STATIONARY SUNSPOT EQUILIBRIA

The Case of Small Fluctuations Around a Deterministic Steady State*

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ABSTRACT

It is shown that stationary sunspot equilibria exist near a deterministic steady state of a stationary economy not subject to exogenous shocks, if and only if there exists a continuum of (non-stationary) perfect foresight equilibria converging asymptotically to that steady state. This allows us to demonstrate the existence of stationary sunspot equilibria in a much wider range of economic models than have been treated in the literature to date; in particular, stationary overlapping generations exchange economies are treated in some generality.

Demonstration of the above necessary and sufficient condition requires consideration of a broader class of stationary stochastic processes than the finite-state Markov processes assumed in most previous literature on stationary sunspot equilibria. Our main technique is an implicit function theorem for Banach spaces. This sort of local analysis establishes that conclusions regarding the bounded solutions of linear rational expectations models can be rigorously applied to nonlinear models as well, in the case of small fluctuations around a deterministic steady state.

This paper presents a necessary and sufficient condition for the existence of stationary sunspot equilibria (s.s.e.) near a deterministic steady state of a stationary economy not subject to exogenous stochastic shocks. Briefly, the condition is that there exist a continuum of non-stationary deterministic equilibria converging asymptotically to the deterministic steady state in question. The latter condition reduces to a condition upon the derivatives of the functions used in defining perfect foresight equilibrium, evaluated at the steady state. The result thus extends a conclusion of Blanchard and Kahn (1980) regarding a class of linear rational expectations models. Previous examples of exact intertemporal equilibrium models (hence involving nonlinear equilibrium conditions) in which such a result was demonstrated, using techniques of less general applicability than those introduced here, include the overlapping generations examples studied by Azariadis (1981), Farmer and Woodford (1984), and Guesnerie (1986), and the cash-in-advance example of Woodford (1986a).[1]

This result allows conclusions about the existence of stationary sunspot equilibria to be derived from existing results regarding the existence of continua of convergent non-stationary deterministic (perfect foresight) equilibria.[2] As a consequence, it is established that s.s.e. exist for many more types of infinite horizon economies than the small number of examples published to date might suggest. Several examples are given in section 3. In particular, it is shown that s.s.e. can exist for a much broader class of stationary overlapping generations economies than that considered by Azariadis (1981), Azariadis and Guesnerie (1982, 1984), Spear (1984), Guesnerie (1986), and Grandmont (1986). On the other hand, s.s.e. can be shown not to exist for other open sets of economies. The existence result contributes to the development of a theory of speculative instability in infinite horizon economies,[3] while the (local) non-existence result is relevant for the development of a theory of stabilization policy for the prevention of such fluctuations.

Section 1 introduces notation and defines the class of stationary models with which we are concerned. Section 2 states and proves the central result concerning existence of s.s.e. near a steady state. Section 3 presents examples of classes of stationary economies to which the result applies. Finally, section 4 discusses issues raised by previous treatments of stationary sunspot equilibria in the context of our results.

1. Stationary Rational Expectations Equilibria: Notation and Definitions

We wish to consider the existence, for a stationary infinite horizon economic model, of equilibria in which endogenous state variables respond to the realization of a "sunspot variable," i.e. (following Cass and Shell (1983)), a random variable that conveys no information about technology, preferences, or endowments, and so does not directly enter the equilibrium conditions for the state variables. However, agents are assumed to be able to observe the realizations of the variable, and to take it into account in making their decisions if they choose to. If a rational expectations equilibrium exists in which agents respond to such a variable, it is a "sunspot equilibrium". Such equilibria may be considered to represent situations in which speculation may be destabilzing, even when agents optimize and have rational expectations.[4] Our focus in this paper will be entirely upon <u>stationary</u> sunspot equilibria, i.e., equilibria in which the endogenous variables follow a stationary stochastic process, as equilibria of that kind are of particular interest as candidate models of the sort of repetitive fluctuations associated with "business cycles".[5]

We accordingly restrict our attention to economies in which both the equilibrium conditions and the exogenous "sunspot" process have a stationary structure. Let us suppose that the sunspot variable each period is a random variable u_t, taking values in a set U with a sigma field Σ. We suppose that U contains at least two points. Let U^∞ denote the product of an infinite sequence of copies of U, and let Σ^∞ be the product sigma field (Loeve, 1977, p. 137). Elements $u^t = (u_t, u_{t-1}, \ldots)$ of U^∞ represent infinite histories of realizations of the sunspot variable, with the most recent listed first. We can represent a stationary stochastic process for the sunspot variable by a probability measure $\pi: \Sigma^\infty \to [0, 1]$. For any sets $A \in \Sigma$, $S \in \Sigma^\infty$, define $AS = \{as \in U^\infty | \ a \in A, \ s \in S\}$, where as denotes a sequence whose first element is a, whose second element is the first element of s, and so on. Then the stationary character of the stochastic process is reflected by the following assumption, asserting shift-invariance of the measure π:

(A.1) For any $S \in \Sigma^\infty$, $\pi(US) = \pi(S)$.

Since for any $A \in \Sigma$, $\pi(AS) = 0$ if $\pi(S) = 0$, the map $\pi(A\cdot): \Sigma^\infty \to [0, 1]$ is π-continuous, and hence, by the Radon-Nikodym theorem (Loeve, 1977, p. 133), a measurable function $\pi(A|\cdot): U^\infty \to [0, 1]$ exists, such that

$$\pi(AS) = \int_S \pi(A|u^{t-1})d\pi(u^{t-1})$$

Here $\pi(A|u^{t-1})$ indicates the probability that $u_t \in A$, given a history u^{t-1}. For each $u^{t-1} \in U^\infty$, $\pi(\cdot|u^{t-1})$ is a probability measure on (U,Σ). Thus π defines a Markov process on U^∞, with a time invariant transition function. However, starting with the measure π on (U^∞,Σ^∞) implies a stronger sense of stationarity for the sunspot process than if we were simply to start with a Markov process on U^∞. In particular, it implies that the conditional distribution for u_t given a history u^0 converges to the same invariant distribution (corresponding to the measure $\pi(\cdot U^\infty)$) as $t \to \infty$, regardless of the value of u^0.

Let us suppose that the endogenous state variables x_t of the model we are interested in take values in $\mathfrak{R}^n$. Then we can represent stationary stochastic process for these variables, measurable with respect to the sunspot process (at present, we assume that there is no intrinsic uncertainty in the model under consideration), by measurable functions from U^∞ to $\mathfrak{R}^n$. Let E denote the Banach space of essentially bounded, measurable functions $\phi: U^\infty \to \mathfrak{R}^n$, with the L_∞ norm

$$\| \phi \| = \operatorname*{ess\ sup}_{u \in U^\infty} |\phi(u)|.$$

(See Dunford and Schwartz, 1958, secs. III.1.11 and IV.2.19 for definition, and sec. III.6.14 for proof that it is a Banach space.) Such functions represent stationary stochastic processes for x_t, not only in that the dependence of x^t upon the history u_t is time-invariant, but also in that any such function ϕ induces a measure $\pi^*(\cdot;\phi)$ on the Borel subsets of $\mathfrak{R}^n$ (Loeve, 1977, p. 168), representing an asymptotic invariant distribution for x_t. (The induced measure is defined by $\pi^*(X;\phi) = \pi(\{u^t \in U^\infty | \phi(u^t) \in X\})$ for X a Borel subset of $\mathfrak{R}^n$.) It should also be noted that the class of such functions allows for a very wide range of possible statistical properties (serial correlation, etc.) for the variables x_t, for any given underlying sunspot process u_t; even if u_t is i.i.d., arbitrarily complex auto-correlation functions for $x_t = \phi(u^t)$ are possible.

The use of the L_∞ topology deserves brief comment. This means that when we speak of other stationary equilibria (i.e., elements $\phi \in E$) that are "close" to the steady state (a constant function, taking the value x^* for all u^t) we mean that $x_t = \phi(u^t)$ remains within a neighborhood of x^* for all u_t (except perhaps a set of measure zero); we do not consider random variables to be "close" to the steady state that take values far from x^* with a positive probability, however small. One advantage of this is that for the purpose of the local analysis undertaken here, we need not even define the equilibrium conditions for the state variables x_t outside a neighborhood of X. In order to use an implicit function theorem for Banach spaces to determine the existence or not of other nearby equilibria, we need an operator defined on an open set in a Banach space of random variables. But if we let E(X) denote the set of functions in E such that $\phi(u^t) \in X$ (a.e.), for x an open subset of $\Re^n$, then under the L_∞ norm topology, E(X) is an open subset of E. Hence it is enough for the operator defining equilibrium to be defined on E(X), which in turn only requires equilibrium conditions defined for $x_t \in X$. Another advantage of this topology is that we are able to obtain necessary and sufficient conditions for local uniqueness with a clear interpretation; as is discussed below in section 4.C, stationary sunspot equilibria may exist "near" a steady state even when our necessary condition does not hold, under a more inclusive definition of what count as "nearby" stochastic processes for the endogenous variables.

Let E_1 denote the Banach space of essentially bounded, measurable functions $\nu: U \to \Re^n$, again with the L_∞ norm, and let $E_1(X)$ denote the open subset of functions $\nu: U \to X$. We will use such functions to represent the way in which agents in period t expect the value of x_{t+1} to depend upon the realization of u_{t+1}. We need only consider expectations regarding the distribution of values that may be taken by x_{t+1} that are of this form in defining equilibrium, since in any rational expectations equilibrium agents correctly expect the distribution of values for x_{t+1} given by $x_{t+1} = \phi(u_{t+1}, u_t, \ldots)$. For any $\phi \in E$, let $\phi(\cdot, u^t)$ denote the section of ϕ at $(\cdot, u^t)$, defined by

$$\phi(u_{t+1}, u^t) = \phi(u_{t+1} u^t)$$

Then $\phi(\cdot, u^t) \in E_1$ for all u^t, since a section of a measurable function is measurable (Loeve, 1977, p.135). Thus to every possible equilibrium stochastic process $\phi \in E$, there corresponds a rational expectation function $\phi(\cdot, u^t) \in E_1$.

We consider stationary economies with equilibrium conditions of the following form. Let x_t be a vector of real-valued endogenous state variables, taking values in X, an open subset of $\Re^n$. Then we assume that the period t state variables are determined by equilibrium conditions of the form

(1.1) $$f(x_{t-1}, \nu_{t-1}, \eta_{t-1}; x_t, \nu_t, \eta_t) = 0$$

where f has the following properties:

(A.2) $f: (X \times E_1(X) \times \Pi(U))^2 \to \Re^n$ is bounded, has continuous (Frechet) derivatives with respect to each of the arguments $(x_{t-1}, v_{t-1}; x_t, v_t)$, and is a continuous function of $(\eta_{t-1}; \eta_t)$.

Here $\Pi(U)$ denotes the space of countably additive probability measures on U, endowed with the the topology generated by the norm of total variation (Dunford and Schwartz, 1958, secs. III.1.4 and IV.2.16).

The measure η_{t-1} represents agents' expectations at time t-1 regarding the distribution of values from which u_t will be drawn, and the function

ν_{t-1} their expectations at time t-1 regarding the way in which x_t will depend upon the realization of u_t; (ν_{t-1}, η_{t-1}) thus indicate agents' expectations at time t-1 regarding the distribution of values for x_t. The pair (ν_t, η_t) represent the corresponding expectations at time t regarding period t+1. Because u_t is a "sunspot" variable, it does not affect any agent's decision problem directly, but only insofar as it affects expectations regarding future values of the endogenous variables x_t. Accordingly, agents' actions in period t do not depend upon either ν_t or η_t except through the distribution of values for x_{t+1} implied by them; hence we also assume the following.

(A.3) For $\nu \in E_1(X)$, $\eta \in \Pi(U)$, let $\eta^*(\nu)$ denote the measure on X induced from η by the function ν. Then the function f depends upon $(\nu_{t-1}, \eta_{t-1}; \nu_t, \eta_t)$ only through the induced measures $\eta^*_{t-1}(\nu_{t-1})$ and $\eta^*_t(\nu_t)$. That is, if $\eta^*_1(\nu_1) = \eta^*_3(\nu_3)$ and $\eta^*_2(\nu_2) = \eta^*_4(\nu_4)$, then $f(x_1, \nu_1, \eta_1; x_2, \nu_2, \eta_2) = f(x_1, \nu_3, \eta_3; x_2, \nu_4, \eta_4)$ for all $x_1, x_2 \in X$.

In a rational expectation equilibrium, of course, agents will never have expections (η_{t-1}, η_t) except of the form $(\pi(\cdot|u^{t-1}), \pi(\cdot|u^t))$, for some history $u^t \in U^\infty$; but it is necessary to assume that f is defined for, and a continuous function of, measures of a broader class, in order to prove Lemma 2 of section 2.

The equilibrium conditions (1.1) are thus time invariant functions of only the endogenous variables and expectations of future values of those variables; it follows that the economy is not subject to any exogenous shocks.[6] (Only in this case, in general, is a deterministic steady state equilibrium possible, near which to undertake our local analysis. But see section 3.D.) Nonetheless, (1.1) may have solutions in which x_t responds to the realization of u_t, and hence is random.

Expectations in period $t-1$ are allowed to affect the determination of temporary equilibrium in period t, in that actions taken in the previous period (on the basis of those expectations) may affect the conditions for equilibrium in the current period. Inclusion of the past expectations is thus a substitute for increasing the number of state variables x_t. It is shown in section 3.A that a general stationary overlapping generations exchange economy -- with an arbitrary finite number of goods per period and an arbitrary finite number of agent types per generation -- yields equilibrium conditions of the form (1.1), where x_t is the vector of goods prices in period t. If the expectations (ν_{t-1}, η_{t-1}) were not to be included as arguments of f, one would have to add to the set of state variable x_t an additional vector of variables for each agent type indicating the consumption choices of young agents of that type (insofar as these affect the preferences of the same agents when old) and the amount saved or borrowed by young agents of that type.

Finally, it should be noted that in writing (1.1) we assume that all agents have identical expectations in any given period. That is, we assume that all agents have the same information set in each period, i.e., each agent observes the realization of u_t in period t.

In the case of stochastic equilibria (i.e., sunspot equilibria), we confine our attention to equilibria that are stationary, in the sense discussed above.

Definition. A stationary rational expectations equilibrium (s.r.e.e.) is a $\phi \in E(X)$ such that

(1.2) $\quad f(\phi(u^{t-1}),\ \phi(\cdot,u^{t-1}),\ \pi(\cdot|u^{t-1});\ \phi(u^t),\ \phi(\cdot,u^t),\ \pi(\cdot|u^t)) = 0$

for all $u^t \in U^\infty$. A s.r.e.e. is a steady state if ϕ is a constant; otherwise, it is a stationary sunspot equilibrium (s.s.e.).

As discussed above, the function ϕ indicates the equilibrium value of x_t for each possible history of sunspot realizations u^t. In the present paper, we are interested solely in the case of X an arbitrarily small neighborhood of a steady state x^*.

Let us assume in addition that f satisfies the following condition.

(A.4) The dependence of f upon η_{t-1} is such that for fixed $(x_{t-1},\nu_{t-1};x_t,\nu_t,\eta_t)$, and for π a measure on U^∞ satisfying (A.1), $f(x_{t-1},\nu_{t-1},\pi(\cdot|u^{t-1});x_t,\nu_t,\eta_t)$ is a measurable function of u^{t-1}. The same is true of the dependence of f upon η_t.

