Notes on Model for "Optimal Inflation Targeting Rules"

Marc P. Giannoni
Columbia University

Michael Woodford
Princeton University

August 11, 2003

Abstract

This note presents some detailed calculations underlying the model used in Giannoni and Woodford (2003, sect. 2, 3). It then considers special cases of the model (cases of no habit persistence, of flexible wages). It finally briefly describes the representation of the model used for the estimation of the structural parameters.

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1 Model

1.1 Household's Problem

We assume that there exists a continuum of households indexed by h and distributed uniformly on the $[0,1]$ interval. Each household h seeks, at date t, to maximize a lifetime expected utility of the form

$$E_t \left\{ \sum_{T=t}^{\infty} \beta^{T-t} \left[u\left(C_T^h - \eta C_{T-1}^h; \xi_T\right) - v\left(H_T^h; \xi_T\right)\right] \right\} \tag{1}$$

where $\beta \in (0,1)$ is the household's discount factor (assumed to be equal for each household), C_t^h is an index of the household's consumption of each of the differentiated goods supplied at time t, and H_t^h is the amount of labor (of type h) that household h supplies at date t. Here we assume that each household specializes in the supply of one type of labor, and that each type of labor is supplied by an equal number of households. The parameter $0 \leq \eta \leq 1$ represents the degree of habit formation. For $\eta > 0$, the household is assumed to derive utility from the excess of consumption at date t relative to some habit stock ηC_{t-1}.[1] The stationary vector ξ_t represents exogenous disturbances to preferences. For each value of ξ, the function $u(\cdot;\xi)$ is assumed to be increasing and concave, while $v(\cdot;\xi)$ is increasing and convex.

We assume, as in Dixit and Stiglitz (1977), that the consumption index

$$C_t^h \equiv \left[\int_0^1 c_t^h(z)^{\frac{\theta_p - 1}{\theta_p}} dz\right]^{\frac{\theta_p}{\theta_p - 1}} \tag{2}$$

aggregates consumption of each good, $c_t^j(z)$, with an elasticity of substitution between goods, $\theta_p > 1$, at each date.

1.1.1 Household's optimal allocation of consumption

Optimal behavior on the part of each household requires first an optimal allocation of consumption spending across differentiated goods at each date, for given level of overall expenditure. We assume that consumption of each good at date t is determined on the basis of information available at date t. Choosing $c_t^h(z)$ for all z to maximize (2) subject to the constraint: $\int_0^1 p_t(z)\, c_t^h(z)\, dz \leq X^h$, where $p_t(z)$ is the price of good z and X^h represents a given level of expected total expenses, yields the usual expression for the optimal consumption of good z:

$$c_t^h(z) = C_t^h \left(\frac{p_t(z)}{P_t}\right)^{-\theta_p},$$

where the price index is defined as

$$P_t \equiv \left[\int_0^1 p_t(z)^{1-\theta_p} dz\right]^{\frac{1}{1-\theta_p}}. \tag{3}$$

[1] Christiano, Eichenbaum and Evans (2001) assume a utility function of the form $u = \log(C_t - \eta C_{t-1})$. Here we allow for a more general utility function. Our specification includes for example functions of the form $u = \frac{(C_t - \eta C_{t-1} + M)^{1-\rho}}{1-\rho}$, where $M \geq 0$ is large enough for the whole term in parenthesis to be positive (for all dates t and all states).

It follows that household h's level of total expenditure is given by $\int_0^1 p_t(z)\, c_t^h(z)\, dz = P_t C_t^h$.

1.1.2 Intertemporal allocation of consumption

While the allocation consumption at date t is chosen at date t, we assume as in Rotemberg and Woodford (1997) that households must choose their index of consumption C_t^h at date $t-2$. Equivalently, we assume that C_t^h is determined at the beginning of period $t-1$, i.e., before the monetary policy shock in $t-1$ is known. This implies that

$$C_t^h = E_{t-2} C_t^h. \tag{4}$$

We assume that financial markets are complete so that risks are efficiently shared. As a result, each household faces a single intertemporal budget constraint. At any date t, the household h faces the budget constraint

$$P_t C_t^h + E_t\left[Q_{t,t+1} A_{t+1}^h\right] \leq A_t^h + w_t(h)\, H_t^h + \Pi_t(h) - T_t \tag{5}$$

where A_t^h denotes the nominal value of the household's beginning-of-period financial wealth (state-contingent bonds held at the end of previous period), $w_t(h)$ is the nominal wage of labor h, $\Pi_t(h)$ denotes the profits from sales of good h, and T_t represent the nominal value of (net) lump-sum taxes. The value of the bond portfolio at the beginning of period $t+1$ is given by A_{t+1}^h, and in the absence of arbitrage opportunities, the value of that portfolio at the end of period t is given by $E_t\left[Q_{t,t+1} A_{t+1}^h\right]$, where $Q_{t,t+1}$ is a stochastic discount factor, and where E_t is the expectation conditional upon the state of the world at date t. Note that the riskless one-period nominal interest rate i_t satisfies

$$\frac{1}{1+i_t} = E_t Q_{t,t+1}. \tag{6}$$

While we allow the household to borrow, we impose a borrowing limit at each date, so that any debt contracted can be repaid with certainty out of future income. The infinite sequence of flow budget constraints and of borrowing limits is equivalent to a single intertemporal budget constraint

$$\sum_{T=t}^{\infty} E_t\left[Q_{t,T} P_T C_T^h\right] \leq \sum_{T=t}^{\infty} E_t Q_{t,T}\left[w_T(h)\, H_T^h + \Pi_T(h) - T_T\right] + A_t^h. \tag{7}$$

Because the problem is the same for each household (the initial level of wealth A_t^h is assumed to differ for any two households in a way that compensates for any difference in their expected labor incomes, and complete financial markets allow complete pooling of idiosyncratic labor income risk thereafter), all households choose identical state-contingent plans for consumption. It follows that $C_t \equiv \int_0^1 C_t^h dj = C_t^h$ for all h and all dates t. We may therefore drop the index h on this variable.

Household's optimal behavior. First, optimal household behavior requires that (7) holds with equality, and that $i_t \geq 0$ at all dates. The optimal intertemporal allocation of consumption is obtained by choosing $\{C_T\}_{T=0}^{\infty}$ where $C_t = E_{t-2} C_t$ for any $t \geq 2$, in order maximize (1) subject to (7), where C_0, C_1 are given. First-order conditions for optimal timing of consumption imply

$$\beta^2 E_{t-2}\left\{u_c\left(C_t - \eta C_{t-1}; \xi_t\right) - \beta\eta u_c\left(C_{t+1} - \eta C_t; \xi_{t+1}\right)\right\} = \Lambda_{t-2} E_{t-2}\left\{Q_{t-2,t} P_t\right\} \tag{8}$$

for each date $t \geq 2$ and each possible state at date $t-2$, where Λ_t represents the household's marginal utility of income at date t. The marginal utilities of income at different dates and in different states must furthermore satisfy

$$\Lambda_t Q_{t,T} = \beta^{T-t} \Lambda_T \tag{9}$$

for any possible state at any date $T \geq t$. Using this, we can rewrite (8) as

$$E_{t-2} \left\{ u_c \left(C_t - \eta C_{t-1}; \xi_t \right) - \beta \eta u_c \left(C_{t+1} - \eta C_t; \xi_{t+1} \right) \right\} = E_{t-2} \left\{ \Lambda_t P_t \right\} \tag{10}$$

at each date $t \geq 2$. Combining (9) with (6), we obtain the Euler equation for the optimal timing of consumption

$$\Lambda_t P_t = \beta E_t \left\{ (1 + i_t) \frac{P_t}{P_{t+1}} \Lambda_{t+1} P_{t+1} \right\}. \tag{11}$$

We will consider log-linear approximations of these relationships about the steady state where the exogenous disturbances take the values $\xi_t = 0$ and where there is no inflation, i.e., $P_t/P_{t-1} = 1$. We let $\bar{C}$, $\overline{\lambda}$,and $\bar{\imath}$ be the constant values of consumption, marginal utility of income $(\Lambda_t P_t)$, and nominal interest rate in that steady state, and define the percent deviations $\hat{C}_t \equiv \log \left(C_t/\bar{C} \right)$, $\hat{\lambda}_t \equiv \log \left(\frac{\Lambda_t P_t}{\bar{\lambda}} \right)$, $\hat{\imath}_t \equiv \log \left(\frac{1+i_t}{1+\bar{\imath}} \right)$, $\pi_t \equiv \log \left(P_t/P_{t-1} \right)$. A log-linear approximation to (10), using (4) yields

$$E_{t-2} \hat{\lambda}_t = -\sigma_c^{-1} (1 - \beta\eta)^{-1} \left[\left(\hat{C}_t - \eta \hat{C}_{t-1} \right) - \beta\eta \left(E_{t-2} \hat{C}_{t+1} - \eta \hat{C}_t \right) + E_{t-2} \left(\beta\eta \bar{C}_{t+1} - \bar{C}_t \right) \right] \tag{12}$$

where

$$\sigma_c \equiv -\frac{u_c}{u_{cc} \bar{C}} > 0$$

represents the intertemporal elasticity of substitution of consumption in the absence of habit formation, and where

$$\bar{C}_t \equiv \sigma_c \frac{u_{c\xi}}{u_c} \xi_t$$

represents exogenous preference shocks. Note that in the log-linear approximations, we do not report the approximation errors, i.e., the residuals of order $O \left(||\xi||^2 \right)$, were $||\xi||$ denotes a bound on the amplitude of the exogenous disturbances.

