

# Doing Without Money: Controlling Inflation in a Post-Monetary World \*

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## Abstract

Central banks now generally agree that conventional monetary aggregates are of little use as targets or even indicators for monetary policy, owing to the instability of money demand relations in economies with well-developed financial markets. But monetary theory has provided little guidance for the analysis of policies that are not formulated in terms of a path for the money supply, and a stable money demand relation is generally assumed as a central element of a theoretical analysis. This paper, instead, shows that it is possible to analyze equilibrium inflation determination without any reference to either money supply or demand, as long as one specifies policy in terms of a “Wicksellian” interest-rate feedback rule.

The paper’s central result is an approximation theorem, showing the existence, for a simple monetary model, of a well-behaved “cashless limit” in which the money balances held to facilitate transactions become negligible. The relations that determine equilibrium inflation in the cashless limit also provide a useful approximate account in the case of an economy in which monetary frictions are present, but small. The approximation remains valid in the case of time variation in the monetary frictions, including variation of a kind that may result in substantial instability of money demand in percentage terms.

Inflation in the cashless limit is shown to be a function of the gap between the “natural rate” of interest, determined by the supply of goods and opportunities for intertemporal substitution, and a time-varying parameter of the interest-rate rule indicating the tightness of monetary policy. Inflation can be completely stabilized, in principle, by adjusting the policy parameter so as to track variation in the natural rate. Under such a regime, instability of money demand has little effect upon equilibrium inflation, and need not be monitored by the central bank.

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# 1 Monetary Policy without Money

Throughout the English-speaking world, at least, central bankers have abandoned the notion that any of the conventional monetary aggregates constitute a suitable intermediate target for monetary policy. This has resulted from the discovery that these aggregates no longer appear to have any very reliable relationship, at least in the short run, with the variables, such as inflation and real activity, about which policymakers actually care,<sup>1</sup> This development is often attributed to the rapid transformation of financial arrangements since the early 1980's, due both to deregulation and financial innovation. But there is reason to fear that the instability of conventional money demand equations may not relate solely to a transitory period of institutional turbulence, in which an old set of arrangements give way to a new, more rational set, that should then be expected to result in stable econometric relationships once again. For from the standpoint of economic theory, there is no reason to believe that there is any uniquely rational or efficient set of arrangements that result in any stable demand for money at all. Instead, because the demand for an asset that is dominated in terms of its purely pecuniary return depends upon the existence of transactions frictions that people benefit, individually and jointly, by overcoming, there is every reason to expect further innovations, due to improvements in information processing and to increased creativity in the evasion of the remaining regulatory constraints, that continually reduce the quantity of the monetary base that needs to be held (on average) in order to carry out a given volume of transactions. The only natural limit to this process is an ideal state of frictionless financial markets, in which there is no positive demand for the monetary base at all, if it is dominated by other financial assets, and no determinate demand for it if it is not.

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<sup>1</sup>See, e.g., Friedman and Kuttner (1996) and Estrella and Mishkin (1996).

A consequence of the apparent instability of money demand has been a widely perceived need for a new way of thinking about the source of guidelines for monetary policy. Much enthusiasm has recently been expressed for “inflation targeting”,<sup>2</sup> under which central banks should have clear targets for an ultimate goal variable (inflation), rather than for an intermediate target that they seek to control only because it is thought to be a reliable means of achieving desirable paths for other variables. Agreement upon a target path for inflation, however, does not settle the question of how a central bank should adjust policy on a day-to-day basis in order to achieve that target. There remains the need for a coherent approach to the design of instrument rules or operating procedures for monetary policy, in a world in which one cannot rely upon any stable money demand function.

Monetary theory has offered surprisingly little guidance in this quest, for the existing literature is almost entirely concerned with the analysis of the consequences of alternative paths (or perhaps, state-contingent rules) for the money supply. The reason for this is not simply that academics have remained attached to models that imply that a constant growth rate for “the money supply” should be optimal, if only one knew how to measure it. The main problem is instead a deeper one. This is that the conventional approach to the problem of the determination of the general level of prices in an economy takes as its starting point the quantity theory of money, according to which the equilibrium price level is that value that makes the real purchasing power of the existing money supply equal to the desired level of real money balances. As a consequence, it is taken for granted that for purposes of analysis, the differences between alternative monetary policies reduce to their different consequences for the path of the money supply, and that it suffices simply to consider

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<sup>2</sup>See, e.g., Leiderman and Svensson (1995), Freedman (1996), and Bernanke and Mishkin (1997).

the theoretical consequences of different such paths. And the existence of a well-defined demand function for real money balances is taken as the *sine qua non* of a coherent model; refinements of the theory have mainly sought to develop increasingly sophisticated models of that relation, taking for granted its centrality to any model of price-level determination and of the consequences of monetary policy. Thus the literature has much to say about the choice among alternative paths for the money supply in a world where the money demand function takes one form or another, but next to nothing to say about the choice among operating procedures that make no reference to any measure of the money supply, in a world where the connection between money balances and transactions of real goods and services is unstable, poorly measured, and increasingly irrelevant.

I wish to argue that this entire approach to the problem of price-level determination is neither necessary nor desirable. Instead, it is possible to analyze the determination of the money prices of goods and services without any reference either to the money supply or to a money demand relation. This might seem not to be possible, for the well-known reason that, when one abstracts from the role of money in exchange, the relations determining the supply of and demand for real goods and services involve only relative prices, and not the absolute (money) prices of any goods. The key to the alternative approach is that equilibrium money prices are determined by the way in which *government policy* depends upon the absolute price level; monetary and/or fiscal policy rules depend upon the the general level of prices, in such a way as to make only a certain price level consistent with equilibrium. In the type of policy regime that I mainly consider here, that I call Wicksellian, it is the monetary policy rule that makes real quantities depend upon the level of money prices. This case is of particular relevance to current discussions of inflation targeting, since the existence of a

target path for the price level suggests that policy actions should depend upon the degree to which the current or expected future price level differs from that target.

It might be suggested, at least from the point of view of abstract monetary theory, that an account of price-level determination along such lines is unsatisfactory, because it obviously cannot apply to a world without government, or at least without any government that intervenes in the functioning of markets in any way. The case of no government, in which there is no central bank, but simply a fixed quantity of money (a durable token with no intrinsic value) that happens to exist, has played an especially prominent role in the literature on monetary theory, doubtless in the belief that one is thereby led to focus on the essential aspects of the problem of price-level determination. I would argue instead that this familiar approach is misguided. It emphasizes a case that is surely of no practical interest, especially if the aim of the theory is to provide advice for central banks. Furthermore, abstraction from features of the world in economic models is justified only when doing so simplifies the analysis, thus facilitating understanding. But abstracting from government policy when analyzing price-level determination does not simplify matters; instead, because of the pathological features of this case, the analysis must give attention to features of the economy (namely, the demand for money balances to facilitate transactions) that are in fact inessential under many other cases of greater practical importance.

I would argue that it is in fact a great disadvantage of the quantity-theoretic approach to price-level determination that it implies that transactions frictions – obstacles to the execution of mutually beneficial trades – are an essential element in price-level determination. I wish to demonstrate, instead, the possibility of a theory of price-level determination that applies equally to to the limiting case in which such frictions become insignificant, and to

argue for the usefulness of analysis of that case, as at least a first approximation for most purposes. Of course, an understanding of price-level determination in this “cashless limit” does not in any way preclude one from also considering how this baseline model is perturbed when account is taken of realistic transactions frictions. But even if that is one’s eventual goal, I believe that understanding of how the more complex model works is likely to be aided by an analysis of the cashless baseline model first – a way of proceeding that is, of course, commonplace in economic analysis. Nor is it obvious that the addition of frictions to take account of the fact that currency and bank reserves are held despite being dominated in return by other assets would be the most important direction of elaboration of the simple baseline model presented here, from the point of view of improving the practical reliability of the conclusions drawn from the model. <sup>3</sup> Indeed, I believe that in monetary theory there have been significant costs of the misplaced emphasis upon the need to model the sources of money demand, to the neglect of other aspects of model specification that matter more, for practical purposes, such as the nature of price adjustment. All too often, the analysis of complicated transactions frictions, intended to provide a more realistic account of the reasons why money is held, has required that other aspects of the model be radically simplified, to the extent that few if any practical questions about monetary policy can be addressed.

Because the central economic relation emphasized by the quantity-theoretic approach, the money demand function, depends for its existence upon market frictions, this approach also makes price-level determination appear to depend upon features of the economy that

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<sup>3</sup>In Rotemberg and Woodford (1997), a more elaborate version of the model of a Wicksellian regime proposed here is used to interpret the co-movements of inflation, output, and interest rates in the U.S. economy since 1980. Several additional complications are added to the model in order to better account for the relations observed among those series – in particular, decision lags in both purchasing decisions and pricing decisions – but these do not include any reference to the role of money in transactions or to the demand for the monetary base.

need not be stable over time. Of course, any structural relation of an economic model that is not an identity is certain to be subject to disturbances.<sup>4</sup> But the money demand function is different in that there are good economic reasons to expect such a relation to be unstable, especially in an economy with sophisticated financial markets and institutions. Furthermore, I shall argue that instability in this relation need not have any consequences for the ability of suitably chosen monetary policy to stabilize inflation, or aggregate demand more generally. In the model developed below, I show that if the cashless limit is a sufficiently good approximation (*i.e.*, if the process of financial innovation has proceeded far enough to make cash balances of sufficiently small importance in carrying out transactions), then fluctuations in money demand (that may, in percentage terms, be substantial) have only negligible effects upon the equilibrium price level under a Wicksellian policy regime, even though this regime does not require the central bank to track the fluctuations in money demand for its implementation. If this is true in practice, then a model of price-level determination that dispenses with reference to that relation and its disturbances is obviously desirable.

Finally, the quantity-theoretic approach to price-level determination is sometimes used to argue that unchecked financial innovation and deregulation are likely to result in macroeconomic instability, or a loss of control over aggregate spending and hence inflation on the part of the central bank.<sup>5</sup> It is thus argued that there is a conflict between the policy goals of microeconomic efficiency in financial arrangements on the one hand, and macroe-

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<sup>4</sup>This is certainly true, for example, of the central relation of the simple model of price-level determination in the cashless limit presented in section 3, the Euler equation relating the planned path of consumption demand to the real interest rate, and indeed I allow for disturbances to this equation in an extension of the model in section 4. The importance of these disturbances in practice is measured by Rotemberg and Woodford (1997), and their nature is a crucial determinant of the account of optimal monetary policy given in that paper.

<sup>5</sup>See, *e.g.*, Cagan (1986) for an expression of concerns of this sort.

conomic stability on the other. The analysis presented here suggests that such concerns are misplaced. It is shown that under a Wicksellian regime, the equilibrium price level remains well-defined in the cashless limit, and near that limit, as just mentioned, disturbances to the transactions technology that cause large fluctuations in money demand have little effect on the price level. Thus there would seem to be no reason to impede progress toward that state of greater efficiency in financial intermediation for the sake of macroeconomic stability.

The plan of the paper is as follows. In section 2, I introduce a simple model of a monetary economy, in which the degree to which money is used as a means of payment is a parameter that may be continuously varied, and made arbitrarily close to zero. The model has quite conventional implications with regard to money demand; in fact, the equilibrium conditions of this model are formally identical to those of a Sidrauski-Brock representative household model with money in the utility function. The only reason here for specification of the details of the transactions technology taken to lie behind that familiar indirect utility function is in order to clarify what I mean by the “cashless limit” of this model. In section 3, I consider price-level determination in the cashless limit, and show that in the case of a particular class of policy regimes, that I call Wicksellian, the model’s equilibrium remains well-behaved in this limit. Furthermore, the equilibrium in the limiting case, which does not depend upon any properties of money demand, can be used to approximate the equilibrium of the model in any case in which the transactions frictions are small enough (though present), including cases in which they are small but subject to significant variation over time, in relative terms. In section 4, I describe more explicitly the factors that determine the path of the price level in the cashless limit of such a regime. I show how it is useful to think of the real disturbances that affect price-level determination as shifts in Wicksell’s (1898) “natural rate” of interest,

and that inflation stabilization depends upon adjusting the nominal interest rate that is the central bank's instrument so that it tracks variations in the natural rate, much as Wicksell advocated. Section 5 concludes.

## 2 Price-Level Determination in a Monetary Economy

Here I offer a simple illustration of how the equilibrium relations involved in price-level determination can remain well-behaved in the limit as the amount of money used in transactions goes to zero. I model the role of money in transactions by assuming a “cash-in-advance constraint” that applies to some transactions; I then consider the consequences of progressively reducing the fraction of transactions to which the constraint applies. In the “cashless limit”, the fraction of transactions subject to the constraint is zero. For certain ways of specifying monetary policy, that I would argue are of practical relevance, the equilibrium path of the price level can be shown to be continuous in this limit. It follows that price-level determination in such “near-cashless” economies can be understood without any reference to the transactions frictions that determine money demand; and that variations in those transactions frictions, that may cause significant variation (in percentage terms) in money demand, need not have significant effects upon the equilibrium price level.

In order to consider continuous variation in the fraction of transactions subject to the cash-in-advance constraint, it is convenient to introduce a continuum of differentiated goods. This specification also has the advantage that, as in Lucas (1980), one can motivate the use of money in transactions as a way of allowing less specialization in consumption. I shall also follow Lucas and Stokey (1987) in assuming that some goods can be purchased without the use of money; the possibility of substitution between “cash” and “credit goods” is then a

simple reason for money demand to be interest-sensitive. The latter aspect of the model is crucial insofar as I wish to consider monetary policies in which the central bank's instrument is a short-term nominal interest rate.

