Firm investment and the composition of debt*

Nicolas Crouzet†
Columbia University

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Abstract
I propose a static model of the joint determination of debt structure and the scale of investment. An entrepreneur finances a project of variable size using internal funds and external borrowing from two types of creditors: banks and public debt markets. The key distinction between the two is that, when liquidation looms, bank loans are easier to restructure than market debt. Absent deadweight losses in liquidation, debt structure is irrelevant to the investment choices of the entrepreneur, and projects are financed by whichever lender has the lowest marginal lending cost. With liquidation losses, I show that investment is financed by a combination of bank and market finance so long as 1) banks have higher marginal lending costs than markets and 2) entrepreneurs’ internal resources are sufficiently small. In that case, the share of bank finance in total investment depends non-monotonically on internal resources: firms with very limited internal resources are increasingly reliant on bank finance to expand investment, while medium-sized firms reduce the contribution of bank finance for each additional marginal unit of equity. I show that, as a result of firms adopting mixed debt structures, asymmetric changes in lending costs lead to large changes in investment at the firm level.

Keywords: banks, bonds, debt structure, financial frictions, investment, productivity risk.

JEL Classification Numbers: E22, E23, G21, G33.

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†Email: nc2371@columbia.edu.
1 Introduction

The most prominent feature of the corporate debt structure in the US is how dramatically it varies, both in the cross-section of firms and over time. Firms use various types of debt instruments, issued to different types of lenders, with diverse covenants and organized within rich priority structures. In the cross-section, recent work by Rauh and Sufi (2010) analyzes the holdings of bank debt, bonds, program debt (such as commercial paper) and mortgage debt of a sample of publicly traded firms. They find that 70% of firm-year observations hold at least two different types of debt instruments on their balance sheets; moreover, the two most prominently used types of debt are bonds (convertible and non-convertible), and bank debt. Over time, aggregates of corporate debt also display dramatic variation. Adrian, Colla, and Shin (2012), echoing the findings of Kashyap, Stein, and Wilcox (1993) on bank debt and commercial paper, document the fact that over the course of the 2007-2009 recession, the fall in aggregate bank debt outstanding was mirrored by a rise in the stock of corporate debt, suggesting that the overall debt structure of firms adapts to changes in the outstanding stock of bank debt.

Little attention has however been paid to the interaction between debt heterogeneity and the real decisions of firms. On the one hand, the literature the link between financial constraints and firm-level investment has typically treats all corporate debt as homogeneous. The large empirical (Fazzari, Hubbard, and Petersen (1988), Hoshi, Kashyap, and Scharfstein (1991), Whited (1992), Kaplan and Zingales (1997), Hubbard (1998)) typically focuses on measures of total leverage, and does ask whether debt structure is an important determinant of firms’ investment decisions. Similarly, the theoretical literature on firm dynamics and limited access to external finance (Cooley and Quadrini (2001), Albuquerque and Hopenhayn (2004), Clementi and Hopenhayn (2006)) explores the link between capital structure and investment dynamics, but again in the context of a single borrower - lender relationship. On the other hand, the extensive theoretical literature on debt heterogeneity (Diamond (1991), Diamond (1993), Rajan (1992), Bolton and Scharfstein (1996), Bolton and Freixas (2000), Park (2000), DeMarzo and Fishman (2007)) explores optimal capital structure with multiple potential lenders, but typically in the context of a firm that must finance a project of fixed size; it is silent on the question of whether, or how, investment size and firm growth could be affected by the debt financing options available to the firm.

This paper asks how debt heterogeneity affects real decisions of firms, and specifically the determination of investment and thus the scale of the projects they undertake. To analyze this question, I study a static model in which an entrepreneur can finance investment by borrowing from two potential sources, which I label "bank" and "market" lenders. The entrepreneur uses her own internal finance, in combination with

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1 See the first column of their table 1, p4250.
external debt, to finance investment in fixed capital; in turn, this capital is the input into a decreasing returns to scale production function. Production is subject to idiosyncratic risk, and the entrepreneur has limited liability; she can default on any debt claims, in which case the project is liquidated. Borrowing is constrained by the fact that liquidation entails inefficient losses, so that debt contracts charge the borrower a "liquidation risk" premium.

The model postulates two main differences between bank and market debt. On the one hand, I assume that bank debt can be easily renegotiated in times of financial distress, while market debt cannot. The view that bank debt is more flexible than market debt underlies much of the literature on debt heterogeneity; Gertner and Scharfstein (1991) and Bolton and Scharfstein (1996) develop its microfoundations, based on the view that dispersed holding of market debt creates a hold-out problem in which individual debt holders’ have limited incentives to agree to a debt restructuring plan. In my model, this flexibility implies that bank debt typically commands a smaller liquidation risk premium than market debt. On the other hand, I assume that the marginal cost of debt issuance is larger for banks that for market lenders. The debt structure chosen by the entrepreneur typically equates the marginal benefit of bank debt flexibility, to the difference in marginal cost of debt issuance between banks and market lenders.

The first prediction of the model is that the optimal debt structure of firms features a mix of bank and market finance. This is consistent with the fact documented by Rauh and Sufi (2010) that firms simultaneously use different types of debt instruments, but it stands in contrast with much of the theoretical literature on debt heterogeneity, which typically finds that the optimal debt structure involves borrowing either from one source or from the other, depending on firms’ characteristics. Here, the fact that investment and scale are endogenous is crucial. Indeed, the very largest firms, which have sufficient internal finance relative to the optimal investment scale and face little liquidation risk, do not find it beneficial to combine bank with market debt, and instead prefer to use only market debt. For the other firms, with more limited internal resources and higher liquidation risk, the marginal benefit of bank flexibility is not negligible relative to the "spread" between bank and market marginal lending costs; this leads to an interior solution for the debt structure.

The second prediction of the model is that the share of bank debt in total debt is a non-monotonic function of firms' internal resources. Among those firms that use a mixed debt structure, the share of bank debt is increasing when internal finance is very small, but starts decreasing when internal finance becomes sufficiently large (and eventually drops to 0 for the largest firms). This non-monotonic pattern arises because the very smallest firms choose an investment policy which exhausts their bank borrowing capacity, so that any marginal increase in internal finance, by expanding borrowing capacity, leads to an increase in bank debt.

\[^2\]Empirical evidence in support of the view that bank debt is easier to restructure is discussed in detail in section 3.
On the other hand, when firms have accumulated sufficient resources, their bank borrowing constraint need not bind; in that case, a marginal increase in internal finance makes bank reduces liquidation losses, and makes bank borrowing less attractive.

The third prediction of the model is that changes in the relative marginal cost of bank and market lending have markedly different effects on firms’ debt structure than either supply shocks that affect bank and market lenders identically, or shocks that affect the firms’ production possibilities, such as productivity shocks. In particular, I show that an increase in banks’ marginal lending costs, relative to market lenders’, causes the debt structure of firms with plentiful internal finance to lower their bank liabilities and increase their market liabilities, while for smaller firms, with more less internal resources, simply reduce overall lending, which initially took mostly the form of bank borrowing. A fall in average productivity, or an increase in the dispersion of productivity, on the other hand, both curtail borrowing from market lenders; additionally, a fall in average productivity reduces bank borrowing, while an increase in the dispersion of productivity increases it.

The model’s predictions match three facts on the debt structure of US firms, in the cross-section and over time. These facts, which I document in a sample of US manufacturing firms including both private and publicly traded firms, echo and supplement the findings of Rauh and Sufi (2010) and Adrian, Colla, and Shin (2012) (who focus on publicly traded, and therefore large firms). First, much as in the model, I show that the bank share of total debt is non-monotonic in firm size: it is increasing for the very smallest firms, and decreasing thereafter\(^3\). Second, over the last three recessions, and especially over the 2007-2009 recession, small firms experienced a very different change in their debt structure than large firms. While small firms saw their overall liabilities decrease, exclusively because of a fall in market debt, large firms substituted market debt for bank debt and did not experience a significant fall in their overall liabilities. From the standpoint of the model, this third fact is consistent with the view that shocks to the relative cost of bank lending played an important role in the in the early stages of the 2007-2009 recession.

This paper contributes to the theoretical literature on debt heterogeneity in two ways. First, while it builds on the “trade-off” theory of debt structure (Gertner and Scharfstein (1991), Rajan (1992), Bolton and Scharfstein (1996)), the paper extends it by endogenizing the choice of investment scale. In this respect, the closest setup to this paper is Hackbarth, Henessy, and Leland (2007), who likewise rely on the assumption that bank debt is more flexible than market debt in restructuring, but do not allow for an endogenous determination of investment scale. Second, it shows that debt heterogeneity can explain why credit supply shocks have different effects on the investment behavior of small and large firms, as documented by Kashyap,

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\(^3\)The data I use focuses on total assets as a measure of size, rather than internal finance; I discuss this issue in more detail in section 4.

The rest of the paper is organized as follows. Section 2 summarizes the three broad facts on the debt structure of US firms mentioned in the introduction. Section 3 describes the model and derives the set of feasible debt structures for a firm with a given level of internal resources. Section 4 studies the optimal debt structure and discusses whether it is consistent with the cross-sectional distribution of debt structure documented in section 2. Section 5 studies some comparative statics of the model, and relates them to changes in the debt structure of small and large firms in recent recessions. Finally, section 6 concludes.

2 Three facts on the debt structure of US firms

In this section, I document the following three facts on the relationship between the size of a firm and its debt structure:

1. The majority of financial liabilities of small firms are bank loans, while the majority of financial liabilities of large firms are non-bank credit instruments (fact 1).

2. On average over the last three recessions, small firms’ financial liabilities fell, entirely as a result of a fall in bank credit; non-bank credit did not change (fact 2).

3. On average over the last three recessions, large firms’ bank liabilities fell, while non-bank liabilities increased (fact 3).

I document these facts in the Quarterly Financial Report of Manufacturing firms (QFR). The QFR is a survey of manufacturing firms that reports aggregates of balance sheets and income statements across size bins. The QFR’s measure of size is assets at book value. One important advantage of the QFR is that it contains information on firms of all sizes, and in particular smaller, privately owned companies. Moreover, it is a quarterly dataset. The drawback is that data is only available in semi-aggregated form. An alternative is to use firm-level data, such as Compustat. The Compustat sample contains more detailed information on all debt instruments used by firms. It is also not restricted to manufacturing firms. However, it contains only annual balance sheet information for publicly traded companies. Thus, it excludes most smaller, non-traded companies, and has lower frequency of observation. Adrian, Colla, and Shin (2012) document the same pattern of debt substitution as fact 3, in that sample, over the last recession. More information on the QFR is given in appendix D.1.
2.1 Measures of the debt structure

Breaking down the liability side of firms’ balance sheets  Along with financial liabilities, the QFR balance sheets contain information on non-financial items, such as accounts payables and various forms of tax liabilities. I restrict my attention to financial liabilities, and more specifically to debt. This excludes, in particular, stockholders’ equity and trade credit, an important component of liabilities, especially for smaller firms. Within the category of financial liabilities, some are current (due in more than one year) and some are non-current (due in one year or less). I construct measures of the composition of debt for both maturity categories. For large firms, the behavior of current debt during recessions differs substantially from that of non-current. The focus of this paper is on the behavior of total liabilities of firms; I do not address changes in the maturity structure.

Bank and non-bank liabilities  The QFR sample contains two subsamples of firms, one of small and medium-sized firms and one of large firms. Smaller firms report their liabilities with less detail than large firms (appendix D.1 contains more information on the differences between short and long form samples). In particular, small firms report as a group all their non-bank financial liabilities, whereas the large firm samples reports separately commercial paper and long-term debt. To maintain comparability across asset size classes, I therefore focus on ”non-bank” liabilities as whole; that includes, for largest firms, both commercial paper and long-term debt. As reported in appendix D.2, the three facts reported here are robust to the exclusion of commercial paper from measures of non-bank liabilities for large firms.

Aggregates of financial liabilities  In the following discussion, I define two debt aggregates for each firm category: bank liabilities, denoted by $CB$ and $TB$; and non-bank financial liabilities, including commercial paper and bonds (denoted by $CNB$ and $TNB$). Variables beginning with a $C$ denote aggregates computed for current liabilities only, while variables beginning with a $T$ denote aggregates computed for total liabilities. I also contruct measures of total financial liabilities ($CFIN$ and $TFIN$), eliminating non-financial liabilities from the aggregate balance sheet measures. Table 3 in appendix D.1 summarizes the construction of these variables.

2.2 Debt structure in the cross-section (fact 1)

Figure 1(a) illustrates the differences in debt structure between small and large firms, by reporting the average bank share of total (bank and non-bank) debt across different asset size bins. Roughly 70% of the debt of small firms is held in the form of bank loans, while 80% of the debt of the largest firms is held in

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4Table 2 in appendix D.1 provides a breakdown of current and non-current financial liabilities in the short and long sample.
(a) **Fact 1:** the composition of debt in the cross-section

(b) **Fact 2:** the debt structure of small firms during recessions

(c) **Fact 3:** the debt structure of large firms during recessions

Figure 1: **Three facts on the debt structure of US manufacturing firms.** In figure 1(a), the left panel graphs current liabilities: the ratios $CB/CFIN$ (bank loans, green line) and $CNB/CFIN$ (non-bank credit, red line). The right panel plots total liabilities: the ratios $TB/FIN$ and $TNB/TFIN$. In figures 2 and 3, the right panels plot current liabilities and the left panels plot total liabilities; see text for details. Series in figures 2 and 3 are smoothed with a 2 by 4 MA smoother before computing growth rates.
the form of non-bank credit. Of these 80%, on average over the last 5 years, 60% were held in the form of commercial paper and bonds. This holds whether one focuses on total or current debt. Thus, the majority of the debt of small firms are bank loans, while the majority of the debt of large firms are non-bank credit instruments (fact 1). Note, however, that the relationship between size and the bank share is not strictly monotonic; for the very smallest firms, the bank share is slightly increasing in size; I come back to this fact in section 4.

2.3 Cyclical changes in debt structure (facts 2 and 3)

Fact 1 emphasizes that debt structure varies substantially in the cross-section of firms. In studying how debt structure changed during the most recent recessions, I will therefore focus on the differences between small and large firms. For creating aggregates of small and large firms, I directly use the fixed bins of asset size provided by the QFR. I define small firms as those with $1bn in assets or less (the first seven size bins of the QFR sample; see table 2), and large firms as those with more than $1bn in assets (the last bin of the QFR sample). This definition of small and large firm categories is transparent, but does not address the fact that, because of trend growth in the book value of assets, large firm bins become more populated over time. In appendix D.2, I consider an alternative definition of small and large firm groups, proposed by Gertler and Gilchrist (1994), which keeps the fraction of total sales accounted for by the small firm group category constant. I show that the three facts documented here are robust to these alternatives definitions of size classes.

I focus on cumulative growth rate of financial liabilities around NBER recession troughs. Using growth rates allows one to abstract from the large differences in levels between liabilities of small and large firms. The series reported in figures 2 and 3 shows averages of cumulative growth rates over the three recession in the sample5. I focus on a window of 1 year before the recession trough to 2 years after the recession trough.

Figure 2 graphs the growth rate in bank loans and non-bank credit for small firms. For each type of liability, the growth rate reported is weighted by its share in total liabilities at the recession trough. For example, for total bank loans, the series reported is the average of \( \gamma_{TB,t} = \frac{TB_0}{TB_0 + TNB_0} \left( \frac{TB_t}{TB_0} - 1 \right) \) over recessions, where the time index is 0 for the quarter corresponding to the trough of a recession. Weighting the growth rates in this fashion conserves additivity: for example, for small firms, \( \gamma_{TB,t} + \gamma_{TNB,t} = \gamma_{TFIN,t} \), where \( \gamma_{TFIN,t} \) is the growth rate of total financial liabilities of small firms around the recession trough. From figure 2, it is clear that for small firms, outstanding bank debt falls and non-bank debt does not change, resulting in a fall in total liabilities (fact 2). This holds regardless of whether one focuses on total liabilities (left panel) or only current liabilities (right panel).

5 Appendix D.4 reports the levels of all the series used.
Figure 3 reports the same series as figure 2, for the aggregate of firms defined as "large" (the last asset size bin of the QFR). The left panel of figure 3 looks at total (current and non-current) financial liabilities. Outstanding bank loans to large firms fall, as did those to small firms. However, the surprising fact to emerge from this figure is that on average over the last few recessions, non-bank liabilities rose. Thus, for large firms, total outstanding banks debt falls, while non-bank debt increases (fact 3). Note that, since the increase in non-bank debt is larger than that of bank debt, total financial liabilities are in fact slightly increasing. The increase in total assets may be the artifact of the accumulation of firms in the larger size bins over time; in fact, as reported in appendix D.2, total liabilities for large firms do not increase for alternative definitions of the large firm category, but the debt substitution pattern remains. Note, importantly, that this pattern is driven by new and/or long-dated liabilities. Indeed, no increase in non-bank current liabilities (those maturing in one year or less) is visible, as reported in the right panel of figure 3.

Since this sample contains only three recessions, one of which, 2007-2009, was substantially larger in magnitude than the others, it is natural to ask whether the facts reported here hold for all three recessions, or whether it reflects mostly the 2007-2009 recession. This question is addressed in appendix D.3, where I report figures similar to 2-3, excluding the last recession. While the patterns of facts 2 and 3 still hold excluding the last recession, there are two differences: first, the overall reduction in bank lending is smaller over these recessions, both for small and large firms; second, substitution towards non-bank credit for large firms is more muted than when including 2012. This brings a caveat to fact 3: it is during the 2007-2009 recession that the changes in the debt structure of large firms were most different from changes in the debt structure of small firms.

Summarizing, in the cross-section, the debt structure of firms is characterized by a negative relationship between firm size and the composition of debt structure (fact 1). During the past three recessions, the debt structure of US firms has shown marked variation. Bank lending to small firms falls, while non-bank credit is unchanged (fact 2). Large firms display a different behavior: bank lending falls, but non-bank credit increases, especially during the last recession (fact 3). In what follows, I develop a model to address the issue of the joint determination of a firm’s overall investment and its debt structure. Sections 3 and 4 describe this model and discuss its ability to replicate fact 1. Section 5 explores the changes in debt structure across firm sizes associated with variation in financial intermediation costs, on the one hand, and variation in firms' production possibilities, on the other. There, I contrast the changes in debt structure predicted by the model to those documented in facts 2 and 3.
3 A model of bank and market financing

This section describes a simple static setup in which an entrepreneur with own equity \( e \) finances a project by borrowing from two lenders: a bank, and the market. The only friction of the model is that there is limited liability; the entrepreneur can choose to default on her debt obligations. However, doing may involve output losses, if the project is liquidated. This motivates the key distinction between bank and market lenders: bank loans can be restructured in times of financial distress, in order to avoid inefficient liquidation losses. Market debt, on the other hand, cannot. I come back to the interpretation of the model’s assumptions below after describing the key elements of the model.

3.1 Production structure and timing

The entrepreneur owns the firm and operates a technology which takes physical assets \( k \) as an input, and produces output:

\[
y = \phi k^\zeta
\]

Here, \( \phi \), the productivity of the technology employed by the entrepreneur, is a random variable, the realization of which is unknown to both the entrepreneur and the lenders at the time when investment in physical assets is carried out. In what follows, I denote the CDF of \( \phi \) by \( F(\cdot) \). \( \zeta \) governs the degree of returns to scale of the technology operated by the entrepreneur. Assets depreciate at rate \( \delta \in [0, 1] \). After production has been carried out and depreciation has taken place, the entrepreneur has resources:

\[
\pi(\phi) = \phi k^\zeta + (1 - \delta)k
\] (1)

I make the following assumptions about the production structure:

**Assumption 1 (Production structure)** *The firm’s production technology has the following characteristics:*

- *Production has decreasing returns to scale: \( \zeta < 1 \);*

- *The productivity shock \( \phi \) is a positive, continuous random variable with density \( f \). Moreover, \( f(0) = 0 \) and the hazard rate of \( \phi \) is strictly increasing.*

The first part of the assumption is standard in models of firm investment, and guarantees that firms have a finite optimal scale of operation. The second part of the assumptions concerns the distribution of productivity shocks. The fact that the shock \( \phi \) is a positive random variable implies that there is a positive lower bound
on resources, \((1 - \delta)k\), so that riskless lending may occur, to the extent that \(\delta < 1\). The increasing hazard rate is a technical assumption which guarantees the unicity of lending contracts\(^6\).