Next, a log-linear approximation of (11) yields

$$\hat{\lambda}_t = E_t \left[\hat{\lambda}_{t+1} + \hat{\imath}_t - \pi_{t+1} \right]. \tag{13}$$

Iterating forward, we obtain

$$\hat{\lambda}_t = \hat{\lambda}_\infty + \sum_{j=0}^{\infty} E_t \left(\hat{\imath}_{t+j} - \pi_{t+j+1} \right)$$

where

$$\hat{\lambda}_\infty = \lim_{T \to \infty} E_t \hat{\lambda}_T$$

is the long run average value of $\hat{\lambda}_t$ under the policy under consideration.

1.2 The Intertemporal IS Equation

We assume that the government purchases an aggregate G_t of the form (2) of all goods in the economy, and thus that the government's demand for each of the good z is given by $g_t(z) = G_t \left(\frac{p_t(z)}{P_t}\right)^{-\theta_p}$. It follows that the demand for good z, $y_t(z) \equiv \int_0^1 y_t^h(z)\,dh = \int_0^1 c_t^h(z)\,dh + g_t(z)$, or

$$y_t(z) = Y_t \left(\frac{p_t(z)}{P_t}\right)^{-\theta_p} \tag{14}$$

where the index of aggregate demand

$$Y_t \equiv \left[\int_0^1 y_t(z)^{\frac{\theta_p - 1}{\theta_p}}\,dz\right]^{\frac{\theta_p}{\theta_p - 1}} \tag{15}$$

satisfies $Y_t = C_t + G_t$. Log-linearizing this, we have $\hat{Y}_t = s_c \hat{C}_t + \hat{G}_t$, where $s_c \equiv \bar{C}/\bar{Y}$, where $\bar{Y}$, $\bar{C} = \bar{Y} - \bar{G}$, are the constant values of output, consumption in the steady state, and $\hat{Y}_t \equiv \log\left(Y_t/\bar{Y}\right)$, $\hat{C}_t \equiv \log\left(C_t/\bar{C}\right)$, $\hat{G}_t \equiv \left(G_t - \bar{G}\right)/\bar{Y}$. Replacing $\hat{C}_t$ with $s_c^{-1}\left(\hat{Y}_t - \hat{G}_t\right)$, we can rewrite (12) as

$$E_{t-2}\hat{\lambda}_t = -\varphi\left(\breve{Y}_t - \breve{g}_t\right) \tag{16}$$

where $\sigma \equiv \sigma_c s_c$,

$$\varphi \equiv \sigma^{-1}\left(1 - \beta\eta\right)^{-1} > 0$$

and

$$\begin{aligned}
\breve{Y}_t &\equiv \left(\hat{Y}_t - \eta\hat{Y}_{t-1}\right) - \beta\eta\left(E_{t-2}\hat{Y}_{t+1} - \eta\hat{Y}_t\right)\\
\breve{g}_t &\equiv \left(g_t - \eta\hat{G}_{t-1}\right) - \beta\eta\left(E_{t-2}g_{t+1} - \eta\hat{G}_t\right)\\
g_t &\equiv \hat{G}_t + s_c E_{t-2}\bar{C}_t,
\end{aligned}$$

Using (13) to solve for $E_{t-2}\left(\hat{\lambda}_t - \hat{\lambda}_{t+1}\right) = E_{t-2}\left(\hat{\imath}_t - \pi_{t+1}\right)$, and differentiating (16), we obtain

$$E_{t-2}\left(\hat{\imath}_t - \pi_{t+1}\right) = \varphi\left[E_{t-2}\left(\breve{Y}_{t+1} - \breve{g}_{t+1}\right) - \left(\breve{Y}_t - \breve{g}_t\right)\right].$$

This can be rewritten as

$$E_{t-2}\left(\hat{\imath}_t - \pi_{t+1}\right) = \varphi\left[E_{t-2}\left(\breve{x}_{t+1} + \breve{Y}_{t+1}^n - \breve{g}_{t+1}\right) - \left(\breve{x}_t + \breve{Y}_t^n - \breve{g}_t\right)\right] \tag{17}$$

where

$$x_t \equiv \hat{Y}_t - \hat{Y}_t^n$$

denotes the "output gap", i.e., the difference between output and the natural rate of output, $\hat{Y}_t^n$ (which we define below), and

$$\begin{aligned}
\breve{x}_t &\equiv \breve{Y}_t - \breve{Y}_t^n\\
\breve{Y}_t^n &\equiv \left(\hat{Y}_t^n - \eta\hat{Y}_{t-1}^n\right) - \beta\eta\left(E_{t-2}\hat{Y}_{t+1}^n - \eta\hat{Y}_t^n\right).
\end{aligned}$$

Taking expectations at date $t-2$ on both sides of (17), we obtain

$$E_{t-2}\left(\hat{\imath}_t - \pi_{t+1}\right) = \varphi\left[E_{t-2}\left(\breve{x}_{t+1} + \breve{Y}^n_{t+1} - \breve{g}_{t+1}\right) - E_{t-2}\left(\breve{x}_t + \breve{Y}^n_t - \breve{g}_t\right)\right] \tag{18}$$

and subtracting this from (17), we observe that

$$\breve{x}_t = E_{t-2}\breve{x}_t + \left(\breve{g}_t - \breve{Y}^n_t\right) - E_{t-2}\left(\breve{g}_t - \breve{Y}^n_t\right). \tag{19}$$

Letting

$$\begin{aligned}
\tilde{Y}_t &\equiv \left(\hat{Y}_t - \eta\hat{Y}_{t-1}\right) - \beta\eta\left(E_t\hat{Y}_{t+1} - \eta\hat{Y}_t\right) \\
\tilde{Y}^n_t &\equiv \left(\hat{Y}^n_t - \eta\hat{Y}^n_{t-1}\right) - \beta\eta\left(E_t\hat{Y}^n_{t+1} - \eta\hat{Y}^n_t\right) \\
\tilde{x}_t &= \tilde{Y}_t - \tilde{Y}^n_t \\
\tilde{g}_t &\equiv \left(g_t - \eta\hat{G}_{t-1}\right) - \beta\eta\left(E_t g_{t+1} - \eta\hat{G}_t\right),
\end{aligned}$$

and noting that $\breve{Y}_t = \tilde{Y}_t + \beta\eta\left(E_t\hat{Y}_{t+1} - E_{t-2}\hat{Y}_{t+1}\right)$, $\breve{x}_t = \tilde{x}_t + \beta\eta\left(E_t x_{t+1} - E_{t-2}x_{t+1}\right)$, and $\breve{g}_t = \tilde{g}_t + \beta\eta\left(E_t g_{t+1} - E_{t-2}g_{t+1}\right)$, we can rewrite (19) as

$$\tilde{x}_t = E_{t-2}\tilde{x}_t + \left(\tilde{g}_t - \tilde{Y}^n_t\right) - E_{t-2}\left(\tilde{g}_t - \tilde{Y}^n_t\right) - \beta\eta\left[E_t\left(x_{t+1} + \hat{Y}^n_{t+1} - g_{t+1}\right) - E_{t-2}\left(x_{t+1} + \hat{Y}^n_{t+1} - g_{t+1}\right)\right]. \tag{20}$$

Here, we will define the natural rate of output Y^n_t as the equilibrium level of output under flexible prices, flexible wages, constant levels of distorting taxes and of desired markups in the labor and product markets, and in the case that all period-t endogenous variables are chosen without delay at date t. Let r^n_t be the equilibrium real rate of interest in that situation as well. In contrast, to simplify the algebra in the "Optimal Inflation Targeting" paper, we define the natural rate of output and of interest in a similar way, except that they are such that the wages, prices and spending decisions are assumed to be predetermined by one period. This implies that the natural rate of output and of interest rate there correspond to $E_{t-1}\hat{Y}^n_t$, and $E_{t-1}\hat{r}^n_t$ here. This doesn't make any difference for our analysis of optimal monetary policy since, as we will see below, it is the forecastable component of the natural rate of output and interest, and not their actual level, that are relevant for optimal monetary policy.

