiv k(Y_t; \xi_t) + \alpha \beta E_t \{\Pi(F_{t+1}, K_{t+1})^{\theta(1+\omega)} K_{t+1}\},$$

where $f(Y; \xi)$ and $k(Y; \xi)$ are defined in (1.16) and (1.17).

Then in the first stage, x_{t_0} and $X_{t_0+1}(\cdot)$ are chosen so as to maximize $\hat{J}[x_{t_0}, X_{t_0+1}(\cdot)](\xi_{t_0})$ over values of x_{t_0} and $X_{t_0+1}(\cdot)$ such that

(i) Π_{t_0} and Δ_{t_0} satisfy (1.20);

(ii) the values

$$F_{t_0} = \hat{F}[x_{t_0}, X_{t_0+1}(\cdot)](\xi_{t_0}), \tag{1.21}$$

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[21]In the notation $\mathcal{F}(\xi_t)$, ξ_t refers to the state of the world at date t, *i.e.*, to a complete specification of all information that is available at that date about both the current exogenous disturbances and the joint probability distribution of all future disturbances. Under the assumption that the state vector ξ_t is Markovian, we can use the same notation ξ_t for a summary of all exogenous disturbances in period t and the state of the world in period t. The argument ξ_t of the value function $V(\Delta_{t-1}, X_t; \xi_t)$ has the same interpretation.

$$K_{t_0} = \hat{K}[x_{t_0}, X_{t_0+1}(\cdot)](\xi_{t_0}) \tag{1.22}$$

satisfy

$$\Pi_{t_0} = \Pi(F_{t_0}, K_{t_0}); \tag{1.23}$$

and

(iii) the choices $(\Delta_{t_0}, X_{t_0+1}) \in \mathcal{F}$ for each possible state of the world ξ_{t_0+1}.
The following result can then be established, as shown in Appendix A.

PROPOSITION 1. Given Δ_{t_0-1}, let the process $\{x_t\}$ be determined by (i) choosing x_{t_0} and state-contingent commitments $X_{t_0+1}(\xi_{t_0+1})$ to solve the first-stage problem just stated, and (ii) for each possible state of the world ξ_{t_0+1}, choosing the evolution of x_t for $t \geq t_0 + 1$ so as to maximize U_{t_0+1}, among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0 + 1$, given Δ_{t_0}, and that are also consistent with the value of $X_{t_0+1}(\xi_{t_0+1})$ determined in the first stage. Then the process $\{x_t\}$ represents a *Ramsey policy;* that is, it maximizes U_{t_0} among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0$, given Δ_{t_0-1}.

The optimization problem in stage two of this reformulation of the Ramsey problem is of the same form as the Ramsey problem itself, except that there are additional constraints associated with the precommitted values for the elements of $X_{t_0+1}(\xi_{t_0+1})$. Let us consider a problem like the Ramsey problem just defined, looking forward from some period t_0, except under the constraints that the quantities X_{t_0} must take certain given values, where $(\Delta_{t_0-1}, X_{t_0}) \in \mathcal{F}(\xi_{t_0})$. This constrained problem can similarly be expressed as a two-stage problem of the same form as above, with an identical stage two problem to the one described above. Stage two of this constrained problem is thus of exactly the same form as the problem itself. Hence the constrained problem has a recursive form, even though the original Ramsey problem did not. This is shown by the following proposition, also proved in Appendix A.

PROPOSITION 2. Given some $(\Delta_{t_0-1}, X_{t_0}) \in \mathcal{F}(\xi_{t_0})$, consider the sequential decision problem in which in each period $t \geq t_0$, $(x_t, X_{t+1}(\cdot))$ are chosen to maximize $\hat{J}[x_t, X_{t+1}(\cdot)](\xi_t)$, subject to constraints (i) – (iii) of the "first stage" problem stated above, given the predetermined state variable Δ_{t-1} and the precommitted values X_t. Then the process $\{x_t\}$ that is chosen in this way is the process that maximizes U_{t_0}

among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0$, given Δ_{t_0-1}, and also consistent with the specified values X_{t_0}.

Our aim here is to characterize policy that solves the constrained optimization problem with which Proposition 2 is concerned *i.e.,* policy that is optimal from some date t onward given precommitted values for X_t.[22] Because of the recursive form of this problem, it is possible for a commitment to a time-invariant policy rule from date t onward to implement an equilibrium that solves the problem, for some specification of the initial commitments X_t. A time-invariant policy rule with this property is said by Woodford (2003b, chap. 7) to be "optimal from a timeless perspective."[23] Such a rule is one that a policymaker that solves a traditional Ramsey problem would be willing to commit to *eventually* follow, though the solution to the Ramsey problem involves different behavior initially, as there is no need to internalize the effects of prior anticipation of the policy adopted for period t_0.[24] One might also argue that it is desirable to commit to follow such a rule immediately, even though such a policy would not solve the (unconstrained) Ramsey problem, as a way of demonstrating one's willingness to accept constraints that one wishes the public to believe that one will accept in the future.

2 A Linear-Quadratic Approximate Problem

In fact, we shall here characterize the solution to this problem (and similarly, derive optimal time-invariant policy rules) only for initial conditions near certain steady-

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[22]The problem considered in Khan *et al.* (2003) also differs from the standard (unconstrained) Ramsey policy problem, in their case by modifying the assumed policy objective to include additional terms similar to those in (4.1) below; again, the modification results in a policy problem with a recursive form. In fact, their recursive policy problem is equivalent to the one characterized in Proposition 2, as their additional terms in the modified objective are essentially the terms in a Lagrangian corresponding our initial commitments.

[23]See also Woodford (1999) and Giannoni and Woodford (2002).

[24]In the present model, Ramsey policy involves an initial positive rate of inflation, even in the absence of any shocks, even though in the long run it involves a commitment to maintain a zero inflation rate on average. This is because welfare is increased by exploiting the Phillips curve to increase output through an inflationary policy initially; but it is not optimal to create the anticipation that one will behave in this way later, owing to the adverse effects of the anticipated inflation on earlier periods' inflation/output tradeoffs. See Woodford (2003b, chap. 7) for further discussion.

state values, allowing us to use local approximations in characterizing optimal policy.[25] We establish that these steady-state values have the property that if one starts from initial conditions close enough to the steady state, and exogenous disturbances thereafter are small enough, the optimal policy subject to the initial commitments remains forever near the steady state. Hence our local characterization describes the *long run* character of Ramsey policy, in the event that disturbances are small enough. Of greater interest here, it describes policy that is optimal from a timeless perspective in the event of small disturbances.

We first must show the existence of a steady state, *i.e.,* of an optimal policy (under appropriate initial conditions) that involves constant values of all variables. To this end we consider the purely deterministic case, in which the exogenous disturbances $\bar{C}_t, G_t, \bar{H}_t, A_t, \mu_t^w, \tau_t$ each take constant values $\bar{C}, \bar{H}, \bar{A}, \bar{\mu}^w, \bar{\tau} > 0, \bar{G} \geq 0$ for all $t \geq t_0$. We wish to find an initial degree of price dispersion Δ_{t_0-1} and initial commitments $X_{t_0} = \bar{X}$ such that the solution to the problem defined in Proposition 2 involves a constant policy $x_t = \bar{x}$, $X_{t+1} = \bar{X}$ each period, in which $\bar{\Delta}$ is equal to the initial price dispersion. We show in Appendix B.2 that the first-order conditions for this problem admit a steady-state solution of this form, and we verify below that (when our parameters satisfy certain bounds) the second-order conditions for a local optimum are also satisfied.

We show that $\bar{\Pi} = 1$(zero inflation), and correspondingly that $\bar{\Delta} = 1$(zero price dispersion).[26] We may furthermore assume without loss of generality that the constant values of $\bar{C}$ and $\bar{H}$ are chosen so that in the optimal steady state, $C_t = \bar{C}$ and $H_t = \bar{H}$ each period.[27]

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[25]Local approximations of the same sort are often used in the literature in numerical characterizations of Ramsey policy. Strictly speaking, however, such approximations are valid only in the case of initial commitments X_{t_0} near enough to the steady-state values of these variables, and the t_0- optimal (Ramsey) policy need not involve values of X_{t_0} near the steady-state values, even in the absence of random disturbances. Khan *et al.* (2003) are able to rigorously employ a local approximation to the first-order conditions that characterize Ramsey policy because they modify the policy objective, assigning initial multipliers of precisely the size needed to make the optimal policy in the absence of disturbances a steady state.

[26]Our conclusion that the optimal steady-state inflation rate is zero can be generalized to other price-setting mechanisms and a more general preference specification, as shown in Benigno and Woodford (2004a), and to the case in which only distorting taxes are available as in Benigno and Woodford (2003).

[27]Note that we may assign arbitrary positive values to $\bar{C}, \bar{H}$ without changing the nature of the

We next wish to characterize the optimal responses to small perturbations of the initial conditions and small fluctuations in the disturbance processes around the above values. To do this, we compute a linear-quadratic approximate problem, the solution to which represents a linear approximation to the solution to the policy problem defined in Proposition 2. An important advantage of this approach is that it allows direct comparison of our results with those obtained in other analyses of optimal monetary stabilization policy. Other advantages are that it makes it straightforward to verify whether the second-order conditions hold that are required in order for a solution to our first-order conditions to be at least a local optimum (see section 3.1), and that it provides us with a welfare measure with which to rank alternative sub-optimal policies, in addition to allowing computation of the optimal policy.

We begin by computing a Taylor-series approximation to our welfare measure (1.8), expanding around the steady-state allocation defined above, in which $y_t(i) = \bar{Y}$ for each good at all times and $\xi_t = 0$ at all times.[28] As a second-order (logarithmic) approximation to this measure, we obtain[29]

$$
\begin{aligned}
U_{t_0} &= \bar{Y}\bar{u}_c \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0}[\Phi\hat{Y}_t - \frac{1}{2}u_{yy}\hat{Y}_t^2 + \hat{Y}_t u_{y\xi}\xi_t - u_{\Delta}\hat{\Delta}_t] \\
&\quad + \text{ t.i.p.} + \mathcal{O}(||\xi||^3),
\end{aligned}
\tag{2.1}
$$

where $\hat{Y}_t \equiv \log(Y_t/\bar{Y})$ and $\hat{\Delta}_t \equiv \log\Delta_t$ measure deviations of aggregate output and the price dispersion measure from their steady-state levels, the term "t.i.p." collects terms that are independent of policy (constants and functions of exogenous disturbances) and hence irrelevant for ranking alternative policies, and $||\xi||$ is a bound on

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implied preferences, as long as the value of λ is appropriately adjusted.

[28]Here the elements of ξ_t are assumed to be $\bar{c}_t \equiv \log(\bar{C}_t/\bar{C}), \bar{h}_t \equiv \log(\bar{H}_t/\bar{H}), a_t \equiv \log(A_t/\bar{A}), \hat{\mu}_t^w \equiv \log(\mu_t^w/\bar{\mu}^w), \hat{G}_t \equiv (G_t - \bar{G})/\bar{Y}$, and $\hat{\tau}_t \equiv (\tau_t - \bar{\tau})/\bar{\tau}$, so that a value of zero for this vector corresponds to the steady-state values of all disturbances. The perturbation $\hat{G}_t$ is not defined to be logarithmic so that we do not have to assume positive steady-state value for this variable.

[29]See Appendix B.3 for details. Our calculations here follow closely those of Woodford (2003b, chap. 6).

the amplitude of our perturbations of the steady state.[30] Here the coefficient

$$\Phi \equiv 1 - \frac{\theta-1}{\theta}\frac{1-\bar{\tau}}{\bar{\mu}^w} < 1$$

measures the steady-state wedge between the marginal rate of substitution between consumption and leisure and the marginal product of labor, and hence the inefficiency of the steady-state output level $\bar{Y}$. The coefficients $u_{yy}, u_{y\xi}$ and u_Δ are defined in Appendix B.3.

In addition, we can take a second-order approximation to equation (1.20) and integrate it to obtain

$$\sum_{t=t_0}^{\infty}\beta^{t-t_0}\ \hat{\Delta}_t = \frac{\alpha}{(1-\alpha)(1-\alpha\beta)}\theta(1+\omega)(1+\omega\theta)\sum_{t=t_0}^{\infty}\beta^{t-t_0}\ \frac{\pi_t^2}{2}+\text{t.i.p.}+\mathcal{O}(||\xi||^3). \quad (2.2)$$

Substituting (2.2) into (2.1), we can then approximate our welfare measure by

$$\begin{aligned}
U_{t_0} &= \bar{Y}\bar{u}_c \cdot E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}[\Phi\hat{Y}_t - \frac{1}{2}u_{yy}\hat{Y}_t^2 + \hat{Y}_t u_{y\xi}\xi_t - \frac{1}{2}u_\pi\pi_t^2] \\
&\quad +\text{t.i.p.} + \mathcal{O}(||\xi||^3), \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (2.3)
\end{aligned}$$

for a certain coefficient $u_\pi > 0$ defined in Appendix B.3. Note that we can now write our stabilization objective purely in terms of the evolution of the aggregate variables $\{\hat{Y}_t, \pi_t\}$ and the exogenous disturbances.

We note that when $\Phi > 0$, there is a non-zero linear term in (2.3), which means that we cannot expect to evaluate this expression to second order using only an approximate solution for the path of aggregate output that is accurate only to first order. Thus we cannot determine optimal policy, even up to first order, using this approximate objective together with approximations to the structural equations that are accurate only to first order. Rotemberg and Woodford (1997) avoid this problem by assuming an output subsidy (*i.e.,* a value $\bar{\tau} < 0$) of the size needed to ensure that

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[30]Specifically, we use the notation $\mathcal{O}(||\xi||^k)$ as shorthand for $\mathcal{O}(||\xi, \hat{\Delta}_{t_0-1}^{1/2}, \hat{X}_{t_0}||^k)$, where in each case hats refer to log deviations from the steady-state values of the various parameters of the policy problem. We treat $\hat{\Delta}_{t_0}^{1/2}$ as an expansion parameter, rather than $\hat{\Delta}_{t_0}$ because (1.20) implies that deviations of the inflation rate from zero of order ϵ only result in deviations in the dispersion measure Δ_t from one of order ϵ^2. We are thus entitled to treat the fluctuations in Δ_t as being only of second order in our bound on the amplitude of disturbances, since if this is true at some initial date it will remain true thereafter. (See Appendix B.3 for further discussion.)

$\Phi = 0$. Here we wish to relax this assumption. We show here that an alternative way of dealing with this problem is to use a second-order approximation to the aggregate-supply relation to eliminate the linear terms in the quadratic welfare measure. We show in Appendix B.4 that to second order, equation (1.19) can be written in the form

$$V_t = \kappa(\hat{Y}_t + c_\xi \xi_t + \frac{1}{2}c_{yy}\hat{Y}_t^2 - \hat{Y}_t c_{y\xi}\xi_t + \frac{1}{2}c_\pi \pi_t^2) + \beta E_t V_{t+1}$$
$$+ \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3), \tag{2.4}$$

for certain coefficients defined in the appendix. Here the notation "s.o.t.i.p." indicates terms independent of policy that are entirely of second or higher order, and we have defined

$$V_t \equiv \pi_t + \frac{1}{2}v_\pi \pi_t^2 + v_z \pi_t Z_t, \tag{2.5}$$

where

$$Z_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t}[z_y \hat{Y}_T + z_\pi \pi_T + z_\xi \xi_T]; \tag{2.6}$$

again the coefficients are defined in Appendix B.4. Note that to first order (2.4) reduces simply to

$$\pi_t = \kappa[\hat{Y}_t + c_\xi \xi_t] + \beta E_t \pi_{t+1}, \tag{2.7}$$

for a certain coefficient $\kappa > 0$. This is the familiar "New Keynesian Phillips curve" relation.

Integrating forward equation (2.4), we obtain a relation of the form

$$V_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \kappa[\hat{Y}_t + \frac{1}{2}c_{yy}\hat{Y}_t^2 - \hat{Y}_t c_{y\xi}\xi_t + \frac{1}{2}c_\pi \pi_t^2] + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \tag{2.8}$$

We can then use (2.8) to write the discounted sum of output terms in (2.3) as a function of purely quadratic terms, up to a residual of third order. As shown in Appendix B.5, we can rewrite (2.3) as

$$U_{t_0} = -\Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{q_\pi}{2}\pi_t^2 + \frac{q_y}{2}(\hat{Y}_t - \hat{Y}_t^*)^2 \right\} + T_{t_0} + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \tag{2.9}$$

where [31]

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[31]In what follows, the following definitions have been used: $\sigma^{-1} \equiv \tilde{\sigma}^{-1}s_C^{-1}$ with $s_C \equiv \bar{C}/\bar{Y}$; $\omega q_t \equiv \nu\bar{h}_t + \phi(1+\nu)a_t$; $g_t \equiv \hat{G}_t + s_C\bar{c}_t$; $\omega_\tau \equiv \bar{\tau}/(1-\bar{\tau})$; $\kappa \equiv (1-\alpha\beta)(1-\alpha)(\omega+\sigma^{-1})/[\alpha(1+\theta\omega)]$.

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$$\Omega \equiv \bar{Y}u_c > 0,$$

$$q_\pi \equiv \frac{\theta}{\kappa}[(\omega + \sigma^{-1}) + \Phi(1 - \sigma^{-1})], \tag{2.10}$$

$$q_y \equiv \omega + \sigma^{-1} + \Phi(1 - \sigma^{-1}) - \frac{\Phi\sigma^{-1}(s_C^{-1} - 1)}{\omega + \sigma^{-1}}, \tag{2.11}$$

$$\hat{Y}_t^* = \omega_1\hat{Y}_t^n - \omega_2\hat{G}_t + \omega_3\hat{\mu}_t^w + \omega_4\hat{\tau}_t, \tag{2.12}$$

and

$$\hat{Y}_t^n \equiv -c_\xi\xi_t = \frac{\sigma^{-1}g_t + \omega q_t - \hat{\mu}_t^w - \omega_\tau\hat{\tau}_t}{(\omega + \sigma^{-1})},$$

in which expressions

$$\omega_1 = q_y^{-1}[(\omega + \sigma^{-1}) + \Phi(1 - \sigma^{-1})],$$

$$\omega_2 = \frac{\Phi s_C^{-1}\sigma^{-1}}{(\omega + \sigma^{-1})^2 + \Phi[(1 - \sigma^{-1})(\omega + \sigma^{-1}) - (s_C^{-1} - 1)\sigma^{-1}]},$$

$$\omega_3 \equiv \frac{1 - \Phi}{(\omega + \sigma^{-1}) + \Phi[(1 - \sigma^{-1}) - (s_C^{-1} - 1)\sigma^{-1}(\omega + \sigma^{-1})^{-1}]},$$

$$\omega_4 \equiv \frac{\omega_\tau}{(\omega + \sigma^{-1}) + \Phi[(1 - \sigma^{-1}) - (s_C^{-1} - 1)\sigma^{-1}(\omega + \sigma^{-1})^{-1}]}.$$

Here $\hat{Y}_t^n$ represents a log-linear approximation to the "natural rate of output," *i.e.,* the flexible-price equilibrium level of output (Woodford, 2003b, chap. 3); in terms of this notation, the log-linear aggregate supply relation (2.7) can be written as

$$\pi_t = \kappa[\hat{Y}_t - \hat{Y}_t^n] + \beta E_t\pi_{t+1}. \tag{2.13}$$

The term $T_{t_0} \equiv \Phi\bar{Y}\bar{u}_c\kappa^{-1}V_{t_0}$ is a transitory component defined in Appendix B.5.

Once again, we are interested in characterizing optimal policy from a timeless perspective. We observe from the form of the structural relations (2.4) and the definition of V_t that the aspects of the expected future evolution of the endogenous variables that affect the feasible set of values for inflation, output in any period t can be summarized (in our second-order approximation to the structural relations) by the expected values of V_{t+1}, Z_{t+1}. Hence the only commitments regarding future outcomes that can be of value in improving stabilization outcomes in period t can be summarized by commitments at t regarding the state-contingent values of those two variables in the following period. It follows that we are interested in characterizing

optimal policy from any date t_0 onward subject to the constraint that given values for V_{t_0}, Z_{t_0} be satisfied,[32] in addition to the constraints represented by the structural equations.