The entrepreneur finances investment in physical assets, \(k\), from three sources: own resources (or equity), \(e\), with which it is initially endowed; bank debt, \(b\), and market debt \(m\). The issuance of outside equity, in particular, is not allowed. The resulting balance sheet constraint is thus simply:

\[
k = e + b + m.
\]

The timing of the model is summarized in figure 2. The model has two periods. At \(t = 0\), the entrepreneur, the bank lender and the market lender agree about a debt structure \((b, m)\), and promised repayments, \(R_b\) to the bank, and \(R_m\) for the market lender. Investment in \(k\) then takes place, and the productivity of the firm, \(\phi\), is realized. At time \(t = 1\), debt payments are settled; that is, the firm can choose to make good on promised repayments, restructure its debt, or proceed to bankruptcy.

Finally, in this section, I only assume that all agents are utility maximizers and have preferences that are weakly increasing in their monetary payoffs. In the next section, I study the case of a risk-neutral entrepreneur; however, all the results presented in this section on the settlement of debt are entirely independent of the assumption of risk-neutrality, and hold for general preference specifications so long as preferences are increasing in payoffs. In particular, the set of feasible debt structures characterized in this section is identical across preference specifications.

### 3.2 Debt settlement

The debt settlement stage takes place once the productivity of the firm has been observed by all parties. I model the debt settlement process as a two-stage game. In the first stage, the entrepreneur can choose between three alternatives, summarized in figure 3: repay in full both its bank and market creditors; make a restructuring offer to the bank; or file for bankruptcy. If the entrepreneur chooses to repay in full both its

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\(^6\)See appendix A for details.
creditors, her payoff is:

$$\pi_P(\phi) = \pi(\phi) - R_m - R_b$$  \hspace{1cm} (2)

while the payoff to the bank and market lender are, respectively, $R_b$ and $R_m$, as initially promised. I next turn to describing each party’s payoff under the two other alternatives, bankruptcy and restructuring.

### 3.2.1 Bankruptcy

If the entrepreneur chooses to file for bankruptcy, the project is terminated and liquidated, and the proceeds from liquidation are distributed to creditors. Once bankruptcy is declared, the entrepreneur has no claim to liquidation proceeds\(^7\); that is, her liquidation payments are assumed to be 0, so that the monetary payoff to the entrepreneur in bankruptcy is:

$$\pi_B(\phi) = 0.$$  \hspace{1cm} (3)

I make the following assumption about the impact of liquidation on output:

**Assumption 2 (Liquidation losses)** Under bankruptcy, the proceeds $\tilde{\pi}(\phi)$ to be distributed to creditors and the entrepreneur are a fraction $\chi$ of the project’s value:

$$\tilde{\pi}(\phi) = \chi \pi(\phi), \quad 0 \leq \chi < 1.$$

\(^7\)This is without loss of generality. Allowing for the entrepreneur to be a residual claimant in bankruptcy would not alter the results, since in the debt settlement stage, bankruptcy would never be declared in states in which the entrepreneur has sufficient resources to repay both lenders. I omit this possibility to alleviate notation.
Liquidation leads to inefficient losses of output; that is, the liquidation value of the project is strictly smaller than the value of the project under restructuring or repayment. Specifically, liquidation losses are equal to \((1 - \chi)\pi(\phi)\). Consistent with the evidence in Bris et al. (2006) discussed below, this assumption captures the fact that bankruptcy proceedings are typically costly, both administratively and because they halt production activities. Moreover, asset values of firms after cash auction proceedings are typically only a fraction of pre-bankruptcy values. This is the key friction in the static model with risk-neutrality: absent bankruptcy losses, lending would be unconstrained, as I will discuss below.

I assume that bankruptcy proceeds are distributed among creditors according to an agreed-upon priority structure, in line with the Absolute Priority Rule (APR) that governs chapter 7 proceedings in the US\(^8\). In this section, I assume that bank debt is senior to market debt\(^9\). Under this priority structure, the payoff to bank lenders and market lenders, are:

\[
\hat{R}^b(\phi) = \min \left( R_b, \chi \pi(\phi) \right),
\]

\[
\hat{R}^m(\phi) = \max \left( \chi \pi(\phi) - R_b, 0 \right).
\]

The first line states that the bank’s payoff in bankruptcy is at most equal to its promised repayment \(R_b\). The second payoff states that market lenders are residual claimants. Note that this formulation does not, a priori, preclude cases in which \(\hat{R}_m(\phi) \geq R_m\), that is, market lenders receiving a residual payment larger than their initial claim. I will however show that this never occurs in the equilibrium of the settlement game.

### 3.2.2 Restructuring

Instead of filing for bankruptcy, I assume that the entrepreneur can enter a private workout process with her creditors. Because going bankrupt implies losses of output, it may sometimes be in the interest of creditors and the entrepreneur to arrive at a compromise. I make the following restriction to the workout process.

**Assumption 3 (Bank debt flexibility)** The entrepreneur may only restructure debt payments owed to the bank, \(R_b\); payments to the market lender, \(R_m\), cannot be renegotiated.

This is the key distinction between bank and market lending in the model; I delay its discussion to the next paragraph, and first describe its implications for the debt settlement process. I assume that the private workout operates as follows: the entrepreneur makes a one-time offer to the bank which takes the form of

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\(^8\)see White (1989) for institutional details on the APR.

\(^9\)I come back to the issue of the optimality of bank seniority in conclusion. It is likely that, in this model, bank seniority is the optimal priority structure from the standpoint of the firm; numerically, it is the case for all the versions of the model which I have explored.
a reduction in promised repayments $l_b(\phi) \leq R_b$. In case the offer is accepted, the bank obtains a payoff of $l_b(\phi)$, and the entrepreneur obtains a payoff of:

$$\pi_R(\phi) = \pi(\phi) - R_m - l_b(\phi).$$

(6)

If, on the other hand, the bank refuses the entrepreneur’s offer, the private workout fails, and the project is liquidated. In this case, the payoff to the bank is given by equation (4). The participation constraint of the bank is thus:

$$l_b \geq \tilde{R}^K_b(\phi).$$

The entrepreneur will choose her restructuring offer to maximize her net payoff under restructuring, subject to the participation constraint of the bank. Formally:

$$\pi_R(\phi) = \max_{l_b} \pi(\phi) - R_m - l_b \quad \text{s.t.} \quad l_b \geq \tilde{R}^K_b(\phi)$$

$$= \begin{cases} 
\pi(\phi) - R_b - R_m & \text{if } R_b \leq \chi \pi(\phi) \\
(1 - \chi)\pi(\phi) - R_m & \text{if } R_b > \chi \pi(\phi)
\end{cases}$$

(7)

Intuitively, this result indicates that the entrepreneur will choose to make a restructuring offer only when its cash on hand is small enough, relative to promised repayments to the bank. Note that the larger the value of $\chi$, the higher the restructuring threshold; that is, bankruptcy losses effectively allows the entrepreneur to extract concessions from the bank.

### 3.2.3 Debt settlement equilibria

Given the realization of $\phi$, the entrepreneur chooses whether to repay, restructure or file for bankruptcy, by comparing her payoffs $\pi_P(\phi)$, $\pi_R(\phi)$ and $\pi_R(\phi)$ under each option. The following proposition describes the resulting perfect equilibria in pure strategies of the debt settlement game described in figure 3. There is a unique equilibrium for each realization of $\phi$; however, the set of possible equilibria, parametrized by $\phi$, depends on the terms of the debt contracts.

**Proposition 1 (Debt settlement equilibria)**

*If $\frac{R_m}{1 - \chi} < \frac{R_b}{\chi}$ (R-contracts), there are some realizations of $\phi$ for which the entrepreneur chooses to use her restructuring option. Specifically, the entrepreneur chooses to repay her creditors when $\pi(\phi) \geq \frac{R_b}{\chi}$; to restructure debt when $\frac{R_m}{1 - \chi} \leq \pi(\phi) < \frac{R_b}{\chi}$; and to file for bankruptcy when $\pi(\phi) < \frac{R_m}{1 - \chi}$. 

If $\frac{R_m}{1 - \chi} \geq \frac{R_b}{\chi}$ (K-contracts), there are no realizations of $\phi$ such that the entrepreneur attempts to*
restructure debt with the bank. Instead, she chooses to repay her creditors when \( \pi(\phi) \geq R_m + R_b \), and otherwise, she files for bankruptcy.

Moreover, in bankruptcy or restructuring, market and bank lenders never obtain more than their promised repayments: \( \tilde{R}_m(\phi) \leq R_m \) and \( \tilde{R}_b(\phi) \leq R_b \), regardless of whether the debt contract is an R-contract or a K-contract.

The proof for this and all following propositions are reported in appendix. Figure 5 illustrates sets of equilibria for each type of contract. In the case of a K-contract (\( \frac{R_b}{\chi} < \frac{R_m}{1-\chi} \)), no restructuring ever occurs, and bankruptcy losses cannot be avoided when the cash on hand of the firm, \( \pi(\phi) \), falls below the threshold at which the firm prefers declaring bankruptcy over repayment, \( R_m + R_b \). This occurs because the stake of the flexible creditors, \( R_b \), is too small for restructuring to bring about gains sufficient for the entrepreneur to avoid default on market debt.

On the other hand, in the case of an R-contract, (\( \frac{R_b}{\chi} \geq \frac{R_m}{1-\chi} \)), the flexibility of bank debt sometimes allows the entrepreneur to make good on its payments on market debt (this corresponds to restructuring region below \( R_m + R_b \) in figure 5). Some R-equilibria will see the entrepreneur exert a degree of bargaining power over the bank: indeed, the bank will be forced to accept a settlement, even though the firm has enough cash on hand to make good on both its promises (this corresponds to the restructuring region above \( R_m + R_b \) in figure 5). This region corresponds to strategic restructurings on the part of the entrepreneur, who takes advantage of the fact that the bank can never extract from her more than its reservation value under restructuring, \( \chi \pi(\phi) \), in any private workout.

### 3.3 Discussion

The model’s fundamental distinction between bank lending and market lending is that bank lenders are capable of flexibility in times when the firm is not able (or willing) to repay her debt. A natural question
is then whether this assumption is borne out in the data. Gilson, John, and Lang (1990) examine a sample of 169 financially distressed firms. They show that about half (80 or 47% of the total) firms successfully restructure outside of formal judicial proceedings, while the other half (89, or 53% of the total) file for bankruptcy. They show that successful restructurings out of court involve, in 90% of cases, a firm that has outstanding bank loans, while only 37.5% of successful restructurings involves firms with outstanding public debt. Moreover, they show that the existence of bank loans is the single most important determinant of whether firms successfully restructure out of court, more so than other firm characteristics such as firm size, age or leverage. A theoretical justification of the observation that bank debt is easier to restructure is developed by Gertner and Scharfstein (1991), who argue that when coordination problems among dispersed holders of public debt lead to a failure to efficiently restructure that debt. Bolton and Scharfstein (1996) also study how free-riding problems can lead to inefficiencies during debt restructuring involving a large number of creditors. The assumption of bank debt flexibility can thus be thought of as a reduced-form manner of capturing the coordination and free-riding problems discussed elsewhere in the literature.

Additionally, the model makes the assumption that the formal liquidation of a project – which occurs only if restructuring has failed – leads to inefficient losses. In the model, this occurs when \( \chi < 1 \); when \( \chi = 1 \), liquidation leads to a transfer of ownership but not losses in values. Bris, Welch, and Zhu (2006) study a sample of 61 chapter 7 and 225 chapter 11 filings between 1995 and 2001. In particular, they report measures of the ratio of pre to post-bankruptcy asset values (excluding legal fees). For chapter 11 proceedings, which provides a legal framework for debt restructuring but does not involve liquidation, the mean of the ratio of pre to post-bankruptcy asset values in their sample is 106.5%; that is, on average, this ratio increases after chapter 11 proceedings. For chapter 7 proceedings, this is not the case: asset values decline as a result after the bankruptcy. Here, Bris, Welch, and Zhu (2006) offer two measures of post-bankruptcy asset values. The first measure is liquidation value of the firm after collateralized lenders have seized the assets to which they had a lien outside of bankruptcy proceedings. With this measure, asset values post-bankruptcy are 17.2% of pre-bankruptcy asset values, on average. The second measure tries attempts to include the value of collateralized assets; post-bankruptcy values are then estimated to represent 80.0% of pre-bankruptcy values.

While the model’s debt settlement stage does not clearly distinguish between private and formal (chapter 11) workouts, it assumes that they are costless, whereas liquidation (chapter 7) is assumed to be costly. In this respect, the model’s assumptions are thus consistent with the results of Bris, Welch, and Zhu (2006). In general, assuming that debt renegotiation (private or under a chapter 11 filing) is costly should not alter the key qualitative results of the model. So long as renegotiation costs are strictly smaller than those associated to liquidation, renegotiation will be beneficial to the firm so long as promised repayment to bank lenders are sufficiently large, as in figure 3.
3.4 Debt pricing and the lending menu

I now turn to describing the pricing of bank and market debt contracts at \( t = 0 \), before the realization of the shock \( \phi \) and the resources \( \pi(\phi) \).

3.4.1 Expected lending returns

Appendix A details the expressions of lenders’ payoff functions \( \tilde{R}_b(\phi) \) and \( \tilde{R}_m(\phi) \) for the debt settlement equilibrium associated to each realization of \( \phi \). Let:

\[
E_i(e, b, m, R_b, R_m) = \int_{0}^{+\infty} \tilde{R}_i(\phi) dF(\phi), \quad i = b, m
\]

denote the gross expected returns for each lender (market or bank) at time \( t = 0 \). The exact expression of \( E_i \) depends on whether the debt contract is an R- or a K- contract. For example, assume that the contract is a K-contract, that is, \( \frac{R_b}{\chi} < \frac{R_m}{1-\chi} \). In this case, the bank will face three possible outcomes, conditional on the realization of \( \pi(\phi) \):

- If \( \phi \geq \phi_\equiv \frac{R_b+R_m-(1-\delta)(e+b+m)}{(e+b+m)} \), that is, above the bankruptcy threshold, the bank will be repayed in full;
- If \( \phi > \phi > \phi_\equiv \frac{R_b-(1-\delta)(e+b+m)}{\chi(e+b+m)} \), the entrepreneur will file for bankruptcy; but the bank, because of its seniority in the priority structure of the bank, will still be repayed in full (market lenders will only receive partial repayments);
- If \( \phi > \phi \), the bank is only partially repayed, and the market lenders receive no payments.

Thus, the expected return function of the bank lender will be given by:

\[
E_b(e, b, m, R_b, R_m) = \left( 1 - F(\phi_\) \right) R_b + R_b \int_{\phi_}^{\phi} dF(\phi) + \chi \int_{0}^{\phi} \pi(\phi) dF(\phi)
\]

Expressions of the expected returns functions of both types of lenders for each type of contract are also reported in appendix A. Importantly, expected lending returns for the bank are independent of \( R_m \), and thus independent of whether the contract is an R- or a K- contract. This result follows from the assumption that the bank may only accept or reject the offer of the firm. Because of this, the bank always accepts its reservation value, the bankruptcy payoff, as restructuring settlement. When the bank is senior in the priority structure, the bankruptcy payoff is therefore independent of \( R_m \), so that the expected returns from bank lending are independent of the value of \( R_m \), and therefore of the condition \( \frac{R_m}{1-\chi} \geq \frac{R_b}{\chi} \).
3.4.2 The lending menu

I make the assumption that both kinds of financial intermediaries are perfectly competitive, so that debt is priced by equating the gross expected return from lending to the lenders’ gross cost of funds. A contract \((R_b, R_m)\) corresponding to debt structure \((b, m)\) will be available to a firm with equity \(e\) if it satisfies the zero profit condition of both lenders.

**Definition 1 (Lending contracts)** The set of lending contracts for debt structure \((b, m)\) at equity \(e\) is the set:

\[
\mathcal{L}(b, m, e) \equiv \left\{ (R_b, R_m) \in \mathbb{R}_+^2 \middle| \begin{array}{l}
E_b(e, b, m, R_b, R_m) = (1 + r_b)b \\
E_m(e, b, m, R_b, R_m) = (1 + r_m)m
\end{array} \right\}
\]

Since \(\mathcal{L}(b, m, e)\) is a subset of \(\mathbb{R}^2\), it is a partially ordered set when endowed by the product order \(\leq_x\). \((\mathcal{L}(b, m, e), \leq_x)\) therefore has at most one least element\(^{11}\). This justifies the following definition:

**Definition 2 (The dominating contract)** The dominating contract for debt structure \((b, m)\) at equity \(e\) is the least element of the partially ordered set \((\mathcal{L}(b, m, e), \leq_x)\) (if it exists).

Finally, the lending menu at equity \(e\) is the set of all debt structures \((b, m)\) such that there exists a dominating contract at \((b, m)\) for equity \(e\):

**Definition 3 (The lending menu)** The lending menu at \(e\) is the set:

\[
\mathcal{S}(e) \equiv \left\{ (b, m) \in \mathbb{R}_+^2 \middle| \begin{array}{l}
\mathcal{L}(b, m, e) \neq \emptyset \\
(\mathcal{L}(b, m, e), \leq_x) \text{ has a least element}
\end{array} \right\}
\]

Several elements of these definitions are worth emphasizing. First, the lending menu \(\mathcal{S}(e)\) of definition 3 is the set of feasible debt structures for the entrepreneur. There are two requirements for a debt structure to be part of the lending menu: first, there must exist lending contracts for that debt structure; second, one of them must be a dominating contract, in the sense of definition 2. Intuitively, the dominating contract has the property that it is (weakly) cheaper, in both dimensions \((R_b, R_m)\), than any other contract that satisfies the lenders’ zero-profit conditions.

Note that, a priori, there is no reason to think that \((\mathcal{L}(b, m, e), \leq_x)\) generically has a least element. It could well be that, for a certain equity level \(e\) and a certain debt structure \((b, m)\), the set of lending contracts contains two elements \((R_b, R_m)\) and \((R'_b, R'_m)\) such that \(R_b > R'_b\) but \(R_m < R'_m\), which cannot be ordered by \(\leq_x\). In that case, \((\mathcal{L}(b, m, e), \leq_x)\) would have no least element. Definition 3 would then exclude \((b, m)\)

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\(^{10}\)This is the partial order defined by \((x_1, y_1) \leq_x (x_2, y_2) \iff x_1 \leq y_1\) and \(x_2 \leq y_2\).

\(^{11}\)This is \((R_b, R_m) \in \mathcal{L}(b, m, e)\) such that \(\forall (\tilde{R}_b, \tilde{R}_m) \in \mathcal{L}(b, m, e), (R_b, R_m) \leq_x (\tilde{R}_b, \tilde{R}_m)\).
from the firms’ feasible set, the lending menu $S(e)$, despite the fact that $L(b, m, e)$ would be non-empty. This would seem to arbitrarily restrict the set of feasible contracts. Fortunately, this is never the case: $(L(b, m, e), \leq_s)$ always has a least element when $L(b, m, e) \neq \emptyset$, so that the lending menu never contains debt structures associated to "ambiguous" contracts\textsuperscript{12}. I discuss this further in the next paragraph.