Consider an economy made up of a continuum of identical infinite-lived households, located on a circle  $S$  of circumference 1. Each point  $j \in S$  indexes both a differentiated consumption good, produced only at that location, and a household that specializes in the production of that good. While each household  $j$  produces only good  $j$ , it consumes all of the goods. Each household seeks to maximize a lifetime utility given by

$$\sum_{t=0}^{\infty} \beta^t u(C_t), \quad (2.1)$$

where  $u$  is an increasing, strictly concave, twice-differentiable function,  $0 < \beta < 1$  is a discount factor, and  $C_t$  is an index of purchases of all of the differentiated goods, defined by

$$C_t \equiv \left[ \int_S c_t(i)^\gamma di \right]^{1/\gamma} \quad (2.2)$$

for some  $0 < \gamma < 1$ .<sup>6</sup> I assume in addition that preferences satisfy

$$\lim_{C \rightarrow 0} u'(C) = \infty, \quad (2.3)$$

so that consumption is necessarily desired in each period.

The payments technology works as follows. If a household  $j$  purchases some of the good produced by a household  $i$  that is sufficiently close to its own location, the transaction generates a charge against an account that is settled at the end of the period, without any need for the buyer to hold cash balances. On the other hand, in the case of transactions between more distant locations, a third-party billing technology is used. The latter technology requires the

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<sup>6</sup>This is a constant-elasticity-of-substitution aggregator, with an elasticity of substitution among goods given by  $1/(1 - \gamma)$ .

buyer to hold cash balances at the end of the period sufficient to cover all charges made against this account during the period; the amount charged will then be delivered to the seller, at the beginning of the subsequent period. Thus the second kind of transactions, but not the first, require the buyer to maintain cash balances; they are also more costly for the seller, in that the seller receives payment only at the beginning of the following period, and thus not in time to invest the proceeds in an interest-earning account at the end of the period in which the sale occurs. The advantage of the second technology is that it is assumed to require less ability on the part of the seller to verify the creditworthiness of the buyer, which instead becomes the responsibility of the third party.<sup>7</sup> I assume furthermore that the first technology (informal credit) is used in the case of transactions for which  $|i - j| \leq (1 - \alpha)/2$ , while the second (cash settlement) must be used when  $|i - j| > (1 - \alpha)/2$ , where  $0 < \alpha \leq \gamma$  is a parameter indicating the fraction of goods that can only be purchased using cash.<sup>8</sup>

Because payment for a sale using the “cash settlement” technology is received by the seller at a later point in time, a seller will not in general be indifferent between the two types of payments, and so may charge more in the case of a “cash settlement” sale. We may think of this as occurring through the addition of an interest charge to accounts that are settled only at the beginning of the following period. We may then let  $p_t(j)$  denote the price (in units of money) charged for good  $j$  in period  $t$ , to either kind of buyer, and  $R_t^s(j)$  be

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<sup>7</sup>It may be wondered why I do not simply assume, as in Lucas and Stokey (1987), that the second type of transactions require immediate payment in cash, out of money balances of the buyer. This more familiar form of “cash-in-advance” constraint would serve equally well to generate an equilibrium demand for money balances to facilitate transactions. However, the more complex setup proposed here has several advantages when it comes to the interpretation of the equilibrium conditions eventually obtained. For one, the equilibrium conditions obtained here are identical to those of a standard Sidrauski-Brock model, as is shown below. This allows direct comparability of the discussion here to that in papers such as Woodford (1995, 1996). Furthermore, the equilibrium conditions obtained from the “cashless limit” of this model conform to the standard timing conventions for a model with no cash-in-advance constraint, which is not true in the case of the Lucas-Stokey model.

<sup>8</sup>Here, for any two points  $i, j$  on the circle,  $|i - j|$  measures the arc distance between them.

the gross interest charge imposed in the case of “cash settlement” purchases of that good. Finally, let  $y_{ht}(j)$  be the quantity sold in period  $t$  of good  $j$  using payments technology  $h$ , for  $h = 1, 2$ . Then household  $j$  is subject to the following budget constraints. First, its end-of-period financial assets each period satisfy

$$M_t + B_t = W_t - T_t + p_t(j)y_{1t}(j) - \int_{|i-j| \leq (1-\alpha)/2} p_t(i)c_t(i)di, \quad (2.4)$$

where  $M_t$  represents end-of-period money balances,  $B_t$  the nominal value of net bond holdings (that are negative if the household is a borrower),  $W_t$  the nominal value of beginning-of-period financial wealth (the value of both money and maturing bonds, after settlement of all period  $t - 1$  transactions),  $T_t$  the nominal value of net tax collections (that are negative in the case of transfers from the government), and  $c_t(i)$  the quantity purchased of good  $i$ . Here the final two terms in (2.4) represent the household’s receipts and expenditures respectively, using payments technology 1 (informal credit), because these charges are settled by the end of period  $t$ . The household’s initial wealth  $W_0$  is given as an initial condition; in subsequent periods, it is given by

$$W_{t+1} = M_t + R_t B_t + R_t^s(j)p_t(j)y_{2t}(j) - \int_{|i-j| > (1-\alpha)/2} R_t^s(i)p_t(i)c_t(i)di, \quad (2.5)$$

where  $R_t$  is the gross nominal interest rate on a one-period bond purchased in period  $t$ . Note that the final two terms in (2.5) represent the household’s receipts and expenditures in connection with period  $t$  transactions using payments technology 2, which are settled only at the beginning of period  $t + 1$ . It is this difference in settlement times that accounts for the fact that, in general, a seller will demand interest on cash-settlement transactions.

Use of the “cash settlement” payments technology requires the household to hold money

balances, that satisfy

$$M_t \geq \int_{|i-j| > (1-\alpha)/2} R_t^s(i) p_t(i) c_t(i) di. \quad (2.6)$$

Equation (2.6) is the “cash-in-advance constraint” in our model, that accounts for households’ willingness to hold money, even when bonds pay a positive interest rate. Finally, the household’s bond issues are subject to a borrowing constraint. It is natural to require that

$$W_{t+1} \geq - \sum_{s=t+1}^{\infty} \delta_{t+1,s} [Y_s - T_s], \quad (2.7)$$

where  $Y_s$  denotes the maximum attainable value of period  $s$  sales revenues  $p_s y_{1s}(j) + R_s^s p_s y_{2s}(j) / R_s$ , given the household’s productive capacity in period  $s$ , and  $\delta_{t,s}$  is the discount factor defining the present value at  $t$  of nominal income received in period  $s \geq t$ ,

$$\delta_{t,s} \equiv \prod_{j=t}^{s-1} R_j^{-1}.$$

Constraint (2.7) allows a household to borrow any quantity that it is possible for it to eventually repay, by refraining from consumption for a sufficient time, but not more. Such a constraint makes sense given the requirement that the household’s consumption plan satisfy

$$c_t(i) \geq 0 \quad (2.8)$$

for all goods in all periods. The household’s total sales to the two types of buyers are assumed to be subject to a constraint

$$y_{1t}(j) + y_{2t}(j) \leq y_t, \quad (2.9)$$

where  $y_t > 0$  is an exogenous capacity. If we assume competition in the supply of each of the differentiated goods, then the household takes  $p_t(j)$  and  $R_t^s(j)$  as given; (2.9) then implies a given value for  $Y_t$  each period, independent of household plans. The right-hand side of (2.7)

is thus determined by market conditions, independent of the household's plans.<sup>9</sup> Note that this borrowing limit may equivalently be expressed as a bound upon household bond issues,

$$B_t \geq -R_t^{-1}[M_t + R_t^s(j)p_t(j)y_{2t}(j) - \int_{|i-j| > (1-\alpha)/2} R_t^s(i)p_t(i)c_t(i)di] - \sum_{s=t+1}^{\infty} \delta_{t,s}[Y_s - T_s].$$

Finally, constraints (2.6) and (2.7) together imply that expenditure is subject to an intertemporal budget constraint of the form

$$\sum_{s=t}^{\infty} \delta_{t,s}E_s \leq W_t + \sum_{s=t}^{\infty} \delta_{t,s}[Y_s - T_s], \quad (2.10)$$

where

$$E_t \equiv \int_{|i-j| \leq (1-\alpha)/2} p_t(i)c_t(i)di + \int_{|i-j| > (1-\alpha)/2} R_t^s(i)p_t(i)c_t(i)di$$

measures total nominal expenditure in period  $t$ .

The representative household's problem is then to choose paths for the  $\{c_t(i)\}_{i \in S}$ , and for the variables  $y_{1t}$ ,  $y_{2t}$ ,  $M_t$ ,  $B_t$ , so as to maximize (2.1), subject to the constraints (2.8), (2.9), (2.5), (2.4), (2.6), and (2.7).<sup>10</sup> In these constraints, the price sequences  $\{p_t(i), R_t^s(i)\}_{i \in S}$  and  $R_t$  are taken as given, as are the capacity sequence  $y_t$ , the sequence of net tax obligations  $T_t$ , and the initial wealth  $W_0$ . A perfect foresight equilibrium is then a set of price sequences, and associated quantity decisions, such that the quantity decisions are optimal given the prices, and markets clear at all dates. I shall consider only the case of a symmetric equilibrium, in which  $W_0$  is the same for all households, the prices  $p_t(i)$ ,  $R_t^s(i)$  are the same for all goods  $i$  at each date, and hence all quantity decisions by the households are similarly the same (if goods are identified by their relative position on the circle, rather than their absolute

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<sup>9</sup>If the infinite sum does not converge, there will be no borrowing limit; in this case, there can be no optimal plan for the household, because there is no bound upon the level of consumption that can be financed. Thus convergence of the sum is a requirement for equilibrium.

<sup>10</sup>Note that the optimal consumption plan can be more simply defined as the one that maximizes (2.1) subject to (2.8) and (2.10), given initial wealth  $W_0$ .

position). In fact, because of the symmetry of (2.2), each household will choose to purchase the same quantity  $c_{1t}$  of every good that can be purchased using informal credit, and the same quantity  $c_{2t}$  of every good that must be purchased using cash settlement. Market clearing then means that  $M_t = M_t^s$ , where  $M_t^s$  is the money supply; that  $B_t = B_t^s$ , where  $B_t^s$  is the net supply of bonds by the government; and that (assuming there are no purchases of goods by the government)  $(1 - \alpha)c_{1t} = y_{1t}$ ,  $\alpha c_{2t} = y_{2t}$ . The evolution of the variables  $M_t^s$  and  $B_t^s$  will depend upon the monetary and fiscal policy of the government, to be further specified below, as will the sequence  $T_t$ .

This model of the use of money in exchange can in fact be shown to result in equilibrium conditions that are identical to those of a standard Sidrauski-Brock model. Note first that in any equilibrium, one must have  $R_t^s(i) = R_t$  for each good, in order for household  $i$  to be willing to supply the good to buyers who use each of the two possible payments technologies. (Since each household demands a positive quantity of each good, in the case of any finite relative prices, there is no possibility of an equilibrium in which goods are purchased using only one of the payments technologies.) In this case, and also specializing to the case of a symmetric equilibrium, the budget constraints (2.4) and (2.5) may be written in the simpler form

$$M_t + \tilde{B}_t = W_t - T_t + p_t y_t - p_t c_t \quad (2.11)$$

and

$$W_{t+1} = M_t + R_t \tilde{B}_t, \quad (2.12)$$

where

$$\tilde{B}_t \equiv B_t + p_t y_{2t} - p_t c_{2t}$$

and

$$c_t \equiv \int_{i \in S} c_t(i) di = (1 - \alpha)c_{1t} + \alpha c_{2t}. \quad (2.13)$$

Here  $\tilde{B}_t$  is a measure of the household's end-of-period financial wealth in addition to its cash balances, that takes account of the period  $t$  transactions that have not yet been settled as of the end of the period;  $c_t$  is a measure of the household's total consumption demand, not in terms of the utility-relevant aggregate  $C_t$  defined in (2.2), but in terms of the resource cost of supplying the household's consumption. The household's objective (2.1) may equivalently be written as

$$\sum_{t=0}^{\infty} \beta^t U(c_t, M_t/p_t), \quad (2.14)$$

where the indirect utility  $U(c_t, M_t/p_t)$  represents the maximum attainable value of  $u(C_t)$ , when the household's consumption purchases are subject to the two constraints

$$\int_{i \in S} c_t(i) di \leq c_t,$$

$$\int_{|i-j| > (1-\alpha)/2} c_t(i) di \leq M_t/p_t,$$

in addition to (2.8). Note that the constraint set for this maximization is non-empty as long as

$$c_t, M_t \geq 0. \quad (2.15)$$

The indirect utility function  $U(c, m)$  is easily seen to be well-defined on this set. It is strictly increasing in  $c$ , and non-decreasing in  $m$ ; it is strictly increasing in  $m$  as well, for all  $m/c$  below a critical value  $\bar{\mu} > 0$  (at which the cash-in-advance constraint ceases to bind), but independent of  $m$  for all larger values. It is concave in the two arguments, strictly concave in  $c$ , and strictly concave in both arguments in the region  $m/c < \bar{\mu}$ . Finally, it is continuously differentiable, and has continuous second derivatives except at the critical value of  $m/c$ .

The representative household's optimization problem may then be equivalently stated as the choice of sequences  $c_t, M_t$  and  $\tilde{B}_t$  so as to maximize (2.14), subject to the constraints (2.15), (2.11), (2.12), and (2.7), given the price sequences  $p_t, R_t$  and initial wealth  $W_0$ . The conditions for market clearing can be written in terms of these variables as  $M_t = M_t^s$ ,  $B_t = B_t^s$ , and  $c_t = y_t$ . These are just the conditions for a perfect foresight monetary equilibrium in a Sidrauski-Brock model, with a single good with an exogenous supply of  $y_t$ , and with the utility function of the representative household being given by (2.14). The point of our rather circuitous derivation of this familiar model is that there will now be a natural way in which to consider the consequences of innovations in the payments mechanism that reduce the need for cash balances; we shall simply consider the effects of a reduction in the parameter  $\alpha$ , holding fixed the rest of our specification.