But given predetermined values for V_{t_0} the value of the transitory component T_{t_0} is predetermined. Hence, over the set of admissible policies, higher values of (2.9) correspond to lower values of

$$L_{t_0} \equiv E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{q_\pi}{2} \pi_t^2 + \frac{q_y}{2} (\hat{Y}_t - \hat{Y}_t^*)^2 \right\}. \tag{2.14}$$

It follows that we may rank policies in terms of the implied value of the discounted quadratic loss function L_{t_0}. Because this loss function is purely quadratic (*i.e.,* lacking linear terms), it is possible to evaluate it to second order using only a first-order approximation to the equilibrium evolution of inflation and output under a given policy. Hence the log-linear approximate structural relation (2.7) (or equivalently, (2.13)) is sufficiently accurate for our purposes. Similarly, it suffices that we use log-linear approximations to the variable V_{t_0} in describing the initial commitments, which are given by $\hat{V}_{t_0} = \pi_{t_0}$. Then an optimal policy from a timeless perspective is a policy from date t_0 onward that minimizes the quadratic loss function L_{t_0} subject to the constraints implied by the linear structural relation (2.13) holding in each period $t \geq t_0$ and subject also to the constraints that a certain predetermined value for $\hat{V}_{t_0}$ be achieved.[33] This last constraint may equivalently be expressed as a constraint on the initial inflation rate,

$$\pi_{t_0} = \bar{\pi}_{t_0}. \tag{2.15}$$

(The definition of the constraint value $\bar{\pi}_{t_0}$ under a policy that is optimal from a timeless perspective is discussed further in Woodford, 2003b, chap. 7, sec. 2.1.)

The policy objective L_{t_0} now depends only on the evolution of the inflation rate and the welfare-relevant output gap

$$y_t \equiv \hat{Y}_t - \hat{Y}_t^*.$$

---

[32]Note that a specification of initial values for these two variables corresponds, in our quadratic approximation to the structural equations, to a specification of initial values for the variables F_{t_0}, K_{t_0} in section 1. The local quadratic approximation to the constraints implied by the initial commitments discussed in section 1 is treated in Appendix B.6.

[33]The constraint associated with a predetermined value for Z_{t_0} can be neglected, in a first-order characterization of optimal policy, because the variable Z_t does not appear in the first-order approximation to the aggregate-supply relation.

It is useful to write the linear constraints implied by our model's structural equations in terms of the welfare-relevant output gap as well. The aggregate-supply relation (2.13) can be alternatively expressed as

$$\pi_t = \kappa y_t + \beta E_t \pi_{t+1} + u_t, \tag{2.16}$$

where u_t is a composite "cost-push" term, indicating the degree to which the exogenous disturbances preclude simultaneous stabilization of inflation and the welfare-relevant output gap. In terms of our previous notation for the exogenous disturbances in the model, this is given by

$$
\begin{aligned}
u_t &\equiv \kappa(\hat{Y}_t^* - \hat{Y}_t^n) \\
&= \kappa(\omega_1 - 1)\hat{Y}_t^n - \kappa\omega_2 \hat{G}_t + \kappa\omega_3 \hat{\mu}_t^w + \kappa\omega_4 \hat{\tau}_t.
\end{aligned}
$$

It is important for the discussion below to note that pure markup shocks are not the only source of movements in the cost-push term u_t.

We have thus shown that an objective for policy of the form (2.14), as discussed in the introduction, can indeed be justified on welfare-theoretic grounds. This requires that the "output gap" in such an objective be interpreted in the way defined here, *i.e.,* as the percentage deviation of output from a variable target level of output that depends on the evolution of exogenous disturbances of many sorts. (There is thus no reason, in general, for the welfare-theoretic target level of output to correspond to a smooth trend.) We have also seen that exogenous disturbances may indeed preclude simultaneous stabilization of inflation and the welfare-relevant output gap; the extent to which this is true depends on the degree of variability of the disturbance term u_t defined above. We now turn to the consequences of this characterization for the nature of optimal policy.

3   Optimal Inflation Stabilization

We now use our linear-quadratic approximate policy problem to characterize optimal policy in the event of small enough disturbances. We begin by establishing conditions under which the second-order conditions for loss minimization are satisfied, so that the first-order conditions determine a loss-minimizing policy, and hence approximate at least a local welfare maximum. These are also conditions under which welfare

cannot be increased (at least locally) by arbitrary randomization of policy. We then use the first-order conditions to characterize the optimal responses of inflation and output to exogenous disturbances, and discuss the conditions under which optimal policy corresponds to complete price stability.

3.1   Conditions for the Desirability of Policy Randomization

We have shown in the previous section that our approximate policy problem consists of choosing processes $\{\pi_t, \hat{Y}_t\}$ for dates $t \geq t_0$ to minimize the loss function L_{t_0} defined in (2.14), subject to the constraint that the log-linear approximate aggregate supply relation (2.16) hold each period, and that the initial inflation rate satisfy a constraint of the form (2.15). We first consider whether a solution to the first-order conditions associated with this problem necessarily represents a loss minimum. This is necessarily true if the loss function is convex, as it will be if $q_\pi, q_y > 0$; but as we shall see, our approximate loss function is not necessarily (globally) convex, yet our LQ approximation may nonetheless suffice to characterize (locally) optimal policy. Here we examine the somewhat weaker conditions under which this will still be true.

As a closely related question, we consider the issue of whether purely random policy — randomization of policy by the monetary authority, uncorrelated with any random variation in economic "fundamentals" — can be welfare-improving. Again, in the case of a convex loss function, of the kind conventionally assumed in analyses of monetary stabilization policy with *ad hoc* objectives, it can be shown that arbitrary randomization is never optimal. But if our approximate loss function need not be convex, the answer is not obvious, and Dupor (2003) exhibits a general-equilibrium model with sticky prices in which randomization of monetary policy can be welfare-improving. Here we use our LQ approximation method to establish general conditions under which a result like Dupor's will obtain in a model with Calvo-style staggered pricing.

Both questions turn on the positive definiteness of a certain quadratic form defined by the coefficients of the LQ problem. Suppose that $\{\pi_t, \hat{Y}_t\}$ are stochastic processes consistent with both the equilibrium relation (2.16) at all dates $t \geq t_0$ and the initial constraint (2.15), and let us then consider the perturbed processes

$$\tilde{\pi}_t \equiv \pi_t + \psi_t^\pi, \qquad \tilde{Y}_t \equiv \hat{Y}_t + \psi_t^y, \tag{3.1}$$

for some stochastic processes $\{\psi_t^\pi, \psi_t^y\}$. Each of these stochastic processes $\{x_t\}$ is

assumed to be such that

$$E_{t_0} \sum_{t=t_0}^{} \beta^{t-t_0} x_t^2 < \infty, \tag{3.2}$$

so that the loss function L_{t_0} is well-defined for both the original and the perturbed processes. The perturbed processes will also represent a possible rational-expectations equilibrium consistent with (2.15) if the processes $\{\psi_t^\pi, \psi_t^y\}$ satisfy

$$\psi_t^\pi = \kappa \psi_t^y + \beta E_t \psi_{t+1}^\pi \tag{3.3}$$

for all $t \geq t_0$, and

$$\psi_{t_0}^\pi = 0. \tag{3.4}$$

Now consider the Hilbert space $\mathcal{H}$ of stochastic processes $\psi \equiv \{\psi_t^\pi, \psi_t^y\}$ for dates $t \geq t_0$ satisfying the bounds (3.2) for $x = \psi^\pi, \psi^y$.[34] Then the quadratic form

$$L(\psi) \equiv E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[\frac{q_\pi}{2} \psi_t^{\pi 2} + \frac{q_y}{2} \psi_t^{y 2} \right] \tag{3.5}$$

is well defined for any processes $\psi \in \mathcal{H}$. Furthermore, let the linear subspace $\mathcal{H}_1$ be the set of processes $\psi \in \mathcal{H}$ that satisfy (3.4) in addition to satisfying (3.3) for each $t \geq t_0$. Then the quadratic form (3.5) is positive definite on the subspace $\mathcal{H}_1$ if $L(\psi) > 0$ for any processes $\psi \in \mathcal{H}_1$ that are not identically zero (i.e., equal to zero almost surely at all dates). This is the critical condition for both of the issues with which we are concerned, as indicated in the following proposition.

PROPOSITION 3. Randomization of monetary policy increases the expected losses L_{t_0} — and hence is locally welfare-reducing in the exact problem as well — if and only if the quadratic form (3.5) is positive definite on the subspace $\mathcal{H}_1$. Furthermore, if and only if this is true, processes $\{\pi_t, \hat{Y}_t\}$ that satisfy the first-order conditions for the LQ optimization problem [discussed further in section 3.3] represent a loss minimum, and hence an approximation to (at least a local) welfare maximum in the exact problem.

Furthermore, the necessary and sufficient conditions for (3.5) to be positive definite on $\mathcal{H}_1$ reduce to the following: q_π and q_y are not *both* equal to zero; and *either*

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[34]This can be shown to be a Hilbert space if the inner product of two processes ψ^1, ψ^2 is defined as $E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} [\psi_t^{1,\pi} \psi_t^{2,\pi} + \psi_t^{1,y} \psi_t^{2,y}]$.

(i) $q_y \geq 0$ and

$$q_\pi + (1 - \beta^{1/2})^2 \kappa^{-2} q_y > 0, \tag{3.6}$$

holds, or (ii) $q_y \leq 0$ and

$$q_\pi + (1 + \beta^{1/2})^2 \kappa^{-2} q_y > 0, \tag{3.7}$$

holds.

The proof is given in Appendix A.

Note that in the case that both $q_y, q_\pi \geq 0$, (3.6) is satisfied as long as at least one coefficient is strictly positive; thus the case of a convex loss function is one in which the second-order conditions are necessarily satisfied and randomization of policy is necessarily welfare-reducing. However, Proposition 3 shows that the requirement of convexity of the loss function can be weakened while retaining these results.

In fact, in the case of isoelastic functional forms, convexity is likely to obtain for quantitatively reasonable parameter values, even if it is not a necessary consequence of the general assumptions made above. In the isoelastic case, q_y and q_π are given by (2.11) and (2.10) respectively. It follows from this expression and our general assumptions that $q_\pi > 0$, though it remains possible in the isoelastic case for q_y to be negative. Furthermore, one observes that a necessary condition for q_y to be negative is that $s_C < 1/2$, or alternatively that $s_G > 1/2$, which is larger share of government purchases in total demand than is typical of industrial economies.

Even if $q_y < 0$, Proposition 3 shows that randomization of policy will still be welfare-reducing, as long as

$$q_y \geq -\frac{\kappa^2 q_\pi}{(1 + \beta^{1/2})^2}. \tag{3.8}$$

Violation of this bound requires an even more extreme role of the government in the economy, though it remains a technical possibility, consistent with our general neoclassical assumptions.[35] We show elsewhere (Benigno and Woodford, 2004a) that it is possible for randomization to be welfare-improving without such an extremely large share of government purchases in total demand, in the case of more general

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[35]For given values $0 < \beta < 1$, $\omega \geq 0$, $\sigma^{-1} > 0$, $\Phi > 0$, $\kappa > 0$, and $\theta > 1$, choice of a value of s_G close enough to 1 — and hence a value of s_C close enough to zero — will make q_y an arbitrarily large negative quantity, while q_π and the other expressions on the right-hand side of (3.8) remain finite. Hence it is possible to find parameter values for which (3.8) is violated.

functional forms. Nonetheless, this possibility seems to be of more theoretical than practical interest.

3.2 The Case for Price Stability

Under certain circumstances, our characterization of the approximate loss function yields immediate conclusions regarding the nature of optimal policy. These are the conditions under which optimal policy involves complete stabilization of the inflation rate at zero, *i.e.,* complete price stability. While the conditions under which this is exactly true are fairly special, they are nonetheless of interest, insofar as price stability may be a good approximation to optimal policy as long as the conditions are not too grossly violated.

The quadratic loss function L_{t_0} defined in (2.14) is clearly minimized by a policy under which inflation is zero at all times if two conditions are met: (i) the coefficients of the loss function satisfy $q_y, q_\pi > 0$; and (ii) the exogenous terms $\hat{Y}_t^n$ and $\hat{Y}_t^*$ coincide at all times. Condition (ii) implies that a policy under which inflation is zero at all times will also involve $\hat{Y}_t = \hat{Y}_t^*$ at all times, as a consequence of (2.16).[36] Condition (i) then implies that such an equilibrium necessarily achieves the lowest possible value for expected losses, since expected losses are zero and the loss function is necessarily non-negative.

In fact, condition (i) can be weakened; it suffices that q_y and q_π satisfy the conditions stated in Proposition 3. In Appendix A we establish the following result.

PROPOSITION 4. Suppose that $\hat{Y}_t^n = \hat{Y}_t^*$ at all times, and that the conditions stated in Proposition 3 are satisfied. Then the policy that uniquely minimizes L_{t_0} is the one under which $\pi_t = 0$ at all times, regardless of the realizations of the exogenous disturbances [as long as these are small enough to make such an equilibrium possible].

This means that in the exact model as well, a policy under which inflation is zero at all times is optimal from a timeless perspective. That is, under the initial constraint that $\pi_{t_0} = 0$, expected utility is maximized by a policy under which $\pi_t = 0$ for all

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[36]Here we assume that a policy under which inflation is zero at all times is *feasible.* In the model proposed here, this is necessarily the case as long as disturbances are small enough, so that nominal interest rate required for an equilibrium with zero inflation is non-negative at all times.

$t \geq t_0$.

The condition that $\hat{Y}_t^n = \hat{Y}_t^*$ at all times, assumed in Proposition 4, is not quite so special a situation as might be imagined. It is consistent with the existence of a number of distinct types of independent disturbances, as long as certain model parameters take special values. Comparing the definitions of $\hat{Y}_t^n$ and $\hat{Y}_t^*$ above, one sees that [for the isoelastic case considered in section 2] both expressions will be affected to exactly the same extent by technology shocks, by shocks to household impatience to consume, and by shocks to the disutility of labor supply, in the case that $\omega_1 = 1$. This condition in turn holds if $\Phi(s_C^{-1} - 1) = 0$, which holds if *either* $\Phi = 0$ or $s_G = 0$. Furthermore, both expressions are affected to exactly the same extent by variations in government purchases as well, if in addition $\omega_2 = 0$, which holds if $\Phi = 0$. However, variations in the wage markup or in the level of distorting taxes necessarily affect the two expressions differently, except in a special case that would imply that they are no longer affected in the same way by any disturbances to tastes or technology. We thus obtain the following result.

PROPOSITION 5. Consider a model with the isoelastic functional forms (1.3) – (1.4), and parameter values $\omega \geq 0, \sigma^{-1} > 0$, and suppose that there are random fluctuations in the composite disturbance term $\omega q_t + \sigma^{-1}\bar{c}_t$. [This is generally true if either preferences or technology are random.] Then $\hat{Y}_t^n = \hat{Y}_t^*$ at all times — so that the "cost-push" term in the aggregate-supply relation (2.16) is zero at all times — if and only if (i) there are no random variations in the wage markup or the tax rate ($\hat{\mu}_t^w = \hat{\tau}_t = 0$ at all times); and (ii) *either* (a) the steady-state level of output is efficient ($\Phi = 0$) or (b) there are no government purchases ($G_t = 0$ at all times).

The result that there is no "cost-push" term in the aggregate-supply relation in the case that $\Phi = 0$, as long as there are no markup fluctuations or variations in the level of distorting taxes, has already been obtained in Woodford (2003b, chap. 6), following Rotemberg and Woodford (1997). Here there is also a simple intuition for the fact that price stability is optimal, first stated by Goodfriend and King (1997): the model is one in which, if prices were perfectly flexible, the equilibrium allocation of resources would be optimal. Even with staggered price adjustment, a policy that achieves zero inflation at all times leads to an equilibrium allocation of resources that is the same as if prices were flexible; hence the policy is optimal.

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More interesting is the conclusion that even when the steady-state is inefficient ($\Phi > 0$), a policy of complete price stability is *still* optimal (from a timeless perspective[37]) in the isoelastic case, as long as there are no government purchases. (The absence of government purchases is actually necessary in order for this case to be isoelastic in the relevant sense; for it is only if $G_t = 0$ that (1.3) implies that the marginal utility of income will be an isoelastic function of the level of output Y_t, and not simply of the level of consumption C_t.)

This result provides an analytical explanation of certain numerical results obtained by Khan *et al.* (2003) in a closely related model.[38] Khan *et al.* assume isoelastic functional forms, as we have, and also calibrate their model so that in the steady state there are no government purchases ($s_G = 0$), even though they consider the effects of small departures of G_t from the steady-state value of zero. When they consider the optimal policy response to a technology shock, and use a linearization method[39] to compute a linear approximation to the optimal response — i.e., to compute the derivative of the optimal paths with respect to the amplitude of the technology shock, evaluated at the case of a zero disturbance (the steady state) — they are in effect computing a linear approximation to optimal policy in a model in which there are no government purchases, since they compute a perturbation which involves no change in the level of government purchases to a steady state with no government purchases.

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[37]That is, it is optimal among policies that satisfy an initial commitment (2.15) with $\bar{\pi}_{t_0} = 0$, though it is not optimal in the absence of such a constraint, except when $\Phi = 0$. See the comparison of Ramsey policy to timelessly optimal policy in the low-Φ case treated in Woodford, 2003b, chap. 7, sec. 1.1. We are not able to give a similar analysis here of the initial dynamics under (unconstrained) Ramsey policy when Φ is large using our linear-quadratic approach, for the initial inflation rate under Ramsey policy is (in the low-Φ approximation) proportional to Φ, and so no longer small enough for our local analysis to be accurate unless $\Phi = \mathcal{O}(||\xi||)$.

[38]The model considered by Khan *et al.*, in the variant that abstracts from monetary frictions, is essentially the same as ours, except for a different form of staggering of pricing decisions: in their model, the probability that a price is revised each period depends on the number of periods since the last revision of that price, rather than being a constant as in the Calvo model. We discuss the consequences of this more general form of staggering in Benigno and Woodford (2004a).

[39]The method that they use to compute a linear approximation to optimal policy involves first writing the exact (nonlinear) first-order conditions that characterize optimal policy, then linearizing these first-order conditions, and solving the linearized equations. This method yields an identical linear approximation to optimal policy as the solution to our LQ problem though, as we have explained in section 2, we believe there are advantages to proceeding from an LQ approximate policy problem.

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In fact, Khan *et al.* find that the optimal response to a technology shock involves no change in the inflation rate (which continues to equal zero, the optimal steady-state inflation rate in their model as in ours[40]), and a response of output that is the same as would occur in a model with flexible prices (i.e., $\hat{Y}_t = \hat{Y}_t^n$).[41] This is just what Propositions 4 and 5 would imply for our model.

Instead, they find that the optimal response to a variation in government purchases involves some change in the inflation rate, and an output response that differs slightly from the flexible-price equilibrium response. This too is what our analysis would predict, in the case that $\Phi > 0$. Thus our results provide analytical insight into the reason for the numerical results obtained by Khan *et al.* for a particular numerical calibration, which allows us to better understand their degree of generality. On the one hand, we find that their conclusion with regard to technology shocks does not depend on their precise parameter values, except the choice to assume that $s_G = 0$. However, our analysis also indicates that they would not have obtained the same result under a more realistic calibration in which $s_G > 0$; so this simplification was not innocuous. Our further analysis in Benigno and Woodford (2004a) also shows that their result would not obtain, in general, in the case of non-isoelastic functional forms, even under the assumption that $s_G = 0$.

3.3 Optimal Responses to "Cost-Push" Disturbances

While in the previous section we have described cases in which complete price stability is optimal, we have also found that this is exactly true only in fairly special cases, when we allow (realistically) for a distorted steady state. In general, the "cost-push" term u_t will be non-zero. This is obviously true if there is time variation in the size of tax distortions or in wage markups, since disturbances of this kind affect the flexible-price equilibrium level of output while they are irrelevant for the efficient allocation of resources. But our results above show that even if there are no disturbances of those types, shocks to tastes or technology, or variations in government purchases, *also* generally give rise to fluctuations in the cost-push term. In any such case, it is not possible simultaneously to fully stabilize both inflation and the welfare-relevant

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[40]Here we mean the version of their model that abstracts from monetary frictions, as we do here. In their full model, the optimal steady-state inflation rate is slightly negative.

