Credit Frictions and Optimal Monetary Policy: Technical Appendix*

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1 The economy

1.1 Intertemporal allocation of expenditure

1.1.1 Households: Preferences

Households are *ex ante* identical, but have heterogeneous preferences at any point in time, owing to independent fluctuations in their preferences. Each aims to maximize an expected discounted utility

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[u^{\tau_t(i)} \left(c_t \left(i \right); \xi_t \right) - \int_0^1 v^{\tau_t(i)} \left(h_t \left(j, i \right); \xi_t \right) dj \right], \tag{1.1}$$

where $\tau_t(i) \in \{b, s\}$ indicates household i's type in period t, with the utility from consumption given by

$$u^\tau \left(c_t \left(i \right); \xi_t \right) \equiv \frac{\left[c_t \left(i \right) \right]^{1-\sigma_\tau^{-1}} \left(\bar{C}_t^\tau \right)^{\sigma_\tau^{-1}}}{1 - \sigma_\tau^{-1}}, \tag{1.2}$$

and the disutility of labor given by

$$v^\tau \left(h_t \left(j, i \right); \xi_t \right) \equiv \frac{\psi_\tau}{1+\nu} \left[h_t \left(j, i \right) \right]^{1+\nu} \bar{H}_t^{-\nu}, \tag{1.3}$$

for $\tau = b, s$.

Assumptions:

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*The views expressed in this paper are those of the authors and do not necessarily reflect the position of the Federal Reserve Bank of New York or the Federal Reserve System.

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- There is a continuum of differentiated goods; $c_t\,(i)$ is a Dixit-Stiglitz aggegator of the household's purchases of differentiated goods indexed by j,

$$c_t(i) \equiv \left[\int_0^1 c_t(j,i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}} \tag{1.4}$$

  for some constant elasticity of substitution $\theta > 1$.

- Each household supplies a continuum of different types of specialized labor, indexed by j, that are hired by firms in different sectors of the economy.

- Stochastic evolution of a household's type:

  – $\tau_t\,(i)$ evolves as an independent two-state Markov chain for each i.

  – Each period, with probability $0 \leq \delta < 1$ the household's type remains the same, while with probability $1 - \delta$ a new type is drawn.

  – When a new type is drawn, it is b with probability π_b and s with probability π_s, where $0 < \pi_b,\, \pi_s < 1$, $\pi_b + \pi_s = 1$.

  – If and only if household i draws a new type in period t, it has access to the insurance agency at the beginning of period t (after learning that it will draw a new type, but before learning what its new type is).

Note that each household's consumption and labor supply (as well as its saving or borrowing) in any period t will in general depend both on the economy's aggregate state $s^t \equiv (\ldots, s_{t-1}, s_t)$, where s_t is the vector of (aggregate) disturbances (such as the preference states ξ_t) in period t, and on the individual household's type history $\tau^t(i)$. We shall represent $\tau^t(i)$ by a sequence of elements drawn from the set $\{0, b, s\}$; the element 0 is entered for any period in which the household's type does not change, while b (or s) is entered for any period in which the household draws a new type that is b (or s). (The household's current type, $\tau_t(i)$, is then indicated by the most recent element of $\tau^t(i)$ that is not equal to 0.) We use this notation for the type history so that it also indicates the periods in which the household has access to the insurance agency. In expressions such as (1.1), $c_t(i)$ should be interpreted as referring to a state-contingent consumption level $c(s^t; \tau^t(i))$, and similarly for expressions such as $h_t(j, i)$. When there is no risk of ambiguity, we shall sometimes omit the household index i, and write for example $c(s^t; \tau^t)$. In fact, since we assume that households are *ex ante* identical, in equilibrium $c(s^t; \tau^t(i))$ is the same for all households i with the same type history τ^t.

Finally, it is worth noting that the variable $c_t(i)$ has two possible interpretations (corresponding to the two possible interpretations of the resources used by the intermediary sector, discussed below in section 1.5.2). In the case that the intermediaries purchase produced goods (in addition to the purchases of households and government), $c_t(i)$ is simply the consumption of household i (of the composite good). Under the alternative interpretation (discussed further below), under which intermediation has a utility cost rather than using produced goods, $c_t(i)$ should instead be interpreted as indicating the household's purchases *net of its work* for the intermediary; but utility is still a function of $c_t(i)$ in this case, rather than of the household's purchases of goods. Subject to this stipulation about the interpretation of the variables, (1.1) applies in either case, and the same is true of the expressions given below for the household's budget constraint, the optimal choice of the variable $c_t(i)$, and so on.

1.1.2 Households: Budget constraints

The nominal assets of household i at the beginning of any period t, after insurance payments (if any) and distributions by intermediaries, are given by

$$A_t(i) = [B_{t-1}(i)]^+ \left(1 + i_{t-1}^d\right) + [B_{t-1}(i)]^- \left(1 + i_{t-1}^b\right) + T_t(i), \tag{1.5}$$

where $B_{t-1}(i)$ denotes the household's nominal assets at the end of period $t-1$,

$$[B_{t-1}(i)]^+ \equiv \max [B_{t-1}(i), 0],$$
$$[B_{t-1}(i)]^- \equiv \min [B_{t-1}(i), 0],$$

i_{t-1}^d is the riskless one-period nominal interest rate on deposits, i_{t-1}^b is the nominal interest rate on riskless one-period loans, and $T_t(i)$ denotes the household's nominal transfers from the insurance agency (if i has access in period t).

Note that in writing (1.5) we assume that $i_t^b \geq i_t^d$ each period. Were this not to be the case, households would face an arbitrage opportunity, and choose both to borrow and deposit unbounded amounts; hence there can obviously be no equilibrium of that kind. Given the inequality, it is optimal for a household to deposit an amount equal to its positive asset level (if end-of-period assets are positive) and to borrow an amount equal to (the absolute value of) its negative asset level (if end-of-period assets are negative), so that beginning-of-period assets the following period are given by (1.5).

The end-of-period net asset position of household i is given by

$$B_t(i) = A_t(i) + \int W_t(j)\, h_t(j,i)\, dj + D_t^f + D_t^{int} + T_t^g - P_t c_t(i), \tag{1.6}$$

where W_t denotes the nominal wage for labor of type j, D_t^f denotes the nominal distribution by goods-producing firms (integrating over all of the sectors j, and the same for each household i, as there is no trading in shares of the firms), D_t^{int} similarly denotes the nominal distribution by intermediaries (also the same for each household), and T_t^g denotes net nominal (lump-sum) government transfers (also the same for each i).

As noted above, in expressions such as (1.5) and (1.6), $B_t(i)$ actually means $B(s^t; \tau^t(i))$. Aggregate variables such as D_t^f instead mean $D^f(s^t)$, as the values are the same for a given household regardless of its type history.

In addition to influencing the evolution of its net asset position through adjustment of its consumption expenditure and labor supply each period, the household also chooses its state-contingent transfers from the insurance agency, in an *ex ante* insurance market at some initial date t_0.[1] The state-contingent transfers $\{T(s^t;\tau^t(i))\}$ that the household arranges through the insurance agency must satisfy the following constraints:

- $T(s^t;(\tau^{t-1},0)) = 0$; i.e., there is no access to insurance transfers except in periods in which the household draws a new type.

- $T(s^t;(\tau^{t-1},b)) = T(s^t;(\tau^{t-1},s))$; i.e., the transfer cannot be contingent upon the household's new type. In fact, it is useful to introduce the notation $\sigma(\tau^{t-1})$ for the type history after it is learned that a household (that had type history τ^{t-1} through period $t-1$) will have access to the insurance agency in period t, but before its new type is drawn. We can then write $T^\dagger(s^t;\tau^{t-1}) \equiv T(s^t;\sigma(\tau^{t-1}))$ to indicate the transfer received in the state $(s^t;\sigma(\tau^{t-1}))$, regardless of what the new type $\tau_t(i)$ turns out to be.

- The transfers must satisfy an intertemporal budget constraint

$$\sum_{t=t_0}^{\infty}\sum_{s^t}\sum_{\tau^{t-1}}\pi(s^t)\pi(\tau^{t-1})Q(s^t)T^\dagger(s^t;\tau^{t-1}) = 0, \tag{1.7}$$

  where $\pi(s^t)$ is the *ex ante* probability of reaching the aggregate state s^t in period t and $\pi(\tau^{t-1})$ is the *ex ante* probability of reaching the type history τ^{t-1} in period $t-1$,[2] and $Q(s^t)$ is a stochastic discount factor indicating the relative prices in the *ex ante* market of transfers to different possible future states.

The relative prices of transfers to different states satisfy $Q(s^t) > 0$ in all states, and the value depends only on the aggregate state, and not the type history of the household receiving the payment. These relative prices are determined in a competitive market, which clears if and only if

$$\sum_{\tau^{t-1}}\pi(\tau^{t-1})\int T^\dagger(s^t;\tau^{t-1})(i)di = 0 \tag{1.8}$$

in each aggregate state s^t, so that the net transfer from the insurance agency to households is zero. (Here we use the fact that the *ex ante* probability of a type history τ^{t-1} in period $t-1$ for any individual household is also the fraction of households with access to the insurance agency in period t that have type history $\sigma(\tau^{t-1})$.)

The household's asset accumulation plan must also satisfy an asymptotic constraint, that implies that any borrowing must eventually be repaid (so that "Ponzi schemes" are not

[1] We suppose that this initial date may have occurred prior to the date 0 from which utility is calculated in (1.1). The reason is that while we assume that all households had identical circumstances at the initial date t_0, we do not necessarily assume that they have identical financial claims at the beginning of period 0; households may have different financial claims as a consequence of different type histories prior to date 0. Similarly, when we say that all households with the same "type history" have the same wealth, we do not necessarily mean only their type history *since date 0.*

[2] The *ex ante* probability of reaching the type history $\sigma(\tau^{t-1})$ is $(1-\delta)\pi(\tau^{t-1})$, but the form of (1.7) is unaffected if we divide out the constant factor $(1-\delta)$ from each term.

possible). Looking forward from any date t at which household i has access to the insurance agency, we require that

$$\liminf_{T\to\infty} E_t^\dagger Q_{\tilde{T}(T)} A^\dagger_{\tilde{T}(T)}(i) \geq 0, \tag{1.9}$$

using the notation $E_t^\dagger[\cdot]$ for the expectation conditional on the state $(s^t;\sigma(\tau^{t-1}))$, and letting the random date $\tilde{T}(T)$ be the first date (equal to T or later) at which the household has access to the insurance agency, along any type history. It suffices to impose such a constraint looking forward from the states in which the household has access to the insurance agency, since at any time the household expects to again have access to the insurance agency with probability 1; and it makes the most sense to state the constraint only for those states, since (owing to the credit frictions) present values are only unambiguously defined when discounting wealth in some state $(s^{\tilde{T}};\sigma(\tau^{\tilde{T}-1}))$ back to some prior state $(s^t;\sigma(\tau^{t-1}))$. Constraints (1.5)–(1.9) then completely describe the ways in which the household can shift purchasing power across dates and states of the world.

It is sometimes useful to measure the household's *total financial wealth*, taking account of the present value of expected future transfers from the insurance agency in addition to the household's net claims on the intermediary sector.[3] Thus in any state $(s^t;\sigma(\tau^{t-1})$ in which a household has access to the insurance agency, we can define

$$\hat{A}_t^\dagger(i) \equiv A_t^\dagger(i) + Q_t^{-1} \sum_{s=t+1}^{\infty} E_t^\dagger[Q_s T_s], \tag{1.10}$$

where the infinite sum necessarily has a finite value owing to (1.7). The borrowing constraint (1.9) implies that a similar asymptotic condition must be satisfied by the evolution of $\{\hat{A}_t^\dagger(i)\}$, and vice versa, so that it does not matter in terms of which measure of financial wealth we express the borrowing constraint.

1.1.3 Households: Euler equations

The first-order conditions for optimal intertemporal allocation of household consumption expenditure imply that

$$\lambda_t(i) = u_c^{\tau_t(i)}\left(c_t(i);\xi_t\right), \tag{1.11}$$

where $\lambda_t(i)$ is household i's marginal utility of additional real income (in units of the composite good) in period t. (The form of period utility function assumed implies that there is always an interior solution for the level of consumption in each state of the world, so that the marginal utility of income and the marginal utility of consumption are necessarily equal.) Our observations above about $c_t(i)$ then apply as well to $\lambda_t(i)$: it is a function of $(s^t;\tau_t(i))$, and since all households are *ex ante* identical and choose identical consumption plans contingent on their type histories, we can simply write $\lambda_t(i) = \lambda(s^t;\tau_t(i))$, where the function $\lambda(s^t;\tau^t)$ is independent of i.

[3] Note that there are no claims to future payments by or to intermediaries, since we assume that all contracts with intermediaries are one-period contracts.

An optimal *ex ante* choice of the state-contingent insurance transfers will furthermore require that between any two states (defined to include the household's type history), the first-order condition

$$\frac{\lambda^\dagger(s^T;\tau^{T-1})}{\lambda^\dagger(s^t;\tau^{t-1})} = \beta^{t-T}\frac{Q(s^T)}{Q(s^t)}\frac{P(s^T)}{P(s^t)} \tag{1.12}$$

must be satisfied, where

$$\lambda^\dagger(s^t;\tau^{t-1}) \equiv \pi_b\lambda(s^t;(\tau^{t-1},b)) + \pi_s\lambda(s^t;(\tau^{t-1},s))$$

is the marginal utility of real income in a state $(s^t;\sigma(\tau^{t-1}))$ in which the household has access to the insurance agency, and $P(s^t)$ is the state-contingent value of P_t, the nominal price of the composite good. Note that the relative prices on the right-hand side of (1.12) are independent of the type histories; hence it follows that $\lambda^\dagger(s^t;\tau^{t-1})$ must in fact be independent of the type history τ^{t-1}. (This is a consequence of risk-sharing among the *ex ante* identical households through the insurance agency.) Thus we can use the simpler notation $\lambda^\dagger(s^t)$[4] for the marginal utility of real income when the aggregate state is s^t and a household has access to the insurance agency (but has not yet drawn its new type). Note that the marginal utilities of all households with access to the insurance agency in state s^t must be the *same*, until they learn their new types.

Finally, optimal choice of the quantity to deposit with or borrow from intermediaries in any state $(s^t;\tau^t)$ requires that

$$\lambda(s^t;\tau^t) \geq \beta E_t\left[\frac{1+i_t^d}{\Pi_{t+1}}\left\{\delta\lambda(s^{t+1};(\tau^t,0)) + (1-\delta)\lambda^\dagger(s^{t+1})\right\}\right], \tag{1.13}$$

$$\lambda(s^t;\tau^t) \leq \beta E_t\left[\frac{1+i_t^b}{\Pi_{t+1}}\left\{\delta\lambda(s^{t+1};(\tau^t,0)) + (1-\delta)\lambda^\dagger(s^{t+1})\right\}\right], \tag{1.14}$$

where $\Pi_t \equiv P_t/P_{t-1}$ is the gross rate of inflation. Here condition (1.13) follows from the possibility of (non-negative) saving at the riskless rate i_t^d and (1.14) follows from the possibility of (non-negative) borrowing at the riskless rate i_t^b. Moreover, (1.13) must hold with equality in any state in which $B_t(i) > 0$, while (1.14) must hold with equality in any state in which $B_t(i) < 0$. Hence one of three situations must obtain in any state: either $B_t(i) > 0$ and $\lambda_t(i)$ is equal to the right-hand side of (1.13); $B_t = 0$ and $\lambda_t(i)$ is bounded between the right-hand side of (1.13) and the right-hand side of (1.14); or $B_t(i) < 0$ and $\lambda_t(i)$ is equal tot he right-hand side of (1.14).

The complementary slackness relations between the sign of $B_t(i)$ and the signs of the Euler inequalities imply that in any period, the household's saving/borrowing decision must satisfy

$$\lambda_t(i)\frac{B_t(i)}{P_t} = \beta E_t\left[\lambda_{t+1}(i)\frac{A_{t+1}-T_{t+1}(i)}{P_{t+1}}\right], \tag{1.15}$$

[4] In the text, this variable is called simply λ_t, which is possible without ambiguity as the variable called $\lambda(s^t;\tau^t)$ here is never referred to without a type index superscript.

regardless of whether it saves or borrows. This allows us to write present-value budget relations for the household, in which the multipliers $\{\lambda_t(i)\}$ play the role of a household-specific stochastic discount factor allowing present values of random income streams to be defined. (See Proposition 2 below.)

1.1.4 Households: Asymptotic wealth accumulation

In addition to the first-order conditions stated in the previous subsection, optimality of the household's intertemporal expenditure plan requires that a transversality condition be satisfied. This is most simply established if we restrict attention to equilibria that satisfy certain bounds. Looking forward from any date t, let

$$v_t(i) \equiv E_t \sum_{T=t}^{\infty} \beta^{T-t} \left[u^{\tau_T(i)}\left(c_T\left(i\right); \xi_T\right) - \int_0^1 v^{\tau_T(i)}\left(h_T\left(j, i\right); \xi_T\right) dj \right] \tag{1.16}$$

be household i's *continuation utility* in a given equilibrium. In addition, let $\underline{v}_t^\tau$ instead be the maximum attainable continuation utility for a household of type $\tau_t(i) = \tau$ under autarchy, *i.e.,* if assets brought into the period are $A_t(i) = 0$, and the household has no access to either the intermediaries or the insurance agency from date t onward. (Note that for either value of τ, $\underline{v}_t^\tau$ depends only on the aggregate state s^t.) We shall restrict attention to equilibria that satisfy the following bound.

Assumption 1 *In equilibrium, there exists $K < \infty$ such that for each household i,*

$$v_t(i) \leq \underline{v}_t^{\tau_t(i)} + K \tag{1.17}$$

at all times.

We do not bother to establish general conditions under which this assumption can be shown to hold, since in the paper we only characterize equilibria which are small perturbations of a deterministic steady state (stationary equilibria in which aggregate shocks are small), and in the case of those equilibria Assumption 1 is obviously satisfied, as both v_t and $\underline{v}_t^\tau$ fluctuate over bounded intervals.

We can then establish necessity of the following transversality condition.

Proposition 1 *In equilibrium,*

$$\lim_{T \to \infty} E_t^\dagger Q_{\tilde{T}(T)} A_{\tilde{T}(T)}^\dagger(i) = 0, \tag{1.18}$$

looking forward from any date t at which household i has access to the insurance agency.

Proof. We prove the result by contradiction. Suppose instead that

$$\limsup_{T \to \infty} E_t^\dagger Q_{\tilde{T}(T)} A_{\tilde{T}(T)}^\dagger > 0, \tag{1.19}$$

looking forward from some state $(s^t; \sigma(\tau^{t-1}))$. The intertemporal budget constraint (1.7) implies that each household's plan must also satisfy

$$\lim_{T\to\infty} \sum_{s=T+1}^{\infty} E_t^{\dagger}[Q_s T_s] = 0,$$

so that (1.19) implies that one must also have

$$\limsup_{T\to\infty} E_t^{\dagger} Q_{\tilde{T}(T)} \hat{A}^{\dagger}_{\tilde{T}(T)} > 0,$$

where $\hat{A}^{\dagger}_{\tilde{T}}$ is defined in (1.10). Then there exists $\epsilon > 0$ and a sequence of dates $\{T_n\}$ such that

$$E_t^{\dagger} Q_{\tilde{T}(T_n)} \hat{A}^{\dagger}_{\tilde{T}(T_n)} \geq \epsilon P(s^t) Q(s^t)$$

for each date T_n.

We show that this cannot be an optimal plan by exhibiting an alternative plan from date t onward that yields higher continuation utility (and requires no change in behavior prior to date t, or along histories in which the state $(s^t; \sigma(\tau^{t-1}))$ is never reached). For any T belonging to the sequence $\{T_n\}$, consider the following alternative plan: (i) at date t, increase the household's transfer from the insurance agency $T^{\dagger}(s^t; \tau^{t-1})$ by the amount

$$\delta \equiv \frac{1}{Q(s^t)} E_t^{\dagger}[Q_{\tilde{T}(T)} \max(\hat{A}^{\dagger}_{\tilde{T}(T)}, 0)] \geq \epsilon P(s^t),$$

but also increase consumption by amount $\delta/P(s^t)$, whichever type the household draws in period t, while leaving labor supply and end-of-period assets unchanged; (ii) at all dates $t < t' < \tilde{T}(T)$, make no change in the household's plan with regard to consumption, labor supply, transfers from the insurance agency, or end-of-period assets; (iii) at date $\tilde{T}(T)$, if $\hat{A}^{\dagger}(s^{\tilde{T}}; \tau^{\tilde{T}-1}) \leq 0$, make no change in the household's plan at dates $t' \geq \tilde{T}(T)$ either; (iv) if instead $\hat{A}^{\dagger}(s^{\tilde{T}}; \tau^{\tilde{T}-1}) > 0$, make the following changes in the household's plan for dates $t' \geq \tilde{T}(T)$: (a) reduce the household's transfer from the insurance agency $T^{\dagger}(s^{\tilde{T}}; \tau^{\tilde{T}-1})$ by the amount of $A^{\dagger}(s^{\tilde{T}}; \tau^{\tilde{T}-1})$, so that beginning-of-period assets are instead equal to zero; (b) at each date $t' > \tilde{T}(T)$ at which the household again has access to the insurance agency, set the planned transfer from the insurance agency to zero; and (c) at each date $t' \geq \tilde{T}(T)$, change the household's consumption, labor supply, and saving/borrowing decisions to those corresponding to the *optimal autarchy plan* at that date (which depends only on the aggregate state $s^{t'}$ at that date and the household's current type $\tau_{t'}$). This is a feasible plan, because the present value of the adjustments to the transfers from the insurance agency is zero (so that (1.7) continues to be satisfied); consumption expenditure in period t is increased by the amount of the increased transfer from the insurance agency, so that no other changes in the household's plan need to be made in order to satisfy the flow budget constraint (1.6); and in the event that $\hat{A}^{\dagger}_{\tilde{T}(T)} > 0$ under the original plan, one switches to a plan under which

$A_{t'}(i) = 0$ and $B_{t'}(i) = 0$ for all $t' \geq \tilde{T}(T)$, so that the consumption and labor supply that are optimal under autarchy are consistent with the flow budget constraint (1.6) in each period.

Let us consider the effect on the household's continuation utility

$$v^{\dagger}(s^t; \tau^{t-1}) \equiv E_t^{\dagger}[v_t(s^t; \tau^t)]$$

of this change in the household's plan. For each state $(s^{\tilde{T}(T)}; \tau^{\tilde{T}(T)-1})$ at which $\hat{A}^{\dagger}(s^{\tilde{T}(T)}; \tau^{\tilde{T}(T)-1}) > 0$ under the original plan, the continuation utility $v_{\tilde{T}(T)}(i)$ is replaced by $\underline{v}^{\tau}_{\tilde{T}(T)}$, for either of the possible draws τ of the household's new type in period $\tilde{T}(T)$. It then follows from (1.17) that the continuation utility is reduced by at most K in any such state. For each state $(s^{\tilde{T}(T)}; \tau^{\tilde{T}(T)-1})$ at which $\hat{A}^{\dagger}(s^{\tilde{T}(T)}; \tau^{\tilde{T}(T)-1}) \leq 0$ under the original plan, there is instead no change in continuation utility. Hence the total reduction in $v^d ag(s^t; \tau^{t-1})$ due to the changes in the plan at dates $t' \geq T$ is at most equal to $\beta^{T-t}K$. At the same time, consumption is increased in period t by an amount $\delta / P(s^t)$, that is at least of size ϵ, regardless of the new type that is drawn in period t. This increases $v^{\dagger}(s^t; \tau^{t-1})$ by an amount that is at least equal to

$$\tilde{\delta} \equiv E_t^{\dagger}[u(c_t(i) + \epsilon; \xi_t) - u(c_t(i); \xi_t)] > 0.$$

Hence there is a net increase in $v^{\dagger}(s^t; \tau^{t-1})$ of at least the magnitude $\tilde{\delta} - \beta^{T-t}K$. Since there exist dates T in the sequence $\{T_n\}$ that are arbitrarily far in the future, it is possible to choose T so that

$$\tilde{\delta} - \beta^{T-t}K > 0,$$

in which case the alternative plan has a higher continuation utility. (Note that neither $\tilde{\delta}$ nor K depends on the value of T.) Since the household's plan is not changed at any dates prior to t or along histories in which the state $(s^t; \sigma(\tau^{t-1}))$ is never reached, it follows that (1.1) is higher as well, and the original plan cannot have been optimal among those plans satisfying the household's budget constraints.