The percent deviations from steady-state $\tilde{Y}^n_t$ and $\hat{r}^n_t$ satisfy an equation of the form (18) in which $\tilde{x}_t = 0$ at all dates, and the expectations are taken at date t. The natural rate of interest is thus given by

$$\hat{r}^n_t = \varphi E_t\left[\left(\tilde{Y}^n_{t+1} - \tilde{Y}^n_t\right) - \left(\tilde{g}_{t+1} - \tilde{g}_t\right)\right]. \tag{21}$$

Finally, taking expectations at $t-2$ on both sides of (21), subtracting from (18) on both sides, and noting that $E_{t-2}\breve{x}_t = E_{t-2}\tilde{x}_t$ and so on, we obtain an expression for the forecastable component of the output gap

$$E_{t-2}\tilde{x}_t = E_{t-2}\tilde{x}_{t+1} - \varphi^{-1}E_{t-2}\left(\hat{\imath}_t - \pi_{t+1} - \hat{r}^n_t\right). \tag{22}$$

The demand side of the model can be summarized by equations (20) and (22). While (20) relates the output gap to its forecastable component and exogenous variables, equation (22) determines the forecastable component of the output gap as a function of its expected future value and a real interest-rate gap.

As we will see below, it is convenient to define

$$\mu_t \equiv \hat{\lambda}_t - \varphi\left(\tilde{g}_t - \tilde{Y}_t\right) \tag{23}$$

which corresponds to the discrepancy between the (log) marginal utility of real income and the (log) marginal utility of consumption. Note that (16) implies that $E_{t-2}\mu_t = 0$. Moreover, if consumption at period t was chosen without delay at period t, then we would have $\mu_t = 0$. Using (13) and (16), we obtain

$$\begin{aligned} E_{t-1}\hat{\lambda}_t &= E_{t-1}\left(\hat{\imath}_t - \pi_{t+1}\right) + E_{t-1}\hat{\lambda}_{t+1} \\ &= E_{t-1}\left(\hat{\imath}_t - \pi_{t+1}\right) + \varphi E_{t-1}\left(\tilde{g}_{t+1} - \tilde{Y}_{t+1}\right) \end{aligned}$$

Combining this with (23), we obtain

$$E_{t-1}\mu_t = E_{t-1}\left(\hat{\imath}_t - \pi_{t+1}\right) + \varphi E_{t-1}\left[\left(\tilde{g}_{t+1} - \tilde{g}_t\right) - \left(\tilde{Y}_{t+1} - \tilde{Y}_t\right)\right]. \tag{24}$$

1.3 Optimal Wage Setting

As in Erceg et al. (2000), Amato and Laubach (2001), and Woodford (2003, ch. 3), we assume that there is a single economy-wide labor market. The producers of all goods hire the same kinds of labor and face the same wages. Capital is assumed to be completely immobile. Firm z is the monopolistic supplier of good z, which it produces according to the production function

$$y_t(z) = A_t F\left(\bar{K}, H_t(z)\right) \equiv A_t f\left(H_t(z)\right)$$

where $f' > 0$, $f'' < 0$, the variable $A_t > 0$ is an exogenous technology factor, and capital is assumed to be fixed so that labor is the only variable input. The labor used to produce each good z is a CES aggregate

$$H_t(z) \equiv \left[\int_0^1 H_t^h(z)^{\frac{\theta_w - 1}{\theta_w}} dh\right]^{\frac{\theta_w}{\theta_w - 1}} \tag{25}$$

for some $\theta_w > 1$, where $H_t^h(z)$ is the labor of type h that is hired to produce a given good z. The demand for labor of type h by firm z is obtained by maximizing the index (25) for a given level of wage payments. It is given by

$$H_t^h(z) = H_t(z)\left(\frac{w_t(h)}{W_t}\right)^{-\theta_w} \tag{26}$$

where $w_t(h)$ is again the nominal wage of labor of type h and the wage index

$$W_t \equiv \left[\int_0^1 w_t(h)^{1-\theta_w} dh\right]^{\frac{1}{1-\theta_w}}.$$

Note that the wage bill of firm z is $\int_0^1 w_t(h)\, H_t^h(z)\, dh = W_t H_t(z)$.

We assume that the wage for each type of labor is set by the monopoly supplier of that type, who is ready to supply as many hours of work as is demanded at that wage. Integrating across firms, the demand for labor faced by household h is $H_t^h = H_t \left(\frac{w_t(h)}{W_t}\right)^{-\theta_w}$ where $H_t \equiv \int_0^1 H_t(z)\, dz$.

Each worker of type h is in a situation of monopolistic competition, and sets a wage $w_t(h)$ under the assumption that the wage has a negligible impact on the wage index W_t. Moreover, as in Calvo, we assume that each wage is reoptimized with a fixed probability $1-\alpha_w$ each period. However, as in Christiano, Eichenbaum and Evans (2001), and Woodford (2002, ch. 3) if a wage is not reoptimized, it is adjusted according to the indexation rule

$$\log w_t(h) = \log w_{t-1}(h) + \gamma_w \pi_{t-1}$$

for some $0 \leq \gamma_w \leq 1$. A worker of type h who chooses a new wage $w_t(h)$ at date t, expects to have a wage $w_t(h)(P_{T-1}/P_{t-1})^{\gamma_w}$ with probability α_w^{T-t} at any date $T \geq t$. In the face of such a wage, the worker will face a demand of $H_T \left(\frac{w_t(h)(P_{T-1}/P_{t-1})^{\gamma_w}}{W_T}\right)^{-\theta_w}$ for its work. We assume that the newly chosen wage that comes into effect in period t, w_t^*, is chosen at the end of period $t-1$, i.e., on the basis of information available at date $t-1$. This wage is thus chosen to maximize

$$E_{t-1}\left\{\sum_{T=t}^{\infty}(\alpha_w\beta)^{T-t}\left[\Lambda_T(1+\tau_w)\,w_t(h)\left(\frac{P_{T-1}}{P_{t-1}}\right)^{\gamma_w} H_T\left(\frac{w_t(h)(P_{T-1}/P_{t-1})^{\gamma_w}}{W_T}\right)^{-\theta_w}\right.\right.$$
$$\left.\left. -v\left(H_T\left(\frac{w_t(h)(P_{T-1}/P_{t-1})^{\gamma_w}}{W_T}\right)^{-\theta_w};\xi_T\right)\right]\right\}$$

As in Amato and Laubach (2001), we add a subsidy $0 \leq \tau_w < 1$ for employment that offsets the effect on imperfect competition in labor markets on the steady-state level of output. The first-order condition for this problem can be expressed as

$$\begin{aligned} 0 \;=\; & E_{t-1}\left\{\sum_{T=t}^{\infty}(\alpha_w\beta)^{T-t}\left[H_T(w_t^*)^{-\theta_w-1}\left(\frac{(P_{T-1}/P_{t-1})^{\gamma_w}}{W_T}\right)^{-\theta_w}(1-\theta_w)\right]\times\right. \\ & \left.\left[\Lambda_T(1+\tau_w)\,w_t^*\left(\frac{P_{T-1}}{P_{t-1}}\right)^{\gamma_w} - \mu_w v_h\left(H_T\left(\frac{w_t^*(P_{T-1}/P_{t-1})^{\gamma_w}}{W_T}\right)^{-\theta_w};\xi_T\right)\right]\right\} \end{aligned}$$

where $\mu_w \equiv \theta_w/(\theta_w - 1) > 1$. Note that the optimal wage w_t^* is the same for all h, since the function $v(H,\xi)$ is the same for all h, and since the problem faced by each worker who can adjust the wage at date t is the same. The previous equation determines implicitly the optimal wage w_t^*. The aggregate wage is then given by

$$W_t \equiv \left[\alpha_w\left(W_{t-1}\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_w}\right)^{1-\theta_w} + (1-\alpha_w)(w_t^*)^{1-\theta_w}\right]^{\frac{1}{1-\theta_w}}.$$