Additionally, note that the definition of the set of lending contracts allows for the possibility that lenders have different cost of funds, $r_b \neq r_m$. This has no bearing on the results of this section on the structure of the lending menu, but it matters for the optimal debt structure decision of the firm. I therefore discuss the importance of this assumption more at length in the following section.

Finally, while the definition of competitive contracts only require promised repayments to be positive, they in fact also satisfy $R_m \geq (1 + r_m)m$ and $R_b \geq (1 + r_b)b$; that is, lenders never ask for promised repayments which imply a yield below their marginal cost of funds. Intuitively, since the expected value of total (bankruptcy and non-bankruptcy) claims has to equal the cost of funds for each lender, it cannot be the case that both are strictly smaller or strictly larger that that cost of funds. As bankruptcy claims are strictly smaller than non-bankruptcy claims, it must therefore be the case that non-bankruptcy claims exceed the costs of funds.

### 3.4.3 Some general properties of the lending menu

The following proposition describes in more detail the structure of the lending menu, emphasizing the fact that it can be explicitly partitioned between contracts leading to $B$-equilibria and contracts leading to $R$-equilibria.

**Proposition 2 (A partition of the lending menu in the general case)** The lending menu $S(e)$ can be partitioned as:

$$S(e) = S_R(e) \cup S_K(e), \quad S_R(e) \cap S_K(e) = \emptyset,$$

where:

$$\tilde{S}_K(e) \equiv \left\{ (b, m) \in \mathbb{R}_+^2 \mid \begin{array}{l}
0 \leq \frac{(1+r_b)b + (1+r_m)m}{(e+b+m)x} \leq \hat{M}(e + b + m) + (1 - \delta)(e + b + m)^{1-\zeta} \\
R_l(b, m, e) \geq \frac{R_b(b,m,e)}{\chi} 
\end{array} \quad (c - joint) \right\},$$

$$\tilde{S}_R(e) \equiv \left\{ (b, m) \in \mathbb{R}_+^2 \mid \begin{array}{l}
0 \leq \frac{(1+r_b)b}{\chi(e+b+m)} \leq \hat{E}(\phi) + (1 - \delta)(e + b + m)^{1-\zeta} \\
0 \leq \frac{R_m(b,m,e)}{1-x} \leq \frac{R_b(b,m,e)}{1-x} 
\end{array} \quad (c - bank) \right\}.$$

\textsuperscript{12}This result is proved in appendix A, as part of the proof of proposition 2.
and:

\[ S_R(e) = \hat{S}_R(e), \quad S_K(e) = \hat{S}_K(e) \setminus \left( \hat{S}_R(e) \cap \hat{S}_K(e) \right). \]

The sets \( S_R(e) \) and \( S_K(e) \) are non-empty, compact and connected subsets of \( \mathbb{R}^2_+ \). Moreover, given an equity level \( e \):

- The dominating contract associated to \( (b, m) \) is an R-contract, if and only if, \( (b, m) \in S_R(e) \);
- The dominating contract associated to \( (b, m) \) is a K-contract, if and only if, \( (b, m) \in S_K(e) \);

Expressions for the functions \( \hat{M}(e + b + m) \), \( \hat{I}(e + b + m) \), \( R_l(b, m, e) \), \( R_m,b_l(b, m, e) \) and \( R_0(b, m, e) \) are given in appendix A.

There are two important elements in this proposition. First, the lending menu is the union of two subsets, \( \hat{S}_K(e) \) and \( \hat{S}_R(e) \), which correspond, respectively, with the set of debt structures for which there exists a K-contract (the set \( \hat{S}_K(e) \)) and the set of debt structures consistent for which there exists an R-contract (the set \( \hat{S}_R(e) \)). The intersection of these two sets is however not empty; that is, there are debt structures for which there exists both a K- and an R- contract. However, the proposition establishes that for such debt structures, the R-contract is always the dominating contract. The intuition for this result is straightforward. Imagine that there are two contracts, \( (R_b, R_m) \) and \( (\tilde{R}_b, \tilde{R}_m) \), associated to a debt structure \( (b, m) \): an R-contract, \( \frac{R_b}{\chi} \geq \frac{R_m}{1-\chi} \), and a K-contract, \( \frac{\tilde{R}_b}{\chi} < \frac{\tilde{R}_m}{1-\chi} \). As discussed previously, because of bank debt seniority, banks’ gross expected returns from lending are the same for both contracts, so that so \( R_b = \tilde{R}_b \). Therefore, \( \frac{\tilde{R}_m}{1-\chi} < \frac{R_m}{\chi} = \frac{R_b}{\chi} \geq \frac{R_m}{1-\chi} \), so that the two contracts can be ordered, and the R-contract dominates.

Second, each of the two subsets \( \hat{S}_K(e) \) and \( \hat{S}_R(e) \) can be described as the intersection of the sets defined three inequality constraints. The inequalities \( (c – \text{bank}) \), \( (c – \text{market}) \) and \( (c – \text{joint}) \) correspond, respectively, to borrowing constraints with respect to market, bank lending and total lending. The borrowing constraint with respect to bank lending, for example, can be rewritten as:

\[ 0 \leq (1 + r_b)b \leq \chi \left( \mathbb{E}(\phi)(e + b + m)^\delta + (1 - \delta)(e + b + m) \right) = \chi \int_0^{+\infty} \pi(\phi)dF(\phi). \]

The first inequality states that firms are assumed not to be allowed to lend. The second inequality states that \( (1 + r_b)b \) cannot exceed the maximum expected return on the which the bank can achieve by lending to the firm. This maximum is attained when the threshold for \( \phi \) below which the firm renegotiates, which is implicitly defined by \( \frac{R_b}{\chi} = \pi(\phi) \) as per proposition 1, is \( +\infty \). This implies that when the bank borrowing constraint is binding (that is, when the debt structure is such that \( (c – \text{bank}) \) holds with equality), the entrepreneur always renegotiates her loan with the bank, so that the banks’ effective repayment is \( \chi \pi(\phi) \).
∀\(\phi\). The banks’ expected repayment is then a fraction \(\chi\) of total expected output. Thus, when the firm is at her bank borrowing constraint, the bank effectively holds a claim to a constant fraction of the project’s value.

This is in contrast to the market borrowing constraint (\(c - \text{market}\)). This borrowing constraint also states that \((1 + r_m)m\) cannot exceed the maximum expected repayment which market lenders can achieve. Contrary to the banks’ case, this expected repayment is not monotonic in the threshold for \(\phi\) below which the entrepreneur goes bankrupt, which is implicitly defined by \(\frac{R_m}{1 - \chi} = \pi(\phi)\). This is a direct consequence of the fact that there are losses associated with going bankrupt, that is, \(\chi < 1\). These losses (along with the technical part of assumption 1) imply that expected returns of market lenders are increasing in \(R_m\) when \(R_m\) is small (so that bankruptcy probability and bankruptcy losses are small), but decreasing in \(R_m\) when \(R_m\) is large (so that bankruptcy probabilities and bankruptcy losses are large). The value \(\hat{I}(e + b + m) < \mathbb{E}(\phi)\) correspond to the bankruptcy threshold which maximizes expected returns for market lenders, which trade off bankruptcy losses with higher promised repayments. If instead there are no bankruptcy losses, \(\chi = 1\), expected returns would never decrease in promised repayments \(R_m\), and the market borrowing constraint would take the same form as the banks’; that is, we would have \(\hat{I}(e + b + m) = \mathbb{E}(\phi)\). Likewise, in the case of a K-contract, renegotiation is never used for any realization \(\phi\); bankruptcy losses are therefore unavoidable. For the same reasons as in the case of the R-contract, this implies that the total surplus from lending has an interior maximum in the bankruptcy threshold, which corresponds to the value \(\hat{M}(e + b + m) < \mathbb{E}(\phi)\). Thus, the ability to renegotiate debt is a crucial determinant of borrowing constraints in this model.

Proposition 2 does not provide a full characterization of the lending menu. However, some general results can be established about the type of debt structures which are to be found in each of the subsets \(S_K(e)\) and \(S_R(e)\). This is the object of the following corollary to proposition 2.

**Corollary 3 (Debt structure and the lending menu)** For a given level of equity \(e\) and a debt structure \((b, m) \in S(e)\), let \(s = \frac{b}{b + m}\) denote bank debt as a fraction of total debt. Define:

\[
\mathfrak{s}_R = \frac{1}{1 + \frac{1 - \chi}{\chi (1 + r_m)}},
\]

and let \(\mathfrak{s}_K(e)\), \(1 \geq \mathfrak{s}_K(e) \geq \mathfrak{s}_R\), be defined as in appendix A. Then:

- if \(1 \geq s \geq \mathfrak{s}_K(e)\), then \((b, m) \in S_R(e)\);
- if \(\mathfrak{s}_R > s \geq 0\), then \((b, m) \in S_K(e)\).

---

13Appendix A contains a detailed proof of this statement.
This corollary indicates that the set of debt structures $S_R(e)$ contains no debt structures $(b, m)$ with "too much" market debt: there can indeed be no debt structures such that $\frac{b}{b+m} < \overline{s}_R$ in that set. In particular, when $s = 0$ (a pure market contract), the associated contract must be a $K$-contract. This is intuitive: in that case, there can be no renegotiation from the part of the entrepreneur, since no bank has taken part in lending. The surprising result, however, is that the debt structure of firms must contains a sufficient fraction of bank debt for renegotiation to be a possibility in the equilibrium of the debt settlement game. Otherwise, the gains associated with the renegotiation of bank liabilities are never sufficient to repay in full the market lenders in case of bankruptcy, so that bankruptcy can never be avoided. Accordingly, the threshold $s_R$ below which the debt structure ceases to allow for renegotiation is increasing in $\chi$: the larger the bankruptcy losses (the smaller $\chi$), the larger the renegotiation gains, and the more market debt the firm can take on as a fraction of total debt. On the contrary, in the limit where $\chi = 1$, there are no renegotiation gains, and $s_R = 1$, so that there are no debt structures associated with $R$-contracts.

Likewise, the corollary also indicates that the set $S_K(e)$ contains no debt structures particularly tilted towards bank debt; a pure bank contract, $s = 1$, must be associated to an $R$-contract, since in the absence of market debt, it is always beneficial for the entrepreneur to renegotiate down debt payments (provided her productivity $\phi$ is sufficiently low). Thus, the model indicates that debt structures tilted towards bank debt tend to be associated with contracts leading to debt renegotiations, whereas debt structures tilted towards maker debt tend to be associated with contracts where debt renegotiations are not an equilibrium outcome.

### 3.4.4 An analytical characterization of the lending menu when $\delta = 1$

I next turn to a particular case in which the lending menu can be characterized analytically.

**Proposition 4** Let $(d, s) \in \mathbb{R}^+ \times [0, 1]$ denote the debt structure, with $d = b + m$ and $s = \frac{b}{b+m}$. The sets $S_R(e)$ and $S_K(e)$ can be parametrized as:

\[
S_R(e) = \left\{ (d, s) \in \mathbb{R}_+ \times [\overline{s}_R, 1] \middle| 0 \leq d \leq \overline{d}_R(s, e) \right\},
\]

\[
S_K(e) = \left\{ (d, s) \in \mathbb{R}_+ \times [0, \overline{s}_K] \middle| d_R(s, e) \leq d \leq \overline{d}_K(s, e) \right\}.
\]

Here, $\overline{s}_R$ is defined as in corollary 3, and:

\[
\overline{s}_B = \frac{1}{1 + \left( \frac{1}{\chi} - 1 \right) \frac{1+r_b}{1+r_m}}
\]

Expression for the borrowing limits $\overline{d}_R(s, e)$, $\overline{d}_K(s, e)$ and the constant $\chi$ are given in
Moreover, \( \frac{\partial s_R}{\partial s_R} (s, e) < 0 \), while \( \frac{\partial d_K}{\partial s} (s, e) \geq 0 \), if and only if, \( s_R \leq s \leq s_{R,M} < 1 \), where:

\[
s_{R,M} = \frac{1}{1 + \frac{1 - \chi}{\chi} \frac{1 + r_b}{1 + r_m} \mathbb{E}(\sigma)}.
\]

Figure 5 depicts the lending menu \( S(e) \) when \( \delta = 1 \), using the results of proposition 4\(^{14}\). The lending menu is plotted in \((b, m)\) rather than \((d, s)\) space, but there is a simple correspondence between the two: each value of \( s \) corresponds to a straight line passing through 0 and with slope \( \frac{1 - s}{s} \), and along each of these lines, total debt \( d \) increases.

Note first that the bank share of external liabilities \( s = \frac{b}{b+m} \) spans the lending menu, as it varies from \( s = 0 \) (which corresponds to the the vertical axis) to \( s = 1 \) (which corresponds to the horizontal axis). As emphasized in the general case, the lending menu contains only elements in \( S_K(e) \) if \( s \leq s_K \), that is, to the left of the solid black line with slope \( \frac{1 - s_K}{s_K} \); and only elements in \( S_R(e) \) is \( s > s_K(e) \), that is, to the right of the solid black line with slope \( \frac{1 - s_K}{s_K} \).

\(^{14}\)Other model parameters, and in particular the shock distribution, are identical to those of the baseline calibration of the model, discussed in section 5. In this example, \( e = 10 \); model parameters are choosen in such a way that a firm with equity \( e = 100 \) is indifferent between borrowing from market lenders and having no debt.
The set \( S_K(e) \) corresponds to the light gray area of the graph. This set comprises debt structures such that the bank share of external liabilities \( s = \frac{b}{b + m} \) ranges from 0 to \( \bar{s}_K \) (corresponding to the solid black line with slope \( \frac{1 - \bar{s}_K}{\bar{s}_K} \)). For this range of bank shares, the upper bound on borrowing using a K-contract, \( \bar{d}_R(s, e) \), is the dotted line at the boundary of \( S_K(e) \). This frontier corresponds to the pairs \((b, m)\) such that condition \((c - \text{joint})\) is binding, and it is downward sloping. Intuitively, this indicates that bank and market borrowing, for this type of contract, are substitutes, in the sense that a marginal increase in bank borrowing tightens the market borrowing constraint (and conversely). This is because for K-contracts, given the seniority of bank debt, a marginally larger amount of bank liabilities makes it less likely that market liabilities will be repayed. Note that for \( s < \bar{s}_R \), the lower bound on total debt for debt structures belonging to \( S_K(e) \) is 0, while for \( \bar{s}_R \leq s \leq \bar{s}_K \), it coincides with the upper bound on total debt of the other set of debt structures, \( S_R(e) \). Appendix A shows that this boundary corresponds to the conditions \((\text{frontier} - R)\) and \((\text{frontier} - K)\), which exactly coincide in the range \( \bar{s}_R \leq s \leq \bar{s}_K \). For this range of debt structures, a particular bank share \( s \) can therefore correspond to either a K-contract (if total borrowing is small enough, i.e., below \( \bar{d}_R(s, e) \)), or to an R-contract (if total borrowing is large enough, i.e., above \( \bar{d}_R(s, e) \)).

The set \( S_R(e) \) corresponds to the dark gray area of the graph. In this region, the upper bound on total borrowing is associated to the dashed blacked line \( \bar{d}_R(s, e) \). For debt structures such that \( \bar{s}_K \leq s \leq s_{R,M} \), this dotted line corresponds to debt structures for which the constraint \((c - \text{market})\) is binding, while for debt structures such that \( s \geq s_{R,M} \), this line corresponds to the bank borrowing constraint \((c - \text{bank})\). The slope of the frontiers thus indicate that and increase in the bank share loosens the market borrowing constraint so long as the bank borrowing constraint is not binding, and thus leads to higher total borrowing limits. In that region, the lending menu thus exhibits "complementarity", contrary to the part of the lending menu associated with K-contracts.

In this section, I have proposed a model of market and bank debt pricing based on the view that bank debt is easier to renegotiate than market debt, and I have derived the set of feasible debt contracts for an entrepreneur with own equity \( e \) implied by the model. This set was derived under the assumptions that credit markets were perfectly competitive and that entrepreneurs’ utility was increasing in monetary payoffs. The key insight from the analysis of feasible debt structures is that renegotiation of debt in times of financial distress is only desirable for firms that choose to hold a sufficient amount of bank debt relative to market debt. I next turn to drawing the implications of these findings for the optimal choice of debt structure.
4 The optimal choice of debt structure

This section addresses the question of which debt structure, among those that are feasible, an entrepreneur with own equity \( e \) will choose to finance investment. Furthermore, I explore how both the optimal share of borrowing financed by banks \( (s = \frac{b}{b + m}) \) as well as the optimal total amount of borrowing \( b + m \) vary with own equity \( e \).

Throughout the section, I maintain two assumptions. First, I assume that assets fully depreciate at the end of period 1, that is, \( \delta = 1 \). Given the static nature of the model, this is a natural assumption, and it furthermore simplifies the analytical characterization of the optimal debt structure. It is however not crucial to any of the results below. The second assumption I maintain in this section is that the entrepreneur is risk-neutral. While risk-neutrality can be viewed as a benchmark case, it influences strongly the results, by linking closely profit maximization and maximization of total surplus from investment, as I analyze below.

Assumption 4 (Assumptions on the entrepreneur’s problem) The entrepreneur is risk-neutral, and her assets completely depreciate after productivity is realized and output is produced: \( \delta = 1 \).

4.1 The entrepreneur’s profit function

Under assumption 4, the optimal debt structure of an entrepreneur with own equity \( e \) solves:

\[
\hat{\pi}(e) = \max_{b, m \in S(e)} \mathbb{E} \left[ \hat{\pi}(\phi; e, b, m) \right],
\]

where \( \hat{\pi}(\phi; e, b, m) \) denotes the profits accruing to the entrepreneur, conditional on the debt structure \( (b, m) \) and therefore the associated contract \( (R_b, R_m) \), and the realization of the shock \( \phi \).

Proposition 5 For \( (b, m) \in S(e) \), the objective function of the entrepreneur is given by:

\[
\mathbb{E} \left[ \hat{\pi}(\phi; e, b, m) \right] = \mathbb{E}(\pi(\phi)) - (1 + r_b)b - (1 + r_m)m - (1 - \chi) \int_0^{\phi(e, b, m)} \pi(\phi)dF(\phi),
\]

where:

\[
\phi(e, b, m) = \begin{cases} 
R_K(b, m, e) & \text{if } (b, m) \in S_K(e) \\
R_m(b, m, e) & \text{if } (b, m) \in S_R(e)
\end{cases}
\]

This result is a consequence of risk-neutrality of the lenders and the entrepreneur: under risk-neutrality, profit maximization for the entrepreneur is equivalent to the maximization of total expected surplus, net of
the losses incurred in case liquidation is carried out. The threshold \( \phi(e, b, m) \) indeed coincides, for each type of contract (K or R), to the liquidation threshold. In particular, in the absence of bankruptcy costs (that is, when \( \chi = 1 \)), profit maximization for the firm is equivalent to total surplus maximization. In this case, it is clear that the optimal debt structure is always a corner solution, with the entrepreneur borrowing only from the lender with the smallest cost of funds, as described in the lemma below.