We have thus obtained a contradiction to our assumption that in an optimal plan (1.19) holds. It follows that under any optimal plan, we must instead have

$$\limsup_{T \to \infty} E_t^{\dagger} Q_{\tilde{T}(T)} A^{\dagger}_{\tilde{T}(T)} \leq 0.$$

But this together with (1.9) implies that (1.18) must hold. ■

This result allows us to write a present-value relation between a household's total financial wealth and the value of its subsequent expenditure in excess of non-financial income, defined as

$$X_t(i) \equiv P_t c_t(i) - \int W_t(j)\, h_t(j,i)\, dj - D_t^f - D_t^{int} - T_t^g.$$

Proposition 2 *In equilibrium,*

$$\lambda_t^{\dagger}(i) \frac{\hat{A}_t^{\dagger}(i)}{P_t} = E_t^{\dagger} \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \lambda_s(i) \frac{X_s(i)}{P_s} \right\}, \qquad (1.20)$$

looking forward from any date t at which household i has access to the insurance agency.

Proof. Condition (1.6) together with (1.15) implies that

$$\lambda_t(i)\frac{A_t(i)}{P_t} = \lambda_t(i)\frac{X_t(i)}{P_t} + \beta E_t\left[\lambda_{t+1}(i)\frac{A_{t+1} - T_{t+1}(i)}{P_{t+1}}\right].$$

Summing this relation between date t and the random date $\tilde{T}(T)$, for any $T > t$, yields

$$\begin{aligned}\lambda_t(i)\frac{A_t(i)}{P_t} + E_t\left\{\sum_{s=t+1}^{\tilde{T}(T)} \beta^{s-t}\lambda_s(i)\frac{T_s(i)}{P_s}\right\} \quad &= \quad E_t\left\{\sum_{s=t}^{\tilde{T}(T)-1} \beta^{s-t}\lambda_s(i)\frac{X_s(i)}{P_s}\right\} \\ &\quad +E_t\left\{\beta^{\tilde{T}(T)-t}\lambda_{\tilde{T}(T)}(i)\frac{A_{\tilde{T}(T)}(i)}{P_t}\right\}.\end{aligned}$$

Letting t be a date at which the household has access to the insurance agency, and taking the conditional expectation of all terms with respect to the state $(s^t; \sigma(\tau^{t-1}))$ before the household's new period-t type is learned, we obtain

$$\begin{aligned}&\lambda_t^\dagger(i)\frac{A_t^\dagger(i)}{P_t} + E_t^\dagger\left\{\sum_{s=t+1}^{\tilde{T}(T)} \beta^{s-t}\lambda_s^\dagger(i)\frac{T_s^\dagger(i)}{P_s}\right\} \\ &\qquad = E_t^\dagger\left\{\sum_{s=t}^{\tilde{T}(T)-1} \beta^{s-t}\lambda_s(i)\frac{X_s(i)}{P_s}\right\} + E_t^\dagger\left\{\beta^{\tilde{T}(T)-t}\lambda_{\tilde{T}(T)}(i)\frac{A_{\tilde{T}(T)}(i)}{P_t}\right\}. \qquad (1.21)\end{aligned}$$

We further observe that using (1.12), the second term on the left-hand side must be equal to

$$\frac{\lambda_t^\dagger(i)}{Q_t}E_t^\dagger\left\{\sum_{s=t+1}^{\tilde{T}(T)} Q_s T_s^\dagger(i)\right\}. \qquad (1.22)$$

Since $\tilde{T}(T)$ is a period in which the household has access to the insurance agency (and draws a new type), and the value of $A_{\tilde{T}(T)}(i)$ is the same regardless of the new type drawn, the final term on the right-hand side of (1.21) can equivalently be written

$$E_t^\dagger\left\{\beta^{\tilde{T}(T)-t}\lambda_{\tilde{T}(T)}^\dagger(i)\frac{A_{\tilde{T}(T)}^\dagger(i)}{P_t}\right\}.$$

Moreover, (1.12) implies that this must in turn equal

$$\frac{\lambda_t^\dagger(i)}{Q_t}E_t^\dagger[Q_{\tilde{T}(T)}A_{\tilde{T}(T)}^\dagger(i)].$$

The transversality condition (1.18) implies that this quantity must have a limiting value of zero as T is made unboundedly large. Since (1.7) implies that (1.22) must have a well-defined limit as T is made unboundedly large as well, it follows that the first term on the right-hand side on (1.21) must also have a well-defined limit as T is made large. Hence we can take the limit of each term in (1.21) as T goes to infinity, obtaining (1.20). ■

This establishes a present-value relation between a household's total financial wealth (at the beginning of any period in which it has access to the insurance agency) and its planned future expenditure in excess of its non-financial income (in which we count both its labor income and the distributions that it expects to receive from the firms, all of which income is unrelated to any transactions with either intermediaries or the insurance agency). Note that the marginal utility of income of the individual household (which generally depends on its type history, in addition to the aggregate state) must be used as a "personal stochastic discount factor" in defining the present value of income and expenditure in states in which the household does not have access to the insurance agency (or any other opportunity to trade state-contingent claims).

1.1.5 Aggregate behavior of the household sector

Aggregation is simplified if we restrict attention to equilibria in which certain more special assumptions are satisfied.

Assumption 2 *In equilibrium, in each period in which $\tau_t(i) = s$, household i chooses $B_t(i) > 0$, and in each period in which $\tau_t(i) = b$, it chooses $B_t(i) < 0$. Hence at any point in time, every type s household is a saver and every type b household is a borrower.*

Assumption 3 *In equilibrium,*

$$\frac{1+i_t^d}{\Pi_{t+1}} \leq \frac{1+i_t^b}{\Pi_{t+1}} \leq 1+r^* \tag{1.23}$$

at all times, where the bound r^* satisfies

$$\beta\delta(1+r^*) < 1. \tag{1.24}$$

Moreover, there exists a bound $\lambda^* < \infty$ such that

$$0 < \lambda_t(i) < \lambda^* \tag{1.25}$$

at all times, for every household i.

Below, we state more primitive assumptions about the model parameterization under which these assumptions hold as long as the exogenous random disturbances are small enough in amplitude. (This requires only that we verify that certain inequalities are satisfied in the deterministic steady state implied by the model.) We also verify that the assumptions are satisfied by the calibrated parameter values used in the numerical results presented in the paper.

We are then able to obtain an important aggregation result relied upon in the text.

Proposition 3 *In any equilibrium consistent with Assumptions 2 and 3, the marginal utility of income $\lambda(s^t;\tau^t)$ depends only on the household's type τ_t in the current period, and is independent of the household's prior type history.*

Proof. In any equilibrium consistent with Assumption 3, we observe that

$$\lim_{j\to\infty}(\beta\delta)^j E_t\left[\left\{\prod_{k=1}^{j}\frac{1+i^b_{t+k-1}}{\Pi_{t+k}}\right\}\lambda(s^{t+j};(\tau^t,0,\ldots,0))\right]=0, \tag{1.26}$$

looking forward from any state $(s^t;\tau^t)$ at date t, and the same is true *a fortiori* if we replace i^b_{t+k-1} by i^d_{t+k-1} in each term. And in any equilibrium consistent with Assumption 2, we observe that at any point in time, (1.13) must hold with equality in the case of any type history τ^t such that $\tau_t=s$, while (1.14) must hold with equality in the case of any type history such that $\tau_t=b$.

It then follows that in the case of any type history τ^t that implies that the household's current type is b,

$$\begin{aligned}\lambda(s^t;\tau^t) &= \beta(1-\delta)E_t\left[\frac{1+i^b_t}{\Pi_{t+1}}\lambda^\dagger(s^{t+1})\right]+\beta\delta E_t\left[\frac{1+i^b_t}{\Pi_{t+1}}\lambda(s^{t+1};(\tau^t,0))\right]\\ &= \beta(1-\delta)E_t\left[\frac{1+i^b_t}{\Pi_{t+1}}\lambda^\dagger(s^{t+1})\right]+\beta^2\delta(1-\delta)E_t\left[\frac{1+i^b_t}{\Pi_{t+1}}\frac{1+i^b_{t+1}}{\Pi_{t+2}}\lambda^\dagger(s^{t+2})\right]\\ &\quad+(\beta\delta)^2E_t\left[\frac{1+i^b_t}{\Pi_{t+1}}\frac{1+i^b_{t+1}}{\Pi_{t+2}}\lambda(s^{t+2};(\tau^t,0,0))\right].\end{aligned}$$

Continuing recursively in this way, using the fact that (1.14) holds with equality to substitute for the final term of each successive stage, and using (1.26) to guarantee convergence of the infinite sequence, we find that

$$\lambda(s^t;\tau^t)=\sum_{j=0}^{\infty}\beta(\beta\delta)^j(1-\delta)E_t\left[\left\{\prod_{k=0}^{j}\frac{1+i^b_{t+k}}{\Pi_{t+k+1}}\right\}\lambda^\dagger(s^{t+k+1})\right].$$

Note that the right-hand side of this expression involves only the forecasted evolution of the future aggregate states, and so depends only on the aggregate state s^t. It follows that $\lambda(s^t;\tau^t)$ is the same for all type histories for which the current type is b. Similarly, the fact that (1.13) holds with equality for all type s households can be used to establish that $\lambda(s^t;\tau^t)$ is the same for all type histories for which the current type is s. ■

Thus the marginal utility of income of all type τ households is the same at any point in time, for $\tau=b,s$. We can denote the common marginal utility of type b households by $\lambda^b(s^t)$, and the common marginal utility of type s households by $\lambda^s(s^t)$. It then follows from (1.11) that the level of consumption expenditure by all type τ households is also the same at any point in time, and we can introduce the notation $c^\tau(s^t)$ for the common expenditure level of type τ households, for $\tau = b, s$. It similarly follows from the first-order conditions for optimal household labor supply (discussed in section xx below) that there must be a common level of labor supply by type τ households at any point in time, and correspondingly a common level of wage income. These results are useful in aggregating the expenditure and labor supply decisions of the households in order to obtain dynamic equations for aggregate expenditure and aggregate labor supply; in fact, we need only to aggregate the variables describing the decisions of the two types.

As stated in the text, the Euler equations for optimal expenditure by either type τ satisfy[5]

$$\lambda_t^\tau = u_c^\tau\left(c_t^\tau; \xi_t\right), \tag{1.27}$$

$$\lambda_t^\tau = \beta E_t\left[\frac{1+i_t^\tau}{\Pi_{t+1}}\left\{\delta\lambda^\tau(s^{t+1}) + (1-\delta)\lambda^\dagger(s^{t+1})\right\}\right] \tag{1.28}$$

at all times, where in the case $\tau = s$ the notation i_t^τ is understood to mean the deposit rate i_t^d, and where in each period

$$\lambda^\dagger(s^t) = \pi_b\lambda^b(s^t) + \pi_s\lambda^s(s^t).$$

(Here, as in the text, we use the shorthand λ_t^τ for $\lambda^\tau(s^t)$ and $\lambda_t^\dagger$ for $\lambda^\dagger(s^t)$.[6])

The result that the equilibrium expenditure of a household of type τ at any time is independent of the household's prior type history similarly implies that after any insurance transfer, a household's post-transfer assets $A^\dagger(s^t; \tau^{t-1})$ must also be independent of its prior type history τ^{t-1}. Since both $\lambda_t(i)$ and $X_t(i)$ have been shown not to depend on a household's type history prior to period t, it follows from (1.20) that $\hat{A}_t^\dagger(i)$ will be independent of household i's type history. That is, there will be a common value for total financial wealth $\hat{A}_t^\dagger$ for all households that access the insurance agency at a given time, that depends only on the aggregate state s^t at that time. Moreover, while there will in general be some indeterminacy of the way in which the total wealth $\hat{A}_t^\dagger$ is decomposed into assets outside the insurance agency as opposed to expected future insurance transfers, since the same total wealth is used to support an identical planned continuation path for expenditure, we can without loss of generality assume that these households also have the same beginning-of-period assets $A_t^\dagger(i)$ and the same expectations regarding future state-contingent insurance transfers. Under this assumption, there is also a common value $A_t^\dagger$ that depends only on s^t, that represents the beginning-of-period assets $A_t(i)$ for any household i with access to the insurance agency in period t.

1.1.6 Equilibrium end-of-period wealth distribution

It follows from the analysis of the previous subsection that (in an equilibrium consistent with Assumptions 1-3) a household's end-of-period asset position $B_t(i)$ will depend only on

[5] Note that writing ξ_t without a superscript for the type does not mean that we assume that aggregate shocks must affect the utilities of the two types in the same way. We allow ξ_t to be a vector of disturbances, some components of which may affect the preferences of only one type. Indeed, in the paper we present results for disturbances to the utility of consumption of only a single type.

[6] In the text, we further simplify by using the notation λ_t for the variable here called $\lambda_t^\dagger$.

the household's type and the number of periods since it has last had access to the insurance agency (which will also be the number of periods since it drew that type). Letting $B_t^{\tau(j)}$ be the assets of a type τ household that last had access to the insurance agency in period $t-j$, for $\tau \in \{b, s\}$, $j = 0, 1, 2, \ldots$, we can completely describe the evolution of the distribution of wealth by specifying the evolution of the quantities $\{B^\tau(j)_t\}$.

Integrating (1.5) over all of the households i with access to the insurance agency in period t, we find that their common level of beginning-of-period assets must equal

$$A_t^\dagger = P_{t-1}[s_{t-1}(1 + i_{t-1}^d) - b_{t-1}(1 + i_{t-1}^b)], \tag{1.29}$$

where

$$s_t \equiv \int [B_t(i)]^+ \, di/P_t$$

denotes aggregate real (end-of-period) private saving and

$$b_t \equiv -\int [B_t(i)]^- \, di/P_t$$

denotes aggregate real (end-of-period) private borrowing. Here we use the fact that the households which have access to the insurance agency in period t are a uniform random sample from among all households, so that the total saving and borrowing by these households at the end of period $t-1$ is exactly the same (per capita) as that of the aggregate economy, and the fact that the average transfer from the insurance agency must be zero, by (1.8).

It then follows from (1.6) that the end-of-period assets of a household that has access to the insurance agency in period t and then draws type τ will equal

$$B_t^{\tau(0)} = A_t^\dagger + W_t^\tau - P_t c_t^\tau + D_t^f + D_t^{int} + T_t^g, \tag{1.30}$$

for $\tau \in \{b, s\}$, where

$$W_t^\tau \equiv \int_0^1 W_t(j) h_t^\tau(j) dj$$

denotes the total wage earnings of each type τ household in period t.[7] The end-of-period assets of a household that has not had access to the insurance agency in the current period will correspondingly be given by

$$B_t^{\tau(j+1)} = B_{t-1}^{\tau(j)}(1 + i_{t-1}^\tau) + W_t^\tau - P_t c_t^\tau + D_t^f + D_t^{int} + T_t^g, \tag{1.31}$$

for any $j \geq 0$.

Aggregate saving and borrowing can then be obtained by summing these quantities,

$$s_t = \sum_{j=0}^{\infty} \pi_s (1-\delta)\delta^j B_t^{s(j)}/P_t, \tag{1.32}$$

[7] These are common to all households of a given type because equilibrium labor supply depends only on the household's current type, as explained further in section 1.2 below.

$$b_t = -\sum_{j=0}^{\infty} \pi_b (1-\delta)\delta^j B_t^{b(j)}/P_t. \tag{1.33}$$

It is a requirement for equilibrium that the infinite sums in (1.32) and (1.33) must converge, when the individual terms are defined by (1.30)–(1.31).

It follows from (1.30)–(1.31) that sufficient conditions for Assumption 2 to hold are that

$$P_t c_t^b - W_t^b > A_t^\dagger + D_t^f + D_t^{int} + T_t^g > P_t c_t^s - W_t^s, \tag{1.34}$$

$$P_t c_t^b - W_t^b > D_t^f + D_t^{int} + T_t^g > P_t c_t^s - W_t^s \tag{1.35}$$

at all times. The inequalities in (1.34) are equivalent to the requirement that $B_t^{s(0)} > 0$ and $B_t^{b(0)} < 0$. The inequalities in (1.35) are sufficient conditions to establish for arbitrary $j \geq 0$ that $B_t^{s(j+1)} > 0, B_t^{b(j+1)} < 0$ if one assumes that $B_t^{s(j)} > 0, B_t^{b(j)} < 0$. Hence if (1.34)–(1.35) are jointly satisfied, one can show by a recursive argument that $B_t^{s(j)} > 0, B_t^{b(j)} < 0$ for all $j \geq 0$.

1.1.7 Fiscal Transfers by Government and Firms

Government is assumed to purchase a quantity G_t each period of the composite good defined by (1.4). These are assumed to be financed through some combination of a proportional sales tax τ_t on all output sold by firms, lump-sum taxation (assumed to apply uniformly to all households), and borrowing. We assume for simplicity that all government debt is one-period riskless nominal debt, so that government debt and deposits with the intermediaries are perfect substitutes from the standpoint of savers (hence our assumption of only a single interest rate available on savings in (1.5)). We specify fiscal policy by exogenous sequences $\{G_t\}$ for government purchases, $\{\tau_t\}$ for the sales tax rate, and $\{b_t^g\}$ for the real value of the (end-of-period) government debt. The implied value of the net lump-sum transfers T_t^g is then determined as a residual, by the government's flow budget constraint,

$$T_t^g = \tau_t P_t Y_t - P_t G_t + B_t^g - (1 + i_{t-1}^d) B_{t-1}^g, \tag{1.36}$$

where $B_t^g \equiv P_t b_t^g$ is the *nominal* value of government debt and we use the fact that the equilibrium interest yield on government debt must be the same as that on deposits.

The per capita distribution D_t^f each period by goods-producing firms is equal to the aggregate sales revenues of the firms, net of sales taxes, minus the aggregate wage bill of the firms,

$$D_t^f = (1-\tau_t) P_t Y_t - \pi_b W_t^b - \pi_s W_t^s. \tag{1.37}$$

Here we use the fact that, because the composition of the demand for individual goods (on the part of all final consumers: households, government, and intermediaries) corresponds to the cost-minimizing way of achieving a given quantity of the composite good, total sales revenues are equal simply to $P_t Y_t$, as well as the fact that the total wage bill of the firm sector is equal to the total wage income of all households. It is moreover worth noting that in both (1.36) and (1.37), total sales revenues of the firms must equal total expenditure by households, government and intermediaries. Thus we may substitute

$$Y_t = \pi_b c_t^b + \pi_s c_t^s + G_t + \Xi_t \tag{1.38}$$

for Y_t in either of these equations, where the variable Ξ_t indicates the resources consumed by the intermediary sector (see section 1.5 below).

Finally, as explained further in section 1.5 below, the profit distribution by intermediaries each period will equal

$$D_t^{int}/P_t = d_t - b_t - \Xi_t, \tag{1.39}$$

where d_t is the real value of one-period deposits with the intermediaries in period t, b_t is the real value of one-period loans extended by the intermediaries (which must equal aggregate household borrowing, defined in (1.33), and Ξ_t again represents the real resources consumed in the activity of loan origination. There are no profits to distribute in the period in which the loan and deposit contracts mature, as we assume that an intermediary lends an amount that just suffices to allow it to repay what it owes its depositors out of the proceeds from loan repayments,

$$(1 + i_t^b)b_t = (1 + i_t^d)d_t. \tag{1.40}$$

Given the latter relation, (1.39) can alternatively be written

$$D_t^{int}/P_t = \omega_t b_t - \Xi_t, \tag{1.41}$$

where $\omega_t \geq 0$ is the spread between borrowing rates and deposit rates, defined by

$$1 + i_t^b = (1 + \omega_t)(1 + i_t^d). \tag{1.42}$$

Aggregate deposits d_t must furthermore be related to aggregate household saving (defined in (1.32)) through the equilibrium relation

$$s_t = d_t + b_t^g. \tag{1.43}$$

Comparison of (1.40) and (1.43) with (1.29) allows us to write

$$A_t^\dagger = B_{t-1}^g(1 + i_{t-1}^d). \tag{1.44}$$

This is because the aggregate financial wealth of households at the beginning of any period (not counting the value of expected net transfers from the insurance agency, or future distributions from firms) will equal the value of maturing government debt, which represents the aggregate per capita wealth of the households that access the insurance agency, and that is uniformly redistributed among those households.

We can use equations (1.36)– (1.38), (1.41) and (1.44) to eliminate T_t^g, D_t^f, D_t^{int}, and $A_t^\dagger$ from the inequalities (1.34)–(1.35). We observe that the two inequalities (1.34) both hold if and only if

$$(c_t^b - c_t^s) - (w_t^b - w_t^s) > \max\{\pi_s^{-1}(\omega_t b_t + b_t^g), -\pi_b^{-1}(\omega_t b_t + b_t^g)\},$$

where $w_t^\tau \equiv W_t^\tau/P_t$ for $\tau = b, s$. Under the assumption that government debt is always non-negative under the policies that we consider, the first of the two terms on the right-hand side is necessarily at least as large as the second, so that the condition can be written more simply as

$$(c_t^b - c_t^s) - (w_t^b - w_t^s) > \pi_s^{-1}(\omega_t b_t + b_t^g). \tag{1.45}$$

Similarly, the two inequalities (1.35) both hold if and only if

$$(c_t^b - c_t^s) - (w_t^b - w_t^s) > \max\{\pi_s^{-1}(\omega_t b_t + b_t^g - (1+r_t^d)b_{t-1}^g), -\pi_b^{-1}(\omega_t b_t + b_t^g - (1+r_t^d)b_{t-1}^g)\}, \tag{1.46}$$

where

$$1 + r_t^d \equiv (1 + i_{t-1}^d)/\Pi_t \tag{1.47}$$

is the *ex post* real return on deposits that mature in period t. We verify below that the bounds (1.45)–(1.46) hold at all times in our numerical examples, so that Assumption 2 is satisfied. (We show that for our numerical parameter values, both inequalities hold in the deterministic steady state; it then follows that they also both at all times in the case of small enough random disturbances.) Note that both conditions are satisfied in the event that there is sufficient heterogeneity in the equilibrium levels of expenditure of the two types of household (in excess of any heterogeneity in their levels of wage income).