We will log-linearize the above conditions under the assumption that W_t/P_t, P_t/P_{t-1}, and w_t^*/W_t will remain close to their steady-state values $\bar{\omega}, 1$, and 1 respectively. An approximation to the previous equation yields

$$\hat{w}_t^* = \frac{\alpha_w}{1-\alpha_w}\left(\pi_t^w - \gamma_w \pi_{t-1}\right), \tag{27}$$

where $\hat{w}_t^* \equiv \log\left(w_t^*/W_t\right)$, $\pi_t^w \equiv \log\left(W_t/W_{t-1}\right)$, and $\pi_t \equiv \log\left(P_t/P_{t-1}\right)$. As an approximation to the first-order condition above, we obtain

$$0 = E_{t-1}\left\{\sum_{T=t}^{\infty}\left(\alpha_w\beta\right)^{T-t}\left[\left(1+\nu\theta_w\right)\left(\hat{w}_t^* - \sum_{\tau=t+1}^{T}\left(\pi_\tau^w - \gamma_w\pi_{\tau-1}\right)\right) + \hat{\omega}_T - \hat{v}_T\right]\right\}$$

where $\nu \equiv \frac{v_{hh}\bar{H}}{v_h}$, $\hat{H}_t \equiv \log\left(H_t/\bar{H}\right)$, and where

$$\hat{\omega}_t \equiv \log\left(\frac{W_t/P_t}{\bar{\omega}}\right)$$

represents the percent deviation of the real wage from steady-state, and

$$\hat{v}_t \equiv \nu\hat{H}_t + \frac{v_{h\xi}}{v_h}\xi_t - \hat{\lambda}_t \tag{28}$$

is the average (over different types of labor) of the deviation from steady state of the log of $v_h\left(.,\xi_t\right)/\Lambda_t P_t$, the marginal rate of substitution between labor and consumption at date t. Solving the log-linearized FOC for $\hat{w}_t^* = E_{t-1}\hat{w}_t^*$, we obtain

$$\hat{w}_t^* = E_{t-1}\left\{\left[\sum_{T=t}^{\infty}\left(\alpha_w\beta\right)^{T-t}\left[\left(\pi_T^w - \gamma_w\pi_{T-1}\right) + \frac{1-\alpha_w\beta}{1+\nu\theta_w}\left(\hat{v}_T - \hat{\omega}_T\right)\right]\right] - \left(\pi_t^w - \gamma_w\pi_{t-1}\right)\right\}$$

Quasi-differentiating this and using (27), we get finally

$$\left(\pi_t^w - \gamma_w\pi_{t-1}\right) = \xi_w E_{t-1}\left(\hat{v}_t - \hat{\omega}_t\right) + \beta E_{t-1}\left(\pi_{t+1}^w - \gamma_w\pi_t\right) \tag{29}$$

where

$$\xi_w \equiv \frac{\left(1-\alpha_w\right)\left(1-\alpha_w\beta\right)}{\alpha_w\left(1+\nu\theta_w\right)} > 0.$$

Log-linearizing the production function and assuming that the steady-state value of A_t is 1 yields

$$\hat{y}_t(z) = a_t + \phi^{-1}\hat{H}_t(z)$$

where $\hat{y}_t(z) \equiv \log\left(y_t(z)/\bar{Y}\right)$, $a_t \equiv \log\left(A_t\right)$, $\hat{H}_t(z) \equiv \log\left(H_t(z)/\bar{H}\right)$, and $\phi \equiv \frac{f\left(\bar{H}\right)}{\bar{H}f'\left(\bar{H}\right)} > 1$. Aggregating over firms z and solving for $\hat{H}_t$ yields

$$\hat{H}_t = \phi\left(\hat{Y}_t - a_t\right).$$

Substituting this into (28), we obtain

$$\hat{v}_t = \omega_w\left(\hat{Y}_t - a_t\right) + \frac{v_{h\xi}}{v_h}\xi_t - \hat{\lambda}_t$$

where $\omega_w \equiv \nu\phi = \frac{v_{hh}\bar{H}}{v_h}\frac{f}{\bar{H}f'} > 0$. Using (23) to substitute for $\hat{\lambda}_t$, we can express $\hat{v}_t$ as

$$\hat{v}_t = \omega_w\left(\hat{Y}_t - a_t\right) + \frac{v_{h\xi}}{v_h}\xi_t - \mu_t - \varphi\left(\tilde{g}_t - \tilde{Y}_t\right). \tag{30}$$

1.4 Optimal Price Setting

As in Calvo, we assume that each price is reoptimized with a fixed probability $1-\alpha_p$ each period. However, as in Woodford (2002, ch. 3) if a price is not reoptimized, it is adjusted according to the indexation rule

$$\log p_t(z) = \log p_{t-1}(z) + \gamma_p \pi_{t-1}$$

for some $0 \leq \gamma_p \leq 1$. As each supplier who is allowed to change its price in period t faces the same decision problem, each of them chooses the same optimal price p_t^*. It follows that the price index (3) satisfies

$$P_t = \left[(1-\alpha_p) p_t^{*1-\theta_p} + \alpha_p \left(P_{t-1}\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_p}\right)^{1-\theta_p}\right]^{\frac{1}{1-\theta_p}}.$$

Log-linearizing this yields

$$\hat{p}_t^* = \frac{\alpha_p}{1-\alpha_p}\pi_t^{qd} \tag{31}$$

where $\hat{p}_t^* \equiv \log(p_t^*/P_t)$, and $\pi_t^{qd} \equiv \pi_t - \gamma_p \pi_{t-1}$.

We suppose that prices are chosen one period in advance, i.e., at date $t-1$, while the purchase of each individual good is chosen at date t. Any supplier who may set a reoptimized price in period t chooses its new price p_t^* to maximize the expected present discounted value of future profits given by

$$E_{t-1}\left\{\sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T}\left[\Pi_T(p_t)\right]\right\}$$

where nominal profits in period T are

$$\Pi_T(p) = (1+\tau_p)\, p \left(\frac{P_{T-1}}{P_{t-1}}\right)^{\gamma_p} \left(\frac{p\,(P_{T-1}/P_{t-1})^{\gamma_p}}{P_T}\right)^{-\theta_p} Y_T - W_t f^{-1}\left(\left(\frac{p\,(P_{T-1}/P_{t-1})^{\gamma_p}}{P_T}\right)^{-\theta_p} \frac{Y_T}{A_T}\right),$$

where $0 \leq \tau_p < 1$ is a subsidy for output that offsets the effect on imperfect competition in goods markets on the steady-state level of output.

The first-order condition for this problem can be expressed as

$$\begin{aligned} 0 \quad = \quad & E_{t-1}\left\{\sum_{T=t}^{\infty} (\alpha_p\beta)^{T-t}\left[\frac{\Lambda_T}{\Lambda_t}(p_t^*)^{-\theta_p-1}\left(\frac{(P_{T-1}/P_{t-1})^{\gamma_p}}{P_T}\right)^{-\theta_p}(1-\theta_p) P_T Y_T\right]\times\right. \\ & \left.\left[(1+\tau_p)\frac{p_t^*}{P_T}\left(\frac{P_{T-1}}{P_{t-1}}\right)^{\gamma_p} - \mu_p \frac{W_T/P_T}{A_T f'\left(f^{-1}\left(\left(\frac{p_t^*(P_{T-1}/P_{t-1})^{\gamma_p}}{P_T}\right)^{-\theta_p}\frac{Y_T}{A_T}\right)\right)}\right]\right\} \end{aligned}$$

where $\mu_p \equiv \theta_p/(\theta_p - 1) > 1$. Note that the optimal price p_t^* is the same for all firms, since the problem faced by each firm who can adjust the wage at date t is the same. Log-linearizing this equation around a steady state in which $(1+\tau_p) = \mu_p \bar{\omega}/f'$, we obtain

$$0 = E_{t-1}\left\{\sum_{T=t}^{\infty}(\alpha_p\beta)^{T-t}\left[(1+\omega_p\theta_p)\left(\hat{p}_t^* - \sum_{\tau=t+1}^{T}\pi_\tau^{qd}\right) - \hat{\omega}_T - \hat{\psi}_T\right]\right\}$$

where $\omega_p \equiv -f''\bar{Y}/\left(f'\right)^2$, $\hat{p}_t^* \equiv \log\left(p_t^*/P_t\right)$, $\hat{\omega}_t \equiv \log\left(\frac{W_t/P_t}{\bar{\omega}}\right)$, and

$$\hat{\psi}_t \equiv \omega_p \hat{Y}_t - \left(1+\omega_p\right) a_t$$

is minus the average (across firms) deviation of log marginal product of labor from its steady-state. Solving the log-linearized first-order condition for $\hat{p}_t^* = E_{t-1}\hat{p}_t^*$, we obtain

$$\hat{p}_t^* = E_{t-1}\left\{\left[\sum_{T=t}^{\infty}\left(\alpha_p\beta\right)^{T-t}\left[\pi_T^{qd} + \frac{1-\alpha_p\beta}{1+\omega_p\theta_p}\left(\hat{\omega}_T + \hat{\psi}_T\right)\right]\right] - \pi_t^{qd}\right\}$$

Quasi-differentiating the last equation and using (31), we get

$$\pi_t - \gamma_p\pi_{t-1} = \xi_p E_{t-1}\left(\hat{\psi}_t + \hat{\omega}_t\right) + \beta E_{t-1}\left(\pi_{t+1} - \gamma_p\pi_t\right) \tag{32}$$

where

$$\xi_p \equiv \frac{\left(1-\alpha_p\right)\left(1-\alpha_p\beta\right)}{\alpha_p\left(1+\omega_p\theta_p\right)} > 0.$$

Equation (32) can be further rewritten as

$$\pi_t - \gamma_p\pi_{t-1} = \kappa_p E_{t-1}\left(\hat{Y}_t - \hat{Y}_t^n\right) + \xi_p E_{t-1}\left(\hat{\omega}_t - \hat{\omega}_t^n\right) + \beta E_{t-1}\left(\pi_{t+1} - \gamma_p\pi_t\right)$$

where $\kappa_p \equiv \xi_p\omega_p$, and $\hat{Y}_t^n$ represents the percentage deviation of the natural rate of output (i.e., the equilibrium level of output under flexible prices and wages, and period-t consumption is decided at date t) from steady-state, and

$$\hat{\omega}_t^n \equiv \left(1+\omega_p\right) a_t - \omega_p\hat{Y}_t^n$$

represents the percent deviation from steady-state value $\bar{\omega}$ of the "natural real wage", i.e., the equilibrium real wage when both wages and prices are fully flexible.