Under this assumption, (1.2) may be written $\Psi(\phi) = 0$, where for any $\phi \in E(X)$, $\Psi(\phi)$ is defined by the left hand side of (1.2). Condition (A.4) insures that $\Psi(\phi)$ is measurable, so that Ψ: $E(X) \to E$. It follows from (A.2) that Ψ has a continuous (Frechet) derivative. The properties of this derivative map $D\Psi$ are crucial for the analysis of local uniqueness in section 2.

Our main result asserts a relationship between the existence of stationary sunspot equilibria near a steady state and the existence of non-stationary deterministic equilibria near it. Non-stationary deterministic equilibria are defined as follows. For any $x \in \mathbb{R}^n$, let $\tilde{x}$ denote the element of E_1 such that $\tilde{x}(u) = x$ for all $u \in U$. (We will also use $\hat{x}$ for the element of E such that $\hat{x}(u') = x$ for all $u' \in U^\infty$.) Then define

(1.3) $F(x_1, x_2, x_3, x_4) \equiv F(x_1, \tilde{x}_2, \eta_1; x_3, \tilde{x}_4, \eta_2)$

where because of (A.3) this expression is independent of (η_1, η_2). It follows from the assumptions on f that $F: X^4 \to \Re^n$ is a C^1 function.

Definition. A perfect foresight equilibrium (p.f.e.) is a sequence $\{x_t\}_{t=-\infty}^{\infty}$ with $x_t \in X$ for all t, such that

(1.4) $$F(x_{t-1}, x_t, x_t, x_{t+1}) = 0$$

for all t.

A steady state, i.e., a constant sequence $x_t = x^*$ for all t, where x^* satisfies $F(x^*, x^*, x^*, x^*) = 0$, is one kind of p.f.e. We are interested in whether or not there exist other (non-stationary) p.f.e. near a given steady state, i.e., such that x_t remains within a neighborhood of x^* for all t.

Under conditions that will be generically valid in applications of interest, no such nearby non-stationary equilibria exist, when the neighborhood of x^* is made small enough; non-stationary p.f.e. may converge to the steady state as $t \to \infty$, or as $t \to -\infty$, but must diverge from it in at least one direction. However, robust examples of the following state of affairs may exist.

Definition. Perfect foresight equilibrium is indeterminate near a steady state if there exists a manifold of dimension greater than n of sequences $\{x_t\}_{t=0}^{\infty}$ satisfying (1.4) for $t \geq 1$, all of which converge to the steady state as $t \to \infty$.

In such a case, for a generic set of initial connditions in period $t=0$, representing the determinants of equilibrium in period $t=0$ given a particular history of the economy up until that period, and consistent with a constant sequence $x_t = x^*$ for $t \geq 0$, there will exist a continuum of perfect foresight equilibria for periods $t \geq 0$, all consistent with the given set of initial conditions and all remaining within an arbitrarily small neighborhood of x^* for all $t \geq 0$.[7] It is indeterminacy of perfect foresight equilibrium in this sense that turns out to be a necessary and sufficient condition for s.s.e. to exist arbitrarily close to a steady state.

We assume that the derivatives of F satisfy the following regularity conditions:

(A.5) At each steady state x^*, Det $D_4F \neq 0$, and the $2n \times 2n$ matrix

$$M(x^*) = \begin{bmatrix} -(D_4F)^{-1}(D_2F + D_3F) & -(D_4F)^{-1}D_1F \\ I & 0 \end{bmatrix} \quad (1.5)$$

has no eigenvalues with modulus exactly equal to one.

Here all derivatives of F are evaluated at (x^*, x^*, x^*, x^*). Conditions (A.5), which are quite standard,[8] can be shown to hold for a generic stationary overlapping generations economy. They imply that the steady states are hyperbolic fixed points (Hirsch and Smale, 1974, p. 187) of the dynamical system defined by (1.4); this allows us to determine whether p.f.e. is indeterminate near a steady state x^* solely by reference to the matrix $M(x^*)$.

<u>Proposition</u>. Perfect foresight equilibrium is indeterminate at a steady state x^* if and only if the number eigenvalues of $M(x^*)$ with modulus less than one is greater than n.[9]

Hence the relationship between indeterminacy of p.f.e. and the existence of s.s.e. may be demonstrated by establishing a connection between the eigenvalues of $M(x^*)$ and the existence of s.s.e.

The existence of such a relationship depends upon a further regularity condition upon the derivatives of F.

(A.6) For any steady state x^*, let W denote the stable subspace of $M(x^*)$, i.e., the set of $v \in \Re^{2n}$ such that $M(x^*)^t v \to 0$ as $t \to \infty$; and let K denote the kernel of the linear operator $[D_4F \; D_3F]$, where the derivatives are evaluated at (x^*, x^*, x^*, x^*). Then

$$\dim W \cap K = \max(\dim W - n, 0)$$

This assumption is less familiar, as it does not pertain to the conditions defining perfect foresight equilibrium (only the sum $D_2F + D_3F$ matters for the definition of p.f.e., not either matrix separately). However, like (A.3), it holds generally for the applications of interest to us. Condition (A.5) guarantees that $\dim K = n$. Then (A.6) asserts that the intersection of W and K is transversal (Guillemin and Pollack, 1974, pp. 30-31) if transversality is consistent with a nonempty intersection, and that it consists only of a single point (the zero vector) if not. Hence it is obvious that (A.6) should hold generically.

2. Local Uniqueness of Stationary Equilibria

Let us suppose that there exists a steady state $x^* \in X$, i.e., that $\hat{x}^*$ (the function taking always the value x^*) is a zero of Ψ. We are interested in whether this zero of Ψ is isolated in E, i.e., whether, for a neighborhood $N \subset E$ of $\hat{x}^*$ chosen sufficiently small, $\hat{x}^*$ is the unique $\phi \in N$ such that $\Psi(\phi) = 0$. The basic results used to address such a question are the following.

Inverse Function Theorem. (Berger, 1977, 3.1.5.) Let f be a C^1 mapping defined in a neighborhood of some point x_0 of a Banach space X, with range in a Banach space Y. Then if $Df(x_0)$ is a linear homeomorphism of X onto Y, there exists a neighborhood $U(x_0)$ of x_0 such that x_0 is the unique solution to $f(x) = f(x_0)$ in $U(x_0)$. Furthermore, there exists a unique solution to $f(x) = y$ in $U(x_0)$, for any Y sufficiently close to $f(x_0)$.

Implicit Function Theorem. (Berger, 1977, 3.1.10.) Let X, Y, Z be Banach spaces, and let f be a continuous mapping of a neighborhood of (x_0, y_0) in $X \times Y$ into Z. Then if $D_y f(x_0, y_0)$ exists, is continuous in x, and is a linear homeomorphism of Y onto Z, there is a unique continuous mapping $g: U \to Y$, defined on some neighborhood U of x_0, such that $g(x_0) = y_0$ and $f(x,g(x)) = f(x_0,y_0)$ for all $x \in U$.

Corollary. (Berger, 1977, 3.1.11.) If in addition to the hypothesis of the implicit function theorem, $D_x f$ exists and is continuous for (x,y) near (x_0,y_0), then the function g is continuously differentiable, and its derivative is $Dg(x) = -[D_y f(x,g(x))]^{-1} D_x f(x,g(x))$.

Evidently, local uniqueness of the s.r.e.e. $\hat{x}^*$ depends upon whether $D\Psi(\hat{x}^*)$ is a linear homeomorphism (i.e., has a bounded inverse). The following operators are useful in representing $D\Psi$ and its inverse.

Lemma 1. The linear operators A,B defined by

$$(2.1) \qquad A\phi(u^t) \equiv \int \phi(u_{t+1}, u^t) d\pi(u_{t+1} | u^t)$$

$$(2.2) \qquad B\phi(u^t) \equiv \phi(u^{t-1})$$

for $\phi \in E$ both map E into itself. Both are also bounded (i.e., continuous) linear operators, since

$$(2.3) \qquad \| A\phi \| \leq \| \phi \|$$

$$(2.4) \qquad \| B\phi \| = \| \phi \|$$

for all $\phi \in E$.

Proof: As noted in section 1, $\phi \in E$ implies that $\phi(\cdot, u^t) \in E_1$ (i.e., is measurable and essentially bounded) for any choice of u^t, so that the latter function is integrable (Loeve, 1977, p. 121). Hence the expression on the right hand side of (2.1) is well-defined. Property (2.3) follows immediately from the definition, so $A\phi$ is essentially bounded. It remains only to show that $A\phi$ is measurable. Consider first the case of ϕ a simple function (ibid., p. 107). In this case $A\phi$ is a sum of the form $\Sigma_j x_j \pi(S_j | \cdot)$, where the sum is over a finite set, $x_j \in \mathfrak{R}^n$ for each j, and $S_j \in B$ for each j. As noted in section 1, $\pi(S_j | \cdot)$ is measurable if $S_j \in B$, hence a sum of such functions is measurable as well. But then any non-negative measurable function ϕ can be expressed as the limit of a non-decreasing sequence of non-negative simple functions (ibid., p. 109), and the integral of such a function is defined as the limit of the integrals of the simple functions (ibid., p. 119). Hence $A\phi$ is in this case the limit of a sequence of measurable functions, and so itself measurable (ibid., p. 114). Finally, any measurable ϕ can be written as the sum of a non-negative measurable function and a non-positive measurable functions (ibid., p.109); then $A\phi$ for such a function is a sum of two measurable function, and so itself measurable. Therefore $A: E \to E$.

If ϕ is measurable and $X \subset \mathfrak{R}^n$ is a Borel set, $\phi^{-1}(X) \in \Sigma^\infty$. But then $(B\phi)^{-1}(X) = U[\phi^{-1}(X)] \in \Sigma^\infty$ as well, so that $B\phi$ is also measurable. Property (2.4) follows from (A.1) and the definition of the L_∞ norm. Hence ϕ essentially bounded implies $B\phi$ essentially bounded, and $B: E \to E$. Q.E.D.

We can now give an explicit representation for $D\Psi$, evaluated at a constant function.

Lemma 2. For any $x \in X$,

$$D\Psi(\hat{x}) = D_1F \cdot B + D_2F \cdot BA + D_3F + D_4F \cdot A \tag{2.5}$$

where the derivatives of F are evaluated at (x, x, x, x), and the operators A, B are defined in (2.1), (2.2).

Proof: It follows immediately from the definition of Ψ that $D\Psi$, evaluated at some $\phi \in E$ and applied to some $\psi \in E$, yields a function which, evaluated at u^t, is equal to

$$D\Psi(\phi)\psi(u^t) = D_1f \cdot \psi(u^{t-1}) + D_2f \cdot \psi(\cdot, u^{t-1}) + D_3f\psi(u^t) + D_4f \cdot \psi(\cdot, u^t)$$

where D_1f, D_2f, D_3f, and D_4f represent the derivatives of f with respect to x_{t-1}, ν_{t-1}, x_t, and ν_t respectively (to use the notation for the arguments given in (1.1)), evaluated at $(\phi(u^{t-1}),\ \phi(\cdot,u^{t-1}),\ \pi(\cdot|u^{t-1});\ \phi(u^t),\ \phi(\cdot,u^t),$ $\pi(\cdot|u^t))$. Furthermore, it follows immediately from the definition of F that D_1f and D_3f, evaluated at $(x,\ \tilde{x},\ \eta_1;\ x,\ \tilde{x},\ \eta_2)$, for any $\eta_1,\ \eta_2 \in \Pi(U)$, are equal to D_1F and D_3F respectively, evaluated at (x, x, x, x). It remains only to demonstrate that D_2f and D_4f, evaluated at that same point, and applied to some $\nu \in F_1$, yield

$$(2.6)\qquad \begin{aligned} D_2f\cdot\nu &= D_2F \cdot \int\nu(u)d\eta_1(u) \\ D_4f\cdot\nu &= D_4F \cdot \int\nu(u)d\eta_2(u) \end{aligned}$$

in order to derive (2.5). Since both of the above relations hold for the same reason, it suffices that we derive (2.6).

Consider first the case of ν of the form $\nu = yI_S$, where $y \in \mathbb{R}^n$, $S \in \Sigma$, and I_S is the indicator function

$$I_S(u) = \begin{cases} 1 & \text{if} \quad u \in S \\ 0 & \text{if} \quad u \notin S \end{cases}$$

Condition (A.3) implies that $f(x,\ \tilde{x},\ \eta_1;\ x,\ \tilde{x} + \nu,\ \eta_2)$ depends upon ν only through the induced measure $\eta_2^*(\tilde{x} + \nu)$, which measure assigns probability $\eta_2(S)$ to the value $x + y$ and probability $1 - \eta_2(S)$ to the value x. Thus f depends only upon y and $\eta_2(S)$, and hence the same is true of $D_4f\cdot\nu$. Because the derivative is a linear operator, one must have

$$(2.7)\qquad D_4f\cdot\nu = G(\eta_2(S))y$$

where G is a matrix-valued function. Because of (A.2), f is a continuous function of $\eta(S)$, and so G must be as well. Now consider $S_1,\ S_2 \in \Sigma$ such that $S_1 \cup S_2 = S$, and write $\nu_j = yI_{S_j}$, $j = 1, 2$. Since $\nu = \nu_1 + \nu_2$, one must have $D_4F\cdot\nu = D_4F\cdot\nu_1 + D_4F\cdot\nu_2$, and hence

(2.8) $$G(\eta_2(S)) = G(\eta_2(S_1)) + G(\eta_2(S_2))$$

Let us recall that f is defined for arbitrary $\eta_2 \in \Pi(U)$, not just η_2 of the form $\pi(\cdot | u^t)$ for some $u^t \in U^\infty$. Therefore (2.7) holds for arbitrary η_2. Hence, even if U consists of only two points, by varying η_2 it is possible for $\eta_2(S)$ to take any value in the interval $[0, 1]$, and so G must be defined on the entire interval. Condition (2.8) and continuity then imply that G is a linear function. But when $S = U$, $\nu = \tilde{y}$, in which case it follows from (1.3) that $D_4f \cdot \nu = D_4Fy$. Hence $G(1) = D_4F$, and so

$$D_4f \cdot \nu = D_4F \cdot y\eta_2(S)$$

for any ν of the form $\nu = yI_S$.

Thus (2.6) holds for ν of this form. But any simple function ν is a sum of functions of this kind, and, as D_4f must be a linear operator and so is the integral, it follows that (2.6) holds for any simple function. One then proceeds from simple functions to arbitrary measurable functions as in the proof of Lemma 1. Hence (2.6) holds for any $\nu \in E_1$. Q.E.D.

Determining whether $D\Psi(\hat{x}^*)$ is a linear homeomorphism amounts to determining whether, for an arbitrary $\psi \in E$, there exists a unique $\phi \in E$ satisfying $D\Psi \cdot \phi = \psi$. Lemma 2 implies that this amounts to a consideration of the stationary solutions of the "linear rational expectations model"

(2.9) $$[D_1F \cdot B + D_2F \cdot BA + D_3F + D_4F \cdot A]\phi = \psi$$

Accordingly, the same techniques used to address the issues of existence and uniqueness of stationary solutions for linear models suffice to address the issue of <u>local</u> uniqueness in the case of <u>nonlinear</u> models as well, near a deterministic steady state.