Before turning to this, however (in the next section), let us recall the standard analysis of price-level determination by the path of the money supply in such a model. I shall begin by characterizing optimal household behavior. Note first that there is no solution to the representative household's optimum problem unless

$$R_t \geq 1 \tag{2.16}$$

in each period. For otherwise, a household can earn arbitrage profits by issuing debt (choosing a more negative value for  $B_t$ ) and using the proceeds to increase its money holdings  $M_t$ , and the borrowing limit (2.7) does not limit the extent to which it may do this. Thus (2.16) is a requirement for equilibrium; I shall assume that it holds in stating the following necessary and sufficient conditions for household optimization.

Now let us consider the optimal allocation of consumption spending across the different

goods  $i \in S$ , given a desired consumption index  $C_t$ . Since the cost of each “credit good” is  $p_t$ , while the cost each “cash good” is  $R_t p_t$ , it follows from the form of (2.2) that the household will purchase the same quantities  $c_{1t}$  of each “credit good” and  $c_{2t}$  of each “cash good”, where

$$c_{kt} = \theta_k(R_t)C_t \quad (2.17)$$

for  $k = 1, 2$ , and

$$\begin{aligned} \theta_1(R) &\equiv \left[ (1 - \alpha)R^{\frac{\gamma}{1-\gamma}} + \alpha \right]^{-1/\gamma}, \\ \theta_2(R) &\equiv \left[ (1 - \alpha) + \alpha R^{\frac{-\gamma}{1-\gamma}} \right]^{-1/\gamma}. \end{aligned}$$

Given (2.17), the cash-in-advance constraint (2.6) requires money balances of at least

$$M_t \geq \alpha R_t \theta_2(R_t) p_t C_t \quad (2.18)$$

in each period. Because no interest is paid on cash balances, in any period in which  $R_t > 1$ , optimization requires that (2.18) hold with equality.

We have thus characterized the optimal allocation of the household’s consumption and portfolio each period, *given* its choice of the consumption index  $C_t$ . An optimal choice of the time path  $\{C_t\}$ , in turn, must satisfy the intertemporal Euler equation

$$\frac{u'(C_t)}{p_t} \theta_1(R_t)^{\gamma-1} = \beta R_t \frac{u'(C_{t+1})}{p_{t+1}} \theta_1(R_{t+1})^{\gamma-1} \quad (2.19)$$

for each period  $t \geq 0$ . (The Inada condition (2.3) allows us to ignore the possibility of a corner solution.) Equation (2.19) differs from the form that is familiar for a non-monetary representative-household model due to the effects of the requirement that cash be used to settle some transactions; the factor  $\theta_1(R_t)$  represents the marginal increase in the index  $C_t$  that can be obtained per  $p_t$  units of additional expenditure.<sup>11</sup>

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<sup>11</sup>Note that the constant-elasticity-of-substitution form (2.2) implies that  $\partial C / \partial c_1 = (c_1 / C)^{\gamma-1}$ .

An optimal consumption plan must also exhaust the intertemporal budget constraint. Thus (2.10) must hold with equality, looking forward from any period  $t \geq 0$ . It is also necessary that the right-hand side of (2.10) be a finite quantity, since otherwise (2.7) imposes no limit upon borrowing. This implies that the household's consumption plan must satisfy

$$\sum_{s=t}^{\infty} \delta_{t,s} E_s < \infty. \quad (2.20)$$

Given this, exhaustion of the intertemporal budget constraint is equivalent to the requirement that the household's accumulation of wealth satisfy the transversality condition

$$\lim_{T \rightarrow \infty} \delta_{t,T} W_T = 0. \quad (2.21)$$

A complete set of necessary conditions for household optimization is thus given by the requirements that (2.16), (2.17), (2.18), and (2.19) hold each period, with either (2.16) or (2.18) holding with equality, and the requirements that (2.20) hold, and that (2.10) hold with equality (or, equivalently, that (2.21) hold). These conditions can also be shown to be sufficient for optimality.

Sequences  $\{p_t, R_t\}$  then constitute an intertemporal (or perfect foresight) equilibrium if the optimal household plan associated with them clears all markets. Goods market clearing requires that  $c_t = y_t$  at all times, where  $c_t$  is the *resource* cost of the representative household's purchases, defined in (2.13). (To simplify, I assume that the government consumes no goods.) Using (2.17), this can be written as

$$C_t = \omega(R_t) y_t, \quad (2.22)$$

where

$$\omega(R) \equiv [(1 - \alpha)\theta_1(R) + \alpha\theta_2(R)]^{-1} = \frac{[(1 - \alpha)R^{\gamma/1-\gamma} + \alpha]^{1/\gamma}}{[(1 - \alpha)R^{1/1-\gamma} + \alpha]}.$$

Equilibrium also requires that the desired money holdings of the representative household,  $M_t$ , equal the money supply, and that the desired net financial wealth of the representative household,  $W_t$ , equal the net supply of outside financial assets by the government.

In the familiar quantity-theoretic approach to price-level determination, one solves for the equilibrium paths  $\{p_t, R_t\}$  as a function of the expected path of the money supply  $\{M_t\}$ . Consider a policy regime that is defined by a path  $\{M_t\}$  for the money supply, with  $M_t > 0$  at all dates. (For simplicity, I do not introduce fractional-reserve banking, and so this is simply the path of the monetary base, supplied by the central bank.) Suppose that there is no government debt at any time, and that as a result fiscal policy consists simply of the net lump-sum transfers required to increase or decrease the money supply each period as specified. Thus net tax collections each period are given by  $T_t = -(M_t - M_{t-1})$ . A familiar special case of such a regime is that in which there is no government at all, but a fixed quantity of money  $M_t = M > 0$  simply exists and is passed from hand to hand.

In the case of such a policy regime, the equilibrium conditions can be expressed as follows. Substituting (2.22) into (2.18), one obtains the “money market clearing relation”

$$M_t/p_t \geq \mu(R_t)y_t, \tag{2.23}$$

where again this must hold with equality in any period in which  $R_t > 1$ . Here

$$\mu(R) \equiv \alpha R \theta_2(R) \omega(R) = \frac{\alpha R}{\left[ (1 - \alpha) R^{1/(1-\gamma)} + \alpha \right]}.$$

In the case that  $\alpha \leq \gamma$  (as assumed here), one observes that  $\mu(R)$  is a monotonically decreasing function, varying from a value of  $\alpha$  when  $R = 1$ , to an arbitrarily small positive quantity for large enough  $R$ . Thus money demand is uniquely defined and positive for any positive nominal interest rate.

In the quantity-theoretic literature, equation (2.23) is viewed as determining the price level. Even supposing that  $M_t$  is exogenously given, however, the solution for  $p_t$  depends upon the endogenous nominal interest rate  $R_t$ . This is determined by the Euler equation (2.19). Substituting (2.22) into the latter equation, one obtains

$$\frac{u'(\omega(R_t)y_t)}{p_t}\theta_1(R_t)^{\gamma-1} = \beta R_t \frac{u'(\omega(R_{t+1})y_{t+1})}{p_{t+1}}\theta_1(R_{t+1})^{\gamma-1}. \quad (2.24)$$

We can similarly express equilibrium condition (2.20) in terms of the path of interest rates only. Note first that (2.17) implies that

$$E_t = p_t C_t [(1 - \alpha)\theta_1(R_t) + \alpha R_t \theta_2(R_t)] = p_t C_t \theta_1(R_t)^{1-\gamma}.$$

Furthermore, (2.24) implies that each period's discount factor  $\delta_{t,T}$  is proportional to

$$\beta^T \frac{u'(\omega(R_T)y_T)}{p_T} \theta_1(R_T)^{\gamma-1}. \quad (2.25)$$

Substituting these two expressions, along with (2.22), into (2.20) then yields

$$\sum_{T=t}^{\infty} \beta^T u'(\omega(R_T)y_T) \omega(R_T) y_T < \infty. \quad (2.26)$$

Finally, in equilibrium under such a policy regime, one must have  $B_t = 0$  at all times, and hence  $W_t = M_{t-1}$ . Substituting this into (2.21), together with (2.25), one obtains

$$\lim_{T \rightarrow \infty} \beta^T u'(\omega(R_T)y_T) \theta_1(R_T)^{\gamma-1} \frac{M_{T-1}}{p_T} = 0. \quad (2.27)$$

Equations (2.16), (2.23), (2.24), (2.26), and (2.27), together with the requirement that at least one of (2.16) and (2.23) holds with equality each period, then comprise a complete system of equilibrium conditions to determine the sequences  $\{p_t, R_t\}$ .

We can reduce this system to a difference equation for the price level as follows. Because of the monotonicity of  $\mu(R)$  noted above, one can invert (2.23) to obtain the “liquidity preference” relation for the nominal interest rate

$$R_t = \rho(p_t y_t / M_t), \quad (2.28)$$

where  $\rho(v) = 1$  for all  $v \leq \alpha^{-1}$ , and a monotonically increasing function for all larger values of  $v$ , increasing without bound as  $v$  does. Substituting this into (2.24) and (2.27), we obtain

$$F(M_t/p_t; y_t) = \beta(M_t/M_{t+1})G(M_{t+1}/p_{t+1}; y_{t+1}), \quad (2.29)$$

$$\sum_{T=t}^{\infty} \beta^T H(M_T/p_T; y_T) < \infty, \quad (2.30)$$

and

$$\lim_{T \rightarrow \infty} \beta^T F(M_T/p_T; y_T) = 0, \quad (2.31)$$

where

$$F(m; y) \equiv \rho(y/m)^{-1}G(m; y),$$

$$G(m; y) \equiv u'(\omega(\rho(y/m))y)\theta_1(\rho(y/m))^{\gamma-1}m,$$

$$H(m; y) \equiv u'(\omega(\rho(y/m))y)\omega(\rho(y/m))y.$$

A perfect foresight equilibrium is then just a positive price-level sequence  $\{p_t\}$  that satisfies the first-order nonlinear difference equation (2.29) each period, and satisfies the bounds implied by (2.30) and (2.31).

As is well-known, such a system may not uniquely determine the equilibrium path of the price level.<sup>12</sup> However, under certain circumstances it does. In such a case, the equilibrium

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<sup>12</sup>See, e.g., Brock (1975), Obstfeld and Rogoff (1983), or Woodford (1994).

price level at each date is given by a function of the current and expected future money supplies, that depends upon real disturbances, such as (here) the current and expected future supply of goods:

$$p_t = p(M_t, M_{t+1}, M_{t+2}, \dots; y_t, y_{t+1}, y_{t+2}, \dots). \quad (2.32)$$

It is obvious from the form of (2.29) and (2.31) that when such a unique solution exists, the function  $p$  is homogeneous of degree one in the arguments  $(M_t, M_{t+1}, M_{t+2}, \dots)$ .

### 3 Wicksellian Policies and the Cashless Limit

I now consider price-level determination in an economy of the kind just described, in the limiting case in which the number of goods that must be purchased using cash becomes arbitrarily small. Formally, we simply consider the consequences of making the parameter  $\alpha$  arbitrarily small. More generally, there is no reason to restrict attention to the case in which the parameter  $\alpha$  is constant over time. We may instead suppose that there may be a different value  $\alpha_t > 0$  each period; the functions  $\theta_k(R)$ ,  $\omega(R)$ , and  $\mu(R)$  defined in the previous section are all then time-varying (or may be written with  $\alpha_t$  as a second argument). Using the sup norm to measure the size of a given sequence  $\alpha \equiv \{\alpha_t\}$ ,

$$|\alpha| \equiv \sup_t |\alpha_t|, \quad (3.1)$$

we are then interested in price-level determination in economies for which  $|\alpha|$  is arbitrarily small.

It is easily seen that, in the case of a policy regime of the kind just considered, with an exogenous path for the money supply, the conditions determining the equilibrium price level will not be well-defined in such a limit. In particular, let us suppose that the path of the

money supply is such that  $M_t \geq \underline{M}$  for all  $t$ , for some lower bound  $\underline{M} > 0$ . Then one can show that there is no hope of the existence of a function  $p$  of the kind referred to in (2.32) that is continuous in  $\alpha$  for  $\alpha$  near the zero sequence.

Note that the sequence of equilibrium conditions (2.29) can be written abstractly in the form

$$\Phi(p; M, y; \alpha) = 0, \tag{3.2}$$

where  $p, M, y$  refer to the sequences of values for the variables  $p_t, M_t, y_t$  in periods  $t = 0, 1, 2, \dots$ , and  $\Phi$  maps a four-tuple of infinite sequences into an infinite sequence. For any  $t \geq 0$ , the  $t$  element of this vector-valued function is given by

$$\Phi_t(p; M, y; \alpha) \equiv F(M_t/p_t; y_t, \alpha_t) - \beta(M_t/M_{t+1})G(M_{t+1}/p_{t+1}; y_{t+1}, \alpha_{t+1}).$$

Now let us fix positive sequences  $M$  and  $y$ , and consider whether it is possible to express the perfect foresight equilibrium price sequence as a continuous function,  $p = f(\alpha)$ , on a domain that includes  $\alpha = 0$ . When dealing with infinite sequences, it is necessary to specify what one means by continuity, and so I shall stipulate the topology defined by the sup norm, defined in (3.1) for sequences  $\alpha$ , and by

$$|p - p'| \equiv \sup_t |\log p_t - \log p'_t| \tag{3.3}$$

in the case of price sequences.