Defining w_t as the log real wage, and w_t^n as the log of the "natural real wage", and noting that $w_t - w_t^n = \left(\hat{\omega}_t + \log\bar{\omega}\right) - \left(\hat{\omega}_t^n + \log\bar{\omega}\right) = \hat{\omega}_t - \hat{\omega}_t^n$, we can rewrite the above inflation equation as

$$\pi_t - \gamma_p\pi_{t-1} = \kappa_p E_{t-1}x_t + \xi_p E_{t-1}\left(w_t - w_t^n\right) + \beta E_{t-1}\left(\pi_{t+1} - \gamma_p\pi_t\right). \tag{33}$$

1.5 Natural Rate of Output

We now turn to the determination of the natural rate of output, i.e., the equilibrium level of output under flexible prices, flexible wages, constant levels of distorting taxes and of desired markups in the labor and product markets, and in the case that all period-t endogenous variables are chosen without delay at date t. As mentioned above, the measure of the natural rate of output considered in the "Optimal Inflation Targeting" paper is similar, except that it is such that the wages, prices and spending decisions are assumed to be predetermined by one period. This implies that the natural rate of output there corresponds to $E_{t-1}\hat{Y}_t^n$. Again, as indicated above, this doesn't make any difference for our analysis of optimal monetary policy, since, as further indicated below, it is the forecastable component of the natural rate of output, and not its actual level, that is relevant for optimal monetary policy.

First, note that the first-order condition for the optimal supply of labor by household h is given by

$$\frac{v_h\left(H_t^h;\xi_t\right)}{\Lambda_t P_t}=\frac{w_t\left(h\right)}{P_t} \tag{34}$$

at all dates t.

Next, the firm's profits are given by

$$\begin{aligned}\Pi_t\left(z\right) &\equiv (1+\tau_p)\,p_t\left(z\right)y_t\left(z\right)-W_tH_t\left(z\right)\\ &= (1+\tau_p)\,p_t\left(z\right)^{1-\theta_p}P_t^{\theta_p}Y_t-W_tf^{-1}\left(p_t\left(z\right)^{-\theta_p}P_t^{\theta_p}Y_t/A_t\right).\end{aligned}$$

In the case that prices are perfectly flexible, the optimal pricing decision for the firm z, i.e., the price that would maximize profits at each period is given by

$$p_t\left(z\right)=\frac{\mu_p}{1+\tau_p}\frac{W_t}{A_tf'\left(f^{-1}\left(y_t\left(z\right)/A_t\right)\right)}$$

where the desired markup $\mu_p\equiv\frac{\theta_p}{\theta_p-1}$. Using (14), we note that the relative supply of good z must in turn satisfy

$$\left(\frac{y_t\left(z\right)}{Y_t}\right)^{-1/\theta_p}=\frac{\mu_p}{1+\tau_p}\frac{W_t}{P_t}\frac{1}{A_tf'\left(f^{-1}\left(y_t\left(z\right)/A_t\right)\right)}.$$

Because all wages are the same in the case of flexible wages, $w_t\left(h\right)=W_t$ and $H_t^h=H_t$ for all h. Thus when wages and prices are flexible, all sellers supply a quantity Y_t^n satisfying

$$1=\frac{\mu_p}{1+\tau_p}\frac{v_h\left(f^{-1}\left(Y_t^n/A_t\right);\xi_t\right)}{\lambda_t^n}\frac{1}{A_tf'\left(f^{-1}\left(Y_t^n/A_t\right)\right)} \tag{35}$$

where λ_t^n denotes the marginal utility of income at date t in the case of flexible prices, flexible wages, and in the case that consumption at date t is decided at date t. Note that in steady-state, (35) reduces to

$$\frac{v_h}{\bar{\lambda}f'}=\frac{1+\tau_p}{\mu_p}\equiv 1-\Phi,$$

where Φ is a measure of inefficiency in the economy, at steady-state. As in Woodford (2003), we will assume that Φ is of order $O\left(||\xi||\right)$. Using furthermore (10), we observe that in the steady state, $u_c\left(1-\beta\eta\right)=\bar{\lambda}$, so that

$$v_h=\left(1-\Phi\right)\left(1-\beta\eta\right)u_cf'. \tag{36}$$

Log-linearizing (35) about this steady-state and solving for $\hat{Y}_t^n$ yields

$$\omega\hat{Y}_t^n=\left(1+\omega\right)a_t-\frac{v_{h\xi}}{v_h}\xi_t+\hat{\lambda}_t^n \tag{37}$$

where $\hat{\lambda}_t^n\equiv\log\left(\lambda_t^n/\bar{\lambda}\right).$

In the case of flexible prices and wages, and in the case that consumption decisions for period t are made without delay at period t, the variable μ_t defined as $\mu_t\equiv\hat{\lambda}_t-\varphi\left(\tilde{g}_t-\tilde{Y}_t\right)$ is equal to zero at all dates. It thus follows that

$$\hat{\lambda}_t^n=-\varphi\left(\tilde{Y}_t^n-\tilde{g}_t\right) \tag{38}$$

Using this to substitute for $\hat{\lambda}_t^n$ in (37), we obtain

$$E_t\left\{\left[\omega+\varphi\left(1-\eta L\right)\left(1-\beta\eta L^{-1}\right)\right]\hat{Y}_t^n\right\}=E_t\left(\left(1+\omega\right)a_t-\frac{v_{h\xi}}{v_h}\xi_t+\varphi\tilde{g}_t\right) \tag{39}$$

which implicitly determines the natural rate of output.

1.5.1 Rewriting the wage inflation equation

Using (37), we can rewrite the average deviation from steady state of the log marginal rate of substitution between labor and consumption as follows

$$\hat{v}_t=\omega_w x_t+\varphi\tilde{x}_t-\mu_t+\hat{\omega}_t^n$$

where once again

$$\hat{\omega}_t^n\equiv\left(1+\omega_p\right)a_t-\omega_p\hat{Y}_t^n.$$

represents the percent deviation from steady-state value $\bar{\omega}$ of the "natural real wage", i.e., the equilibrium real wage when both wages and prices are fully flexible, and consumption at date t is chosen at that date. Noting again that $w_t-w_t^n=\hat{\omega}_t-\hat{\omega}_t^n$, it follows that the wage-inflation equation (29) can be rewritten as

$$\begin{aligned}\left(\pi_t^w-\gamma_w\pi_{t-1}\right) \quad = \quad & \xi_w E_{t-1}\left(\omega_w x_t+\varphi\tilde{x}_t\right)-\xi_w E_{t-1}\mu_t \\ & +\xi_w E_{t-1}\left(w_t^n-w_t\right)+\beta E_{t-1}\left(\pi_{t+1}^w-\gamma_w\pi_t\right)\end{aligned}$$

where $E_{t-1}\mu_t$ is given by (24).