1.1.8 Dynamics of Aggregate Private Debt

The law of motion for aggregate (real) private debt b_t can then be obtained by summing equation (1.30), multiplied by $-\pi_b(1-\delta)P_t^{-1}$, and equation (1.31) for each value of $j \geq 0$, multiplied by $-\pi_b(1-\delta)\delta^{j+1}P_t^{-1}$. One obtains

$$b_t = \delta(1+r_t^b)b_{t-1} + \pi_b\pi_s[(c_t^b - c_t^s) - (w_t^b - w_t^s)] - \pi_b[b_t^g - (1+r_t^d)b_{t-1}^g] - \pi_b\omega_t b_t - \pi_b(1-\delta)(1+r_t^d)b_{t-1}^g,$$

where r_t^b is the *ex post* real return on loans maturing in period t, defined analogously to (1.47), and again using (1.33) to define aggregate private debt, and employing the same substitutions as in the previous subsection to replace terms such as D_t^f, D_t^{int}, and T_t^g in (1.30)–(1.31).

If we suppose that the credit spread ω_t is a non-decreasing function $\omega_t(b_t)$ of the volume of intermediated credit, as explained in section 1.5, we can solve this equation for b_t, obtaining

$$b_t = \Phi_t^{-1}\left(\pi_b\pi_s[(c_t^b - c_t^s) - (w_t^b - w_t^s)] + \delta(1 + r_t^d)[(1 + \omega_{t-1}(b_{t-1}))b_{t-1} + \pi_b b_{t-1}^g] - \pi_b b_t^g\right), \tag{1.48}$$

where $\Phi_t(\cdot)$ is the monotonic (and hence invertible) function defined by

$$\Phi_t(b) \equiv [1 + \pi_b\omega_t(b)]\, b,$$

and we have used (1.42) to substitute for $1 + i_t^b$. Equation (1.48) determines the evolution of aggregate private credit b_t from date zero onward, given an initial condition b_{-1} for private credit, and the evolution of the real deposit rate r_t^d, the real public debt b_t^g, the expenditure-imbalance measure $(c_t^b - c_t^s) - (w_t^b - w_t^s)$, and the exogenous determinants of the function $\omega_t(\cdot)$, discussed further in section 1.5.

1.1.9 Interest Rates and Aggregate Demand

We can now summarize the equilibrium relations that determine the path for the policy rate required to support a given evolution of aggregate expenditure. Given the evolution of the marginal utilities $\{\lambda_t^b, \lambda_t^s\}$, aggregate demand for produced goods is equal to

$$Y_t = \pi_b c^b(\lambda_t^b; \xi_t) + \pi_s c^s(\lambda_t^s; \xi_t) + G_t + \Xi_t, \tag{1.49}$$

where where for $\tau = b, s$, $c^\tau(\lambda_t^\tau; \xi_t)$ is the function obtained by inverting (1.27). Given our isoelastic utility specification, these functions take the specific form

$$c_t^\tau = \bar{C}_t^\tau \cdot (\lambda_t^\tau)^{\sigma_\tau}.$$

The evolution of the marginal utilities of income λ_t^τ is in turn related to the path of interest rates through the pair of equilibrium relations (1.28) for $\tau = b, s$. Using (1.42) to substitute for i_t^b in these equations, we obtain a pair of equations for the evolution of λ_t^b and λ_t^s, given the paths of the real deposit return r_t^d and the credit spread ω_t. The system consisting of these equations together with (1.49) provide a set of equations to determine the joint evolution of the variables $Y_t, \lambda_t^b, \lambda_t^s$, given the evolution of r_t^d, ω_t, and Ξ_t; or alternatively, to determine the joint evolution of $r_t^d, \lambda_t^b, \lambda_t^s$, given the evolution of Y_t, ω_t, and Ξ_t.

The relation thus established between the path of real interest rates on the one hand and the level of real aggregate demand (or output) on the other generalizes the relation summarized by the "intertemporal IS equation" of the basic New Keynesian model, to take account of heterogeneity and credit frictions. Note that in the case that the two types have identical preferences and there are no credit frictions (so that $\omega_t = \Xi_t = 0$), $\lambda_t^b = \lambda_t^s$, which is in turn simply the marginal utility of consumption of the representative household, given by $u_c(c_t; \xi_t)$, where c_t is the common level of consumption; there is a single real interest rate $r_t^d = r_t^b = r_t$; and the system of equations reduces to the single Euler equation

$$u_c(Y_t - G_t; \xi_t) = \beta E_t[(1 + r_{t+1}) u_c(Y_{t+1} - G_{t+1}; \xi_{t+1})]$$

of the basic New Keynesian model. While this relation is complicated in the present model by the presence of heterogeneity and credit frictions, the essential character of the equilibrium relation between the path of Y_t and the path of r_t^d remains the same; the credit frictions are mainly a source of additional disturbances to the "IS equation," as is made clear by the log-linear approximation presented in section 3.

1.2 Labor supply

Suppose that firms in industry j hire labor of type j from both "borrower" and "saver" households. Then, for each type, we get

$$v_h^\tau\left(h_t^\tau(j); \xi_t\right) = \frac{\lambda_t^\tau}{\mu_t^w} w_t(j), \tag{1.50}$$

where $w_t(j)$ is the real wage, μ_t^w is a wage markup. With preferences

$$v(h_t(j);\xi_t) \equiv \frac{\psi_\tau}{1+\nu}[h_t(j)]^{1+\nu}\bar{H}_t^{-\nu},$$

we get

$$h_t^\tau(j) = \bar{H}_t\left(\frac{\lambda_t^\tau}{\psi_\tau}\frac{w_t(j)}{\mu_t^w}\right)^{1/\nu}, \tag{1.51}$$

for $\tau = b, s$.

The market clearing condition

$$\pi_b h_t^b(j) + (1-\pi_b)h_t^s(j) = h_t(j) \tag{1.52}$$

and

$$h_t(j) = \bar{H}_t\left(\frac{w_t(j)}{\mu_t^w}\right)^{1/\nu}\left[\pi_b\left(\frac{\lambda_t^b}{\psi_b}\right)^{1/\nu} + (1-\pi_b)\left(\frac{\lambda_t^s}{\psi_s}\right)^{1/\nu}\right]$$

or, equivalently,

$$h_t(j) = \bar{H}_t\left(\frac{\tilde{\lambda}_t}{\psi}\frac{w_t(j)}{\mu_t^w}\right)^{1/\nu}, \tag{1.53}$$

with

$$\tilde{\lambda}_t \equiv \psi\left[\pi_b\left(\frac{\lambda_t^b}{\psi_b}\right)^{\frac{1}{\nu}} + (1-\pi_b)\left(\frac{\lambda_t^s}{\psi_s}\right)^{\frac{1}{\nu}}\right]^\nu, \tag{1.54}$$

$$\psi^{-\frac{1}{\nu}} \equiv \pi_b\psi_b^{-\frac{1}{\nu}} + (1-\pi_b)\psi_s^{-\frac{1}{\nu}}. \tag{1.55}$$

Note: these definitions are normalized so that if $\lambda_t^b = \lambda_t^s$, $\tilde{\lambda}_t = \lambda_t^b = \lambda_t^s$, and if $\psi_b = \psi_s$, then $\psi = \psi_b = \psi_s$. Also, $\tilde{\lambda}_t$ is a homogeneous of degree 1 functions of $(\lambda_t^b, \lambda_t^s)$, and ψ is a homogeneous degree 1 function of (ψ_b, ψ_s).

Real wage is determined by

$$w_t(j) = \psi\mu_t^w\left(\frac{h_t(j)}{\bar{H}_t}\right)^\nu\tilde{\lambda}_t^{-1}. \tag{1.56}$$

1.3 Firms

Continuum of firms operating in environment typical of Calvo pricing.

Technology

$$Y_t(i) = Z_t h_t(i)^{1/\phi}, \tag{1.57}$$

implying that labor demand is

$$h_t(i) = \left(\frac{Y_t(i)}{Z_t}\right)^\phi. \tag{1.58}$$

Labor market equilibrium for industry j implies

$$w_t(j) = \psi\mu_t^w\tilde{\lambda}_t^{-1}\left(\left(\frac{Y_t(j)}{Z_t}\right)^{\phi}\frac{1}{\bar{H}_t}\right)^{\nu}, \tag{1.59}$$

and demand for each variety of goods

$$Y_t(i) = Y_t\left(\frac{p_t(i)}{P_t}\right)^{-\theta}. \tag{1.60}$$

The profits function for a given firm in industry j is:

$$\Pi\left(p, p^j, P; Y, \tilde{\lambda}, \xi\right) \equiv (1-\tau)\, pY\,(p/P)^{-\theta} - \psi\mu^w\tilde{\lambda}^{-1}\left(\left(\frac{Y\,(p^j/P)^{-\theta}}{Z}\right)^{\phi}\frac{1}{\bar{H}}\right)^{\nu} P\left(\frac{Y\,(p/P)^{-\theta}}{Z}\right)^{\phi}, \tag{1.61}$$

where p is the price of individual firm, p^j the common price in industry j, P is the aggregate price, Y is the aggregate level of output and μ^w is a markup shock.

A firm's objective is to maximize the contribution to the average utility of shareholders (share in firms can't be traded) so that the relevant stochastic discount factor is

$$Q_{t,T} = \beta^{T-t}\frac{\lambda_T}{\lambda_t}\frac{P_t}{P_T}. \tag{1.62}$$

The FOC is

$$0 = E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\frac{\Lambda_T}{P_T}\left[(1-\tau_T)\,Y_T\left(\frac{p_t}{P_T}\right)^{-\theta} - \mu^p(1+\omega_y)\,\psi\mu_T^w\tilde{\lambda}_T^{-1}\bar{H}_T^{-\nu}\left(\frac{Y_T}{Z_T}\right)^{1+\omega_y}\left(\frac{p_t}{P_T}\right)^{-\theta(1+\omega_y)-1}\right], \tag{1.63}$$

where $\mu^p \equiv \theta/(\theta-1)$, $\omega_y \equiv \phi(1+\nu) - 1$ and

$$\Lambda_t = \pi_b\lambda_t^b + (1-\pi_b)\,\lambda_t^s \tag{1.64}$$

We then get

$$\begin{aligned} p_t^{1+\theta\omega_y} &= \frac{E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\Lambda_T\left[\mu^p(1+\omega_y)\,\psi\mu_T^w\tilde{\lambda}_T^{-1}\bar{H}_T^{-\nu}\left(\frac{Y_T}{Z_T}\right)^{1+\omega_y}P_T^{\theta(1+\omega_y)}\right]}{E_t\sum_{T=t}^{\infty}(\alpha\beta)_T^{T-t}\Lambda\left[(1-\tau_T)\,Y_TP_T^{\theta-1}\right]} \\ \Leftrightarrow\quad \left(\frac{p_t}{P_t}\right)^{1+\theta\omega_y} &= \frac{E_t\sum_{T=t}^{\infty}(\alpha\beta)_T^{T-t}\Lambda\left[\mu^p(1+\omega_y)\,\psi\mu_T^w\tilde{\lambda}_T^{-1}\bar{H}_T^{-\nu}\left(\frac{Y_T}{Z_T}\right)^{1+\omega_y}\left(\frac{P_T}{P_t}\right)^{\theta(1+\omega_y)}\right]}{E_t\sum_{T=t}^{\infty}(\alpha\beta)_T^{T-t}\Lambda\left[(1-\tau_T)\,Y_T\left(\frac{P_T}{P_t}\right)^{\theta-1}\right]} \end{aligned}$$

so that

$$\frac{p_t}{P_t} = \left(\frac{K_t}{F_t}\right)^{\frac{1}{1+\omega_y\theta}} \tag{1.65}$$

with

$$K_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \Lambda_T \left[\mu^p (1+\omega_y) \psi \mu_T^w \tilde{\lambda}_T^{-1} \bar{H}_T^{-\nu} \left(\frac{Y_T}{Z_T}\right)^{1+\omega_y} \left(\frac{P_T}{P_t}\right)^{\theta(1+\omega_y)}\right], \quad (1.66)$$

$$F_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \Lambda_T \left[(1-\tau_T) Y_T \left(\frac{P_T}{P_t}\right)^{\theta-1}\right]. \quad (1.67)$$

Further write it in recursive form:

$$K_t = \Lambda_t \mu^p (1+\omega_y) \psi \mu_t^w \tilde{\lambda}_t^{-1} \bar{H}_t^{-\nu} \left(\frac{Y_t}{Z_t}\right)^{1+\omega_y} + \alpha\beta E_t \left[\Pi_{t+1}^{\theta(1+\omega_y)} K_{t+1}\right] \quad (1.68)$$

$$F_t = \Lambda_t (1-\tau_t) Y_t + \alpha\beta E_t \left[\Pi_{t+1}^{\theta-1} F_{t+1}\right] \quad (1.69)$$

where the law of motion of prices is

$$1 = \alpha \left(\Pi_t^{-1}\right)^{1-\theta} + (1-\alpha) \left(\frac{K_t}{F_t}\right)^{\frac{1-\theta}{1+\omega_y\theta}},$$

or, equivalently,

$$\frac{1-\alpha\Pi_t^{\theta-1}}{1-\alpha} = \left(\frac{F_t}{K_t}\right)^{\frac{\theta-1}{1+\omega_y\theta}}. \quad (1.70)$$

1.4 Income distribution

The total real wage bill is

$$\begin{aligned} \int w_t(j) H_t(j)\, dj &= \int \frac{\mu_t^w \psi}{\tilde{\lambda}_t} \left(\frac{H_t(j)}{\bar{H}_t}\right)^{\nu} H_t(j)\, dj \\ &= \frac{\mu_t^w \psi}{\tilde{\lambda}_t} \bar{H}_t^{-\nu} \int H_t(j)^{1+\nu}\, dj. \end{aligned}$$

Using (1.58) and (1.60) we get

$$\int H_t(j)^{1+\nu}\, dj = \left(\frac{Y_t}{Z_t}\right)^{\phi(1+\nu)} \int \left(\frac{p_t(j)}{P_t}\right)^{-\theta\phi(1+\nu)} dj \quad (1.71)$$

hence

$$\int w_t(j) H_t(j)\, dj = \frac{\psi \mu_t^w}{\tilde{\lambda}_t} \bar{H}_t^{-\nu} \left(\frac{Y_t}{Z_t}\right)^{1+\omega_y} \Delta_t, \quad (1.72)$$

with Δ_t, a measure of price dispersion, defined as

$$\Delta_t \equiv \int_0^1 \left(\frac{p_t(j)}{P_t}\right)^{-\theta(1+\omega_y)} dj, \quad (1.73)$$

and its law of motion given by

$$\Delta_t = \alpha \Delta_{t-1} \Pi_t^{\theta(1+\omega_y)} + (1-\alpha) \left(\frac{K_t}{F_t} \right)^{\frac{-\theta(1+\omega_y)}{1+\omega_y \theta}},$$

or, equivalently,

$$\Delta_t = \alpha \Delta_{t-1} \Pi_t^{\theta(1+\omega_y)} + (1-\alpha) \left(\frac{1-\alpha \Pi_t^{\theta-1}}{1-\alpha} \right)^{\frac{\theta(1+\omega_y)}{\theta-1}}. \tag{1.74}$$

Type τ share of each type of labor is

$$\left(\frac{\psi}{\psi_\tau} \frac{\lambda_t^\tau}{\tilde{\lambda}_t} \right)^{\frac{1}{\nu}}.$$

Hence this is also the type τ share of the wage bill. This implies that the wage income differential is

$$w_t^b - w_t^s = \frac{\left(\frac{\lambda_t^b}{\psi_b} \right)^{\frac{1}{\nu}} - \left(\frac{\lambda_t^s}{\psi_s} \right)^{\frac{1}{\nu}}}{\left(\frac{\tilde{\lambda}_t}{\psi} \right)^{\frac{1}{\nu}}} \frac{\psi \mu_t^w}{\tilde{\lambda}_t \bar{H}_t^\nu} \left(\frac{Y_t}{Z_t} \right)^{1+\omega_y} \Delta_t. \tag{1.75}$$

1.5 Financial intermediation

1.5.1 Financial intermediation problem

Suppose that in order to make legitimate loans in real quantity b_t, a bank must also make loans in real quantity $\chi_t(b_t)$ to fraudulent borrowers. We assume that $\chi_t'(b_t) \geq 0$, $\chi_t''(b_t) \geq 0$, $\chi_t(0) = 0$. ($\chi_t(b)$ may be strictly convex, because of reduced accuracy of screening the larger the volume of lending relative to the banks capacity, e.g. the available time of its managers.) The bank cannot tell the legitimate borrowers and fraudulent borrowers apart, and so must treat them equally. However, the bank is able to predict the fraction of its loans that will turn out to be fraudulent, and so correctly predicts that loan repayments in period $t+1$ will total only $P_t b_t \left(1 + i_t^b\right)$, even though the loans extended had value $P_t \left[b_t + \chi_t(b_t)\right]$.

The opportunity to make a fraudulent loan contract is assumed to arrive randomly to all households with equal probability, regardless of the households current type. Thus each household has additional real income each period equal to $\chi_t(b_t)$, its earnings from fraud. Each household also chooses how many legitimate loan contracts to enter into, understanding that these loans must be repaid; only type b households choose to enter legitimate loan contracts in equilibrium.

A bank also has real resource costs of loan origination of an amount $\Xi_t(b_t)$ in period t (costs paid in the period when loans are originated). We assume that $\Xi_t'(b_t) \geq 0$, $\Xi_t''(b_t) \geq 0$, $\Xi_t(0) = 0$. (Strict convexity of $\Xi_t(b)$ would indicate increasing costs owing to a capacity constraint, e.g. the scarcity of available managerial time.)

A bank collects deposits d_t in the largets quantity that can be repaid from the proceeds of its loans (anticipating that some fraction of those will prove to be fraudulent). Any excess funds received from depositors that are not lent out or used to pay the resource costs of loan origination are distributed immediately to shareholders. Thus real distributions in period t equal

$$\Pi_t^{int} = d_t - b_t - \chi_t (b_t) - \Xi_t (b_t) . \tag{1.76}$$

Since deposits d_t satisfy $\left(1 + i_t^d\right) d_t = \left(1 + i_t^b\right) b_t$, $d_t = (1 + \omega_t) b_t$, and real distributions by intermediaries equal

$$\Pi_t^{int} = \omega_t b_t - \chi_t (b_t) - \Xi_t (b_t) . \tag{1.77}$$

The income flow to households (D_t^{int}) should also include households' earnings from fraud; hence

$$D_t^{int} = P_t \left[\omega_t b_t - \Xi_t (b_t)\right] . \tag{1.78}$$

Competitive loan pricing: a bank that can lent at a spread ω_t will choose b_t to maximize Π_t^{int}, leading to the following F.O.C.

$$\omega_t - \chi_t' (b_t) - \Xi_t' (b_t) = 0. \tag{1.79}$$

hence in equilibrium, competition between banks leads to an equilibrium credit spread

$$\omega_t = \omega_t (b_t) \equiv \chi_t' (b_t) + \Xi_t' (b_t) . \tag{1.80}$$

Thus $\chi_t (b_t)$ plays the role of a "markup" factor that can cause credit spreads in excess of the marginal resource cost of loan origination.

Further assume that

$$\chi_t (b_t) \equiv \tilde{\chi}_t b_t^{1+\varkappa}, \tag{1.81}$$

$$\Xi_t (b_t) \equiv \tilde{\Xi}_t b_t^{\eta}, \tag{1.82}$$

with $\varkappa \geq 0$ and $\eta \geq 1$. Under this parametrization the spread is

$$\omega_t = (1 + \varkappa) \tilde{\chi}_t b_t^{\varkappa} + \eta \tilde{\Xi}_t b_t^{\eta - 1} \tag{1.83}$$

1.5.2 Alternative interpretations of financial intermediation costs

We can interpret the cost of financial intermediation in one of two ways:

- a quantity of the composite produced good that is used in the activity of banking
- a quantity of a distinct type of labor that happens to be a perfect substitute for consumption in the utility of households

In the first case we simply consider:

- $\Xi_t (b_t)$: quantity of the composite produced good that is used in the activity of banking

- $c_t(i)$: real consumption
- Utility defined in terms of real consumption, $c_t(i)$
- Aggregate expenditure:
$$Y_t = G_t + \int c_t(i)\,di + \Xi_t(b_t)$$

In the second case we consider:

- $\Xi_t(b_t)$: quantity of distinctive labor used in financial intermediation
- $c_t(i)$: consumption purchases net of work for intermediary
- $\tilde{c}_t(i)$: gross consumption purchases
- Utility defined in terms of $c_t(i) \equiv \tilde{c}_t(i) - \Xi_t(b_t)$
- Aggregate expenditure:

$$\begin{aligned} Y_t &= G_t + \int \tilde{c}_t(i)\,di \\ &= G_t + \int c_t(i)\,di + \Xi_t(b_t) \end{aligned}$$

In both cases the profit function for the financial intermediaries is exactly the same.