It is not surprising, then, that our main result is simply a local version (for a slightly different class of models) of the necessary and sufficient condition given by Blanchard and Kahn (1980) for the existence of stationary sunspot equilibria as solutions to linear rational expectations models. They show that a linear model of the form (2.9) has bounded solutions in which "sunspot" variables matter if and only if the matrix $M(x^*)$ defined in (A.5) has more than n eigenvalues of modulus less than one.[10] We show below that this is also exactly the case in which there exist stationary sunspot equilibria arbitrarily close to a deterministic steady state at which the derivatives in (2.9) are evaluated, in the case of a nonlinear model.

Following Blanchard and Kahn, we resolve (2.9) into three separate sets of equations. First, applying the operator A to both sides of (2.9) yields

$$[D_1F + (D_2F + D_3F)\cdot A + D_4F\cdot A^2]\phi \quad = \quad A\psi \tag{2.10}$$

Here we use the fact that $AB = I$ (the identity operator), and the fact that A (like B) commutes with matrices. Applying the operator $(I - BA)$ to both sides of (2.9) yields

$$(I - BA)[D_3F + D_4F\cdot A]\phi \quad = \quad (I - BA)\psi \tag{2.11}$$

It is easily shown that (2.10) and (2.11) jointly imply (2.9); hence we may replace (2.9) by the system (2.10) - (2.11).

It is also useful to rewrite (2.10) in the form

$$A\xi \quad = M(x^*)\xi + \begin{bmatrix}(D_4F)^{-1}A\psi \\ 0\end{bmatrix} \tag{2.12}$$

where

$$\xi \equiv \begin{bmatrix} A\phi \\ \phi \end{bmatrix}$$

(We can invert D_4F because of (A.5). Also note that, by an abuse of notation, the same letter A is used both for an operator mapping $\Re^n$-valued functions into $\Re^n$-valued functions, and for the operator mapping $\Re^{2n}$-valued functions into $\Re^{2n}$-valued functions.) Finally, (A.5) guarantees that a direct-sum decomposition of $M(x^*)$ is possible, $M(x^*) = C^{-1}JC$, where J is block diagonal

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

and where J_1 is a matrix whose eigenvalues all have modulus less than one, and J_2 is a matrix whose eigenvalues all have modulus greater than one. (For example, we may put $M(x^*)$ in real canonical form. See Hirsch and Smale, 1974, pp. 129-130.) Note that J_1 is $k \times k$, and J_2 is $(2n-k) \times (2n-k)$, where $k = \dim W$, i.e., the number of eigenvalues of $M(x^*)$ with modulus less than one. If we write

$$C\xi \equiv \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \quad C \begin{bmatrix} (D_4F)^{-1}A\psi \\ 0 \end{bmatrix} \equiv \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

where θ_1, λ_1 are $\Re^k$-valued functions and θ_2, λ_2 are $\Re^{2n-k}$-valued functions, then (2.12) decomposes into the two sets of equations

$$A\theta_1 = J_1\theta_1 + \lambda_1 \tag{2.13}$$

$$A\theta_2 = J_2\theta_2 + \lambda_2 \tag{2.14}$$

It is also useful to rewrite (2.11) as an equation for θ_1 and θ_2 instead of ϕ. If we write

$$[D_4F \quad D_3F]C^{-1} = [N_1 \quad N_2]$$

then (2.11) becomes

$$(I - BA)(N_1\theta_1 + N_2\theta_2) = (I - BA)\psi$$

Substituting (2.13), this may be rewritten

(2.15) $$N_1(I - J_1B)\theta_1 = (I - BA)(\psi - N_2\theta_2) + N_1B\lambda_1$$

Accordingly, (2.9) may be replaced by the three separate sets of equations (2.13), (2.14) and (2.15).

The following lemma is crucial in indicating the number of independent restrictions represented by (2.15).

Lemma 3. Assumption (A.6) implies that rank $N_1 = \min(n, k)$.

Proof: The stable subspace W consists of all vectors $y \in \mathfrak{R}^{2n}$ of the form

$$y = C^{-1}\begin{bmatrix} e \\ 0 \end{bmatrix}$$

where $e \in \mathfrak{R}^k$. If $N_1e = 0$, y also belongs to K. Therefore $\dim \ker N_1 = \dim (W \cap K)$. By (A.6), this equals $\max (k-n,0)$. Then rank $N_1 = k - \dim \ker N_1 = \min (n, k)$. Q.E.D.

Lemma 3 indicates the role of (A.6) in our analysis. It is also further clarifies the likehood that (A.6) should hold generically for applications of interest, since the rank of a generic matrix is equal to the minimum of the number of its rows and the number of its columns.

The following lemma is useful in computing the solutions of linear rational expectations models.

Lemma 4. Let H be a matrix all of whose eigenvalues have modulus less than one. Then $\Sigma_{j=0}^{\infty} \|H^j\| < \infty$.

Proof: H can be written as the sum of a semisimple matrix S and a nilpotent matrix N, that commute with one another (Hirsch and Smale, 1974, p. 116). Then there exists a finite integer k such that $N^k = 0$, and for any $j \geq k-1$,

$$\begin{aligned} H^j &= (S + N)^j \\ &= \Sigma_{h=0}^{j} \binom{j}{h} S^{j-h} N^h \\ &= \Sigma_{h=0}^{k-1} \binom{j}{h} S^{j-h} N^h \end{aligned}$$

The triangle inequality then implies (for all $j \geq 0$)

$$\begin{aligned} \|H^j\| &\leq \Sigma_{h=0}^{k-1} \binom{j}{h} \|S\|^{j-h} \|N\|^h \\ &\leq \|S\|^j \Sigma_{h=0}^{k-1} \frac{j^h}{h!} \|S\|^{-h} \|N\|^h \end{aligned}$$

Now for $P(j)$ any finite order polynomial in j, and for any $\mu > 1$, there exist a finite K such that $P(j) < K\mu^j$ for all $j \geq 0$. If $\rho < 1$ is the maximum modulus among the eigenvalues of H, then there exists a metric for which $\|S\| = \rho$, and it is possible to choose $1 < \mu < \rho^{-1}$ and $K < \infty$ such that

$$\sum_{h=0}^{k-1} \frac{j^h}{h!} \|S\|^{-h} \|N\|^h < K\mu^j$$

for all $j \geq 0$. Then $\|H^j\| < K(\rho\mu)^j$, and $\Sigma_{j=0}^{\infty} \|H^j\| < \dfrac{K}{1-\rho\mu} < \infty$. Q.E.D.

The following result illustrates the application of Lemma 4 to the solution of linear rational expectations models.

Lemma 5. Let E^m denote the Banach space of essentially bounded measurable functions $\theta: U^{\infty} \rightarrow \mathbb{R}^m$, again with the L_{∞} norm topology, for m any positive integer. Then for any given $\lambda_2 \in E^{2n-k}$, (2.14) has a unique solution $\theta_2 \in E^{2n-k}$, given by $\theta_2 = T_2\lambda_2$, where the operator T_2 is defined by

$$T_2 = -\Sigma_{j=0}^{\infty} (J_2^{-1})^{j+1} A^j$$

Proof: Since all eigenvalues of J_2 have modulus greater than one, J_2^{-1} exists and all its eigenvalues have modulus less than one. Lemma 4 (together with (2.3)) then implies that T_2 is well defined and maps E^{2n-k} into itself, since, by the triangle inequality

$$\begin{aligned}
\|T_2\lambda_2\| &\le \Sigma_{j=0}^{\infty}\|(J_2^{-1})^{j+1}A^j\lambda_2\| \\
&\le \Sigma_{j=0}^{\infty}\|(J_2^{-1})^{j+1}\|\ \|A^j\lambda_2\| \\
&\le \Sigma_{j=0}^{\infty}\|(J_2^{-1})^{j+1}\|\ \|\lambda_2\| \\
&< \infty
\end{aligned}$$

whenever $\|\lambda_2\| < \infty$. One may verify by substitution that $\theta_2 = T_2\lambda_2$ satisfies (2.14). Finally, let θ_2 be any solution to (2.14). Then $A(\theta_2 - T_2\lambda_2) = J_2(\theta_2 - T_2\lambda_2)$. It follows that $A^j(\theta_2 - T_2\lambda_2) = J_2^j(\theta_2 - T_2\lambda_2)$, and so that $(J_2^{-1})^jA^j(\theta_2 - T_2\lambda_2) = (\theta_2 - T_2\lambda_2)$, for all positive integers j. By Lemma 4, there must exist a j such that $\|(J_2^{-1})^j\| < 1$. Then, if $\|\theta_2 - T_2\lambda_2\| > 0$, we must have (again using (2.3))

$$\begin{aligned}
\|\theta_2 - T_2\lambda_2\| &= \|(J_2^{-1})^jA^j(\theta_2 - T_2\lambda_2)\| \\
&< \|A^j(\theta_2 - T_2\lambda_2)\| \\
&\le \|\theta_2 - T_2\lambda_2\|
\end{aligned}$$

which is a contradiction. Therefore $\|\theta_2 - T_2\lambda_2\| = 0$, and $\theta_2 = T_2\lambda_2$. Thus $T_2\lambda_2$ is the unique essentially bounded solution. Q.E.D.

Using this sequence of preliminary lemmas, we may now present our main result.

<u>Theorem 1</u>. A steady state $\hat{x}^*$ is a locally unique s.r.e.e. (i.e., there exists a neighborhood $N \subset \mathbb{R}^n$ of x^* such that there exists no other

s.r.e.e. with $\phi(u^t) \in N$ for almost all u^t) if and only if the number of eigenvalues of $M(x^*)$ with modulus less than one is no greater than n. If the number of such eigenvalues exceeds n, then there exist stationary sunspot equilibria arbitrarily close to the steady state (i.e., for any neighborhood N of x^*, there exist s.s.e. with $\phi(u^t) \in N$ for almost all u^t). Thus s.s.e. exist arbitrarily close to a steady state if and only if perfect foresight equilibrium is indeterminate at the steady state.

Proof: The last sentence follows from the characterization of the indeterminacy of p.f.e. in section 1. We consider in turn the three possible cases: dim W exactly equal to, less than, or greater than n, respectively.

(i) Suppose that $M(x^*)$ has exactly n eigenvalues with modulus less than one (and, because of (A.5), exactly n with modulus greater than one). That is, suppose that dim $W = k = n$. In this case we show that $D\Psi(\hat{x}^*)$ is a linear homeomorphism. The inverse function theorem then implies that $\hat{x}^*$ is a locally unique s.r.e.e.

$D\Psi$ is a linear homeomorphism if and only if (2.9) has a unique solution $\phi \in E$ for every $\psi \in E$. As detailed above, (2.9) is equivalent to (2.13), (2.14) and (2.15). By Lemma 5, the unique solution to (2.14) is $\theta_2 = T_2\lambda_2$. Substituting this into (2.15), and making use of the fact that N_1 is invertible (by Lemma 3, since $k = n$), we obtain

$$\theta_1 - J_1B\theta_1 = N_1^{-1}[(I - BA)(\psi - N_2T_2\lambda_2)] + B\lambda_1 \tag{2.16}$$

The unique $\theta_1 \in E$ satisfying (2.16) is $\theta_1 = T_1\psi$, where

$$T_1\psi \equiv \Sigma_{j=0}^{\infty} J_1^j B^j [B\lambda_1 + (I - BA)\, N_1^{-1}(\psi - N_2T_2\lambda_2)]$$

Since all eigenvalues of J_1 have modulus less than one, Lemma 4 (together with (2.4)) implies that T_1 is well defined and maps E into itself, and that $\|T_1\| < \infty$, so that T_1 is continuous as well. One may verify by substitution that $\theta_1 = T_1\psi$ satisfies (2.16). Finally, let θ_1 be any solution to (2.16). Then $(\theta_1 - T_1\psi) = J_1B(\theta_1 - T_1\psi)$, and similarly $(\theta_1 - T_1\psi) = J_1{}^jB^j(\theta_1 - T_1\psi)$ for any positive integer j. As in the proof of Lemma 5, Lemma 4 and (2.4) imply a contradiction unless $\|\theta_1 - T_1\psi\| = 0$, so that $\theta_1 = T_1\psi$.

Furthermore, any solution to (2.16) also satisfies (2.13). For application of the operator A to both sides of (2.16) yields exactly (2.13), since $AB = I$. Accordingly, the unique $\phi \in E$ satisfying (2.13), (2.14), and (2.15) is given by

$$\phi = [0 \ I]C^{-1}\begin{bmatrix} T_1\psi \\ T_2\lambda_2 \end{bmatrix}$$

Thus one constructs the inverse map $[D\Psi]^{-1}$. As T_1 and T_2 have been shown to be continuous linear operators, $[D\Psi]^{-1}$ is as well. Therefore the inverse function theorem applies.

(ii) Consider instead the case $k = \dim W < n$. Again $\theta_2 = T_2\lambda_2$ is the unique solution to (2.14). Select k equations from the set of n equations (2.15), so that the corresponding k rows of N_1 form an invertible matrix $\tilde{N}_1$. (Lemma 3 guarantees that such a selection is possible.) These k equations can be put in the form (2.16), where N_1 is replaced by $\tilde{N}_1$, N_2 is replaced by the corresponding k rows of N_2, and ψ is replaced by the corresponding k elements of ψ. Then a unique solution is obtained as above. Accordingly, if one neglects the remaining n-k equations of (2.15), the inverse function theorem implies that $\hat{x}^*$ is a locally unique s.r.e.e.

A fortiori, there exist no other nearby solutions of the complete set of equilibrium conditions.

(iii) Consider now the case $k = \dim W > n$, in which perfect foresight equilibrium is indeterminate near a steady state. Again $\theta_2 = T_2\lambda_2$ is the unique solution to (2.14).

Now let us add to the set of equilibrium conditions $\Psi(\phi) = 0$ an auxiliary set of $k-n$ conditions of the form

$$(I - BA)Q\theta_1 - \mu z = 0 \tag{2.17}$$

where Q is a $(k-n) \times k$ matrix, μ is a scalar, and $z \neq 0$ is some element of Z, the Banach space of bounded measurable functions $z: U^\infty \to \mathbb{R}^{n-k}$ with the property $Az = 0$. (Such a z may be obtained by setting $z = (I - BA)y$ for some $y \in E^{k-n}$; z represents the innovation in the stationary stochastic process y.) Furthermore, let us choose Q so that the $k \times k$ matrix

$$\tilde{N}_1 \equiv \begin{bmatrix} N_1 \\ Q \end{bmatrix}$$

has full rank. (Lemma 3 guarantees that such a Q may be chosen.)

Conditions (2.17) together with the equilibrium conditions define a map $\tilde{\Psi}_\mu: E(X) \to E \times Z$. The map $\tilde{\Psi}_\mu$ is just the product of the map $\Psi: E(X) \to E$ defined by (1.2) and the linear map $\Phi_\mu: E(X) \to Z$ defined by $\Phi_\mu(\phi) = (I - BA)Q\theta_1 - \mu z$. Since Φ_μ is a bounded linear map, it is continuous and has a continuous derivative. Hence $\tilde{\Psi}_\mu$ has a continuous derivative.

Consider first the case $\mu = 0$. Then $\phi = \hat{x}^*$ solves $\tilde{\Psi}_0(\phi) = 0$. One can show furthermore that this solution is locally isolated, using the inverse function theorem, by showing that $D\tilde{\Psi}_0$ has a bounded inverse. The proof is as in case (i) above. But the mere fact that $\hat{x}^*$ is a locally isolated zero of $\tilde{\Psi}_0$ does not mean that it must be a locally isolated zero of Ψ, since $\tilde{\Psi}_0$ contains the auxiliary conditions Φ_0 as well.