1.6 Summary of the Model

1.6.1 Structural equations

On the demand side, the forecastable component of the output gap $x_t\equiv\hat{Y}_t-\hat{Y}_t^n$ satisfies

$$E_{t-2}\tilde{x}_t=E_{t-2}\tilde{x}_{t+1}-\varphi^{-1}E_{t-2}\left(\hat{\imath}_t-\pi_{t+1}-\hat{r}_t^n\right)$$

where $\tilde{x}_t\equiv\left(x_t-\eta x_{t-1}\right)-\beta\eta\left(E_t x_{t+1}-\eta x_t\right).$ The output gap then relates to the expected output gap through

$$\tilde{x}_t=E_{t-2}\tilde{x}_t+\left(\tilde{g}_t-\tilde{Y}_t^n\right)-E_{t-2}\left(\tilde{g}_t-\tilde{Y}_t^n\right)-\beta\eta\left[E_t\left(x_{t+1}+\hat{Y}_{t+1}^n-g_{t+1}\right)-E_{t-2}\left(x_{t+1}+\hat{Y}_{t+1}^n-g_{t+1}\right)\right].$$

On the supply side, price inflation is given by

$$\pi_t-\gamma_p\pi_{t-1}=\xi_p\omega_p E_{t-1}x_t+\xi_p E_{t-1}\left(w_t-w_t^n\right)+\beta E_{t-1}\left(\pi_{t+1}-\gamma_p\pi_t\right)$$

and wage inflation is given by

$$\pi_t^w-\gamma_w\pi_{t-1}=\xi_w E_{t-1}\left(\omega_w x_t+\varphi\tilde{x}_t\right)-\xi_w E_{t-1}\mu_t+\xi_w E_{t-1}\left(w_t^n-w_t\right)+\beta E_{t-1}\left(\pi_{t+1}^w-\gamma_w\pi_t\right)$$

where the expected discrepancy between the (log) marginal utility of real income the (log) marginal utility of consumption satisfies

$$E_{t-1}\mu_t = E_{t-1}\left(\hat{\imath}_t - \pi_{t+1}\right) + \varphi E_{t-1}\left[\left(\tilde{g}_{t+1} - \tilde{g}_t\right) - \left(\tilde{Y}_{t+1} - \tilde{Y}_t\right)\right].$$

The (log) real wage then satisfies

$$w_t = w_{t-1} + \pi_t^w - \pi_t.$$

The composite exogenous variables of the model are then defined as follows. The natural rate of interest is given by

$$\hat{r}_t^n = \varphi E_t\left[\left(\tilde{Y}_{t+1}^n - \tilde{Y}_t^n\right) - \left(\tilde{g}_{t+1} - \tilde{g}_t\right)\right]$$

where

$$\begin{aligned}
\tilde{Y}_t^n &\equiv \left(\hat{Y}_t^n - \eta\hat{Y}_{t-1}^n\right) - \beta\eta\left(E_t\hat{Y}_{t+1}^n - \eta\hat{Y}_t^n\right) \\
\tilde{g}_t &\equiv \left(g_t - \eta\hat{G}_{t-1}\right) - \beta\eta\left(E_t g_{t+1} - \eta\hat{G}_t\right) \\
g_t &\equiv \hat{G}_t + s_c E_{t-2}\bar{C}_t.
\end{aligned}$$

The natural rate of output is defined implicitly by

$$E_t\left\{\left[\omega + \varphi\left(1-\eta L\right)\left(1-\beta\eta L^{-1}\right)\right]\hat{Y}_t^n\right\} = E_t\left(\left(1+\omega\right)a_t - \frac{v_{h\xi}}{v_h}\xi_t + \varphi\tilde{g}_t\right),$$

and the log of the “natural real wage” is given by

$$w_t^n \equiv \left(1+\omega_p\right)a_t - \omega_p\hat{Y}_t^n + \bar{\omega},$$

where $\bar{\omega}$ is the steady-state value of the log of the “real wage”.

1.6.2 Welfare loss function

As shown in Appendix B.2. of the paper, the model-based welfare loss function is of the form

$$E_0\sum_{t=0}^{\infty}\beta^t\left[\lambda_p\left(\pi_t - \gamma_p\pi_{t-1}\right)^2 + \lambda_w\left(\pi_t^w - \gamma_w\pi_{t-1}\right)^2 + \lambda_x\left(x_t - \delta x_{t-1} - \hat{x}^*\right)^2\right]$$

where

$$\begin{aligned}
\lambda_p &\equiv \frac{\theta_p\xi_p^{-1}}{\theta_p\xi_p^{-1} + \theta_w\phi^{-1}\xi_w^{-1}} > 0, \qquad \lambda_w \equiv \frac{\theta_w\phi^{-1}\xi_w^{-1}}{\theta_p\xi_p^{-1} + \theta_w\phi^{-1}\xi_w^{-1}} > 0 \\
\lambda_x &\equiv \frac{\vartheta\varphi}{\theta_p\xi_p^{-1} + \theta_w\phi^{-1}\xi_w^{-1}} > 0.
\end{aligned}$$

We now show that whether the natural rate of output or the output gap is predetermined by one period (as it is assumed to be in the paper) or not (as is the case in this note) does not play any role for optimal monetary policy. In fact, we can rewrite the output gap, as

$$\begin{aligned} x_t &\equiv \hat{Y}_t - \hat{Y}_t^n \\ &= s_c \hat{C}_t + \hat{G}_t - \hat{Y}_t^n \\ &= v_{t-2} + s_{t-1} + \zeta_t \end{aligned}$$

where

$$v_{t-2} \equiv s_c \hat{C}_t + E_{t-2}\left(\hat{G}_t - \hat{Y}_t^n\right)$$

is an endogenous variable determined at date $t-2$ (since consumption plans are assumed to be determined two periods in advance), and

$$\begin{aligned} s_{t-1} &\equiv E_{t-1}\left(\hat{G}_t - \hat{Y}_t^n\right) - E_{t-2}\left(\hat{G}_t - \hat{Y}_t^n\right) \\ \zeta_t &\equiv \left(\hat{G}_t - \hat{Y}_t^n\right) - E_{t-1}\left(\hat{G}_t - \hat{Y}_t^n\right) \end{aligned}$$

are exogenous variables satisfying $E_{t-2}s_{t-1} = 0$ and $E_{t-1}\zeta_t = 0$. (Note that in the paper $\zeta_t = 0$ at all dates, as x_t is assumed to be determined at date $t-1$.)

Using this, the term relative to the output gap in the welfare loss function can be rewritten as

$$\begin{aligned} E_0 \sum_{t=0}^{\infty} \beta^t \lambda_x \left(x_t - \delta x_{t-1} - \hat{x}^*\right)^2 &= \lambda_x \left(x_0 - \delta x_{-1} - \hat{x}^*\right) + \lambda_x \beta \left(x_1 - \delta x_0 - \hat{x}^*\right) + \lambda_x \beta^2 E_0 \sum_{t=0}^{\infty} \beta^t \left(x_{t+2} - \delta x_{t+1} - \hat{x}^*\right)^2 \\ &= \lambda_x \beta^2 E_0 \sum_{t=0}^{\infty} \beta^t \left(x_{t+2} - \delta x_{t+1} - \hat{x}^*\right)^2 + tip \\ &= \lambda_x \beta^2 E_0 \sum_{t=0}^{\infty} \beta^t \left[v_t + s_{t+1} + \zeta_{t+2} - \delta\left(v_{t-1} + s_t + \zeta_{t+1}\right) - \hat{x}^*\right]^2 + tip \\ &= \lambda_x \beta^2 E_0 \sum_{t=0}^{\infty} \beta^t \left[\left(v_t - \delta v_{t-1} - \delta s_t - \hat{x}^*\right) + \left(s_{t+1} + \zeta_{t+2} - \delta \zeta_{t+1}\right)\right]^2 + tip \\ &= \lambda_x \beta^2 E_0 \sum_{t=0}^{\infty} \beta^t \Big[\left(v_t - \delta v_{t-1} - \delta s_t - \hat{x}^*\right)^2 + \left(s_{t+1} + \zeta_{t+2} - \delta \zeta_{t+1}\right)^2 \\ &\quad + 2\left(v_t - \delta v_{t-1} - \delta s_t - \hat{x}^*\right)\left(s_{t+1} + \zeta_{t+2} - \delta \zeta_{t+1}\right)\Big] + tip \\ &= \lambda_x \beta^2 E_0 \sum_{t=0}^{\infty} \beta^t \left(v_t - \delta v_{t-1} - \delta s_t - \hat{x}^*\right)^2 + tip \end{aligned}$$

where *tip* denotes terms independent of policy adopted at date 0, such as exogenous variables or endogenous variables determined before date 0. Note that to obtain the last line, we used the fact that s_t and ζ_t are unforecastable, so that $E_0 z_t s_{t+1} = E_0\left[z_t\left(E_t s_{t+1}\right)\right] = 0$ and $E_0 z_t \zeta_{t+1} = E_0\left[z_t\left(E_t \zeta_{t+1}\right)\right] = 0$ for any date $t \geq 0$, and any variable z_t determined at date t or earlier.

We observe from the last expression that whether the output gap is predetermined or not, i.e., whether $\zeta_t = 0$ at all dates or not, is irrelevant for the welfare loss function. Since it is furthermore only the forecastable component of the output gap, $E_{t-1}x_t$, that is relevant for the determination of price inflation and wage inflation in the structural equations of the model, it doesn't matter, for the purpose of optimal monetary policy, whether the output gap is predetermined or not.