1.5.3 Dynamics of private debt

Combining law of motion of debt stated in equation (1.48) with the wage income differential (1.75) allows us to write

$$\begin{aligned} (1+\pi_b\omega_t)\,b_t &= \pi_b(1-\pi_b)\,B\left(\lambda_t^b, \lambda_t^s, Y_t, \Delta_t; \xi_t\right) - \pi_b b_t^g \\ &\quad + \delta\left[b_{t-1}(1+\omega_{t-1}) + \pi_b b_{t-1}^g\right]\frac{1+i_{t-1}^d}{\Pi_t} \end{aligned} \tag{1.84}$$

with

$$\begin{aligned} B\left(\lambda_t^b, \lambda_t^s, Y_t, \Delta_t; \xi_t\right) &\equiv \bar{C}_t^b\left(\lambda_t^b\right)^{-\sigma_b} - \bar{C}_t^s\left(\lambda_t^s\right)^{-\sigma_s} \\ &\quad - \left[\left(\frac{\lambda_t^b}{\psi_b}\right)^{\frac{1}{\nu}} - \left(\frac{\lambda_t^s}{\psi_s}\right)^{\frac{1}{\nu}}\right]\left(\frac{\tilde{\lambda}_t}{\psi}\right)^{-\frac{1+\nu}{\nu}} \mu_t^w \bar{H}_t^{-\nu}\left(\frac{Y_t}{Z_t}\right)^{1+\omega_y}\Delta_t \end{aligned} \tag{1.85}$$

1.6 Welfare function

When considering optimal policy we assume that the central planner maximizes the average household welfare, where each individual's utility is

$$U_t^{\tau_t(i)}(i) = u^{\tau_t(i)}(c_t(i);\xi_t) - \int_0^1 v^{\tau_t(i)}(h_t(j;i);\xi_t)\,dj$$

Using the labor market equilibrium we get

$$h_t(j;i) = \left(\frac{\lambda_t^{\tau_t(i)}}{\psi_{\tau_t(i)}}\frac{\psi}{\tilde{\lambda}_t}\right)^{1/\nu} H_t(j) \tag{1.86}$$

hence

$$\begin{aligned}
\int_0^1 v^{\tau_t(i)}(h_t(j;i);\xi_t)\,dj &= \frac{\pi_b\psi_b\left(\frac{\lambda_t^b}{\psi_b}\right)^{\frac{1+\nu}{\nu}} + (1-\pi_b)\,\psi_s\left(\frac{\lambda_t^s}{\psi_s}\right)^{\frac{1+\nu}{\nu}}}{1+\nu}\left(\frac{\psi}{\tilde{\lambda}_t}\right)^{\frac{1+\nu}{\nu}}\bar{H}_t^{-\nu}\int_0^1 H_t(j)^{1+\nu}\,dj \\
&= \frac{\psi}{1+\nu}\left(\frac{\tilde{\Lambda}_t}{\tilde{\lambda}_t}\right)^{\frac{1+\nu}{\nu}}\bar{H}_t^{-\nu}\left(\frac{Y_t}{Z_t}\right)^{1+\omega_y}\Delta_t
\end{aligned}$$

and the welfare function is

$$\begin{aligned}
\tilde{U}_t &= \pi_b\frac{\left(c_t^b\right)^{1-\sigma_b^{-1}}\left(\bar{C}_t^b\right)^{\sigma_b^{-1}}}{1-\sigma_b^{-1}} + (1-\pi_b)\frac{\left(c_t^s\right)^{1-\sigma_s^{-1}}\left(\bar{C}_t^s\right)^{\sigma_s^{-1}}}{1-\sigma_s^{-1}} \\
&\quad -\frac{\psi}{1+\nu}\left(\frac{\tilde{\lambda}_t}{\tilde{\Lambda}_t}\right)^{-\frac{1+\nu}{\nu}}\bar{H}_t^{-\nu}\left(\frac{Y_t}{Z_t}\right)^{1+\omega_y}\Delta_t
\end{aligned} \tag{1.87}$$

with

$$\tilde{\Lambda}_t^{\frac{1+\nu}{\nu}} \equiv \psi^{\frac{1}{\nu}}\left[\pi_b\psi_b^{-\frac{1}{\nu}}\left(\lambda_t^b\right)^{\frac{1+\nu}{\nu}} + (1-\pi_b)\,\psi_s^{-\frac{1}{\nu}}\left(\lambda_t^s\right)^{\frac{1+\nu}{\nu}}\right] \tag{1.88}$$

1.7 All equations

The objective:

$$\tilde{U}_t = \pi_b\frac{\left(\lambda_t^b\right)^{1-\sigma_b}\bar{C}_t^b}{1-\sigma_b^{-1}} + (1-\pi_b)\frac{\left(\lambda_t^s\right)^{1-\sigma_s}\bar{C}_t^s}{1-\sigma_s^{-1}} - \frac{\psi}{1+\nu}\left(\frac{\tilde{\lambda}_t}{\tilde{\Lambda}_t}\right)^{-\frac{1+\nu}{\nu}}\bar{H}_t^{-\nu}\left(\frac{Y_t}{Z_t}\right)^{1+\omega_y}\Delta_t \tag{1.89}$$

The equations describing the economy are summarized below:

$$0 = \left(1+i_t^d\right)(1+\omega_t)\,\beta E_t\left[\left[\delta+(1-\delta)\,\pi_b\right]\frac{\lambda_{t+1}^b}{\Pi_{t+1}} + (1-\delta)(1-\pi_b)\frac{\lambda_{t+1}^s}{\Pi_{t+1}}\right] - \lambda_t^b \tag{1.90}$$

$$0 = \left(1+i_t^d\right)\beta E_t\left[(1-\delta)\,\pi_b\frac{\lambda_{t+1}^b}{\Pi_{t+1}} + \left[\delta + (1-\delta)(1-\pi_b)\right]\frac{\lambda_{t+1}^s}{\Pi_{t+1}}\right] - \lambda_t^s \tag{1.91}$$

$$0 = \Lambda\left(\lambda_t^b,\lambda_t^s\right)\mu^p\left(1+\omega_y\right)\psi\mu_t^w\tilde{\lambda}\left(\lambda_t^b,\lambda_t^s\right)^{-1}\bar{H}_t^{-\nu}\left(\frac{Y_t}{Z_t}\right)^{1+\omega_y} + \alpha\beta E_t\left[\Pi_{t+1}^{\theta(1+\omega_y)}K_{t+1}\right] - K_t \tag{1.92}$$

$$0 = \Lambda\left(\lambda_t^b,\lambda_t^s\right)(1-\tau_t)\,Y_t + \alpha\beta E_t\left[\Pi_{t+1}^{\theta-1}F_{t+1}\right] - F_t \tag{1.93}$$

$$\begin{aligned} 0 \;=\; & \pi_b(1-\pi_b)\,B\left(\lambda_t^b,\lambda_t^s,Y_t,\Delta_t;\xi_t\right) - \pi_b b_t^g \\ & +\delta\left[b_{t-1}(1+\omega_{t-1}) + \pi_b b_{t-1}^g\right]\frac{1+i_{t-1}^d}{\Pi_t} - (1+\pi_b\omega_t)\,b_t \end{aligned} \tag{1.94}$$

$$0 = \pi_b\bar{C}_t^b\left(\lambda_t^b\right)^{-\sigma_b} + (1-\pi_b)\,\bar{C}_t^s\left(\lambda_t^s\right)^{-\sigma_s} + \tilde{\Xi}_t b_t^\eta + G_t - Y_t \tag{1.95}$$

$$0 = \alpha\Delta_{t-1}\Pi_t^{\theta(1+\omega_y)} + (1-\alpha)\left(\frac{1-\alpha\Pi_t^{\theta-1}}{1-\alpha}\right)^{\frac{\theta(1+\omega_y)}{\theta-1}} - \Delta_t \tag{1.96}$$

$$0 = \frac{1-\alpha\Pi_t^{\theta-1}}{1-\alpha} - \left(\frac{F_t}{K_t}\right)^{\frac{\theta-1}{1+\omega_y\theta}} \tag{1.97}$$

$$0 = 1 + (1+\varkappa)\,\tilde{\chi}_t b_t^\varkappa + \eta\tilde{\Xi}_t b_t^{\eta-1} - (1+\omega_t) \tag{1.98}$$

Auxiliary:

$$\begin{aligned} B\left(\lambda_t^b,\lambda_t^s,Y_t,\Delta_t;\xi_t\right) \;\equiv\; & \bar{C}_t^b\left(\lambda_t^b\right)^{-\sigma_b} - \bar{C}_t^s\left(\lambda_t^s\right)^{-\sigma_s} \\ & -\left[\left(\frac{\lambda_t^b}{\psi_b}\right)^{\frac{1}{\nu}} - \left(\frac{\lambda_t^s}{\psi_s}\right)^{\frac{1}{\nu}}\right]\left(\frac{\tilde{\lambda}_t}{\psi}\right)^{-\frac{1+\nu}{\nu}}\mu_t^w\bar{H}_t^{-\nu}\left(\frac{Y_t}{Z_t}\right)^{1+\omega_y}\Delta_t \end{aligned} \tag{1.99}$$

$$\Lambda\left(\lambda_t^b,\lambda_t^s\right) \equiv \pi_b\lambda_t^b + (1-\pi_b)\,\lambda_t^s \tag{1.100}$$

$$\tilde{\lambda}\left(\lambda_t^b,\lambda_t^s\right) \equiv \psi\left[\pi_b\left(\frac{\lambda_t^b}{\psi_b}\right)^{\frac{1}{\nu}} + (1-\pi_b)\left(\frac{\lambda_t^s}{\psi_s}\right)^{\frac{1}{\nu}}\right]^\nu \tag{1.101}$$

$$\tilde{\Lambda}\left(\lambda_t^b,\lambda_t^s\right) \equiv \psi^{\frac{1}{1+\nu}}\left[\pi_b\psi_b^{-\frac{1}{\nu}}\left(\lambda_t^b\right)^{\frac{1+\nu}{\nu}} + (1-\pi_b)\,\psi_s^{-\frac{1}{\nu}}\left(\lambda_t^s\right)^{\frac{1+\nu}{\nu}}\right]^{\frac{\nu}{1+\nu}} \tag{1.102}$$

$$c_t^b = \bar{C}_t^b\left(\lambda_t^b\right)^{-\sigma_b} \tag{1.103}$$

$$c_t^s = \bar{C}_t^s\left(\lambda_t^s\right)^{-\sigma_s} \tag{1.104}$$

2 Steady state

2.1 List of equations in steady state

Full list of equations in steady state:

$$0=\left(1+\bar{r}^{d}\right)\left(1+\bar{\omega}\right)\beta\left[\left[\delta+\left(1-\delta\right)\pi_{b}\right]\bar{\lambda}^{b}+\left(1-\delta\right)\left(1-\pi_{b}\right)\bar{\lambda}^{s}\right]-\bar{\lambda}^{b} \tag{2.1}$$

$$0=\left(1+\bar{r}^{d}\right)\beta\left[\left(1-\delta\right)\pi_{b}\bar{\lambda}^{b}+\left[\delta+\left(1-\delta\right)\left(1-\pi_{b}\right)\right]\bar{\lambda}^{s}\right]-\bar{\lambda}^{s} \tag{2.2}$$

$$0=\Lambda\left(\bar{\lambda}^{b},\bar{\lambda}^{s}\right)\mu^{p}\left(1+\omega_{y}\right)\psi\bar{\mu}^{w}\tilde{\lambda}\left(\bar{\lambda}^{b},\bar{\lambda}^{s}\right)^{-1}\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{Z}}\right)^{1+\omega_{y}}-\left[1-\alpha\beta\bar{\Pi}^{\theta(1+\omega_{y})}\right]\bar{K} \tag{2.3}$$

$$0=\Lambda\left(\bar{\lambda}^{b},\bar{\lambda}^{s}\right)\left(1-\bar{\tau}\right)\bar{Y}-\left[1-\alpha\beta\bar{\Pi}^{\theta-1}\right]\bar{F} \tag{2.4}$$

$$\begin{aligned}0 &= \pi_{b}\left(1-\pi_{b}\right)B\left(\bar{\lambda}^{b},\bar{\lambda}^{s},\bar{Y},\bar{\Delta};0\right)-\pi_{b}\bar{b}^{g}\\ &\quad+\delta\left[\bar{b}\left(1+\bar{\omega}\right)+\pi_{b}\bar{b}^{g}\right]\left(1+\bar{r}^{d}\right)-\left(1+\pi_{b}\bar{\omega}\right)\bar{b}\end{aligned} \tag{2.5}$$

$$0=\pi_{b}\bar{C}^{b}\left(\bar{\lambda}^{b}\right)^{-\sigma_{b}}+\left(1-\pi_{b}\right)\bar{C}^{s}\left(\bar{\lambda}^{s}\right)^{-\sigma_{s}}+\tilde{\Xi}\bar{b}^{\eta}+\bar{G}-\bar{Y} \tag{2.6}$$

$$0=\alpha\bar{\Delta}\bar{\Pi}^{\theta(1+\omega_{y})}+\left(1-\alpha\right)\left(\frac{1-\alpha\bar{\Pi}^{\theta-1}}{1-\alpha}\right)^{\frac{\theta\left(1+\omega_{y}\right)}{\theta-1}}-\bar{\Delta} \tag{2.7}$$

$$0=\frac{1-\alpha\bar{\Pi}^{\theta-1}}{1-\alpha}-\left(\frac{\bar{F}}{\bar{K}}\right)^{\frac{\theta-1}{1+\omega_{y}\theta}} \tag{2.8}$$

$$0=1+\left(1+\varkappa\right)\bar{\chi}\bar{b}^{\varkappa}+\eta\bar{\Xi}\bar{b}^{\eta-1}-\left(1+\bar{\omega}\right) \tag{2.9}$$

Auxiliary:

$$\begin{aligned}B\left(\bar{\lambda}^{b},\bar{\lambda}^{s},\bar{Y},\bar{\Delta};0\right) &= \bar{C}^{b}\left(\bar{\lambda}^{b}\right)^{-\sigma_{b}}-\bar{C}^{s}\left(\bar{\lambda}^{s}\right)^{-\sigma_{s}}\\ &\quad-\frac{\left(\frac{\bar{\lambda}^{b}}{\psi_{b}}\right)^{\frac{1}{\nu}}-\left(\frac{\bar{\lambda}^{s}}{\psi_{s}}\right)^{\frac{1}{\nu}}}{\left(\frac{\tilde{\lambda}\left(\bar{\lambda}^{b},\bar{\lambda}^{s}\right)}{\psi}\right)^{\frac{1+\nu}{\nu}}}\bar{\mu}^{w}\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{Z}}\right)^{1+\omega_{y}}\bar{\Delta}\end{aligned} \tag{2.10}$$

$$\Lambda\left(\bar{\lambda}^{b},\bar{\lambda}^{s}\right)=\pi_{b}\bar{\lambda}^{b}+\left(1-\pi_{b}\right)\bar{\lambda}^{s} \tag{2.11}$$

$$\tilde{\lambda}\left(\bar{\lambda}^{b},\bar{\lambda}^{s}\right)=\psi\left[\pi_{b}\left(\frac{\bar{\lambda}^{b}}{\psi_{b}}\right)^{\frac{1}{\nu}}+\left(1-\pi_{b}\right)\left(\frac{\bar{\lambda}^{s}}{\psi_{s}}\right)^{\frac{1}{\nu}}\right]^{\nu} \tag{2.12}$$

$$\tilde{\Lambda}\left(\bar{\lambda}^b, \bar{\lambda}^s\right) = \psi^{\frac{1}{1+\nu}} \left[\pi_b \psi_b^{-\frac{1}{\nu}} \left(\bar{\lambda}^b\right)^{\frac{1+\nu}{\nu}} + (1-\pi_b)\, \psi_s^{-\frac{1}{\nu}} \left(\bar{\lambda}^s\right)^{\frac{1+\nu}{\nu}}\right]^{\frac{\nu}{1+\nu}} \tag{2.13}$$

$$\bar{c}^b = \bar{C}^b \left(\bar{\lambda}^b\right)^{-\sigma_b} \tag{2.14}$$

$$\bar{c}^s = \bar{C}_t^s \left(\bar{\lambda}^s\right)^{-\sigma_s} \tag{2.15}$$

2.2 Zero inflation steady state

We consider first the solution to steady state in which we simply assume zero inflation. For simplification of the analysis consider the following definitions

$$\begin{aligned}
s_c &\equiv \pi_b s_b + (1-\pi_b)\, s_s, \\
s_b &\equiv \bar{c}^b / \bar{Y}, \\
s_s &\equiv \bar{c}^s / \bar{Y}, \\
s_c^{bs} &\equiv s_b / s_s, \\
\bar{\sigma} &\equiv \pi_b s_b \sigma_b + (1-\pi_b)\, s_s \sigma_s, \\
\sigma_{bs} &\equiv \sigma_b / \sigma_s, \\
\rho_b &\equiv \bar{b} / \bar{Y}, \\
s_\Xi &\equiv \bar{\Xi}\left(\bar{b}\right) / \bar{Y}, \\
\rho_b^g &\equiv \bar{b}^g / \bar{Y}, \\
s_g &\equiv \bar{G} / \bar{Y}, \\
\psi_{bs} &\equiv \psi_b / \psi_s.
\end{aligned}$$

Without any loss of generality we calibrate the following values:

$$\bar{Y} = 1,$$

$$\psi = 1.$$

The values of s_c, s_b/s_s and σ_b/σ_s are set according to the calibration described in the paper.

For the interest rate we have:

$$1 + \bar{r}^d = \beta^{-1} \frac{(\delta+1) + \bar{\omega}\left[\delta + (1-\delta)\,\pi_b\right] - \sqrt{\left\{(\delta+1) + \bar{\omega}\left[\delta + (1-\delta)\,\pi_b\right]\right\}^2 - 4\delta\,(1+\bar{\omega})}}{2\delta\,(1+\bar{\omega})}. \tag{2.16}$$

(Note that if $\bar{\omega} = 0$, this reduces to $1 + \bar{r}^d = \beta^{-1}$.) We use this steady-state relation to calibrate β, given assumed values for δ, π_b, $\bar{\omega}$ and $\bar{r}^d$.

We can also write

$$1 + \bar{\imath}^d = 1 + \bar{r}^d. \tag{2.17}$$

The markup is calibrated so that $\bar{\chi}$ and $\bar{\Xi}$ insure that the equation defining $\bar{\omega}$ is satisfied. We consider four cases:

- Exogenous, takes resources: set $\eta = 1$, $\varkappa = 0$ and $\bar{\bar{\chi}} = 0$, implying that $\bar{\bar{\Xi}} = \bar{\omega}$ and $s_\Xi = \bar{\omega}\rho_b$.
- Exogenous, no resources: $\eta = 1$, $\varkappa = 0$ and $\bar{\bar{\Xi}} = 0$, implying that $\bar{\bar{\chi}} = \bar{\omega}$ and $s_\Xi = 0$.
- Endogenous, takes resources: set $\eta > 1$, $\varkappa > 0$ and $\bar{\chi} = 0$, implying that $\bar{\Xi}'\left(\bar{b}\right) = \bar{\omega}$, $s_\Xi = \frac{\bar{\omega}}{\eta}\rho_b$ and $\bar{\bar{\Xi}} = \bar{\omega}/\left(\eta\bar{b}^{\eta-1}\right)$.
- Endogenous, no resources: set $\varkappa > 0$ and $\bar{\Xi} = 0$, implying that $\bar{\chi}'\left(\bar{b}\right) = \bar{\omega}$, $s_\Xi = 0$ and $\bar{\bar{\chi}} = \bar{\omega}/\left((1+\varkappa)\,\bar{b}^{\varkappa}\right)$.

Furthermore we can write, from one of the Euler equations:

$$\bar{\lambda}^b = \bar{\Omega}\bar{\lambda}^s, \tag{2.18}$$

where

$$\bar{\Omega} \equiv \frac{1 - \left(1+\bar{r}^d\right)\beta\left[\delta + (1-\delta)(1-\pi_b)\right]}{\left(1+\bar{r}^d\right)\beta(1-\delta)\pi_b}. \tag{2.19}$$

Given the assumption that we calibrate ψ_{bs} and ψ, we can then write

$$\psi_s = \psi\left[\pi_b\psi_{bs}^{-\frac{1}{\nu}} + (1-\pi_b)\right]^\nu, \tag{2.20}$$

and $\psi_b = \psi_{bs}\psi_s$.

This implies that, given ψ_b and ψ_s, we get

$$\Lambda\left(\bar{\lambda}^b, \bar{\lambda}^s\right) = \left[\pi_b\bar{\Omega} + (1-\pi_b)\right]\bar{\lambda}^s, \tag{2.21}$$

$$\tilde{\lambda}\left(\bar{\lambda}^b, \bar{\lambda}^s\right) = \psi\left[\pi_b\bar{\Omega}^{\frac{1}{\nu}}\psi_b^{-\frac{1}{\nu}} + (1-\pi_b)\,\psi_s^{-\frac{1}{\nu}}\right]^\nu\bar{\lambda}^s, \tag{2.22}$$

$$\tilde{\Lambda}\left(\bar{\lambda}^b, \bar{\lambda}^s\right) = \psi^{\frac{1}{1+\nu}}\left[\pi_b\psi_b^{-\frac{1}{\nu}}\bar{\Omega}^{\frac{1+\nu}{\nu}} + (1-\pi_b)\,\psi_s^{-\frac{1}{\nu}}\right]^{\frac{\nu}{1+\nu}}\bar{\lambda}^s. \tag{2.23}$$

Using $\bar{F} = \bar{K}$,

$$(1-\bar{\tau}) = \mu^p(1+\omega_y)\,\psi\mu_t^w\tilde{\lambda}\left(\bar{\lambda}^b, \bar{\lambda}^s\right)^{-1}\frac{\bar{H}^{-\nu}}{\bar{Z}^{1+\omega_y}},$$

hence

$$\bar{\lambda}^s = \frac{\mu^p(1+\omega_y)\,\mu_t^w\frac{\bar{H}^{-\nu}}{\bar{Z}^{1+\omega_y}}}{(1-\bar{\tau})\left[\pi_b\bar{\Omega}^{\frac{1}{\nu}}\psi_b^{-\frac{1}{\nu}} + (1-\pi_b)\,\psi_s^{-\frac{1}{\nu}}\right]^\nu}. \tag{2.24}$$

Given our calibration of s_c and s_c^{bs}, we can write:

$$s_s = \frac{s_c}{\pi_b s_c^{bs} + (1-\pi_b)}, \tag{2.25}$$

and $s_b = s_c^{bs} s_c$.

The resources constraint implies

$$1 - s_c - s_g = \frac{\bar{\omega}}{\eta}\rho_b. \tag{2.26}$$

Zero inflation implies that

$$\bar{\Delta} = 1. \tag{2.27}$$

The debt equation is

$$\left[1 + \pi_b\bar{\omega} - \delta\left(1+\bar{\omega}\right)\left(1+\bar{r}^d\right)\right]\rho_b = \pi_b\left(1-\pi_b\right)\frac{B\left(\bar{\lambda}^b, \bar{\lambda}^s, \bar{Y}, \bar{\Delta}; 0\right)}{\bar{Y}} - \pi_b\rho_b^g\left[1-\delta\left(1+\bar{r}^d\right)\right],$$

with

$$\frac{B\left(\bar{\lambda}^b, \bar{\lambda}^s, 1, 1; 0\right)}{\bar{Y}} = s_b - s_s - \frac{\bar{\Omega}^{\frac{1}{\nu}}\psi_b^{-\frac{1}{\nu}} - \psi_s^{-\frac{1}{\nu}}}{\pi_b\bar{\Omega}^{\frac{1}{\nu}}\psi_b^{-\frac{1}{\nu}} + \left(1-\pi_b\right)\psi_s^{-\frac{1}{\nu}}}\frac{1-\bar{\tau}}{\mu^p\left(1+\omega_y\right)},$$

implying that

$$\rho_b = \frac{\pi_b\left(1-\pi_b\right)\left(s_b - s_s - \frac{\bar{\Omega}^{\frac{1}{\nu}}\psi_b^{-\frac{1}{\nu}} - \psi_s^{-\frac{1}{\nu}}}{\pi_b\bar{\Omega}^{\frac{1}{\nu}}\psi_b^{-\frac{1}{\nu}} + \left(1-\pi_b\right)\psi_s^{-\frac{1}{\nu}}}\frac{1-\bar{\tau}}{\mu^p\left(1+\omega_y\right)}\right) - \pi_b\rho_b^g\left[1-\delta\left(1+\bar{r}^d\right)\right]}{1+\pi_b\bar{\omega} - \delta\left(1+\bar{\omega}\right)\left(1+\bar{r}^d\right)}, \tag{2.28}$$

which is then used to solve for real debt according to $\bar{b} = \rho_b\bar{Y}$. Given $\bar{b}$ and s_c, the real resources equation (2.26) determines $s_g = 1 - s_c - \frac{\bar{\omega}}{\eta}\rho_b$.