[41]King and Wolman (1999) obtain a similar conclusion in a model where government purchases are not considered at all.

output gap; the optimal trade-off between the two stabilization objectives generally involves some degree of variation in both variables in response to disturbances.

In order to consider optimal policy in this more general case, it suffices that we specify the stochastic process for fluctuations in the composite cost-push term $\{u_t\}$; the underlying source of those fluctuations does not matter, at least as far as the optimal fluctuations in inflation and in the welfare-relevant output gap are concerned. (The optimal responses of other variables, such as output, employment, or private consumption, will instead generally depend on what kind of real disturbances have occurred.) It follows from the approximation introduced in section 2 that a log-linear approximation to the optimal evolution of inflation and the output gap are given by the processes $\{\pi_t, y_t\}$ that minimize L_{t_0}, subject to the constraints that the aggregate-supply relation (2.16) be satisfied each period, and that the initial inflation rate satisfy a constraint of the form (2.15). The solution to this problem plainly depends only on the stochastic evolution of the composite cost-push term. Thus from this point we make treat the specification of the transitory fluctuations $\{u_t\}$ as a primitive.

The form of the optimization problem just stated is the same as in a model where the steady state is assumed to be efficient ($\Phi = 0$); the only differences made by allowing Φ to be positive have to do with the expressions that we have derived for q_π and q_y as functions of underlying model parameters, the expression for u_t as a function of underlying disturbances, and the definition of the welfare-relevant output gap y_t. The solution to the problem is therefore the same (in the case of a given $\{u_t\}$ process and given values of q_π and q_y) as in the $\Phi = 0$ case treated in Woodford (2003b, chap. 7).[42] We recall here some of the main results presented there, which directly apply to the present case as well.

The first-order conditions for the optimization problem just stated are of the form

$$q_\pi \pi_t + \varphi_t - \varphi_{t-1} = 0, \tag{3.9}$$

$$q_y y_t - \kappa \varphi_t = 0, \tag{3.10}$$

for each $t \geq t_0$, where φ_t is the Lagrange multiplier associated with the constraint (2.16) in period t. Bounded processes $\{\pi_t, y_t, \varphi_t\}$ that satisfy (2.16) and (3.9) – (3.10) for each $t \geq t_0$ and are consistent with the initial condition (2.15) represent an optimum. Using (3.9) to eliminate π_t and (3.10) to eliminate y_t,[43] (2.16) becomes an

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[42]See also Clarida, Gali and Gertler (1999) for analysis of an LQ problem of this form.

equation for the evolution of the multiplier

$$\beta q_y E_t \varphi_{t+1} - [(1+\beta)q_y + \kappa^2 q_\pi]\varphi_t + q_y \varphi_{t-1} = q_\pi q_y u_t. \tag{3.11}$$

The initial condition (2.15) can similarly be expressed as a constraint on the path of the multipliers

$$\varphi_{t_0} - \varphi_{t_0-1} = -q_\pi \bar{\pi}_{t_0}. \tag{3.12}$$

An optimum can then be described by a bounded process $\{\varphi_t\}$ for all dates $t \geq t_0 - 1$ that satisfies (3.11) for each $t \geq t_0$ and is also consistent with (3.12).

Equation (3.11) has a unique bounded solution consistent with (3.12) if and only if the characteristic equation

$$\beta q_y \mu^2 - \left[(1+\beta)q_y + \kappa^2 q_\pi\right]\mu + q_y = 0 \tag{3.13}$$

has exactly one root such that $|\mu| < 1$. This requires that the characteristic equation have real roots, exactly one of which lies in the interval between -1 and 1; this in turn is true if and only if[44] $q_\pi \neq 0$ and

$$\frac{q_y}{q_\pi} > -\frac{\kappa^2}{2(1+\beta)}. \tag{3.14}$$

Note that in the case that $\Phi = 0$ (treated in Woodford, 2003b, chap. 7), this condition is necessarily satisfied, since in that case $q_\pi, q_y > 0$. We then obtain the following result.

PROPOSITION 6. Suppose that $q_\pi \neq 0$, and that (3.14) is satisfied in addition to the conditions listed in Proposition 3. Then in the case of any small enough value of $\bar{\pi}_{t_0}$, and any sufficiently tightly bounded fluctuations in the cost-push disturbance process $\{u_t\}$, the solution to the optimization problem stated in Proposition 2 involves fluctuations $\{\pi_t, y_t\}$ that remain forever within any given neighborhood of the steady-state values $(0, 0)$. These optimal dynamics are furthermore approximated (arbitrarily well, in the case of tight enough bounds on $\bar{\pi}_{t_0}$ and on the amplitude of the cost-push

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[43]Here we assume that both $q_\pi, q_y \neq 0$. Note that if either q_π or q_y happens to equal zero, optimal policy is easily characterized: it consists simply of the complete stabilization of the variable with the non-zero weight in the loss function.

[44]Note that while we have assumed $q_y \neq 0$ in the above derivation, (3.11), (3.12) and (3.13) are also correct even when $q_y = 0$.

terms) by the log-linear dynamics corresponding to the unique bounded solution to equations (2.16), (3.9) and (3.10) consistent with initial condition (2.15).

This solution is obtained by solving (3.9) and (3.10) for π_t and y_t respectively, where the multiplier process $\{\varphi_t\}$ is specified recursively by the relation

$$\varphi_t = \mu\varphi_{t-1} - q_\pi \sum_{j=0}^{\infty} \beta^j \mu^{j+1} E_t u_{t+j}. \tag{3.15}$$

Here μ is the root of (3.13) that satisfies $-1 < \mu < 1$, and the initial value φ_{t_0-1} is chosen so that that the solution is consistent with (2.15).

The proof follows exactly the same lines as in the case with $\Phi = 0$ treated in Woodford (2003b, chap. 7). Further details are given there of how one may compute the value of φ_{t_0-1} corresponding to a given initial commitment (2.15), and examples are given there of self-consistent initial commitments associated with policy that is optimal "from a timeless perspective."

In the isoelastic case, as discussed above, $q_\pi > 0$. One can then show furthermore that condition (3.14) implies condition (3.8), though the former condition is stronger.[45] Hence it suffices that (3.14) hold in order for Proposition 6 to apply. Since this is necessarily satisfied if $q_y \geq 0$, it also follows from our discussion above that if $s_G \leq 1/2$, the condition is necessarily satisfied. Thus in the isoelastic case, Proposition 6 necessarily applies, unless government purchases are a large share of total output. (But once again, it remains possible for the condition not to hold; indeed, it is possible for (3.14) to fail even though (3.8) is satisfied.)

As an example of the implications of Proposition 6, consider the case of exogenous fluctuations in the level of government purchases, according to a first-order autoregressive process of the form

$$\hat{G}_t = \rho_G \hat{G}_{t-1} + \epsilon_t^G, \tag{3.16}$$

---

[45]Whenever (3.8) is satisfied, so that a bounded solution to the first-order conditions would correspond to an optimum, there is necessarily no more than one bounded solution. However, there might be *no* bounded solution, as the optimal policy might involve mildly explosive dynamics. This is the case in which (3.8) is satisfied though (3.14) is not. We do not wish to consider such cases here, as our local LQ approximation to the policy problem could not be guaranteed to remain an accurate approximation in such a case. Hence we shall require that the stronger condition (3.14) be satisfied. In the case of an *exact* LQ problem, this condition would not be required in order for (3.11) to determine a well-defined optimal policy.

where $0 \leq \rho_G < 1$ and $\{\epsilon_t^G\}$ is an i.i.d., bounded, mean-zero exogenous shock process. It follows from the definition of the cost-push term in section 2 that in this case, $u_t = \gamma_G \hat{G}_t$, with a coefficient

$$\gamma_G \equiv -\kappa \Phi \frac{\sigma^{-1}}{(\omega + \sigma^{-1})q_y}.$$

In this case, (3.15) reduces to

$$\varphi_t = \mu \varphi_{t-1} + \phi_G \hat{G}_t,$$

where

$$\phi_G \equiv -\frac{q_\pi \mu \gamma_G}{1 - \beta \mu \rho_G}.$$

It then follows that an innovation ϵ_t^G to the level of government purchases affects the current level and expected future path of the Lagrange multiplier by an amount

$$E_t \varphi_{t+j} - E_{t-1} \varphi_{t+j} = \frac{\mu^{j+1} - \rho_G^{j+1}}{\mu - \rho_G} \phi_G \epsilon_t^G$$

for each $j \geq 0$. Given this impulse response for the multiplier, (3.9) – (3.10) can be used to derive corresponding impulse responses for prices and the output gap,

$$E_t p_{t+j} - E_{t-1} p_{t+j} = -\frac{1}{q_\pi} \frac{\mu^{j+1} - \rho_G^{j+1}}{\mu - \rho_G} \phi_G \epsilon_t^G, \tag{3.17}$$

$$E_t y_{t+j} - E_{t-1} y_{t+j} = \frac{\kappa}{q_y} \frac{\mu^{j+1} - \rho_G^{j+1}}{\mu - \rho_G} \phi_G \epsilon_t^G, \tag{3.18}$$

where in (3.17) we use the notation $p_t \equiv \log P_t$.

If we further specialize to the case in which $\bar{G} = 0$, so that $s_C = 1$ (as in the calibration of Khan *et al.*, 2003), then in the case of any $\Phi > 0$ we have

$$q_y = \omega + \Phi + \sigma^{-1}(1 - \Phi) > 0,$$

$$q_\pi = \frac{\theta}{\kappa} q_y > 0,$$

as a consequence of which one can show that $0 < \mu < 1$. We also observe in this case that $\gamma_G < 0$, as a result of which $\phi_G > 0$. It then follows that each of the coefficients of the impulse response function (3.17) is negative, while each of the coefficients of the impulse response function (3.18) is positive. That is, an unexpected increase in

government purchases results in a decrease in prices and an increase in the (welfare-relevant) output gap; both impulse responses return asymptotically to zero, without ever overshooting their long-run levels.

This provides us with an analytical explanation of the results of Khan *et al.* (2003) in a closely related model. They also find that the optimal response to an increase in government purchases involves a temporary reduction in prices, together with a greater contraction of private consumption (and a smaller increase in output) than would occur in the flexible-price equilibrium, or than would result from a monetary policy that completely stabilized inflation. Our analytical results here yield the same conclusion. Because $\gamma_G < 0$, an increase in government purchases causes a negative "cost-push shock," meaning that it is not possible to maintain $\hat{Y}_t$ equal to $\hat{Y}_t^*$ without deflation (as $\hat{Y}_t^*$ rises less than does the natural rate $\hat{Y}_t^n$). The optimal tradeoff between the objectives of inflation stabilization and output-gap stabilization requires one to accept some deflation, though not as much as would be required to maintain $\hat{Y}_t$ equal to $\hat{Y}_t^*$.

This involves an increase in the welfare-relevant output gap, and since $\hat{Y}_t^* = \psi_G \hat{G}_t$, where

$$\psi_G = \frac{\sigma^{-1}}{\omega + \sigma^{-1}} \frac{\omega + \sigma^{-1}(1 - \Phi)}{\omega + \Phi + \sigma^{-1}(1 - \Phi)} > 0,$$

the target level of output also increases; hence output increases relative to trend in response to such a shock. Nonetheless, optimal policy involves output temporarily *lower* than the flexible-price equilibrium level $\hat{Y}_t^n$, as found by Khan *et al.* The price-level response (3.17) implies that $E_t p_{t+1}$ falls by an amount that is $\mu + \rho_G < 2$ times as large as the decline in p_t; hence $E_t \pi_{t+1}$ does not decline by as much as does π_t (if it falls at all). It then follows from (2.13) that $\hat{Y}_t - \hat{Y}_t^n$ must fall in response to a positive innovation ϵ_t^G. Thus output rises less (at least in the period of the shock) under optimal policy than it would in a flexible-price equilibrium; or alternatively, consumption falls by more than it would in a flexible-price equilibrium, as reported by Khan *et al.* Our results for a model with Calvo pricing are thus qualitatively similar to theirs for a model with an alternative form of staggering of price changes, and we are also able to obtain precise analytical expressions for the size of the effects in question.

4 Evaluation of Sub-Optimal Policy Rules

We have noted in the introduction that an advantage of our linear-quadratic approach over the method used by authors such as Khan *et al.* (2003) to obtain a local log-linear characterization of optimal policy is that our approach also allows us to evaluate the welfare consequences of alternative policy rules, as long as these rules imply paths for the endogenous variables that remain forever close enough to the optimal steady state in the event of small enough stochastic disturbances.[46] For example, one may wish to determine which rule is best from among some class of "simple" policy rules, all of which are sub-optimal. Here we illustrate how this can be done, taking as our application the computation of the optimal policy from among the class of policies that are "purely forward-looking" in the sense of Woodford (2003b, chap. 7).

4.1 A Generalized Quadratic Loss Function

It might seem natural to evaluate alternative rules in terms of the implied value of the ex ante expected utility of the representative household (1.8), in the equilibrium determined by the policy together with the constraints (1.14), (1.15), (1.19) and (1.20) for each $t \geq t_0$, given the initial price dispersion $\Delta_{t_0 - 1}$. However, it might happen that rules other than the optimal policy from a "timeless perspective" characterized above would do better than that policy under this criterion. For optimal policy from a timeless perspective does not achieve the unconstrained maximum of criterion (1.8); it only maximizes (1.8) among those policies consistent with certain initial commitments regarding the values of F_{t_0} and K_{t_0}. Nor would it be appealing to define the optimal rule among some suboptimal family as the rule that maximizes (1.8) among those policies in the family that result in the specified values for F_{t_0} and K_{t_0}. For there might be *no* rules within the simple family consistent with that commitment; or — because of the stationarity of the simple policies, as in the example considered here — a constraint of that kind on initial behavior might also amount to a severe constraint on policy forever, contrary to the spirit of our introduction of the initial commitments.

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[46]The linear-quadratic approach developed here *cannot,* for example, be correctly used to characterize equilibrium inflation under discretionary policy, for in the case of large steady-state distortions, discretionary policy results in a high rate of inflation even in the absence of shocks. This problem is avoided in the analysis of discretionary policy in Woodford (2003b, chap. 7) by the assumption there that $\Phi = \mathcal{O}(||\xi||)$.

We note, however, that the constrained optimization problem characterized in Proposition 2 can alternatively be described as the choice of a process $\{x_t\}$ for $t \geq t_0$ to maximize the alternative objective

$$
\begin{aligned}
\hat{U}_{t_0} \quad \equiv \quad & E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} U(Y_t, \Delta_t; \xi_t) - \phi_{2,t_0-1}^* \alpha(\Pi_{t_0}^{(\theta-1)} F_{t_0} - \Pi_{t_0}^{*\,(\theta-1)} F_{t_0}^*) + \\
& -\phi_{3,t_0-1}^* \alpha(\Pi_{t_0}^{\theta(1+\omega)} K_{t_0} - (\Pi_{t_0}^*)^{\theta(1+\omega)} K_{t_0}^*),
\end{aligned} \tag{4.1}
$$

among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0$, given Δ_{t_0-1}.[47] Here $F_{t_0}^*$ is the initial commitment regarding the value of F_{t_0} in the problem characterized in Proposition 2, $K_{t_0}^*$ is the initial commitment regarding the value of K_{t_0}, and $\Pi_{t_0}^*$ is the initial inflation rate

$$
\Pi_{t_0}^* \equiv \Pi(F_{t_0}^*, K_{t_0}^*)
$$

implied by these commitments. The initial commitments in the problem characterized in Proposition 2 can alternatively be written in the form

$$
\Pi_{t_0}^{\theta-1} F_{t_0} = \Pi_{t_0}^{*(\theta-1)} F_{t_0}^*,
$$

$$
\Pi_{t_0}^{\theta(1+\omega)} K_{t_0} = \Pi_{t_0}^{*\theta(1+\omega)} K_{t_0}^*,
$$

as discussed in Appendix B.1. The constants $\phi_{2,t_0-1}^*, \phi_{3,t_0-1}^*$ are Lagrange multipliers associated with these two constraints on the problem characterized in Proposition 2, and their values depend on the values of the initial commitments, along with other parameters defining the problem. An advantage of writing the modified objective in this particular form is that the problem of maximizing (4.1) subject to the constraints (1.19) and (1.20) results in exactly the same Lagrangian as the one given in Appendix B.1 for the problem characterized in Proposition 2.

The objective (4.1) can be evaluated in the case of any policy that results in paths for the endogenous variables that satisfy certain bounds, and the optimal policy consistent with the particular initial commitments $X_{t_0}^*$ maximizes this objective among all such policies. Hence we propose to also use the criterion (4.1) to rank alternative suboptimal policies, where the multipliers ϕ_{j,t_0-1}^* are the ones associated with the optimal policy problem.

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[47]This alternative way of posing the policy problem can be the basis for an alternative recursive formulation, as discussed by Khan *et al.* (2003).

Once again, in the case of rules that imply paths for the endogenous variables that remain always near the optimal steady state in the event of small enough shocks, it suffices to compute the value of a second-order Taylor series approximation to (4.1) to compare alternative policies. Similar calculations as have been used above to derive the quadratic loss function for the optimal policy problem now yield

$$
\begin{aligned}
\hat{U}_{t_0} &= -\Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{q_\pi}{2} \pi_t^2 + \frac{q_y}{2} (\hat{Y}_t - \hat{Y}_t^*)^2 \right\} + T_{t_0} + \bar{\phi}_2^* \frac{(1-\alpha\beta)(\omega + \sigma^{-1})}{\kappa} \bar{F} V_{t_0} \\
&\quad + \hat{\phi}_{t_0-1}^{*\prime} Q \hat{X}_{t_0} + \text{t.i.p.} + \mathcal{O}(||\xi||^3),
\end{aligned}
\tag{4.2}
$$

where the definitions of $q_\pi, q_y, Y_t^*, T_{t_0}$, and V_{t_0} are the same as in (2.9); $\bar{F}$ is the steady-state value of F_t; $\bar{\phi}_2^*$ is the steady-state value of the Lagrange multiplier associated with the initial commitments (*i.e.*, the value of the multiplier in the case that the initial commitments are consistent with the optimal steady state);[48]

$$
\hat{\phi}_{t_0-1}^* \equiv \left[\begin{array}{c} \hat{\phi}_{2,t_0-1}^* \\ \hat{\phi}_{3,t_0-1}^* \end{array} \right]
$$

is the vector of log deviations of the multipliers ϕ_{j,t_0-1}^* from their steady-state values; $\hat{X}_{t_0}$ is the vector of log deviations of the elements of X_{t_0} from their steady-state values under the policy under consideration, and Q is a matrix of coefficients defined in Appendix B.6.

Note that all terms in (4.2) are the same as in (2.9), except the last term on the first line and the first term on the second line; these latter terms represent a quadratic approximation to the additional terms in $\hat{U}_{t_0}$ relative to U_{t_0}. But in fact one can show that the last two terms on the first line cancel, so that (4.2) can be written more simply as

$$
\begin{aligned}
\hat{U}_{t_0} &= -\Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{q_\pi}{2} \pi_t^2 + \frac{q_y}{2} (\hat{Y}_t - \hat{Y}_t^*)^2 \right\} + \hat{\phi}_{t_0-1}^{*\prime} Q \hat{X}_{t_0} \\
&\quad + \text{t.i.p.} + \mathcal{O}(||\xi||^3).
\end{aligned}
\tag{4.3}
$$

We thus have a criterion to evaluate that is *purely quadratic*; because there are no linear terms, we can evaluate $\hat{U}_{t_0}$ to second-order accuracy using only a log-linear

---

[48]See Appendix B.2 for further discussion of the steady-state values of both endogenous variables and Lagrange multipliers.

approximation to the equilibrium dynamics under a given policy.[49] Hence a linear-quadratic approximation also yields a correct ranking of alternative "simple" policy rules. The correct quadratic criterion to use in ranking alternative policies is actually (4.3) rather than (2.9); but this criterion reduces to the earlier one in the case of policies that satisfy the initial commitments, since the vector $\hat{X}_{t_0}$ takes the same value (the precommitted value) in the case of all such policies.