2 Special Cases

2.1 Absence of Habit Persistence

In the absence of habit persistence, $\eta = 0$ so that $\tilde{x}_t = x_t$, $\tilde{Y}_t = \hat{Y}_t$, $\tilde{Y}_t^n = \hat{Y}_t^n$, and $\tilde{g}_t = g_t$. In addition, the parameter φ reduces to $\varphi = \sigma^{-1}$. It follows that the equations of the model reduce to the following. On the demand side, the forecastable component of the output gap $x_t \equiv \hat{Y}_t - \hat{Y}_t^n$ satisfies

$$E_{t-2}x_t = E_{t-2}x_{t+1} - \varphi^{-1}E_{t-2}\left(\hat{\imath}_t - \pi_{t+1} - \hat{r}_t^n\right)$$

and the output gap relates to the expected output gap through

$$x_t = E_{t-2}x_t + \left(g_t - \hat{Y}_t^n\right) - E_{t-2}\left(g_t - \hat{Y}_t^n\right).$$

On the supply side, price inflation is given by

$$\pi_t - \gamma_p\pi_{t-1} = \xi_p\omega_p E_{t-1}x_t + \xi_p E_{t-1}\left(w_t - w_t^n\right) + \beta E_{t-1}\left(\pi_{t+1} - \gamma_p\pi_t\right)$$

and wage inflation is given by

$$\pi_t^w - \gamma_w\pi_{t-1} = \xi_w\left(\omega_w + \varphi\right)E_{t-1}x_t - \xi_w E_{t-1}\mu_t + \xi_w E_{t-1}\left(w_t^n - w_t\right) + \beta E_{t-1}\left(\pi_{t+1}^w - \gamma_w\pi_t\right)$$

where the expected discrepancy between the (log) marginal utility of real income the (log) marginal utility of consumption satisfies

$$E_{t-1}\mu_t = E_{t-1}\left(\hat{\imath}_t - \pi_{t+1}\right) + \varphi E_{t-1}\left[\left(g_{t+1} - g_t\right) - \left(\hat{Y}_{t+1} - \hat{Y}_t\right)\right].$$

The (log) real wage then satisfies

$$w_t = w_{t-1} + \pi_t^w - \pi_t.$$

The composite exogenous variables of the model are then defined as follows. The natural rate of interest is given by

$$\hat{r}_t^n = \varphi E_t\left[\left(\hat{Y}_{t+1}^n - \hat{Y}_t^n\right) - \left(g_{t+1} - g_t\right)\right]$$

where

$$g_t \equiv \hat{G}_t + s_c E_{t-2}\bar{C}_t,$$

the natural rate of output reduces to

$$\hat{Y}_t^n = \left(\omega + \varphi\right)^{-1}\left(\left(1 + \omega\right)a_t - \frac{v_{h\xi}}{v_h}\xi_t + \sigma^{-1}g_t\right).$$

and the log of the "natural real wage" is given by

$$w_t^n \equiv \left(1 + \omega_p\right)a_t - \omega_p\hat{Y}_t^n + \bar{\omega}.$$

In this case, it follows from section B.2 of the technical appendix that the loss function reduces to

$$E_0\sum_{t=0}^{\infty}\beta^t\left[\lambda_p\left(\pi_t - \gamma_p\pi_{t-1}\right)^2 + \lambda_w\left(\pi_t^w - \gamma_w\pi_{t-1}\right)^2 + \lambda_x\left(x_t - x^*\right)^2\right]$$

where

$$\lambda_p \equiv \frac{\theta_p \xi_p^{-1}}{\theta_p \xi_p^{-1} + \theta_w \phi^{-1} \xi_w^{-1}} > 0, \qquad \lambda_w \equiv \frac{\theta_w \phi^{-1} \xi_w^{-1}}{\theta_p \xi_p^{-1} + \theta_w \phi^{-1} \xi_w^{-1}} > 0$$

are the same as in the complete model, but where λ_x reduces to

$$\lambda_x \equiv \frac{\omega + \varphi}{\theta_p \xi_p^{-1} + \theta_w \phi^{-1} \xi_w^{-1}} > 0,$$

as the numerator satisfies $\vartheta\varphi = \frac{\beta}{2}\left(\chi + \sqrt{\chi^2}\right)\varphi = \beta\chi\varphi = \omega + \varphi$.

2.2 Flexible Wages

In the presence of habit persistence but in the case of perfectly flexible wages, as in the Boivin and Giannoni (2003) model, each wage is reoptimized at each period so that $\alpha_w = 0$ so that $\xi_w^{-1} = 0$. It follows from the wage inflation equation that

$$\begin{aligned} E_{t-1}\left(w_t^n - w_t\right) &= \xi_w^{-1}\left[\pi_t^w - \gamma_w \pi_{t-1} - \beta E_{t-1}\left(\pi_{t+1}^w - \gamma_w \pi_t\right)\right] - E_{t-1}\left(\omega_w x_t + \varphi \tilde{x}_t\right) + E_{t-1}\mu_t \\ &= -E_{t-1}\left(\omega_w x_t + \varphi \tilde{x}_t\right) + E_{t-1}\mu_t \end{aligned}$$

Combining this with the price inflation equation, we obtain an aggregate supply equation of the form

$$\begin{aligned} \pi_t - \gamma_p \pi_{t-1} &= \xi_p \omega_p E_{t-1} x_t + \xi_p E_{t-1}\left(w_t - w_t^n\right) + \beta E_{t-1}\left(\pi_{t+1} - \gamma_p \pi_t\right) \\ &= \xi_p E_{t-1}\left(\omega x_t + \varphi \tilde{x}_t\right) - \xi_p E_{t-1}\mu_t + \beta E_{t-1}\left(\pi_{t+1} - \gamma_p \pi_t\right), \end{aligned}$$

where $\omega \equiv \omega_p + \omega_w$. As discussed in section 1.5 of the text, this equation may be equivalently expressed as

$$\pi_t - \gamma_p \pi_{t-1} = \kappa E_{t-1}\left[\left(x_t - \delta x_{t-1}\right) - \beta\delta\left(x_{t+1} - \delta x_t\right)\right] - \xi_p E_{t-1}\mu_t + \beta E_{t-1}\left(\pi_{t+1} - \gamma_p \pi_t\right)$$

where $0 \leq \delta \leq \eta$ is the smaller root of the quadratic equation

$$\eta\varphi\left(1 + \beta\delta^2\right) = \left[\omega + \varphi\left(1 + \beta\eta^2\right)\right]\delta$$

and

$$\kappa \equiv \xi_p \eta\varphi/\delta > 0.$$

Again, the expected discrepancy between the (log) marginal utility of real income the (log) marginal utility of consumption satisfies

$$E_{t-1}\mu_t = E_{t-1}\left(\hat{\imath}_t - \pi_{t+1}\right) + \varphi E_{t-1}\left[\left(\tilde{g}_{t+1} - \tilde{g}_t\right) - \left(\tilde{Y}_{t+1} - \tilde{Y}_t\right)\right].$$

On the demand side, the structural equations remain the same as in the complete model, namely, the forecastable component of the output gap $x_t \equiv \hat{Y}_t - \hat{Y}_t^n$ satisfies

$$E_{t-2}\tilde{x}_t = E_{t-2}\tilde{x}_{t+1} - \varphi^{-1} E_{t-2}\left(\hat{\imath}_t - \pi_{t+1} - \hat{r}_t^n\right)$$

where $\tilde{x}_t \equiv (x_t - \eta x_{t-1}) - \beta\eta\,(E_t x_{t+1} - \eta x_t)$. The output gap then relates to the expected output gap through

$$\tilde{x}_t = E_{t-2}\tilde{x}_t + \left(\tilde{g}_t - \tilde{Y}_t^n\right) - E_{t-2}\left(\tilde{g}_t - \tilde{Y}_t^n\right) - \beta\eta\left[E_t\left(x_{t+1} + \hat{Y}_{t+1}^n - g_{t+1}\right) - E_{t-2}\left(x_{t+1} + \hat{Y}_{t+1}^n - g_{t+1}\right)\right].$$

The natural rate of interest is given by

$$\hat{r}_t^n = \varphi E_t\left[\left(\tilde{Y}_{t+1}^n - \tilde{Y}_t^n\right) - (\tilde{g}_{t+1} - \tilde{g}_t)\right]$$

where

$$\begin{aligned}
\tilde{Y}_t^n &\equiv \left(\hat{Y}_t^n - \eta\hat{Y}_{t-1}^n\right) - \beta\eta\left(E_t\hat{Y}_{t+1}^n - \eta\hat{Y}_t^n\right) \\
\tilde{g}_t &\equiv \left(g_t - \eta\hat{G}_{t-1}\right) - \beta\eta\left(E_t g_{t+1} - \eta\hat{G}_t\right) \\
g_t &\equiv \hat{G}_t + s_c E_{t-2}\bar{C}_t,
\end{aligned}$$

and the natural rate of output is defined implicitly by

$$E_t\left\{\left[\omega + \varphi\,(1-\eta L)\left(1-\beta\eta L^{-1}\right)\right]\hat{Y}_t^n\right\} = E_t\left((1+\omega)\,a_t - \frac{v_{h\xi}}{v_h}\xi_t + \varphi\tilde{g}_t\right).$$