Furthermore, we set $\bar{\sigma}$, hence

$$\sigma_s = \frac{\bar{\sigma}}{\pi_b s_b \sigma_{bs} + \left(1-\pi_b\right)s_s}, \tag{2.29}$$

$$\sigma_b = \sigma_{bs}\sigma_s. \tag{2.30}$$

Finally,

$$\bar{C}^b = s_b\left(\bar{\lambda}^b\right)^{\sigma_b}, \tag{2.31}$$

$$\bar{C}^s = s_b\left(\bar{\lambda}^s\right)^{\sigma_s}, \tag{2.32}$$

$$\bar{K} = \frac{\Lambda\left(\bar{\lambda}^b, \bar{\lambda}^s\right)\mu^p\left(1+\omega_y\right)\psi\mu^w\tilde{\lambda}\left(\bar{\lambda}^b, \bar{\lambda}^s\right)^{-1}\frac{\bar{H}^{-\nu}}{\bar{Z}^{1+\omega_y}}}{1-\alpha\beta}, \tag{2.33}$$

$$\bar{F} = \frac{\Lambda\left(\bar{\lambda}^b, \bar{\lambda}^s\right)\left(1-\bar{\tau}\right)}{1-\alpha\beta}. \tag{2.34}$$

We further set ψ_b/ψ_s such that the labor supply is the same in steady state, which implies that

$$\frac{\bar{\lambda}^b}{\psi_b} = \frac{\bar{\lambda}^s}{\psi_s} \Leftrightarrow \frac{\bar{\lambda}^b}{\bar{\lambda}^s} = \frac{\psi_b}{\psi_s} \Rightarrow \frac{\psi_b}{\psi_s} = \bar{\Omega}, \tag{2.35}$$

hence

$$\psi_s = \left[\pi_b \bar{\Omega}^{-\frac{1}{\nu}} + (1-\pi_b)\right]^{\nu}, \tag{2.36}$$

$$\psi_b = \bar{\Omega}\psi_s, \tag{2.37}$$

$$\bar{\lambda}^s = \frac{\mu^p (1+\omega_y)\,\mu^w \frac{\bar{H}^{-\nu}}{\bar{Z}^{1+\omega_y}}}{(1-\bar{\tau})\left[\pi_b \bar{\Omega}^{\frac{1}{\nu}} \psi_b^{-\frac{1}{\nu}} + (1-\pi_b)\,\psi_s^{-\frac{1}{\nu}}\right]^{\nu}}, \tag{2.38}$$

$$\bar{\lambda}^b = \bar{\Omega}\bar{\lambda}^s, \tag{2.39}$$

$$\Lambda\left(\bar{\lambda}^b, \bar{\lambda}^s\right) = \pi_b \bar{\lambda}^b + (1-\pi_b)\,\bar{\lambda}^s. \tag{2.40}$$

2.3 Optimal steady state

The central planner maximizes the social welfare function (1.89) subject to the laws of motion of the economy given by (1.90)-(1.98). Assign Lagrangian multipliers φ_1, ..., φ_9 to those equations respectively. Then the F.O.C. of the central planner in steady state are the following ones.

2.3.1 FOC

FOC w.r.t. i_t^d

$$0 = \bar{\varphi}_1 \bar{\lambda}^b + \bar{\varphi}_2 \bar{\lambda}^s + \bar{\varphi}_5 \delta\beta \left[\bar{b}\,(1+\bar{\omega}) + \pi_b \bar{b}^g\right]\left(1+\bar{r}^d\right). \tag{2.41}$$

FOC w.r.t. Π_t

$$\begin{aligned} 0 \;=\; & -\beta^{-1}\bar{\varphi}_1\bar{\lambda}^b - \beta^{-1}\bar{\varphi}_2\bar{\lambda}^s - \bar{\varphi}_5\delta\left[\bar{b}\,(1+\bar{\omega}) + \pi_b\bar{b}^g\right]\left(1+\bar{r}^d\right) \\ & +\bar{\varphi}_3\alpha\theta\,(1+\omega_y)\,\bar{\Pi}^{\theta(1+\omega_y)}\bar{K} + \bar{\varphi}_4\alpha\,(\theta-1)\,\bar{\Pi}^{\theta-1}\bar{F} \\ & +\bar{\varphi}_7\theta\,(1+\omega_y)\,\alpha\bar{\Delta}\bar{\Pi}^{\theta(1+\omega_y)} - \bar{\varphi}_7\theta\,(1+\omega_y)\,\alpha\bar{\Pi}^{\theta-1}\left(\frac{1-\alpha\bar{\Pi}^{\theta-1}}{1-\alpha}\right)^{\frac{\theta(1+\omega_y)}{\theta-1}-1} \\ & -\bar{\varphi}_8\frac{\alpha\,(\theta-1)}{1-\alpha}\bar{\Pi}^{\theta-1}. \end{aligned} \tag{2.42}$$

FOC w.r.t. Δ_t

$$\begin{aligned} 0 &= \frac{\psi}{1+\nu}\left(\frac{\tilde{\lambda}}{\bar{\tilde{\Lambda}}}\right)^{-\frac{1+\nu}{\nu}}\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y} \\ &+\bar{\varphi}_5\pi_b\left(1-\pi_b\right)\left[\left(\frac{\bar{\lambda}^b}{\psi_b}\right)^{\frac{1}{\nu}}-\left(\frac{\bar{\lambda}^s}{\psi_s}\right)^{\frac{1}{\nu}}\right]\left(\frac{\tilde{\lambda}}{\psi}\right)^{-\frac{1+\nu}{\nu}}\mu^w\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y} \\ &+\bar{\varphi}_7\left(1-\alpha\beta\bar{\Pi}^{\theta(1+\omega_y)}\right). \end{aligned} \tag{2.43}$$

FOC w.r.t. K_t

$$0=-\bar{\varphi}_3\left(1-\alpha\bar{\Pi}^{\theta(1+\omega_y)}\right)+\bar{\varphi}_8\frac{\theta-1}{1+\omega_y\theta}\left(\frac{1-\alpha\bar{\Pi}^{\theta-1}}{1-\alpha}\right)\bar{K}^{-1}. \tag{2.44}$$

FOC w.r.t. F_t

$$0=-\bar{\varphi}_4\left(1-\alpha\bar{\Pi}^{\theta-1}\right)-\bar{\varphi}_8\frac{\theta-1}{1+\omega_y\theta}\frac{1-\alpha\bar{\Pi}^{\theta-1}}{1-\alpha}\bar{F}^{-1} \tag{2.45}$$

FOC w.r.t. Y_t

$$\begin{aligned} 0 &= -\left(1+\omega_y\right)\frac{\psi}{1+\nu}\left(\frac{\tilde{\lambda}}{\bar{\tilde{\Lambda}}}\right)^{-\frac{1+\nu}{\nu}}\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y}\bar{\Delta}\bar{Y}^{-1} \\ &+\bar{\varphi}_4\Lambda\left(\bar{\lambda}^b,\bar{\lambda}^s\right)\left(1-\bar{\tau}\right)-\bar{\varphi}_6 \\ &+\bar{\varphi}_3\left(1+\omega_y\right)\frac{\Lambda\left(\bar{\lambda}^b,\bar{\lambda}^s\right)}{\tilde{\lambda}\left(\bar{\lambda}^b,\bar{\lambda}^s\right)}\mu^p\left(1+\omega_y\right)\psi\mu^w\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y}\bar{Y}^{-1} \\ &-\bar{\varphi}_5\pi_b\left(1+\omega_y\right)\left(1-\pi_b\right)\left[\left(\frac{\bar{\lambda}^b}{\psi_b}\right)^{\frac{1}{\nu}}-\left(\frac{\bar{\lambda}^s}{\psi_s}\right)^{\frac{1}{\nu}}\right]\left(\frac{\tilde{\lambda}}{\psi}\right)^{-\frac{1+\nu}{\nu}}\mu^w\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y}\bar{\Delta}\bar{Y}^{-1}. \end{aligned} \tag{2.46}$$

FOC w.r.t. λ_t^b

$$
\begin{aligned}
0 \;=\;& \pi_b \frac{1-\sigma_b}{1-\sigma_b^{-1}} \left(\bar{\lambda}^b\right)^{-\sigma_b} \bar{C}^b \qquad (2.47)\\
&+\pi_b \frac{\psi}{\nu} \left(\frac{\psi}{\psi_b} \frac{\bar{\lambda}^b}{\tilde{\lambda}}\right)^{\frac{1}{\nu}} \left(\bar{\lambda}^b\right)^{-1} \left[\left(\frac{\tilde{\lambda}}{\tilde{\Lambda}}\right)^{-\frac{1+\nu}{\nu}} - \frac{\bar{\lambda}^b}{\tilde{\lambda}}\right] \bar{H}^{-\nu} \left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y} \bar{\Delta}\\
&-\bar{\varphi}_1 + \bar{\varphi}_1 \left(1+\bar{r}^d\right)(1+\bar{\omega})\left[\delta + (1-\delta)\pi_b\right] + \bar{\varphi}_2 \left(1+\bar{r}^d\right)(1-\delta)\pi_b\\
&+\bar{\varphi}_3 \mu^p (1+\omega_y)\psi \mu^w \pi_b \left[\tilde{\lambda}^{-1} - \left(\frac{\psi}{\psi_s}\frac{\bar{\lambda}^s}{\tilde{\lambda}}\right)^{\frac{1}{\nu}} \left(\bar{\lambda}^s\right)^{-1} \frac{\Lambda}{\tilde{\lambda}}\right] \bar{H}^{-\nu} \left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y}\\
&+\bar{\varphi}_4 (1-\tau)\bar{Y}\pi_b - \bar{\varphi}_5 \pi_b (1-\pi_b)\sigma_b \bar{C}^b \left(\bar{\lambda}^b\right)^{-\sigma_b - 1} - \bar{\varphi}_6 \sigma_b \pi_b \bar{C}^b \left(\bar{\lambda}^b\right)^{-\sigma_b-1}\\
&+\bar{\varphi}_5 \pi_b (1-\pi_b) \left(\frac{\tilde{\lambda}}{\psi}\right)^{-1} \left(\frac{\psi}{\psi_b}\frac{\bar{\lambda}^b}{\tilde{\lambda}}\right)^{\frac{1}{\nu}} \left(\bar{\lambda}^b\right)^{-1} \left[1 - \frac{1+\nu}{\nu}\left(\frac{\psi}{\psi_s}\frac{\bar{\lambda}^s}{\tilde{\lambda}}\right)^{\frac{1}{\nu}}\right] \mu^w \bar{H}^{-\nu} \left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y} \bar{\Delta}.
\end{aligned}
$$

FOC w.r.t. λ_t^s

$$
\begin{aligned}
0 \;=\;& (1-\pi_b) \frac{1-\sigma_s}{1-\sigma_s^{-1}} \left(\bar{\lambda}^s\right)^{-\sigma_s} \bar{C}^s \qquad (2.48)\\
&+(1-\pi_b) \frac{\psi}{\nu} \left(\frac{\psi}{\psi_s}\frac{\bar{\lambda}^s}{\tilde{\lambda}}\right)^{\frac{1}{\nu}} \left[\left(\frac{\tilde{\lambda}}{\tilde{\Lambda}}\right)^{-\frac{1+\nu}{\nu}} - \frac{\bar{\lambda}^s}{\tilde{\lambda}}\right] \left(\bar{\lambda}^s\right)^{-1} \bar{H}^{-\nu} \left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y} \bar{\Delta}\\
&+\bar{\varphi}_1 \left(1+\bar{r}^d\right)(1+\bar{\omega})(1-\delta)(1-\pi_b) - \bar{\varphi}_2 + \bar{\varphi}_2 \left(1+\bar{r}^d\right)(1-\delta)\pi_b\\
&+\bar{\varphi}_3 \mu^p (1+\omega_y)\psi\mu^w (1-\pi_b) \left[\tilde{\lambda}^{-1} - \left(\frac{\psi}{\psi_s}\frac{\bar{\lambda}^s}{\tilde{\lambda}}\right)^{\frac{1}{\nu}} \left(\bar{\lambda}^s\right)^{-1} \frac{\Lambda}{\tilde{\lambda}}\right] \bar{H}^{-\nu} \left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y}\\
&+\bar{\varphi}_4 (1-\pi_b)(1-\bar{\tau})\bar{Y}\\
&+\bar{\varphi}_5 \pi_b (1-\pi_b) \left(\frac{\tilde{\lambda}}{\psi}\right)^{-1} \left(\frac{\psi}{\psi_s}\frac{\bar{\lambda}^s}{\tilde{\lambda}}\right)^{\frac{1}{\nu}} \left(\bar{\lambda}^s\right)^{-1} \left[\frac{1+\nu}{\nu}\left(\frac{\psi}{\psi_b}\frac{\bar{\lambda}^b}{\tilde{\lambda}}\right)^{\frac{1}{\nu}} - 1\right] \mu^w \bar{H}^{-\nu} \left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y} \bar{\Delta}\\
&+\bar{\varphi}_5 \sigma_s \pi_b (1-\pi_b) \bar{C}^s \left(\bar{\lambda}^s\right)^{-\sigma_s-1} - \bar{\varphi}_6 (1-\pi_b) \bar{C}^s \left(\bar{\lambda}^s\right)^{-\sigma_s-1}.
\end{aligned}
$$

FOC w.r.t. b_t

$$
0 = -(1+\pi_b\bar{\omega})\bar{\varphi}_5 + \bar{\varphi}_5 \delta\beta \left(1+\bar{r}^d\right)(1+\bar{\omega}) + \bar{\varphi}_6 \Xi'\left(\bar{b}\right) + \bar{\varphi}_9 \left[\bar{\chi}''\left(\bar{b}\right) + \bar{\Xi}''\left(\bar{b}\right)\right]. \qquad (2.49)
$$

FOC w.r.t. ω_t

$$
0 = \bar{\varphi}_1 \frac{\bar{\lambda}^b}{1+\bar{\omega}} + \bar{\varphi}_5 \left[\beta\delta\left(1+\bar{r}^d\right) - \pi_b\right]\bar{b} - \bar{\varphi}_9. \qquad (2.50)
$$

2.3.2 Simplification

We can thus write the FOC as follows.

FOC w.r.t. i_t^d

$$0 = \varpi_{i^d,1}\bar{\varphi}_1 + \varpi_{i^d,2}\bar{\varphi}_2 + \varpi_{i^d,5}\bar{\varphi}_5, \tag{2.51}$$

with

$$\varpi_{i^d,1} \equiv \bar{\lambda}^b,$$
$$\varpi_{i^d,2} \equiv \bar{\lambda}^s,$$
$$\varpi_{i^d,5} \equiv \delta\beta\left[\bar{b}\left(1+\bar{\omega}\right)+\pi_b\bar{b}^g\right]\left(1+\bar{r}^d\right).$$

FOC w.r.t. Π_t

$$0 = \varpi_{\Pi,3}\bar{\varphi}_3 + \varpi_{\Pi,4}\bar{\varphi}_4 + \varpi_{\Pi,7}\bar{\varphi}_7 + \varpi_{\Pi,8}\bar{\varphi}_8, \tag{2.52}$$

with

$$\varpi_{\Pi,3} \equiv \alpha\theta\left(1+\omega_y\right)\bar{\Pi}^{\theta(1+\omega_y)}\bar{K},$$
$$\varpi_{\Pi,4} \equiv \alpha\left(\theta-1\right)\bar{\Pi}^{\theta-1}\bar{F},$$
$$\varpi_{\Pi,7} \equiv \theta\left(1+\omega_y\right)\alpha\left[\bar{\Delta}\bar{\Pi}^{\theta(1+\omega_y)} - \bar{\Pi}^{\theta-1}\left(\frac{1-\alpha\bar{\Pi}^{\theta-1}}{1-\alpha}\right)^{\frac{\theta(1+\omega_y)}{\theta-1}-1}\right],$$
$$\varpi_{\Pi,8} \equiv -\frac{\alpha\left(\theta-1\right)}{1-\alpha}\bar{\Pi}^{\theta-1},$$

where I used the fact that $\varpi_{\Pi,1} = -\beta^{-1}\varpi_{i^d,1}$, $\varpi_{\Pi,2} = -\beta^{-1}\varpi_{i^d,2}$, $\varpi_{\Pi,5} = -\beta^{-1}\varpi_{i^d,5}$ and (2.51).

FOC w.r.t. Δ_t

$$0 = \varpi_{\Delta,0} + \varpi_{\Delta,5}\bar{\varphi}_5 + \varpi_{\Delta,7}\bar{\varphi}_7, \tag{2.53}$$

with

$$\varpi_{\Delta,0} \equiv \frac{\psi}{1+\nu}\left(\frac{\tilde{\lambda}}{\tilde{\Lambda}}\right)^{-\frac{1+\nu}{\nu}}\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{Z}}\right)^{1+\omega_y},$$
$$\varpi_{\Delta,5} \equiv \pi_b\left(1-\pi_b\right)\left[\left(\frac{\bar{\lambda}^b}{\psi_b}\right)^{\frac{1}{\nu}} - \left(\frac{\bar{\lambda}^s}{\psi_s}\right)^{\frac{1}{\nu}}\right]\left(\frac{\tilde{\lambda}}{\psi}\right)^{-\frac{1+\nu}{\nu}}\mu^w\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{Z}}\right)^{1+\omega_y},$$
$$\varpi_{\Delta,7} \equiv 1-\alpha\beta\bar{\Pi}^{\theta(1+\omega_y)}.$$

FOC w.r.t. K_t

$$0 = \varpi_{K,3}\bar{\varphi}_3 + \varpi_{K,8}\bar{\varphi}_8, \tag{2.54}$$

with

$$\varpi_{K,3} \equiv -\left(1-\alpha\bar{\Pi}^{\theta(1+\omega_y)}\right),$$
$$\varpi_{K,8} \equiv \frac{\theta-1}{1+\omega_y\theta}\left(\frac{1-\alpha\bar{\Pi}^{\theta-1}}{1-\alpha}\right)\bar{K}^{-1}.$$

FOC w.r.t. F_t

$$0 = \varpi_{F,4}\bar{\varphi}_4 + \varpi_{F,8}\bar{\varphi}_8, \tag{2.55}$$

with

$$\varpi_{F,4} \equiv -\left(1 - \alpha\bar{\Pi}^{\theta-1}\right),$$

$$\varpi_{F,8} \equiv -\frac{\theta-1}{1+\omega_y\theta}\frac{1-\alpha\bar{\Pi}^{\theta-1}}{1-\alpha}\bar{F}^{-1}.$$

FOC w.r.t. Y_t

$$0 = \varpi_{Y,0} + \varpi_{Y,3}\bar{\varphi}_3 + \varpi_{Y,4}\bar{\varphi}_4 + \varpi_{Y,5}\bar{\varphi}_5 + \varpi_{Y,6}\bar{\varphi}_6, \tag{2.56}$$

with

$$\varpi_{Y,0} \equiv -\left(1+\omega_y\right)\frac{\psi}{1+\nu}\left(\frac{\tilde{\lambda}}{\tilde{\Lambda}}\right)^{-\frac{1+\nu}{\nu}}\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{A}}\right)^{1+\omega_y}\bar{\Delta}\bar{Y}^{-1},$$

$$\varpi_{Y,3} \equiv \frac{\Lambda\left(\bar{\lambda}^b,\bar{\lambda}^s\right)}{\tilde{\lambda}\left(\bar{\lambda}^b,\bar{\lambda}^s\right)}\mu^p\left(1+\omega_y\right)^2\psi\mu^w\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{Z}}\right)^{1+\omega_y}\bar{Y}^{-1},$$

$$\varpi_{Y,4} \equiv \Lambda\left(\bar{\lambda}^b,\bar{\lambda}^s\right)\left(1-\bar{\tau}\right),$$

$$\varpi_{Y,5} \equiv -\pi_b\left(1+\omega_y\right)\left(1-\pi_b\right)\left[\left(\frac{\bar{\lambda}^b}{\psi_b}\right)^{\frac{1}{\nu}} - \left(\frac{\bar{\lambda}^s}{\psi_s}\right)^{\frac{1}{\nu}}\right]\left(\frac{\tilde{\lambda}}{\psi}\right)^{-\frac{1+\nu}{\nu}}\mu^w\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{Z}}\right)^{1+\omega_y}\bar{\Delta}\bar{Y}^{-1},$$

$$\varpi_{Y,6} \equiv -1.$$

FOC w.r.t. λ_t^b

$$0 = \varpi_{\lambda^b,0} + \varpi_{\lambda^b,1}\bar{\varphi}_1 + \varpi_{\lambda^b,2}\bar{\varphi}_2 + \varpi_{\lambda^b,3}\bar{\varphi}_3 + \varpi_{\lambda^b,4}\bar{\varphi}_4 + \varpi_{\lambda^b,5}\bar{\varphi}_5 + \varpi_{\lambda^b,6}\bar{\varphi}_6, \tag{2.57}$$

with

$$\begin{aligned}\varpi_{\lambda^b,0} \quad\equiv\quad & \pi_b\frac{1-\sigma_b}{1-\sigma_b^{-1}}\left(\bar{\lambda}^b\right)^{-\sigma_b}\bar{C}^b \\ & +\pi_b\frac{\psi}{\nu}\left(\frac{\psi}{\psi_b}\frac{\bar{\lambda}^b}{\tilde{\lambda}}\right)^{\frac{1}{\nu}}\left(\bar{\lambda}^b\right)^{-1}\left[\left(\frac{\tilde{\lambda}}{\tilde{\Lambda}}\right)^{-\frac{1+\nu}{\nu}} - \frac{\bar{\lambda}^b}{\tilde{\lambda}}\right]\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{Z}}\right)^{1+\omega_y}\bar{\Delta},\end{aligned}$$

$$\varpi_{\lambda^b,1} \equiv \left(1+\bar{r}^d\right)\left(1+\bar{\omega}\right)\left[\delta + \left(1-\delta\right)\pi_b\right] - 1,$$

$$\varpi_{\lambda^b,2} \equiv \left(1+\bar{r}^d\right)\left(1-\delta\right)\pi_b,$$

$$\varpi_{\lambda^b,3} \equiv \mu^p\left(1+\omega_y\right)\psi\mu^w\pi_b\left[\tilde{\lambda}^{-1} - \left(\frac{\psi}{\psi_s}\frac{\bar{\lambda}^s}{\tilde{\lambda}}\right)^{\frac{1}{\nu}}\left(\bar{\lambda}^s\right)^{-1}\frac{\Lambda}{\tilde{\lambda}}\right]\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{Z}}\right)^{1+\omega_y},$$

$$\varpi_{\lambda^b,4} \equiv (1-\tau)\,\bar{Y}\pi_b,$$

$$\begin{aligned}\varpi_{\lambda^b,5} \;\equiv\; & -\pi_b\,(1-\pi_b)\,\sigma_b\bar{C}^b\left(\bar{\lambda}^b\right)^{-\sigma_b-1} \\ & +\pi_b\,(1-\pi_b)\left(\frac{\tilde{\lambda}}{\psi}\right)^{-1}\left(\frac{\psi}{\psi_b}\frac{\bar{\lambda}^b}{\tilde{\lambda}}\right)^{\frac{1}{\nu}}\left(\bar{\lambda}^b\right)^{-1}\left[1-\frac{1+\nu}{\nu}\left(\frac{\psi}{\psi_s}\frac{\bar{\lambda}^s}{\tilde{\lambda}}\right)^{\frac{1}{\nu}}\right]\mu^w\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y}\bar{\Delta},\end{aligned}$$

$$\varpi_{\lambda^b,6} \equiv -\sigma_b\pi_b\bar{C}^b\left(\bar{\lambda}^b\right)^{-\sigma_b-1}.$$