Because any given element  $\Phi_t$  depends only upon a finite number of elements of  $p$  and  $\alpha$ , it is evident that the mapping  $\Phi$  is continuous in the sup norm topology if and only if for each  $t$ ,  $\Phi_t$  is a continuous function of  $p_t, p_{t+1}, \alpha_t$ , and  $\alpha_{t+1}$ . The definitions given in the previous section make it trivial to verify that  $\theta_k, \omega$ , and  $mu$  are each continuous functions

of  $R$  and  $\alpha$ , for all  $R \geq 1, 0 \leq \alpha \leq \gamma$ , and that  $\rho$  is a continuous function of  $v$  and  $\alpha$ , for all  $v \geq 0, 0 \leq \alpha \leq \gamma$ . These results together with the definitions of  $F$  and  $G$  then imply that  $\Phi_t$  is indeed a continuous function for each  $t$ , and hence that  $\Phi$  is a continuous map of sequences to sequences. But then, if there exists the hypothesized function  $f$ , it must satisfy  $\Phi(f(\alpha); m, y; \alpha) = 0$  for all positive sequences  $\alpha$  near enough to zero. If  $f$  is continuous, then  $p^* \equiv f(0)$  must satisfy

$$\lim_{\alpha \rightarrow 0} f(\alpha) = p^*; \quad (3.4)$$

because  $\Phi$  is continuous, this in turn implies that  $\Phi(p^*; m, y; 0) = 0$ , and so that the sequence  $\{p_t^*\}$  satisfies the sequence of equations (2.29) in the case that  $\alpha_t = 0$  each period.

But in the limiting case  $\alpha = 0$ , the distortions resulting from the cash-in-advance constraint vanish. We have simply  $\theta_1(R) = 1$ ,  $\theta_2(R) = R^{-1/1-\gamma}$ ,  $\omega(R) = 1$ , and  $\mu(R) = 0$  for all  $R \geq 1$ , and hence  $\rho(v) = 1$  for all  $v \geq 0$ . In this limit, the Euler equation (2.29) takes the simple form

$$\frac{u'(y_t)}{p_t} = \beta \frac{u'(y_{t+1})}{p_{t+1}}, \quad (3.5)$$

which is just the familiar asset-pricing equation for a non-monetary economy, applied to an asset that pays no dividend and has an exchange value (in terms of the consumption good) of  $1/p_t$ . Equilibrium condition (2.31) similarly takes the form

$$\lim_{T \rightarrow \infty} \beta^T u'(y_T) M_T / p_T = 0. \quad (3.6)$$

Now if the sequence  $\{p_t^*\}$  satisfies (3.5), it follows that

$$\lim_{T \rightarrow \infty} \beta^T u'(y_T) / p_T^* = u'(y_0) / p_0^* > 0.$$

Then if  $M_T \geq \underline{M}$  for all  $T$ ,

$$\lim_{T \rightarrow \infty} \beta^T u'(y_T) M_T / p_T^* \geq u'(y_0) \underline{M} / p_0^* > 0,$$

and (3.6) is violated. Thus such a solution to equations (2.29) does not represent an equilibrium.<sup>13</sup> One furthermore observes that any positive price sequence  $p$  that is close enough to  $p^*$ , in the sense of the norm (3.3), must fail to satisfy (3.6) as well. Then (3.4) would imply that  $f(\alpha)$  cannot represent an equilibrium, in the case of small enough positive sequences  $\alpha$ ; and thus we obtain a contradiction.

This result makes intuitive sense, of course, from the point of view of the quantity-theoretic approach to price-level determination; as the fraction of transactions that require cash falls to zero, desired real money balances fall to zero as well, and so one should expect the equilibrium price level to cease to be defined. The failure of the solution to be continuous near  $\alpha = 0$  simply means that one cannot a solution for the limiting case as an approximation to the small- $\alpha$  case; instead, the use of money in transactions is intrinsic to the model's ability to determine an equilibrium price level.

However, this conclusion is not more generally valid; it is actually rather special to the type of monetary policy just considered, in which the path of the money supply is exogenously specified – a case that is of rather little practical relevance, despite the attention given to it in the theoretical literature. As an example of an alternative way of specifying monetary policy, let us consider what may be called a “Wicksellian” regime, after the proposal of Wicksell

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<sup>13</sup>This is just the familiar result that an equilibrium with valued fiat money is not possible in a representative household model, when money is in positive net supply, and no monetary “frictions” are present. Such an equilibrium, were it to exist, would represent a pricing “bubble”, since the money never pays dividends; and its non-existence is simply a special case of the more general proposition that in such a model there can be no equilibrium pricing bubbles in the case of any assets in positive net supply (Santos and Woodford, 1997).

(1907). Wicksell argued that price stability could be achieved under a fiat money regime by a policy that made no attempt to regulate the quantity of banknotes in circulation, but instead required the central bank to adjust nominal interest rates in response to deviations of the price level from the desired level, “lowering them when prices are getting low, and raising them when prices are getting high”.<sup>14</sup>

This suggests consideration of a rule for monetary policy that specifies the short-term nominal interest rate as a function of the price level,

$$R_t = \phi(p_t), \tag{3.7}$$

where  $\phi$  is a strictly increasing, continuous function, defined for arbitrary positive prices, such that  $\phi(p) \geq 1$  for all  $p > 0$ . In specifying monetary policy in terms of a rule for the nominal interest rate, we recognize that as a matter of actual central banking practice, the direct instrument of monetary policy is almost invariably a short-term nominal interest rate (such as the Federal funds rate in the U.S.), even when this instrument is adjusted in a manner intended to control the growth of a monetary aggregate. The central bank controls this rate by intervening in the market for short-term nominal debt; we may imagine that it simply stands ready to exchange debt for money in arbitrary quantities at the price that it has decided upon. It will not be possible for the central bank to bring about an interest rate  $R_t < 1$ , since, as noted earlier, this would be inconsistent with equilibrium, owing to the arbitrage opportunity that it would create. But any non-negative interest rate is attainable. Since we have shown that desired money holdings equal  $\mu(R_t)p_t y_t$ , where  $\mu(R)$  is monotonically decreasing in  $R$ , and positive for all finite  $R$ , it follows that the central bank

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<sup>14</sup>For another recent analysis of price-level determination under a Wicksellian regime, see Fuhrer and Moore (1995).

can bring about any desired non-negative nominal interest rate by an appropriate adjustment of the composition of government liabilities between money and bonds.

A complete specification of the policy regime requires that we specify fiscal policy as well. One simple specification that results in equilibrium conditions that are well-behaved in the cashless limit is to assume that the government maintains a target level  $W > 0$  for the total nominal value of government liabilities, and adjusts net tax collections  $T_t$  as necessary to ensure that  $W_{t+1} = W$  at the beginning of the following period. Assuming that the economy starts with an initial condition  $W_0 = W$ , this is equivalent to assuming that the government runs a conventional budget deficit (*i.e.*, inclusive of nominal interest payments on government debt) of zero each period:

$$D_t \equiv (R_t - 1)B_t - T_t = 0. \quad (3.8)$$

We now consider perfect foresight equilibrium under such a regime. Condition (2.16) no longer need be considered as a separate equilibrium condition, as it follows from (3.7). Condition (2.23) still matters, but it simply determines the equilibrium path of the endogenous money supply, once we have solved for the equilibrium sequences  $\{p_t, R_t\}$ . Since it necessarily implies a positive sequence for the money supply, it can be neglected in listing the conditions that determine the other sequences. We are thus left with (2.21), (2.24), (2.26), and (3.7) as equilibrium conditions. Substituting (2.25) and the path of total government liabilities implied by (3.8) into (2.21), this becomes

$$\lim_{T \rightarrow \infty} \beta^T u'(\omega(R_T)y_T)\theta_1(R_T)^{\gamma-1} \frac{W}{p_T} = 0. \quad (3.9)$$

The complete set of necessary and sufficient conditions for sequences  $\{p_t, R_t\}$  to constitute an equilibrium are then (2.24), (2.26), (3.7), and (3.9).

Just as in the case of the exogenous-money regime, we can substitute (3.7) to eliminate the variable  $R_t$  from the other equilibrium conditions. We again obtain a nonlinear difference equation in the price level, together with bounds on the asymptotic behavior of the price level. These may again be written in the form

$$F(p_t; y_t) = \beta G(p_{t+1}; y_{t+1}), \quad (3.10)$$

$$\sum_{T=t}^{\infty} \beta^T H(p_T; y_T) < \infty, \quad (3.11)$$

$$\lim_{T \rightarrow \infty} \beta^T F(p_T; y_T) = 0, \quad (3.12)$$

where now

$$F(p; y) \equiv G(p; y)/\phi(p),$$

$$G(p; y) \equiv u'(\omega(\phi(p))y)p^{-1}\theta_1(\phi(p))^{\gamma-1},$$

$$H(p; y) \equiv u'(\omega(\phi(p))y)\omega(\phi(p))y.$$

Again, these equations may or may not suffice to determine a unique equilibrium price sequence; the analysis of this question (which is not a particular concern here) proceeds along lines similar to those followed in the more familiar case of an exogenous-money policy. Indeed, in the special case that the rule (3.7) is of the form

$$\phi(p) = \rho(p/k),$$

for some constant  $k > 0$ , equilibrium conditions (3.10), (3.11), and (3.12) are *identical* to those associated with an exogenous-money policy, in which the path of the money supply is given by  $M_t = ky_t$  for all  $t$ .<sup>15</sup>

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<sup>15</sup>Note that this correspondence between a particular Wicksellian policy and a particular exogenous path for the money supply holds only for a particular value of  $\alpha > 0$ . It is for this reason that the conclusions that we reach about the properties of the equilibrium conditions as we vary  $\alpha$  need not be the same.

Despite this close similarity in the form of the equilibrium conditions, the theory of price-level determination under the Wicksellian regime is qualitatively different in one important respect: the equilibrium conditions (3.10) – (3.12) remain well-behaved in the cashless limit. Specifically, we can show the following.

**Proposition 1** *Consider a Wicksellian policy regime defined by (3.7) and (3.8), and suppose that  $\phi$  is a continuously differentiable function, at least on a neighborhood of some price level  $p^* > 0$ , at which  $\phi(p^*) = \beta^{-1}$ ,  $\phi'(p^*) > 0$ . Let  $y^* > 0$  be any constant level of productive capacity. Then there exist neighborhoods  $U$  of  $p^*$ ,  $V$  of  $y^*$ , and  $W$  of  $0$ , and a function  $f$ , assigning a price sequence  $p = f(y; \alpha)$  with  $p_t \in U$  for all  $t$  to any capacity sequence  $y$  with  $y_t \in V$  for all  $t$  and any parameter sequence  $\alpha$  with  $\alpha \in W$  for all  $t$ , such that*

- (i)  *$f$  is a continuous function in the sup norm topology,<sup>16</sup> in both arguments; and*
- (ii) *for any sequence  $y$  and for any sequence  $\alpha$  with  $\alpha_t > 0$  for all  $t$ ,  $p = f(y; \alpha)$  is a perfect foresight equilibrium price sequence, and the unique equilibrium sequence with the property that  $p_t \in U$  for all  $t$ .*

The proposition does not assert that equilibrium is unique, but it is at least *locally* unique. The important property of this equilibrium, for present purposes, is that it is defined even in the cashless limit, and that the function  $f$  is *continuous* in this limit. Thus does not simply mean that the price sequence

$$p_t = \hat{p}(y_t, y_{t+1}, \dots)$$

defined by  $\hat{p}(y) = f(y, 0)$  is a solution to the equilibrium conditions in the  $\alpha = 0$  limit. It also means that this solution *approximates* a perfect foresight equilibrium in the case of any

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<sup>16</sup>Here we intend the norm defined in (3.3) for capacity sequences as well.

parameter sequence in which  $\alpha_t$  is small enough for all  $t$ , to a degree of accuracy (in terms of the norm (3.3)) that can be made arbitrarily great by placing a low enough bound on the size of the parameter sequence  $\alpha$  (in terms of the norm (3.1)).

The proposition can be proved as follows. Once again, the difference equation (3.10) may be written as an equation system of the form  $\Phi(p; y; \alpha) = 0$ , where  $\Phi$  maps a triple of infinite sequences into an infinite sequence, the  $t$  element of which is given by

$$\Phi_t(p; y; \alpha) \equiv F(p_t; y_t, \alpha_t) - \beta G(p_{t+1}; y_{t+1}, \alpha_{t+1}).$$

Now let us restrict attention to the definition of this mapping for sequences  $p$  and  $y$  such that  $p_t$  belongs forever to some bounded neighborhood  $P$  of  $p^*$ , and  $y_t$  belongs forever to some bounded neighborhood  $Y$  of  $y^*$ ,<sup>17</sup> and suppose that  $P$  is chosen so that  $\phi$  is continuously differentiable everywhere on  $P$ . Then it is possible to choose a neighborhood  $A$  of 0 such that  $F(p; y; \alpha)$  and  $G(p; y; \alpha)$  are each well-defined for all triples  $(p, y, \alpha) \in P \times Y \times A$ , continuously differentiable on this set, and bounded.<sup>18</sup> Then the mapping  $\Phi$  maps triples of bounded sequences  $(p, y, \alpha) \in P^\infty \times Y^\infty \times A^\infty$  into bounded sequences, where  $X^\infty$  represents the product of a countably infinite sequence of copies of the bounded interval  $X$ , and is continuously differentiable in the sequences  $(p, y, \alpha)$ , in the sup norm topology.

We next observe that in the cashless limit ( $\alpha_t = 0$  for all  $t$ ) of our equilibrium conditions, the price sequence in which  $p_t = p^*$  for all  $t$  solves each of the equilibrium conditions (3.10) – (3.12). This is because, in the cashless limit, these equilibrium conditions become

$$\frac{u'(y_t)}{p_t \phi(p_t)} = \beta \frac{u'(y_{t+1})}{p_{t+1}}, \quad (3.13)$$

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<sup>17</sup>In each case, by a “bounded” neighborhood we mean that  $|\log p - \log p^*|$  and  $|\log y - \log y^*|$  are bounded.

<sup>18</sup>Note that the expressions written earlier for functions such as  $\theta_k(R; \alpha)$  are well-defined and continuously differentiable for small negative values of  $\alpha$ , even though negative values of  $\alpha$  have no meaning in terms of our model.

$$\sum_{T=t}^{\infty} \beta^T u'(y_T) y_T < \infty, \quad (3.14)$$

$$\lim_{T \rightarrow \infty} \beta^T u'(y_T) p_T^{-1} = 0, \quad (3.15)$$

which conditions are obviously satisfied by the constant sequences just mentioned. Thus we observe a crucial difference between this sort of policy regime and the exogenous-money regime: it is possible for a solution to exist to the cashless limit of the equilibrium conditions.