Because the criterion (4.3) involves the multipliers $\hat{\phi}^*_{t_0-1}$, it depends on the particular values that are assumed for the initial commitments $X^*_{t_0}$. In the spirit of our proposal to analyze optimal policy from a "timeless perspective," we propose to rank alternative policy rules according to the unconditional expected value of $\hat{U}_{t_0}$, averaging over all the initial states that the economy might be in at date t_0, and assuming in each case initial commitments of the kind that would exist in that state in the case that an optimal policy had already been followed for a sufficiently long period of time. More specifically, we find a function $X(\xi_t, \xi_{t-1}, \ldots)$ that is *self-consistent,* in the sense that for any initial history $(\xi_{t_0}, \xi_{t_0-1}, \ldots)$, if the economy begins with that history and the initial commitments are given by $X^*_{t_0} = X(\xi_{t_0}, \xi_{t_0-1}, \ldots)$, then the optimal recursive commitment characterized in Proposition 2 involves $X_t = X(\xi_t, \xi_{t-1}, \ldots)$ in each period $t \geq t_0$. Associated with the optimization problem defined by these initial commitments will be Lagrange multipliers

$$\phi_{t_0-1} = \phi(\xi_{t_0-1}, \xi_{t_0-2}, \ldots), \tag{4.4}$$

that depend only on the history through period $t_0 - 1$.[50] Our proposal is to evaluate (4.3) for a given policy rule using the multipliers ϕ_{t_0-1} given by (4.4). We furthermore propose to rank alternative rules according to the implied value of $\mathrm{E}[\hat{U}_{t_0}]$, where the unconditional expectation operator integrates over alternative possible initial histories $(\xi_{t_0}, \xi_{t_0-1}, \ldots)$.

---

[49]Some authors prefer to rank alternative simple policy rules according to the associated value of $\mathrm{E}[U_{t_0}]$ rather than $\mathrm{E}[\hat{U}_{t_0}]$. In this case, a correct ranking of alternative rules is *not* possible on the basis of a solution of the log-linearized structural equations under the candidate rules, because of the presence of the term T_{t_0} in (2.9), which can only be evaluated to second-order accuracy using a second-order accurate solution for the evolution of the endogenous variables.

[50]The fact that they depend only on the history through period $t_0 - 1$ depends on the initial commitments having been chosen in a self-consistent manner, so that the Lagrange multipliers ϕ_{t_0-1} coincide with the multipliers associated with an optimal policy problem looking forward from date $t_0 - 1$.

In the case that the initial commitments are chosen in this way, we show in Appendix B.6 that our quadratic criterion can alternatively be expressed as

$$
\begin{aligned}
\hat{U}_{t_0} \;=\; & -\Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{q_\pi}{2} \pi_t^2 + \frac{q_y}{2} y_t^2 \right\} + \Omega \varphi_{t_0-1}^{*} (\pi_{t_0} - \pi_{t_0}^{*}) \\
& + \text{t.i.p.} + \mathcal{O}(\|\xi\|^3), \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (4.5)
\end{aligned}
$$

where $\varphi_{t_0-1}^{*} = -\Phi \kappa^{-1} \hat{\phi}_{2,t_0-1}^{*}$, and $\pi_{t_0}^{*}$ is the value of $\log \Pi_{t_0}^{*}$ implied by the self-consistent commitments $X_{t_0}^{*}$. We can accordingly rank alternative sub-optimal rules according to the loss function

$$
L_{t_0}^{r} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{q_\pi}{2} \pi_t^2 + \frac{q_y}{2} y_t^2 \right\} - \varphi_{t_0-1}^{*} (\pi_{t_0} - \pi_{t_0}^{*}), \quad\quad (4.6)
$$

generalizing (2.14). (Once again, the final term is zero in the case of all policies consistent with the self-consistent commitments, so that policies of that kind are correctly ranked by the previous loss function (2.14).)

Note that this is exactly the same criterion that we would use to rank alternative policies from a timeless perspective if we were to assume an exact linear-quadratic problem defined by loss function (2.14) and structural equation (2.16). Because of the role of expected inflation in the constraint, we would in that case have concluded that an optimal policy from a timeless perspective does not minimize (2.14) subject to (2.16) being satisfied for each $t \geq t_0$; instead, we would need to impose an additional constraint (2.15), as in the discussion in section 2. Alternatively, the optimal policy would minimize a modified loss function of the form (4.6), where $\varphi_{t_0-1}^{*}$ would represent a Lagrange multiplier associated with the constraint (2.15) on the initial inflation rate. This would suggest ranking alternative suboptimal policies according the value of (4.6); in fact, we have shown that this is correct, to second order in the amplitude of the exogenous disturbances.

One can also show that, to first order, $\varphi_{t_0-1}^{*}$ is just the time $t_0 - 1$ Lagrange multiplier associated with the aggregate-supply constraint in the LQ problem discussed in section 3. In particular, we have shown that in the solution to this problem, $\{\varphi_t\}$ satisfies (3.15). We can thus obtain a solution for $\varphi_{t_0}^{*}$ as a function of the initial history $(\xi_{t_0}, \xi_{t_0-1}, \ldots)$ by integrating (3.15).

4.2 The Optimal Non-Inertial Plan

As an application of the proposed criterion for ranking alternative suboptimal policies, we give a local characterization of the optimal non-inertial policy in our model, in the case of small enough shocks. By a *non-inertial* policy we mean one in which the central bank's behavior is a function only of the economy's current state (more precisely, only of those aspects of the current state that matter for the determination of current and/or future values of the endogenous variables $\{x_t\}$ that matter for welfare.[51] If policy results in a determinate equilibrium, this must also be one in which the endogenous variables x_t are functions only of the current state.

If we assume that the process for u_t is Markovian, with a law of motion of the form

$$u_t = \rho_u u_{t-1} + \varepsilon_t, \tag{4.7}$$

where $|\rho_u| < 1$ and $\{\varepsilon_t\}$ is a mean-zero i.i.d. random sequence, then the current state of the economy is summarized by the two variables (u_t, Δ_{t-1}). It follows that under any non-inertial (purely forward-looking) policy, the variables π_t, Y_t, and Δ_t will all be time-invariant functions of (u_t, Δ_{t-1}). It furthermore follows from the form of the approximate criterion (4.6) that in order to evaluate this criterion to second order, it suffices that we characterize the evolution of the two variables $\{\pi_t, Y_t\}$ to first order. Since any dependence of these variables on Δ_{t-1} would represent at most a second-order term,[52] a completely general description of the possible equilibrium dynamics, accurate to first order, can be written in the form

$$\pi_t = \bar{\pi} + f_\pi u_t, \tag{4.8}$$

$$y_t = \bar{y} + f_y u_t, \tag{4.9}$$

where the coefficients $\bar{\pi}, \bar{y}, f_\pi, f_y$ remain to be determined.

It further follows from (2.16) that the coefficients must satisfy the restrictions

$$(1 - \beta)\bar{\pi} = \kappa \bar{y} \tag{4.10}$$

---

[51]The proposed restriction is analogous to the common restriction to "Markovian strategies" in the literature on policy games. Our definition follows Woodford (2003a; 2003b, chap. 7).

[52]Here we assume, as earlier, that the initial price dispersion $\Delta_{t_0-1} = \mathcal{O}(||\xi||^2)$. In the case of a larger initial dispersion of prices, the optimal dynamics of inflation and the output gap would depend on the degree of price dispersion in the way discussed by Yun (2005).

$$(1 - \beta\rho_u)f_\pi = 1 + \kappa f_y \tag{4.11}$$

in any possible equilibrium. There is thus only a two-parameter family of possible non-inertial policies, to the order of accuracy that matters for our calculations.[53] The assumptions required for the validity of our local expansions are satisfied by all policies in this simple family, as long as $\bar{\pi}$ and $\bar{y}$ are small (*i.e.*, of order $\mathcal{O}(\|\xi\|)$).

Evaluation of (4.6) requires that we compute $\varphi^*_{t_0-1}$ for each possible initial history of the exogenous disturbances. We can do this using (3.15) and (4.7) to obtain

$$
\begin{aligned}
\varphi^*_t &= \mu\varphi^*_{t-1} - q_\pi \frac{\mu}{1 - \rho_u\beta\mu} u_t \\
&= -q_\pi \frac{\mu}{1 - \rho_u\beta\mu} \sum_{j=0}^\infty \mu^j u_{t-j}.
\end{aligned} \tag{4.12}
$$

Moreover, using expressions (4.8) – (4.9) for the evolution of the endogenous variables, together with (4.7), we can compute the values of the inflation and output gap terms in (4.6) as functions of u_{t_0}. In this way we obtain

$$E[L^r_{t_0}] = \frac{(q_y\bar{y}^2 + q_\pi\bar{\pi}^2)}{2(1-\beta)} + \frac{q_y f_y^2 + q_\pi f_\pi^2}{2} \frac{\sigma_u^2}{(1-\beta)} + q_\pi f_\pi \frac{\rho_u\mu}{(1 - \rho_u\beta\mu)(1 - \rho_u\mu)}\sigma_u^2, \tag{4.13}$$

through calculations that are further explained in Appendix B.6.

The optimal non-inertial plan is then described by coefficients $\bar{\pi}, \bar{y}, f_\pi, f_y$ that minimize (4.13) under the restrictions (4.10) – (4.11).[54] The solution to this problem is given by

$$\bar{\pi} = \bar{x} = 0,$$

$$f_y = -\frac{\kappa q_\pi}{[(1 - \beta\rho_u)^2 q_y + \kappa^2 q_\pi]}\left[1 + \frac{\rho_u\mu(1 - \beta)(1 - \beta\rho_u)}{(1 - \beta\rho_u\mu)(1 - \rho_u\mu)}\right],$$

$$f_\pi = \left[\frac{(1 - \beta\rho_u)\kappa^{-2}q_y - (1 - \beta)(1 - \beta\rho_u\mu)^{-1}(1 - \rho_u\mu)^{-1}\rho_u\mu q_\pi}{[(1 - \beta\rho_u)^2\kappa^{-2}q_y + q_\pi]}\right].$$

---

[53]These two parameters correspond to the choice of the average inflation target and a parameter that indexes the point that one chooses on the tradeoff between inflation stabilization and output-gap stabilization in response to cost-push shocks.

[54]This differs from the definition given in Woodford (2003b, chap. 7, sec. 3.1), though the present definition, like the one given there, implies that $\bar{\pi} = \bar{y} = 0$. The definition proposed here seems to us conceptually superior, as a single criterion (4.6) is used to determine both the optimal deterministic component of policy and the optimal responses to shocks.

Here we have characterized the best possible equilibrium outcome that can be achieved by *any* purely forward-looking policy rule. A variety of types of policy rules might be adopted in order to achieve this outcome. For example, the central bank might commit itself to a "flexible inflation targeting rule" of the form

$$\pi_t + \phi y_t = 0,$$

where the optimal weight on the output gap in the target criterion is given by

$$\phi = -\frac{f_\pi}{f_y}.$$

Alternatively (in the case that fluctuations in the natural rate of interest are also Markovian), it might commit itself to a Taylor-type interest-rate feedback rule. The method that can be used to compute the coefficients of a Taylor rule consistent with the above equilibrium outcome are discussed in Woodford (2003b, chap. 7, sec. 3.2).

5 Extensions

We have provided rigorous welfare-theoretic foundations for the form of linear-quadratic policy problem postulated in Clarida *et al.* (1999), among many other recent studies, in terms of the maximization of the expected utility of the representative household in a canonical "new Keynesian" model with monopolistic competition and staggered price-setting of the kind introduced by Calvo (1983). We have furthermore shown that this is possible even without the special assumption relied upon by Rotemberg and Woodford (1997) and Woodford (2002), according to which an output subsidy offsets the steady-state distortions that would otherwise result from the existence of market power on the part of the suppliers of differentiated goods. We find that a linear-quadratic policy problem of the same form is obtained even in the case of a distorted steady, indeed, one that may be substantially distorted, as a result of the tax system as well as market power. With a few caveats (such as the theoretical possibility of a failure of the coefficient q_y to be positive), we find that the conclusions of studies such as Clarida *et al.* (1999) regarding optimal monetary policy continue to apply in this case.

In Benigno and Woodford (2004a), we show that these conclusions can be generalized still further. In particular, we show that the special isoelastic functional forms

for preferences and technology assumed here are not necessary, except to simplify our calculations. In the case of completely general differentiable functions, we show that it is possible to derive a quadratic approximation to expected utility of the form (2.14); the only difference is that in the general case the expressions for the coefficients q_π, q_y, and the definition of the exogenous target level of output $\hat{Y}_t^*$ are more complicated. Proposition 3 continues to state the correct second-order conditions for the linear-quadratic optimization problem; but in the general case, it is theoretically possible for q_π as well as q_y to be negative, and there are additional theoretically possible cases in which the second-order conditions fail to hold. (We nonetheless continue to regard the cases in which the SOCs fail to hold as being of little practical interest.) In the general case there are also additional ways in which exogenous disturbances can give rise to "cost-push" terms in the aggregate-supply relation; for example, it is no longer true, in general, that a technology shock gives rise to no cost-push term, even in the case that $\bar{G} = 0$. Thus the case in which price stability is exactly optimal appears an even more special case; yet it remains true that for empirically realistic parameterizations, an optimal policy will involve only very small departures from a zero inflation rate.

We also show that a similar linear-quadratic policy problem can be defined in the case of staggering schemes other than Calvo's, *i.e.*, when the probability of revision of a given price is not independent of the length of time that it has been in effect; we discuss a more general framework that can deal with cases such as the fixed-length price commitments considered in Chari *et al.* (2000) and the more complex parameterization assumed by Khan *et al.* (2003). In this case, the welfare-theoretic loss function is no longer as simple as (2.14). However, it can still be expressed as a sum of squared price-differential terms (that all equal zero if and only if the aggregate price index never varies) and a squared output-gap term, so that once again price stability is optimal if and only if there are no "cost-push" disturbances to the aggregate-supply relation, and the sources of cost-push disturbances are essentially the same as in the case of Calvo pricing.

In Benigno and Woodford (2004b), we extend the present framework by allowing for sticky wages as well as prices. This allows us to generalize the welfare analysis of Erceg *et al.* (2000), again without relying upon the output and employment subsidies assumed by those authors, following the lead of Rotemberg and Woodford (1997). Again we find that even in the case of a distorted steady state, we can derive a

purely quadratic loss function, though this now includes a term proportional to the squared rate of nominal wage growth, in addition to the terms present in (2.14). As emphasized by Erceg *et al.,* this implies that in general complete stabilization of the inflation rate is not optimal. We find furthermore that in the case of a distorted steady state, the tensions among the three alternative stabilization objectives represented by the three terms in the welfare-theoretic loss function are greater than is indicated by the numerical results of Erceg *et al.* under the assumption of an efficient steady state.

The method used here has also proven fruitful for the analysis of monetary stabilization policy in open economies. In Benigno and Benigno (2004), the present analysis is extended to the case of a two-country model. In the case of an open economy, the device used by Rotemberg and Woodford (1997) is unavailable even in the presence of subsidies that offset the distortions due to market power, since it is no longer possible to express the consumption of the representative household by an exact function of domestic production and express utility in terms of the level of production only. The linear terms in the Taylor series expansion for the utility of the representative household of each country can nonetheless be eliminated using the method illustrated here, allowing derivation of a purely quadratic objective for each country that approximates the expected utility of its representative household. De Paoli (2004) similarly derives a utility-based loss function for a small open economy integrated with the rest of the world through complete financial markets. She shows that the objective function includes a target for the real exchange rate, and characterizes the optimal policy in this context.

Finally, in Benigno and Woodford (2003), we extend the present analysis to consider the jointly optimal determination of monetary and fiscal policy. The tax-rate process $\{\tau_t\}$ is considered to be freely chosen by a fiscal authority, rather than treated as exogenous as in this paper, and lump-sum taxes are assumed not to exist, so that an intertemporal solvency condition for the government becomes an additional constraint on possible state-contingent paths for the economy. The welfare-theoretic stabilization objective is again shown to be of the form (2.14), though the coefficients q_π, q_y and the target output process $\{\hat{Y}_t^*\}$ are defined somewhat differently, owing to the existence of the additional constraint.

The nature of the tensions between inflation stabilization and output-gap stabilization are also somewhat different when fiscal considerations are taken into account.

On the one hand, fluctuations in the cost-push term u_t do not necessarily imply any conflict between the two stabilization goals, as another policy instrument (variation in the tax rate τ_t) can be used to offset cost-push shocks. But on the other hand, there will be a conflict between the two goals, even in the absence of any cost-push effects, to the extent that shocks cause variations in the requirements for intertemporal government solvency (variations in "fiscal stress"). Hence the case in which complete price stability is optimal is found to be even more restrictive. Nonetheless, inflation stabilization is found to be an important goal (for *both* monetary and fiscal policy), and in our numerical analysis of the optimal response to fiscal disturbances, we conclude that inflation should fluctuate very little under an optimal policy.

Benigno and De Paoli (2005) extend this analysis to the case of a small open economy of the kind treated by De Paoli (2004). Ferrero (2005) studies a currency-area model with centralized monetary policy and independent fiscal authorities and characterizes the policy objective function for the whole area in order to determine the optimal decisions of monetary and fiscal policies. He also compares the jointly optimal plan with the rules that would be followed if fiscal authorities are constrained to maintain balanced national budgets. In all of these cases, a linear-quadratic approach proves insightful in characterizing both the appropriate aims of stabilization policy and the nature of optimal policy.

A   Proofs of Propositions

PROPOSITION 1. Given Δ_{t_0-1}, let the process $\{x_t\}$ be determined by (i) choosing x_{t_0} and state-contingent commitments $X_{t_0+1}(\xi_{t_0+1})$ to solve the first-stage problem stated in section 1.2, and (ii) for each possible state of the world ξ_{t_0+1}, choosing the evolution of x_t for $t \geq t_0+1$ so as to maximize U_{t_0+1}, among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0 + 1$, given Δ_{t_0}, and that are also consistent with the value of $X_{t_0+1}(\xi_{t_0+1})$ determined in the first stage. Then the process $\{x_t\}$ represents a *Ramsey policy*; that is, it maximizes U_{t_0} among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0$, given Δ_{t_0-1}.

PROOF: First, note that the process $\{x_t\}$ associated with the solution to the two-stage problem is a feasible plan for the Ramsey problem; that is, it satisfies (1.19) and (1.20) for each $t \geq t_0$, given Δ_{t_0-1}. For conditions (1.19) and (1.20) are satisfied for $t = t_0$ as a consequence of conditions (i) – (iii) of the first-stage problem, while they are satisfied for all dates $t \geq t_0 + 1$ as a consequence of the constraints on the second-stage problem. It then remains to show that there cannot be any other process $\{\tilde{x}_t\}$ that also satisfies all of the constraints of the Ramsey problem, and that attains a higher level of ex ante expected utility U_{t_0}.

The proof is by contradiction. Suppose that there exists such a process $\{\tilde{x}_t\}$, and let $\tilde{X}_{t_0+1}(\cdot)$ be the implied state-contingent values for X_{t_0+1} in each of the possible states of the world at date $t_0 + 1$, let $\tilde{U}_{t_0+1}(\xi_{t_0+1})$ be the utility looking forward from any given state of the world at date $t_0 + 1$ under that plan, and let $\tilde{U}_{t_0}$ be the implied level of ex ante expected utility under the plan. By hypothesis, $\tilde{U}_{t_0} > U_{t_0}$, where the latter quantity represents the level of ex ante expected utility implied by the solution to the two-stage problem.

Note then that the values $(\tilde{x}_{t_0}, \tilde{X}_{t_0+1}(\cdot))$ satisfy conditions (i) – (iii) of the first-stage problem. It is then possible to define $\hat{J}[\tilde{x}_{t_0}, \tilde{X}_{t_0+1}(\cdot)](\xi_{t_0})$. Because the process $\{\tilde{x}_t\}$ for $t \geq t_0 + 1$ is one possible plan consistent with (1.19) and (1.20) for each $t \geq t_0 + 1$, given $\tilde{\Delta}_{t_0}$, and also consistent with the precommitment $\tilde{X}_{t_0+1}(\xi_{t_0+1})$ in each possible state of the world at date $t_0 + 1$, we must have

$$V(\tilde{\Delta}_{t_0}, \tilde{X}_{t_0+1}; \xi_{t_0+1}) \geq \tilde{U}_{t_0+1}(\xi_{t_0+1})$$

for each possible state ξ_{t_0+1}. It follows from this that

$$\hat{J}[\tilde{x}_{t_0}, \tilde{X}_{t_0+1}(\cdot)](\xi_{t_0}) \geq \tilde{U}_{t_0}.$$

But then

$$\hat{J}[\tilde{x}_{t_0}, \tilde{X}_{t_0+1}(\cdot)](\xi_{t_0}) > U_{t_0},$$

which contradicts the assumption that the process $\{x_t\}$ solves the first-stage optimization problem. Hence no such alternative process $\{\tilde{x}_t\}$ can exist, and the process $\{x_t\}$ represents a Ramsey policy.