FOC w.r.t. λ_t^s

$$0 = \varpi_{\lambda^s,0} + \varpi_{\lambda^s,1}\bar{\varphi}_1 + \varpi_{\lambda^s,2}\bar{\varphi}_2 + \varpi_{\lambda^s,3}\bar{\varphi}_3 + \varpi_{\lambda^s,4}\bar{\varphi}_4 + \varpi_{\lambda^s,5}\bar{\varphi}_5 + \varpi_{\lambda^s,6}\bar{\varphi}_6, \tag{2.58}$$

with

$$\begin{aligned}\varpi_{\lambda^s,0} \;\equiv\; & (1-\pi_b)\,\frac{1-\sigma_s}{1-\sigma_s^{-1}}\left(\bar{\lambda}^s\right)^{-\sigma_s}\bar{C}^s \\ & +(1-\pi_b)\,\frac{\psi}{\nu}\left(\frac{\psi}{\psi_s}\frac{\bar{\lambda}^s}{\tilde{\lambda}}\right)^{\frac{1}{\nu}}\left[\left(\frac{\tilde{\lambda}}{\tilde{\Lambda}}\right)^{-\frac{1+\nu}{\nu}}-\frac{\bar{\lambda}^s}{\tilde{\lambda}}\right]\left(\bar{\lambda}^s\right)^{-1}\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y}\bar{\Delta},\end{aligned}$$

$$\varpi_{\lambda^s,1} \equiv \left(1+\bar{r}^d\right)(1+\bar{\omega})\,(1-\delta)\,(1-\pi_b)\,,$$

$$\varpi_{\lambda^s,2} \equiv \left(1+\bar{r}^d\right)(1-\delta)\,\pi_b - 1,$$

$$\varpi_{\lambda^s,3} \equiv \mu^p\,(1+\omega_y)\,\psi\mu^w\,(1-\pi_b)\left[\tilde{\lambda}^{-1}-\left(\frac{\psi}{\psi_s}\frac{\bar{\lambda}^s}{\tilde{\lambda}}\right)^{\frac{1}{\nu}}\left(\bar{\lambda}^s\right)^{-1}\frac{\Lambda}{\tilde{\lambda}}\right]\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y},$$

$$\varpi_{\lambda^s,4} \equiv (1-\pi_b)\,(1-\bar{\tau})\,\bar{Y},$$

$$\begin{aligned}\varpi_{\lambda^s,5} \;\equiv\; & \sigma_s\pi_b\,(1-\pi_b)\,\bar{C}^s\left(\bar{\lambda}^s\right)^{-\sigma_s-1} \\ & \pi_b\,(1-\pi_b)\left(\frac{\tilde{\lambda}}{\psi}\right)^{-1}\left(\frac{\psi}{\psi_s}\frac{\bar{\lambda}^s}{\tilde{\lambda}}\right)^{\frac{1}{\nu}}\left(\bar{\lambda}^s\right)^{-1}\left[\frac{1+\nu}{\nu}\left(\frac{\psi}{\psi_b}\frac{\bar{\lambda}^b}{\tilde{\lambda}}\right)^{\frac{1}{\nu}}-1\right]\mu^w\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y}\bar{\Delta},\end{aligned}$$

$$\varpi_{\lambda^s,6} \equiv -(1-\pi_b)\,\bar{C}^s\left(\bar{\lambda}^s\right)^{-\sigma_s-1}.$$

FOC w.r.t. b_t

$$0 = \varpi_{b,5}\bar{\varphi}_5 + \varpi_{b,6}\bar{\varphi}_6 + \varpi_{b,9}\bar{\varphi}_9, \tag{2.59}$$

with

$$\varpi_{b,5} \equiv \delta\beta\left(1+\bar{r}^d\right)(1+\bar{\omega}) - (1+\pi_b\bar{\omega})\,,$$

$$\varpi_{b,6} \equiv \Xi'\left(\bar{b}\right),$$

$$\varpi_{b,9} \equiv \bar{\chi}''\left(\bar{b}\right) + \bar{\Xi}''\left(\bar{b}\right).$$

FOC w.r.t. ω_t

$$0 = \varpi_{\omega,1}\bar{\varphi}_1 + \varpi_{\omega,5}\bar{\varphi}_5 + \varpi_{\omega,9}\bar{\varphi}_9, \tag{2.60}$$

with

$$\varpi_{\omega,1} \equiv \frac{\bar{\lambda}^b}{1+\bar{\omega}},$$

$$\varpi_{\omega,5} \equiv \beta\delta\left(1+\bar{r}^d\right) - \pi_b,$$

$$\varpi_{\omega,9} \equiv -1.$$

2.3.3 Simplified FOC list

Summarize the equations as:

$$0 = \varpi_{\Pi,3}\bar{\varphi}_3 + \varpi_{\Pi,4}\bar{\varphi}_4 + \varpi_{\Pi,7}\bar{\varphi}_7 + \varpi_{\Pi,8}\bar{\varphi}_8, \tag{2.61}$$

$$0 = \varpi_{i^d,1}\bar{\varphi}_1 + \varpi_{i^d,2}\bar{\varphi}_2 + \varpi_{i^d,5}\bar{\varphi}_5, \tag{2.62}$$

$$0 = \varpi_{\Delta,0} + \varpi_{\Delta,5}\bar{\varphi}_5 + \varpi_{\Delta,7}\bar{\varphi}_7, \tag{2.63}$$

$$0 = \varpi_{K,3}\bar{\varphi}_3 + \varpi_{K,8}\bar{\varphi}_8, \tag{2.64}$$

$$0 = \varpi_{F,4}\bar{\varphi}_4 + \varpi_{F,8}\bar{\varphi}_8, \tag{2.65}$$

$$0 = \varpi_{Y,0} + \varpi_{Y,3}\bar{\varphi}_3 + \varpi_{Y,4}\bar{\varphi}_4 + \varpi_{Y,5}\bar{\varphi}_5 + \varpi_{Y,6}\bar{\varphi}_6, \tag{2.66}$$

$$0 = \varpi_{\lambda^b,0} + \varpi_{\lambda^b,1}\bar{\varphi}_1 + \varpi_{\lambda^b,2}\bar{\varphi}_2 + \varpi_{\lambda^b,3}\bar{\varphi}_3 + \varpi_{\lambda^b,4}\bar{\varphi}_4 + \varpi_{\lambda^b,5}\bar{\varphi}_5 + \varpi_{\lambda^b,6}\bar{\varphi}_6, \tag{2.67}$$

$$0 = \varpi_{\lambda^s,0} + \varpi_{\lambda^s,1}\bar{\varphi}_1 + \varpi_{\lambda^s,2}\bar{\varphi}_2 + \varpi_{\lambda^s,3}\bar{\varphi}_3 + \varpi_{\lambda^s,4}\bar{\varphi}_4 + \varpi_{\lambda^s,5}\bar{\varphi}_5 + \varpi_{\lambda^s,6}\bar{\varphi}_6, \tag{2.68}$$

$$0 = \varpi_{b,5}\bar{\varphi}_5 + \varpi_{b,6}\bar{\varphi}_6 + \varpi_{b,9}\bar{\varphi}_9, \tag{2.69}$$

$$0 = \varpi_{\omega,1}\bar{\varphi}_1 + \varpi_{\omega,5}\bar{\varphi}_5 + \varpi_{\omega,9}\bar{\varphi}_9. \tag{2.70}$$

Notice that, conditional on the endogenous variables this is a system of 10 linear equations and 9 unknowns.

We will now show that setting $\bar{\Pi} = 1$ satisfies the above system and that one of the equations drops out (is always satisfied for all $\{\bar{\varphi}_i\}$) hence we will end up with a linear equation system of 9 equations and 9 unknowns.

2.3.4 Zero inflation is solution to the FOC w.r.t. inflation variable

We further know that with zero inflation, $\varpi_{\Pi,7} = 0$. We can also solve (2.64) and (2.65) to get

$$\bar{\varphi}_3 = -\frac{\varpi_{K,8}}{\varpi_{K,3}}\bar{\varphi}_8, \tag{2.71}$$

$$\bar{\varphi}_4 = -\frac{\varpi_{F,8}}{\varpi_{F,4}}\bar{\varphi}_8, \tag{2.72}$$

and plug it into 2.61

$$0 = -\frac{\varpi_{K,8}}{\varpi_{K,3}}\varpi_{\Pi,3} - \frac{\varpi_{F,8}}{\varpi_{F,4}}\varpi_{\Pi,4} + \varpi_{\Pi,8}. \tag{2.73}$$

As long as this is a true statement we know that $\bar{\Pi} = 1$ is a solution to the optimal steady state and that (2.61) drops out of the system of linear equations.

Let's evaluate (2.73):

$$\begin{aligned} 0 &= -\frac{\frac{\theta-1}{1+\omega_y\theta}\bar{K}^{-1}}{-(1-\alpha)}\alpha\theta\left(1+\omega_y\right)\bar{K} - \frac{-\frac{\theta-1}{1+\omega_y\theta}\bar{F}^{-1}}{-(1-\alpha)}\alpha\left(\theta-1\right)\bar{F} - \frac{\alpha\left(\theta-1\right)}{1-\alpha} \\ &= \left(1+\omega_y\theta\right) - \theta\left(1+\omega_y\right) + \left(\theta-1\right), \end{aligned}$$

which is indeed a true proposition.

2.3.5 Solution to Lagrangian multipliers

For the subsystem of steady state conditions (2.1)-(2.9) we already characterized the solution previously. Let us turn now into the solution of the remainder of the system.

Use (2.64) to write

$$\bar{\varphi}_3 = \varpi_{3,8}\bar{\varphi}_8, \tag{2.74}$$

with

$$\varpi_{3,8} \equiv \frac{\theta-1}{\left(1-\alpha\right)\left(1+\omega_y\theta\right)}\bar{K}^{-1}.$$

Use (2.65) and (2.8) to write

$$\bar{\varphi}_4 = -\bar{\varphi}_3. \tag{2.75}$$

Use (2.63) to write

$$\bar{\varphi}_7 = \varpi_{7,0} + \varpi_{7,5}\bar{\varphi}_5, \tag{2.76}$$

with

$$\varpi_{7,0} \equiv -\frac{\pi_b\left(1-\pi_b\right)}{1-\alpha\beta}\left[\left(\frac{\bar{\lambda}^b}{\psi_b}\right)^{\frac{1}{\nu}} - \left(\frac{\bar{\lambda}^s}{\psi_s}\right)^{\frac{1}{\nu}}\right]\left(\frac{\tilde{\lambda}}{\psi}\right)^{-\frac{1+\nu}{\nu}}\mu^w\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y},$$

$$\varpi_{7,5} \equiv -\frac{1}{1-\alpha\beta}\frac{\psi}{1+\nu}\left(\frac{\tilde{\lambda}}{\tilde{\Lambda}}\right)^{-\frac{1+\nu}{\nu}}\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y}.$$

Use (2.62) to write

$$\bar{\varphi}_5 = \bar{\Omega}\varpi_{5,2}\bar{\varphi}_1 + \varpi_{5,2}\bar{\varphi}_2, \tag{2.77}$$

with

$$\varpi_{5,2} \equiv -\frac{\bar{\lambda}^s}{\delta\beta\left(1+\bar{r}^d\right)\left[\bar{b}\left(1+\bar{\omega}\right)+\pi_b\bar{b}^g\right]}.$$

Use (2.70) to write

$$0 = \frac{\bar{\lambda}^b}{1+\bar{\omega}}\bar{\varphi}_1 + \left[\delta\beta\left(1+\bar{r}^d\right)-1\right]\bar{b}\bar{\varphi}_5 - \bar{\varphi}_9,$$

and further use (2.77) to simplify to

$$\bar{\varphi}_9 = \varpi_{9,1}\bar{\varphi}_1 + \varpi_{9,2}\bar{\varphi}_2, \tag{2.78}$$

with

$$\varpi_{9,1} \equiv \left[\frac{1}{1+\bar{\omega}} - \frac{\left[\delta\beta\left(1+\bar{r}^d\right)-1\right]\bar{b}}{\delta\beta\left(1+\bar{r}^d\right)\left[\bar{b}\left(1+\bar{\omega}\right)+\pi_b\bar{b}^g\right]}\right]\bar{\lambda}^b,$$

$$\varpi_{9,2} \equiv -\frac{\left[\delta\beta\left(1+\bar{r}^d\right)-1\right]\bar{b}}{\delta\beta\left(1+\bar{r}^d\right)\left[\bar{b}\left(1+\bar{\omega}\right)+\pi_b\bar{b}^g\right]}\bar{\lambda}^s.$$

Use (2.69) to write

$$0 = \varpi_{b,5}\bar{\varphi}_5 + \varpi_{b,6}\bar{\varphi}_6 + \varpi_{b,9}\bar{\varphi}_9,$$

$\Leftrightarrow$

$$\bar{\varphi}_6 = -\frac{\varpi_{b,5}}{\varpi_{b,6}}\bar{\varphi}_5 - \frac{\varpi_{b,9}}{\varpi_{b,6}}\bar{\varphi}_9.$$

and using (2.77) and (2.78) we can write

$$\bar{\varphi}_6 = \varpi_{6,1}\bar{\varphi}_1 + \varpi_{6,2}\bar{\varphi}_2, \tag{2.79}$$

with

$$\varpi_{6,1} \equiv -\frac{\varpi_{b,5}\varpi_{5,2}\bar{\Omega} + \varpi_{b,9}\varpi_{9,1}}{\varpi_{b,6}},$$

$$\varpi_{6,2} \equiv -\frac{\varpi_{b,5}\varpi_{5,2} + \varpi_{b,9}\varpi_{9,2}}{\varpi_{b,6}}.$$

Use (2.66), (2.74), (2.75), (2.77) and (2.79) to write

$$\bar{\varphi}_8 = \varpi_{8,0} + \varpi_{8,1}\bar{\varphi}_1 + \varpi_{8,2}\bar{\varphi}_2, \tag{2.80}$$

with

$$\varpi_{8,0} \equiv -\frac{\varpi_{Y,0}}{\varpi_{Y,8}},$$

$$\varpi_{8,1} \equiv \frac{\varpi_{6,1} - \varpi_{Y,5}\varpi_{5,1}}{\varpi_{Y,8}},$$

$$\varpi_{8,2} \equiv \frac{\varpi_{6,2} - \varpi_{Y,5}\varpi_{5,2}}{\varpi_{Y,8}}.$$

Use (2.67) and the above results to write:

$$\bar{\varphi}_1 = \varpi_{1,0} + \varpi_{1,2}\bar{\varphi}_2, \tag{2.81}$$

with

$$\varpi_{1,0} \equiv -\frac{\varpi_{\lambda^b,0} + \left(\varpi_{\lambda^b,3} - \varpi_{\lambda^b,4}\right)\varpi_{3,8}\varpi_{8,0}}{\varpi_{\lambda^b,1} + \left(\varpi_{\lambda^b,3} - \varpi_{\lambda^b,4}\right)\varpi_{3,8}\varpi_{8,1} + \varpi_{\lambda^b,5}\bar{\Omega}\varpi_{5,2} + \varpi_{\lambda^b,6}\varpi_{6,1}},$$

$$\varpi_{1,2} \equiv -\frac{\varpi_{\lambda^b,2} + \left(\varpi_{\lambda^b,3} - \varpi_{\lambda^b,4}\right)\varpi_{3,8}\varpi_{8,2} + \varpi_{\lambda^b,5}\varpi_{5,2} + \varpi_{\lambda^b,6}\varpi_{6,2}}{\varpi_{\lambda^b,1} + \left(\varpi_{\lambda^b,3} - \varpi_{\lambda^b,4}\right)\varpi_{3,8}\varpi_{8,1} + \varpi_{\lambda^b,5}\bar{\Omega}\varpi_{5,2} + \varpi_{\lambda^b,6}\varpi_{6,1}}.$$

Finally use (2.68) and the above results to write:

$$\bar{\varphi}_2 = -\frac{\varpi_{2,0} + \varpi_{2,1}\varpi_{1,0}}{\varpi_{2,2} + \varpi_{2,1}\varpi_{1,2}}, \tag{2.82}$$

with

$$\varpi_{2,0} \equiv \varpi_{\lambda^s,0} + \left(\varpi_{\lambda^s,3} - \varpi_{\lambda^s,4}\right)\varpi_{3,8}\varpi_{8,0},$$

$$\varpi_{2,1} \equiv \left[\varpi_{\lambda^s,1} + \left(\varpi_{\lambda^s,3} - \varpi_{\lambda^s,4}\right)\varpi_{3,8}\varpi_{8,1} + \varpi_{\lambda^s,5}\bar{\Omega}\varpi_{5,2} + \varpi_{\lambda^s,6}\varpi_{6,1}\right],$$

$$\varpi_{2,2} \equiv \varpi_{\lambda^s,2} + \left(\varpi_{\lambda^s,3} - \varpi_{\lambda^s,4}\right)\varpi_{3,8}\varpi_{8,2} + \varpi_{\lambda^s,5}\varpi_{5,2} + \varpi_{\lambda^s,6}\varpi_{6,2}.$$

Now we could use (2.82) to solve (2.81) for $\bar{\varphi}_1$, and use both $\bar{\varphi}_2$ and $\bar{\varphi}_1$ to solve for $\bar{\varphi}_8$, $\bar{\varphi}_6$ and φ_5 and then use these to solve for the remaining ones.

3 Log-linear equations

In this section we present all the log-linear relations of the model, in which we linearize around the zero inflation steady state.

3.1 Full system

The full system of log-linear equation is given by:

$$\hat{\lambda}_t^b = \hat{\imath}_t^d + \hat{\omega}_t - E_t\pi_{t+1} + \chi_b E_t\hat{\lambda}_{t+1}^b + \left(1 - \chi_b\right)E_t\hat{\lambda}_{t+1}^s, \tag{3.1}$$

$$\hat{\lambda}_t^s = \hat{\imath}_t^d - E_t\pi_{t+1} + \left(1 - \chi_s\right)E_t\hat{\lambda}_{t+1}^b + \chi_s E_t\hat{\lambda}_{t+1}^s, \tag{3.2}$$

$$\begin{aligned} \hat{K}_t &= (1-\alpha\beta)\left[\hat{\Lambda}_t - \widehat{\bar{\lambda}}_t + \hat{\mu}_t^w - \nu\bar{h}_t + \left(1+\omega_y\right)\left(\hat{Y}_t - z_t\right)\right] \\ &\quad + \alpha\beta E_t\left[\theta\left(1+\omega_y\right)\pi_{t+1} + \hat{K}_{t+1}\right], \end{aligned} \tag{3.3}$$

$$\hat{F}_t = (1-\alpha\beta)\left[\hat{\Lambda}_t - \hat{\tau}_t + \hat{Y}_t\right] + \alpha\beta E_t\left[(\theta-1)\,\pi_{t+1} + \hat{F}_{t+1}\right], \tag{3.4}$$

$$\begin{aligned}(1+\pi_b\bar{\omega})\,\hat{b}_t &= \pi_b(1-\pi_b)\,\rho_b^{-1}\hat{B}_t - \pi_b(1+\bar{\omega})\,\hat{\omega}_t \\ &+\delta\left(1+\bar{r}^d\right)\left[(1+\bar{\omega}) + \pi_b\rho_b^g/\rho_b\right]\left(\hat{i}_{t-1}^d - \pi_t\right) \\ &+\delta\left(1+\bar{r}^d\right)(1+\bar{\omega})\left(\hat{b}_{t-1} + \hat{\omega}_{t-1}\right) \\ &-\pi_b\rho_b^{-1}\left[\hat{b}_t^g - \delta\left(1+\bar{r}^d\right)\hat{b}_{t-1}^g\right],\end{aligned} \tag{3.5}$$

$$\hat{Y}_t = \pi_b s_b\left(\bar{c}_t^b - \sigma_b\hat{\lambda}_t^b\right) + (1-\pi_b)\,s_s\left(\bar{c}_t^s - \sigma_s\hat{\lambda}_t^s\right) + \hat{\bar{\Xi}}_t + \eta s_\Xi\hat{b}_t + \hat{G}_t, \tag{3.6}$$

$$\hat{\Delta}_t = \alpha\hat{\Delta}_{t-1}, \tag{3.7}$$

$$\pi_t = \frac{1-\alpha}{\alpha}\frac{1}{1+\omega_y\theta}\left(\hat{K}_t - \hat{F}_t\right), \tag{3.8}$$

$$\hat{\omega}_t = \frac{(1+\varkappa)\,\bar{\chi}\bar{b}^{\varkappa}}{1+\bar{\omega}}\left(\frac{\zeta_\chi}{\bar{\chi}}\hat{\tilde{\chi}}_t + \varkappa\hat{b}_t\right) + \frac{\eta\bar{\Xi}\bar{b}^{\eta-1}}{1+\bar{\omega}}\left(\frac{\zeta_\Xi}{\bar{\Xi}}\hat{\tilde{\Xi}}_t + (\eta-1)\,\hat{b}_t\right). \tag{3.9}$$

Auxiliary equations:

$$\begin{aligned}\hat{B}_t &= s_b\left(\bar{c}_t^b - \sigma_b\hat{\lambda}_t^b\right) - s_s\left(\bar{c}_t^s - \sigma_s\hat{\lambda}_t^s\right) \\ &-\psi\bar{\tilde{\lambda}}^{-1}\bar{\mu}^w\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y}\bar{Y}^{-1}\frac{1}{\nu}\left[\left(\frac{\psi}{\psi_b}\frac{\bar{\lambda}^b}{\bar{\tilde{\lambda}}}\right)^{\frac{1}{\nu}}\left(\hat{\lambda}_t^b - \hat{\tilde{\lambda}}_t\right) - \left(\frac{\psi}{\psi_s}\frac{\bar{\lambda}^s}{\bar{\tilde{\lambda}}}\right)^{\frac{1}{\nu}}\left(\hat{\lambda}_t^s - \hat{\tilde{\lambda}}_t\right)\right] \\ &-\psi\bar{\tilde{\lambda}}^{-1}\bar{\mu}^w\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y}\bar{Y}^{-1}\left[\left(\frac{\psi}{\psi_b}\frac{\bar{\lambda}^b}{\bar{\tilde{\lambda}}}\right)^{\frac{1}{\nu}} - \left(\frac{\psi}{\psi_s}\frac{\bar{\lambda}^s}{\bar{\tilde{\lambda}}}\right)^{\frac{1}{\nu}}\right]\times \\ &\times\left[\hat{\mu}_t^w - \nu\bar{h}_t - \hat{\tilde{\lambda}}_t + (1+\omega_y)\left(\hat{Y}_t - z_t\right) + \hat{\Delta}_t\right],\end{aligned} \tag{3.10, 3.11}$$

$$\hat{\tilde{\lambda}}_t = \pi_b\left(\frac{\psi}{\psi_b}\frac{\bar{\lambda}^b}{\bar{\tilde{\lambda}}}\right)^{\frac{1}{\nu}}\hat{\lambda}_t^b + (1-\pi_b)\left(\frac{\psi}{\psi_s}\frac{\bar{\lambda}^s}{\bar{\tilde{\lambda}}}\right)^{\frac{1}{\nu}}\hat{\lambda}_t^s, \tag{3.12}$$

$$\hat{\Lambda}_t = \pi_b\frac{\bar{\lambda}^b}{\bar{\lambda}}\hat{\lambda}_t^b + (1-\pi_b)\frac{\bar{\lambda}^s}{\bar{\lambda}}\hat{\lambda}_t^s, \tag{3.13}$$

$$\hat{\tilde{\Lambda}}_t = \pi_b\frac{\psi_b^{-\frac{1}{\nu}}\left(\bar{\lambda}^b\right)^{\frac{1+\nu}{\nu}}}{\psi^{-\frac{1}{\nu}}\bar{\tilde{\Lambda}}^{\frac{1+\nu}{\nu}}}\hat{\lambda}_t^b + (1-\pi_b)\frac{\psi_s^{-\frac{1}{\nu}}\left(\bar{\lambda}^s\right)^{\frac{1+\nu}{\nu}}}{\psi^{-\frac{1}{\nu}}\bar{\tilde{\Lambda}}^{\frac{1+\nu}{\nu}}}\hat{\lambda}_t^s, \tag{3.14}$$

$$\hat{c}_t^b = \bar{c}_t^b - \sigma_b\hat{\lambda}_t^b, \tag{3.15}$$

$$\hat{c}_t^s = \bar{c}_t^s - \sigma_s\hat{\lambda}_t^s. \tag{3.16}$$

The exogenous variables all follow an AR(1) process as follows:

$$\xi_t = \rho_\xi \xi_{t-1} + \varepsilon_t \tag{3.17}$$

In the above equations we consider the following definitions

$$\hat{i}_t^\tau \equiv \ln\left(\left(1+i_t^\tau\right)/\left(1+\bar{i}_t^\tau\right)\right), \tag{3.18}$$

$$\hat{\omega}_t \equiv \ln\left(\left(1+\omega_t\right)/\left(1+\bar{\omega}\right)\right), \tag{3.19}$$

$$\pi_t \equiv \ln \Pi_t, \tag{3.20}$$

$$\bar{\lambda}_t^\tau \equiv \ln\left(\lambda_t^\tau/\bar{\lambda}^\tau\right), \tag{3.21}$$

$$\hat{Y}_t \equiv \ln\left(Y_t/\bar{Y}\right), \tag{3.22}$$

$$\hat{F}_t \equiv \ln\left(F_t/\bar{F}\right), \tag{3.23}$$

$$\hat{K}_t \equiv \ln\left(K_t/\bar{K}\right), \tag{3.24}$$

$$\hat{b}_t \equiv \ln\left(b_t/\bar{b}\right), \tag{3.25}$$

$$\bar{h}_t \equiv \ln\left(\bar{H}_t/\bar{H}\right), \tag{3.26}$$

$$z_t \equiv \ln\left(Z_t/\bar{Z}\right), \tag{3.27}$$

$$\hat{\tau}_t \equiv -\log\left(\left(1-\tau_t\right)/\left(1-\bar{\tau}\right)\right), \tag{3.28}$$

$$\hat{b}_t^g \equiv \left(b_t^g - \bar{b}\right)/\bar{Y}, \tag{3.29}$$

$$\bar{c}_t^\tau \equiv \ln\left(\bar{C}_t^\tau/\bar{C}^\tau\right), \tag{3.30}$$

$$\hat{\mu}_t^w \equiv \ln\left(\mu_t^w/\bar{\mu}^w\right), \tag{3.31}$$

$$\hat{G}_t \equiv \left(G_t - \bar{G}\right)/\bar{Y}, \tag{3.32}$$

$$\hat{\tilde{\Xi}}_t \equiv \frac{\bar{b}^\eta}{\bar{Y}}\left(\tilde{\Xi}_t - \bar{\tilde{\Xi}}\right), \tag{3.33}$$

$$\widehat{\tilde{\chi}}_t \equiv \left(1+\varkappa\right)\bar{b}^\varkappa\left(\tilde{\chi}_t - \bar{\tilde{\chi}}\right), \tag{3.34}$$

and

$$\chi_\tau \equiv \beta\left(1+\bar{r}^\tau\right)\left[\delta + \left(1-\delta\right)\pi_\tau\right]. \tag{3.35}$$