In this case, because $\xi_w^{-1} = 0$, it follows from section B.2 of the technical appendix that the loss function reduces to

$$E_0\sum_{t=0}^{\infty}\beta^t\left[\left(\pi_t - \gamma_p\pi_{t-1}\right)^2 + \lambda_x\left(x_t - \delta x_{t-1} - \hat{x}^*\right)^2\right]$$

where

$$\lambda_x \equiv \frac{\vartheta\varphi}{\theta_p\xi_p^{-1}} = \frac{\xi_p\eta\varphi/\delta}{\theta_p} = \frac{\kappa}{\theta_p} > 0.$$

3 Model's Representation for Estimation

We estimate the model by matching the model-based impulse response functions to unexpected monetary shocks with those estimated from a VAR. Because shocks other than the unexpected disturbances to the policy rule (2.1) of the text, ε_t, have no effect on these impulse response functions, we can eliminate these shocks from the model. Thus we may set $g_t = \tilde{g}_t = \hat{Y}_t^n = \tilde{Y}_t^n = \hat{\omega}_t^n = \hat{r}_t^n = 0$, so that we can replace x_t with $\hat{Y}_t$. The model reduces to the following set of equations.

First we have a policy rule of the form

$$\begin{aligned} 0 &= -\hat{\imath}_t + \phi_{\pi 0}\pi_t + \phi_{\pi 1}\pi_{t-1} + \phi_{\pi 2}\pi_{t-2} + \phi_{w0}\hat{\omega}_t + \phi_{w1}\hat{\omega}_{t-1} + \phi_{w2}\hat{\omega}_{t-2} \\ &\quad + \phi_{y0}\hat{Y}_t + \phi_{y1}\hat{Y}_{t-1} + \phi_{y2}\hat{Y}_{t-2} + \phi_{i1}\hat{\imath}_{t-1} + \phi_{i2}\hat{\imath}_{t-2} + \phi_{i3}\hat{\imath}_{t-3} + \varepsilon_t \end{aligned} \tag{40}$$

where the coefficients are the ones estimated from the VAR.

The inflation equation can be rewritten as:

$$\beta E_t \pi_{t+2} + \xi_p E_t \hat{\omega}_{t+1} = \left(1 + \beta\gamma_p\right)\pi_{t+1} - \gamma_p \pi_t - \kappa_p \hat{Y}_{t+1} \tag{41}$$

The wage inflation equation can be rewritten as

$$\begin{aligned} \beta E_t \pi_{t+2}^w - \xi_w E_t \hat{\omega}_{t+1} &= \xi_w E_t \mu_{t+1} + \pi_{t+1}^w + \beta\gamma_w \pi_{t+1} + \xi_w \varphi\beta\eta\hat{Y}_{t+2} - \gamma_w \pi_t - \xi_w\left[\omega_w + \varphi\left(1+\beta\eta^2\right)\right]\hat{Y}_{t+1} + \xi_w\varphi\eta\hat{Y}_t \\ &= \xi_w E_t \mu_{t+1} + \pi_{t+1}^w + \beta\gamma_w \pi_{t+1} + \xi_w \beta\psi^{-1}\tilde{\eta}\hat{Y}_{t+2} - \gamma_w \pi_t - \xi_w\left[\omega_w + \psi^{-1}\right]\hat{Y}_{t+1} + \xi_w\psi^{-1}\tilde{\eta}\hat{Y}_t \end{aligned}$$

where

$$\begin{aligned} \psi^{-1} &\equiv \varphi\left(1+\beta\eta^2\right) = \frac{\left(1+\beta\eta^2\right)}{\sigma\left(1-\beta\eta\right)} > 0 \\ 0 &\leq \tilde{\eta} \equiv \frac{\eta}{1+\beta\eta^2} \leq (1+\beta)^{-1}. \end{aligned}$$

The equation determining $E_t\mu_{t+1}$ can then be rewritten as

$$E_t\pi_{t+2} - \beta\varphi\eta E_t\hat{Y}_{t+3} - E_t\hat{\imath}_{t+1} = -E_t\mu_{t+1} - \varphi\left(1+\beta\eta+\beta\eta^2\right)\hat{Y}_{t+2} + \varphi\left(1+\beta\eta^2+\eta\right)\hat{Y}_{t+1} - \varphi\eta\hat{Y}_t$$

or equivalently as

$$E_t\pi_{t+2} - \beta\psi^{-1}\tilde{\eta}E_t\hat{Y}_{t+3} - E_t\hat{\imath}_{t+1} = -E_t\mu_{t+1} - \psi^{-1}\left(1+\beta\tilde{\eta}\right)\hat{Y}_{t+2} + \psi^{-1}\left(1+\tilde{\eta}\right)\hat{Y}_{t+1} - \psi^{-1}\tilde{\eta}\hat{Y}_t.$$

The intertemporal IS equation determining the forecastable component of the output gap can be more compactly written as

$$E_t\mu_{t+2} = 0.$$

The real wage, in turn, satisfies

$$E_t\hat{\omega}_{t+1} = \hat{\omega}_t + \pi_{t+1}^w - \pi_{t+1}.$$

To solve the model, we need the additional 17 identities:

$$
\begin{aligned}
E_t\left[\hat{\imath}_{t+2}\right] &= E_t\hat{\imath}_{t+2} \\
E_t\left[\hat{\imath}_{t+1}\right] &= E_t\hat{\imath}_{t+1} \\
E_t\left[\hat{Y}_{t+2}\right] &= \hat{Y}_{t+2} \\
E_t\left[\pi_{t+1}\right] &= \pi_{t+1} \\
E_t\left[\pi^w_{t+1}\right] &= \pi^w_{t+1} \\
E_t\left[\hat{\imath}_{t}\right] &= \hat{\imath}_{t} \\
E_t\left[\hat{Y}_{t+1}\right] &= \hat{Y}_{t+1} \\
E_t\left[\pi_{t}\right] &= \pi_{t} \\
E_t\left[\pi^w_{t}\right] &= \pi^w_{t} \\
E_t\left[\hat{\imath}_{t-1}\right] &= \hat{\imath}_{t-1} \\
E_t\left[\hat{Y}_{t}\right] &= \hat{Y}_{t} \\
E_t\left[\pi_{t-1}\right] &= \pi_{t-1} \\
E_t\left[\pi^w_{t-1}\right] &= \pi^w_{t-1} \\
E_t\left[\hat{\imath}_{t-2}\right] &= \hat{\imath}_{t-2} \\
E_t\left[\hat{Y}_{t-1}\right] &= \hat{Y}_{t-1} \\
E_t\left[\hat{\omega}_{t}\right] &= \hat{\omega}_{t} \\
E_t\left[\hat{\omega}_{t-1}\right] &= \hat{\omega}_{t-1}.
\end{aligned}
$$

In total, we have 23 equations and 23 endogenous variables. The system can then be written in first-order form as follows

$$
AE_t\begin{bmatrix} z_{t+1} \\ Z_{t+1} \end{bmatrix} = B\begin{bmatrix} z_{t} \\ Z_{t} \end{bmatrix} + C\varepsilon_t
$$

where z_t is a vector of non-predetermined endogenous variables and Z_t is a vector of predetermined endogenous variables given by

$$z_t = \begin{bmatrix} E_t\hat{\imath}_{t+2} \\ E_t\hat{\imath}_{t+1} \\ E_t\hat{Y}_{t+2} \\ E_t\pi_{t+1} \\ E_t\pi^w_{t+1} \\ \hat{\imath}_t \\ E_t\mu_{t+1} \end{bmatrix}, \quad \text{and} \quad Z_t = \begin{bmatrix} E_t\hat{Y}_{t+1} \\ \pi_t \\ \pi^w_t \\ \hat{\omega}_t \\ \hat{\imath}_{t-1} \\ \hat{Y}_t \\ \pi_{t-1} \\ \pi^w_{t-1} \\ \hat{\imath}_{t-2} \\ \hat{Y}_{t-1} \\ \pi_{t-2} \\ \pi^w_{t-2} \\ \hat{\imath}_{t-3} \\ \hat{Y}_{t-2} \\ \hat{\omega}_{t-1} \\ \hat{\omega}_{t-2} \end{bmatrix}.$$

Such a dynamic system can then be solved using standard techniques (e.g., King and Watson, 1998), and impulse response functions to an unexpected increase in ε_t can finally be computed.

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