**Lemma 1 (The optimal debt structure in the absence of liquidation losses)**  Assume there are no liquidation losses, that is: \( \chi = 1 \). Then:

- If \( r_b < r_m \), the solution to the entrepreneur’s problem is:

  \[
  b^*(e) = k_b^* - e, \quad m^*(e) = 0, \quad \pi^*(e) = (1 + r_b) \left( \frac{1 - \zeta}{\zeta} k_b^* + e \right),
  \]

  \[
  k_b^* = \left( \frac{\mathbb{E}(\phi)}{1 + r_b} \right)^{\frac{1}{1 - \zeta}}.
  \]

- If \( r_b > r_m \), the solution to the entrepreneur’s problem is:

  \[
  b^*(e) = 0, \quad m^*(e) = k_m^* - e, \quad \pi^*(e) = (1 + r_m) \left( \frac{1 - \zeta}{\zeta} k_m^* + e \right),
  \]

  \[
  k_m^* = \left( \frac{\mathbb{E}(\phi)}{1 + r_m} \right)^{\frac{1}{1 - \zeta}}.
  \]

- If \( r_b = r_m = r \), the entrepreneur is indifferent between all debt structures \((b^*(e), m^*(e))\) such that

  \[
  b^*(e) + m^*(e) + e = k^*, \quad \text{where} \quad k^* = \left( \frac{\mathbb{E}(\phi)}{1 + r} \right)^{\frac{1}{1 - \zeta}}.
  \]

### 4.2 Lenders’ cost of funds

Lemma 1 emphasizes that the relative cost of funds of lenders is a crucial determinant of the optimal debt structure; when \( \chi = 1 \), the entrepreneur only borrows from the lender with the smallest cost of funds.

When \( \chi < 1 \), the entrepreneur’s expected profits depend not only on the relative cost of funds, but also on the magnitude of expected liquidation losses, the second term in the right hand side of equation (8). Debt structure plays a role in determining the magnitude of these losses, because it affects how often the project must be liquidated at the debt settlement stage. Generically, for a comparable total amount of lending \( b + m \), debt structures \((b, m) \in \mathcal{S}_R(e)\) will be liquidated less often. Indeed, the very point of an R-contract is that, by restructuring bank debt, it offers the entrepreneur a means of avoiding liquidation even when her resources fall below ”natural” liquidation threshold, that is, when productivity is such that \( \pi(\phi) < R_m + R_b \).
Thus, by choosing an R-contract, the entrepreneur will reduce expected liquidation losses. At the same time, corollary 3 indicates that debt structures \((b, m) \in S_R(e)\) typically feature more bank debt; that is, \(s = \frac{b}{b+m}\) is closer to 1. By borrowing more from bank lenders, the entrepreneur thus reduces her expected liquidation losses.

Thus, intuitively, the optimal debt structure should limit expected liquidation losses, while at the same time it should feature as much debt as possible from the lender that has the smallest cost of funds, as suggested by lemma 1. It is clear that these objectives need not conflict with one another. If \(r_b \leq r_m\), borrowing more from bank lenders allows the entrepreneur both to minimize her liquidation losses, and to gain from the smaller cost of funds of banks. Given a particular scale of total borrowing \(b+m\), when \(r_b \leq r_m\), the entrepreneur should always try to use as much bank finance as possible. This intuition is formalized in the following proposition.

**Proposition 6 (The optimal debt structure when \(r_b \leq r_m\))** Assume that banks have a lower marginal cost of funds than markets, that is, \(r_b \leq r_m\). Then, the optimal debt structure of an entrepreneur with equity \(e\) either features no borrowing from markets, \(m^*(e) = 0\), or, is such that the bank borrowing constraint, \((c - \text{bank})\), is binding.

Thus, when \(r_b \leq r_m\), the only reason for which an entrepreneur would want to issue liabilities to market lenders is that she has already exhausted her borrowing capacity from bank lenders. In that case, only firms with little own equity \(e\) will issue market liabilities, since for large levels of \(e\), the bank borrowing constraint is less likely to bind. In turn, this would imply that larger firms are less likely to issue market debt; in particular, the very largest firms would issue only bank debt. When \(r_b \leq r_m\), the model thus leads to a counterfactually negative relationship between own equity and size, on the one hand, and bank debt’s fraction of total external liabilities, on the other. This motivates the following assumption.

**Assumption 5 (Lending costs)** Bank lenders have a larger marginal cost of funds:

\[
r_b > r_m.
\]

A possible interpretation for this assumption is the following. Lenders both have the same marginal cost of funds, \(1 + r\), but banks incur an additional costs \(c(b)\) per dollar lent. This cost could arise for two reasons. First, there might be due diligence costs associated to obtaining information that allows the bank to restructure the entrepreneur’s liability when profits are low, as in Rajan (1992). Although this information acquisition, and the problems of information revelation associated with it, are not modelled here, this cost could be seen as a reduced-form way of modelling them. Second, the bank may face different balance sheet
restrictions than market lenders; in particular, banks may face tighter capital requirement. The function \( c(b) \) may then stand for the costs of issuing additional bank equity in order to meet those capital requirements. With these additional lending costs \( c(b) \), the bank’s costs of funds becomes \((1 + r)b + c(b)b\). In particular, if \( c(b) = c \), this formulation of the model is equivalent to the one developed above, with \( r_m = r \) and \( r_b = r + c \). Overall, assumption 5 is not only necessary for the model to deliver relevant comparative statics; it is also consistent with the view that bank loans is more costly process, per unit of debt issue, than the issuance of market debt.

4.3 Which type of debt structure does the entrepreneur choose?

With the assumption that \( r_b > r_m \), there is tension between the two objectives that an optimally chosen debt structure should pursue, that is, limiting liquidation losses while at the same time borrowing as much as possible from lenders with the smallest possible cost of funds. On the one hand, to limit liquidation losses, the entrepreneur should choose a debt structure \((b, m) \in S_R(e)\) leading to an R-contract; on the other, doing so involves borrowing a large fraction of total liabilities from banks, which is costly. There is thus a trade-off between the flexibility afforded by bank debt and the lower marginal cost of market debt, which the optimal debt structure reflects.

This trade-off does not necessarily lead to a debt structure in which the entrepreneur borrows both from market lenders and from bank lenders. The following proposition emphasizes that this depends on the extent of her own resources. When the entrepreneur has sufficient own equity \( e \), the gains associated to bank debt flexibility are always outweighed by the gains from borrowing from the cheaper source of funds, the market.

Proposition 7 (Market finance vs. mixed finance) Assume that banks have a larger marginal cost of funds than markets, that is, \( r_b > r_m \). Let \((b^*(e), m^*(e))\) denote the optimal debt structure of an entrepreneur with equity \( e \). There exists \( \bar{e} > 0 \) such that:

- if \( e > \bar{e} \), \((b^*(e), m^*(e)) \in S_R(e)\); moreover, the optimal debt structure features "pure market finance":

\[
m^*(e) > 0 \quad , \quad b^*(e) = 0;
\]

- if \( e < \bar{e} \), \((b^*(e), m^*(e)) \in S_K(e)\); moreover, the optimal debt structure features "mixed finance":

\[
m^*(e) \geq 0 \quad , \quad b^*(e) > 0.
\]

This proposition states that there exists a switching point \( \bar{e} \) in equity \( e \), which dictates the type of debt
structure that the entrepreneur will choose. For those entrepreneurs with $e < \tau$, it is optimal to choose a debt structure in $S_R(e)$, which allows for restructuring. On the other hand, for entrepreneurs with $e \geq \tau$, it is optimal to choose a debt structure in $S_K(e)$; and in fact, it is optimal to choose a debt structure with only market debt.

There are two intuitions behind this proposition. First, among debt structures $(b, m) \in S_K(e)$, the entrepreneur always prefers those with no bank debt at all. This is because, under a K-contract, restructuring never occurs; so the additional costs of borrowing from banks come with no added benefits of flexibility in restructuring. The two types of debt are equivalent in terms of their effect on liquidation losses, so that part of the trade-off is irrelevant, and the entrepreneur simply issues all her liabilities with whichever lender has the smallest marginal cost of funds. Under assumption 5, market lenders have the cheapest cost of funds, so that the optimal debt structure always has $b = 0$, that is, lies on the y-axis in the lending menu depicted in figure 5. If instead $r_b \leq r_m$, then the optimal debt structure within $S_K(e)$ would have been located at the frontier between the two sets $S_K(e)$ and $S_R(e)$, and would be locally dominated by some debt structure $(b, m) \in S_R(e)$ allowing for restructuring.

The second aspect of the proposition is the fact that entrepreneurs with little own equity $e$ prefer debt structures which allow for restructuring, that is, prefer debt structure $(b, m) \in S_R(e)$. Here, the assumption of decreasing returns to scale plays an important role. To understand this, it is useful to think of the case of no liquidation costs. In that case, decreasing returns imply that projects have an optimal size $e + b + m = k^*_m$, using the notation of lemma 2. In that case, leverage ratios $\frac{b + m}{e} = \frac{k^*_m - e}{e}$ are decreasing with equity $e$. In turn, a higher leverage ratio implies a higher probability of liquidation. This is because, for two different levels of equity $e$ and $e' > e$, the distribution of profits is identical (and given by the distribution of $\phi(k^*_m \xi)$), but promised repayment $R_b$ and $R_m$ are larger for $e$ than for $e'$, because more total borrowing is needed to with less equity, that is, $k^* - e > k^* - e'$. Thus, absent liquidation costs, the smaller an entrepreneur’s own equity, the higher her probability of being liquidated.

With liquidation costs, the optimal investment scale $k^*$ cannot necessarily be reached for any level of equity $e$, so that two entrepreneurs with different levels of equity need not choose the same total scale $k = e + b + m$; in particular, entrepreneurs with smaller amounts of equity $e$ may be limited to debt structures allowing only total a maximum total scale $\overline{K}(e) < k^*$. However, to the extent that the elasticity of total borrowing to equity is sufficiently small (in particular, strictly smaller than 1), it will still be the case that leverage ratios decrease with equity. Thus, a logic similar to the case without liquidation costs applies, and an entrepreneur with little own equity has a higher probability of being liquidated at the debt settlement stage.

Thus, for entrepreneurs with small $e$, the gains from choosing a debt structure that allows for debt
Figure 6: Cross section of the profit function for two different sizes of internal funds $e$ and total investment $k$: small $e$ and $k$ (left), large $e$ and $k$ (right).

restructuring, that is, a debt structure which is an element of $S_R(e)$, are large, relative to the costs of using this type of debt structure (which arise because $r_b > r_m$). On the contrary, with large $e$, the entrepreneur needs little outside funding; lending is therefore close to risk-free, the benefits of debt flexibility are negligible, and debt structures allowing for restructuring become unappealing to the firm. The switching point $\bar{e}$ thus corresponds to the level of equity such that the optimal debt structure within $S_R(e)$ (which allows for restructuring) and the optimal debt structure within $S_K(e)$ (which does not) leave the entrepreneur indifferent.

This logic is further illustrated in figure 6. The left panel represents a section of the profit function of the entrepreneur along a line corresponding to the optimal total borrowing $d^*(e)$ associated with equity level $e$. This corresponds, graphically, to a line with slope $-1$ in the lending menu depicted in figure 5. The region marked with a $K$ corresponds to debt structures in $S_K(e)$. In this region, debt structures are tilted towards market debt, and the optimum $\hat{\pi}_B(e)$ within that region is attained for $b = 0$, the leftmost point on the graph; this corresponds to the first point discussed above. However, the global maximum of the entrepreneur’s profit function, $\hat{\pi}_B(e)$ corresponds to the local maximum in region marked with an R, which contains the debt structures in $S_R(e)$ along the line $b + m = d^*(e)$. In this case, $e$ is sufficiently small that the benefits of flexibility outweigh the costs of borrowing from banks rather than markets; the resulting debt structure is mixed, between bank and market finance. The right panel of the figure looks at a similar cross-section, for a value of internal funds $e' > e$. In this case, the leftmost point in the region of K-contracts is the global optimum; the firm then chooses a pure market debt structure. The switching point $\bar{e}$ corresponds to the case in which the two local maxima of the regions associated with K and R contracts are equal: $\hat{\pi}_K(e) = \hat{\pi}_R(e)$.

Summarizing, the first important prediction of the model is that firms with large amounts of own equity $e$ will choose to finance themselves through market debt, while firms with small amounts of internal finance...
will rely on a mix of market and bank debt dominated by bank debt.

4.4 The optimal debt structure under R-contracts

Next, I turn to characterizing more precisely the nature of the optimal debt structure when \( e < \bar{e} \). In that case, following proposition 7, the optimal debt structure \((b^*(e), m^*(e))\) is an element of \( S_P(e) \), and is therefore associated to an R-contract. The key results are summarized in the following proposition.

Proposition 8 (The optimal debt structure when \( e \leq \bar{e} \)) Assume \( r_b > r_m \). Consider an entrepreneur with equity \( e < \bar{e} \) and let \( s^*(e) = \frac{b^*(e)}{b^*(e) + m^*(e)} \) denote the fraction of total liabilities that are bank debt in her optimal debt structure, and let \( d^*(e) = b^*(e) + m^*(e) \) denote total borrowing. Then, there exists \( e < \bar{e} \) such that:

- For \( 0 \leq e < e \), the bank borrowing constraint is binding at the optimal debt structure, \( \frac{\partial s^*}{\partial e} > 0 \) and \( \frac{\partial d^*}{\partial e} \);
- For \( e \leq e \leq \bar{e} \), the optimal debt structure of the firm satisfies:

\[
s^*(e) = 1 - \frac{\Gamma(\chi)}{1 + r_m k - e}, \quad d^*(e) = k - e
\]

where the expression of \( \Gamma(\chi) \) and \( k \) are given in appendix B. In particular, \( \frac{\partial s^*}{\partial e} \leq 0 \) and \( \frac{\partial d^*}{\partial e} \leq 0 \).

4.4.1 The bank share

Proposition 8 states that the optimal debt structure is such that the optimal bank share \( s^*(e) \) is non-monotonic in the entrepreneur’s own equity, \( e \). This is illustrated in figure 7, which plots \( s^*(e) \) as a function of \( e \).

To understand why the banking share of liabilities has a non-monotonic relationship with equity, it is useful to note that the derivative of the objective function of the entrepreneur with respect to the share of bank debt, \( s = \frac{b}{b + m} \), using equation (8), is given by:

\[
\frac{\partial E(\tilde{\pi})}{\partial s} = (1 - \chi)\pi(\tilde{\phi}) \frac{\partial f(\tilde{\phi})}{\partial s} \left(-\frac{\partial \phi}{\partial s}\right) - (r_b - r_m)
\]

The first term in this expression, \((1 - \chi)\pi(\phi) \frac{\partial f(\phi)}{\partial s} \left(-\frac{\partial \phi}{\partial s}\right)\), represents the marginal decrease in expected liquidation losses associated with an increase in the share of bank borrowing, keeping total borrowing fixed. Note that \( \frac{\partial \phi}{\partial s} < 0 \), that is, an increase in bank borrowing keeping total borrowing fixed leads to a reduction in expected liquidation losses, as previously discussed. The second term in this expression, \((r_b - r_m)\), represents the increase in lending costs associated with a marginal increase in bank borrowing, again keeping
When equity is sufficiently small \((e < \ell)\), the bank share is increasing in equity because the bank borrowing constraint binds at the optimal debt structure; an increase in equity, by loosening the bank borrowing constraint, will necessarily lead to more bank borrowing. The bank borrowing constraint itself is binding because, when equity is small, marginal reductions in expected liquidation losses associated to more bank borrowing are large. In particular, when \(e < \ell\), any debt structure \((b, m) \in S_R(e)\) is such that:

\[
(1 - \chi)\pi(\phi)f(\phi)\left(-\frac{\partial \phi}{\partial s}\right) > r_b - r_m.
\]

Under this condition, at any debt structure, the marginal value of more bank debt is strictly larger than the associated costs, \(r_b - r_m\)\(^{15}\). However, note that this need not imply that \(s = 1\); even though the firm exhausts it bank borrowing capacity, it may still find it profitable to borrow from the market.

Note that, as discussed previously, when the bank borrowing constraint is binding, bank debt is always renegotiated by the firm, regardless of the realization of the productivity \(\phi\). In this case, the bank always receives a fraction \(\chi\) of the output produced by the firm. Thus, the optimal debt structure, when equity is

\(^{15}\)The equity level \(\ell\) in fact solves \((r_b - r_m) = -(1 - \chi)\pi(\phi)f(\phi)\frac{\partial \phi}{\partial s}\), with the right hand side is evaluated at \(s = s_{R,M}\) and \(d = d_{R}(s_{R,M}, \hat{e})\); see appendix B for details.
sufficiently small, is such that the bank contract essentially has the feature of an equity contract. Namely, the bank effectively agrees to receive a constant fraction of the firms’ output. This contract is beneficial to the entrepreneur because it allows her to avoid very frequent liquidation, given her high leverage.

For a sufficiently large level of equity, \( e \geq \underline{e} \), the bank borrowing constraint becomes loose at the optimal debt structure. This is the second case described in the proposition. In that case, the optimal debt structure of the firm satisfies:

\[
(1 - \chi)\pi(\phi) f(\phi) \left( -\frac{\partial \phi}{\partial s} \right) = r_b - r_m
\]

The marginal benefits of bank and market lending are exactly equalized, and the optimal debt structure is an interior point of the set \( S_K(e) \). Expression (9) is the analytical solution to this first order condition; in particular, it indicates that bank’s share of total debt is decreasing, as a function of own equity. This result comes from the fact that an increase in equity reduces the marginal impact of bank lending on total expected liquidation losses; formally, \( \frac{\partial^2 \phi}{\partial e \partial s} < 0 \). As own equity increases, gains from bank borrowing relative to market borrowing thin out, as the optimal leverage ratio falls and liquidation becomes less likely. Note, additionally, that for a given level of equity \( e \), the bank’s share of total borrowing is increasing in \( r_m \). Moreover, appendix B shows that \( \frac{\partial \Gamma}{\partial \chi} > 0 \), so that, when liquidation losses increase (\( \chi \) decreases), optimal debt structure becomes more tilted toward bank debt for firms with equity \( \underline{e} < e < \overline{e} \).

### 4.4.2 Total borrowing

The second important result from proposition 8 is that total borrowing, \( d^*(e) \), is non-monotonic in own equity \( e \). This is reported in the two top panels of figure 8, which plot optimal bank borrowing \( b^*(e) \) and optimal market borrowing \( m^*(e) \). The two middle panels of figure 8 plot total borrowing \( d^*(e) = b^*(e) + m^*(e) \) and total investment \( k^*(e) = e + b^*(e) + m^*(e) \) as a function of equity, while the two bottom ones report leverage ratios.

For small values of equity, total borrowing is increasing, so that total assets are also increasing. Like in the case of the bank share, total borrowing is increasing for small values of equity because in that range, the bank borrowing constraint binds. Total borrowing increases mostly as a result of the increase in bank borrowing associated to the loosening of the borrowing constraint. In that range of equities, most borrowing originates from the bank. Moreover, since total borrowing increases with equity, total assets also do. In that region, the leverage ratio of the firm is increasing with equity.

On the other hand, for values of equity above \( \underline{e} \) (but below \( \overline{e} \)), borrowing falls one for one with equity, so total assets are constant and equal to \( k^* \). On this range, for every dollar increase in own equity, the firm chooses to retire one dollar of bank debt; the amount of borrowing from market lenders, on the other hand,
is unchanged. The leverage ratio of the firm is now decreasing with equity. Using the results of proposition 8, the amount of market borrowing is given by:

\[ m^*(e) = \frac{\Gamma(\chi)}{1 + r_m} \xi \]

As shown in appendix B, the function \( \Gamma(\chi) \) is increasing with \( \chi \), so that lower liquidation losses (higher \( \chi \)) lead to more market borrowing. This result, as others reported in this section, does not depend on whether there is full depreciation of assets or not; that is, when \( \delta = 1 \), total borrowing is still constant over a certain range of equity. However, there is no simple analytical characterization of borrowing and the debt structure in that case.