Let us consider now the zeroes of $\tilde{\Psi}_\mu$, for μ varying over some open interval $J \subset \mathbb{R}$, containing a neighborhood of zero. We can define a map $\tilde{\Psi}: E(X) \times J \to E \times Z$ by $\tilde{\Psi}(\phi,\mu) = \tilde{\Psi}_\mu(\phi)$. Furthermore, $\tilde{\Psi}$ is continuous in both arguments, and $D_\phi\tilde{\Psi}$ exists and is a continuous function of μ. (In fact, it does not even depend upon μ, since $D_\phi\tilde{\Psi} = D\Psi \times D_\phi\Phi_\mu$, Ψ does not depend upon μ, and

$$D_\phi\Phi_\mu = (I - BA)Q[I\ 0]C\begin{bmatrix} A \\ I \end{bmatrix}$$

does not depend upon μ either.) If, as just asserted, $D\tilde{\Psi}_0(\hat{x}^*)$ is a linear homeomorphism, then $D_\phi\tilde{\Psi}(\hat{x}^*,0)$ is a linear homeomorphism as well, and it follows from the implicit function theorem that there exists a neighborhood H of zero in J, a neighborhood N of x^* in X, and a continuous function $\tilde{\phi}: H \to E(N)$ such that $\tilde{\phi}(\mu)$ is the unique zero of $\tilde{\Psi}_\mu$ in $E(N)$ for each $\mu \in H$, and such that $\tilde{\phi}(0) = \hat{x}^*$.

But note that a zero of $\tilde{\Psi}_\mu$ for any μ is a zero of Ψ. Hence each of the set of functions $\tilde{\phi}(\mu)$, for $\mu \in H$, represents a stationary rational expectations equilibrium. Furthermore, for any $\mu \neq 0$, and any $x \in X$, $\tilde{\Psi}_\mu(\hat{x}) \neq 0$, since $z \neq 0$ by assumption. Hence for each $\mu \neq 0$, $\tilde{\phi}(\mu)$ is not a constant function, i.e., it is a stationary sunspot equilibrium. Thus there exists a continuum of s.s.e. Furthermore, because $\tilde{\phi}$ is a continuous function of μ, for $\mu \neq 0$ in a sufficiently small neighborhood of zero, $\tilde{\phi}(\mu)$ remains almost always within an arbitrarily small neighborhood of x^*.

Thus we can show that there exist s.s.e. arbitrarily close to the steady state in this case. It remains only to prove that $D\tilde{\Psi}_0$ has a bounded inverse, as asserted above. We do this constructively, by exhibiting the unique, essentially bounded, measurable function ϕ that solves (2.9) and

$$(I - BA)Q\theta_1 = y \tag{2.18}$$

for arbitrary $(\psi,y) \in E \times Z$. Again one can replace (2.9) by (2.13), (2.14) and (2.15). Again $\theta_2 = T_2\lambda_2$ is the unique solution to (2.14). Adjoining conditions (2.18) to (2.15) yields

$$\tilde{N}_1(I - J_1B)\theta_1 = (I - BA)(\tilde{\psi} - \tilde{N}_2T_2\lambda_2) + \tilde{N}_1B\lambda_1 + \tilde{y} \tag{2.19}$$

where $\tilde{N}_1$ is defined above, and where

$$\tilde{N}_2 \equiv \begin{bmatrix} N_2 \\ 0 \end{bmatrix}$$

$$\tilde{\psi} \equiv \begin{bmatrix} \psi \\ 0 \end{bmatrix}$$

$$\tilde{y} \equiv \begin{bmatrix} 0 \\ y \end{bmatrix}$$

where in each case the upper block has n rows and the lower block $n-k$ rows. Since we have chosen Q so that $\text{Det } \tilde{N}_1 \neq 0$, (2.19) can be put in a form analogous to (2.16). The unique solution can be shown to be

$$\theta_1 = \Sigma_{j=0}^{\infty} J_1{}^j B^j [B\lambda_1 + (I - BA)\ \tilde{N}_1^{-1}(\tilde{\psi} - \tilde{N}_2T_2\lambda_2) + \tilde{N}_1^{-1}\tilde{y}]$$

using the same argument as in case (i), and again this solution can be shown to satisfy (2.13) as well. Thus the above solution for θ_1, together with $\theta_2 = T_2\lambda_2$, define the inverse operator $[D\tilde{\Psi}_0]^{-1}$. As before the operator is easily shown to be bounded. Q.E.D.

In fact it is possible to analyze local uniqueness of s.r.e.e. using these techniques without relying upon Lemma 3 (and hence without assumption (A.6)), at the cost of some additional complexity. Consider again, for example, the case $k = n$. Assumption (A.6) requires in this case that $K \cap W = \{0\}$; but suppose instead that $K \cap W$ is an m-dimensional linear subspace, for some $m > 0$. Then there exists an m-dimensional space of vectors $f \in \Re^n$ such that $f'N_1 = 0$. Hence (2.15) implies

$$(I - BA)(f'\psi - f'N_2T_2\lambda_2) = 0 \tag{2.20}$$

for all f in this space, where we have again substituted the solution $\theta_2 = T_2\lambda_2$ to (2.14). Conditions (2.20) are a set of m linear restrictions upon ψ; thus $D\Psi$ does not map E onto itself, but rather onto a linear subspace Y of E, consisting of the $\psi \in E$ that satisfy (2.20). This, however, is no obstacle to the application of the inverse function theorem. We wish to know whether there is a unique solution $\phi \in E$, for a given ψ in Y. If so, Y would be homeomorphic to E, and one would again be able to prove that the steady state is a locally isolated s.r.e.e. But in fact there is not a unique solution ϕ, as can be shown using the method of case (iii) above. Let the equations

$$(I - BA)(\hat{N}_1\theta_1 + \hat{N}_2\theta_2) = (I - BA)\hat{\psi}$$

represent a selection of $n-m$ linearly independent conditions from among the n conditions (2.11); because of (2.20), these $n-m$ equations contain the entire content of (2.11). Then adjoin to these an auxiliary set of conditions of the form (2.17), where Q is an $m \times n$ matrix chosen so that

$$\text{Det}\begin{bmatrix}\hat{N}_1 \\ 'Q\end{bmatrix} \neq 0$$

gain one can show that there exists a bounded solution ϕ for each choice of μ, and that even when $\psi = 0$, $\phi \neq 0$ for $\mu \neq 0$, so that there are multiple solutions. As in case (iii) above, this implies the existence of s.s.e. arbitrarily close to the steady state, despite the fact that perfect foresight equilibrium is locally determinate near that same steady state.

More generally, reasoning of this sort allows one to show that, if all of our previous assumptions hold, with the possible exception of (A.6), stationary sunspot equilibria exist arbitrarily close to a steady state if and only if $\dim (W \cap K) > 0$. If (A.6) does not hold, one can have $\dim (W \cap K) > 0$ even when $k \leq n$, and it is in these cases that s.s.e. exist near a steady state even though p.f.e. is determinate. Since (A.6) seems likely to hold generically in all applications of interest, we have excluded this case in the statement of Theorem 1.

It is important to note that both the case $k > n$ and the case $k \leq n$ can occur for robust examples of stationary economies. For example, in the case of the stationary overlapping generations economies treated in section 3.A below, one can show that there exists an open set of economies in which k takes any value betwenn 1 and 2n. Thus Theorem 1 establishes that <u>neither</u> local uniqueness of s.r.e.e. nor local indeterminacy is a generic property of stationary economic models.

The methods used to prove Theorem 1 also immediately yield additional results concerning local uniqueness of s.r.e.e. in the case of sufficiently small exogenous shocks to the economic fundamentals. Consider, for example, a one-parameter family of stationary economic models, each defined by a set of equilibrium conditions of the form

$$f_\gamma(x_{t-1}, \nu_{t-1}, \eta_{t-1}; x_t, \nu_t, \eta_t; u^t) = 0 \qquad (2.21)$$

where f has continuous derivatives with respect to $(x_{t-1}, \nu_{t-1}; x_t, \nu_t)$ and a parameter γ taking values in an open interval $\Gamma \subset \Re$. Let us suppose futhermore that f_γ has no dependence upon u^t only for $\gamma = 0$. Thus for $\gamma \neq 0$, f_γ represents a model in which the stochastic process u_t represents some shock to economic "fundamentals" (e.g., an endowment shock), rather than a "sunspot" variable. However, as γ approaches zero, the shock to fundamentals is made progressively smaller, until, in the limit, it becomes a sunspot variable. We can then prove the following about s.r.e.e. near a steady state of the economy not subject to exogenous shocks, for economies in the family with γ near zero.

<u>Theorem 2.</u> Consider a smooth one-parameter family of stationary economies of the form (2.21), where the economy corresponding to $\gamma = 0$ is not subject to exogenous shocks, and has a steady state equilibrium x^* with all the properties assumed in Theorem 1, but where the exogenous shocks u_t affect the equilibrium conditions directly if $\gamma \neq 0$. If perfect foresight equilibrium is indeterminate at the steady state of the $\gamma = 0$ economy, then in the case of sufficiently small exogenous shocks (i.e., $\gamma \neq 0$ sufficiently small) one will have both

(i) a continuum of s.r.e.e. near the steady state of the $\gamma = 0$ economy, in all of which the endogenous variables depend only upon the history of the exogenous shocks, <u>and</u>

(ii) a continuum of s.r.e.e. near the steady state of the $\gamma = 0$ economy in which the endogenous variables respond to the realizations of a stationary sunspot process, as well as to the exogenous shocks that affect the equilibrium conditions directly.

On the other hand, if $\dim W = n$ at the steady state of the $\gamma = 0$ economy, there exists a unique s.r.e.e. near that steady state for any perturbed economy with γ sufficiently small, and that s.r.e.e. does not involve response to any sunspot variables. If $\dim W < n$, no s.r.e.e. exists close to the steady state of the $\gamma = 0$ economy, for any of the economies with small $\gamma \neq 0$, in the case of generic perturbation of the $\gamma = 0$ economy.

Proof: If $\dim W = n$, then $D\Psi_0$ is a linear homeomorphism (as shown in the proof of Theorem 1). The implicit function theorem then implies the existence of a unique s.r.e.e. (i.e., zero of Ψ_γ) in a neighborhood of the steady state, for each γ in a certain neighborhood of zero. Since the proof guarantees the existence of a unique solution regardless of the number of "sunspot" variables that are included (along with the exogenous shocks that directly affect the equilibrium conditions) in the set of random variables u_t, the unique solution must depend only upon the minimum set of exogenous variables u_t that it is possible to include without changing the equilibrium conditions (2.21), i.e., only upon the exogenous shocks and not upon any sunspot variables.

If $\dim W < n$, then, as shown above, $[D\Psi_0(\hat{x}^*)]^{-1}\psi$ does not exist for most $\psi \in E$; one has to drop some of the conditions in the set $D\Psi_0 \cdot \phi = \psi$ in order for an inverse to exist. Suppose that again these conditions are dropped; then the implicit function theorem implies the existence of a unique solution near the steady state to the reduced set of equilibrium conditions, for each γ in a neighborhood of zero. Let that solution be denoted $\tilde{\phi}(\gamma)$. Then let $g_\gamma(\phi) = 0$ be one of the equilibrium conditions that was dropped, where $g_\gamma: E \rightarrow \Re$. It is evident that for a generic one-parameter family of economies, one will have $D_\gamma g_\gamma(\tilde{\phi}(\gamma))|_{\gamma=0} \neq 0$, so that $g_\gamma(\tilde{\phi}(\gamma)) \neq 0$ for any $\gamma \neq 0$ in a neighborhood of zero. Accordingly, no s.r.e.e. exists near the steady state for $\gamma \neq 0$ in that neighborhood.

Finally, if $\dim W > n$, then, as in proof of Theorem 1, one can adjoin to the equilibrium conditions an additional set of conditions, indexed by a parameter μ; let the augmented set of equilibrium conditions be written $\tilde{\Psi}_{\gamma\mu}(\phi) = 0$. When the additional conditions are chosen as in the proof of Theorem 1, $\hat{x}^*$ is a zero of $\tilde{\Psi}_{00}$, and $D\tilde{\Psi}_{00}(\hat{x}^*)$ is a linear homeomorphism. The implicit function theorem in this case implies the existence of a unique ϕ close to $\hat{x}^*$ satisfying $\tilde{\Psi}_{\gamma\mu}(\phi) = 0$, for each choice of (γ,μ) close enough to $(0,0)$. But then, for each $\gamma \neq 0$ close enough to zero, there exists a continuum of s.r.e.e. near the steady state, corresponding to alternative values of μ. If the auxiliary conditions do not explicitly involve any sunspot variables, then none of these s.r.e.e are sunspot equilibria — each equilibrium represents the endogenous state variables as a measurable function of the history of exogenous shocks. On the other hand, if the auxiliary conditions do involve sunspot variables for all $\mu \neq 0$, as in the proof of Theorem 1, then the s.r.e.e. corresponding to $\mu \neq 0$ will be s.s.e. Thus both continua of stationary equilibria, in which no sunspot variables matter, and continua of stationary sunspot equilibria exist near the steady state for $\gamma \neq 0$ close enough to zero. Q.E.D.

Hence s.r.e.e. need not be locally unique, even if one rules out sunspot equilibria from consideration. The sorts of economic structures for which this occurs (in the case of an economy subject to only small exogenous shocks, and when one is interested in equilibria involving only small fluctuations) are exactly those for which p.f.e. is indeterminate in the absence of the exogenous shocks. Furthermore, even when none of the equilibria are sunspot equilibria, the existence of a multiplicity of s.r.e.e. of this sort can be regarded as indicating a type of instability. For consider the equilibria corresponding to a given choice of $\mu \neq 0$, for γ varying in a neighborhood of zero. As $\gamma \to 0$, the amplitude of the fluctuations, as measured, say, by ess sup $|\phi(u^t) - \int \phi d\pi|$, remains bounded away from zero. Accordingly, for sufficiently small γ, one must regard the equilibrium fluctuations (in this particular equilibrium) as being disproportionate to the size of the shock to fundamentals that occurs, even though it is not a sunspot equilibrium. Cases of "over-response" to real shocks of this sort are likely more interesting examples of instability resulting from self-fulfilling expectations than the pure sunspot case;[11] the significance of examples of sunspot equilibria is simply that they provide a particularly dramatic demonstration that "over-response" to shocks may be consistent with rational expectations equilibrium.

The condition given in Theorem 2 for existence of a unique s.r.e.e. near the steady state of the economy not subject to shocks, in the case of small exogenous shocks, is also of no small interest. Deterministic models in which dim W = n at the unique steady state often arise in economic dynamics; for example, Levhari and Liviatan (1972) and Scheinkman (1976) show that optimal growth models have this property (under relatively ordinary assumptions regarding preferences and technologies), in the case of a sufficiently low rate of time preference. (Cases in which stationary overlapping generations economies have this property are discussed in section 3.A.) Theorem 2 implies that it is possible to introduce small stochastic shocks into such models and have a unique s.r.e.e. continue to exist near the steady state of the deterministic model. Furthermore, the stochastic properties of that s.r.e.e. are well approximated (in the case of sufficiently small shocks) by the solution to the "linear rational expectations model"

$$D_\phi \Psi_0(\hat{x}^*)\cdot(\phi - \hat{x}^*) = -D_\gamma \Psi_0(\hat{x}^*)\cdot\gamma$$

Thus Theorem 2 allows a rigorous application to a wide variety of exact, nonlinear models of the techniques for solving and estimating linear rational expectations models that are widely used by macroeconomists, but that have heretofore only been known to be applicable to rigorously grounded models in the case of extremely special functional forms, such as the linear-quadratic objective functions assumed by Hansen and Sargent (1980).[12]

3. Applications

In this section we provide examples of stationary economic models that yield equilibrium conditions of the form (1.1), and hence to which the results of section 2 apply.