3.2 Simplified log-linear system

Aggregate demand

Define

$$\hat{\Omega}_t \equiv \hat{\lambda}_t^b - \hat{\lambda}_t^s, \tag{3.36}$$

$$\hat{\lambda}_t \equiv \pi_b \hat{\lambda}_t^b + (1-\pi_b)\,\hat{\lambda}_t^s, \tag{3.37}$$

$$\hat{\imath}_t^{avg} \equiv \pi_b \left(\hat{\imath}_t^d + \hat{\omega}_t\right) + (1-\pi_b)\,\hat{\imath}_t^d = \hat{\imath}_t^d + \pi_b \hat{\omega}_t, \tag{3.38}$$

so that we can combine the two Euler equations into

$$\hat{\Omega}_t = \hat{\omega}_t + \hat{\delta} E_t \hat{\Omega}_{t+1}, \tag{3.39}$$

$$\hat{\lambda}_t = \hat{\imath}_t^{avg} - E_t \pi_{t+1} + E_t \hat{\lambda}_{t+1} - \psi_\Omega E_t \hat{\Omega}_{t+1}, \tag{3.40}$$

with

$$\hat{\delta} \equiv \chi_b + \chi_s - 1, \tag{3.41}$$

$$\psi_\Omega \equiv \pi_b (1-\chi_b) - (1-\pi_b)(1-\chi_s). \tag{3.42}$$

Define

$$s_c \bar{c}_t \equiv \pi_b s_b \bar{c}_t^b + (1-\pi_b)\, s_s \bar{c}_t^s, \tag{3.43}$$

$$g_t \equiv s_c \bar{c}_t + \hat{G}_t, \tag{3.44}$$

so that we can write the AD as

$$\hat{Y}_t = g_t + \hat{\Xi}_t - \bar{\sigma}\left(\hat{\lambda}_t + s_\Omega \hat{\Omega}_t\right), \tag{3.45}$$

with

$$s_\Omega \equiv \pi_b (1-\pi_b) \frac{s_b \sigma_b - s_s \sigma_s}{\bar{\sigma}}. \tag{3.46}$$

We can thus solve for $\hat{\lambda}_t$

$$\hat{\lambda}_t = -\bar{\sigma}^{-1}\left(\hat{Y}_t - g_t - \hat{\Xi}_t\right) - s_\Omega \hat{\Omega}_t, \tag{3.47}$$

and plug this into the average Euler to get the IS relation

$$\begin{aligned} \hat{Y}_t \;=\; & E_t \hat{Y}_{t+1} - \bar{\sigma}\left(\hat{\imath}_t^{avg} - E_t \pi_{t+1}\right) - E_t \Delta g_{t+1} - E_t \Delta \hat{\Xi}_{t+1} \\ & - \bar{\sigma} s_\Omega \hat{\Omega}_t + \bar{\sigma}\left(s_\Omega + \psi_\Omega\right) E_t \hat{\Omega}_{t+1}, \end{aligned} \tag{3.48}$$

with

$$\Delta g_t \equiv g_t - g_{t-1}, \tag{3.49}$$

$$\Delta \hat{\Xi}_t \equiv \hat{\Xi}_t - \hat{\Xi}_{t-1}. \tag{3.50}$$

Aggregate supply

Combine the equation defining inflation and those defining $\hat{F}_t$ and $\hat{K}_t$ to get

$$\pi_t = \beta E_t \pi_{t+1} + \xi \left[\omega_y \hat{Y}_t - \hat{\bar{\lambda}}_t + \hat{\mu}^w_t - \nu \bar{h}_t + \hat{\tau}_t - (1+\omega_y) z_t \right], \tag{3.51}$$

with

$$\xi \equiv \frac{1-\alpha}{\alpha} \frac{1-\alpha\beta}{1+\omega_y \theta}. \tag{3.52}$$

Further notice that

$$\hat{\bar{\lambda}}_t = (\gamma_b - \pi_b) \hat{\Omega}_t + \hat{\lambda}_t, \tag{3.53}$$

with

$$\gamma_b \equiv \pi_b \left(\frac{\psi}{\psi_b} \frac{\bar{\lambda}^b}{\bar{\bar{\lambda}}} \right)^{\frac{1}{\nu}}. \tag{3.54}$$

Using (3.47) and (3.53), we can thus write the Phillips curve as

$$\begin{aligned} \pi_t \;=\; & \beta E_t \pi_{t+1} + \xi \left[\left(\omega_y + \bar{\sigma}^{-1} \right) \hat{Y}_t - \bar{\sigma}^{-1} \left(g_t + \hat{\Xi}_t \right) + \hat{\mu}^w_t - \nu \bar{h}_t + \hat{\tau}_t - (1+\omega_y) z_t \right] \\ & + \xi \left(s_\Omega + \pi_b - \gamma_b \right) \hat{\Omega}_t, \end{aligned}$$

or, equivalently,

$$\pi_t = \beta E_t \pi_{t+1} + u_t + \kappa \left(\hat{Y}_t - \hat{Y}^n_t \right) - \xi \bar{\sigma}^{-1} \hat{\Xi}_t + \xi \left(s_\Omega + \pi_b - \gamma_b \right) \hat{\Omega}_t, \tag{3.55}$$

with

$$\hat{Y}^n_t \equiv \left(\omega_y + \bar{\sigma}^{-1} \right)^{-1} \left[\bar{\sigma}^{-1} g_t + \nu \bar{h}_t + (1+\omega_y) z_t \right], \tag{3.56}$$

$$u_t \equiv \xi \left(\hat{\mu}^w_t + \hat{\tau}_t \right), \tag{3.57}$$

$$\kappa \equiv \xi \left(\omega_y + \bar{\sigma}^{-1} \right). \tag{3.58}$$

Law of motion of debt

Consider the equations determining $\hat{\omega}_t$

$$\hat{\omega}_t = \omega_b \hat{b}_t + \omega_\chi \hat{\tilde{\chi}}_t + \omega_\Xi \hat{\tilde{\Xi}}_t, \tag{3.59}$$

with

$$\omega_b \equiv \frac{\varkappa \bar{\chi}' \left(\bar{b} \right) + \eta (\eta - 1) \frac{s_\Xi}{\rho_b}}{1+\bar{\omega}}, \tag{3.60}$$

$$\omega_\chi \equiv \frac{1}{1+\bar{\omega}}, \tag{3.61}$$

$$\omega_\Xi \equiv \frac{1}{1+\bar{\omega}} \frac{\eta}{\rho_b}. \tag{3.62}$$

We can now simplify the expression of $\hat{B}_t$

$$\begin{aligned}\hat{B}_t &= s_c\bar{c}_t + \frac{B_\Omega}{\pi_b(1-\pi_b)}\hat{\Omega}_t + \frac{B_\lambda}{\pi_b(1-\pi_b)}\hat{\lambda}_t \\ &\quad -\frac{\tilde{B}_u}{\pi_b(1-\pi_b)}\left[\hat{\mu}_t^w - \nu\bar{h}_t + (1+\omega_y)\left(\hat{Y}_t - z_t\right)\right],\end{aligned} \tag{3.63}$$

with

$$\tilde{B}_\Omega \equiv \psi\bar{\tilde{\lambda}}^{-1}\bar{\mu}^w\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y}\bar{Y}^{-1}\frac{1}{\nu}\gamma_b(1-\gamma_b), \tag{3.64}$$

$$\tilde{B}_u \equiv \psi\bar{\tilde{\lambda}}^{-1}\bar{\mu}^w\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{\bar{Z}}}\right)^{1+\omega_y}\bar{Y}^{-1}(\gamma_b - \pi_b), \tag{3.65}$$

$$B_\Omega \equiv s_\Omega\pi_b - s_b\sigma_b\pi_b(1-\pi_b) - \tilde{B}_\Omega + \tilde{B}_u(\gamma_b - \pi_b), \tag{3.66}$$

$$B_\lambda \equiv \tilde{B}_u - s_\Omega. \tag{3.67}$$

Now plug in the law of motion of $\hat{b}_t$

$$\begin{aligned}\hat{b}_t &= \frac{\delta\left(1+\bar{r}^d\right)}{1+\pi_b\bar{\omega}}\left[(1+\bar{\omega}) + \pi_b\frac{\rho_b^g}{\rho_b}\right]\left(\hat{\imath}_{t-1}^d - \pi_t\right) \\ &\quad +\pi_b(1-\pi_b)\frac{s_c}{\rho_b(1+\pi_b\bar{\omega})}\bar{c}_t - \frac{\tilde{B}_u}{\rho_b(1+\pi_b\bar{\omega})}\left[\hat{\mu}_t^w - \nu\bar{h}_t + (1+\omega_y)\left(\hat{Y}_t - z_t\right)\right] \\ &\quad +\frac{B_\Omega}{\rho_b(1+\pi_b\bar{\omega})}\hat{\Omega}_t + \frac{B_\lambda}{\rho_b(1+\pi_b\bar{\omega})}\hat{\lambda}_t - \frac{\pi_b(1+\bar{\omega})}{1+\pi_b\bar{\omega}}\hat{\omega}_t \\ &\quad +\frac{\delta\left(1+\bar{r}^d\right)(1+\bar{\omega})}{1+\pi_b\bar{\omega}}\left(\hat{b}_{t-1} + \hat{\omega}_{t-1}\right) - \frac{\pi_b}{\rho_b(1+\pi_b\bar{\omega})}\left[\hat{b}_t^g - \delta\left(1+\bar{r}^d\right)\hat{b}_{t-1}^g\right],\end{aligned}$$

and using (3.47) we can further write

$$\begin{aligned}\hat{b}_t &= \varrho_r\left(\hat{\imath}_{t-1}^d - \pi_t\right) + \varrho_Y\hat{Y}_t + \varrho_\Omega\hat{\Omega}_t + \varrho_\omega\hat{\omega}_t + \varrho_b\left(\hat{b}_{t-1} + \hat{\omega}_{t-1}\right) \\ &\quad +\varrho_\xi\left[\pi_b(1-\pi_b)s_c\bar{c}_t + B_\lambda\bar{\sigma}^{-1}\left(g_t + \hat{\bar{\Xi}}_t\right) - \tilde{B}_u\left[\hat{\mu}_t^w - \nu\bar{h}_t - (1+\omega_y)z_t\right]\right] \\ &\quad -\pi_b\varrho_\xi\left[\hat{b}_t^g - \delta\left(1+\bar{r}^d\right)\hat{b}_{t-1}^g\right],\end{aligned} \tag{3.68}$$

with

$$\varrho_r \equiv \frac{\delta\left(1+\bar{r}^d\right)}{1+\pi_b\bar{\omega}}\left[(1+\bar{\omega}) + \pi_b\frac{\rho_b^g}{\rho_b}\right], \tag{3.69}$$

$$\varrho_Y \equiv -\varrho_\xi\left(\tilde{B}_u(1+\omega_y) + B_\lambda\bar{\sigma}^{-1}\right), \tag{3.70}$$

$$\varrho_\Omega \equiv \varrho_\xi(B_\Omega - B_\lambda s_\Omega), \tag{3.71}$$

$$\varrho_\omega \equiv -\frac{\pi_b(1+\bar{\omega})}{1+\pi_b\bar{\omega}}, \tag{3.72}$$

$$\varrho_\xi \equiv \frac{1}{\rho_b\left(1+\pi_b\bar{\omega}\right)}, \tag{3.73}$$

$$\varrho_b \equiv \frac{\delta\left(1+\bar{r}^d\right)\left(1+\bar{\omega}\right)}{1+\pi_b\bar{\omega}}. \tag{3.74}$$

3.3 Complete simplified system of log-linear equations

We can write the required equations as

$$\hat{\imath}_t^{avg} = \hat{\imath}_t^d + \pi_b\hat{\omega}_t, \tag{3.75}$$

$$\hat{\Omega}_t = \hat{\omega}_t + \hat{\delta}E_t\hat{\Omega}_{t+1}, \tag{3.76}$$

$$\begin{aligned}\hat{Y}_t \;=\;& E_t\hat{Y}_{t+1} - \bar{\sigma}\left(\hat{\imath}_t^{avg} - E_t\pi_{t+1}\right) - E_t\Delta g_{t+1} - E_t\Delta\hat{\Xi}_{t+1} \\ & -\bar{\sigma}s_\Omega\hat{\Omega}_t + \bar{\sigma}\left(s_\Omega+\psi_\Omega\right)E_t\hat{\Omega}_{t+1},\end{aligned} \tag{3.77}$$

$$\pi_t = \beta E_t\pi_{t+1} + u_t + \kappa\left(\hat{Y}_t - \hat{Y}_t^n\right) - \xi\bar{\sigma}^{-1}\hat{\Xi}_t + \xi\left(s_\Omega+\pi_b-\gamma_b\right)\hat{\Omega}_t, \tag{3.78}$$

$$\hat{\omega}_t = \omega_b\hat{b}_t + \omega_\chi\hat{\tilde{\chi}}_t + \omega_\Xi\hat{\hat{\Xi}}_t, \tag{3.79}$$

$$\begin{aligned}\hat{b}_t \;=\;& \varrho_r\left(\hat{\imath}_{t-1}^d - \pi_t\right) + \varrho_Y\hat{Y}_t + \varrho_\Omega\hat{\Omega}_t + \varrho_\omega\hat{\omega}_t + \varrho_b\left(\hat{b}_{t-1} + \hat{\omega}_{t-1}\right) \\ & +\varrho_\xi\left[\pi_b\left(1-\pi_b\right)s_c\bar{c}_t + B_\lambda\bar{\sigma}^{-1}\left(g_t + \hat{\Xi}_t\right) - \tilde{B}_u\left[\hat{\mu}_t^w - \nu\bar{h}_t - \left(1+\omega_y\right)z_t\right]\right] \\ & -\pi_b\varrho_\xi\left[\hat{b}_t^g - \delta\left(1+\bar{r}^d\right)\hat{b}_{t-1}^g\right],\end{aligned} \tag{3.80}$$

with

$$\hat{Y}_t^n \equiv \left(\omega_y + \bar{\sigma}^{-1}\right)^{-1}\left[\bar{\sigma}^{-1}g_t + \nu\bar{h}_t + \left(1+\omega_y\right)z_t\right], \tag{3.81}$$

$$s_c\bar{c}_t = \pi_b s_b\bar{c}_t^b + \left(1-\pi_b\right)s_s\bar{c}_t^s, \tag{3.82}$$

$$g_t = s_c\bar{c}_t + \hat{G}_t, \tag{3.83}$$

$$u_t \equiv \xi\left(\hat{\mu}_t^w + \hat{\tau}_t\right), \tag{3.84}$$

$$\Delta g_t \equiv g_t - g_{t-1}, \tag{3.85}$$

$$\Delta\hat{\hat{\Xi}}_t \equiv \hat{\hat{\Xi}}_t - \hat{\hat{\Xi}}_{t-1}. \tag{3.86}$$

$$\xi_t = \rho_\xi\xi_{t-1} + \varepsilon_t \tag{3.87}$$

4 Quadratic approximation of the welfare function

4.1 Welfare function

The loss function can be written as

$$\tilde{U}_t = \pi_b \frac{\left(\bar{C}_t^b\right)^{\sigma_b^{-1}} \left(c_t^b\right)^{1-\sigma_b^{-1}}}{1-\sigma_b^{-1}} + (1-\pi_b) \frac{\left(\bar{C}_t^s\right)^{\sigma_s^{-1}} \left(c_t^s\right)^{1-\sigma_s^{-1}}}{1-\sigma_s^{-1}} - \frac{\psi}{1+\nu} \left(\frac{\tilde{\lambda}_t}{\tilde{\Lambda}_t}\right)^{-\frac{1+\nu}{\nu}} \bar{H}_t^{-\nu} \left(\frac{Y_t}{\tilde{A}_t}\right)^{1+\omega_y} \Delta_t,$$

where I used

$$\lambda_t^b = \left(\bar{C}_t^b\right)^{\sigma_b^{-1}} \left(c_t^b\right)^{-\sigma_b^{-1}},$$

$$\lambda_t^s = \left(\bar{C}_t^s\right)^{\sigma_s^{-1}} \left(c_t^s\right)^{-\sigma_s^{-1}}.$$

4.2 Quadratic approximation: efficient steady state and no spread

Efficient steady state implies that

$$\Phi = 0,$$

and

$$(1-\bar{\tau}) = \mu^p \mu^w.$$

Zero spread implies

$$\bar{\lambda}^b = \bar{\lambda}^s = \Lambda\left(\bar{\lambda}^b, \bar{\lambda}^s\right) = \tilde{\lambda}\left(\bar{\lambda}^b, \bar{\lambda}^s\right) = \tilde{\Lambda}\left(\bar{\lambda}^b, \bar{\lambda}^s\right) = \bar{\lambda},$$

$$\bar{\Omega} = 1,$$

$$0 = \bar{\chi}'\left(\bar{b}\right) + \bar{\Xi}'\left(\bar{b}\right).$$

Further consider that $\bar{\chi}'\left(\bar{b}\right) = \bar{\Xi}'\left(\bar{b}\right) = 0$.