PROPOSITION 2. Given some $(\Delta_{t_0-1}, X_{t_0}) \in \mathcal{F}(\xi_{t_0})$, consider the sequential decision problem in which in each period $t \geq t_0$, $(x_t, X_{t+1}(\cdot))$ are chosen to maximize $\hat{J}[x_t, X_{t+1}(\cdot)](\xi_t)$, subject to constraints (i) – (iii) of the "first stage" problem stated above, given the predetermined state variable Δ_{t-1} and the precommitted values X_t. Then the process $\{x_t\}$ that is chosen in this way is the process that maximizes U_{t_0} among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0$, given Δ_{t_0-1}, and also consistent with the specified values X_{t_0}.

PROOF: Consider the problem of choosing a process $\{x_t\}$ to maximize U_{t_0} among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0$, given Δ_{t_0-1}, and also consistent with the specified values X_{t_0}. This is the same kind of optimization problem as in Proposition 1, except for the additional constraint that X_{t_0} take the specified values. Using a proof exactly analogous to the one used to establish Proposition 1, one can show that this problem is equivalent to a two-stage problem in which (i) one chooses x_{t_0} and state-contingent commitments $X_{t_0+1}(\xi_{t_0+1})$ to solve the first-stage problem stated in section 1.2, except with the additional stipulation that equations (1.21) – (1.22) are satisfied by the specified values for X_{t_0}; and (ii) for each possible state of the world ξ_{t_0+1}, one chooses the evolution of x_t for $t \geq t_0+1$ so as to maximize U_{t_0+1}, among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0 + 1$, given Δ_{t_0}, and that are also consistent with the value of $X_{t_0+1}(\xi_{t_0+1})$ determined in the first stage. This establishes that in the optimal plan, $(x_{t_0}, X_{t_0+1}(\cdot))$ solve a "first stage" problem of the kind described in the proposition.

Note furthermore that the "second stage" problem here is exactly the same form of optimization problem as the one considered in the proposition. One can then use the same proof to show that it is itself equivalent to a two-stage problem of the same

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kind. This then implies that in the optimal plan, $(x_{t_0+1}, X_{t_0+2}(\cdot))$ solve a "first stage" problem of the kind described in the proposition. The same argument can be applied, iteratively $(t-t_0+1$ times), to establish that for any period $t \geq t_0$, in the optimal plan $(x_t, X_{t+1}(\cdot))$ solve a "first stage" problem of the kind described in the proposition.

Now suppose that for each $t \geq t_0$, $(x_t, X_{t+1}(\cdot))$ are chosen to solve the "first stage" problem described in the proposition, given the solution for previous periods, as assumed in the hypothesis. It follows from the argument just given that in any period t, the vector x_t chosen in this way coincides (for every possible history) with the one that would be chosen under an optimal plan, as asserted by the proposition.

PROPOSITION 3. Randomization of monetary policy increases the expected losses (2.14) — and hence is locally welfare-reducing in the exact problem as well — if and only if the quadratic form (3.5) is positive definite on the subspace $\mathcal{H}_1$. Furthermore, if and only if this is true, processes $\{\pi_t, \hat{Y}_t\}$ that satisfy the first-order conditions for the LQ optimization problem represent a loss minimum, and hence an approximation to (at least a local) welfare maximum in the exact problem.

Furthermore, the necessary and sufficient conditions for (3.5) to be positive definite on $\mathcal{H}_1$ reduce to the following: q_π and q_y are not *both* equal to zero; and *either* (i) $q_y \geq 0$ and
$$q_\pi + (1 - \beta^{1/2})^2 \kappa^{-2} q_y > 0, \tag{A.1}$$
holds, or (ii) $q_y \leq 0$ and
$$q_\pi + (1 + \beta^{1/2})^2 \kappa^{-2} q_y > 0, \tag{A.2}$$
holds.

PROOF: (1) We begin by considering the second-order conditions for optimality, *i.e.*, the conditions under which a solution to the first-order conditions (3.9)–(3.10) will represent a loss minimum. Let $\{\pi_t, \hat{Y}_t\}$ be any stochastic processes in $\mathcal{H}$ consistent with both the equilibrium relation (2.16) at all dates $t \geq t_0$ and the initial constraint (2.15), and then consider the perturbed processes $\{\tilde{\pi}_t, \tilde{Y}_t\}$ defined by (3.1) for some stochastic processes $\{\psi_t^\pi, \psi_t^y\} \in \mathcal{H}_1$. Because the perturbation processes are assumed to satisfy (3.3) and (3.4), both the original processes $(\pi, \hat{Y})$ and the perturbed processes $(\tilde{\pi}, \tilde{Y})$ represent rational-expectations equilibria consistent with (2.15). It also follows from our hypotheses that both pairs of stochastic processes

belong to $\mathcal{H}$, and hence that the loss function (2.14) is well-defined for each pair of processes.

Let $L(\pi, \hat{Y})$ denote the value of (2.14) in the case of the original processes and $L(\tilde{\pi}, \tilde{Y})$ the value in the case of the perturbed processes. Then

$$L(\tilde{\pi}, \tilde{Y}) = L(\pi, \hat{Y}) + E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} [q_\pi \pi_t \psi_t^\pi + q_y \hat{Y}_t \psi_t^y] + E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[\frac{q_\pi}{2} \psi_t^{\pi 2} + \frac{q_y}{2} \psi_t^{y 2} \right].$$
(A.3)

Suppose furthermore that the original processes $\{\pi_t, \hat{Y}_t\}$ satisfy the first-order conditions for a loss minimum (3.9)–(3.10). This implies that the middle term on the right-hand side of (A.3) must equal zero, for *any* processes $\{\psi_t^\pi, \psi_t^y\}$ satisfying (3.2) – (3.4). A solution to the first-order conditions is then a loss minimum if and only if

$$E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[\frac{q_\pi}{2} \psi_t^{\pi 2} + \frac{q_y}{2} \psi_t^{y 2} \right] > 0$$
(A.4)

for any processes $\{\psi_t^\pi, \psi_t^y\}$ satisfying (3.2) – (3.4), other than the trivial case in which $\psi_t^\pi = \psi_t^y = 0$ for all t almost surely. Thus the first- and second-order conditions are jointly necessary and sufficient for a pair of processes $(\pi, \hat{Y}) \in \mathcal{H}$ to represent a loss minimum in the LQ problem; they also imply that the solution $\{\pi_t, \hat{Y}_t\}$ approximates an equilibrium that maximizes expected utility at least *locally* in the exact policy problem.

(2) The second-order conditions (A.4) are also necessary and sufficient in order for arbitrary randomization of policy to be welfare-reducing, at least locally. For suppose that $\{\tilde{\pi}_t, \tilde{Y}_t\}$ are some equilibrium processes consistent with (2.15), (2.16) and (3.2), which depend non-trivially on the realization of a "sunspot" variable at some date $t > t_0$. Then let $\{\pi_t, \hat{Y}_t\}$ be the processes obtained by averaging the processes $\{\tilde{\pi}_t, \tilde{Y}_t\}$ over the alternative sunspot states with the same values of all "fundamental" disturbances. The processes $\{\pi_t, \hat{Y}_t\}$ will then also satisfy (2.16) for all $t \geq t_0$, (2.15) and (3.2). Defining the processes $\{\psi_t^\pi, \psi_t^y\}$ by relations (3.1), one notes that

$$E_{t_0} \pi_t \psi_t^\pi = E_{t_0} \hat{Y}_t \psi_t^y = 0$$

for all $t > t_0$. Hence the middle term on the right-hand side of (A.3) is equal to zero. Then if the second-order conditions (A.4) hold, it follows that $L(\pi, \hat{Y}) < L(\tilde{\pi}, \tilde{Y})$. Since a lower-loss equilibrium can be found in the case of any equilibrium $\{\tilde{\pi}_t, \tilde{Y}_t\}$ that involves arbitrary randomization, optimal policy cannot involve such randomization.

(3) Conversely, suppose that the second-order conditions do *not* hold. Then there exist processes $\{\psi_t^\pi, \psi_t^y\}$, not both equal to zero almost surely at all times, such that the expression in (A.4) is less than or equal to zero. Since condition (A.4) depends only on the serial correlation properties of the processes $\{\psi_t^\pi, \psi_t^y\}$, and not on their relation to any fundamental sources of uncertainty, we may suppose that they are "sunspot" variables, distributed independently of the fundamental disturbances. We may furthermore suppose that they have *ex ante* mean zero, *i.e.,* that

$$E_{t_0}\psi_t^\pi = E_{t_0}\psi_t^y = 0 \tag{A.5}$$

for all $t \geq t_0$.[55]

Now consider any equilibrium processes $\{\pi_t, \hat{Y}_t\}$ consistent with (2.15) and (3.2), and the perturbed processes $\{\tilde{\pi}_t, \tilde{Y}_t\}$ defined by (3.1), where $\{\psi_t^\pi, \psi_t^y\}$ are the sunspot processes just discussed. The perturbed processes represent another possible equilibrium consistent with (2.15) and (3.2), one involving arbitrary randomization. Furthermore, because the processes $\{\psi_t^\pi, \psi_t^y\}$ are distributed independently of the processes $\{\pi_t, \hat{Y}_t\}$,

$$E_{t_0}\pi_t\psi_t^\pi = E_{t_0}\pi_t E_{t_0}\psi_t^\pi = 0,$$

and likewise for $E_{t_0}\hat{Y}_t\psi_t^y$. It follows that the middle term on the right-hand side of (A.3) must equal zero. Then the hypothesis that (A.4) does not hold implies that $L(\tilde{\pi}, \tilde{Y}) \leq L(\pi, \hat{Y})$, so that arbitrary randomization is not welfare-reducing. Thus the second-order condition is also necessary for this not to be possible.

(4) It remains to consider the algebraic conditions on the parameters of the LQ optimization problem under which (A.4) holds for all stochastic processes $\psi \in \mathcal{H}_1$ that are not equal to zero at all times almost surely. We first show that this is

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[55]Let $\{\hat{\psi}_t^\pi, \hat{\psi}_t^y\}$ be any sunspot processes (not almost surely equal to zero at all times) that satisfy (3.3) for all $t \geq t_0$ and (3.4), as well as (3.2), such that the expression in (A.4) is less than or equal to zero. By hypothesis, some such processes exist. Then consider the alternative sunspot processes such that $\tilde{\psi}_{t_0}^\pi = \tilde{\psi}_{t_0}^y = 0$, while the joint distribution of the processes $\{\tilde{\psi}_t^\pi, \tilde{\psi}_t^y\}$ for $t \geq t_0 + 1$ is identical to the joint distribution of the processes $\{\hat{\psi}_t^\pi, \hat{\psi}_t^y\}$ for $t \geq t_0$, under a time shift of one period. Finally, let $\psi_{t_0}^\pi = \psi_{t_0}^y = 0$, and $\psi_t^\pi = \sigma_{t_0+1}\tilde{\psi}_t^\pi$, $\psi_t^y = \sigma_{t_0+1}\tilde{\psi}_t^y$ for all $t \geq t_0 + 1$, where σ_{t_0+1} is another independently distributed sunspot variable, realized at date $t_0 + 1$, and taking the value -1 or 1, each with probability 1/2. Then the processes $\{\psi_t^\pi, \psi_t^y\}$ are also sunspot processes, not almost surely equal to zero at all times, that satisfy (3.3) for all $t \geq t_0$ and (3.4), as well as (3.2), and such that the expression in (A.4) is less than or equal to zero. In addition, the new processes $\{\psi_t^\pi, \psi_t^y\}$ necessarily satisfy (A.5), even if the original processes $\{\hat{\psi}_t^\pi, \hat{\psi}_t^y\}$ did not.

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equivalent to the positive definiteness of a corresponding quadratic form defined for deterministic sequences. Let $\bar{\mathcal{H}}$ be the Hilbert space of complex-valued sequences $\{\bar{\psi}_t^\pi, \bar{\psi}_t^y\}$ such that

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} |x_t|^2 < \infty \tag{A.6}$$

for $x = \bar{\psi}^\pi, \bar{\psi}^y$, and let $\bar{\mathcal{H}}_1$ be the subspace of $\bar{\mathcal{H}}$ consisting of sequences that in addition satisfy

$$\bar{\psi}_t^\pi = \kappa \bar{\psi}_t^y + \beta \bar{\psi}_{t+1}^\pi \tag{A.7}$$

for all $t \geq t_0$.[56] Then we shall establish that (A.4) holds for all (real-valued) stochastic processes $\{\psi_t^\pi, \psi_t^y\} \in \mathcal{H}_1$ that are not equal to zero at all times almost surely if and only if

$$\bar{L}(\bar{\psi}) \equiv \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[\frac{q_\pi}{2} |\bar{\psi}_t^\pi|^2 + \frac{q_y}{2} |\bar{\psi}_t^y|^2 \right] > 0 \tag{A.8}$$

for any complex-valued (deterministic) sequences $\{\bar{\psi}_t^\pi, \bar{\psi}_t^y\} \in \bar{\mathcal{H}}_1$ that are not equal to zero at all times.

We begin by showing that (A.4) holding on $\mathcal{H}_1$ implies that (A.8) must hold on $\bar{\mathcal{H}}_1$. We show this by contradiction. Suppose instead that that there exists a pair of sequences $\{\bar{\psi}_t^\pi, \bar{\psi}_t^y\} \in \bar{\mathcal{H}}_1$, not both equal to zero at all dates, for which (A.8) does not hold. If a pair of complex-valued sequences of this kind exist, we can also find a pair of real-valued sequences. For any $\bar{\psi} \in \bar{\mathcal{H}}_1$ can be written as

$$\bar{\psi} = \bar{\psi}^{re} + i \bar{\psi}^{im},$$

where $\bar{\psi}^{re}, \bar{\psi}^{im}$ are real-valued sequences, and it can be shown that $\bar{\psi}^{re}, \bar{\psi}^{im}$ are both real-valued elements of $\bar{\mathcal{H}}_1$. Furthermore, one observes that

$$\bar{L}(\bar{\psi}) = \bar{L}(\bar{\psi}^{re}) + \bar{L}(\bar{\psi}^{im}).$$

Then as by hypothesis $\bar{L}(\bar{\psi}) \leq 0$, it follows that $\bar{L} \leq 0$ for at least one of the real-valued sequences as well. Thus we may assume without loss of generality that $\bar{\psi}$ is a real-valued sequence.

Then we can define a real-valued sunspot process $\psi_{t_0}^\pi = \psi_{t_0}^y = 0$, and $\psi_t^\pi = \sigma_{t_0+1} \bar{\psi}_{t-1}^\pi$, $\psi_t^y = \sigma_{t_0+1} \bar{\psi}_{t-1}^y$ for all $t \geq t_0+1$, where σ_{t_0+1} is an independently distributed

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[56]Note that in the definition of the subspace $\bar{\mathcal{H}}_1$, we do not require that a condition analogous to (2.15) be satisfied.

sunspot variable, realized at date $t_0 + 1$, and taking the value -1 or 1, each with probability 1/2. Then the processes $\{\psi_t^\pi, \psi_t^y\}$ satisfy (3.2), are not almost surely equal to zero at all times, satisfy (3.3) for all $t \geq t_0$, and satisfy (3.4), but are such that the left-hand side of (A.4) is less than or equal to zero. Thus (A.4) would not hold for all processes $\psi \in \mathcal{H}_1$. It follows that if (A.4) holds on $\mathcal{H}_1$, (A.8) must hold for all complex-valued sequences $\bar{\psi} \in \bar{\mathcal{H}}_1$.

(5) Conversely, one can also show that (A.8) holding on $\bar{\mathcal{H}}_1$ implies that (A.4) must hold on $\mathcal{H}_1$. Let any process $\psi \in \mathcal{H}_1$ be decomposed as

$$\psi_t = \sum_{j=0}^{t-t_0} \psi_t^{(j)},$$

where $\psi_t^{(0)} \equiv E_{t_0}\psi_t$ and $\psi_t^{(j)} \equiv E_{t_0+j}\psi_t - E_{t_0+j-1}\psi_t$, for each $j \geq 1$. Note that this implies that $\psi_t^{(j)} = 0$ for all $t_0 \leq t < t_0 + j$, and that the entire sequence $\{\psi_t^{(j)}\}$ is known with certainty at date $t_0 + j$. It then follows that

$$E_{t_0}x_t^2 = \sum_{j=0}^{t-t_0} E_{t_0}x_t^{(j)2} \tag{A.9}$$

for $x = \psi^\pi, \psi^y$, from which it follows that if the process $\{x_t\}$ satisfies (3.2), the process $\{x_t^{(j)}\}$ must also satisfy (3.2), for each $j \geq 0$. This in turn implies that for any j, the sequences of values $\{\psi_t^{(j)}\}$ for $t \geq t_0 + j$ satisfies (A.6) almost surely. Furthermore, if for any $j \geq 0$ we define the sequence $\bar{\psi}^{(j)}$ by $\bar{\psi}_t^{(j)} = \psi_{t+j}^{(j)}$ for all $t \geq t_0$, then the fact that (by hypothesis) the process ψ satisfies (3.3) furthermore implies that the sequences $\bar{\psi}^{(j)}$ each such satisfy (A.7) almost surely.[57] Thus for each $j \geq 0$, the sequence $\bar{\psi}^{(j)}$ belongs almost surely to $\bar{\mathcal{H}}_1$. Furthermore, there exists at least one j for which $\bar{\psi}^{(j)}$ is not almost surely equal to zero.

It follows from (A.9) that

$$L(\psi^\pi, \psi^y) = \sum_{j=0}^{\infty} \beta^j E_{t_0}\bar{L}(\bar{\psi}^{(j)}). \tag{A.10}$$

Since by hypothesis (A.8) holds for all elements of $\bar{\mathcal{H}}_1$, $L(\bar{\psi}^{(j)}) \geq 0$ for all j, and the inequality is strict in the case of those j (of which there must be at least one,

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[57]The value of the sequence $\bar{\psi}^{(j)}$ is known with certainty at date $t_0 + j$. The "almost surely" refers to the *ex ante* probability distribution over possible states of the world at date $t_0 + j$.

with positive probability) for which $\psi^{(j)} \neq 0$. Thus the sum on the right-hand side of (A.10) must be positive, from which it follows that ψ satisfies (A.4), as was to be proven.

(6) Our problem thus reduces to a search for necessary and sufficient conditions under which (A.8) must be satisfied by all complex-valued sequences $\bar{\psi} \in \bar{\mathcal{H}}_1$. We can show that this is equivalent to a related problem that arises in connection with the optimal control of a purely backward-looking system, so that classical results can be applied. Let $\bar{\mathcal{H}}_2$ be the subspace of $\bar{\mathcal{H}}_1$ consisting of those sequences that satisfy the additional condition $\psi_{t_0}^\pi = 0$. We shall establish that (A.8) holds for all sequences $\psi \in \bar{\mathcal{H}}_1$ if and only if it holds for all sequences in $\bar{\mathcal{H}}_2$. It is obvious, of course, that if (A.8) holds on $\bar{\mathcal{H}}_1$ it must hold on $\bar{\mathcal{H}}_2$. It remains to show that the converse is true as well.

For any complex number ψ_0, let us define[58]

$$V(\psi_0) \equiv \min_{\bar{\psi} \in \bar{\mathcal{H}}_1} \bar{L}(\bar{\psi}) \qquad \text{s.t.} \quad \bar{\psi}_{t_0}^\pi = \psi_0.$$

We establish the following properties of the function V. First, we note that if $\bar{\psi}$ is an element of $\bar{\mathcal{H}}_1$ consistent with initial condition ψ_0, then the complex conjugate sequence $\bar{\psi}^\dagger$ is an element of $\bar{\mathcal{H}}_1$ consistent with initial condition $\psi_0^\dagger$. Then since $\bar{L}(\bar{\psi}^\dagger) = \bar{L}(\bar{\psi})$, it follows that $V(\psi_0^\dagger) \leq V(\psi_0)$. The same argument can be used to show that $V(\psi_0) \leq V(\psi_0^\dagger)$, and so we conclude that $V(\psi_0^\dagger) = V(\psi_0)$ for all ψ_0. An argument of exactly the same form shows that $V(-\psi_0) = V(\psi_0)$ for all ψ_0.