A. Stationary Overlapping Generations Exchange Economies

Azariadis (1981) shows that stationary sunspot equilibria can exist in a stationary overlapping generations economy in which fiat money is used as a store of value. However, his method of proving existence of s.s.e. cannot be extended to models in which any predetermined variables exist, as the two-state s.s.e. assumed by his method cannot exist in such models. (See section 4.B for further discussion.) Hence it cannot treat overlapping generations models with more than one agent type per generation (in which case the demands of the old in a given period depend upon the distribution of money balances across different agents) or with more than one good per period and preferences that are not additively separable between periods (in which case the demands of the old depend upon their consumption in youth). Theorem 1, however, allows us to establish that s.s.e. can exist in models of those kinds as well.

In this section we generalize the model considered by Azariadis, Guesnerie, Grandmont, and Spear to allow for an arbitrary finite number of goods per period, and an arbitrary number of agent types per generation with arbitrary smooth preferences. As in their model, we assume that all agents live for two consecutive periods, that all goods are perishable, that there is no production, and that fiat money is the sole asset. The class of economies with which we are concerned is thus that treated by Grandmont and Hildenbrand (1974), except that here endowments are non-stochastic, and demand functions are differentiable (at least in a neighborhood of the deterministic steady state of interest). Conditions for the existence of multiple perfect foresight equilibria near a steady state in such models have been derived by Kehoe and Levine (1984, 1985) and Kehoe et al. (1986).

Let there be n perishable consumption goods each period; and let each generation consist of H agent types, indexed $h = 1,\ldots,H$. (Generalization to infinite sets of agents types is trivial.) Each agent h has a von Neumann-Morgenstern utility function u^h, defined over pairs of n-vectors $(y,z) \in Y^h \times Z^h$. Here y denotes the vector of excess demands in the first period of life (i.e., consumption demand in excess of endowment, with a negative quantity denoting excess supply), and z the vector of net excess demand in the second period of life; Y^h and Z^h are subsets of $\Re^n$. We assume the following properties for u^h:

(B.1) $u^h: Y^h \times Z^h \to \Re$ is C^2; $Du^h \gg 0$ and D^2u^h is negative definite, for all (y,z).

Because fiat money is the only asset, the budget constraint for an agent of type h born in period t is

(3.1) $$p_t'y_t^h + p_{t+1}'z_{t+1}^h \leq 0$$

where $p_t \in \mathbb{R}^n_{++}$ is the vector of money prices of goods in period t. This represents an independent budget constraint for each possible realization of p_{t+1}; z_{t+1}^h is chosen only in period $t+1$, after p_{t+1} has been realized. There is no requirement that money holdings from the first to the second period of life (equal to $-p_t'y_t^h$) need be non-negative; agents may borrow, but only by promising to pay a fixed nominal amount, independent of the realization of any period $t+1$ random variables. (The treatment below is applicable even to a case in which aggregate outside assets are negative.)

Kehoe and Levine (1984) establish that under standard boundary conditions on preferences, there exists a monetary steady state, i.e., a constant price vector p^*, and constant excess demands (y^{h*}, z^{h*}), for $h = 1,\ldots, H$, such that (y^{h*},z^{h*}) represent optimal consumption demands for an agent of type h given constant prices p^*, and such that $\Sigma_{h=1}^{H}(y^{h*}+z^{h*}) = 0$. (The steady state does not, however, necessarily involve a positive quantity of outside assets; this would require in addition that $\Sigma_{h=1}^{H} p^{*\prime}z^{h*} > 0$.)

As we are concerned here only with equilibria that remain near such a steady state, we need only define demand functions for prices in some compact set $K \subset \mathbb{R}^n_{++}$ containing a neighborhood of p^*. For any agent type h, let Y^h and Z^h be compact sets containing neighborhoods of y^{h*} and z^{h*} respectively. Then the excess demand function in old age $z^h(y, p_t, p_{t+1})$ can be defined as the $z \in Z^h$ that maximizes $u^h(y,z)$ subject to (3.1). By a standard result,[13] z^h is well defined and is a C^1 function on $Y^h \times K \times K$, at least if the sets Y^h and K are chosen small enough while Z^h is chosen large enough so that the maximizing value is never on a boundary of Z^h.

In order to show that the demand function in youth is also differentiable, we use the following result.

Lemma 6. Let $g\colon K \to \Re$ have a continuous first derivative, where $K \subset \Re^n$ is a compact set. Then the function $G\colon E_1(K) \times \Pi(U) \to \Re$ defined by

$$(3.2) \qquad G(\nu,\eta) = \int g(\nu(u))d\eta(u)$$

is a continuous function of η and depends upon η in such a way that $G(\nu, \pi(\cdot|u^t))$ is a measurable function of u^t. Furthermore, G has a (Frechet) derivative with respect to ν, given by

$$(3.3) \qquad D_\nu G\cdot\mu = \int Dg(\nu(u))\mu(u)d\eta(u)$$

for any $\mu \in E_1$. This derivative is continuous in both arguments of G.

Proof: Because g is continuous on a compact set K, it is bounded and hence integrable; therefore the integral in (3.2) is well defined. For any $\eta, \eta' \in \Pi(U)$, it follows from the definition of the norm of total variation that

$$|G(\nu,\eta') - G(\nu,\eta)| \le \|g\| \, \|\eta'-\eta\|$$

and so for g a bounded function, G must be a continuous function of η. The proof that $G(\nu, \pi(\cdot|u^t))$ is measurable proceeds in the same manner as the proof that $A\phi(u^t)$ is measurable in Lemma 1.

Since Dg is also continuous on K, the integral in (3.3) is also well defined. The proof that $D_\nu G$ is a continuous function of η proceeds as in the case of G itself. The uniform continuity theorem implies that for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, x' \in X$, $|x'-x| < \delta$ implies $|Dg(x') - Dg(x)| < \epsilon$. Then for any sequence $\{\nu^j\}$ of elements of E_1, if $\nu^j \to \nu$, then for any $\delta > 0$, there exists an N such that for all $j > N$, $|\nu^j(u) - \nu(u)| < \delta$ for almost every u, from which it follows that $|Dg(\nu^j(u)) - Dg(\nu(u))| < \epsilon$ for almost every u. Therefore, for any $\epsilon > 0$, there exists an N such that for all $j > N$,

$$\begin{aligned} |D_\nu G(\nu^j)\cdot\mu - D_\nu G(\nu)\cdot\mu| &= |\int[Dg(\nu^j(u)) - Dg(\nu(u))]\mu(u)d\eta(u)| \\ &\le \int|Dg(\nu^j(u)) - Dg(\nu(u))||\mu(u)|d\eta(u) \\ &< \epsilon\int|\mu(u)|d\eta(u) \le \epsilon\,\|\mu\| \end{aligned}$$

Thus $\nu^j \to \nu$ implies $\|D_\nu G(\nu^j) - D_\nu G(\nu)\| \to 0$, and $D_\nu G$ is a continuous function of ν.

In order to establish that the operator defined in (3.3) is indeed the derivative of G, we must show that for any sequence $\{\nu^j\}$, if $\nu^j \to \nu$, then for any $\epsilon > 0$, there exist an N such that for all $j > N$,

$$|G(\nu^j,\eta) - G(\nu,\eta) - D_\nu G(\nu)(\nu^j - \nu)| < \epsilon\,\|\nu^j-\nu\|$$

As above, for any $\epsilon > 0$, there is a $\delta > 0$ such that $|x'-x| < \delta$ implies $|Dg(x') - Dg(x)| < \epsilon$. By the mean value theorem, for any $x, x' \in X$, there exists an $\hat{x}$, such that each coordinate of $\hat{x}$ lies between the corresponding coordinates of x and x', and such that $g(x') - g(x) = Dg(\hat{x})\cdot(x'-x)$.

If $|x'-x| < \delta$, it follows that $|\hat{x}-x| < \delta$ as well, so that

$$\begin{aligned} |g(x') - g(x) - Dg(x)(x'-x)| &= |[Dg(\hat{x}) - Dg(x)]\cdot(x'-x)| \\ &\leq |Dg(\hat{x}) - Dg(x)||x'-x| \\ &< \epsilon |x'-x| \end{aligned}$$

But for any $\delta > 0$, there exists N such that for all $j > N$, $|\nu^j(u) - \nu(u)| < \delta$ for almost all u, which then implies that

$$|g(\nu^j(u)) - g(\nu(u)) - Dg(\nu(u))\cdot(\nu^j(u) - \nu(u))| < \epsilon |\nu^j(u) - \nu(u)|$$

for almost all u. But this implies that for all $j > N$,

$$\begin{aligned} |G(\nu^j,\eta)-G(\nu,\eta)-D_\nu G(\nu)\cdot(\nu^j-\nu)| &= |\int[g(\nu^j(u))-g(\nu(u))-Dg(\nu(u))(\nu^j(u)-\nu(u))]d\eta(u)| \\ &\leq \int|g(\nu^j(u))-g(\nu(u))-Dg(\nu(u))(\nu^j(u)-\nu(u))|d\eta(u) \\ &< \epsilon \int|\nu^j(u) - \nu(u)|d\eta(u) \\ &\leq \epsilon \|\nu^j - \nu\| \end{aligned}$$

which is just (3.4). Q.E.D.

Now let us define

$$\lambda^h(y, p_t, p_{t+1}) = \frac{D_z^j u^h(y, z^h(y, p_t, p_{t+1}))}{p_{t+1}^j}$$

for j any one of the n goods, where $D_z^j u^h$ denotes the derivative of u^h with respect to the j^{th} component of z. It follows from the first order conditions for optimization by an old agent that the right hand side is independent of which good j is chosen. Let us then define

$$v^h(y, p_t, p_{t+1}) = D_y u^h(y, z^h(y, p_t, p_{t+1})) - \lambda^h(y, p_t, p_{t+1})p_t$$

It follows from the stated properties of u^h and z^h that v^h is a C^1 function of all three arguments. Finally, define

$$w^h(y;\, p,\, \nu,\, \eta) \;=\; \int v^h(y,\, p,\, \nu(u))d\eta(u)$$

Lemma 6 implies that $w^h\colon Y^h \times K \times E_1(K) \times \Pi(U) \to \Re^n$ is a continuous function of η such that $w^h(y;\, p,\, \nu,\, \pi(\cdot|u^t))$ is measurable with respect to u^t. Lemma 6 also implies that w^h has a continuous derivative with respect to ν. Finally, w^h depends upon $(\nu,\, \eta)$ only through the induced probability measure $\eta^*(\nu)$, since we might equivalently have written

$$w^h(y;\, p,\, \nu,\, \eta) \;=\; \int v^h(y,\, p,\, p')d\eta^*(\nu(p'))$$

A young agent of type h, facing a vector of current goods prices p and having expectations $(\nu,\, \eta)$ of the distribution of possible goods prices in the following period, chooses an excess demand $y \in Y^h$ to maximize $\int u^h(y,\, z^h(y,\, p,\, \nu(u)))d\eta(u)$. Let the solution to this problem be denoted $y^h(p;\, \nu,\, \eta)$. For prices (and expected prices) close enough to the steady state price vector p^*, y^h is completely characterized by the first order conditions

$$\int [D_y u^h + D_z u^h D_y z^h] d\eta \;=\; 0 \tag{3.6}$$

where the arguments of $D_y u^h$ and $D_z u^h$ are $(y,\, z^h(y,\, p,\, \nu(u)))$ and the arguments of $D_y z^h$ are $(y,\, p,\, \nu(u))$. It follows from (3.5) that $D_z u^h D_y z^h = \lambda^h \nu(u)' D_y z^h$. Furthermore, since (3.1) always holds with equality,

$$p_1' y + p_2' z^h(y,\, p_1,\, p_2) \;=\; 0$$

for all $(y,\, p_1,\, p_2)$, differentiation of which with respect to y yields

$$p_2' D_y z^h(y, p_1, p_2) = -p_1$$

Therefore (3.6) may be written

$$(3.7) \qquad w^h(y;\ p,\ \nu,\ \eta) = 0$$

Then $y^h(p;\ \nu,\ \eta)$ solves (3.7).

We know that $w^h(y^{h*};\ p^*,\ \tilde{p}^*,\ \eta) = 0$ for arbitrary $\eta \in \Pi(U)$. As noted above, w^h has a continuous derivative with respect to y. It is easily shown that for any $a \in \Re^n$,

$$a' D_y w^h(y^{h*}; p^*, \tilde{p}^*, \eta) a$$
$$= a' D^2_{yy} u^h a + (p^{*\prime}[D^2_{zz} u^h]^{-1} p^*)^{-1} (a' D^2_{yz} u^h [D^2_{zz} u^h]^{-1} p^* - a' p^*)^2 < 0$$

where the derivatives of u^h are evaluated at $(y^{h*}, \tilde{z}^{h*})$. Thus $D_y w^h(y^{h*}; p^*, \tilde{p}^*, \eta)$ is a negative definite matrix, given (B.1), and hence non-singular. The implicit function theorem then implies that for P a sufficiently small neighborhood of p^*, there exists a unique function $y^h(p; \nu, \eta)$ that solves (3.7) and maps $P \times E_1(P) \times \Pi(U)$ into a neighborhood of y^{h*}. Furthermore, y^h inherits the following properties of w^h: it is continuous in $(p;\ \nu,\ \eta)$, depends upon $(\nu,\ \eta)$ only through the induced measure $\dot{\eta}^*(\nu)$, and depends upon η in such a way that $y^h(p, \nu, \pi(\cdot|u^t))$ is a measurable function of u^t. Finally, because w^h also has continuous derivatives with respect to p and ν, the corollary to the implicit function theorem implies that y^h has continuous derivatives with respect to p and ν as well.

A goods price vector $p_t \in P$ then constitutes a temporary equilibrium, given expectations $(\nu_t,\ \eta_t) \in E_1(P) \times \Pi(U)$ and previous period's prices and expectations $(p_{t-1},\ u_{t-1},\ \eta_{t-1}) \in P \times E_1(P) \times \Pi(U)$, if and only if

(3.8) $\Sigma_{h=1}^{H} y^h(p_t;\ \nu_t,\ \eta_t) + \Sigma_{h=1}^{H} z^h(y^h(p_{t-1},\ \nu_{t-1},\ \eta_{t-1}),\ p_{t-1},\ p_t) = 0.$

Let the left hand side of (3.8) be denoted $f(p_{t-1},\ \nu_{t-1},\ \eta_{t-1};\ p_t,\ \nu_t,\ \eta_t)$. Equations (3.8) represent a set of equilibrium conditions of the form (1.1), and it follows from the discussion above of the properties of y^h and z^h that f satisfies (A.2), (A.3), and (A.4).

Assumption (A.5), however, is not satisfied. For the demand functions are all homogenous degree zero in prices, i.e.,

$$z^h(y,\ \lambda p_1,\ \lambda p_2) = z^h(y,\ p_1,\ p_2)$$
$$y^h(\lambda p;\ \lambda \nu,\ \eta) = y^h(p;\ \nu,\ \eta)$$

for any $\lambda > 0$. This implies that $F(\lambda p_1,\ \lambda p_2,\ \lambda p_3,\ \lambda p_4) = F(p_1,\ p_2,\ p_3,\ p_4)$ for any $\lambda > 0$, which in turn implies

$$[D_1F + D_2F + D_3F + D_4F]p^* = 0$$

by Euler's theorem, where the derivatives of F are evaluated at $(p^*,\ p^*,\ p^*,\ p^*)$. This in turn implies that

$$M(p^*)\begin{bmatrix} p^* \\ p^* \end{bmatrix} = \begin{bmatrix} p^* \\ p^* \end{bmatrix}$$

so that $M(p^*)$ has an eigenvalue equal to 1.