### 4.5 The relationship between equity, the bank share, and total borrowing

The results from the two previous subsections show that the bank share is, broadly speaking, high for entrepreneurs with little own equity, and low for firms with large own equity. More precisely, for firms with
own equity below \( e \leq \tau \), the bank share is non-monotonic, but overall relatively large, and in fact sufficiently so for the bank borrowing constraint to bind for entrepreneurs with very little own equity. When \( e > \tau \), on the other hand, any debt issuance takes the form of market borrowing, as entrepreneurs find that the costs associated to borrowing from banks completely outweigh benefits of debt flexibility.

This paints a picture which is, at first glance, consistent with the first of the three empirical facts established in section 2. However, the model predicts a relationship between own equity \( e \) and the bank share of liabilities, while fact 1 relates total assets to the bank share of liabilities. As discussed previously, the model does not predict a strictly monotonic relationship between size \( k(e) = e + b^*(e) + m^*(e) \) and equity \( e \). One cannot therefore infer from the model’s predicted relationship between equity and the bank share, a similar link between total assets and the bank share.

This need not be viewed as a shortcoming of the model. First, this is a static model, in which \( b \) and \( m \) represent new debt issuance, which are to be retired after a single period. In particular, the entrepreneur does not inherit legacy assets or long-lived debt that needs to be serviced. In reality, much of the balance sheet of the firms in the sample analyzed in section 2 is constituted of liabilities maturing over long periods of time. It is also this lack of history-dependence which accounts for the fact that above the equity level \( \tau \), the firm switches to a debt structure entirely composed of market borrowing. On the face of it, this is not a realistic feature of the model, but it should be interpreted as indicating that new debt issuances will take the form of market debt once equity has reach the threshold \( \tau \). In the conclusion, I discuss the extent to which a dynamic model may fully account for the link between size and the bank share.

Second, the assumption of risk neutrality is crucial to obtain the result that total borrowing is constant. With risk aversion, a marginal increase in equity would not necessarily need a marginal fall in bank borrowing, to the extent that bank lending not only reduces expected liquidation losses, but also reduces the variance of the payoff to the entrepreneur.

Third, fact 1 concerns the complete distribution of firms across asset sizes. The fact that firms are distributed across different asset sizes may originate from other reasons than initial differences in own equity. Besides history-dependence, one source of firm size heterogeneity that has received particular attention in the literature are differences in the (average) marginal productivity of capital, \( E(\phi) \). Firms with different productivity levels both operate at different scales, and face different borrowing constraint. I come back to the link between marginal productivity, equity and the optimal debt structure in the next section.

Summarizing, I have shown that the model outlined in section 3, in which debt structure is the result of a trade-off between bank flexibility and the relative cost of funds of lenders, predicts a broadly decreasing relationship between a entrepreneur’s own equity \( e \) and the share of bank debt employed in her optimal debt
structure. The bank share is in fact increasing for the very smallest firms, after which the entrepreneur retires bank debt as her equity increases, first progressively, then – as equity reaches the threshold $\tau$ – abruptly switching to market debt only. This is qualitatively consistent with the first fact documented in section 2, which documented the fact that small firms tend to have a debt structure dominated by bank debt, while large firms mostly borrow from markets. I next turn to analyzing the dependence of debt structure on other factors than equity, and in particular, the cost of funds of borrowers, and the distribution of entrepreneur’s productivity shocks, $F(.)$.

5 Comparative statics of the debt structure

In this section, I ask how the optimal debt structure of firms changes in response to changes in fundamentals. I focus, in particular, on the effects of changes in the cost of funds of lenders, $r_b$ and $r_m$, and of changes in the distribution of productivity shocks $F(.)$.

While I provide some general results on these comparative statics, most of the discussion focuses on the comparison between numerical solutions to the model. In all calibrations, the baseline case is as follows, illustrated here. First, I assume full depreciation, $\delta = 1$. Second, I choose a degree of returns to scale of $\zeta = 0.92$, in line with the gross output estimates of Basu and Fernald (1997) for the US manufacturing sector. Third, I use the upper range of the estimates of Bris et al. (2006) for the gap between pre and post-bankruptcy asset values, and set $\chi = 0.60$, implying losses of 40% of output in liquidation. Finally, I assume that $\phi$ follows a Weibull distribution. In the baseline calibration, the location parameter of the distribution is normalized so that a firm with internal finance $e = 100$ is indifferent between borrowing from the market and using only its internal funds to invest in physical assets. Finally, in the baseline calibration, the cost of funds of bank and market lenders are set to $r_m = 5\%$ and $r_b = 6\%$, respectively. This calibration is summarized in ?, along with the other calibrations discussed in this section.

5.1 The effect of an increase in banks’ cost of lending

I first ask what patterns of change in debt structure the model predicts, when banks’ cost of funds increase relative to market lenders’. This simple comparative static is meant to capture the effect of an adverse shock to banks’ borrowing capacities on firms’ debt structures. The following proposition provides a general result on the effect of an increase in the spread $r_b - r_m$.

16 Unlike the log-normal distribution commonly used in the literature, this distribution has the advantage of having an increasing hazard rate, in accordance with assumption 1.
Figure 9: The effect of an increase in the spread \( r_b - r_m \) on bank share’s of total debt. The black line is the bank share in the baseline calibration; the grey line is the bank share after an increase in the spread \( r_b - r_m \).

Proposition 9 (The threshold between mixed and market finance) The threshold \( \tau \) is a decreasing function of the lending cost of banks \( r_b \) and of the spread \( r_b - r_m \).

This proposition implies that any reduction in the spread between bank and market will induce a switch from mixed to market finance for firms with sufficiently high equity. Specifically, for any \( (r_b, r_m) \) and \( (\hat{r}_b, \hat{r}_m) \) such that \( r_b - r_m < \hat{r}_b - \hat{r}_m \), all firms with equity levels \( \bar{e}_{r_b-r_m} < e < \bar{e}_{r_b-r_m} \) will choose a mixed debt structure under the spread \( r_b - r_m \), but will move to borrowing only from market lenders under the higher spread \( \hat{r}_b - \hat{r}_m \).

To illustrate this, figure 9 reports the optimal bank share as a function of equity, in the baseline case \( r_m = 5\% \) and \( r_b = 6\% \), and in a case where \( \hat{r}_m = r_m = 5\% \) but \( \hat{r}_b = 7\% \). The red line corresponds to the optimal debt structure under the high spread, while the green line corresponds to the optimal debt structure under the low spread. In this case, the increase in spread occurs because of an increase in banks’ cost of funds; the markets’ cost of funds, however is unchanged. For this reason, any firm with equity above \( \bar{e}_{r_b-r_m} \) is unaffected by the change in borrowing costs. For firms with equity \( e < \bar{e}_{r_b-r_m} \), several features of the change in optimal debt structure are noticeable.

First, as indicated by proposition 10, the threshold for switching to bank finance decreases and consequently, firms with a sufficiently large level of equity switch to a debt structure with only market debt.
This substitution pattern, for with large levels of equity, is consistent with fact 2; bank debt is substituted for market debt. As reported in figure 10, total borrowing for these firms falls; the substitution between bank and market debt is less than one for one. Fact 2, on the other hand, indicates that the substitution of bank for market credit is at least one for one. As discussed section 2, it is likely that, in the data, the magnitude of debt substitution for large firms is overstated by upwards reclassification. Additionally, the model omits important alternative forms of financing than could also serve as substitutes to bank debt, in particular short-term instruments such as commercial paper, which is an important channel of adjustment for firms in times of tight money (see Acharya and Schnabl (2010))\(^\text{17}\), and are therefore likely to account for a share of substitution away from bank debt. With these caveats in mind, while the model seems to predict substitution away from bank debt for firms with a relatively large amount of own equity, consistently with fact 2, it may also understate the extent of this substitution.

Second, for firms with equity levels below \(e_{\hat{r}_b - \hat{r}_m}\), the main effect of the increase in spreads is a large reduction in bank borrowing. Firms that are not at their bank borrowing constraint reduce significantly there is very little change in market debt. In fact, market borrowing somewhat falls. the amount borrowed from banks. Constrained firms, however, experience a smaller reduction in bank credit. Market borrowing changes very little. It increases slightly for firms that are not at their bank borrowing constraint, while, for firms whose borrowing constraint tightens as a result of the increase in banks’ borrowing costs, it falls. Overall, the pattern of a large reduction of bank credit for firms with small levels of internal finance is consistent with fact 3.

\(^{17}\)Because commercial paper is no separated from other non-bank liabilities for all asset size classes in the QFR, it was included in non-bank liabilities in the construction of fact 2).
The model thus predicts that an increase in the spread between the marginal cost of funds for banks and the market should firms with sufficiently large amounts of internal finance switch entirely to market finance. For firms with smaller amounts of internal finance, bank borrowing should fall, while market borrowing should remain mostly unchanged. These two patterns are qualitatively consistent with facts 2 and 3.

An interesting questions is whether the model predicts that an joint increase in the lending costs of both types of lenders has similar consequences on firms’ debt structure as the increase in the spread. Figures 13 and 14 in appendix C adress this question. There, I document the effect of an increase of $r_m$ from 5% to 6% and of $r_b$ from 6% to 7%. Figure 13 corresponds to the calibration reported under ”High cost levels (1)” in table 1. In this calibration, to keep things comparable to the baseline case, I also reduce the location parameter of the productivity distribution in such a way that the maximum equity level $e^{18}$ is unchanged under the new calibration. In this case, the joint increase in the level of borrowing costs leaves borrowing and the bank share almost completely unchanged. By contrast, figure 14 reports the debt structure when the location parameter of productivity is kept fixed, according to the calibration reported under ”High cost levels (2)” in table 1. In this case, the maximum equity level $e$ falls substantially, as the unconstrained size of the entrepreneur’s project declines (see lemma 1). As a result, borrowing in both types of debt falls. Particularly, overall borrowing falls for the largest firms – in contrast with both fact 3 and with the model’s prediction in the case of an increase in the spread $r_b - r_m$.

This exercise suggests that while joint increases in lenders’ costs of fund result in an overall fall in borrowing, it does so uniformly across the spectrum of equity levels. Some degree of asymmetry in the increase in lending costs, that is, some variation in the spread $r_b - r_m$, is thus needed to account for facts 2 and 3.

5.2 The effect of changes in the distribution of firms’ productivity

I next turn to discussing the effects of a change in the distribution of productivity shocks, $F(.)$, on the optimal debt structure.

5.2.1 A fall in average productivity

I first look at the impact of a fall in the average productivity of firms, $E(\phi)$. Specifically, I consider a change in the location and scale parameters of Weibull distribution such that average productivity, $E(\phi)$, falls by 1% relative to the baseline calibration, while the second moment of productivity is unchanged. The corresponding calibration is reported in table 1, under the column ”Low average productivity”.

\footnote{That is, the equity level at which firms are indifferent between borrowing from the market and lending risk-free.}
The resulting changing in the optimal debt structure is reported in figure 11, and figure ?? in appendix C reports changes in borrowing in more detail. Much as a joint increase in lenders’ cost of funds, the fall in average productivity has similar effects on borrowing both from banks and from the markets: optimal borrowing in both types of debt is strictly smaller, at all levels of equity. Note, additionally, that the switching threshold ε moves to the left, since the maximum size of the firm also falls. Again, the change in debt structure brought about by a fall in average productivity does not lead to the type of substitution across types of debt documented in fact 3 for large firms.

5.2.2 An increase in the dispersion of productivity

Next, I look at the impact of an increase in the second central moment of the distribution of productivity, keeping the first moment constant. In figure 12, I report the result of increasing the coefficient of variation of the distribution by 10 %, relative to the baseline calibration, while keeping the first moment constant. The corresponding calibration is reported in table 1, under the column “High productivity dispersion”.

Note that, contrary to the change in the first moment of the distribution, this change does not affect the maximum size of the firm, so that the range of relevant equity levels for the firm is unchanged. The result of the increase in dispersion is different for market and bank lending: market lending decreases uniformly, for all levels of equity, while bank debt is relatively unchanged. As higher dispersion generically increases expected liquidation losses, the threshold for switching towards market finance increases. All these patterns – the rightward shift of the threshold, the overall fall in market liabilities issued by entrepreneurs, and the limited changes in the amount of bank debt issued – do not match the patterns documented in facts 2 and 3. Rather, they indicate that an increase in the dispersion of productivity should lead all firms to increase the importance of bank debt in their debt structure.
Summarizing, I have shown that changes in the optimal debt structure across equity levels are markedly different depending on whether one focuses on shocks to the spread between borrowing costs of lenders, or shocks to either the overall level of borrowing costs or the entrepreneurs’ production possibilities. An increase in the spread leads to a pattern of debt substitution for larger firms, and debt reduction for smaller firms, consistently with facts 2 and 3. On the other hand, a joint in increase in borrowing costs, a fall in average productivity and the dispersion of productivity lead to a decline in market borrowing across all equity levels. Bank borrowing declines with a decline in average productivity or a joint increase in borrowing costs, but increases with an increase in the dispersion of productivity. Thus, the model suggests that the changes in debt structure documented in facts 2 and 3 require increases in the spread $r_b - r_m$ in order to be rationalized.

6 Conclusion

In this paper, I proposed a static model of the joint determination of investment and debt structure for a firm with access to both bank debt and market debt. The model builds on the trade-off theory of the debt structure, according to which a firm’s debt structure reflects a trade-off between the flexibility afforded by bank debt in times of financial distress, and the higher marginal costs of lending of banks. The model extends existing studies by allowing the entrepreneur to choose the scale of the project she operates, and thus explicitly modelling the link between investment and debt structure choices.

I showed that this model is consistent with the fact that small firms borrow mostly from banks, while the debt structure of large firms consists predominantly of market debt. Additionally, the model predicts that in response to an increase in banks’ cost of funds, relative to market lenders’, firms with sufficient internal finance should optimally substitute their bank liabilities for market debt, while firms with less access to
internal finance reduce bank borrowing without increasing market liabilities. This pattern is consistent with changes in the debt structure of US firms, small and large, over the last three recessions, and particularly after the 2007-2009 recession.

This paper suggests three avenues for future research. First, the model emphasizes the importance of lenders’ costs of funds $r_b$ and $r_m$ as determinants of the debt structure in the cross-section, and suggests that an increase in the spread $r_b - r_m$ may account well for changes in the debt structure of US firms. An important question is then whether it is possible to construct an empirical counterpart to this spread, and whether it changes substantially over the business cycle. As emphasized by the model, interest rates on loans or bond yields are clearly only upper bounds on these costs since they incorporate premia (liquidation risk premia, in the case of the model). Second, the model postulates that bank debt is senior to market debt. While, empirically, bank debt tends to be placed on top of firms’ priority structures, the rationale for this a subject of debate in much of the literature on debt heterogeneity; in fact, in some setups, such as Rajan (1992), the optimal debt structure makes bank debt junior to non-monitored debt. In the context of the model developed in this paper, numerical results suggest the alternative priority structure, whereby bank debt is junior, would not be preferred by the firm, if it were it be given the ability to choose. An important question is whether this result can be generally established, and in particular whether it always holds for any equity level. Third, because of the generality of the results on set of feasible debt structures, the model I developed in this paper can be used to study the dynamic implications of debt heterogeneity for firm investment and growth. I leave all three topics to future research.

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19Hackbath, Hennessy, and Leland (2007), in the context of their model with fixed investment size, prove that bank debt seniority is generally preferable.
References


A Appendix to section 3

A.1 Proof of proposition 1

Proof of proposition 1. Note first that the payoff to the entrepreneur under the optimal restructuring offer, given by equation (7), can be rewritten as:

\[ \pi_R(\phi) = \pi(\phi) - \min(\chi \pi(\phi), R_b) - R_m. \] (10)

First, assume that the repayments promised to the bank and market lenders satisfy:

\[ \frac{R_b}{\chi} > \frac{R_m}{1 - \chi}. \] (11)

Since:

\[ R_m + R_b = \chi \frac{R_b}{\chi} + (1 - \chi) \frac{R_m}{1 - \chi}, \]

condition (11) implies that:

\[ \frac{R_b}{\chi} > R_m + R_b > \frac{R_m}{1 - \chi}. \]

The threshold \( \frac{R_b}{\chi} \) is the one where restructuring becomes profitable relative to repayment for the entrepreneur, while the threshold \( R_m + R_b \) is the ”natural” bankruptcy threshold – the one that would obtain absent the possibility of private workouts with the bank. Given condition (11), consider the three following possibilities:

- if \( \pi(\phi) \geq \frac{R_b}{\chi} \), using equation (10), the payoff of the entrepreneur under restructuring is identical to that under repayments; that is, the firm cannot lower its repayments in a private workout. Moreover, as \( \pi(\phi) > \frac{R_b}{\chi} > R_m + R_b \), repayment is preferable to bankruptcy. Thus, the entrepreneur chooses full repayment.

- if \( \frac{R_b}{\chi} > \pi(\phi) \geq \frac{R_m}{1 - \chi} \), the entrepreneur prefers restructuring to repayment since:

\[ \pi_R(\phi) = \pi(\phi) - \chi \pi(\phi) - R_m > \pi(\phi) - R_b - R_m. \]

Moreover, the entrepreneur also prefers restructuring to bankruptcy, since when \( \pi(\phi) > \frac{R_m}{1 - \chi} \),

\[ \pi_R(\phi) > 0 = \pi_B(\phi). \]
Thus, the entrepreneur chooses to restructure her debt with the bank,

- if $\frac{R_m}{1-\chi} > \pi(\phi)$, the entrepreneur now prefers bankruptcy to restructuring, as the gains achieved under restructuring would still not be sufficient to repay market creditors in full and end up with a strictly positive amount of cash on hand. Furthermore, since under condition (11), $\pi(\phi) < \frac{R_m}{\chi} < R_m + R_b$, the entrepreneur also prefers bankruptcy to repayment. Thus, in this case, the entrepreneur chooses to file for bankruptcy.

This establishes the first part of the proposition on the structure of R-equilibria. Additionally, note that in this type of equilibrium, the payoff to market lenders under bankruptcy is always 0, because bankruptcy occurs only when $\pi(\phi) < \frac{R_m}{1-\chi} < \frac{R_b}{\chi}$.

Next, consider the case:

$$\frac{R_b}{\chi} \leq \frac{R_m}{1 - \chi},$$

which implies that:

$$\frac{R_b}{\chi} < R_m + R_b < \frac{R_m}{\chi}.$$

Under condition (12), consider the three following possibilities:

- if $\pi(\phi) \geq R_m + R_b$, repayment is preferable to bankruptcy. Moreover, because in this case $\pi(\phi) \geq R_m + R_b > \frac{R_b}{\chi}$, restructuring cannot lead to any gains for the entrepreneur. Thus, the entrepreneur chooses repayment.

- if $R_m + R_b > \pi(\phi) \geq \frac{R_b}{\chi}$, bankruptcy is preferable to repayment, and moreover, restructuring still cannot achieve any gains relative to repayment; thus, the entrepreneur chooses bankruptcy.

- if $\frac{R_b}{\chi} > \pi(\phi)$, restructuring can now achieve gains relative to repayment. However, because $\frac{R_m}{1-\chi} > \frac{R_b}{\chi} > \pi(\phi)$ the payoff under restructuring satisfies:

$$\pi_R(\phi) = (1 - \chi)\pi(\phi) - R_m < 0 = \pi_B(\phi).$$

Thus, the entrepreneur still chooses bankruptcy.