Similarly, if $\bar{\psi}_1$ is an element of $\bar{\mathcal{H}}_1$ consistent with initial condition $\psi_{0,1}$, and $\bar{\psi}_2$ is an element of $\bar{\mathcal{H}}_1$ consistent with $\psi_{0,2}$, then for any real number $0 < \lambda < 1$, one observes that the sequence $\lambda \bar{\psi}_1 + (1-\lambda)\bar{\psi}_2$ is an element of $\bar{\mathcal{H}}_1$ consistent with initial condition $\lambda \psi_{0,1} + (1-\lambda)\psi_{0,2}$. Because $\bar{L}$ is a convex function,

$$\bar{L}(\lambda \bar{\psi}_1 + (1-\lambda)\bar{\psi}_2) \leq \lambda \bar{L}(\bar{\psi}_1) + (1-\lambda)\bar{L}(\bar{\psi}_2), \tag{A.11}$$

from which it follows that

$$V(\lambda \psi_{0,1} + (1-\lambda)\psi_{0,2}) \leq \lambda V(\psi_{0,1}) + (1-\lambda)V(\psi_{0,2}). \tag{A.12}$$

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[58]It is easily shown that the set of sequences $\bar{\psi} \in \bar{\mathcal{H}}_1$ consistent with any given initial value ψ_0 is non-empty. If there is no lower bound on the value of $\bar{L}$ on this set, the value of $V(\psi_0)$ is defined to be $-\infty$.

One thus establishes that V is a convex function of ψ_0. Furthermore, the inequality in (A.11) is strict unless $\bar{\psi}_1 = \bar{\psi}_2$, from which it follows that the inequality in (A.12) is strict unless $\psi_{0,1} = \psi_{0,2}$ or $V(\psi_{1,0}) = V(\psi_{2,0}) = -\infty$. Thus V is a strictly convex function, if there exists any ψ_0 for which $V(\psi_0) > -\infty$.

We have established that if there exists any ψ_0 for which $V(\psi_0) > -\infty$, V is a strictly convex function of ψ_0 with the properties that $V(\psi_0^\dagger) = V(\psi_0)$ and $V(-\psi_0) = V(\psi_0)$ for all ψ_0. It is easily shown that any such function must reach its unique minimum at $\psi_0 = 0$. Hence $V(\psi_0) > 0$ for all $\psi_0 \neq 0$ if and only if $V(\psi_0) \geq 0$. It then follows that $\bar{L}(\bar{\psi}) > 0$ for all non-zero $\bar{\psi} \in \bar{\mathcal{H}}_1$ if and only if the same inequality holds for all non-zero $\bar{\psi} \in \bar{\mathcal{H}}_1$ that satisfy the initial condition $\bar{\psi}_{t_0}^\pi = 0$, *i.e.*, all non-zero $\bar{\psi} \in \bar{\mathcal{H}}_2$. This is what we have sought to establish.

(7) Our problem now reduces to a search for necessary and sufficient conditions under which (A.8) must be satisfied by all complex-valued sequences $\bar{\psi} \in \bar{\mathcal{H}}_2$. This is just the second-order condition for optimality in the problem of minimizing

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{q_\pi}{2} \pi_t^2 + \frac{q_y}{2} y_t^2 \right\}, \tag{A.13}$$

subject to the constraints that the deterministic sequences $\{\pi_t, y_t\}$ satisfy (3.2) and the law of motion

$$\pi_{t+1} = \beta^{-1}[\pi_t - \kappa y_t], \tag{A.14}$$

starting from a given initial condition for the predetermined state variable π_{t_0}. (Note that (A.14) is just a deterministic version of (2.16), except that we now treat inflation as a predetermined state variable, so that the constraint (A.14) is no longer forward-looking.)

This problem is of the type studied by Telser and Graves (1972). We can write our problem as the minimization of a loss function of the form

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} x_t' B x_t$$

where

$$x_t \equiv \begin{bmatrix} \pi_t \\ y_t \end{bmatrix}, \qquad B \equiv \begin{bmatrix} q_\pi & 0 \\ 0 & q_y \end{bmatrix},$$

subject to a law of motion of the form

$$A(L)x_t = 0$$

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for all $t \geq t_0$, where
$$A(L) \equiv [1 \ 0] + [-\beta^{-1} \ \beta^{-1}\kappa]L.$$

Then by Theorems 5.1 and 5.3 of Telser and Graves, the second-order condition for this problem is satisfied — i.e., (A.8) is satisfied by all complex-valued sequences $\bar{\psi} \in \bar{\mathcal{H}}_2$ — if and only if the determinant of the bordered Hermitian matrix[59]

$$M(\theta) \equiv \begin{bmatrix} 0 & A(\beta^{1/2}e^{-i\theta}) \\ A'(\beta^{1/2}e^{i\theta}) & B \end{bmatrix}$$

is negative for all $-\pi \leq \theta \leq \pi$.

In our case,

$$\det M(\theta) = -q_\pi \beta^{-1}\kappa^2 - q_y(1 - 2\beta^{-1/2}\cos\theta + \beta^{-1}),$$

so that the SOC reduces to the requirement that

$$\min_\theta \left\{ q_\pi \beta^{-1}\kappa^2 + q_y(1 - 2\beta^{-1/2}\cos\theta + \beta^{-1}) \right\} > 0. \tag{A.15}$$

If $q_y \geq 0$, the minimum value of the term in curly braces occurs when $\theta = 0$, in which case the term in parentheses is equal to $(1 - \beta^{-1/2})^2$. If instead $q_y \leq 0$, the minimum value occurs when $\theta = \pm\pi$, in which case the term in parentheses is equal to $(1 - \beta^{-1/2})^2$. Making the appropriate substitution in each of the two cases, we find that (A.15) is equivalent to the inequalities stated in the proposition.

PROPOSITION 4. Suppose that $\hat{Y}_t^n = \hat{Y}_t^*$ at all times, and that the conditions stated in Proposition 3 are satisfied. Then the policy that uniquely minimizes L_{t_0} is the one under which $\pi_t = 0$ at all times, regardless of the realizations of the exogenous disturbances [as long as these are small enough to make such an equilibrium possible].

Proof. If $\hat{Y}_t^n = \hat{Y}_t^*$ at all times, then $u_t = 0$ at all times. The first-order necessary conditions for an optimum[60] are then

$$q_\pi \pi_t + \varphi_t - \varphi_{t-1} = 0,$$

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[59]In the way that Telser and Graves define the matrix $M(\theta)$, $2B$ appears as the lower right block rather than B, but this makes no difference for the second-order conditions that are implied. Note that replacing B by $B/2$ in the loss function does not change the optimization problem at all.

[60]See section 3.3 for further discussion of these.

$$q_y y_t - \kappa \varphi_t = 0,$$

together with

$$\pi_t - \kappa y_t - \beta E_t \pi_{t+1} = 0,$$

each of which must hold for all $t \geq t_0$. If there is an additional constraint of the form (2.15), then this condition also be satisfied by an optimum; if there is no such constraint, then one must adjoin the additional condition

$$\varphi_{t_0-1} = 0.$$

In the case that disturbances are small enough, a policy under which $\pi_t = 0$ at all times is feasible, since the nominal interest rate required for this equilibrium is non-negative at all times. Moreover this policy satisfies the above necessary conditions for an optimum, as all of these conditions are observed to be satisfied in the case that $\pi_t = y_t = \varphi_t = 0$ for all t. Proposition 3 implies that the second-order conditions are also satisfied, and that the zero-inflation policy represents a unique loss minimum for all $t \geq t_0$, among those policies consistent with an initial commitment $\bar{\pi}_{t_0} = 0$.

The Kuhn-Tucker theorem then implies that the zero-inflation policy also minimizes

$$L_{t_0} - \varphi_{t_0-1} \pi_{t_0},$$

for some value of the multiplier φ_{t_0-1}, subject only to the constraint that the paths $\{\pi_t, y_t\}$ represent a rational-expectations equilibrium. The first-order conditions for this alternative minimization problem are easily seen to be identical to the conditions written above, from which we observe that the value of the multiplier is zero. Hence the zero-inflation policy minimizes L_{t_0}, even when the value of π_{t_0} is unconstrained.

Proposition 5 is proved in the text, and Proposition 6 is directly analogous to the results derived in Woodford (2003b, chap. 7).

B   Additional Appendices

B.1   Lagrangian for the optimal policy problem

Here we present a Lagrangian for the optimal policy problem characterized in Proposition 2 (*i.e.*, the problem in which policy is constrained by initial commitments). We recall that the problem is to choose a process $\{x_t\}$ for dates $t \geq t_0$ to maximize (1.8) among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0$, given Δ_{t_0-1}, and also consistent with the precommitted values $X_{t_0}^*$.[61]

We wish to express the Lagrangian in the discounted stationary form

$$E_{t_0} \sum_{t=0}^{\infty} \beta^{t-t_0} g(\tilde{\phi}_t; \tilde{x}_t), \tag{B.1}$$

where $\tilde{x}_t$ is a vector of endogenous variables (that may include more variables than just x_t), $\tilde{\phi}_t$ is a vector of Lagrange multipliers, and the function $g(\cdot)$ takes the same form at each date, so that we obtain a time-invariant system of first-order conditions (which in turn make the definition of an optimal steady state straightforward). The constraints corresponding to the precommitments are not naturally in this form, since the definitions of (1.14) – (1.15) of F_t and K_t involve a different discount factor than the objective (1.8). However, we can obtain a system of constraints of a suitable form by writing them as constraints on the joint evolution of the processes $\{x_t, X_t\}$.

We can write (1.19) more compactly as

$$K_t p(\Pi_t)^{\frac{1+\omega\theta}{\theta-1}} = F_t, \tag{B.2}$$

where we define

$$p(\Pi_t) \equiv \left(\frac{1 - \alpha\Pi_t^{\theta-1}}{1 - \alpha} \right).$$

This is a first constraint on the possible joint values of x_t and X_t. Definitions (1.14) and (1.15) respectively imply that

$$F_t = (1 - \tau_t)f(Y_t; \xi_t) + \alpha\beta E_t[\Pi_{t+1}^{\theta-1} F_{t+1}], \tag{B.3}$$

$$K_t = k(Y_t; \xi_t) + \alpha\beta E_t[\Pi_{t+1}^{\theta(1+\omega)} K_{t+1}]. \tag{B.4}$$

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[61]Here we use asterisks to denote the precommitted values, as we wish, in our Lagrangian, to distinguish between the precommitted values for the vector X_{t_0} and the value of that vector under an arbitrary policy.

Conversely, any paths $\{x_t, X_t\}$ that satisfy (B.3) – (B.4) together with certain bounds must satisfy (1.14) – (1.15). Since we are here only interested in the case in which optimal policy involves paths for the endogenous variables that remain forever near a steady state to be defined below, we may impose bounds such that conditions (B.3) – (B.4) are equivalent to (1.14) – (1.15). Finally, (1.20) can be written as

$$\Delta_t = \alpha \Delta_{t-1} \Pi_t^{\theta(1+\omega)} + (1-\alpha)p(\Pi_t)^{\frac{\theta(1+\omega)}{\theta-1}}. \tag{B.5}$$

Our problem can then equivalently be stated as the choice of processes $\{x_t, X_t\}$ for dates $t \geq t_0$ (satisfying bounds sufficient to guarantee that the expressions on the right-hand sides of (1.14) and (1.15) are well-defined) to maximize (1.8) among all of the paths consistent with (B.2) – (B.5) for each $t \geq t_0$, given Δ_{t_0-1}, and also consistent with the precommitted values $X_{t_0}^*$.

The constraints implied by the initial commitments can be written as $X_{t_0} = X_{t_0}^*$. Alternatively, we can write the constraints in the form

$$\Pi_{t_0}^{\theta-1} F_{t_0} = \Pi_{t_0}^{*(\theta-1)} F_{t_0}^*, \tag{B.6}$$

$$\Pi_{t_0}^{\theta(1+\omega)} K_{t_0} = \Pi_{t_0}^{*\theta(1+\omega)} K_{t_0}^*, \tag{B.7}$$

where $\Pi_{t_0}^* \equiv \Pi(F_{t_0}^*, K_{t_0}^*)$ is the initial inflation rate implied by the precommitments. One can easily show that (B.6) – (B.7) imply that $X_{t_0} = X_{t_0}^*$, so that this is an equivalent system of initial commitments. The alternative form is useful in obtaining the desired discounted stationary form for the Lagrangian.

We introduce Lagrange multipliers ϕ_{1t} through ϕ_{4t} corresponding to constraints (B.2) through (B.5) respectively for each $t \geq t_0$,, and multipliers $-\alpha\phi_{2,t_0-1}^*$ and $-\alpha\phi_{3,t_0-1}^*$ for constraints (B.6) and (B.7) respectively. (This last choice of notation is necessary in order to achieve the desired stationary form.) A Lagrangian for the optimization problem just defined then can be written as

$$
\begin{aligned}
\mathcal{L}_{t_0} = {} & E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ U(Y_t, \Delta_t) + \phi_{1,t}[K_t p(\Pi_t)^{\frac{1+\omega\theta}{\theta-1}} - F_t] \right. \\
& + \phi_{2,t}[F_t - (1-\tau_t)f(Y_t; \xi_t) - \alpha\beta(\Pi_{t+1}^{\theta-1} F_{t+1})] \\
& + \phi_{3,t}[K_t - k(Y_t; \xi_t) - \alpha\beta(\Pi_{t+1}^{\theta(1+\omega)} K_{t+1})] \\
& \left. + \phi_{4,t}[\Delta_t - \alpha\Delta_{t-1}\Pi_t^{\theta(1+\omega)} - (1-\alpha)p(\Pi_t)^{\frac{\theta(1+\omega)}{\theta-1}}] \right\} \\
& - \phi_{2,t_0-1}^*\alpha[\Pi_{t_0}^{\theta-1} F_{t_0} - \Pi_{t_0}^{*(\theta-1)} F_{t_0}^*] - \phi_{3,t_0-1}^*\alpha[\Pi_{t_0}^{\theta(1+\omega)} K_{t_0} - \Pi_{t_0}^{*\theta(1+\omega)} K_{t_0}^*].
\end{aligned}
$$

Rearranging terms, we see that (up to a constant) this is of the desired form (B.1), where

$$
\begin{aligned}
g(\phi_{t-1}, \phi_t; x_{t-1}, x_t, X_t) &= U(Y_t, \Delta_t) + \phi_{1,t}[K_t p(\Pi_t)^{\frac{1+\omega\theta}{\theta-1}} - F_t] \\
&\quad + \phi_{2,t}[F_t - (1 - \tau_t) f(Y_t; \xi_t)] - \alpha\phi_{2,t-1}[\Pi_t^{\theta-1} F_t] \\
&\quad + \phi_{3,t}[K_t - k(Y_t; \xi_t)] - \alpha\phi_{3,t-1}[\Pi_t^{\theta(1+\omega)} K_t] \\
&\quad + \phi_{4,t}[\Delta_t - \alpha\Delta_{t-1}\Pi_t^{\theta(1+\omega)} - (1 - \alpha) p(\Pi_t)^{\frac{\theta(1+\omega)}{\theta-1}}].
\end{aligned}
$$

B.2   The optimal deterministic steady state

Here we show the existence of an optimal steady state, *i.e.,* of a solution to the recursive policy problem defined in Proposition 2 (under appropriate initial conditions) that involves constant values of all variables, in the case that there are no stochastic disturbances. Thus we consider a deterministic problem in which the exogenous disturbances $\bar{C}_t$, G_t, $\bar{H}_t$, A_t, μ_t^w, τ_t each take constant values $\bar{C}$, $\bar{H}$, $\bar{A}$, $\bar{\mu}^w$, $\bar{\tau} > 0$ and $\bar{G} \geq 0$ for all $t \geq t_0$. We wish to find an initial degree of price dispersion Δ_{t_0-1} and initial commitments $X_{t_0} = \bar{X}$ such that the recursive problem involves a constant policy $x_{t_0} = \bar{x}$, $X_{t+1} = \bar{X}$ each period, in which $\bar{\Delta}$ is equal to the initial price dispersion.

This is just a deterministic version of the problem discussed in Appendix B.1, and it can be analyzed using a deterministic version of the Lagrangian defined there. The first-order conditions of the maximization problem are then obtained by differentiating the Lagrangian. The one with respect to Y_t is

$$
U_y(Y_t, \Delta_t) - (1 - \tau_t) f_y(Y_t)\phi_{2t} - k_y(Y_t)\phi_{3t} = 0; \tag{B.8}
$$

that with respect to Δ_t is

$$
U_\Delta(Y_t, \Delta_t) + \phi_{4t} - \alpha\beta\Pi_{t+1}^{\theta(1+\omega)}\phi_{4,t+1} = 0; \tag{B.9}
$$

that with respect to Π_t is

$$
\frac{1+\omega\theta}{\theta-1} p(\Pi_t)^{\frac{(1+\omega\theta)}{\theta-1}-1} p_\pi(\Pi_t) K_t \phi_{1,t} - \alpha(\theta-1)\Pi_t^{\theta-2} F_t \phi_{2,t-1}
$$
$$
-\theta(1+\omega)\alpha\Pi_t^{\theta(1+\omega)-1} K_t \phi_{3,t-1} +
$$
$$
-\theta(1+\omega)\alpha\Delta_{t-1}\Pi_t^{\theta(1+\omega)-1}\phi_{4t} - \frac{\theta(1+\omega)}{\theta-1}(1-\alpha) p(\Pi_t)^{\frac{(1+\omega\theta)}{\theta-1}} p_\pi(\Pi_t)\phi_{4t} = 0; \quad \text{(B.10)}
$$

that with respect to F_t is

$$-\phi_{1t} + \phi_{2t} - \alpha\Pi_t^{\theta-1}\phi_{2,t-1} = 0; \tag{B.11}$$

and finally, that with respect to K_t is

$$p(\Pi_t)^{\frac{1+\omega\theta}{\theta-1}}\phi_{1t} + \phi_{3t} - \alpha\Pi_t^{\theta(1+\omega)}\phi_{3,t-1} = 0; \tag{B.12}$$

We search for a solution to these first-order conditions in which $\Pi_t = \bar{\Pi}$, $\Delta_t = \bar{\Delta}$, $Y_t = \bar{Y}$ at all times. A steady-state solution of this kind also requires that the Lagrange multipliers take constant values. We furthermore conjecture the existence of a solution in which $\bar{\Pi} = 1$, as stated in the text. Note that such a solution implies that $\bar{\Delta} = 1$, $p(\bar{\Pi}) = 1$, $p_\pi(\bar{\Pi}) = -(\theta-1)\alpha/(1-\alpha)$, and $\bar{K} = \bar{F}$. Using these substitutions, we find that (the steady-state version of) each of the first-order conditions (B.8) – (B.12) is satisfied if the steady-state values satisfy

$$[(1 - \bar{\tau})f_y(\bar{Y}) - k_y(\bar{Y})]\phi_2 = U_y(\bar{Y}, 1),$$

$$(1 - \alpha\beta)\phi_4 = -U_\Delta(\bar{Y}, 1),$$

$$\phi_1 = (1 - \alpha)\phi_2,$$

$$\phi_3 = -\phi_2.$$

These equations can obviously be solved (uniquely) for the steady-state multipliers, given any value $\bar{Y} > 0$.

Similarly, (the steady-state versions of) the constraints (B.2) – (B.5) are satisfied if

$$(1 - \bar{\tau})u_c(\bar{Y} - \bar{G}) = \frac{\theta}{\theta-1}\bar{\mu}^w v_y(\bar{Y}), \tag{B.13}$$

$$\bar{K} = \bar{F} = (1 - \alpha\beta)^{-1}k(\bar{Y}),$$

Equation (B.13) can be solved for the steady-state value $\bar{Y}$.

B.3 A second-order approximation to utility (equations (2.1), (2.2) and (2.3))

We derive here equations (2.1) — (2.3) in the main text, taking a second-order approximation to (equation (1.8)) following the treatment in Woodford (2003b, chap.