We can then generalize this solution to the case of nearby sequences  $y$  and  $\alpha$  using the implicit function theorem.<sup>19</sup> Note that with the sup norm topology, the linear space of bounded sequences represents a Banach space, often denoted  $l_{\infty}$ . Furthermore, in this topology, sets such as  $P^{\infty}$ ,  $Y^{\infty}$ , and  $A^{\infty}$  are open subsets of the space of bounded sequences. Thus the mapping  $\Phi$  is defined on a product of open subsets of Banach spaces, and takes values in another Banach space. Furthermore,  $\Phi(p^*; y^*; 0) = 0$ , where we now use  $p^*$  to denote the sequence in which  $p_t = p^*$  for all  $t$ , and similarly with  $y^*$ . The implicit function theorem then states that if the partial derivative  $D_p \Phi(p^*; y^*; 0)$  satisfies certain regularity conditions, there exists a neighborhood  $N$  of the pair of sequences  $(y^*, 0)$ , and a continuous function  $f : N \rightarrow P^{\infty}$ , such that  $f(y^*; 0) = p^*$ , and such that for all  $(y, \alpha) \in Q$ ,  $f(y; \alpha)$  satisfies

$$\Phi(f(y; \alpha); y; \alpha) = 0. \quad (3.16)$$

Furthermore, for any  $N$  small enough, there exists a neighborhood  $S$  of the sequence  $p^*$ , such that  $f(N) \subset Q$ , and  $f(y; \alpha)$  is the unique solution to (3.16) belonging to the set  $S$ . Given any such neighborhoods  $N$  and  $S$ , we can choose neighborhoods  $U$  of the scalar

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<sup>19</sup>See, e.g., Lang (1983), pp. 131-132, for a version applicable to general Banach spaces, that can be used in the present instance. See Santos and Bona (1989) for an early application to comparative statics in an intertemporal equilibrium model.

$p^*$ ,  $V$  of the scalar  $y^*$ , and  $W$  of the scalar 0, such that  $U^\infty \subset S$  and  $V^\infty \times W^\infty \subset N$ . Because of the continuity of  $f$ , by choosing  $V$  and  $W$  small enough, we can ensure that  $f(V^\infty \times W^\infty) \subset U^\infty$ . With these choices,  $f$  is a continuous function with the range and domain specified in the proposition, that identifies the unique solution to (3.10) such that  $p_t \in U$  for all  $t$ . Moreover, because  $U$ ,  $V$  and  $W$  are bounded intervals,  $\{F(p_t; y_t; \alpha_t)\}$  and  $\{H(p_t; y_t; \alpha_t)\}$  are bounded sequences, as long as  $p_t \in U$ ,  $y_t \in V$ , and  $\alpha_t \in W$  for all  $t$ . Hence (3.11) and (3.12) are satisfied by any sequences confined to those neighborhoods, and so for any such sequences  $y$  and  $\alpha$ ,  $f(y; \alpha)$  represents a solution to all of the equilibrium conditions (3.10) – (3.12), and hence a perfect foresight equilibrium. And as it is the unique solution to (3.10) such that  $p_t \in U$  for all  $t$ , it is the unique equilibrium with that property.

It remains to discuss the conditions on the derivative  $D_p \Phi(p^*; y^*; 0)$  required in order for the implicit function theorem to apply. It is necessary that this derivative be a continuous linear mapping (of bounded sequences into bounded sequences) with a continuous inverse.<sup>20</sup> The derivative is easily computed; for  $\{\hat{p}_t\}$  any bounded sequence of percentage deviations in prices,  $D_p \Phi \cdot \hat{p} = z$ , where the sequence  $z$  is given by

$$z_t = \Phi_{p1} \hat{p}_t + \Phi_{p2} \hat{p}_{t+1} \tag{3.17}$$

for  $t = 0, 1, 2, \dots$ . Here  $\Phi_{p1}$  is the partial derivative of  $\Phi_t$  with respect to  $\log p_t$ , and  $\Phi_{p2}$  the partial derivative with respect to  $\log p_{t+1}$ ,<sup>21</sup> evaluated at  $(p^*; y^*; 0)$ . (Note that these partial derivatives are the same for all  $t$ .) It is obvious from (3.17) that  $z$  is a bounded sequence if  $\hat{p}$  is; furthermore, there exists a finite bound  $C$  such that  $|z| \leq C|\hat{p}|$  for all

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<sup>20</sup>In the case of a finite number  $n$  of conditions to determine  $n$  endogenous variables, this reduces to the familiar condition that the Jacobian matrix be non-singular, and hence invertible. The condition is less trivial in the case of an infinite-dimensional problem.

<sup>21</sup>Partial derivatives are here understood in the usual sense of finite multivariate calculus, because  $\Phi_t$  depends on only a finite number of the elements of  $p$ ,  $y$ , and  $\alpha$ .

$\hat{p}$ , given by  $C = |\Phi_{p1}| + |\Phi_{p2}|$ . Thus  $D_p\Phi$  is a continuous linear mapping. The inverse mapping identifies the bounded sequence  $\hat{p}$  that satisfies the difference equation (3.17), for any bounded sequence  $z$ . Since (3.17) implies that

$$\hat{p}_t = \sum_{j=0}^{T-t-1} \lambda^j \Phi_{p1}^{-1} z_{t+j} + \lambda^{T-t} \hat{p}_T$$

for arbitrary  $T > t$ , where  $\lambda \equiv -\Phi_{p2}/\Phi_{p1}$ , it follows that if  $|\lambda| < 1$ , there is a unique bounded solution  $\hat{p}$ , given by

$$\hat{p}_t = \sum_{j=0}^{\infty} \lambda^j \Phi_{p1}^{-1} z_{t+j}. \quad (3.18)$$

In this case, (3.18) defines the mapping  $[D_p\Phi]^{-1}$ .<sup>22</sup> Furthermore, there exists a finite bound  $K$  such that  $|\hat{p}| \leq K|z|$ , given by  $K = [(1 - \lambda)\Phi_{p1}]^{-1}$ . Thus  $[D_p\Phi]^{-1}$  is a continuous linear mapping as well, and the implicit function theorem may be applied.

Finally, we must verify that  $|\lambda| < 1$ . Differentiating  $\Phi_t$ , we find that

$$\Phi_{p1} = F_p(p^*; y^*; 0) = -(1 + \epsilon_\phi)F(p^*; y^*; 0),$$

$$\Phi_{p2} = -\beta G_p(p^*; y^*; 0) = F(p^*; y^*; 0),$$

where  $\epsilon_\phi \equiv p^* \phi'(p^*)/\phi(p^*)$ , and hence that  $\lambda = 1/(1 + \epsilon_\phi)$ . Under the assumption that  $\phi'(p^*) > 0$ ,  $\epsilon_\phi > 0$ , and  $0 < \lambda < 1$ . Thus  $[D_p\Phi]^{-1}$  exists and is a continuous linear mapping, the implicit function theorem applies, and the proposition is established.

Thus, in the case of a Wicksellian regime of the kind specified in the proposition, there exists a well-defined perfect foresight equilibrium path for the price level, even in the cashless limit, and this equilibrium approximates an equilibrium (the unique one in which the price level never diverges very far from the constant level  $p^*$ ) of an economy in which the fraction

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<sup>22</sup>In the case that  $\lambda \geq 1$ , multiple bounded sequences  $\hat{p}$  lead to the same sequence  $z$ , and so no inverse mapping exists.

of purchases that must be made using cash is small enough. The equilibrium in the cashless limit is a solution to equations (3.13) and (3.15), which equations make no reference to any aspect of the transactions technology.<sup>23</sup> (For example, they depend neither upon  $\alpha$  nor upon the function (2.2) that determines the substitutability of goods that must be purchased using the different transactions technologies.) Thus there is in fact no necessity to describe the way in which cash facilitates transactions, as I did in the previous section, in order to derive these equations, and hence to obtain a system of equilibrium conditions that describe price level determination in the cashless limit. One might instead have simply written down the equilibrium conditions

$$u'(y_t) = \beta \frac{R_t p_t}{p_{t+1}} u'(y_{t+1}), \quad (3.19)$$

and

$$\lim_{T \rightarrow \infty} \beta^T u'(y_T) \frac{W_T}{p_T} = 0, \quad (3.20)$$

which correspond to conditions (2.19) and (2.21) in the cashless limit. Combining them with the policy rules (3.7) and (3.8), we would have directly obtained equations (3.13) and (3.15).

Note that equations (3.19) and (3.20) are not simply the cashless limit of the equilibrium conditions in our monetary economy; they are also the equilibrium conditions that must be satisfied by the real interest rate and real financial wealth in a completely non-monetary economy. Thus they could easily be derived by abstractly entirely from the use of money in transactions. The only reason that I have described the system consisting of these equations together with the policy rules as determining the price level in the “cashless limit” of a monetary economy – rather than simply saying that they describe price-level determination in an economy where cash is not needed for transactions, is that it is not clear that a central

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<sup>23</sup>Here I drop equation (3.14), since it places no restrictions upon the endogenous variables.

bank should have any way of implementing the policy rule (3.7) when money is not used at all, even though it can implement such a rule in a monetary economy no matter how close to zero the sequence  $\{\alpha_t\}$  may be.<sup>24</sup> Nonetheless, once it is granted that rules such as (3.7) represent possible monetary policies, there is no further need to consider the role of money in the economy.<sup>25</sup>

Equations (3.19) and (3.20) are also conditions for equilibrium in a Sidrauski-Brock monetary model, in which the period utility function  $U(c, m)$  is additively separable in its two arguments, as is often assumed for convenience. The assumption of additive separability, while mathematically convenient, is not very palatable on theoretical grounds; it is hard to think of a plausible underlying transactions technology that would lead to such separability. Nonetheless, the conclusion that is reached on that basis, that monetary variables can be neglected in writing the Euler equation that relates real rates of return to intertemporal resource allocation, is regarded as sensible by many economists, which doubtless accounts for the popularity of the assumption. I would agree that it is sensible to ignore monetary variables in the Euler equation, at least when analyzing an economy with highly-developed financial institutions. But the proper justification for this is not that additive separability holds, or is even a reasonable approximation, but rather that this is what one obtains in the

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<sup>24</sup>The latter conclusion depends upon the fact that, under our assumed transactions technology, there is still *some* positive level of money balances that are demanded at any given nominal interest rate, no matter how small  $\alpha_t$  may be. Under other transactions technologies, it might not be possible for the central bank to force short-term nominal interest rates above a finite ceiling (the interest rate at which money demand falls to zero), and that ceiling might in turn shrink to zero as the transactions constraints are made sufficiently unimportant. In such a case, it would not be possible to hold fixed the rule (3.7) as the transactions constraints are made negligible.

<sup>25</sup>More precisely, the only thing that matters about the money demand function is the constraint that it places upon the class of rules (3.7) that can be implemented, which in the present case amounts solely to the restriction (2.16). This condition, however, is largely independent of the details of the transactions technology. Condition (2.16) is necessary in any model in which additional cash balances do not make transactions any more difficult to arrange; it is also *sufficient* for implementability of the interest-rate policy in a broad range of models.

cashless limit.

We have seen that the solution to equations (3.13) and (3.15) approximates the equilibrium path of the price level, for any sequence  $\{\alpha_t\}$  every element of which is small enough. It is worth noting that this allows for time-variation in money demand that may be quite substantial, in relative terms. Note that equilibrium condition (2.23) implies that <sup>26</sup>

$$\frac{M_t}{\alpha_t p_t y_t} = R_t \theta_2(R_t; \alpha_t) \omega(R_t; \alpha_t). \quad (3.21)$$

The right-hand side of this equation is a continuous function of both  $R_t$  and  $\alpha_t$ ; it has a well-defined limit as  $\alpha_t$  approaches zero, equal to  $R_t^{-\gamma/1-\gamma}$ . Taking logarithms of both sides of (3.21), then, we can write

$$\log M_t = \log \alpha_t + \log p_t y_t - \frac{\gamma}{1-\gamma} \log R_t, \quad (3.22)$$

neglecting a term of order  $|\alpha|$  on the right-hand side. Now the fluctuations in  $\log \alpha_t$  from period to period may be arbitrarily large, even though  $|\alpha|$  is small. Thus our equations characterizing price-level determination in the cashless limit may be a good approximation, even in an economy subject to instability in the transactions technology that causes there to be large residuals in a logarithmic money demand equation.