This difficulty can be avoided in the following manner. Equations (3.8), together with the fact that the budget constraints (2.1) always hold with equality, imply that in any equilibrium

$$\Sigma_{h=1}^{H}\ p_t' z_t^h = \Sigma_{h=1}^{H}\ p_{t+1}' z_{t+1}^h$$

This conserved quantity is in fact M, the constant quantity of fiat money in existence, since all savings must be held in the form of money. Thus we have

(3.9) $$\Sigma_{h=1}^{H} p_t' z_t^h = M$$

for all t. Using this relation, one can eliminate one of the demand functions from (3.8). Let it be the aggregate demand of the old for the n^{th} good. Then if $\tilde{y}^h$, $\tilde{z}^h$, $\tilde{p}$ denote the vectors y^h, z^h, p with the n^{th} components deleted, and y^{hn} and p^n denote the n^{th} components of y^h and p, equations (3.8) may be replaced by the following system of n equilibrium conditions each period:

(3.10a) $$\Sigma_{h=1}^{H} \tilde{y}^h(p_t; \nu_t, \eta_t) + \Sigma_{h=1}^{H} \tilde{z}^h(y^h(p_{t-1}, \nu_{t-1}, \eta_{t-1}), p_{t-1}, p_t) = 0$$

(3.10b) $$\Sigma_{h=1}^{H} y^{hn}(p_t; \nu_t, \eta_t) + [M - \Sigma_{h=1}^{H} \tilde{p}_t' \tilde{z}^h(y^h(p_{t-1}; \nu_{t-1}, \eta_{t-1})]/p_t^n = 0$$

Equations (3.10) imply both (3.8) and (3.9). For multiplication of (3.10a) on the left by $\tilde{p}_t'$, (3.10b) by p_t^n, and summing yields

$$\Sigma_{h=1} p_t' y_t^h = -M$$

which in turn implies (3.9) for period $t+1$; and (3.10b) and (3.9) together imply the n^{th} equation of (3.8).

Let us write $g(p_{t-1}, \nu_{t-1}, \eta_{t-1}; p_t, \nu_t, \eta_t)$ for the n functions on the left hand side of (3.10), and let G be formed from g in the way that F is formed from f in (1.3). Furthermore, let $M[G]$ denote the matrix formed from derivatives of G in the way that the matrix $M(p^*)$ (henceforth to be called $M[F]$) is formed from derivatives of F. Then g, like f, satisfies assumptions (A.2), (A.3), and (A.4). Because equation (3.10b) is not homogeneous in prices (unlike the n^{th} equation of (3.8)), $M[G]$ need not have an eigenvalue equal to one. However, as noted above

$$p_3' G(p_1, p_2, p_3, p_4) = p_3' \Sigma_{h=1}^{H} y^h(p_3; \tilde{p}_4, \eta) + M$$

As the right hand side of this does not depend upon p_1 or p_2, one obtains $p_3'D_1G = p_3'D_2G = 0$, from which it follows that

$$[p^{*\prime}D_4G \cdot p^{*\prime}D_3G]\, M[(G] = 0$$

Accordingly, one eigenvalue of $M[G]$ is zero. It can be shown that the other 2n-1 eigenvalues are identical to the 2n-1 eigenvalues of $M[F]$ other than one.

At this point we may make use of the analysis of Kehoe and Levine (1984) of the derivatives of F at such a steady state. Kehoe and Levine show, first, that $\text{Det } D_4F \neq 0$ generically. Since $D_4G = D_4F$, equations (3.10) satisfy the first clause of assumption (A.5), that $\text{Det } D_4G \neq 0$. Hence $M[G]$ is well defined in the generic case. Kehoe and Levine also show that, generically, $M[F]$ has only one eigenvalue of modulus equal to one (the one due to homogeneity of the demand functions). Accordingly, $M[G]$ has no eigenvalues of modulus one in the generic case, and (A.5) is generically valid for equilibrium conditions (3.10).

The method of analysis employed by Kehoe and Levine can also be used to demonstrate that (A.6) holds generically (and likewise (A.7), discussed in section 4.A below). For the result of Debreu (1974) implies that any C^1 perturbation of the aggregate perfect foresight demand functions that continues to satisfy homogeneity and Walras' Law represents demand behavior that can be generated by optimizing agents (assuming $H \geq 2n$). Since homogeneity and Walras' Law do not require the intersection of W and K to be larger than that specified in (A.6), in the case of any economy for which (A.6) happens not to hold, it is always possible to construct an arbitrarily small perturbation of the demand functions such that (A.6) does hold.

Thus, in the generic case, a stationary overlapping generations exchange economy satisfying (B.1) gives rise to a set of equilibrium conditions (3.10) satisfying all the assumptions of section 1. Accordingly Theorem 1 applies. We then obtain the following result.

Proposition. For any number n of goods per period, and for an arbitrary number $H \geq 2n$ of agent types per generation, there exists an open set of stationary overlapping generations exchange economies satisfying (B.1), for each of which there exists a deterministic monetary steady state near which there exist an infinite number of stationary sunspot equilibria. On the other hand, there also exists an open set of such economies for each of which there exists a steady state near which there no s.s.e.

This result follows immediately (using Theorem 1) from the results of Kehoe and Levine (1984, 1985). They show, for arbitrary n and $H \geq 2n$, that for each integer k between 1 and $2n$, there exists an open set of economies with a steady state at which $\dim W = k$. (Note that $\dim W \geq 1$, because one eigenvalue of $M[G]$ is zero.) Hence there is necessarily an open set for which $\dim W \geq n$, regardless of the size of n.

Theorem 1 can also be used to derive more specific results relating the existence or not of s.s.e. near a steady state to properties of the demand functions. For example, the following general result is established.

Proposition. Suppose that the perfect foresight demand functions exhibit the property of gross substitutability. That is, suppose that if $(p_1, p_2) \geq (p_1', p_2')$, and $p_1^j = p'^j_2$ (respectively, $p_2^j = p'^j_2$) for some $j \in \{1,\ldots,n\}$, then

$$\Sigma_{h=1}^{H} y^h(p_1, \tilde{p}_2, \eta) \geq \Sigma_{h=1}^{H} y^h(p_1', \tilde{p}_2', \eta)$$

(respectively,

$$\Sigma_{h=1}^{H} z^h(y^h(p_1, \tilde{p}_2, \eta), p_1, p_2) \geq \Sigma_{h=1}^{H} y^h(p_1', \tilde{p}_2', \eta), p_1', p_2')),$$

with strict inequality if $(p_1, p_2) \neq (p_1', p_2')$. Then no stationary sunspot equilibria exist near any monetary steady state with $M > 0$, whereas there exist an infinite number of s.s.e. near any monetary steady state with $M < 0$.

This result follows immediately from the result of Kehoe et al. (1986), who show that $\dim W = n$ at any steady state with $M > 0$, and that $\dim W = n + 1$ at any steady state with $M < 0$.

A similar analysis is possible of non-monetary overlapping generation economies. In this case the typical steady state (i.e., equilibrium with a stationary deterministic allocation of resources) does not involve a constant price level; instead, prices in terms of the unit of account may either grow or contract at a constant rate. This requires, however, only a trival modification of the notation introduced above. Again one finds an open set of economies for which s.s.e. exist near a steady state, and an open set for which there is a steady state near which there are no s.s.e. In the non-monetary case, gross substitutability implies that no s.s.e. exist near any steady state. These results again follow from the perfect foresight analysis of these economies by Kehoe and Levine (1984, 1985) and Kehoe et al. (1986).

B. An Overlapping Generations Model with Capital

Theorem 1 can also be used to show the existence of stationary sunspot equilibria in overlapping generations models with assets other than fiat money. Here we present a simple example, the perfect foresight equilibria of which are analyzed by Reichlin (1986).

Suppose that (as in Azariadis (1981)), agents supply labor in the first period of life and consume in the second, and all agents born in period t wish to maximize the expected value of $u(c_{t+1}) - v(n_t)$, where c_{t+1} is consumption by the old in period $t+1$, n_t is labor supply by the young in period t, u and v are C^2 functions, and $u' > 0$, $u'' < 0$, $v' > 0$, $v'' > 0$ for all values of their arguments. Finally, let the C^1 functions U and V be defined by $U(c) \equiv cu'(c)$, $V(n) \equiv vn'(v)$, and let $U'(c) > 0$ for all c, so that demands exhibit gross substitutability. Then in the case of a one-sector fixed-coefficients production technology, in which one unit of capital plus m units of labor yield $a > 1$ units of produced good (which may be consumed immediately or used as capital in the following period), a rational expectations equilibrium is a stochastic process for the per capita capital stock k_t satisfying the equilibrium condition

$$\int [U(ak_t - \nu_t(u)) - V(mk_{t-1})] d\eta_t(u) = 0 \tag{3.12}$$

where ν_t indicates the way in which k_{t+1} will depend upon the realization of u_{t+1}. This equilibrium condition is of the form (1.1) and satisfies all the assumptions of section 1, by Lemma 6.

Reichlin shows that there is a unique steady state capital stock k^* for any such economy. He shows that for arbitrary preferences of this sort and arbitrary $m > 0$, there exists an $a^* > 1$ such that for any $a \in (1,a^*)$, there exists a continuum of p.f.e. converging to the steady state (i.e., $\dim W = 2$). Theorem 1 then implies that stationary sunspot equilibria exist near the steady state in this case.[14] Likewise, Reichlin shows that $\dim W = 0$ if $a > a^*$, and so there exist no s.s.e. near the steady state in this case. This example indicates that gross substitutability does not rule out s.s.e.

near a steady state in the case of a non-monetary overlapping generations economy, if production is introduced.

C. A Monetary Economy with Infinite Lived Agents

Theorem 1 also applies to many stationary economies with infinite lived agents. As an example, we consider the monetary model of Lucas and Stokey (1984), but assume that both endowments and the money supply are constant. In this model, there are two consumption goods each period, a "cash good", purchases of which are subject to a cash-in-advance constraint, and a "credit good", purchases of which are not. Each agent has an endowment producing one unit of either good, one unit of endowment producing one unit of either good. Each agent seeks to maximize the expected value of $\Sigma_{t=0}^{\infty} \beta^t U(c_{1t}, c_{2t})$, where $0 < \beta < 1$, c_{1t} is consumption in period t of the cash good, c_{2t} is consumption of the credit good, and U is a strictly increasing, strictly concave utility function, twice continuously differentiable. The budget constraints of a representative agent in period t are then

$$p_t c_{1t} \leq M_t$$

$$M_{t+1} \leq M_t + p_t[y - c_{1t} - c_{2t}]$$

where M_t represents money holdings at the beginning of period t, and p_t the price of goods in period t. In equilibrium, the demands of the representative agent must be such that $c_{1t} + c_{2t} = y$ and $M_{t+1} = M$ in every period, where M is the constant money supply.

Lucas and Stokey show that a rational expectations equilibrium corresponds to a stochastic process for the variable $v_t = c_{1t} D_2 U(c_{1t}, c_{2t})$ that satisfies

(3.13) $$v_t = \beta\int h(\nu_t(u))d\eta_t(u)$$

where h is a function defined by Lucas and Stokey in terms of the utility function U (we supress the argument y as endowments are here assumed to be constant),[15] and ν_t indicates the way in which v_{t+1} is expected to depend upon u_{t+1}. (Compare equation (3.7) of Lucas and Stokey.) Given a stochastic process for v_t that satisfies (3.13), it is possible to uniquely reconstruct stochastic processes for c_{1t}, c_{2t} and p_t, so that the equilibrium is completely specified. The shadow prices associated with the various constraints may also be uniquely determined; for example, it can be shown that the cash-in-advance constraint binds in period t if and only if v_t is less than a critical value $\bar{v}$.

The existence of a steady state, i.e., a $v^* > 0$ such that $v^* = \beta h(v^*)$, is easily established under standard boundary assumptions on U. It can also be shown that at any such steady state, $v^* < \bar{v}$, so that the cash-in-advance constraint binds. We are interested in the existence of other s.r.e.e. in which v_t remains within an arbitrarily small neighborhood of v^*, from which it follows that we are interested only in equilibria in which the cash-in-advance constraint always binds. Lucas and Stokey (who assume only that U is C^1) establish that h is a continuous function; one can similarly show that if U is C^2, h is C^1 at all points except $\bar{v}$. Hence, if our neighborhood of v^* is chosen small enough, h is C^1 everywhere in it. Then (3.13) is an equilibrium condition of the form (1.1), and satisfies all the assumptions of section 1, by Lemma 6. It follows from Theorem 1 that stationary sunspot equilibria exist arbitrarily close to a steady state if and only if $|Dh(v^*)| > \beta^{-1}$. This is a condition upon the first and second derivatives of U, evaluated at the steady state consumption allocation; it can be shown that even if both goods are normal goods, there exists an open set of utility functions for which the inequality holds, as well as an open set for which it does not.[16]

Examples of cash-in-advance economies with infinite lived agents and capital accumulation are presented in Woodford (1986a, 1986b). All of these examples involve equilibrium conditions of the form

$$\int g(k_{t-1}, k_t, \nu_t(u))d\eta_t(u) = 0 \tag{3.14}$$

where g is a C^1 function, k_t is the period t capital stock, and ν_t describes expectations regarding k_{t+1}, so that Theorem 1 is again applicable. One finds in all of these examples as well that s.s.e. may or may not exist near the steady state, depending upon parameters of the model.

D. Multiple Equilibria and Non-neutrality of Money

The method introduced in section 2 has applications other than to the determination of conditions for the existence of stationary sunspot equilibria. It is sometimes the case, even when the variables u_t represent exogenous shocks that affect the equilibrium conditions, that there exists a stationary equilibrium in which certain of the state variables remain constant. An example is provided by models in which certain real variables are unaffected by money supply shocks. In such a case the methods of section 2 may be employed to determine whether there exist other s.r.e.e. near the one exhibiting the "neutrality" property, in which that property fails.

For example, consider again the model of Lucas and Stokey (1984), but now let u_t be a shock to the rate of growth of the money supply. Specifically, suppose that if M_t is the per capita money supply at the end of period t, then each agent receives a lump sum transfer equal to $(u_{t+1}-1)M_t$ at the beginning of period $t+1$. Then Lucas and Stokey show that (3.13) becomes

$$v_t = \beta\int u^{-1}h(\nu_t(u))d\eta_t(u) \tag{3.15}$$

In the case that u_t is an i.i.d. variable with support $U \subset \mathbb{R}_+$ such that $E(u^{-1}) < \beta^{-1}$, there continues to exist a stationary equilibrium in which v_t is constant, while p_t remains always proportional to the money supply (and hence responds to the current realization of u_t). This equilibrium is the one presented by Lucas and Stokey in their discussion of this example (Example 2 in section 5 of their paper). According to them, the neutrality of money supply shocks in this example illustrates a general property of equilibria of their model: "The current rate of money growth plays no direct role in determining the current allocation -- only expectations about money growth ... matter" (p.18).