6). We start by approximating the expected discounted value of the utility of the representative household

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[u(Y_t; \xi_t) - \int_0^1 v(y_t(i); \xi_t) di \right]. \tag{B.14}$$

First we note that

$$\int_0^1 v(y_t(i); \xi_t) di = \frac{\lambda}{1+\nu} \frac{Y_t^{1+\omega}}{A_t^{1+\omega} \bar{H}_t^{\nu}} \Delta_t = v(Y_t; \xi_t) \Delta_t$$

where Δ_t is the measure of price dispersion defined in the text. We can then write (B.14) as

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[u(Y_t; \xi_t) - v(Y_t; \xi_t) \Delta_t \right]. \tag{B.15}$$

The first term in (B.15) can be approximated using a second-order Taylor expansion around the steady state defined in the previous section as

$$
\begin{aligned}
u(Y_t; \xi_t) &= \bar{u} + \bar{u}_c \tilde{Y}_t + \bar{u}_\xi \xi_t + \frac{1}{2} \bar{u}_{cc} \tilde{Y}_t^2 + \bar{u}_{c\xi} \xi_t \tilde{Y}_t + \frac{1}{2} \xi_t' \bar{u}_{\xi\xi} \xi_t + \mathcal{O}(||\xi||^3) \\
&= \bar{u} + \bar{Y} \bar{u}_c \cdot (\hat{Y}_t + \frac{1}{2} \hat{Y}_t^2) + \bar{u}_\xi \xi_t + \frac{1}{2} \bar{Y} \bar{u}_{cc} \hat{Y}_t^2 \\
&\quad + \bar{Y} \bar{u}_{c\xi} \xi_t \hat{Y}_t + \frac{1}{2} \xi_t' \bar{u}_{\xi\xi} \xi_t + \mathcal{O}(||\xi||^3) \\
&= \bar{Y} u_c \hat{Y}_t + \frac{1}{2} [\bar{Y} \bar{u}_c + \bar{Y}^2 \bar{u}_{cc}] \hat{Y}_t^2 - \bar{Y}^2 \bar{u}_{cc} g_t \hat{Y}_t + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\
&= \bar{Y} \bar{u}_c \left\{ \hat{Y}_t + \frac{1}{2} (1 - \sigma^{-1}) \hat{Y}_t^2 + \sigma^{-1} g_t \hat{Y}_t \right\} \\
&\quad + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \tag{B.16}
\end{aligned}
$$

where a bar denotes the steady-state value for each variable, a tilde denotes the deviation of the variable from its steady-state value (e.g., $\tilde{Y}_t \equiv Y_t - \bar{Y}$), and a hat refers to the log deviation of the variable from its steady-state value (e.g., $\hat{Y}_t \equiv \ln Y_t / \bar{Y}$). We use ξ_t to refer to the entire vector of exogenous shocks,

$$\xi_t' \equiv \begin{bmatrix} \hat{G} & g_t & q_t & \hat{\mu}_t^w & \hat{\tau}_t \end{bmatrix},$$

in which $\hat{G}_t \equiv (G_t - \bar{G})/\bar{Y}$, $g_t \equiv \hat{G}_t + s_C \bar{c}_t$, $\omega q_t \equiv \nu \bar{h}_t + \phi(1+\nu)a_t$, $\hat{\mu}_t^w \equiv \ln \mu_t^w / \bar{\mu}^w$, $\hat{\tau}_t \equiv (\tau_t - \bar{\tau})/\bar{\tau}$, $\bar{c}_t \equiv \ln \bar{C}_t / \bar{C}$, $a_t \equiv \ln A_t / \bar{A}$, $\bar{h}_t \equiv \ln \bar{H}_t / \bar{H}$. Moreover, we use the definitions $\sigma^{-1} \equiv \tilde{\sigma}^{-1} s_C^{-1}$ with $s_C \equiv \bar{C}/\bar{Y}$. We have used the Taylor expansion

$$Y_t / \bar{Y} = 1 + \hat{Y}_t + \frac{1}{2} \hat{Y}_t^2 + \mathcal{O}(||\xi||^3)$$

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to get a relation for $\tilde{Y}_t$ in terms of $\hat{Y}_t$. Finally the term "t.i.p." denotes terms that are independent of policy, and may accordingly be suppressed as far as the welfare ranking of alternative policies is concerned.

We may similarly approximate $v(Y_t; \xi_t)\Delta_t$ by

$$
\begin{aligned}
v(Y_t; \xi_t)\Delta_t &= \bar{v} + \bar{v}(\Delta_t - 1) + \bar{v}_y(Y_t - \bar{Y}) + \bar{v}_y(\Delta_t - 1)(Y_t - \bar{Y}) + (\Delta_t - 1)\bar{v}_\xi \xi_t \\
&\quad + \frac{1}{2}\bar{v}_{yy}(Y_t - \bar{Y})^2 + (Y_t - \bar{Y})\bar{v}_{y\xi}\xi_t + \frac{1}{2}\xi_t'\bar{v}_{\xi\xi}\xi_t + \mathcal{O}(||\xi||^3) \\
&= \bar{v}(\Delta_t - 1) + \bar{v}_y\bar{Y}\left(\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2\right) + \bar{v}_y(\Delta_t - 1)\bar{Y}\hat{Y}_t + (\Delta_t - 1)\bar{v}_\xi \xi_t \\
&\quad + \frac{1}{2}\bar{v}_{yy}\bar{Y}^2\hat{Y}_t^2 + \bar{Y}\hat{Y}_t\bar{v}_{y\xi}\xi_t + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \qquad (\text{B.17}) \\
&= \bar{v}_y\bar{Y}[\frac{\Delta_t - 1}{1 + \omega} + \hat{Y}_t + \frac{1}{2}(1 + \omega)\hat{Y}_t^2 + (\Delta_t - 1)\hat{Y}_t - \omega\hat{Y}_t q_t \\
&\quad - \frac{\Delta_t - 1}{1 + \omega}\omega q_t] + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \qquad (\text{B.18})
\end{aligned}
$$

We further note that a Taylor approximation to (1.20), of first order in $\hat{\Delta}_t$ and of second order in π_t, takes the form

$$
\hat{\Delta}_t = \alpha\hat{\Delta}_{t-1} + \frac{\alpha}{1 - \alpha}\theta(1 + \omega)(1 + \omega\theta)\frac{\pi_t^2}{2} + \mathcal{O}(||\xi||^3), \qquad (\text{B.19})
$$

which involves no linear terms in inflation. It follows that as long as $\hat{\Delta}_{t_0-1} = \mathcal{O}(||\xi||^2)$,[62] (B.19) implies that $\hat{\Delta}_t = \mathcal{O}(||\xi||^2)$ for all $t \geq t_0$. Then since

$$
\Delta_t = 1 + \hat{\Delta}_t + \mathcal{O}(|\hat{\Delta}_t|^2),
$$

it follows that $\Delta_t - 1 = \mathcal{O}(||\xi||^2)$ for all $t \geq t_0$ as well.

Substituting this into (B.18) yields

$$
v(Y_t; \xi_t)\Delta_t = (1 - \Phi)\bar{Y}u_c\left\{\frac{\hat{\Delta}_t}{1 + \omega} + \hat{Y}_t + \frac{1}{2}(1 + \omega)\hat{Y}_t^2 - \omega\hat{Y}_t q_t\right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \qquad (\text{B.20})
$$

where we have used the steady state relation $\bar{v}_y = (1 - \Phi)\bar{u}_c$ to replace $\bar{v}_y$ by $(1 - \Phi)\bar{u}_c$, and where

$$
\Phi \equiv 1 - \left(\frac{\theta - 1}{\theta}\right)\left(\frac{1 - \bar{\tau}}{\bar{\mu}^w}\right) < 1
$$

[62]Note that equations (2.1), (2.2) and (2.3) in the text are correct only under this assumption. It should be recalled that in footnote 47 of the text, we have defined the bound $||\xi||$ so as to ensure that this is true.

measures the inefficiency of steady-state output $\bar{Y}$. Combining (B.16) and (B.20), we then obtain equation (2.1) in the text,

$$
\begin{aligned}
U_{t_0} \; = \; & \bar{Y}\bar{u}_c \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0}[\Phi\hat{Y}_t - \frac{1}{2}u_{yy}\hat{Y}_t^2 + \hat{Y}_t u_{y\xi}\xi_t - u_\Delta\hat{\Delta}_t] \\
& + \; \text{t.i.p.} + \mathcal{O}(||\xi||^3),
\end{aligned}
\tag{B.21}
$$

where

$$
\begin{aligned}
u_{yy} \; &\equiv \; (\omega + \sigma^{-1}) - \Phi(1+\omega), \\
u_{y\xi}\xi_t \; &\equiv \; [\sigma^{-1}g_t + (1-\Phi)\omega q_t], \\
u_\Delta \; &\equiv \; \frac{(1-\Phi)}{1+\omega}.
\end{aligned}
$$

We finally observe that (B.19) can be integrated to obtain

$$
\hat{\Delta}_t = \alpha^{t-t_0+1}\hat{\Delta}_{t_0-1} + \frac{\alpha}{(1-\alpha)(1-\alpha\beta)}\theta(1+\omega)(1+\omega\theta)\sum_{s=t_0}^{t}\alpha^{t-s}\frac{\pi_s^2}{2} + \mathcal{O}(||\xi||^3). \tag{B.22}
$$

Multiplying this by β^{t-t_0} and summing over t, we obtain expression (2.2) in the text, where "t.i.p." refers to a multiple of $\hat{\Delta}_{t_0-1}$. By substituting this expression for the term $\sum_{t=t_0}^{\infty}\beta^{t-t_0}\hat{\Delta}_t$ in (B.21), we obtain equation (2.3) in the text, in which we further define

$$
\kappa \equiv \frac{(1-\alpha\beta)(1-\alpha)}{\alpha}\frac{(\omega+\sigma^{-1})}{(1+\theta\omega)}, \qquad u_\pi \equiv \frac{\theta(\omega+\sigma^{-1})(1-\Phi)}{\kappa}. \tag{B.23}
$$

B.4 A second-order approximation to the AS equation (equations (2.4), (2.7), and (2.8))

The AS relation can be written exactly as

$$
\log\left(1 - \frac{\alpha\Pi_t^{\theta-1}}{1-\alpha}\right) = \frac{\theta-1}{1+\omega\theta}(\log K_t - \log F_t). \tag{B.24}
$$

A second-order Taylor series for the left-hand side of (B.24) takes the form

$$
\log\left(1 - \frac{\alpha\Pi_t^{\theta-1}}{1-\alpha}\right) = \frac{\alpha}{1-\alpha}(\theta-1)\left\{\pi_t + \frac{1}{2}\frac{\theta-1}{1-\alpha}\pi_t^2 + \mathcal{O}(||\xi||^3).\right\} \tag{B.25}
$$

It remains to derive similar second-order approximations for $\log K_t$ and $\log F_t$ on the right-hand side.

The definitions of K_t and F_t imply second-order expansions

$$\hat{K}_t + \frac{1}{2}\hat{K}_t^2 + \mathcal{O}(||\xi||^3) = (1-\alpha\beta)E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left[\hat{k}_{t,T} + \frac{1}{2}\hat{k}_{t,T}^2\right] + \mathcal{O}(||\xi||^3) \quad \text{(B.26)}$$

$$\hat{F}_t + \frac{1}{2}\hat{F}_t^2 + \mathcal{O}(||\xi||^3) = (1-\alpha\beta)E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left[\hat{f}_{t,T} + \frac{1}{2}\hat{f}_{t,T}^2\right] + \mathcal{O}(||\xi||^3) \quad \text{(B.27)}$$

where $\hat{k}_{t,T}$ and $\hat{f}_{t,T}$ are given by

$$\hat{k}_{t,T} \equiv \hat{k}_T + \theta(1+\omega)\sum_{s=t+1}^{T}\pi_s$$

$$\hat{f}_{t,T} \equiv \hat{f}_T + (\theta-1)\sum_{s=t+1}^{T}\pi_s$$

and we use the definitions

$$\hat{k}_T \equiv (1+\omega)\hat{Y}_T - \omega q_T + \hat{\mu}_T^w \quad \text{(B.28)}$$

$$\hat{f}_T \equiv \hat{S}_T + \hat{Y}_T - \tilde{\sigma}^{-1}(\hat{C}_T - \bar{c}_T) \quad \text{(B.29)}$$

$$\hat{S}_T \equiv \log(1-\tau_t)/(1-\bar{\tau}). $$

Substituting (B.25) into (B.24) yields

$$\pi_t + \frac{1}{2}\frac{\theta-1}{1-\alpha}\pi_t^2 = \frac{1-\alpha}{\alpha}\frac{1}{1+\omega\theta}(\hat{K}_t - \hat{F}_t) + \mathcal{O}(||\xi||^3). \quad \text{(B.30)}$$

Note that to first order, this reduces to

$$\pi_t = \frac{1-\alpha}{\alpha}\frac{1}{1+\omega\theta}(\hat{K}_t - \hat{F}_t) + \mathcal{O}(||\xi||^2) \quad \text{(B.31)}$$

$$= \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}\frac{1}{1+\omega\theta}E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left[\hat{k}_{t,T} - \hat{f}_{t,T}\right] + \mathcal{O}(||\xi||^2). \quad \text{(B.32)}$$

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We can use (B.26)–(B.27) to obtain a second-order expansion for the right-hand side of (B.30). Subtracting (B.27) from (B.26), we obtain

$$
\begin{aligned}
\hat{K}_t - \hat{F}_t &= (1-\alpha\beta)E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left[(\hat{k}_{t,T}-\hat{f}_{t,T})+\frac{1}{2}(\hat{k}_{t,T}^2-\hat{f}_{t,T}^2)\right]\\
&\quad -\frac{1}{2}(\hat{K}_t^2-\hat{F}_t^2)+\mathcal{O}(||\xi||^3)\\
&= (1-\alpha\beta)E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left[(\hat{k}_{t,T}-\hat{f}_{t,T})+\frac{1}{2}(\hat{k}_{t,T}^2-\hat{f}_{t,T}^2)\right] \quad (B.33)\\
&\quad -\frac{1}{2}(\hat{K}_t-\hat{F}_t)(\hat{K}_t+\hat{F}_t)+\mathcal{O}(||\xi||^3)\\
&= (1-\alpha\beta)E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left[(\hat{k}_{t,T}-\hat{f}_{t,T})+\frac{1}{2}(\hat{k}_{t,T}^2-\hat{f}_{t,T}^2)\right] \quad (B.34)\\
&\quad -\frac{1}{2}(1-\alpha\beta)\frac{\alpha}{(1-\alpha)}(1+\omega\theta)\pi_t Z_t+\mathcal{O}(||\xi||^3),
\end{aligned}
$$

where in passing from (B.33) to (B.34) we have used (B.31) to substitute for $(\hat{K}_t-\hat{F}_t)$ in the second term on the right-hand side, and

$$
(\hat{K}_t+\hat{F}_t)=(1-\alpha\beta)Z_t+\mathcal{O}(||\xi||^2)
$$

to substitute for $(\hat{K}_t+\hat{F}_t)$, in which expression we define

$$
Z_t \equiv E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left[\hat{k}_{t,T}+\hat{f}_{t,T}\right]. \tag{B.35}
$$

We can use the definitions of $\hat{k}_{t,T}$ and $\hat{f}_{t,T}$ to further expand the first term on the right-hand side of (B.34). We obtain

$$
\begin{aligned}
E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left[\hat{k}_{t,T}-\hat{f}_{t,T}\right] &= E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left[\hat{k}_T-\hat{f}_T\right]+(1+\omega\theta)E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\sum_{s=t+1}^{T}\pi_s\\
&= E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left[\hat{k}_T-\hat{f}_T\right]+\frac{(1+\omega\theta)}{1-\alpha\beta}\mathcal{P}_t, \quad (B.36)
\end{aligned}
$$

where

$$
\mathcal{P}_t \equiv E_t\sum_{T=t+1}^{\infty}(\alpha\beta)^{T-t}\pi_T, \tag{B.37}
$$

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and

$$\frac{1}{2}E_t \sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left[\hat{k}_{t,T}^2 - \hat{f}_{t,T}^2\right] = \frac{1}{2}E_t \sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left[\hat{k}_T^2 - \hat{f}_T^2\right]$$

$$+ E_t \sum_{T=t}^{\infty}(\alpha\beta)^{T-t}[\theta(1+\omega)\hat{k}_T + (1-\theta)\hat{f}_T] \cdot \sum_{s=t+1}^{T}\pi_s$$

$$+ \frac{1}{2}(2\theta + \theta\omega - 1)(1+\theta\omega)E_t \sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left(\sum_{s=t+1}^{T}\pi_s\right)^2$$

$$= \frac{1}{2}E_t \sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left[\hat{k}_T^2 - \hat{f}_T^2\right] + E_t \sum_{T=t+1}^{\infty}(\alpha\beta)^{T-t}\pi_T N_T$$

$$+ \frac{1}{2}\frac{(2\theta + \theta\omega - 1)(1+\theta\omega)}{(1-\alpha\beta)}E_t \sum_{T=t+1}^{\infty}(\alpha\beta)^{T-t}\pi_T(\pi_T + 2\mathcal{P}_T)$$

where

$$N_t \equiv E_t \sum_{T=t}^{\infty}(\alpha\beta)^{T-t}[\theta(1+\omega)\hat{k}_T + (1-\theta)\hat{f}_T]. \tag{B.38}$$

Substituting these expressions into (B.34), we obtain

$$\hat{K}_t - \hat{F}_t = (1-\alpha\beta)E_t \sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\left[(\hat{k}_T - \hat{f}_T) + \frac{1}{2}(\hat{k}_T^2 - \hat{f}_T^2)\right] +$$

$$+ (1+\omega\theta)E_t \sum_{T=t+1}^{\infty}(\alpha\beta)^{T-t}\pi_T + (1-\alpha\beta)E_t \sum_{T=t+1}^{\infty}(\alpha\beta)^{T-t}\pi_T N_T +$$

$$\frac{1}{2}(2\theta + \theta\omega - 1)(1+\theta\omega)E_t \sum_{T=t+1}^{\infty}(\alpha\beta)^{T-t}\pi_T(\pi_T + 2\mathcal{P}_T)$$

$$- \frac{1}{2}(1-\alpha\beta)\frac{\alpha}{(1-\alpha)}(1+\omega\theta)\pi_t Z_t + \mathcal{O}(\|\xi\|^3). \tag{B.39}$$

This can be written recursively as

$$\hat{K}_t - \hat{F}_t + \frac{1}{2}\frac{\alpha(1-\alpha\beta)(1+\omega\theta)}{(1-\alpha)}\pi_t Z_t = (1-\alpha\beta)\left[(\hat{k}_t - \hat{f}_t) + \frac{1}{2}(\hat{k}_t^2 - \hat{f}_t^2)\right] +$$

$$+ \alpha\beta(1+\omega\theta)E_t\pi_{t+1} + (1-\alpha\beta)\alpha\beta E_t\pi_{t+1}N_{t+1} +$$

$$\frac{1}{2}(2\theta + \theta\omega - 1)(1+\theta\omega)E_t\pi_{t+1}(\pi_{t+1} + 2\mathcal{P}_{t+1}) +$$

$$+ \alpha\beta E_t\left[\hat{K}_{t+1} - \hat{F}_{t+1} + \frac{1}{2}\frac{\alpha(1-\alpha\beta)(1+\omega\theta)}{(1-\alpha)}\pi_{t+1}Z_{t+1}\right] +$$

$$+ \mathcal{O}(\|\xi\|^3) \tag{B.40}$$

Using (B.30) to substitute for $\hat{K}_t - \hat{F}_t$ in (B.40), we obtain

$$\pi_t + \frac{1}{2}\frac{\theta-1}{1-\alpha}\pi_t^2 + \frac{1}{2}(1-\alpha\beta)\pi_t Z_t = \frac{1-\alpha}{\alpha}\frac{(1-\alpha\beta)}{(1+\omega\theta)}\left[(\hat{k}_t - \hat{f}_t) + \frac{1}{2}(\hat{k}_t^2 - \hat{f}_t^2)\right] +$$

$$+(1-\alpha)\beta E_t\pi_{t+1} + (1-\alpha)\beta\frac{(1-\alpha\beta)}{(1+\theta\omega)}E_t\pi_{t+1}N_{t+1} +$$

$$\frac{1}{2}(2\theta + \theta\omega - 1)(1-\alpha)\beta E_t\pi_{t+1}(\pi_{t+1} + 2\mathcal{P}_{t+1}) +$$

$$+\alpha\beta E_t\left[\pi_{t+1} + \frac{1}{2}\frac{\theta-1}{1-\alpha}\pi_{t+1}^2 + \frac{1}{2}(1-\alpha\beta)\pi_{t+1}Z_{t+1}\right] +$$

$$+\mathcal{O}(||\xi||^3). \tag{B.41}$$

This is our second-order approximation to the AS relation. Note that to first order, this reduces to

$$\pi_t = \frac{1-\alpha}{\alpha}\frac{(1-\alpha\beta)}{(1+\omega\theta)}(\hat{k}_t - \hat{f}_t) + \beta E_t\pi_{t+1} + \mathcal{O}(||\xi||^2), \tag{B.42}$$

as could also have been obtained directly from (B.32).