First-differencing (3.22), we obtain

$$g_{Mt} = g_{pt} + g_{yt} - \frac{\gamma}{1-\gamma} g_{Rt} + g_{\alpha t}, \quad (3.23)$$

where  $g_{Xt} \equiv \log(X_t/X_{t-1})$  denotes the percentage growth rate of a variable  $X$ . If we hold fixed the sequence of growth rates  $\{g_{\alpha t}\}$  while letting the absolute level of the parameters  $\alpha_t$  in each period approach zero, then we obtain (3.23) as a well-defined money demand equation

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<sup>26</sup>Here we restrict attention to the case of an equilibrium in which  $R_t > 1$  each period.

in first-differenced form, valid in the cashless limit (and once again, approximately correct for any economy in which cash balances are sufficiently unimportant). This equation can be used to derive the predictions of the model for the fluctuations in the growth rate of the money supply under a Wicksellian regime, should one care about that variable. (The predictions for this variable, of course, depend upon the details of the transactions technology, and upon the time path of the disturbances  $g_{at}$ .) Note that this provides a rigorous justification for analyzing the determinants of money growth by simply appending a money demand equation to our previous system of equilibrium conditions, without having to modify those other equations (such as the Euler equation) to take account of the use of money for transactions. Of course, this kind of equation system can also be derived from a Sidrauski-Brock model with an additively-separable period utility function.<sup>27</sup> However, the cashless limit defined here provides a better justification, for several reasons. One is the implausibility of additive separability, noted above. Another is that additive separability has strong consequences for the form of the money demand equation, that need not be imposed when (3.23) is derived from the cashless limit. In a Sidrauski-Brock model with  $U(c, m) = v_1(c) + v_2(m)$ , one obtains a money demand equation of the form

$$\frac{v_2'(M_t/p_t)}{v_1'(c_t)} = \frac{R_t - 1}{R_t}.$$

This equation implies a unit income elasticity of money demand, as in (3.22), only if  $v_1'(c)$  and  $v_2'(m)$  are each isoelastic functions, with the same constant elasticity with respect to their respective arguments. Let that common elasticity be  $-1/\sigma$ , where  $\sigma > 0$  is then the

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<sup>27</sup>That sort of derivation is given, for example, in Woodford (1996) and McCallum and Nelson (1997).

intertemporal elasticity of substitution in consumption. One then obtains

$$\log M_t = \log p_t y_t - \sigma \log \left( \frac{R_t - 1}{R_t} \right).$$

One can approximate this by a log-linear form, as in (3.22), in the case of small enough fluctuations in the interest rate. But then the elasticity with respect to  $R_t$  is equal to  $-\sigma/(R^* - 1)$ , where  $R^*$  is the value around which one log-linearizes. Since  $\sigma$  is a parameter of the Euler equation (3.19), the interest elasticity of money demand is not a free parameter; it is instead completely determined by the parameters of the Euler equation and by the average level of the nominal interest rate. In the derivation of (3.23) in the cashless limit, by contrast, the interest elasticity is given by  $-\gamma/(1 - \gamma)$ . There is no theoretical restriction upon the size of this parameter (only its sign), and since the parameter  $\gamma$  has no effect upon any other equilibrium conditions in the cashless limit, one is free to assign the interest elasticity a value that is consistent with empirical studies of money demand.<sup>28</sup>

We have thus far demonstrated the existence of a well-behaved cashless limit only in the case of a monetary policy rule of the form (3.7), with  $\phi(p)$  an increasing function. Such a policy implies that the absolute price level matters to the central bank, and not simply the rate of inflation. In fact, in a modern context, a central bank is more likely to be concerned with control of the rate of inflation, allowing arbitrarily large eventual deviations in the price level to occur as the cumulative effect of many small errors in control of the inflation rate. This seems, at least, to be the way in which inflation targets are determined by those

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<sup>28</sup>By contrast, the assumption of additive separability forces one to assume an interest elasticity that is unrealistically large, given historical experience. Thus, for example, the simulations reported in Woodford (1996) involve implausibly large responses of money balances, and hence of seignorage revenues, to fiscal disturbances, because the model, which assumes additive separability, is “calibrated” by the choice of realistic values for the intertemporal elasticity of substitution in consumption, the rate of time preference, and the income elasticity of money demand. These values imply an interest elasticity of 19, which is two or three times as large as would be realistic for that parameter in a model of the US economy.

central banks that currently practice “inflation targeting”.<sup>29</sup> However, similar results can be obtained for a Wicksellian regime of that kind as well. Suppose that we consider, instead of (3.7), a monetary policy rule of the form

$$R_t = \pi_t \phi(\pi_t), \quad (3.24)$$

where  $\pi_t \equiv p_t/p_{t-1}$ , and  $\phi(\pi)$  is a continuous function with the property that  $\phi(\pi) = 1/\pi$  for all  $0 < \pi \leq \underline{\pi}$ , while  $\phi(\pi)$  is a monotonically increasing function for all  $\pi \geq \underline{\pi}$ , where  $\underline{\pi}$  is some rate of growth satisfying  $\underline{\pi} > \beta$ . This is again a rule that satisfies (2.16), but now the interest rate depends upon the rate of inflation, rather than the price level. We shall be concerned here with equilibria in which inflation is in the range for which  $\phi$  is an increasing function; in that region, (3.24) says that increases in inflation result in nominal interest rate increases that more than compensate for the erosion of the real return on bonds by the additional inflation.<sup>30</sup> This is the sort of behavior that Taylor (1993) argues describes US monetary policy in recent years.<sup>31</sup> In this case, we can obtain the following variant of our previous result.

**Proposition 2** *Consider a Wicksellian policy regime defined by (3.24) and (3.8), and suppose that  $\phi$  is a continuously differentiable function, at least on a neighborhood of some rate of inflation  $\pi^* > \beta$ , at which  $\phi(\pi^*) = \beta^{-1}$ ,  $\phi'(\pi^*) > 0$ . Let  $y^* > 0$  be any constant level of productive capacity. Then there exist neighborhoods  $U$  of  $\pi^*$ ,  $V$  of  $y^*$ , and  $W$  of  $0$ , and a*

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<sup>29</sup>It can also be justified on the basis of a welfare analysis of the distortions created by inflation variability in a model with nominal price rigidities, as in Rotemberg and Woodford (1997).

<sup>30</sup>We are not able to extend this sort of behavior to arbitrary rates of deflation, because the central bank cannot implement a rule that violates (2.16).

<sup>31</sup>The much-discussed “Taylor rule” involves a function  $\phi$  that varies with the level of real activity, as well as inflation, but we omit that here for simplicity. Because  $y_t$  is an exogenous state variable in the present model, consideration of the actual “Taylor rule” would not change any of our conclusions.

function  $f$ , assigning an inflation sequence  $\pi = f(y; \alpha)$  with  $\pi_t \in U$  for all  $t$  to any capacity sequence  $y$  with  $y_t \in V$  for all  $t$  and any parameter sequence  $\alpha$  with  $\alpha \in W$  for all  $t$ , such that

- (i)  $f$  is a continuous function in the sup norm topology,<sup>32</sup> in both arguments; and
- (ii) for any sequence  $y$  and for any sequence  $\alpha$  with  $\alpha_t > 0$  for all  $t$ ,  $\pi = f(y; \alpha)$  is a perfect foresight equilibrium inflation sequence, and the unique equilibrium sequence with the property that  $\pi_t \in U$  for all  $t$ .

The proof proceeds along the same lines as in the case of Proposition 1. Under the policy regime described in the proposition, a sequence of inflation rates  $\{\pi_t\}$  constitutes a perfect foresight equilibrium if and only if it satisfies the equilibrium conditions

$$F(\pi_t; y_t) = \beta G(\pi_{t+1}; y_{t+1}), \quad (3.25)$$

$$\sum_{T=t}^{\infty} \beta^T H(\pi_T; y_T) < \infty, \quad (3.26)$$

$$\lim_{T \rightarrow \infty} \prod_{t=0}^T (\beta/\pi_t) F(\pi_T; y_T) = 0, \quad (3.27)$$

where now

$$F(\pi; y) \equiv G(\pi; y)/\phi(\pi),$$

$$G(\pi; y) \equiv u'(\omega(\pi\phi(\pi))y)\pi^{-1}\theta_1(\pi\phi(\pi))^{\gamma-1},$$

$$H(p; y) \equiv u'(\omega(\pi\phi(\pi))y)\omega(\pi\phi(\pi))y.$$

Again these equations are well-behaved in the cashless limit, and one observes that in the case that  $y_t = y^*$  and  $\alpha_t = 0$  for all  $t$ , the sequence  $\pi_t = \pi^*$  for all  $t$  represents a solution to all three equations. (Once again, we shall also use  $\pi^*$  to denote this sequence.)

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<sup>32</sup>Here we use the norm (3.3) for sequences  $\pi$  as well.

One observes that, if the variables  $\pi_t, y_t$  and  $\alpha_t$  are confined to small enough bounded intervals, containing neighborhoods of  $\pi^*, y^*$  and 0 respectively, the functions  $F(\pi_t; y_t, \alpha_t)$  and  $H(\pi_t; y_t, \alpha_t)$  will be bounded, and one will have  $0 < \beta/\pi_t < \lambda$  for all  $t$ , for some  $\lambda < 1$ . Then (3.26) and (3.27) will necessarily be satisfied. Thus it suffices to show that there exists a locally unique solution to (3.25) for any sequences  $y$  and  $\alpha$  close enough to  $y^*$  and 0, and that it is described by a continuous function  $\pi = f(y; \alpha)$ . Again, the sequence of equilibrium conditions (3.25) can be written in the form  $\Phi(\pi; y; \alpha) = 0$ , where  $\Phi$  is a continuously differentiable mapping of triples of bounded sequences into bounded sequences. Again, application of the implicit function theorem gives the desired result, and turns upon the properties of the derivative mapping  $D_\pi \Phi(\pi^*; y^*; 0)$ .<sup>33</sup> For  $\{\hat{\pi}_t\}$  any bounded sequence of deviations in inflation (*i.e.*,  $\hat{\pi}_t \equiv \log(\pi_t/\pi^*)$ ), the derivative is given by  $D_p \Phi \cdot \hat{p} = z$ , where the sequence  $z$  satisfies

$$z_t = \Phi_{\pi_1} \hat{\pi}_t + \Phi_{\pi_2} \hat{\pi}_{t+1} \quad (3.28)$$

for  $t = 0, 1, 2, \dots$ . Here  $\Phi_{\pi_1}$  is the partial derivative of  $\Phi_t$  with respect to  $\log \pi_t$ , and  $\Phi_{\pi_2}$  the partial derivative with respect to  $\log \pi_{t+1}$ , evaluated at  $(\pi^*; y^*; 0)$ . Again, this defines a continuous linear mapping with a continuous inverse if  $|\lambda| < 1$ , where  $\lambda \equiv -\Phi_{\pi_2}/\Phi_{\pi_1}$ . Since in the present case

$$\Phi_{\pi_1} = F_\pi(\pi^*; y^*; 0) = -(1 + \epsilon_\phi)F(\pi^*; y^*; 0),$$

$$\Phi_{\pi_2} = -\beta G_\pi(\pi^*; y^*; 0) = F(\pi^*; y^*; 0),$$

we again obtain  $\lambda = 1/(1 + \epsilon_\phi)$ . Thus once again, under the assumption that  $\phi'(p^*) > 0$ , we obtain  $0 < \lambda < 1$ , the implicit function theorem applies, and the proposition is established.

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<sup>33</sup>As before, we use the notation  $D_\pi \Phi$  to denote the partial derivatives with respect to variations in  $\log \pi_t$  for each date  $t$ , rather than partial derivatives with respect to  $\pi_t$ .

The Wicksellian policy rules described in Propositions 1 and 2 hardly exhaust the cases in which a well-behaved cashless limit exists. In particular, it will be observed that in the proofs of both propositions, the assumption that  $\phi$  is an increasing function plays a crucial role. Without that assumption, perfect foresight may be indeterminate (*i.e.*, not even locally unique). This does not mean that there is no equilibrium in the cashless limit, but simply that there is no *single* equilibrium in the cashless limit that approximates *all* of the equilibria of a small- $\alpha$  economy, even all of those that remain forever within a neighborhood of the steady-state values for the endogenous variables. Furthermore, even with a monetary policy rule of such a kind – say, a pure interest-rate peg, in which (3.7) is replaced by an exogenous sequence for  $\{R_t\}$  – it is possible for perfect foresight equilibrium to be uniquely determined, if fiscal policy is not of the balanced-budget sort assumed above. In many such cases, the cashless limit is again well-behaved, and approximation results similar to Propositions 1 and 2 can be established.

Consider, for example, the case of an exogenously determined path for the real primary government budget surplus, which in the present context (which already assumes zero government purchases) is equivalent to an exogenous sequence for real net tax collections  $\tau_t \equiv T_t/p_t$ . Using (2.11), (2.12), and (2.23), this implies that total financial wealth (which

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<sup>34</sup>The crucial result required for applicability of the implicit function theorem is the demonstration that perfect foresight equilibrium is determinate (locally unique) near the steady state, in the case of an interest-rate rule (3.24) with  $\phi' > 0$  and a balanced-budget fiscal policy. This result is closely related to that obtained by Schmitt-Grohé and Uribe (1997) for a similar policy regime in the case of a cash-in-advance monetary model of the kind proposed by Lucas and Stokey (1987), and also similar to that obtained by Leeper (1991) for a similar monetary policy, in the case of a Sidrauski-Brock model and a fiscal policy rule that, while not ensuring budget balance each period, also limits the growth of the public debt.

equals total government liabilities) evolves according to

$$W_{t+1} = R_t[W_t - p_t\tau_t] - (R_t - 1)p_t\mu(R_t)y_t. \quad (3.29)$$

In this case, it is no longer true that the transversality condition (2.21) is necessarily satisfied by any sequences for  $\pi_t$  and  $R_t$  that satisfy appropriate bounds, and thus no longer true that even all solutions to (3.10) that satisfy such bounds must constitute perfect foresight equilibria. In fact, one can show <sup>35</sup> that (2.21) is satisfied if and only if

$$\frac{W_t}{p_t} = u'(\omega(R_t)y_t)^{-1}\theta_1(R_t)^{1-\gamma} \sum_{T=t}^{\infty} \beta^{T-t} u'(\omega(R_T)y_T)\theta_1(R_T)^{\gamma-1} [\tau_T + (1-R_T^{-1})\mu(R_T)y_T]. \quad (3.30)$$

Condition (3.30) uniquely determines the price level  $p_t$  at any date, given the nominal value  $W_t$  of financial assets carried into that period, and the expected future paths of  $R_T$ ,  $\tau_T$ , and  $y_T$ , all of which are exogenous series under the regime described. <sup>36</sup> But then condition (3.29) uniquely determines  $W_{t+1}$ , and so (generically) the complete equilibrium sequences are uniquely determined, given an initial condition  $W_0$ .