This establishes the second part of the proposition. Additionally, note that unde the seniority structure assumed, bankruptcy payments to market lenders are $\chi\pi(\phi) - R_b$ if $R_m + R_b > \pi(\phi) \geq \frac{R_b}{\chi}$, and 0 otherwise. In the latter case, this is obviously smaller than $R_m$. In the former case, since bankruptcy only occurs when $\pi(\phi) < R_m + R_b$, we have that

$$\pi(\phi) < R_m + R_b < \frac{R_m + R_b}{\chi}.$$
so that:
\[ \chi \pi(\phi) - R_b < R_m. \]

Therefore, under the assumed priority structure, market lenders never obtain a repayment under bankruptcy that exceeds the promised repayment \( R_m \).

\[ \blacksquare \]

### A.2 Debt pricing

In this section, I describe the payoff and the expected gross return functions of lenders that were omitted from the main text.

#### A.2.1 Payoff functions

Given the description of the equilibria in proposition 1, the payoffs to the lenders and the entrepreneur can be expressed as a function of \( \phi \). I denote them \( \tilde{R}_b(\phi) \) and \( \tilde{R}_m(\phi) \) for the bank and market lenders, respectively, and \( \tilde{\pi}(\phi) \) for the entrepreneur.

In B-equilibria \( \left( \frac{R_b}{\chi} < \frac{R_m}{1-\chi} \right) \), payoffs are given by:

\[
\tilde{R}_b(\phi) = \begin{cases} 
R_b & \text{if } \frac{R_b}{\chi} \leq \pi(\phi) \\
\chi \pi(\phi) & \text{if } \pi(\phi) < \frac{R_b}{\chi}
\end{cases}
\]

\[
\tilde{R}_m(\phi) = \begin{cases} 
R_m & \text{if } R_m + R_b \leq \pi(\phi) \\
\chi \pi(\phi) - R_b & \text{if } \frac{R_b}{\chi} \leq \pi(\phi) < R_m + R_b \\
0 & \text{if } \pi(\phi) < \frac{R_b}{\chi}
\end{cases}
\]

\[
\tilde{\pi}(\phi) = \begin{cases} 
\pi(\phi) - R_b - R_m & \text{if } R_m + R_b \leq \pi(\phi) \\
0 & \text{if } \pi(\phi) < R_m + R_b
\end{cases}
\]

In R-equilibria \( \left( \frac{R_b}{\chi} \geq \frac{R_m}{1-\chi} \right) \), payoffs are given by:

\[
\tilde{R}_b(\phi) = \begin{cases} 
R_b & \text{if } \frac{R_b}{\chi} \leq \pi(\phi) \\
\chi \pi(\phi) & \text{if } \pi(\phi) < \frac{R_b}{\chi}
\end{cases}
\]

\[
\tilde{R}_m(\phi) = \begin{cases} 
R_m & \text{if } \frac{R_m}{1-\chi} \leq \pi(\phi) \\
0 & \text{if } \pi(\phi) < \frac{R_m}{1-\chi}
\end{cases}
\]
\[ \hat{\pi}(\phi) = \begin{cases} 
\pi(\phi) - R_b - R_m & \text{if } \frac{R_b}{\chi} \leq \pi(\phi) \\
(1 - \chi)\pi(\phi) - R_m & \text{if } \frac{R_m}{1-\chi} \leq \pi(\phi) < \frac{R_b}{\chi} \\
0 & \text{if } \pi(\phi) < \frac{R_m}{1-\chi} 
\end{cases} \]

A.2.2 Gross expected returns

I now turn to the gross expected return functions of lenders. Throughout, I use the same change of variables as in the text:

\[ b = ds \]
\[ m = d(1 - s) \]

\( d \) thus denotes the total amount borrowed, \( s \) denotes the fraction borrowed from the bank, and \( 1 - s \) denotes the fraction borrowed from market lenders; clearly \( s \in [0, 1] \) and \( d \geq 0 \). I use the notation:

\[ \bar{E}_i(R_b, R_m; d, s, e) = E_i(e, ds, d(1 - s), R_b, R_m) \]

for the associated gross return of lenders.

**Lemma 2 (Gross expected return functions)** The gross expected return of bank lenders is given by:

\[ \bar{E}_b(R_b; e + d) = \begin{cases} 
R_b & \text{if } \frac{R_b}{\chi} < (1 - \delta)(e + d) \\
\chi\left((e + d)^\zeta G(\phi_b) + (1 - \delta)(e + d)\right) & \text{if } \frac{R_b}{\chi} \geq (1 - \delta)(e + d) 
\end{cases} \]

where:

\[ \phi_b \equiv \frac{R_b - \chi(1 - \delta)(e + d)}{\chi(e + d)^\zeta} \quad \text{and} \quad G(x) \equiv x(1 - F(x)) + \int_0^x \phi dF(\phi) \]

When \( \frac{R_b}{\chi} \geq \frac{R_m}{1-\chi} \), the gross expected return of market lenders is given by:

\[ \bar{E}_{m,R}(R_m; e + d) = \begin{cases} 
R_m & \text{if } \frac{R_m}{1-\chi} < (1 - \delta)(e + d) \\
(1 - \chi)\left((e + d)^\zeta I(\phi_{m,R}; e + d) + (1 - \delta)(e + d)\right) & \text{if } \frac{R_m}{1-\chi} \geq (1 - \delta)(e + d) 
\end{cases} \]

where:

\[ \phi_{m,R} \equiv \frac{R_m - (1 - \chi)(1 - \delta)(e + d)}{(1 - \chi)(e + d)^\zeta} \quad \text{and} \quad I(x; e + d) \equiv x(1 - F(x)) - F(x)(1 - \delta)(e + d)^{1-\zeta}. \]
When $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$, the gross expected return of market lenders is given by:

$$\tilde{E}_{m,B}(R_b, R_m; e + d) = \begin{cases} 
R_m & \text{if } R_m + R_b < (1 - \delta)(e + d) \\
(e + d)^\zeta M(\phi_{m,B}; e + d) + (1 - \delta)(e + d) & \text{if } R_m + R_b \geq (1 - \delta)(e + d) \\
-\tilde{E}_b(R_b; e + d)
\end{cases}$$

where:
$$\phi_{m,B} \equiv \frac{R_b + R_m - (1 - \delta)(e + d)}{(e + d)^\zeta}$$
and
$$M(x; e + d) \equiv (1 - \chi)I(x; e + d) + \chi G(x).$$

**Proof.** In the case $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$, the entrepreneur never defaults when $(1 - \delta)(e + d) > R_m + R_b$, since in that case, the lower bound on her output is greater than the sum of her promised repayments. Expected repayments for the bank and market lenders are therefore equal to promised repayments $R_b$ and $R_m$. When $\frac{R_b}{\chi} < (1 - \delta)(e + d) \leq R_m + R_b$, there may be default on market debt, but even in liquidation bank debt will be repayed in full, so that $\tilde{E}_b = R_b$. Similarly, in the case $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$, there is no default when $(1 - \delta)(e + d) > \frac{R_b}{\chi}$, and no default on market debt if $\frac{R_m}{1-\chi} < (1 - \delta)(e + d) \leq \frac{R_b}{\chi}$. The rest of the expressions follow from the expressions of the payoff functions.

\[\blacksquare\]

### A.2.3 Zero profit conditions

Using the results of lemma 2, the zero profit condition (ZPC) of bank lenders can be written as:

$$\tilde{G}(R_b; e + d) = \frac{(1 + r_b)ds}{\chi(e + d)^\zeta} \quad \text{with} \quad \tilde{G}(R_b; e + d) \equiv \frac{\tilde{E}_b(R_b; e + d)}{\chi(e + d)^\zeta}. \quad (13)$$

**Lemma 3 (The zero profit condition of bank lenders)** Let $(d, s, e)$ be given. Then, there exists a unique solution $R_b(d, s, e)$ to the ZPC of bank lenders, equation (13), if and only if,

$$0 \leq \frac{(1 + r_b)ds}{\chi(e + d)^\zeta} \leq \mathbb{E}(\phi) + (1 - \delta)(e + d)^{1-\zeta}. \quad (14)$$

Moreover, $R_b(d, s, e)$ is given by:

$$R_b(d, s, e) = \begin{cases} 
(1 + r_b)ds & \text{if } 0 \leq \frac{(1 + r_b)ds}{\chi(e + d)^\zeta} < (1 - \delta)(e + d)^{1-\zeta} \\
\chi(1 - \delta)(e + d) & \text{if } (1 - \delta)(e + d)^{1-\zeta} \leq \frac{(1 + r_b)ds}{\chi(e + d)^\zeta} \\
\chi(e + d)^\zeta G^{-1}(y_G(d, s, e)) & \text{if } \mathbb{E}(\phi) + (1 - \delta)(e + d)^{1-\zeta} \end{cases}$$
Here, \( y_G(d,s,e) = \frac{(1+r_m)d(1-s)}{(1-\chi)(e+d)} \), and \( G^{-1}(\cdot) \) denotes the inverse of \( G(\cdot) \), defined on \([0, \mathbb{E}(\phi)]\) with the abuse of notation that \( G^{-1}(\mathbb{E}(\phi)) = +\infty \).

**Proof.** Note that \( \hat{G}(\cdot; e+d) \) is strictly increasing on \( \mathbb{R}_+ \), that \( \hat{G}(0; e+d) = 0 \) and \( \lim_{R_m \to +\infty} \hat{G}(R_m; e+d) = \mathbb{E}(\phi) + (1-\delta)(e+d)^{1-\zeta} \). Similarly, \( G(\cdot) \) is strictly increasing on \( \mathbb{R}_+ \), \( G(0) = 0 \), and \( \lim_{x \to +\infty} G(x) = \mathbb{E}(\phi) \).

Moreover, \( \Delta(0) = \hat{I}(R_0; e+d) = 0 \) and \( \lim_{R_m \to +\infty} \hat{I}(R_m; e+d) = \mathbb{E}(\phi) + (1-\delta)(e+d)^{1-\zeta} \).

**Lemma 4 (The zero profit condition of market lenders when \( \frac{R_m}{\chi} \geq \frac{R_n}{1-\chi} \))** Let \( (d,s,e) \) be given. Then, there exist exactly two solutions, \( R_{m,L}(d,s,e) \leq R_{m,L}(d,s,e) \), to equation (15), if and only if,

\[
0 \leq \frac{(1+r_m)d(1-s)}{(1-\chi)(e+d)^\zeta} \leq I(e+d) + (1-\delta)(e+d)^{1-\zeta},
\]

where \( \hat{I}(e+d) = \hat{I}(\phi_1(e+d); e+d) \) is the maximum of \( I(\cdot; e+d) \), attained at \( \phi_1(e+d) \), the unique (strictly positive) solution to:

\[
1 - F(\phi_1(e+d)) - f(\phi_1(e+d)) (\phi_1(e+d) + (1-\delta)(e+d)^{1-\zeta}) = 0.
\]

Moreover, \( R_{m,L}(d,s,e) \) is given by:

\[
R_{m,L}(d,s,e) = \begin{cases} 
(1+r_m)d(1-s) & \text{if } 0 \leq \frac{(1+r_m)d(1-s)}{(1-\chi)(e+d)^\zeta} < (1-\delta)(e+d)^{1-\zeta} \\
(1-\chi)(1-\delta)(e+d) & \text{if } (1-\delta)(e+d)^{1-\zeta} \leq \frac{(1+r_m)d(1-s)}{(1-\chi)(e+d)^\zeta} \\
+(1-\chi)(e+d)^\zeta I^{-1}(y_1(d,s,e); e+d) & \leq \hat{I}(e+d) + (1-\delta)(e+d)^{1-\zeta}
\end{cases}
\]

where \( y_1(d,s,e) = \frac{(1+r_m)d(1-s)-(1-\chi)(1-\delta)(e+d)}{(1-\chi)(e+d)^\zeta} \), and \( I^{-1}(\cdot; e+d) \) denotes the mapping from \([0, \hat{I}(e+d)]\) to \([0, \phi_1(e+d)]\) such that \( I(I^{-1}(y; e+d); e+d) = y, \forall y \in [0, \hat{I}(e+d)] \).

**Proof.** Note that \( \hat{I}(0; e+d) = 0 \) and \( \lim_{R \to +\infty} \hat{I}(R; e+d) = 0 \). Moreover, \( \frac{\partial \hat{I}}{\partial R_0} = \frac{1}{(1-\chi)(e+d)^\zeta} > 0 \) when \( R_0 < (1-\chi)(1-\delta)(e+d)^{1-\zeta} \), so that \( \hat{I}(\cdot, e+d) \) is increasing in \( R_0 \) on that range. For \( R_0 \geq (1-\chi)(1-\delta)(e+d)^{1-\zeta} \), note that \( \frac{\partial \hat{I}}{\partial R_0} \geq 0 \), if and only if, \( h(\phi_{m,R}) = \frac{\phi_{m,R}}{1-F(\phi_{m,R})} \leq \frac{1}{\phi_{m,R} + (1-\delta)(e+d)^{1-\zeta}} \), with \( \phi_{m,R} \) defined as in lemma 2. Fix \( e \) and \( d \) and let \( \Delta(x) = h(x) - \frac{1}{x + (1-\delta)(e+d)^{1-\zeta}} \). \( h \) is strictly increasing by assumption ???. As the sum of two strictly increasing functions, \( \Delta \) is therefore strictly increasing on \([0, +\infty[\). Moreover, \( \Delta(0) = -\frac{1}{(1-\delta)(e+d)^{1-\zeta}} \) and \( \lim_{x \to +\infty} \Delta(x) = \lim_{x \to +\infty} h(x) \). Since \( h(0) = 0 \) (as
\( f(0) = 0 \) by assumption and \( h \) is strictly increasing, \( \lim_{x \to +\infty} h(x) > 0 \). Thus \( \Delta(x) \) has a unique strictly positive root at \( \phi_{m,R} = \phi_I(e + d) \). In turn, because \( \phi_{m,R} \) is a strictly increasing function of \( R_m \), \( \hat{R}_I(e + d) \equiv (1 - \chi) \left( \phi_I(e + d) + (1 - \delta)(e + d) \right) \) is the global maximum of \( \hat{I}(.; e + d) \). There are moreover no local maxima, so that \( \hat{I}(.; e + d) \) is strictly increasing to the left of \( \hat{R}_I(e + d) \) and strictly decreasing to the right. (Note however that \( I(.; e + d) \) and \( \hat{I}(.; e + d) \) need not be concave). This proves that there are exactly two solutions \( \tilde{R}_{m,l}(d, s, e) \leq \hat{R}_I(e + d) \leq \tilde{R}_{m,l}(d, s, e) \) to equation (15) under the conditions given in the lemma. The expression for \( R_{m,l}(d, s, e) \) follows from recognizing that, for the same reasons as \( \hat{I}(.; e + d) \), \( I(.; e + d) \) is strictly increasing on \([0, \phi_I(e + d)]\) and strictly decreasing thereafter. ■

Finally, when \( \frac{R_m}{\chi} > \frac{R_e}{\chi} \), when the zero profit condition of the bank holds, the zero profit condition of market lenders can be rewritten as:

\[
\tilde{M}(R_b + R_m; e + d) = \frac{(1 + r_m(1 - s) + r_b)d}{(e + d)^{\zeta}}, \tag{17}
\]

where:

\[
\tilde{M}(R; e + d) = \begin{cases} \frac{R}{(e + d)^{\zeta}} & \text{if } R < (1 - \delta)(e + d) \\ M(\phi_{m,B}; e + d) + (1 - \delta)(e + d)^{1 - \zeta} & \text{if } R \geq (1 - \delta)(e + d) \end{cases}
\]

and \( \phi_{m,B} \equiv \frac{R - (1 - \delta)(e + d)}{(e + d)^{\zeta}} \).

**Lemma 5 (The zero profit condition of market lenders when \( \frac{R_e}{\chi} < \frac{R_m}{1 - \chi} \))** Let \( (d, s, e) \) be given. Then, there exist exactly two solutions, \( R_I(d, s, e) \leq R_L(d, s, e) \), to equation (17), if and only if,

\[
\chi \mathbb{E}(\phi) + (1 - \delta)(e + d)^{1 - \zeta} \leq \frac{(1 + r_m(1 - s) + r_b s)d}{(e + d)^{\zeta}} \leq \tilde{M}(e + d) + (1 - \delta)(e + d)^{1 - \zeta}, \tag{18}
\]

where \( \tilde{M}(e + d) = M(\phi_M(e + d), e + d) \) is the maximum of \( M(.; e + d) \), attained at \( \phi_M(e + d) > \phi_I(e + d) \), which is the unique (strictly positive) solution to:

\[
1 - F(\phi_M(e + d) - (1 - \chi)f(\phi_M(e + d)) \left( \phi_M(e + d) + (1 - \delta)(e + d)^{1 - \zeta} \right) = 0.
\]

There exists exactly one solution \( R_I(d, s, e) \) to equation (17), if and only if:

\[
0 \leq \frac{(1 + r_m(1 - s) + r_b s)d}{(e + d)^{\zeta}} < \chi \mathbb{E}(\phi) + (1 - \delta)(e + d)^{1 - \zeta},
\]
Moreover, \( R_1(d,s,e) \) is given by:

\[
R_1(d,s,e) = \begin{cases} 
(1 + r_m(1 - s) + r_b s) d & \text{if } 0 < (1 + r_m(1 - s) + r_b s) d < (1 - \delta)(e + d)^{1 - \zeta} \\
+(1 - \delta)(e + d) & \text{if } (1 - \delta)(e + d)^{1 - \zeta} \leq (1 + r_m(1 - s) + r_b s) d \leq (e + d)^{1 - \zeta} \\
(e + d)^{\zeta} M^{-1}(y_M(d,s,e); e + d) & \leq \hat{M}(e + d) + (1 - \delta)(e + d)^{1 - \zeta}
\end{cases}
\]

Here, \( y_M(d,s,e) = \frac{(1 + r_m(1 - s) + r_b s) d - (1 - \delta)(e + d)}{(e + d)^{1 - \zeta}} \), and \( M^{-1}(.; e + d) \) denotes the mapping from \( [0, \hat{M}(d + e)] \) to \( [0, \phi_M(e + d)] \) such that \( M(M^{-1}(y; e + d); e + d) = y \) \( \forall y \in [0, \hat{M}(e + d)] \).

### A.3 The lending menu in the general case

This subsection contains the proof of proposition 2. The proof relies essentially on proposition ??.

This proposition shows that the set \( \mathcal{S}(e) \) can be separated into two subsets, \( \hat{\mathcal{S}}_R(e) \) and \( \hat{\mathcal{S}}_B(e) \). \( \hat{\mathcal{S}}_B(e) \) roughly correspond to debt structures associated with dominating contracts with \( \frac{R_b}{\chi} \geq \frac{R_m}{1 - \chi} \), while \( \hat{\mathcal{S}}_R(e) \) corresponds to debt structures associated with dominating contracts such that \( \frac{R_b}{\chi} < \frac{R_m}{1 - \chi} \). The main difficulty of this proof is that when a debt structure \((d,s)\) is such that \( \mathcal{L}(d,s,e) \neq \emptyset \), \( \mathcal{L}(d,s,e) \) will typically contain multiple pairs \((R_b, R_m)\) (up to four pairs), each of which satisfy the zero profit conditions of lenders. The multiplicity of contracts comes from two sources. First, zero profit conditions of lenders generically have two solutions, as established in lemmas 4-5. Second, there are debt structures for which the zero profit conditions of lenders have solutions with \( \frac{R_b}{\chi} < \frac{R_m}{1 - \chi} \) as well as solutions with \( \frac{R_b}{\chi} \geq \frac{R_m}{1 - \chi} \) (that is, \( \hat{\mathcal{S}}_R(e) \cap \hat{\mathcal{S}}_B(e) \neq \emptyset \)). All this multiplicity creates the potential for contracts that cannot be ordered using the product order. The main contribution of the lemma is to show that these contracts can in all instances be ordered, so that there is always a unique dominating contract, even when the contract menu contains multiple elements.