But there may also exist s.r.e.e. in this example in which v_t (and hence c_{1t} and c_{2t}) respond to the realization of u_t.[17] We wish to determine if any functions $\phi \in E(X)$ exist, for X an arbitrarily small neighborhood of v^*, such that $v_t = \phi(u^t)$ and $\nu_t = \phi(\cdot, u^t)$ satisfy (3.15), and such that $\phi(u^t) \neq v^*$ for some u^t. If we write (3.15), as $\Psi(\phi) = 0$, then, as in section 2, the answer depends upon whether or not $D\Psi(\hat{v}^*)$ is a linear homeomorphism. This amounts to determining whether there exists a unique $\phi \in E$ solving

$$\beta Dh(v^*) \int u^{-1}\phi(u,u^t)d\eta(u) - \phi(u^t) = \psi(u^t) \tag{3.16}$$

for each $\psi \in E$. If we let the linear operator $\tilde{A}: E \to E$ be defined by $\tilde{A}\phi(u^t) = E(u^{-1})^{-1}\int u^{-1}\phi(u,u^t)d\eta(u)$ then (3.16) becomes

$$(3.17) \qquad [\beta Dh(v^*)E(u^{-1})\tilde{A} - I]\phi = \psi$$

But $\tilde{A}$ has all of the properties of A used in the proof of Lemma 5, and so a similiar proof shows that a unique solution $\phi \in E$ exists for (3.17) if

$$(3.18) \qquad |\beta Dh(v^*)E(u^{-1})| < 1$$

In this case, the inverse function theorem implies that the Lucas-Stokey equilibrium is locally unique. On the other hand, if the inequality in (3.18) is reversed, a continuum of nearby s.r.e.e. can be shown to exist, following the proof of Theorem 1 for case (iii), and money supply shocks are non-neutral in all of them except the Lucas-Stokey equilibrium. Stationary sunspot equilibria also exist near the Lucas-Stokey equilibrium in this case.

This example provides an illustration of the way in which the analysis of local uniqueness of s.r.e.e. can be important for questions of stabilization policy. Let us suppose that preferences are such that the function

$$\gamma(c) = \frac{D_2U[D_1U + c(D_{11}^2U - D_{12}^2U)]}{D_1U[D_2U + c(D_{12}^2U - D_{22}^2U)]}$$

where all derivatives are evaluated at $(c, y-c)$, satisfies the conditions: (a) γ is non-increasing in c for all $0 < c < y$, and (b) $\lim_{c \to 0} |\gamma(c)| \leq 1$. (Note that $\gamma(c) < 1$ for all $0 < c < y$ if, for example, U is additively separable, $U = U^1(c_1) + U^2(c_2)$; while both (a) and (b) hold if, for example, U^1 and U^2 are both constant absolute risk aversion utility functions.) Under these assumptions, there is a unique steady state $v^*(E(u^{-1}))$ corresponding to each average rate of growth of the money supply such that $\underline{u}^{-1} < E(u^{-1}) < \beta^{-1}$, where

$$\underline{u} = \lim_{c \to 0} \beta^{-1} \frac{D_2U(c,y-c)}{D_1U(c,y-c)}$$

Furthermore, (a) implies that $\beta Dh(v^*(E(u^{-1})))E(u^{-1})$ is a non-decreasing function of $E(u^{-1})$, while (b) implies that that quantity is always less than 1, and is greater than -1 for $E(u^{-1})$ small enough.

It follows that (3.18) holds for all averages rates of money growth higher than some critical value. Furthermore if (3.18) fails to hold for any average rate of money growth such that $E(u^{-1}) < \beta^{-1}$, non-neutral equilibria and stationary sunspot equilibria necessarily exist near the steady state for low rates of money growth, while a high enough rate of money growth is "stabilizing" in that it rules out undesirable equilibrium fluctuations of this sort.[18] There is accordingly, in such a case, a trade-off between efficiency of the steady state allocation of resources and determinacy of equilibrium. For if one were simply to compare the allocations of resources in the Lucas-Stokey equilibria associated with alternative rates of money growth, steady state expected utility of the representative agent would be seen to be an increasing function of $E(u^{-1})$, and efficiency would require making $E(u^{-1})$ as close as possible to β^{-1}, as in the prescription of Friedman (1969). But for too low a rate of money growth, (3.18) ceases to hold, and there exist fluctuating equilibria in addition to the Lucas-Stokey equilibrium; and these unnecessary fluctuations reduce expected utility.

4. Comments on the Literature

In this section we comment upon some issues raised by previous treatments of stationary sunspot equilibria, from the standpoint of our own results.

A. "Decentralizability" of Sunspot Equilibria

For given $(x_{t-1}, \nu_{t-1}, \eta_{t-1}; \nu_t, \eta_t)$, the set of $x_t \in X$ such that (1.1) holds comprises the set of temporary equilibria. Guesnerie (1986) distinguishes between "informative" and "non-informative" sunspot equilibria, according to whether the sunspot realization u_t affects the equilibrium value of x_t by shifting the set of temporary equilibria (through its effects upon ν_t and η_t), or merely by affecting the selection that is made from the set (which may contain more than one point).[19] In the latter ("non-informative") case, sunspots have no effect upon the way any agents respond to prices; they affect the equilibrium because the "auctioneer" pays attention to them, not because agents do. In such a case, sunspot equilibria might be regarded as "non-decentralizable", i.e., as an artifact of the Walrasian formalism that would not have any analog in a more complete model of competitive markets.[20] In the "informative" case, by contrast, agents' expectations are affected by the sunspot variable; their decision rules are affected, and so as a result are equilibrium prices and allocations. These expectations are "rational", in that the sunspot variable really does (in equilibrium) convey information about the distribution of future state variables; but it does so because of the way future expectations and actions will in turn be conditioned upon the history of sunspot realizations, not because u_t conveys information about preferences, technology, or other "fundamental" factors that affect the form of the function f.

It might appear that sunspots affect equilibrium in our examples in the way Guesnerie characterizes as "non-informative", because of the connection between existence of stationary sunspot equilibria and the existence of a continuum of convergent deterministic equilibria established by Theorem 1. Certainly the existence of such a continuum of deterministic equilibria implies the existence of sunspot equilibria simply because the sunspot variable may select which equilibrium is to occur, and it might be thought that this is all that occurs in our examples, and hence that such sunspot equilibria are subject to a problem of "non-decentralizability".

In fact, however, the sunspot equilibria in our examples are generically of the "informative" sort. Let us impose as an additional regularity assumption the following.

(A.7) At each steady state x^*, $\text{Det } D_3F \neq 0$.

This assumption should hold generically for applications of interest, just as in the case of (A.5) and (A.6); in the case of stationary overlapping generations exchange economies, the regularity analysis of Kehoe and Levine (1984) can be used to establish that (A.7) holds generically. This condition in turn implies that

$$\text{Det } D_3f(x^*, \tilde{x}^*, \eta_1; x^*, \tilde{x}^*, \eta_2) \neq 0$$

for all $\eta_1, \eta_2 \in \Pi(U)$. The implicit function theorem then implies the existence of a unique temporary equilibrium near x^* for any $(x_{t-1}, \nu_{t-1}; \nu_t)$ sufficiently close to $(x^*, \tilde{x}^*; \tilde{x}^*)$. Then in the case of sunspot equilibria in which x_t remains always close to x^* (the only case treated in this paper), fluctuations in x_t in response to sunspot restrictions must represent movement of the equilibrium set rather than a change in the selection of equilibrium from the set.

Guesnerie links "informativeness" of a sunspot equilibrium to statistical properties of the sunspot variable. Specifically, he argues that a sunspot equilibrium in which the endogenous state variables respond to an independently and identically distributed (i.i.d.) sunspot variable must be "non-informative". However, this is because he restricts his attention to sunspot equilibria in which x_t depends only upon the current realization u_t, in which case, if $\{u_t\}$ is i.i.d. and agents have rational expectations, u_t cannot affect agents' expectations regarding the distribution of x_{t+1}, and so cannot affect the temporary equilibrium set. But the argument just given in the case of our s.s.e. holds regardless of whether $\{u_t\}$ is i.i.d or otherwise; for when x_t depends upon the entire history u^t, u_t can affect the expected distribution of values for x_{t+1} even if $\{u_t\}$ is i.i.d. A correct formulation of Guesnerie's proposition would be: a sunspot equilibrium in which $\{x_t\}$ is i.i.d. must be "non-informative". This would cover the cases discussed by Guesnerie, and is also true even for the more general class of sunspot equilibria considered here.

B. Alternative Representations of Stationary Sunspot Equilibria

The formalism used to describe stationary sunspot equilibria in section 1 differs from that used by previous authors such as Azariadis, Guesnerie, Grandmont, and Spear; our reasons for investing in so much new notation are perhaps worthy of brief comment. For one thing, these authors all consider only equilibrium conditions of the form

$$f(x_t, \nu_t, \eta_t) = 0 \tag{4.1}$$

The methods that they develop for proving the existence of s.s.e. accordingly cannot be applied to models in which any predetermined state variables exist; for example, to a model with capital accumulation, or even to overlapping generations exchange economies with non-additively separable preferences (and more than one good per period), or with more than one agent type per generation. The desire to overcome this limitation is one of the main reasons for introduction of a new technique here.

The previous authors also consider only s.s.e. that can be represented by a function $x_t = \nu(u_t)$, where $\nu \in E_1$, i.e., in which the endogenous variables depend only upon the current sunspot state. We instead allow x_t to depend upon the entire history of sunspot realizations, for several reasons. The first is simply that a more general class of equilibria is better; this is particularly true insofar as our local uniqueness result (i.e., local nonexistence of s.s.e.) is concerned. However, the cogency of this consideration is greatly reduced if -- as is generally the approach of the previous authors -- one considers the entire set of s.s.e. corresponding to all different possible sunspot processes, rather than only a single specification of the stochastic process $\{u_t\}$. For all of the s.s.e. that can be represented using our formalism can also be represented as involving dependence only upon the current sunspot realization, if one defines the "current realization" as an element of U^∞ (rather than U), and the sunspot process as a Markov process on U^∞, defined by

$$\mathrm{Prob}(S|u^t) = \pi(A|u^t)$$

where $S \in \Sigma^\infty$ and A is the maximal element of Σ such that $vu^t \in S$ for all $v \in A$.

The more important drawback of limiting oneself to dependence upon the current sunspot realization is that it has led the previous authors to emphasize statistical properties of the sunspot variable among the necessary conditions for the existence of s.s.e. For example, as noted above, Guesnerie (1986) states that "informative" s.s.e. are not possible in the case of an i.i.d. sunspot process. Azariadis (1981) draws attention to the need for negative serial correlation of the sunspot process in order for s.s.e. of the kind he considers to exist for his model. Our result, by contrast, shows that statistical properties of the process $\{u_t\}$ are completely irrelevant to the question of whether or not "informative" s.s.e. exist near a given steady state -- only the derivatives of F evaluated at the steady state matter. Our result indicates that whether or not expectations-driven instability is possible depends upon the economic "fundamentals", not upon the type of extrinsic random variable that happens to be observed by agents.

The previous authors also have assumed that U is a finite set (in most of the papers, there are only two sunspot states). This limitation has a more important effect upon the kind of results obtained than one might expect. For in this case, the restriction of attention to s.s.e. in which endogenous variables depend only upon the current sunspot state becomes a severe restriction upon the type of equilibria that can be considered. For even when U is a finite set, U^∞ has the cardinality of the continuum, and so the reinterpretation of an equilibrium in which x_t depends upon the entire history u^t as one in which x_t depends only upon u_t would require use of a sunspot variable taking values in an uncountably infinite set.

One consequence of restriction of attention to finite-state s.s.e. is that the previous authors find (in the generic case) only a finite number of locally isolated equilibria for any given specification of the sunspot process, although they are able to demonstrate the existence of continua of s.s.e. for a given model by varying the statistical properties of the sunspot process. The fact that equilibria are locally isolated means, in turn, that s.s.e. are not found near a steady state in the generic case; local methods can be used to demonstrate the existence of s.s.e. only in the case of sunspot variables with very special statistical properties. (See, e.g., the treatment of s.s.e. near a steady state in Guesnerie (1986), sec. 5.) And even in those special cases, the conditions under which finite-state s.s.e. exist near a steady state are more restrictive than those under which general s.s.e. exist near it. For example, Guesnerie finds (in the case of the class of models (4.1), for which $M(x^*)$ necessarily has n zero eigenvalues) that two-state s.s.e. can exist arbitrarily close to a steady state only if $M(x^*)$ has a non-zero real eigenvalue with absolute value less than one. Yet Theorem 1 above, which applies to Guesnerie's class of models, implies that s.s.e. exist arbitrarily close to a steady state in such a model whenever $M(x^*)$ has any non-zero eigenvalue with modulus less than one -- i.e., a complex pair with modulus less than one suffices.

It is easily shown that Guesnerie's overly restrictive result is due to his consideration of only finite-state equilibria. For consider a linear rational expectations model

$$(4.2) \qquad E_t x_{t+1} = A x_t$$

where x_t is an n-vector and A an $n \times n$ matrix. (This is the class of linear models that can arise from linearization near a steady state of a model in the class considered by Guesnerie.) It can be shown that a two-state s.s.e. can exist for such a model only if A has a real eigenvalue with absolute value less than one. For suppose that there exists such an equilibrium. Let x^j, $j = 1,2$, be the vectors of endogenous variables in each of the two sunspot states, and let $\pi_{ij} > 0$ denote the probability of transition from state i state j, for $i,j = 1,2$. If we assume that 1 is not a eigenvalue of A (this is implied by (A.5) of section 1), then $x^* = 0$ is the unique deterministic steady state, and so if $x^j \neq 0$ for any j, one has a s.s.e. For a two-state equilibrium, (4.2) becomes

(4.3) $$Ax^i = \Sigma_{j=1}^{2}\pi_{ij}x^j$$

for i=1,2. This in turn implies

$$A(\pi_{21}x^1 + \pi_{12}x^2) = (\pi_{21}x^1 + \pi_{12}x^2)$$

Since 1 is not an eigenvalue of A, it follows that one must have

(4.4) $$\pi_{21}x^1 + \pi_{12}x^2 = 0$$

One can use (4.4) to eliminate x^2 from (4.3), yielding

$$Ax^2 = (\pi_{22} - \pi_{12})x^2$$

Then since $x^2 \neq 0$ in the case of a sunspot equilibrium (for (4.4) implies that if $x^2 = 0$, $x^1 = 0$ as well), $\pi_{22} - \pi_{12}$ must be an eigenvalue of A. This result explains both why Guesnerie finds that s.s.e. can exist near a steady state only for certain transition probabilities, and why A must have a real eigenvalue. Futhermore, $0 < \pi_{12},\pi_{22} < 1$ implies $-1 < \pi_{22}-\pi_{12} < 1$, so the real eigenvalue must have absolute value less than one.