These equilibrium conditions are also well-behaved in the cashless limit; <sup>37</sup> (3.29) and (3.30) become simply

$$W_{t+1} = R_t[W_t - p_t\tau_t], \quad (3.31)$$

$$\frac{W_t}{p_t} = u'(y_t)^{-1} \sum_{T=t}^{\infty} \beta^{T-t} u'(y_T)\tau_T \quad (3.32)$$

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<sup>35</sup>See Woodford (1994) and (1995) for details, and Sims (1994) for a similar result.

<sup>36</sup>Thus under this sort of regime, what I have elsewhere (Woodford, 1995) called a “fiscal theory of the price level” is possible. In this case, the equilibrium conditions fail to be homogeneous degree zero in money prices, not because of the character of the central bank’s reaction function, as in the Wicksellian regimes, but because the predetermined state variable  $W_t$  is specified in nominal terms, and the fiscal policy rule implies that  $W_t/p_t$  affects the intertemporal budget constraint (2.10) of the representative household, contrary to what would be the case under a “Ricardian” fiscal policy. See also Woodford (1996) for further discussion of such regimes.

<sup>37</sup>This was first observed by Sims (1994), in the case of a slightly different transactions technology.

respectively. Equations (3.31) and (3.32) have a unique solution, and the implicit function theorem can again be used to show that the solution to equations (3.29) and (3.30) have is near it for all small enough sequences  $\alpha$ . Analysis of such regimes is not only possible in the cashless limit; use of the limiting equations makes the effects of fiscal shocks on the equilibrium price level more transparent. This is especially true when the monetary policy rule is more complex, and allows the interest rate to be a function of the price level or inflation rate, as in the regime considered in Woodford (1996). Because the right-hand side of (3.30) is independent of the interest rate in the cashless limit, it is easy to see that (3.30) continues to have a unique solution for the price level in the limiting case. The implicit function theorem can then be used to show the existence of at least a locally unique solution in any small- $\alpha$  economy. Thus, once again, the *real* factors that affect the equilibrium price level are brought into sharper focus by abstracting from the role of money in exchange.<sup>38</sup>

## 4 The Determinants of Inflation under a Wicksellian Regime

In the previous section, we have established the existence of a well-defined equilibrium associated with the cashless limit of our model, in the case of a Wicksellian regime of either of two sorts, and have shown that this equilibrium can be used as an approximation to the behavior of the price level under such a regime, in the case of an economy in which the fraction of transactions that require cash is positive but small. We now turn to a more detailed consideration of the theory of price-level determination that is obtained using this approximation.

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<sup>38</sup>As Sims (1994) notes, proceeding to the cashless limit also makes it clear that the effects of fiscal shocks on the price level under such a regime are not due to any connection between the government budget and the path of the money supply due to the contribution of seignorage revenues to the government budget.

In this way we can see how it is possible to discuss the effects both of monetary policy and of other real disturbances upon equilibrium inflation, without any reference to variations in either money demand or the money supply.

Let us first consider the sort of price-level-targeting regime described in Proposition 1. We can slightly generalize (3.7) to allow for time-variation in monetary policy, possibly in response to disturbances to the economy (real or perceived). Thus we now write

$$R_t = 1 + \rho_t[\phi(p_t) - 1], \quad (4.1)$$

where  $\rho_t > 0$  each period, and  $\phi(p)$  has the same properties as in (3.7); note that (4.1) continues to satisfy (2.16) at all times. We shall restrict attention to the case in which  $\rho_t$  belongs always to a small neighborhood of the value 1. We may also generalize the model introduced in section 2, to allow for an additional type of real disturbance, namely, fluctuations in the rate of time preference.<sup>39</sup> Thus we replace (2.1) by

$$\sum_{t=0}^{\infty} [\prod_{s=0}^{t-1} \beta_s] u(C_t), \quad (4.2)$$

where  $\{\beta_t\}$  is a series of time-varying discount factors. We shall restrict attention to the case in which  $\beta_t$  belongs always to a small neighborhood of a value  $\beta^*$  satisfying  $0 < \beta^* < 1$ .

In this extension of the model, a perfect foresight equilibrium is again a price sequence satisfying a set of conditions of the same form as equations (3.10) – (3.12). However, now the factor  $\beta$  in (3.10) must be replaced by  $\beta_t$ ; and the factor  $\beta^T$  in (3.11) and (3.12) must be replaced by  $\prod_{t=0}^{T-1} \beta_t$ . Furthermore, in the definitions of the functions  $F$ ,  $G$ , and  $H$ , the function  $\phi(p)$  must be replaced by the right-hand side of (4.1). The implicit function can be

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<sup>39</sup>This is intended, in our simple model, to stand in for all of the sorts of disturbances, unrelated to monetary policy or to the economy's productive capacity, that affect the desired intertemporal resource allocation at given real interest rates – what in traditional language would be called “IS shocks”.

used as before to extend Proposition 1 to this case. The sequence of equilibrium conditions corresponding to (3.10) can be written in the form  $\Phi(p; y, \beta, \rho; \alpha) = 0$ , where the mapping  $\Phi$  is continuously differentiable with respect to the sequences  $\beta$  and  $\rho$  as well. Using the same argument as before,<sup>40</sup> this equation implicitly defines a continuous function  $f$  such that the sequence  $p = f(y, \beta, \rho; \alpha)$  is the unique equilibrium with the property that  $p_t$  belongs to a certain neighborhood of  $p^*$  for all  $t$ , for any sequences  $y, \beta, \rho$  and  $\alpha$  confined to sufficiently small neighborhoods of the constant values  $y^*, \beta^*, 1$  and  $0$ . Because of the continuity of  $f$  in  $\alpha$ , the function

$$p = f_{lim}(y, \beta, \rho) \equiv f(y, \beta, \rho; 0)$$

can be used as an approximate description of the way the price level responds to fluctuations in productive capacity, preferences with regard to the timing of consumption, and monetary policy. Our goal now is to characterize the way that  $f_{lim}$  depends upon these variables. The following log-linear approximation to  $f_{lim}$  is useful in the case of sufficiently small fluctuations in any of its arguments.

**Proposition 3** *Let preferences be specified by (4.2), and consider a Wicksellian regime of the kind described in Proposition 1, but with (3.7) replaced by (4.1). Then for any sequences  $y, \beta, \rho$  and  $\alpha$  that are close enough (in terms of the sup norm<sup>41</sup>) to the constant sequences  $y^*, \beta^*, 1$  and  $0$ , the locally unique equilibrium price sequence given by the function  $f$  referred to in Proposition 1 can be approximated by*

$$\hat{p}_t = \sum_{j=0}^{\infty} \lambda^{j+1} [\sigma^{-1}(\hat{y}_{t+j+1} - \hat{y}_{t+j}) - \hat{\beta}_{t+j} - \hat{\rho}_{t+j}], \quad (4.3)$$

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<sup>40</sup>Note that the derivative mapping  $D_p \Phi(p^*; y^*, \beta^*, 1; 0)$  is exactly the same as that computed before, with the factor  $\beta$  replaced by  $\beta^*$ , so that again it has the properties required for the implicit function theorem to apply.

<sup>41</sup>Here we use the norm (3.3) for the first three sequences, and (3.1) for  $\alpha$ .

where for each variable,  $\hat{x}_t \equiv \log(x_t/x^*)$  indicates percentage deviations from the constant steady-state value, except that  $\hat{\rho}_t \equiv (1-\beta)\log(\rho_t)$ ,<sup>42</sup> and once again  $\lambda \equiv 1/(1+\epsilon_\phi)$ . Equation (4.3) would become exact with the addition of a remainder term, which term is of at least second order in  $|y - y^*|$ ,  $|\beta - \beta^*|$ , and  $|\rho - 1|$ , and of at least first order in  $|\alpha|$ .

This result is obtained as a further consequence of the implicit function theorem. Because the mapping  $\Phi$  is continuously differentiable, the function  $f$  implicitly defined by it is continuously differentiable as well. Then, by Taylor's theorem,<sup>43</sup> we can write

$$\hat{p} = D_x f(x^*; 0) \cdot \hat{x},$$

where  $x$  is a vector that includes all of the arguments of  $f_{lim}$ ,  $x^*$  is the vector of steady-state values for those variables, and  $D_x f$  denotes the partial derivative of  $\log f$  with respect to the elements of  $\log x$ , and where we omit a residual that is of at least second order in  $|x - x^*|$ , and of at least first order in  $|\alpha|$ . We thus need simply to compute the derivative  $D_x f$ . The implicit function theorem also tells us that this is given by

$$D_x f(x^*; 0) = -[D_p \Phi(p^*; x^*; 0)]^{-1} \cdot D_x \Phi(p^*; x^*; 0).$$

The mapping  $[D_p \Phi]^{-1}$  has already been computed earlier, and is given by (3.18). The mapping  $D_x \Phi$  is easily computed, using the same method as was used earlier to compute  $D_p \Phi$ . One thus obtains (4.3).

Note that this method of computing  $D_x f$  is equivalent to log-linearizing the system of equilibrium conditions corresponding to (3.10) around the steady-state values  $(p^*; x^*)$ , and

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<sup>42</sup>This rescaling of  $\hat{\rho}_t$  implies that now  $\hat{\rho}_t = .01$  indicates a one percentage-point increase in the nominal interest rate that the central bank sets in the case that  $p_t = p^*$ . This is a more natural measure of the size of the vertical shift in the central bank's policy schedule, and is especially convenient for purposes of comparison with the size of shifts in the "natural rate" of interest, below.

<sup>43</sup>For a version of the theorem applicable to general Banach spaces, and thus suitable for our purposes, see Lang, 1983, pp. 115-116.

then solving for the unique bounded solution for the sequence  $\hat{p}$ , given bounded sequences  $\hat{x}$  for the disturbances. In the cashless limit, the relevant equilibrium conditions are obtained by substituting (4.1) into (3.19), to obtain

$$\frac{u'(y_t)}{p_t[1 + \rho_t(\phi(p_t) - 1)]} = \beta_t \frac{u'(y_{t+1})}{p_{t+1}}. \quad (4.4)$$

The log-linear approximation to this nonlinear difference equation (keeping only the terms of first order in  $\hat{p}$  and  $\hat{x}$  in the Taylor-series expansions of both sides) is in turn given by

$$-(1 + \epsilon_\phi)\hat{p}_t - \sigma^{-1}\hat{y}_t - \hat{\rho}_t = \hat{\beta}_t - \hat{p}_{t+1} - \sigma^{-1}\hat{y}_{t+1}. \quad (4.5)$$

Rearranging terms, this may be written

$$\hat{p}_t = \lambda[\sigma^{-1}(\hat{y}_{t+1} - \hat{y}_t) - \hat{\beta}_t - \hat{\rho}_t] + \lambda\hat{p}_{t+1}.$$

Since  $|\lambda| < 1$ , it is then obvious that there is a unique bounded solution for the sequence  $\hat{p}$ , given any bounded sequences of disturbances  $\hat{y}$ ,  $\hat{\beta}$ , and  $\hat{\rho}$ , and that this is given by (4.3). This then represents a first-order Taylor-series approximation to the function  $f_{lim}$ , and the proposition is established.

Note that the solution (4.3) can be written more simply as

$$\hat{p}_t = \sum_{j=0}^{\infty} \lambda^{j+1} [\hat{r}_{t+j} - \hat{\rho}_{t+j}], \quad (4.6)$$

where the variable

$$\hat{r}_t \equiv \sigma^{-1}(\hat{y}_{t+1} - \hat{y}_t) - \hat{\beta}_t \quad (4.7)$$

measures percentage variations in the equilibrium real rate of return (in the cashless limit), which is independent of monetary policy.<sup>44</sup> Fluctuations in the exogenous variable  $\hat{r}_t$  suffice

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<sup>44</sup>Equation (4.7) for variations in the equilibrium real rate can be obtained as a log-linear approximation to the Euler equation (3.19) for the cashless limit.

to summarize the effects of disturbances to either preferences or productive capacity upon equilibrium inflation under this kind of regime.

Equation (4.6) may be given a Wicksellian interpretation. We observe first that a current or expected future “tightening” of monetary policy, in the sense of increase in  $\rho$ , lowers the equilibrium price level, and is thus an appropriate response to perceived inflationary pressures. The central bank *raises* short-term nominal interest rates in order to *lower* the general price level, as Wicksell argued. There are, of course, noteworthy differences between our account and Wicksell’s simpler one. First of all, in a perfect foresight equilibrium, the anticipation of *future* tightening lowers the equilibrium price level immediately. On the other hand, in the present simple model, there are no “inertial” effects of past monetary policy, or of past expectations regarding current or future monetary policy, upon the price level; price-level determination is purely forward-looking. Second, it is important to note that in our account, a “tightening” of policy means a shift up of the entire schedule indicating the interest rate  $R_t$  that the central bank will set for each possible price level  $p_t$ , and not simply an increase in the increase rate. The place where this is especially crucial is in the specification of expectations regarding *future* monetary policy; we have solved for the equilibrium price level (and expected future path of the price level) under a particular stipulation regarding the expected future evolution of the policy *schedule*  $R_t(p_t)$ , and not simply for an expected future path of the interest rate  $R_t$ .<sup>45</sup>

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<sup>45</sup>Under the other assumptions of our model, especially the specification of a balanced-budget fiscal policy, a monetary policy specified simply in terms of a path for the nominal interest rate would not uniquely determine the equilibrium path of the price level; to be precise, there would not even be a *locally* unique solution, in terms of the sup norm topology. This is because of the much-discussed problem of price-level indeterminacy under an interest-rate peg. (For a demonstration that a unique equilibrium price level may result under an alternative specification of fiscal policy, however, see Woodford, 1994, 1995.) This is sometimes argued to imply that Wicksell’s analysis of the connection between interest and prices is incompatible with perfect foresight or rational expectations reasoning (e.g., Laidler, 1991, pp. 132-135). But we show here that this

Equation (4.6) also indicates that the disturbances not originating with the central bank, but that affect the equilibrium price level, can be completely summarized by the expected evolution of the equilibrium real rate of return  $\hat{r}_t$ , which depends purely upon real factors (preferences and productive capacity). The equation implies that prices will rise if there is an exogenous increase in the current or expected future “natural rate” of interest  $\hat{r}_t$ , that is not matched by a corresponding tightening of monetary policy. Alternatively, a loosening of monetary policy will cause prices to rise, unless it has been justified by an exogenous decline in the “natural rate” of interest. This attribution of price-level changes to changes in the relation between the “natural rate” of interest, determined by real factors, and the level of nominal rates determined by central bank policy, of course, recalls Wicksell’s (1898) famous account, though again our theory is forward-looking rather than inertial.<sup>46</sup>

While it is crucial to our analysis that the Wicksellian regime involve a (possibly time-varying) policy schedule that adjusts nominal interest rates in response to changes in the price level, it is not essential that the central bank care about the absolute level of prices, as assumed in (4.1). We might instead, and more realistically under modern conditions, suppose that the central bank’s policy schedule specifies the nominal interest rate as a function of the rate of inflation rather than price level. Generalizing (3.24). let us consider a time-varying monetary policy rule of the form

$$R_t = 1 + \rho_t[\pi_t\phi(\pi_t) - 1], \tag{4.8}$$

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problem is solved by conceiving of monetary policy as specifying a schedule for each date (and, in a stochastic model, each possible state of the world), rather than simply an interest rate. The resulting analysis still has much of the flavor of Wicksell’s.