**Proof.** The proof draws heavily upon the results of lemmas 3-5 and proceeds in three steps:

**Step 1:** Any \((d,s) \in \mathcal{S}(e)\) must be an element of either \( \hat{\mathcal{S}}_B(e) \) or \( \hat{\mathcal{S}}_R(e) \), so that \( \hat{\mathcal{S}}(e) \subset \left( \hat{\mathcal{S}}_B(e) \cup \hat{\mathcal{S}}_R(e) \right) \);

**Step 2-a:** \( \forall(d,s) \in \hat{\mathcal{S}}_R(e), \mathcal{L}(d,s,e) \) has a least element \((R_b, R_m)\) (for the product order \( \geq_x \)), so that \( \hat{\mathcal{S}}_R(e) \subset \hat{\mathcal{S}}(e) \); moreover, this element satisfies \( \frac{R_b}{\chi} \geq \frac{R_m}{1 - \chi} \);

**Step 2-b:** \( \forall(d,s) \in \hat{\mathcal{S}}_B(e) \setminus \left( \hat{\mathcal{S}}_R(e) \cap \hat{\mathcal{S}}_B(e) \right) \), \( \mathcal{L}(d,s,e) \) has a least element \((R_b, R_m)\) (for the product order \( \geq_x \)), so that \( \left( \hat{\mathcal{S}}_B(e) \setminus \left( \hat{\mathcal{S}}_R(e) \cap \hat{\mathcal{S}}_B(e) \right) \right) \subset \hat{\mathcal{S}}(e) \); moreover, the least element satisfies \( \frac{R_b}{\chi} < \frac{R_m}{1 - \chi} \).

**Step 1** Let \((d,s) \in \mathcal{S}(e)\) and let \((R_b, R_m)\) be the associated dominating contract, that is, the least element of \( \mathcal{L}(d,s,e) \) for \( \geq_x \). It must be the case that either \( \frac{R_b}{\chi} < \frac{R_m}{1 - \chi} \) or \( \frac{R_b}{\chi} \geq \frac{R_m}{1 - \chi} \).

**Assume that** \( \frac{R_b}{\chi} < \frac{R_m}{1 - \chi} \). Then \((R_b, R_m) \in \mathcal{L}(d,s,e)\), so that \( \mathcal{L}(d,s,e) \neq \emptyset \). Moreover, \((R_b, R_m)\) must
solve the zero profit conditions (ZPC) of the lenders in the case \( \frac{R_b}{\chi} \leq \frac{R_m}{1-\chi} \), equations (13) and (17). Using the results of lemmas 3 and 5, conditions (c – bank) and (c – joint) are necessary for the existence of solutions to (13) and (17). Thus, \((d, s, e)\) must satisfy these two conditions; moreover, by lemma 3, \(R_b = R_b(d, s, e)\).

If \(\frac{1+r_m(1-s)+r_s d}{(e+d)} \chi < \chi \mathbb{E}(\phi)\), the solution to equation (17) is unique, according to lemma 5. Therefore we must have \(R_m = R_l(d, s, e) - R_b(d, s, e)\). Since, by assumption, \(\frac{R_b}{\chi} < \frac{R_m}{1-\chi}\), we also have \(\frac{R_b}{\chi} < R_m + R_b\).

Therefore, it must also be the case that \(\frac{R_b(d, s, e)}{\chi} < R_l(d, s, e)\), so that condition (frontier – B) must hold.

If, on the other hand, \(\frac{1+r_m(1-s)+r_s d}{(e+d)} \chi \geq \chi \mathbb{E}(\phi)\), there are exactly two solutions to equation (17), which satisfy \(0 \leq R_l(d, s, e) \leq R_L(d, s, e)\). However, \((R_b(d, s, e), R_L(d, s, e) - R_b(d, s, e)) \geq \chi (R_b(d, s, e), R_l(d, s, e) - R_b(d, s, e))\). Since \((R_b, R_m)\) is the least element of \(\mathcal{L}(d, s, e)\), it must be therefore be the case that \(R_m = R_l(d, s, e) - R_b(d, s, e)\). The fact that condition (frontier – B) holds then follows from the same steps as above. This finishes the proof that if \(\frac{R_m}{1-\chi} > \frac{R_b}{\chi}\), then \((d, s) \in \hat{S}_B(e)\).

**Assume that** \(\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}\). Similar to the other case, this implies that \(\mathcal{L}(d, s, e) \neq \emptyset\) and therefore that conditions (c – bank) and (c – market), which are necessary for the existence of a solution to the ZPC of lenders when \(\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}\), must hold. Moreover, by lemma 3, \(R_b = R_b(d, s, e)\).

Under condition (c – market), there exist exactly two solutions \(R_{m,l}(d, s, e) \leq R_{m,l}(d, s, e)\), once of which \(R_m\) must be equal to. Since \((R_b, R_m)\) is the least element of \(\mathcal{L}(d, s, e)\), it must be the case that \(R_m = R_{m,l}(d, s, e)\). Since \(\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}\), this in turn imply that condition (frontier – B) holds. This finishes the proof that if \(\frac{R_m}{1-\chi} \leq \frac{R_b}{\chi}\), then \((d, s) \in \hat{S}_R(e)\).

Thus, \(\mathcal{S}(e) \in (\hat{S}_B(e) \cup \hat{S}_R(e)\).

**Step 2-a:** Let \((d, s) \in \hat{S}_R(e)\). We need to prove that \((d, s) \in \mathcal{S}_e\), that is, that \(\mathcal{L}(d, s, e)\) is non-empty and has a least element. First, under conditions (c – bank) and (c – market), \((R_b(d, s, e), R_{m,l}(d, s, e)) \in \mathcal{L}(d, s, e)\), so that \(\mathcal{L}(d, s, e) \neq \emptyset\). If \(\mathcal{L}(d, s, e)\) contains only \((R_b(d, s, e), R_{m,l}(d, s, e))\), then it is the least element and we are done. Otherwise, let \((R_b, R_m) \in \mathcal{L}(d, s, e)\), \((R_b, R_m) \neq (R_b(d, s, e), R_{m,l}(d, s, e))\). If \((R_b, R_m)\) is such that \(\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}\), then, since \((R_b, R_m) \neq (R_b(d, s, e), R_{m,l}(d, s, e))\), it must be the case that \(R_b = R_b(d, s, e)\) and \(R_m = R_{m,l}(d, s, e)\); therefore, \((R_b, R_m) \geq \chi (R_b(d, s, e), R_{m,l}(d, s, e))\). If, on the other hand, \(\frac{R_b}{\chi} < \frac{R_m}{1-\chi}\), it must still be the case that \(R_b = R_b(d, s, e)\); hence,

\[
\frac{R_m}{1-\chi} > \frac{R_b}{\chi} = \frac{R_b(d, s, e)}{\chi} \geq \frac{R_{m,l}(d, s, e)}{1-\chi}.
\]

Therefore, \((R_b, R_m) \geq \chi (R_b(d, s, e), R_{m,l}(d, s, e))\) in the case \(\frac{R_b}{\chi} < \frac{R_m}{1-\chi}\) as well. Thus, \(\mathcal{L}(d, s, e)\) has a least element, \((R_b(d, s, e), R_{m,l}(d, s, e))\), so that \((d, s) \in \mathcal{S}(e)\). Hence, \(\hat{S}_R(e) \subset \hat{S}(e)\). Note that since the least element is always unique, I have also established the proposition’s claim that the dominating contract (least
element) \((R_b = R_b(d,s,e), R_m = R_{m,l}(d,s,e))\) associated to \((d,s) \in \hat{S}_R(e)\) satisfies \(\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}\).

**Step 2-b:** Let \((d,s) \in \hat{S}_B(e) \setminus (\hat{S}_B(e) \cap \hat{S}_R(e))\). First, under conditions \((c - \text{bank})\) and \((c - \text{joint})\), we have that \((R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e)) \in \mathcal{L}(d,s,e)\), so \(\mathcal{L}(d,s,e) \neq \emptyset\). Consider \((R_b, R_m) \in \mathcal{L}(d,s,e)\), \((R_b, R_m) \neq (R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e))\). If \((R_b, R_m)\) are such that \(\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}\), then, as in step 1, it must be that \((d,s) \in \hat{S}_R(e)\); but we ruled this out by assumption. So, \(\frac{R_b}{\chi} < \frac{R_m}{1-\chi}\). In this case, since \((R_b, R_m) \neq (R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e))\), it must be that \(R_b = R_b(d,s,e)\) and \(R_m = R_l(d,s,e) - R_b(d,s,e)\) (and additionally that \(\frac{1+r_m(1+\epsilon) + r_b s}{(\epsilon + d)^2} > \chi E(\phi)\), since otherwise the solution to the ZPC of market lenders is unique). So, \((R_b, R_m) \geq_{\chi} (R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e))\), and so \((R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e))\) is the least element of \(\mathcal{L}(d,s,e)\). This proves that \((d,s) \in \hat{S}(e)\), and therefore that \(\left(\hat{S}_B(e) \setminus (\hat{S}_B(e) \cap \hat{S}_R(e))\right) \subset \hat{S}(e)\). Note that again because of uniqueness of the least element, I have also established the lemma’s claim that the dominating contract \((R_b = R_b(d,s,e), R_m = R_l(d,s,e) - R_b(d,s,e))\) associated to \((d,s) \in \hat{S}_B(e) \setminus (\hat{S}_B(e) \cap \hat{S}_R(e))\) satisfies \(\frac{R_b}{\chi} < \frac{R_m}{1-\chi}\).

### A.4 The lending menu when \(\delta = 1\)

I next turn to characterizing the set \(\hat{S}_R(e) = \hat{S}_R(e)\) in more detail.

**Proposition 10 (A parametrization of the set \(\hat{S}_R(e)\))** The set \(\hat{S}_R(e)\) can be described as:

\[
\hat{S}_R(e) = \left\{ (d,s) \in \mathbb{R}_+ \times [\underline{s}_R, 1] \mid 0 \leq d \leq \overline{d}_R(s,e) \right\},
\]

where:

\[
\overline{d}_R(s,e) = \begin{cases} 
\overline{d}_{R,F}(s,e) & \text{if } \underline{s}_R \leq s < s_{R,1} \\
\overline{d}_{R,m}(s,e) & \text{if } s_{R,1} \leq s < s_{R,2} \\
\overline{d}_{R,b}(s,e) & \text{if } s_{R,2} \leq s \leq 1 
\end{cases}
\]

The thresholds \(\underline{s}_R < s_{R,1} < s_{R,2} < 1\) are given by:

\[
\underline{s}_R = \frac{1}{1 + \frac{1-\chi}{\chi} \frac{1+r_b}{1+r_m}}
\]

\[
s_{R,1} = \frac{1}{1 + \frac{1-\chi}{\chi} \frac{1+r_b}{1+r_m} \frac{I(\phi)}{E(\phi)}}
\]

\[
s_{R,2} = \frac{1}{1 + \frac{1-\chi}{\chi} \frac{1+r_b}{1+r_m} \frac{I(\phi)}{E(\phi)}}
\]

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while the functions $\bar{d}_{R,F}(s,e)$, $\bar{d}_{R,m}(s,e)$ and $\bar{d}_{R,b}(s,e)$ are implicitly defined by:

$$G^{-1} \left( \frac{(1+r_b)s\bar{d}_{R,F}(s,e)}{\chi(e+d_{R,F}(s,e))} \right) = I^{-1} \left( \frac{(1+r_m)(1-s)\bar{d}_{R,F}(s,e)}{(1-\chi)(e+d_{R,F}(s,e))} \right)$$

$$\frac{(1+r_m)(1-s)\bar{d}_{R,m}(s,e)}{(1-\chi)(e+d_{R,m}(s,e))} = I(\phi_I)$$

$$\frac{(1+r_b)s\bar{d}_{R,b}(s,e)}{\chi(e+d_{R,b}(s,e))} = E(\phi)$$

**Proof.** For this proof, I proceed in four steps:

**Step 1:** If $(d,s) \in \tilde{S}_R(e)$, then $\underline{s}_R \leq s \leq 1$;

**Step 2:** If $\underline{s}_R \leq s < s_{R,1}$, then, $(d,s) \in \tilde{S}_R(e)$, if and only if, condition (frontier – R) is verified;

**Step 3:** If $s_{R,1} \leq s < s_{R,2}$, then, $(d,s) \in \tilde{S}_R(e)$, if and only if, condition (c – market) is verified;

**Step 4:** If $s_{R,2} \leq s \leq 1$, then, $(d,s) \in \tilde{S}_R(e)$, if and only if, condition (c – bank) is verified.

**Step 1:** First, I prove that when $s < \underline{s}_R$, condition (frontier – R) does not hold, so that $\tilde{S}_R(e)$ cannot contain debt structures with $s < \underline{s}_R$. To see this, first note that $\forall 0 \leq x \leq \phi_I$, $G(x) \geq I(x)$, so that $\forall 0 \leq y \leq I(\phi_I)$, $G^{-1}(y) \leq I^{-1}(y)$ (with equality only at $y = 0$). Note moreover that:

$$s < \underline{s}_R \implies \frac{(1+r_b)ds}{\chi(e+d)^c} < \frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^c}$$

Thus,

$$s < \underline{s}_R \implies G^{-1} \left( \frac{(1+r_b)ds}{\chi(e+d)^c} \right) < G^{-1} \left( \frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^c} \right)$$

$$\implies G^{-1} \left( \frac{(1+r_b)ds}{\chi(e+d)^c} \right) < I^{-1} \left( \frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^c} \right).$$

Therefore, $(d,s) \in \tilde{S}_R(e) \implies s \geq \underline{s}_R$.

**Step 2:** Next, I prove that when $(d,s) \in \tilde{S}_R(e)$, $\underline{s}_R \leq s < s_{R,1}$, then only condition (frontier – R) is relevant to the definition of $\tilde{S}_R(e)$; that is,

$$(d,s) \in \tilde{S}_R(e) \text{ and } \underline{s}_R \leq s < s_{R,1} \implies \left( \frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^c} < I(\phi_I) \text{ and } \frac{(1+r_b)ds}{\chi(e+d)^c} < E(\phi) \right). \quad (19)$$
I establish this by proving the contraposition. First, if \( \frac{(1+r_h)ds}{\chi(e+d)\xi} > \mathbb{E}(\phi) \), then \((d,s) \notin \tilde{S}_R(e)\). If \( \frac{(1+r_m)ds}{\chi(e+d)\xi} = \mathbb{E}(\phi) \), then:

\[
\mathbb{E}(\phi) = \frac{(1+r_h)ds}{\chi(e+d)\xi} < \frac{(1+r_h)s_{R,1}d}{\chi(e+d)\xi} = \frac{(1+r_m)(1-s_{R,1})d G(\phi_I)}{(1-\chi)(e+d)\xi I(\phi_I)} < \frac{(1+r_m)(1-s)d G(\phi_I)}{(1-\chi)(e+d)\xi I(\phi_I)}
\]

Therefore:

\[
\frac{(1+r_m)(1-s_{R,1})d}{(1-\chi)(e+d)\xi} > \frac{\mathbb{E}(\phi)}{G(\phi_I)} I(\phi_I) > I(\phi_I) \implies (d,s) \notin \tilde{S}_R(e).
\]

Second, if \( \frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)\xi} > I(\phi_I) \), then \((d,s) \notin \tilde{S}_R(e)\). If \( \frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)\xi} = I(\phi_I) \), then:

\[
I(\phi_I) = \frac{(1+r_h)(1-s)d}{(1-\chi)(e+d)\xi} > \frac{(1+r_m)(1-s_{1,R})d}{(1-\chi)(e+d)\xi} \frac{(1+r_h)s_{R,1}d I(\phi_I)}{\chi(e+d)\xi G(\phi_I)} \frac{(1+r_h)s_{1,R}d I(\phi_I)}{\chi(e+d)\xi G(\phi_I)}
\]

Therefore,

\[
\frac{(1+r_h)ds}{\chi(e+d)\xi} < G(\phi_I) \implies G^{-1}\left(\frac{(1+r_h)ds}{\chi(e+d)\xi}\right) < \phi_I = I^{-1}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)\xi}\right) \implies (d,s) \notin \tilde{S}_R(e),
\]

which finishes proving statement (19). It is then straightforward to show that condition (frontier – R) is equivalent to \( 0 \leq d \leq \tilde{a}_{R,F}(s,e) \).

**Step 3:** Next, I prove that when \((d,s) \in \tilde{S}_R(e), s_{R,1} < s < s_{R,2}\), then only condition \((c - \text{market})\) is relevant to the definition of \(\tilde{S}_R(e)\); that is,

\[
\left((d,s) \in \tilde{S}_R(e) \text{ and } s_{R,1} < s < s_{R,2}\right) \implies \\
\left(\frac{(1+r_h)ds}{\chi(e+d)\xi} < \mathbb{E}(\phi) \text{ and } G^{-1}\left(\frac{(1+r_h)ds}{\chi(e+d)\xi}\right) > I^{-1}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)\xi}\right)\right) \tag{20}
\]
Finally, I prove that when $(d, s) \in \tilde{S}_R(e)$, then:

\[
\frac{(1 + r_b) s d}{\chi(e + d) \zeta} = \frac{1 - \chi}{1 + r_m} \frac{1 + r_b}{1 + r_m} \frac{s}{1 - s} \frac{(1 + r_m)(1 - s)d}{(e + d) \zeta} \leq \frac{1 - \chi}{1 + r_m} \frac{1 + r_b}{1 + r_m} I(\phi_I) < \frac{1 - \chi}{1 + r_m} \frac{s_{R, 2}}{1 - s_{R, 2}} I(\phi_I) = E(\phi)
\]

so that condition $(c - \text{bank})$ is satisfied with strict inequality. I next prove that when $(d, s) \in \tilde{S}_R(e)$ and $s_{R, 1} < s$, then condition $(\text{frontier} - R)$ holds with strict inequality. As a first step, define $\Delta(x) = \frac{G(\phi_I)}{I(\phi_I)} I(x) - G(x)$. Note that $\Delta(0) = \Delta(\phi_I) = 0$. Moreover, $\Delta'(x) = \left(\frac{G(\phi_I)}{I(\phi_I)} - 1\right) (1 - F(x)) - \frac{G(\phi_I)}{I(\phi_I)} f(x)$, so that, following steps similar to the proof of lemma 2, $\Delta(.)$ is increasing on $[0, \phi]$ and decreasing on $[\phi, \phi_I]$, where $\underline{\phi} < \phi_I$ is the unique solution to $\Delta'(x) = 0$. Thus,

\[0 \leq x \leq \phi_I \implies \frac{G(\phi_I)}{I(\phi_I)} I(x) \geq G(x),\]

so that:

\[0 \leq y \leq I(\phi_I) \implies G^{-1}\left(\left(\frac{G(\phi_I)}{I(\phi_I)} y\right) \right) \geq I^{-1}(y)\]

Moreover, using the definition of $s_{R, 1}$, note that:

\[s > s_{R, 1} \implies \frac{(1 + r_b) s d}{\chi(e + d) \zeta} > \frac{G(\phi_I)}{I(\phi_I)} \frac{(1 + r_m)(1 - s)d}{(1 - \chi)(e + d) \zeta}\]

Thus,

\[s > s_{R, 1} \implies G^{-1}\left(\frac{(1 + r_b) s d}{\chi(e + d) \zeta}\right) > G^{-1}\left(\frac{G(\phi_I)}{I(\phi_I)} \frac{(1 + r_m)(1 - s)d}{(1 - \chi)(e + d) \zeta}\right) \geq I^{-1}\left(\frac{(1 + r_m)(1 - s)d}{(1 - \chi)(e + d) \zeta}\right),\]

This finishes establishing statement (20). It is then straightforward to show that condition $(c - \text{market})$ is equivalent to $0 \leq d \leq \tilde{a}_{R, m}(s, e)$.