The first term of the objective is then

$$\begin{aligned}\tilde{U}_t^1 \;=\; & \pi_b \left(\bar{C}^b\right)^{\sigma_b^{-1}} \left(\bar{c}^b\right)^{-\sigma_b^{-1}} \left(c_t^b - \bar{c}^b\right) \\ & -\frac{1}{2}\pi_b \sigma_b^{-1} \left(\bar{C}^b\right)^{\sigma_b^{-1}} \left(\bar{c}^b\right)^{-\sigma_b^{-1}-1} \left(c_t^b - \bar{c}^b\right)^2 + \pi_b \sigma_b^{-1} \left(\bar{C}^b\right)^{\sigma_b^{-1}-1} \left(\bar{c}^b\right)^{-\sigma_b^{-1}} \left(c_t^b - \bar{c}^b\right)\left(\bar{C}_t^b - \bar{C}^b\right) \\ & + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)\end{aligned}$$

$\Leftrightarrow$

$$\tilde{U}_t^1 = \pi_b \bar{\lambda} \bar{c}^b \left[\frac{c_t^b - \bar{c}^b}{\bar{c}^b} - \frac{1}{2}\sigma_b^{-1}\left(\frac{c_t^b - \bar{c}^b}{\bar{c}^b}\right)^2 + \sigma_b^{-1}\left(\frac{c_t^b - \bar{c}^b}{\bar{c}^b}\right)\left(\frac{\bar{C}_t^b - \bar{C}^b}{\bar{C}^b}\right)\right] + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)$$

$\Leftrightarrow$

$$\tilde{U}_t^1 = \pi_b \bar{c}^b \bar{\lambda} \left\{\hat{c}_t^b + \frac{1}{2}\left(1-\sigma_b^{-1}\right)\left[\hat{c}_t^b + (\sigma_b - 1)\,\bar{c}_t^b\right]^2\right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right), \tag{4.1}$$

where I used the fact that

$$c_t^b \equiv \bar{c}^b \exp \hat{c}_t^b \quad \Rightarrow \quad \frac{c_t^b - \bar{c}^b}{\bar{c}^b} = \hat{c}_t^b + \frac{1}{2}\left(\hat{c}_t^b\right)^2 + \mathcal{O}\left(\|\xi\|^3\right)$$
$$\Rightarrow \quad \left(\frac{c_t^b - \bar{c}^b}{\bar{c}^b}\right)^2 = \left(\hat{c}_t^b\right)^2 + \mathcal{O}\left(\|\xi\|^3\right).$$

The second term can equivalently be written as

$$\tilde{U}_t^2 = (1-\pi_b)\,\bar{c}^s\bar{\lambda}\left\{\hat{c}_t^s + \frac{1}{2}\left(1-\sigma_s^{-1}\right)\left[\hat{c}_t^s + (\sigma_s - 1)\,\bar{c}_t^s\right]^2\right\} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right). \tag{4.2}$$

The third term is

$$\begin{aligned}
\tilde{U}_t^3 \;=\; & -\frac{\psi}{1+\nu}\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{Z}}\right)^{1+\omega_y}(1+\omega_y)\left[\frac{Y_t-\bar{Y}}{\bar{Y}} + \frac{1}{2}\omega_y\left(\frac{Y_t-\bar{Y}}{\bar{Y}}\right)^2\right] \\
& -\frac{\psi}{1+\nu}\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{Z}}\right)^{1+\omega_y}(1+\omega_y)\left[-(1+\omega_y)\frac{Z_t-\bar{Z}}{\bar{Z}} - \nu\frac{\bar{H}_t-\bar{H}}{\bar{H}}\right]\frac{Y_t-\bar{Y}}{\bar{Y}} \\
& -\frac{\psi}{1+\nu}\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{Z}}\right)^{1+\omega_y}(\Delta_t - 1) \\
& -\frac{\psi}{1+\nu}\bar{H}^{-\nu}\left(\frac{\bar{Y}}{\bar{Z}}\right)^{1+\omega_y}\frac{1}{2}\frac{1+\nu}{\nu}\pi_b(1-\pi_b)\left(\frac{\psi}{\psi_b}\right)^{\frac{1}{\nu}}\left(\frac{\psi}{\psi_s}\right)^{\frac{1}{\nu}}\left(\frac{\lambda_t^b-\bar{\lambda}}{\bar{\lambda}} - \frac{\lambda_t^s-\bar{\lambda}}{\bar{\lambda}}\right)^2 \\
& + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right),
\end{aligned}$$

and notice that the first derivatives w.r.t. λ_t^b and λ_t^s are zero in steady state. We can further simplify it to:

$$\begin{aligned}
\tilde{U}_t^3 \;=\; & -\bar{\lambda}\bar{Y}\left\{\hat{Y}_t + \frac{1}{2}(1+\omega_y)\,\hat{Y}_t^2 - \omega_y q_t \hat{Y}_t + (1+\omega_y)^{-1}\hat{\Delta}_t\right\} \\
& -\frac{1}{2}\bar{\lambda}\bar{Y}\frac{\pi_b(1-\pi_b)}{\nu(1+\omega_y)}\left(\frac{\psi}{\psi_b}\right)^{\frac{1}{\nu}}\left(\frac{\psi}{\psi_s}\right)^{\frac{1}{\nu}}\hat{\Omega}_t^2,
\end{aligned} \tag{4.3}$$

with

$$q_t \equiv \frac{(1+\omega_y)\,z_t + \nu\bar{h}_t}{\omega_y}. \tag{4.4}$$

Now consider a second order approximation to the market clearing condition:

$$Y_t = \pi_b c_t^b + (1-\pi_b)\,c_t^s + G_t + \Xi_t(b_t),$$

hence

$$\begin{aligned}
\bar{Y}\left(\frac{Y_t-\bar{Y}}{\bar{Y}}\right) \;=\; & \pi_b\bar{c}^b\frac{c_t^b-\bar{c}^b}{\bar{c}^b} + (1-\pi_b)\,\bar{c}^s\frac{c_t^s-\bar{c}^s}{\bar{c}^s} + \bar{Y}\frac{G_t-\bar{G}}{\bar{Y}} + \bar{Y}\hat{\bar{\Xi}}_t\left(1+\eta\hat{b}_t\right) \\
& + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)
\end{aligned}$$

$\Leftrightarrow$

$$\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2 = \pi_b s_b \left(\hat{c}_t^b + \frac{1}{2}\left(\hat{c}_t^b\right)^2 \right) + (1-\pi_b)\, s_s \left(\hat{c}_t^s + \frac{1}{2}\left(\hat{c}_t^s\right)^2 \right) + \hat{G}_t + \hat{\bar{\Xi}}_t \left(1 + \eta \hat{b}_t\right) + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right). \quad (4.5)$$

Further recall that

$$\hat{\Omega}_t \equiv \hat{\lambda}_t^b - \hat{\lambda}_t^s = \sigma_s^{-1}\left(\hat{c}_t^s - \bar{c}_t^s\right) - \sigma_b^{-1}\left(\hat{c}_t^b - \bar{c}_t^b\right),$$

$$\hat{c}_t^b = \bar{c}_t^b - \sigma_b \hat{\lambda}_t^b,$$
$$\hat{c}_t^s = \bar{c}_t^s - \sigma_s \hat{\lambda}_t^s.$$

The first order approximation to the resources constraint yields

$$\hat{Y}_t = \pi_b s_b \hat{c}_t^b + (1-\pi_b)\, s_s \hat{c}_t^s + \hat{G}_t + \hat{\bar{\Xi}}_t + \mathcal{O}\left(\|\xi\|^2\right), \quad (4.6)$$

and using the definition of g_t, we can write

$$\hat{Y}_t - g_t - \hat{\bar{\Xi}}_t = \pi_b s_b \left(\hat{c}_t^b - \bar{c}_t^b\right) + (1-\pi_b)\, s_s \left(\hat{c}_t^s - \bar{c}_t^s\right) + \mathcal{O}\left(\|\xi\|^2\right), \quad (4.7)$$

and

$$\left(\hat{c}_t^b - \bar{c}_t^b\right) = \bar{\sigma}^{-1}\sigma_b \left[\left(\hat{Y}_t - g_t - \hat{\bar{\Xi}}_t\right) - (1-\pi_b)\, s_s \sigma_s \hat{\Omega}_t\right] + \mathcal{O}\left(\|\xi\|^2\right), \quad (4.8)$$

$$\left(\hat{c}_t^s - \bar{c}_t^s\right) = \bar{\sigma}^{-1}\sigma_s \left[\left(\hat{Y}_t - g_t - \hat{\bar{\Xi}}_t\right) + \pi_b s_b \sigma_b \hat{\Omega}_t\right] + \mathcal{O}\left(\|\xi\|^2\right). \quad (4.9)$$

Sum now the first two terms in the utility:

$$\begin{aligned}\tilde{U}_t^1 + \tilde{U}_t^2 = {} & \bar{\lambda}\bar{Y}\left\{\pi_b s_b \left(\hat{c}_t^b + \frac{1}{2}\left(\hat{c}_t^b\right)^2 \right) + (1-\pi_b)\, s_s \left(\hat{c}_t^s + \frac{1}{2}\left(\hat{c}_t^s\right)^2 \right)\right\} \\ & -\frac{1}{2}\bar{\lambda}\bar{Y}\left\{\pi_b s_b \sigma_b^{-1}\left(\hat{c}_t^b - \bar{c}_t^b\right)^2 + (1-\pi_b)\, s_s \sigma_s^{-1}\left(\hat{c}_t^s - \bar{c}_t^s\right)^2\right\} \\ & + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}\tilde{U}_t^1 + \tilde{U}_t^2 = {} & \bar{\lambda}\bar{Y}\left\{\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2 - \hat{\bar{\Xi}}_t\left(1 + \eta\hat{b}_t\right)\right\} \\ & -\frac{1}{2}\bar{\lambda}\bar{Y}\left\{\pi_b s_b \sigma_b \left[\bar{\sigma}^{-1}\left(\hat{Y}_t - g_t - \hat{\bar{\Xi}}_t\right) - (1-\pi_b)\, s_s \sigma_s \bar{\sigma}^{-1}\hat{\Omega}_t\right]^2\right\} \\ & -\frac{1}{2}\bar{\lambda}\bar{Y}\left\{(1-\pi_b)\, s_s \sigma_s \left[\bar{\sigma}^{-1}\left(\hat{Y}_t - g_t - \hat{\bar{\Xi}}_t\right) + \pi_b s_b \sigma_b \bar{\sigma}^{-1}\hat{\Omega}_t\right]^2\right\} \\ & + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}\tilde{U}_t^1+\tilde{U}_t^2 &= \bar{\lambda}\bar{Y}\left\{\hat{Y}_t+\frac{1}{2}\hat{Y}_t^2-\hat{\tilde{\Xi}}_t\left(1+\eta\hat{b}_t\right)\right\} \\ &\quad -\frac{1}{2}\bar{\lambda}\bar{Y}\left\{\bar{\sigma}^{-1}\left(\hat{Y}_t-g_t-\bar{Y}^{-1}\hat{\Xi}_t\right)^2+\frac{[\pi_b s_b\sigma_b]\left[(1-\pi_b)\,s_s\sigma_s\right]}{\bar{\sigma}}\hat{\Omega}_t^2\right\} \\ &\quad +t.i.p.+\mathcal{O}\left(\|\xi\|^3\right),\end{aligned} \tag{4.10}$$

and combine with the third term to get:

$$\begin{aligned}\tilde{U}_t &= \bar{\lambda}\bar{Y}\left\{\hat{Y}_t+\frac{1}{2}\hat{Y}_t^2-\hat{\tilde{\Xi}}_t\left(1+\eta\hat{b}_t\right)\right\} \\ &\quad -\frac{1}{2}\bar{\lambda}\bar{Y}\left\{\bar{\sigma}^{-1}\left(\hat{Y}_t-g_t-\hat{\tilde{\Xi}}_t\right)^2+\frac{[\pi_b s_b\sigma_b]\left[(1-\pi_b)\,s_s\sigma_s\right]}{\bar{\sigma}}\hat{\Omega}_t^2\right\} \\ &\quad -\bar{\lambda}\bar{Y}\left\{\hat{Y}_t+\frac{1}{2}(1+\omega_y)\,\hat{Y}_t^2-\omega_y q_t\hat{Y}_t+(1+\omega_y)^{-1}\,\hat{\Delta}_t\right\} \\ &\quad -\frac{1}{2}\bar{\lambda}\bar{Y}\frac{\pi_b(1-\pi_b)}{\nu(1+\omega_y)}\left(\frac{\psi}{\psi_b}\right)^{\frac{1}{\nu}}\left(\frac{\psi}{\psi_s}\right)^{\frac{1}{\nu}}\hat{\Omega}_t^2 \\ &\quad +t.i.p.+\mathcal{O}\left(\|\xi\|^3\right)\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}\tilde{U}_t &= -\frac{\bar{\lambda}\bar{Y}}{2}\left\{\omega_y\left(\hat{Y}_t-q_t\right)^2+\bar{\sigma}^{-1}\left(\hat{Y}_t-g_t-\hat{\tilde{\Xi}}_t\right)^2+\tilde{\lambda}_\Omega\hat{\Omega}_t^2+\frac{2}{1+\omega_y}\hat{\Delta}_t\right\} \\ &\quad -\bar{\lambda}\Xi_t(b_t)+t.i.p.+\mathcal{O}\left(\|\xi\|^3\right),\end{aligned} \tag{4.11}$$

with

$$\tilde{\lambda}_\Omega\equiv\frac{[\pi_b s_b\sigma_b]\left[(1-\pi_b)\,s_s\sigma_s\right]}{\bar{\sigma}}+\frac{\pi_b(1-\pi_b)}{\nu(1+\omega_y)}\left(\frac{\psi}{\psi_b}\right)^{\frac{1}{\nu}}\left(\frac{\psi}{\psi_s}\right)^{\frac{1}{\nu}}.$$

Further write

$$\begin{aligned}&\omega_y\left(\hat{Y}_t-q_t\right)^2+\bar{\sigma}^{-1}\left(\hat{Y}_t-g_t-\hat{\tilde{\Xi}}_t\right)^2 \\ =\; &\left(\omega_y+\bar{\sigma}^{-1}\right)\hat{Y}_t^2-2\left[\bar{\sigma}^{-1}\left(g_t+\hat{\tilde{\Xi}}_t\right)+\omega_y q_t\right]\hat{Y}_t+t.i.p. \\ =\; &\left(\omega_y+\bar{\sigma}^{-1}\right)\left(\hat{Y}_t-\left(\omega_y+\bar{\sigma}^{-1}\right)^{-1}\left[\bar{\sigma}^{-1}\left(g_t+\hat{\tilde{\Xi}}_t\right)+\omega_y q_t\right]\right)^2 \\ =\; &\left(\omega_y+\bar{\sigma}^{-1}\right)\left(\hat{Y}_t-\hat{Y}_t^n-\left(\omega_y+\bar{\sigma}^{-1}\right)^{-1}\bar{\sigma}^{-1}\hat{\tilde{\Xi}}_t\right)^2.\end{aligned}$$

We can then write

$$\begin{aligned}\tilde{U}_t &= -\frac{\bar{\lambda}\bar{Y}}{2}\left\{\left(\omega_y+\bar{\sigma}^{-1}\right)\left(\hat{Y}_t-\hat{Y}_t^*\right)^2+\tilde{\lambda}_\Omega\hat{\Omega}_t^2+\frac{2}{1+\omega_y}\hat{\Delta}_t\right\}-\bar{\lambda}\bar{Y}\hat{\tilde{\Xi}}_t\left(1+\eta\hat{b}_t\right) \\ &\quad +t.i.p.+\mathcal{O}\left(\|\xi\|^3\right).\end{aligned} \tag{4.12}$$

with

$$\hat{Y}_t^* \equiv \hat{Y}_t^n + \left(\omega_y + \bar{\sigma}^{-1}\right)^{-1} \bar{\sigma}^{-1}\hat{\bar{\Xi}}_t \tag{4.13}$$

Now we can get a second order approximation for Δ_t:

$$\begin{aligned}
(\Delta_t - 1) &= \alpha\left(\Delta_{t-1} - 1\right) \\
&\quad +\alpha\theta\left(1+\omega_y\right)\left(\Pi_t - 1\right) + \alpha\theta\left(1+\omega_y\right)\left[\theta\left(1+\omega_y\right) - 1\right]\frac{1}{2}\left(\Pi_t - 1\right)^2 \\
&\quad -\alpha\theta\left(1+\omega_y\right)\left(\Pi_t - 1\right) \\
&\quad -\alpha\theta\left(1+\omega_y\right)\left[(\theta-2) - \frac{\alpha(\theta-1)}{1-\alpha}\left(\frac{\theta\left(1+\omega_y\right)}{\theta-1} - 1\right)\right]\frac{1}{2}\left(\Pi_t - 1\right)^2 \\
&\quad +\mathcal{O}\left(\|\xi\|^3\right)
\end{aligned}$$

$\Rightarrow$

$$\hat{\Delta}_t = \alpha\hat{\Delta}_{t-1} + \theta\left(1+\omega_y\right)\left(1+\theta\omega_y\right)\frac{\alpha}{1-\alpha}\frac{1}{2}\hat{\pi}_t^2 + \mathcal{O}\left(\|\xi\|^3\right). \tag{4.14}$$

Further notice that

$$\begin{aligned}
&-\bar{\lambda}\bar{Y}\sum_{s=0}^{\infty}\beta^{s-t}\frac{\hat{\Delta}_{t+s}}{1+\omega_y} \\
=\; &-\bar{\lambda}\bar{Y}\sum_{s=0}^{\infty}\beta^{s-t}\left[\frac{\alpha\hat{\Delta}_{t+s-1}}{1+\omega_y} + \theta\left(1+\theta\omega_y\right)\frac{\alpha}{1-\alpha}\frac{1}{2}\hat{\pi}_{t+s}^2\right] + \mathcal{O}\left(\|\xi\|^3\right) \\
=\; &-\bar{\lambda}\bar{Y}\sum_{s=0}^{\infty}\beta^{s-t}\theta\left(1+\theta\omega_y\right)\frac{\alpha}{1-\alpha}\frac{1}{2}\hat{\pi}_{t+s}^2 - \bar{\lambda}\bar{Y}\sum_{s=1}^{\infty}\beta^{s-t}\frac{\alpha\hat{\Delta}_{t+s-1}}{1+\omega_y} - \bar{\lambda}\bar{Y}\frac{\alpha\hat{\Delta}_{t-1}}{1+\omega_y} + \mathcal{O}\left(\|\xi\|^3\right) \\
=\; &-\bar{\lambda}\bar{Y}\sum_{s=0}^{\infty}\beta^{s-t}\theta\left(1+\theta\omega_y\right)\frac{\alpha}{1-\alpha}\frac{1}{2}\hat{\pi}_{t+s}^2 - \alpha\beta\bar{\lambda}\bar{Y}\sum_{s=0}^{\infty}\beta^{s-t}\frac{\hat{\Delta}_{t+s}}{1+\omega_y} + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right)
\end{aligned}$$

$\Rightarrow$

$$-\frac{\bar{\lambda}\bar{Y}}{2}\sum_{s=0}^{\infty}\beta^{s-t}\frac{2}{1+\omega_y}\hat{\Delta}_{t+s} = -\frac{\bar{\lambda}\bar{Y}}{2}\sum_{s=0}^{\infty}\beta^{s-t}\frac{\theta\left(1+\theta\omega_y\right)}{(1-\alpha\beta)}\frac{\alpha}{1-\alpha}\hat{\pi}_{t+s}^2 + t.i.p. + \mathcal{O}\left(\|\xi\|^3\right).$$

so that we can write the period welfare as

$$\begin{aligned}
\tilde{U}_t &= -\frac{\bar{\lambda}\bar{Y}}{2}\left\{\omega_y\left(\hat{Y}_t - \hat{Y}_t^*\right)^2 + \tilde{\lambda}_\Omega\hat{\Omega}_t^2 + \frac{\theta}{\xi}\hat{\pi}_{t+s}^2\right\} - \bar{\lambda}\bar{Y}\eta\hat{\bar{\Xi}}_t\hat{b}_t \\
&\quad +t.i.p. + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned} \tag{4.15}$$

with $\hat{Y}_t^n$ defined in (3.56).

We can then say that maximizing this welfare functional is equivalent to minimizing the following loss function

$$L_t = \hat{\pi}_t^2 + \lambda_y\left(\hat{Y}_t - \hat{Y}_t^*\right)^2 + \lambda_\Omega\hat{\Omega}_t^2 + \lambda_\Xi\hat{\bar{\Xi}}_t\hat{b}_t, \tag{4.16}$$

with

$$\hat{Y}_t^* \equiv \hat{Y}_t^n + \frac{\bar{\sigma}^{-1}}{\omega_y + \bar{\sigma}^{-1}} \hat{\tilde{\Xi}}_t,$$

$$\lambda_y \equiv \frac{\xi}{\theta} \omega_y,$$

$$\lambda_\Omega \equiv \frac{\xi}{\theta} \tilde{\lambda}_\Omega,$$

$$\lambda_\Xi \equiv 2\eta \frac{\xi}{\theta},$$

$$\tilde{\lambda}_\Omega \equiv \frac{[\pi_b s_b \sigma_b] \left[(1-\pi_b) s_s \sigma_s\right]}{\bar{\sigma}} + \frac{\pi_b (1-\pi_b)}{\nu (1+\omega_y)} \left(\frac{\psi}{\psi_b}\right)^{\frac{1}{\nu}} \left(\frac{\psi}{\psi_s}\right)^{\frac{1}{\nu}}.$$

5 Calibration

The paper discusses the strategy for the calibration. Here we present the exact values for all the parameters.

Notice that unless otherwise mentioned, all exogenous disturbances follow an AR(1) process with autocorrelation coefficient equal to ρ_ξ, which in the baseline calibration is set to 0.9. The only expection to this is the autocorrelation coefficient of the monetary policy shock, in the case of a Taylor rule, in which case we consider an autocorrelation coefficient of 0.6.

Exogenous, takes resources The spread in the FF model is exogenous and consumes resources, i. e. $\varkappa = 0$, $\bar{\chi} = 0$, $\bar{\Xi} = \bar{\omega}$ and $\eta = 1$. The full list of parameters is:

ϕ^{-1}	0.75	$1+\bar{\omega}$	$(1.02)^{1/4}$	s_Ξ	0.0159	$\bar{\sigma}^{-1}$	0.16
α	0.66	δ	0.975	ρ_b^g	0	σ	8.9286
ω_y	0.473	π_b	0.5	s_c	0.7	σ_b	13.802
ν	0.1048	ρ_b	3.2	s_b	0.7821	σ_s	2.7604
$(\theta-1)^{-1}$	0.15	η	1	s_s	0.6179	σ_b/σ_s	5
μ^p	1.15	$\varkappa$	0	s_b/s_s	1.2657	$\bar{\lambda}_b/\bar{\lambda}_s$	1.2175
$\bar{r}^d$	0.01	$\bar{\mu}^w$	1	ψ_b	1.1492	$\bar{Z}$	1
β	0.9874	$\bar{\tau}$	0.2	ψ_s	0.9439	$\bar{H}$	1
$\bar{Y}$	1	s_g	0.2841	ψ_b/ψ_s	1.2175	ρ_ξ	0.9

Endogenous, takes resources The spread in the FF model is endogenous and takes resources, i.e. $\eta > 1$ (such that the spread elasticity to debt is 1/4), $\varkappa = 0$ and $\bar{\chi} = 0$.

The full list of parameters is:

ϕ^{-1}	0.75	$1+\bar{\omega}$	$(1.02)^{1/4}$	s_{Ξ}	0.0003	$\bar{\sigma}^{-1}$	0.16
α	0.66	δ	0.975	ρ_b^g	0	σ	8.9286
ω_y	0.473	π_b	0.5	s_c	0.7	σ_b	13.8019
ν	0.1048	ρ_b	3.2	s_b	0.7821	σ_s	2.7604
$(\theta-1)^{-1}$	0.15	η	51.623	s_s	0.6179	σ_b/σ_s	5
μ^p	1.15	$\varkappa$	0	s_b/s_s	1.2657	$\bar{\lambda}_b/\bar{\lambda}_s$	1.2175
$\bar{r}^d$	0.01	$\bar{\mu}^w$	1	ψ_b	1.1492	$\bar{Z}$	1
β	0.9874	$\bar{\tau}$	0.2	ψ_s	0.9439	$\bar{H}$	1
$\bar{Y}$	1	s_g	0.2997	ψ_b/ψ_s	1.2175	ρ_ξ	0.9

6 Models and specifications

In all exercises we consider three versions of the model:

FF Full model with heterogeneous households and a spread between saving and borrowing interest rates.

NoFF Model with heterogeneous households but no spread between saving and borrowing interest rates.

Normal Model without heterogeneous households or spread between saving and borrowing interest rates. This is equivalent to standard New-Keynesian model.

The parametrization differences are:

- in NoFF and RepHH we consider $\beta = \left(1+\bar{r}^d\right)^{-1}$,
- in RepHH we consider $\bar{C}^b = \bar{C}^s = \bar{C}$ and $\sigma_b = \sigma_s = \sigma$.

7 Policy rules

In each case we consider the following alternative policies:[8]

Optimal This is the optimal policy.

Taylor This is the basic Taylor rule

$$\hat{\imath}_t^d = \phi_\pi \pi_t + \frac{\phi_y}{4}\hat{Y}_t + \xi_t^i, \tag{7.1}$$

where $\xi_t^i = \rho\xi_{t-1}^i + \varepsilon_t^i$, $\phi_\pi = 2$, $\phi_y = 1$ and $\rho = 0.75$.

[8] Any variable $\hat{x}_t$ is defined as $\hat{x}_t \equiv \ln(x_t/\bar{x})$ except for the interest rates, which are defined as $\hat{\imath}_t \equiv \log\left((1+i_t)/(1+\bar{\imath})\right)$ and the spread is given by $\hat{\omega}_t \equiv \log\left((1+\omega_t)/(1+\bar{\omega})\right)$.

PiStab Inflation stabilization:

$$\pi_t = 0. \tag{7.2}$$

FlexTarget This is the optimal target criterion proposed in Benigno and Woodford (2005) and discussed in the paper. It takes the form of

$$0 = \pi_t + \lambda_x \left(y_t - y_{t-1}\right), \tag{7.3}$$

where λ_x the optimal weight is defined in Benigno and Woodford (2005), and y_t is the output gap measure relative to the optimal target defined as well in Benigno and Woodford (2005), except that here we replace the autonomous spending disturbance g_t with $g_t + \hat{\tilde{\Xi}}_t$, to reflect the fact that exogenous increases in the intermediation costs increase autonomous expenditures.

References

Benigno, P. and M. Woodford (2005). Inflation stabilization and welfare: The case of a distorted steady state. *Journal of the European Economic Association 3*(6), 1185–1236.