<sup>46</sup>Wicksell’s own analysis of the consequences of a discrepancy between market interest rates and the “natural rate” depends upon backward-looking expectations, especially in his account of the “cumulative process” of inflation set in motion by such a discrepancy.

where  $\phi(\pi)$  has again the properties assumed in (3.24), and  $\{\rho_t\}$  is again an arbitrary positive sequence. Note that (4.8) necessarily satisfies (2.16), and that it reduces to (3.24) in the case that  $\rho_t = 1$  for all  $t$ . As with (4.1), an increase in  $\rho_t$  shifts up the schedule  $R_t(\pi_t)$  at each possible inflation rate  $\pi_t > \underline{\pi}$ . We shall again restrict our attention to the case in which  $\rho_t$  varies over a small neighborhood of the value 1.

The results of Proposition 2 are easily extended to the case in which the policy rule is given by (4.8) and preferences are given by (4.2). We can also easily derive a first-order Taylor-series approximation to the locally unique equilibrium path for inflation described in that proposition, of the kind given for the price-level-targeting regime in Proposition 3.

**Proposition 4** *Let preferences be specified by (4.2), and consider a Wicksellian regime of the kind described in Proposition 2, but with (3.24) replaced by (4.8). Then for any sequences  $y, \beta, \rho$  and  $\alpha$  that are close enough (in terms of the sup norm) to the constant sequences  $y^*, \beta^*, 1$  and  $0$ , the locally unique equilibrium inflation sequence given by the function  $f$  referred to in Proposition 2 can be approximated by*

$$\hat{\pi}_t = \sum_{j=0}^{\infty} \lambda^{j+1} [\hat{r}_{t+j} - \hat{\rho}_{t+j}], \quad (4.9)$$

where again  $\hat{r}_t$  is the series defined in (4.7). Again, equation (4.9) would become exact with the addition of a remainder term, which term is of at least second order in  $|y - y^*|, |\beta - \beta^*|$ , and  $|\rho - 1|$ , and of at least first order in  $|\alpha|$ .

The proof of this proceeds as in the case of Proposition 3. The relevant generalization of difference equation (3.25) reduces in the cashless limit to

$$\frac{u'(y_t)}{1 + \rho_t(\pi_t\phi(\pi_t) - 1)} = \beta_t \frac{u'(y_{t+1})}{\pi_{t+1}}, \quad (4.10)$$

which again simply corresponds to substitution of (4.1) into (3.19), once the latter equation has been generalized to allow for a time-varying discount factor. A log-linear approximation to (4.10) is in turn given by

$$-(1 + \epsilon_\phi)\hat{\pi}_t - \sigma^{-1}\hat{y}_t - \hat{\rho}_t = \hat{\beta}_t - \hat{\pi}_{t+1} - \sigma^{-1}\hat{y}_{t+1}.$$

Note that this is exactly equation (4.5), except with the variable  $\hat{\pi}_t$  replacing  $\hat{p}_t$ . Thus the unique bounded solution for the sequence  $\{\hat{\pi}_t\}$  is identical to that previously obtained for  $\{\hat{p}_t\}$ , and we obtain (4.9) instead of (4.6).

We find, once again, that “tightening” of monetary policy, in the sense of an upward shift in the schedule  $R_t(\pi_t)$ , lowers inflation, as does the expectation of a future tightening. We also find that real disturbances can cause variations in equilibrium inflation, in the absence of any instability in monetary policy (in the sense of shifts in the  $R(\pi)$  schedule), and more specifically, that increases in the “natural rate” of interest, if not countered by a tightening of monetary policy, are inflationary. In this emphasis upon real disturbances, unrelated to monetary frictions, as a source of inflationary impulses, our theory agrees with that of Wicksell; and as he did, we find that it is useful to analyze the effects of these disturbances in a model that abstracts entirely from the use of money in exchange. By shifting attention away from the determinants of money demand, and disturbances affecting them, and instead toward the determinants of the “natural rate” of interest, and disturbances affecting it, I believe that this sort of theory emphasizes the structural relations that are central to an understanding of the kind of policy regimes that are of greatest current interest. And this should become all the more true as the “pure credit economy” that Wicksell considered for analytical purposes becomes a progressively better description of reality.

This model of inflation determination has obvious consequences for the conduct of a monetary policy that aims at inflation stabilization.<sup>47</sup> An equilibrium in which  $\pi_t = \pi^*$  at all times is achieved if and only if  $\rho_t$  is adjusted so as to precisely track the variations in the “natural rate” of interest. The optimal rule from this point of view is uniquely determined only within the class of rules of the form (4.8); there is no demonstration that a rule of this form is the only kind that can work, and indeed we have already seen that rules of the form (4.1) and (4.8) are equally able to achieve complete price stability in principle. Nonetheless, the state-dependent “Taylor rule” is at least a simple example of a suitable rule, and one that would seem to be of practical interest.

Of course, before this insight can be translated into a practical proposal for monetary policy, one needs to develop a more sophisticated model of the determinants of the natural rate than is offered here. Among the more obvious omissions in our model is any consideration of the role in real interest-rate determination of the existence of capital goods and consumer durables; we have also abstracted from decision lags or other adjustment costs in the response of spending decisions to interest rates, and from heterogeneity across households. Finally, we have abstracted from any form of price stickiness, and from any opportunity for demand variations to effect the economy’s output. Still, I believe that this model illustrates how the real determinants of equilibrium real rates of return become the primary sources of inflationary or deflationary pressures under a Wicksellian regime, and of how non-monetary models of intertemporal resource allocation and real interest-rate determination (of which our Euler equation (3.19) represents a simple but canonical example) can be applied to

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<sup>47</sup>Of course, the present simple model, in which inflation has no consequences for real activity, does not explain why policymakers should *want* to stabilize inflation. In an extension of the model with nominal rigidities, however, such a goal can be justified on welfare-economic grounds, along the lines indicated in Rotemberg and Woodford (1997).

the problem of modelling the consequences of monetary policy. Rotemberg and Woodford (1997) present an extension of the present model with sticky prices, decision lags in the response of purchases to real interest-rate changes, and with stochastic processes specified for the real disturbances, which is able to account for a number of quantitative aspects of the comovements of interest rates, inflation, and output, in US time series. That extension is intended to offer a more realistic model, that can be used for quantitative analysis of alternative policy rules; further research will doubtless allow even greater realism in future variants of such a model.

Another practical problem, even given a complete understanding of how the underlying real disturbances affect the “natural rate” in principle, concerns the way in which the central bank is to determine the current state of the economy, in order to implement a state-contingent policy of the kind just hypothesized. One certainly cannot assume that the central bank has direct knowledge of “the natural rate of interest”, or even of more primitive notions such as “the current rate of time preference of the representative household”. In practice, then, policy must rely upon various indicator variables, that may reveal the underlying state of the economy with a greater or lesser degree of precision. The study of the consequences of monetary policy rules using endogenous variables as indicators raises additional complications not present in the above analysis, where the policy shifts  $\{\rho_t\}$  were treated as an exogenous sequence (the idea being that  $\rho_t$  is set as a function of *exogenous* state variables, such as  $\beta_t$  and  $y_t$ ). The way in which  $\rho_t$  is made to depend upon endogenous indicator variables will, in general, affect the equilibrium determination of those variables, and in particular, may undesirably affect their usefulness as indicators of the underlying states to which it is desired that  $\rho_t$  should respond. Furthermore, dependence of  $\rho_t$  upon endogenous variables

makes the proofs of Propositions 3 and 4 invalid; specifically, the derivative mapping  $D_p\Psi$  or  $D_\pi\Psi$  may not be invertible, as required for application of the implicit function theorem, even when the monetary policy rule is otherwise of the kind assumed in those propositions. In such a case, rational expectations equilibrium is not even locally unique. When one extends the analysis to consider stochastic equilibria, one finds a large class of equilibria, even when one restricts attention to equilibria in which the endogenous variables fluctuate within bounded intervals; these include “sunspot” equilibria, in which inflation and other variables respond to arbitrary random events due to self-fulfilling expectations, as well as equilibria in which the response to real disturbances is disproportionate to the magnitude of the change in “fundamentals” such as productive capacity or household marginal rates of substitution.

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Both of these problems are illustrated in the context of a specific proposed implementation of “inflation targeting” in Bernanke and Woodford (1997). Many proponents of inflation targeting have suggested that a desirable policy should respond not simply to current or past inflation, as in the “Taylor rule” (3.24), but especially to indicators that forecast future inflation – perhaps actual forecasts, generated outside the central bank, perhaps asset prices or spreads that are taken to indicate “market expectations” of future inflation, or perhaps simply variables that have been found to be useful in forecasting inflation by the central bank.

<sup>49</sup> This might seem an appealing proposal, for reasons that are independent of any particular model of the way in which inflation is determined. But Bernanke and Woodford show, in the context of a model only slightly more complex than that presented here, that at least

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<sup>48</sup>These consequences are treated in detail in Woodford (1986).

<sup>49</sup>For discussions of the actual or ideal use of forecasts and indicators of forecasts in monetary policy, see, *e.g.*, Hall and Mankiw (1994), King (1994), Sumner (1995), Freedman (1996), and Svensson (1997).

some naive sorts of “forecast targeting” rules have undesirable consequences, both because the response of monetary policy to the forecasts reduces the extent to which they contain information, in equilibrium, about the underlying real disturbances to which the central bank wishes to respond, and because the response of policy to variations in private expectations can make those expectations self-fulfilling. The implication is not that it is impossible for a central bank to infer the underlying real states and adjust policy in response to them, but rather that the design of a policy rule that accomplishes this must make use of a structural model of the way in which these real states determine both inflation and the various available indicator variables. Bernanke and Woodford show how a simple structural model along the lines presented here can be used to this end.<sup>50</sup> While, again, a more detailed model would be necessary in order to derive a proposal that could be recommended for practical use, even their simple exercise suggests that the type of framework proposed here can be used to clarify one’s thought about issues of considerable practical significance for the design of monetary policy rules.

## 5 Conclusion

Ed Prescott used to like to ask, why is monetary economics any more important than postal economics? Sure, there are frictions involved in making payments, but there are frictions involved in getting goods and documents from one place to another, too. So why should economic theorists lavish so much attention upon the problem of modelling the details of the use of money in making payments, while completely neglecting the role in the economy of the postal system?

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<sup>50</sup>The NBER working paper contains more detail on this than does the published paper.

When I first heard this question – I was at that time quite engaged with the foundations of the demand for money, and their implications for the welfare analysis of monetary policy – I regarded it as too frivolous to deserve much thought. For I assumed that Prescott meant to suggest that the level of money prices was of no significance to the economy, and that the conduct of monetary policy was of little importance, so that one could afford to neglect the development of monetary theory. And I felt that these premises could be easily rejected.

But over the years, I have found myself coming around to something like the view that Prescott may have meant to suggest. This is not because I have ceased to believe that price stability matters, or that the proper conduct of monetary policy is one of the most important matters with respect to which economists are called to give expert advice. Instead, it is because I have realized that the project of modelling the fine details of the payments system and the sources of money demand is not essential to the explanation of how money prices are determined, or to the analysis of the effects of alternative monetary policies. And indeed, I now think that, at least in the case of economies with developed financial systems, and in the case of policy regimes that do not lead to serious disruptions of the financial system, one may legitimately abstract from such frictions, at least as a first approximation, just as one abstracts from many other frictions, including those that the postal system exists to deal with.

I have shown in this paper how one can study the effects of monetary policy in the “cashless limit” of a monetary economy, and how one can rigorously justify the use of the results so obtained as an approximation to what one would find in the case of an economy in which monetary frictions exist, but are small. Of course, the mere existence of a well-defined cashless limit is hardly a proof that this approximation is a useful one. The question,

ultimately, is what other aspects of the problem of monetary policy can be brought into better focus, if one dispenses with the complication of modelling the role of money in the economy. I believe that the principal gain is a simpler, and more straightforward view of the way in which the real determinants of equilibrium asset prices and rates of return affect the equilibrium price level, and hence of the way in which these real factors give rise to inflationary and deflationary pressures to which monetary policy should seek to respond, as Wicksell argued a century ago. Analysis of the cashless limit also makes it clear that improvements in the efficiency of financial arrangements, that reduce or destabilize the demand for the monetary base, need not be a source of macroeconomic instability. Once one specifies monetary policy in a way that makes the cashless limit well-behaved, it becomes possible to separate the problems of the desirable regulation of the payments system and of the desirable conduct of monetary policy. Clearer thought about both problems is likely to be the result.

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