We can furthermore eliminate N_t by observing that (B.38) implies that

$$N_t \equiv \frac{1}{2}E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}[(1+\theta\omega)(\hat{k}_t + \hat{f}_t) + (2\theta + \theta\omega - 1)(\hat{k}_t - \hat{f}_t)]$$

$$= \frac{1}{2}E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}[(1+\theta\omega)(\hat{k}_{t,T} + \hat{f}_{t,T}) + (2\theta + \theta\omega - 1)(\hat{k}_{t,T} - \hat{f}_{t,T})]$$

$$-\frac{(2\theta + \theta\omega - 1)(1+\theta\omega)}{(1-\alpha\beta)}\mathcal{P}_t$$

$$= \frac{1}{2}(1+\theta\omega)Z_t + \frac{1}{2}(2\theta + \theta\omega - 1)\frac{(1+\theta\omega)}{1-\alpha}\frac{\alpha}{(1-\alpha\beta)}\pi_t$$

$$-\frac{(2\theta + \theta\omega - 1)(1+\theta\omega)}{(1-\alpha\beta)}\mathcal{P}_t \tag{B.43}$$

By substituting (B.43) into (B.41), we obtain

$$\pi_t + \frac{1}{2}\frac{\theta-1}{1-\alpha}\pi_t^2 + \frac{1}{2}(1-\alpha\beta)\pi_t Z_t = \frac{1-\alpha}{\alpha}\frac{(1-\alpha\beta)}{(1+\omega\theta)}\left[(\hat{k}_t - \hat{f}_t) + \frac{1}{2}(\hat{k}_t^2 - \hat{f}_t^2)\right] +$$

$$+\beta E_t\pi_{t+1} + \frac{1}{2}\beta(1-\alpha\beta)E_t\pi_{t+1}Z_{t+1} +$$

$$\frac{1}{2}(2\theta + \theta\omega - 1)\beta E_t\pi_{t+1}^2 + \alpha\beta E_t\left[\frac{1}{2}\frac{\theta-1}{1-\alpha}\pi_{t+1}^2\right] +$$

$$+\mathcal{O}(||\xi||^3),$$

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which can be rewritten as

$$\pi_t + \frac{1}{2}\frac{\theta-1}{1-\alpha}\pi_t^2 + \frac{1}{2}(1-\alpha\beta)\pi_t Z_t = \frac{1-\alpha}{\alpha}\frac{(1-\alpha\beta)}{(1+\omega\theta)}\left[(\hat{k}_t - \hat{f}_t) + \frac{1}{2}(\hat{k}_t^2 - \hat{f}_t^2)\right]$$

$$+ \beta E_t \pi_{t+1} + \frac{1}{2}\beta(1-\alpha\beta)E_t\pi_{t+1}Z_{t+1} + \beta E_t\left[\frac{1}{2}\frac{\theta-1}{1-\alpha}\pi_{t+1}^2\right]$$

$$+ \frac{1}{2}\theta(1+\omega)\beta E_t\pi_{t+1}^2 + \mathcal{O}(||\xi||^3). \tag{B.44}$$

This is a relation of the form (2.4); it can be integrated forward to obtain a relation of the form (2.8),

$$V_{t_0} = \frac{1-\alpha}{\alpha}\frac{(1-\alpha\beta)}{(1+\omega\theta)}E_t\sum_{t=t_0}^{\infty}(\alpha\beta)^{t-t_0}\left[(\hat{k}_t - \hat{f}_t) + \frac{1}{2}(\hat{k}_t^2 - \hat{f}_t^2)\right] + \frac{1}{2}\theta(1+\omega)E_t\sum_{t=t_0}^{\infty}(\alpha\beta)^{t-t_0}\pi_t^2, \tag{B.45}$$

where

$$V_t \equiv \pi_t - \frac{1}{2}\frac{1-\theta}{1-\alpha}\pi_t^2 + \frac{1}{2}(1-\alpha\beta)\pi_t Z_t + \frac{1}{2}\theta(1+\omega)\pi_t^2. \tag{B.46}$$

Note that this last definition is of the form (2.5) given in the text, where the coefficients are defined as

$$v_\pi \equiv \theta(1+\omega) - \frac{1-\theta}{(1-\alpha)}, \qquad v_z \equiv \frac{(1-\alpha\beta)}{2}.$$

We then obtain the relations given in the text by substituting into the above equations the definitions (B.28) for $\hat{k}_t$ and (B.29) for $\hat{f}_t$. In the expression for $\hat{f}_t$, we can furthermore use a second-order approximation to the identity $Y_t = C_t + G_t$ to solve for $\hat{C}_t$ as a function of $\hat{Y}_t$ and exogenous disturbances,

$$\hat{C}_t = s_C^{-1}\hat{Y}_t - s_C^{-1}\hat{G}_t + \frac{s_C^{-1}(1-s_C^{-1})}{2}\hat{Y}_t^2 + s_C^{-2}\hat{Y}_t\hat{G}_t + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3), \tag{B.47}$$

where "s.o.t.i.p" refers to second-order (or higher) terms independent of policy; the first-order terms have been kept as these will matter for the log-linear aggregate-supply relation that appears as a constraint in our policy problem. We similarly note that

$$\hat{S}_t = -\omega_\tau\hat{\tau}_t + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3), \tag{B.48}$$

where $\omega_\tau \equiv \bar{\tau}/(1-\bar{\tau})$. Equations (B.47) – (B.48) can be used to substitute for $\hat{C}_t$ and $\hat{S}_t$ in (B.29), resulting in an expression for $\hat{f}_t$ that involves only $\hat{Y}_t$ and elements of ξ_t.

Substituting these expressions for $\hat{k}_t$ and $\hat{f}_t$ into (B.44), we obtain equation (2.4) in the text, where we define

$$c_\xi \xi_t \equiv (\omega + \sigma^{-1})^{-1}[-\sigma^{-1}g_t - \omega q_t + \hat{\mu}_t^w + \omega_\tau \hat{\tau}_t],$$

$$c_{yy} \equiv (2 + \omega - \sigma^{-1}) + \sigma^{-1}(1 - s_C^{-1})(\omega + \sigma^{-1})^{-1},$$

$$\begin{aligned} c_{y\xi}\xi_t &\equiv (\omega + \sigma^{-1})^{-1}[-\sigma^{-1}s_C^{-1}\hat{G}_t + \sigma^{-1}(1 - \sigma^{-1})g_t + \omega(1 + \omega)q_t \\ &\quad -(1 + \omega)\hat{\mu}_t^w - (1 - \sigma^{-1})\omega_\tau \hat{\tau}_t], \end{aligned}$$

$$c_\pi \equiv \frac{\theta(1 + \omega)}{\kappa},$$

and κ is again the coefficient defined in (B.23). (Note that $\kappa > 0$, as asserted in the text.) The same substitutions into definition (B.35) allow us to define Z_t by an expression of the form (2.6) given in the text, where

$$z_y \equiv (2 + \omega - \sigma^{-1}) + v_k(\omega + \sigma^{-1}),$$

$$z_\xi \xi_t \equiv \sigma^{-1}(1 - v_k)g_t - \omega(1 + v_k)q_t + (1 + v_k)\hat{\mu}_t^w - \omega_\tau(1 - v_k)\hat{\tau}_t,$$

$$z_\pi \equiv -\frac{(\omega + \sigma^{-1})}{\kappa}v_k,$$

in which expressions we define

$$v_k \equiv \frac{\kappa}{(\omega + \sigma^{-1})}\frac{\alpha}{1 - \alpha\beta}(1 - 2\theta - \omega\theta).$$

To a first-order approximation, equation (2.4) reduces to equation (2.7) given in the text. Finally, the same substitutions for $\hat{k}_t$ and $\hat{f}_t$ into (B.45) yields equation (2.8) in the text, where the term $c_\xi \xi_t$ is now included in terms independent of policy. (Such terms matter when part of the log-linear constraints, as in the case of (2.7), but not when part of the quadratic objective.)

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B.5 Derivation of equation (2.9)

We can multiply equation (2.8) by $\Phi \bar{Y} \bar{u}_c$ and subtract from (2.1) to obtain

$$U_{t_0} = -\bar{Y}\bar{u}_c E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{1}{2} q_y \hat{Y}_t^2 - \hat{Y}_t(u_{y\xi}\xi_t + \Phi c_{y\xi}\xi_t) + \frac{1}{2} q_\pi \pi_t^2 \right\} + T_{t_0} + \text{t.i.p.} + \mathcal{O}(||\xi||^3),$$

where

$$
\begin{aligned}
q_\pi &\equiv u_\pi + \Phi c_\pi \\
&= \frac{\theta(\omega + \sigma^{-1})(1 - \Phi)}{\kappa} + \Phi \frac{\theta(1 + \omega)}{\kappa} \\
&= \frac{\theta}{\kappa}[(\omega + \sigma^{-1}) + \Phi(1 - \sigma^{-1})],
\end{aligned}
$$

$$
\begin{aligned}
q_y &\equiv u_{yy} + \Phi c_{yy} \\
&= (\omega + \sigma^{-1}) - \Phi(1 + \omega) + \Phi(2 + \omega - \sigma^{-1}) + \Phi\sigma^{-1}(1 - s_C^{-1})(\omega + \sigma^{-1})^{-1} \\
&= (\omega + \sigma^{-1}) + \Phi(1 - \sigma^{-1}) + \frac{\Phi\sigma^{-1}(1 - s_C^{-1})}{\omega + \sigma^{-1}}.
\end{aligned}
$$

This can be rewritten in the form (2.9) given in the text, where

$$
\begin{aligned}
\hat{Y}_t^* &\equiv q_y^{-1}[u_{y\xi}\xi_t + \Phi c_{y\xi}\xi_t] \\
&= q_y^{-1}\{\sigma^{-1}g_t + (1 - \Phi)\omega q_t + (\omega + \sigma^{-1})^{-1}\Phi[-\sigma^{-1}s_C^{-1}\hat{G}_t + \sigma^{-1}(1 - \sigma^{-1})g_t + \omega(1 + \omega)q_t \\
&\quad -(1 + \omega)\hat{\mu}_t^w - (1 - \sigma^{-1})\omega_\tau \hat{\tau}_t]\} \\
&= \omega_1 \hat{Y}_t^n - \omega_2 \hat{G}_t + \omega_3 \hat{\mu}_t^w + \omega_4 \hat{\tau}_t,
\end{aligned}
$$

and Ω, $\hat{Y}_t^n$, and the ω_i are defined as in the text.

B.6 The optimal non-inertial plan (equations (4.2), (4.3), (4.6), and (4.13))

We take a second-order approximation of (4.1) around the same steady state as in the calculations above. We thus obtain

$$
\begin{aligned}
\hat{U}_{t_0} &= \bar{Y}\bar{u}_c \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \Phi\hat{Y}_t - \frac{1}{2}u_{yy}\hat{Y}_t^2 + \hat{Y}_t u_{y\xi}\xi_t - u_\Delta\hat{\Delta}_t \right\} + \\
&\quad -\bar{\phi}_2^*\alpha\bar{F}[\hat{F}_{t_0} - \hat{K}_{t_0} - (1+\theta\omega)\pi_{t_0} + \frac{1}{2}(\hat{F}_{t_0}^2 - \hat{K}_{t_0}^2) + \\
&\quad +(\theta-1)\hat{F}_{t_0}\pi_{t_0} - \theta(1+\omega)\hat{K}_{t_0}\pi_{t_0} + \frac{1}{2}((\theta-1)^2 - \theta^2(1+\omega)^2)\pi_{t_0}^2] \\
&\quad -\bar{\phi}_2^*\alpha\bar{F}[((\theta-1)\pi_{t_0} + \hat{F}_{t_0})\hat{\phi}_{2,t_0-1}^* - (\theta(1+\omega)\pi_{t_0} + \hat{K}_{t_0})\hat{\phi}_{3,t_0-1}^*] + \\
&\quad +\text{t.i.p.} + \mathcal{O}(||\xi||^3), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(B.49)}
\end{aligned}
$$

where we have used the facts that in the steady state $\bar{\phi}_2^* = -\bar{\phi}_3^*$ and $\bar{K} = \bar{F}$ as shown in Appendix B.2.

We can simplify the above expression to obtain

$$
\begin{aligned}
\hat{U}_{t_0} &= \bar{Y}\bar{u}_c \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \Phi\hat{Y}_t - \frac{1}{2}u_{yy}\hat{Y}_t^2 + \hat{Y}_t u_{y\xi}\xi_t - u_\Delta\hat{\Delta}_t \right\} + \bar{\phi}_2^* \frac{(1-\alpha\beta)(\omega+\sigma^{-1})}{\kappa}\bar{F}V_{t_0} \\
&\quad -\bar{\phi}_2^*\alpha\bar{F}[((\theta-1)\hat{\phi}_{2,t_0-1}^* - \theta(1+\omega)\hat{\phi}_{3,t_0-1}^*)\pi_{t_0}] - \bar{\phi}_2^*\alpha\bar{F}[\hat{F}_{t_0}\hat{\phi}_{2,t_0-1}^* - \hat{K}_{t_0}\hat{\phi}_{3,t_0-1}^*] + \\
&\quad +\text{t.i.p.} + \mathcal{O}(||\xi||^3), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(B.50)}
\end{aligned}
$$

where we now have used the results

$$
\hat{K}_t - \hat{F}_t = \frac{\alpha}{1-\alpha}(1+\omega\theta)\pi_t + \mathcal{O}(||\xi||^2), \qquad\qquad\qquad\text{(B.51)}
$$

$$
\hat{K}_t + \hat{F}_t = (1-\alpha\beta)Z_t + \mathcal{O}(||\xi||^2), \qquad\qquad\qquad\text{(B.52)}
$$

established in Appendix B.3, and recalled the definitions of V_t and Z_t given by (2.5) and (2.6), respectively.

We can then substitute equation (2.8) into (B.50) to obtain

$$
\begin{aligned}
\hat{U}_{t_0} &= -\Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{q_\pi}{2}\pi_t^2 + \frac{q_y}{2}(\hat{Y}_t - \hat{Y}_t^*)^2 \right\} + T_{t_0} + \bar{\phi}_2^* \frac{(1-\alpha\beta)(\omega+\sigma^{-1})}{\kappa}\bar{F}V_{t_0} + \\
&\quad -\bar{\phi}_2^*\alpha\bar{F}[((\theta-1)\hat{\phi}_{2,t_0-1}^* - \theta(1+\omega)\hat{\phi}_{3,t_0-1}^*)\pi_{t_0}] - \bar{\phi}_2^*\alpha\bar{F}[\hat{F}_{t_0}\hat{\phi}_{2,t_0-1}^* - \hat{K}_{t_0}\hat{\phi}_{3,t_0-1}^*] + \\
&\quad +\text{t.i.p.} + \mathcal{O}(||\xi||^3). \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(B.53)}
\end{aligned}
$$

Then using (B.31) to write π_{t_0} (to first order) as a linear function of $\hat{X}_{t_0}$, the second line of (B.53) can be written as a quadratic form $\hat{\phi}^{*\prime}_{t_0-1} Q \hat{X}_{t_0}$. We thus obtain an expression of the form (4.2) given in the text.

We next recall from Appendix B.2 that $\bar{\phi}^*_2 = -\Phi/[(1-\bar{\tau})(\omega+\sigma^{-1})]$ and that $\bar{F} = (1-\bar{\tau})\bar{u}_c\bar{Y}/(1-\alpha\beta)$, so that

$$T_{t_0} \equiv \Phi\bar{Y}\bar{u}_c\kappa^{-1}V_{t_0} = -\bar{\phi}^*_2\frac{(1-\alpha\beta)(\omega+\sigma^{-1})}{\kappa}\bar{F}V_{t_0}. \tag{B.54}$$

Using this result, (4.2) simplifies to (4.3).

We can furthermore substitute (B.51) and (B.52) into (B.53) to obtain

$$\begin{aligned}
\hat{U}_{t_0} =\ & -\Omega E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left\{\frac{q_\pi}{2}\pi_t^2 + \frac{q_y}{2}(\hat{Y}_t-\hat{Y}_t^*)^2\right\} + \\
& -\bar{\phi}^*_2\alpha\bar{F}[((\theta-1)\hat{\phi}^*_{2,t_0-1} - \theta(1+\omega)\hat{\phi}^*_{3,t_0-1})\pi_{t_0} \\
& +\frac{1}{2}(1-\alpha\beta)Z_t(\hat{\phi}^*_{2,t_0-1}-\hat{\phi}^*_{3,t_0-1}) - \frac{1}{2}\frac{\alpha(1+\theta\omega)}{1-\alpha}(\hat{\phi}^*_{2,t_0-1}+\hat{\phi}^*_{3,t_0-1})] \\
& \text{t.i.p.} + \mathcal{O}(\|\xi\|^3).
\end{aligned} \tag{B.55}$$

We finally observe that a first-order approximation of constraints (B.11) and (B.12) in appendix B.2 implies that

$$\hat{\phi}^*_{2,t} - \alpha\hat{\phi}^*_{2,t-1} = \hat{\phi}^*_{3,t} - \alpha\hat{\phi}^*_{3,t-1}. \tag{B.56}$$

It follows that in the case of any self-consistent function $\phi(\cdot)$ mapping histories of the exogenous disturbances in vectors of Lagrange multipliers, one must have

$$\hat{\phi}^*_{2,t} = \hat{\phi}^*_{3,t} \tag{B.57}$$

for any possible history of disturbances. We can then simplify (B.55) to

$$\begin{aligned}
\hat{U}_{t_0} =\ & -\Omega E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left\{\frac{q_\pi}{2}\pi_t^2 + \frac{q_y}{2}y_t\right\} - \Omega\Phi\kappa^{-1}\hat{\phi}^*_{2,t_0-1}(\pi_{t_0}-\pi_{t_0}^*) + \\
& +\text{t.i.p.} + \mathcal{O}(\|\xi\|^3).
\end{aligned} \tag{B.58}$$

Defining $\varphi_{t_0-1} = -\Phi\kappa^{-1}\hat{\phi}^*_{2,t_0-1}$, we obtain a representation of the form (4.5) given in the text. This welfare criterion is in turn easily seen to vary inversely with the loss function (4.6).

In order to compute the optimal non-inertial plan, we must evaluate (4.6) in the case that inflation and the output gap evolve in accordance with (4.8) – (4.9). Using the restrictions (4.10) – (4.11), we obtain

$$E[L_{t_0}^r] \;=\; E\{E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0}[\tfrac{q_y}{2}(\bar{y} + f_y u_t)^2 + \tfrac{q_\pi}{2}(\bar{\pi} + f_\pi u_t)^2]\} - E\varphi_{t_0-1}(\bar{\pi} + f_\pi u_{t_0}),$$

suppressing a term that is independent of policy. This can in turn be simplified to

$$E[L_{t_0}^r] = \frac{(q_y \bar{y}^2 + q_\pi \bar{\pi}^2)}{2(1-\beta)} + \frac{q_y f_y^2 + q_\pi f_\pi^2}{2} \frac{\sigma_u^2}{(1-\beta)} - f_\pi E\{\varphi_{t_0-1} u_{t_0}\}, \tag{B.59}$$

using the law of motion (4.7) for the process $\{u_t\}$. Finally, using the solution (4.12) for $\varphi_{t_0-1}^*$, we can show that

$$E\{\varphi_{t_0-1}^* u_{t_0}\} = -\frac{q_\pi \rho_u \mu}{(1 - \beta\rho_u\mu)(1 - \rho_u\mu)}\sigma_u^2.$$

Substituting this into (B.59), we obtain the criterion (4.13) given in the text.

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