On the other hand, a linear model of the form (4.2) can have stationary sunspot equilibria even if the only eigenvalues of A less than one in modulus are a complex pair, if one allows x_t to depend upon the complete history of sunspot realizations, or a continuum of possible sunspot states. For suppose that A has a pair of eigenvalues $\rho e^{\pm i\theta}$, where $0 < \rho < 1$, $0 < \theta < \pi$. Then there must exist two linearly independent vectors $u, v \in \Re^n$ such that

$$\begin{bmatrix} Au \\ Av \end{bmatrix} = \begin{bmatrix} \rho\cos\theta & \rho\sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Now suppose that agents observe a two-state sunspot process, with transition probabilities $\pi_{ij} > 0$, $i,j = 1,2$. Let a,b be any two real numbers such that at least one of them is non-zero, and let $e = au + bv$. Then one possible stationary sunspot equilibrium is

$$x_t = \Sigma_{j=0}^{\infty} [c_{t-j} d_{t-j}] \begin{bmatrix} \rho^j \cos j\theta & \rho^j \sin j\theta \\ -\rho^j \sin j\theta & \rho^j \cos j\theta \end{bmatrix}$$

where

$$[c_t \ d_t] = \begin{cases} \pi_{12} \ [a \ b] & \text{if} \quad u_t = 1, \ u_{t-1} = 1 \\ -\pi_{22} \ [a \ b] & \text{if} \quad u_t = 1, \ u_{t-1} = 2 \\ -\pi_{11} \ [a \ b] & \text{if} \quad u_t = 2, \ u_{t-1} = 1 \\ \pi_{21} \ [a \ b] & \text{if} \quad u_t = 2, \ u_{t-1} = 2 \end{cases} ;$$

(In fact, by varying a,b we obtain a two-parameter family of s.s.e.) Thus it is the consideration of a restricted class of s.s.e. that accounts for the difference between Guesnerie's necessary conditions and our own.

C. Sunspot Equilibria and Deterministic Cycles

The previous literature gives a great deal of attention to the existence of deterministic cycles among the perfect foresight equilibria of a given stationary economic model as a necessary and sufficient condition for the existence of stationary sunspot equilibria. Azariadis and Guesnerie (1984) establish, in the case of a very special class of overlapping generations economies, that the existence of two-period cycles is a necessary and sufficient condition for the existence of a two-state s.s.e. (for <u>some</u> two-state sunspot process). Guesnerie (1986) establishes that the existence of two-period cycles is a sufficient condition for the existence of s.s.e. in a broader class of models (essentially those of the form (4.1)), while Grandmont (1986) establishes that it is both necessary and sufficient for the existence of s.s.e. within a much broader class, in the case of the class of models considered by Azariadis and Guesnerie.

The reason that the existence of a two-period cycle is sufficient for the existence of s.s.e. in a model of the form (4.1) is simple.[21] In the case of a two-state s.s.e., (4.1) may be written

$$\tilde{F}(x^1,\ x^2;\ \pi_{11}) = 0$$

$$\tilde{F}(x^2,\ x^1;\ \pi_{22}) = 0$$

where $\tilde{F}(x,\ y;\ p)$ denotes the value of $f(x,\nu,\eta)$ in the case that (ν,η) is a measure assigning probability p to the value x and $(1-p)$ to the value y. (Thus the derivatives of $\tilde{F}$ with respect to $(x,\ y)$ exist and are continuous functions of $(x,\ y;\ p)$.) A two-period cycle is an $x^*,\ x^{**} \in X$ such that $x^* \neq x^{**}$ and $F(x^*,\ x^{**},\ x^{**},\ x^*) = F(x^{**},\ x^*,\ x^*,\ x^{**}) = 0$. But if such an (x^*,x^{**}) exist, then

$$\tilde{F}(x^*,\ x^{**};\ 0) = 0$$
$$\tilde{F}(x^{**},\ x^*;\ 0) = 0$$

Let us suppose furthermore that

$$\text{Det} \begin{bmatrix} D_1\tilde{F}(x^*,x^{**};0) & D_2\tilde{F}(x^*,x^{**};0) \\ D_2\tilde{F}(x^{**},x^*;0) & D_1\tilde{F}(x^{**},x^*;0) \end{bmatrix} \neq 0$$

as can be shown to be generically the case. Then the implicit function theorem implies the existence of continuous functions $x^1(\epsilon_1,\epsilon_2)$, $x^2(\epsilon_1,\epsilon_2)$, defined for all small enough $\epsilon_1, \epsilon_2 \geq 0$, such that $x^1(0,0) = x^*$, $x^2(0,0) = x^{**}$, and that for all small enough $\epsilon_1, \epsilon_2 \geq 0$,

$$\tilde{F}(x^1(\epsilon_1,\epsilon_2),\ x^2(\epsilon_1,\epsilon_2);\ \epsilon_1) = 0$$
$$\tilde{F}(x^2(\epsilon_1,\epsilon_2),\ x^1(\epsilon_1,\epsilon_2);\ \epsilon_2) = 0$$

That is, $(x^1(\epsilon_1,\epsilon_2),\ x^2(\epsilon_1,\epsilon_2))$ is a two-state s.s.e. in the case of a sunspot variable with transition probabilities given by $\pi_{11} = \epsilon_1$, $\pi_{12} = 1-\epsilon_1$, $\pi_{21} = 1-\epsilon_2$, $\pi_{22} = \epsilon_2$.

This result may appear puzzling given our Theorem 1. For on the one hand, it would appear that our method of proof should extend to the analysis of s.r.e.e. near deterministic cycles (as well as steady states), while on the other, the argument shows that for a certain class of models there must be s.s.e. near <u>any</u> two-period cycle, regardless of whether there exists a continuum of p.f.e. converging to the cycle. The contradiction, however, is only apparent. For the two-state s.s.e. just exhibited are not "near" the two-period cycle in the sense of the topology introduced in section 1. In all periods (for ϵ_1, ϵ_2 small enough), x_t is close to <u>either</u> x^* or x^{**}; but x_t is not almost always close to the value it would take in a two-period cycle, since it is not true in any of the s.s.e. just constructed that x_t is always close to x^* in odd periods and close to x^{**} in even periods (or the reverse). One can show that indeterminacy of p.f.e. at the two-period cycle <u>is</u> necessary and sufficient for the existence of <u>periodic</u> s.s.e. near the two-period cycle (i.e., equilibria in which $x_t = \phi^1(u^t)$ for t odd, $x_t = \phi^2(u^t)$ for t even, where ϕ^1 is near $\hat{x}^*$ and ϕ^2 is near $\hat{x}^{**}$ in the L_∞ norm topology).

The fact that our formalism is not well-suited to a demonstration of the connection between s.s.e. and deterministic cycles explored by previous authors does not seem an important drawback, for the connection between indeterminacy of p.f.e. and the existence of s.s.e. developed in section 2 seems to be a much more robust result. The main advantage of the approach via deterministic cycles as a way of characterizing the types of stationary models in which s.s.e. exist is that, in the simple class of models considered by Azariadis and Guesnerie (1984), cycles are not just sufficient but necessary for the global existence of s.s.e., whereas our own necessary condition holds only locally.[22] And a global necessary condition is, of course, a very useful result, particularly for demonstrating that a proposed stabilization policy can in fact prevent the existence of expectations-driven fluctuations. However, the result that cycles are necessary in order for s.s.e. to exist holds only for a very special class of models. Even within the class of stationary overlapping generations exchange economies with a single agent type per generation and a single good per period, the result holds only if preferences satisfy certain boundary conditions assumed by Azariadis and Guesnerie. For example, if agents' utility functions are given by $u(c_1,c_2) = c_1 + (1-\gamma)^{-1}c_2^{1-\gamma}$, and their endowments are $e > 1$ in the first period of life and zero in the second, one can show that perfect foresight equilibrium is indeterminate near the monetary steady state if $\gamma > 2$. Hence Theorem 1 implies that s.s.e exist in that case (and, indeed, two-state s.s.e. of the kind considered by Azariadis and Guesnerie exist in that case). But one can also show that no deterministic cycles exist for this model when $\gamma > 2$. More importantly, the necessary condition ceases to be valid if the model is extended in even small respects. For example, if one adds a government deficit financed by money creation, as in Farmer and Woodford (1984), then s.s.e. can exist even for preferences satisfying "gross substitutability", but no deterministic cycles exist in that case.

Furthermore, even the sufficient condition seems not to be valid in the case of of a broader class of stationary models than is considered by Guesnerie. For example, consider equilibrium conditions of the form

$$(4.5) \qquad E_t g(x_{t-1}, x_t, x_{t+1}) = 0$$

where x_t is a real-valued scalar quantity and $g: \mathfrak{R}^3 \to \mathfrak{R}$ is C^1. (E.g., (3.12) and (3.14) above are of this form.) A two-period cycle exists if there exist $x^*, x^{**} \in \mathfrak{R}$ such that $x^* \neq x^{**}$ and $g(x^*, x^{**}, x^*) = g(x^{**}, x^*, x^{**}) = 0$. Can one still use the implicit function theorem to demonstrate the existence of "nearby" finite state s.s.e. as above? Certainly the existence of a two-period cycle does not imply the existence of "nearby" two-state s.s.e, as in Guesnerie's case, for a two-state s.s.e. is impossible for a (generic) model with a predetermined state variable. The simplest type of s.s.e. that is possible for an open set of models of the form (4.5) is one in which x_t always takes one of two values -- x^1 or x^2 -- but in which the probability that x_{t+1} will equal x^1 depends upon both x_t and x_{t-1}. (The sunspot variable is a four-state Markov process in this case.) Let π_{ijk} denote the probability that $x_{t+1} = x^k$ if $x_t = x^j$ and $x_{t-1} = x^i$; then in the case of an equilibrium of this kind, (4.5) becomes

$$(4.6) \qquad \Sigma_{k=1}^{2} \pi_{ijk} g(x^i, x^j, x^k) = 0$$

for $i,j = 1,2$. Suppose that a two-period cycle (x^*, x^{**}) exists; does this imply (as in the previous case) that a solution to (4.6) exists with $x^1 = x^*$, $x^2 = x^{**}$, if the transition probabilities π_{ijk} take on extreme values? If so, the implicit function theorem would again imply the existence of true s.s.e. (i.e., solutions to (4.6) with $\pi_{ijk} > 0$ for all i, j, k). But this need not be the case. If $\pi_{121} = \pi_{212} = 1$, $\pi_{122} = \pi_{211} = 0$, (4.6) holds for i=1, j=2, and for i=2, j=1. But it is not possible to choose transition probabilities so that $x^1 = x^*$, $x^2 = x^{**}$ satisfies (4.6) in the other cases as well, unless

(4.7a) $$g(x^*, x^*, x^*) \cdot g(x^*, x^*, x^{**}) < 0$$

(4.7b) $$g(x^{**}, x^{**}, x^*) \cdot g(x^{**}, x^{**}, x^{**}) < 0$$

Thus four-state s.s.e. exist "near" a two-period cycle (in the same sense as in Guesnerie's result) if and only if equations (4.7a-b) hold at the two-period cycle. There is no reason why these conditions must hold at all two-period cycles of stationary economic models, and hence it would appear that the existence of a two-period cycle is not sufficient for the existence of stationary sunspot equilibria in models more general than (4.1).

FOOTNOTES

1 The general result presented here was first conjectured in Woodford (1984).

2 See, e.g., Kehoe and Levine (1984, 1985), Muller and Woodford (1985), Woodford (1986b), and Kehoe et al. (1986).

3 For further discussion see Woodford (1986b, 1986d).

4 Such equilibria may be taken as formal representations of the Keynesian idea that entrepreneurial "animal spirits" can be an independent causal factor, in addition to economic "fundamentals" such as technology, consumer preferences, and the like. This interpretation originates with Cass and Shell (1980).

5 See, e.g., Lucas (1977). Stationary sunspot equilibria are also of particular interest as they may represent limit states of disequilibrium learning processes (see Woodford (1986c)), while it is more difficult to imagine how the degree of coordination of beliefs assumed in the case of a non-stationary sunspot equilibrium could come about.

6 But see Theorem 2 in section 2, for the case of exogenous shocks that affect the equilibrium conditions directly.

7 See, e.g., Kehoe and Levine (1985).

8 Again, see Kehoe and Levine (1984, 1985).

9 This can be proved, for example, using the implicit function theorem for Banach spaces stated in section 2. Kehoe and Levine derive the proposition, instead, from the stable manifold theorem for discrete time dynamical systems.

10 We re-derive the result here, however, in order to be precise about the

regularity conditions (such as (A.6)) that must be assumed in order for the result to hold.

11 See, e.g., the discussion of indeterminate response to fiscal policy shocks in Farmer and Woodford (1984). Spear, Srivastava and Woodford (1986) show that coninua of non-sunspot s.r.e.e. may exist in stationary overlapping generations exchange economies (like those discussed in section 3.A below) with stochastic endowments. Their result, based upon an extension of the techique of Farmer and Woodford, applies to small endowment shocks in economies with a steady state (in the absence of endowment shocks) at which dim $W = 2n$, i.e., <u>all</u> eigenvalues of $M(x^*)$ have modulus less than one. The present Theorem 2 shows that dim $W > n$ suffices for a continuum of such equilibria to exist in the case of small enough endowment shocks.

12 From this view of the proper significance of linear rational expectations models, the degree of attention in the literature on linear models given to characterization of <u>explosive</u> as well as bounded solutions to such models would appear to be misplaced, as explosive "solutions" have no counterpart in the local analysis of nonlinear models.

13 Since we only need z^h to be defined for prices in an arbitrarily small neighborhood of p^* and y in an arbitrarily small neighborhood of h^{h*}, the implicit function theorem suffices to prove this. We need not discuss here boundary conditions on u^h that would guarantee the existence of such a function on a larger set.

14 Since in this case dim $W = 2n$, i.e., <u>all</u> eigenvalues of M have modulus less than one, s.s.e. could be shown to exist near the steady state using the technique of Farmer and Woodford (1984). Extensions of that technique directly applicable to a case like Reichlin's are illustrated by Woodford

(1986a) and Spear, Srivastava and Woodford (1986). These methods do not, however, establish that $\dim W = 2$ is also necessary for the existence of s.s.e. near the steady state, as does Theorem 1.

15 Lucas and Stokey impose a number of additional restrictions upon the function U in order to insure that h(v) is uniquely defined for all v; these additional restrictions are not necessary, however, for h(v) to be defined locally or for (3.13) to be necessary and sufficient for an equilibrium in a neighborhood of a steady state.

16 In the case of an additively separable function U, the equilibrium conditions (3.14) are in fact formally analogous to those of the Azariadis (1981) overlapping generations model, so that similar conclusions about the kind of preferences required in order for s.e.e. to exist apply here. For further discussion see Woodford (1986b), sec. 2.A.

17 A result of this kind was first demonstrated, in the case of money supply variations through interest payments on existing money balances and an overlapping generations economy, by Farmer and Woodford (1984).

18 The techniques introduced here, of course, suffer only to show that such a monetary policy rules out s.s.e. involving onnly small fluctuations around the steady state. For a global demonstration that a sufficiently high rate of money growth rules out s.s.e. in a related model, see Grandmont (1986).

19 This is not Guesnerie's exact definition; he calls a sunspot equilibrium "informative" if the sunspot realization u_t conveys information about the probabilities of future realizations $(u_{t+1}, u_{t+2}, \ldots)$. His intention, however, seems to have been the sort of discrimination described here. The definition he gives is plainly only of interest when the endogenous state variables depend only upon the current sunspot state u_t; see further discussion below.

[20] I am indebted to Jean-Michael Grandmont for suggesting this interpretation, in correspondence.

[21] The following simple argument first appeared in Woodford (1984).

[22] It is known that s.s.e. may exist even in a model in which the unique steady state does not have a continuum of p.f.e. converging to it. For example, in the overlapping generations model of Azariadis, there may exist two-period cycles even when the unique steady state is of that sort, and in such a case, the result of Azariadis and Guesnerie (1984) implies that s.s.e. exist, even though our Theorem 1 implies that none of them are near the steady state. See Azariadis and Guesnerie for an example.

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