**Step 4:** Finally, I prove that when $(d, s) \in \tilde{S}_R(e)$, $s_{R, 2} < s \leq 1$, then only condition $(c - \text{bank})$ is relevant to the definition of $\tilde{S}_R(e)$; that is,

\[
\left(\left(\frac{1 + r_m}{1 - \chi} \frac{d(1 - s)}{(e + d) \zeta} < I(\phi_I) \text{ and } G^{-1}\left(\frac{(1 + r_b) s d}{\chi(e + d) \zeta}\right) > I^{-1}\left(\frac{(1 + r_m)(1 - s)d}{(1 - \chi)(e + d) \zeta}\right)\right) \text{ and } (d, s) \in \tilde{S}_R(e) \text{ and } s_{R, 2} < s < 1 \right) \implies
\]

\[\text{ (21) }\]
This is straightforward: note that \( s > s_{R,2} \) immediately implies the first part of statement (21), while the second part obtains because \( s > s_{R,2} > s_{R,1} \). It is then straightforward to show that condition \((c - \text{bank})\) is equivalent to \( 0 \leq d \leq \bar{d}_{R,b}(s,e) \).

For the thresholds \( s_{R,1} \) and \( s_{R,2} \), it is simple to establish that when \( s = s_{R,1} \), conditions \((c - \text{market})\) and \((\text{frontier} - R)\) coincide, and both imply condition \((c - \text{bank})\); while, when \( s = s_{R,2} \), conditions \((c - \text{market})\) and \((c - \text{bank})\) coincide, and imply condition \((\text{frontier} - R)\). This concludes the proof of the proposition.

I next turn to the structure of the set \( \tilde{S}_B(e) \).

**Proposition 11 (A parametrization of the set \( \tilde{S}_B(e) \))** The set \( \tilde{S}_B(e) \) can be parametrized as:

\[
\tilde{S}_B(e) = \left\{ (d,s) \in \mathbb{R}_+ \times [0,\bar{s}_B) \mid d_B(s,e) < d \leq \bar{d}_B(s,e) \right\},
\]

where:
\[
d_B(s,e) = \begin{cases} 
0 & \text{if } 0 \leq s < \bar{s}_R \\
\bar{d}_{B,F}(s,e) & \text{if } \bar{s}_R \leq s \leq \bar{s}_B
\end{cases}
\]

The threshold \( \bar{s}_B \) is given by:
\[
\bar{s}_B = \frac{1}{1 + \left( \frac{1}{\chi} \frac{M(\phi_M)}{G(\phi_M)} - 1 \right) \frac{1 + r_b}{1 + r_m}}
\]

while the functions \( \bar{d}_{B,F}(s,e) \) and \( \bar{d}_{R,m}(s,e) \) are implicitly defined by:
\[
G^{-1} \left( \frac{(1 + r_b)s d_{B,F}(s,e)}{\chi(e + d_{B,F}(s,e))} \right) = M^{-1} \left( \frac{(1 + (1 - s)r_m + sr_b) d_{B,F}}{(e + d_{B,F}(s,e))} \right)
\]
\[
\frac{(1 + (1 - s)r_m + sr_b) \bar{d}_B}{(e + \bar{d}_B(s,e))} = M(\phi_M)
\]

**Proof.** For this proof, I proceed in three steps:

**Step 1:** If \((d,s) \in \tilde{S}_B(e)\), then \( 0 \leq s \leq \bar{s}_B \);

**Step 2:** Let \((d,s) \in \mathbb{R}_+ \times [0,\bar{s}_B]\); then if condition \((c - \text{joint})\) holds, condition \((c - \text{bank})\) also holds;

**Step 3:** Let \((d,s) \in \mathbb{R}_+ \times [0,\bar{s}_B]\); then conditions \((\text{frontier} - R)\) and \((c - \text{joint})\) are equivalent to, respectively, lower and upper bounds on \(d\).
Step 1: I prove the contraposition. Assume that \( s > \bar{s}_B \). Note that in this case,

\[
\frac{(1 + r_b)s}{\chi} \frac{d}{(e + d)^\zeta} > \frac{G(\phi_M)}{M(\phi_M)} [(1 + r_m)(1 - s) + (1 + r_b)s] \frac{d}{(e + d)^\zeta},
\]

for any \( d, e > 0 \). Moreover, using steps analogous to the proof of proposition 8, one can establish that,

\[
\forall 0 \leq y \leq M(\phi_M), \quad G^{-1} \left( \frac{G(\phi_M)}{M(\phi_M)} y \right) \geq M^{-1}(y).\]

Thus,

\[
G^{-1} \left( \frac{(1 + r_b)sd}{\chi(e + d)^\zeta} \right) \geq G^{-1} \left( \frac{G(\phi_M)}{M(\phi_M)} [(1 + r_m)(1 - s) + (1 + r_b)s] \frac{d}{(e + d)^\zeta} \right) \geq M^{-1} \left( [(1 + r_m)(1 - s) + (1 + r_b)s] \frac{d}{(e + d)^\zeta} \right),
\]

where the first line uses the second result above, and the second line uses the first result. This is a violation of \((\text{frontier} - R)\), so \((d, s) \notin \tilde{S}_B(e)\). This proves the result of announced.

Step 2: This result follows from using the fact that, when \( s \leq \tilde{s}_B \),

\[
\frac{(1 + r_b)s}{\chi} \frac{d}{(e + d)^\zeta} \leq \frac{G(\phi_M)}{M(\phi_M)} [(1 + r_m)(1 - s) + (1 + r_b)s] \frac{d}{(e + d)^\zeta}.
\]

When condition \((c - \text{joint})\) holds, we therefore have:

\[
\frac{(1 + r_b)s}{\chi} \frac{d}{(e + d)^\zeta} \leq G(\phi_M) < E(\phi)
\]

where the last inequality follows from the facts that \( G(.) \) is strictly increasing. This shows that condition \((c - \text{bank})\) holds. Condition \((c - \text{bank})\) is therefore irrelevant to the definition of \( \tilde{S}_B(e) \); only conditions \((\text{frontier} - R)\) and \((c - \text{joint})\) matter.

Step 3: I first show that the condition \((\text{frontier} - R)\) is tantamount to a lower bound on \( d \). First, note that if \( 0 \leq \tilde{s}_B \), then for any \( d, e > 0 \):

\[
\frac{(1 + r_b)sd}{\chi(e + d)^\zeta} \leq \frac{(1 + r_m (1 - s) + r_b s)d}{\chi(e + d)^\zeta}
\]

This implies that, when \( \frac{(1 + r_m (1 - s) + r_b s)d}{\chi(e + d)^\zeta} \leq M(\phi_M) \), ie condition \((c - \text{joint})\), then:

\[
G^{-1} \left( \frac{(1 + r_b)sd}{\chi(e + d)^\zeta} \right) \leq G^{-1} \left( \frac{(1 + r_m (1 - s) + r_b s)d}{(e + d)^\zeta} \right) < M^{-1} \left( \frac{(1 + r_m (1 - s) + r_b s)d}{(e + d)^\zeta} \right).
\]
Thus, condition (frontier – R) is automatically verified. When $s_B \leq s \leq \bar{s}_B$, it is straightforward to check that condition (frontier – R) is equivalent to $d \geq d_{B,F}(s,e)$, where $d_{B,F}(s,e)$ is defined in the statement of the proposition. Finally, it is also straightforward to show that condition (c – joint) corresponds to is equivalent to the upper bound $d_B(s,e)$ given in the proposition.

The two previous propositions are useful to characterize the two sets that we are actually interested in, that is, the sets $S_R(e)$ and $S_B(e)$. Recall that $S_R(e) = \tilde{S}_R(e)$, so that the parametrization established in proposition 8 is also characterizes $\tilde{S}_R(e)$. We are left with the task of describing the intersection $\tilde{S}_R(e) \cap \tilde{S}_B(e)$.

This is the object of the following proposition.

**Proposition 12 (A parametrization of the intersection $\tilde{S}_R(e) \cap \tilde{S}_B(e)$)** The intersection $\tilde{S}_R(e) \cap \tilde{S}_B(e)$ can be parametrized as:

$$\tilde{S}_R(e) \cap \tilde{S}_B(e) = \{ (d,s) \in \mathbb{R}_+ \times [\underline{s}_R, \bar{s}_B] \mid d_B(s,e) \leq d \leq \overline{d}_{R \cap B}(s,e) \},$$

where:

$$\overline{d}_{R \cap B}(s,e) = \begin{cases} d_R(s,e) & \text{if } \underline{s}_R \leq s < \underline{s}_{R \cap B} \\ d_B(s,e) & \text{if } \underline{s}_{R \cap B} \leq s \leq \bar{s}_B \end{cases}$$

where the threshold $s_{R \cup B}$ is given by:

$$s_{R \cup B} = \frac{1}{1 + \frac{1}{\min(\pi_k - 
abla \psi, \frac{1}{1+\chi})^{-1} \frac{1+r_b}{1+r_m}}}$$

and satisfies $\underline{s}_R < s_{R \cup B} < s_{R,1}$.

**Proof.** Note first that the intersection $\tilde{S}_B(e) \cap \tilde{S}_R(e)$ may only contain debt structures with $\underline{s}_R \leq s \leq \bar{s}_B$.

Given this, the proof of this theorem proceeds in three steps:

**Step 1:** If $\underline{s}_R \leq s \leq s_{R,1}$, then $d_R(s,e) = d_B(s,e)$, so that $\tilde{S}_B(e) \cap \tilde{S}_R(e)$ contains no elements such that $\underline{s}_R \leq s \leq s_{R,1}$;

**Step 2:** If $s_{R,1} \leq s \leq \bar{s}_B$, then $d_R(s,e) > d_B(s,e)$, so that $\tilde{S}_B(e) \cap \tilde{S}_R(e)$ contains debt structures $(d,s)$ for each $s_{R,1} \leq s \leq \bar{s}_B$;

**Step 3:** There is a unique $s_{R,1} < s_{R \cup B} < \bar{s}_B$ such that $d_B(s,e) \geq d_R(s,e)$, if and only if, $s \leq s_{R \cup B}$. 

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Step 1: When \( s_R \leq s \leq s_{R,1} \), using proposition 8, \( d_R(s,e) \) solves:

\[
I^{-1} \left( \frac{(1 + r_m)(1 - s)d_R(s,e)}{(1 - \chi)(e + d_R(s,e))} \right) = G^{-1} \left( \frac{(1 + r_b)s d_R(s,e)}{\chi(e + d_R(s,e))} \right).
\]

This implies that:

\[
x_1 \equiv \frac{(1 + r_m)(1 - s)d_R(s,e)}{(1 - \chi)(e + d_R(s,e))} \leq I(\phi_I)
\]

\[
x_2 \equiv \frac{(1 + r_b)s d_R(s,e)}{\chi(e + d_R(s,e))} \leq G(\phi_I)
\]

Therefore,

\[
(1 - \chi)x_1 + \chi x_3 = \frac{(1 + r_m)(1 - s) + r_b s d_R(s,e)}{(e + d_R(s,e))} \leq (1 - \chi)I(\phi_I) + \chi G(\phi_I)
\]

\[
= M(\phi_I)
\]

\[
< M(\phi_M)
\]

Thus, \( M^{-1}( (1 - \chi) + x_1 + \chi x_2 ) \) is well-defined. Furthermore:

\[
I^{-1}(x_1) = G^{-1}(x_2) \implies x_2 = G \left( I^{-1}(x_1) \right)
\]

\[
\implies x_2 = x_1 + \int_{0}^{I^{-1}(x_1)} \phi dF(\phi)
\]

\[
\implies \chi x_2 + (1 - \chi)x_1 = M \left( I^{-1}(x_1) \right)
\]

\[
\implies M^{-1} \left( \chi x_2 + (1 - \chi)x_1 \right) = I^{-1}(x_1) = G^{-1}(x_2)
\]

Therefore, \( d_R(s,e) \) also solves:

\[
M^{-1} \left( \frac{(1 + r_m)(1 - s) + r_b s d_R(s,e)}{(e + d_R(s,e))} \right) = G^{-1} \left( \frac{(1 + r_b)s d_R(s,e)}{\chi(e + d_R(s,e))} \right),
\]

so that \( d_R(s,e) = d_B(s,e) \). This concludes the proof of step 1.

Step 2: For \( s_{R,1} \leq s \leq s_B \), \( d_B(s,e) \) solves:

\[
M^{-1} \left( \frac{(1 + r_m)(1 - s) + r_b s d_B(s,e)}{(e + d_B(s,e))} \right) = G^{-1} \left( \frac{(1 + r_b)s d_B(s,e)}{\chi(e + d_B(s,e))} \right),
\]
while \( d_R(s, e) \) solves one of the two conditions:

\[
\frac{(1 + r_m)(1 - s)d_R(s, e)}{(1 - \chi)(e + d_R(s, e))^\zeta} = I(\phi_I)
\]

or

\[
\frac{(1 + r_b)sd_R(s, e)}{\chi(e + d_R(s, e))^\zeta} = \mathbb{E}(\phi).
\]

In order to show that \( d_R(s, e) > d_B(s, e) \), we need to prove that at \( d = d_B(s, e) \), we have:

\[
\frac{(1 + r_m)(1 - s)d_B(s, e)}{(1 - \chi)(e + d_B(s, e))^\zeta} < I(\phi_I)
\]

and

\[
\frac{(1 + r_b)sd_B(s, e)}{\chi(e + d_B(s, e))^\zeta} < \mathbb{E}(\phi).
\]

The second condition follows from the fact that because of the definition of \( d_B(s, e) \), it must be the case that:

\[
x_2 \equiv \frac{(1 + r_b)sd_B(s, e)}{\chi(e + d_B(s, e))^\zeta} \leq G(\phi_M) < \mathbb{E}(\phi).
\]

Furthermore, let:

\[
x_3 = \frac{(1 + r_m(1 - s) + r_b s)d_B(s, e)}{(e + d_B(s, e))^\zeta} = M\left(G^{-1}(x_2)\right)
\]

Then:

\[
x_1 \equiv \frac{(1 + r_m)(1 - s)d_B(s, e)}{(1 - \chi)(d_B(s, e) + e)^\zeta} = \frac{1}{1 - \chi} (x_3 - \chi x_2)
\]

\[
= \frac{1}{1 - \chi} (M\left(G^{-1}(x_2)\right) - \chi x_2)
\]

Note that \( \forall 0 \leq x \leq \phi_M, M(x) = \chi G(x) + (1 - \chi)x(1 - F(x)) \). Therefore:

\[
M\left(G^{-1}(x_2)\right) = \chi x_2 + (1 - \chi)G^{-1}(x_2)\left(1 - F\left(G^{-1}(x_2)\right)\right)
\]

\[
M\left(G^{-1}(x_2)\right) - \chi x_2 = (1 - \chi)G^{-1}(x_2)\left(1 - F\left(G^{-1}(x_2)\right)\right)
\]

\[
x_1 = G^{-1}(x_2)\left(1 - F\left(G^{-1}(x_2)\right)\right) < I(\phi_I)
\]

where the last inequality follows because \( I(x) = x(1 - F(x)) \) is maximized at \( \phi_I \). This concludes the proof of step 2.
**Step 3:** Define \( s_{R \cap B} \) as in the text of the proposition. I first prove that \( \hat{s}_{R,1} < s_{R \cap B} < \hat{s}_{R,2} \). Note that \( s_{R,1} < s_{R \cap B} \), if and only if:

\[
M(\phi_M) > \chi G(\phi_I) + (1 - \chi)I(\phi_I).
\]

This statement is true because \( M(\phi_M) > M(\phi_I) = \chi G(\phi_I) + (1 - \chi)I(\phi_I) \). Next, \( s_{R \cap B} < \hat{s}_{R,2} \), if and only if,

\[
M(\phi_M) > \chi E(\phi) + (1 - \chi)I(\phi_I).
\]

To prove this inequality, define \( \Delta(\chi) = M(\phi_M) - \chi E(\phi) - (1 - \chi)I(\phi_I) \) for \( \chi \in [0,1] \). Note that:

\[
\Delta(0) = \Delta(1) = 0
\]

\[
\frac{\partial \Delta}{\partial \chi} \bigg|_{\chi=0} = G(\phi_I) - E(\phi) < 0
\]

\[
\frac{\partial \Delta}{\partial \chi} \bigg|_{\chi=1} = I(\phi_I) > 0
\]

\[
\frac{\partial^2 \Delta}{\partial \chi^2} = \frac{\partial M}{\partial \chi} f(\phi_M) > 0
\]

Taken together, these properties imply that \( \Delta(\chi) \) is strictly convex on \([0,1]\), and that \( \forall \chi \in [0,1], \Delta(\chi) < 0 \), so that the initial inequality holds, and \( s_{R \cap B} < \hat{s}_{R,2} \).

Next, I prove that \( \overline{d}_B(s,e) \geq \overline{d}_R(s,e) \), if and only if, \( s \leq s_{R \cup B} \). First note that by definition of \( s_{R \cap B} \), \( \overline{d}_B(s_{R \cap B}) = \overline{d}_R(s_{R \cap B}) \). To prove the statement, it is therefore sufficient to show that \( \overline{d}_B(s,e) - \overline{d}_R(s,e) \) is strictly decreasing in \( s \). To show this, I establish that \( \overline{d}_B(s,e) \) is strictly decreasing in \( s \), and that \( \overline{d}_R(s,e) \) is strictly increasing in \( s \). The implicit definition of \( \overline{d}_B(s,e) \) implies that:

\[
\frac{\partial \overline{d}_B(s,e)}{\partial s} = \frac{-\overline{d}_B(s,e)(r_b - r_m)}{1 + (1 - s)r_m + sr_b - \zeta M(\phi_M)(e + \overline{d}_B(s,e))^\zeta} \leq 0
\]

where the last inequality follows from the fact that \( r_b > r_m \). Likewise, using the implicit definition of \( \overline{d}_R(s,e) \), one obtains that:

\[
\frac{\partial \overline{d}_R(s,e)}{\partial s} = \frac{\overline{d}_R(s,e)(e + \overline{d}_R(1 + r_m))}{(1 + r_m)(1 - s)e + (1 - \zeta)I(\phi_I)(e + \overline{d}_R(s,e))^\zeta} > 0
\]

This concludes the proof of step 3.
Step 3 shows that elements of $\tilde{S}_B(e) \cap \tilde{S}_R(e)$ with $s < s_{R\cup B}$ are therefore exactly those with $d_B(s, e) \leq s \leq d_R(s, e)$; while elements of $\tilde{S}_B(e) \cap \tilde{S}_R(e)$ with $s \geq s_{R\cup B}$ are all elements of $\tilde{S}_B(e)$, that is, those with $d_B(s, e) \leq s \leq d_B(s, e)$. This is the main statement of proposition 11.

B Appendix to section 4

Proof of proposition 5. I Assume that $(b = sd, m = (1 - s)d) \in S_K(e)$ and that $((1 + r_m)(1 - s) + (1 + r_b)s)d \geq (1 - \delta)(e + d)$; in that case, using the expression of the return function of the entrepreneur under a K-contract reported in appendix A:

$$\mathbb{E}[\tilde{\pi}(\phi; e, sd, (1 - s)d)] = \left( \int_{\phi_K}^{+\infty} \phi dF(\phi) \right) (e + d)^\zeta - \phi_K (1 - F(\phi_K))(e + d)^\zeta$$

where:

$$\phi_K = \frac{R_K(d, s, e) - (1 - \delta)(e + d)}{(e + d)^\zeta}.$$

The results of lemmas 3-5, we then have that:

$$\mathbb{E}[\tilde{\pi}(\phi; e, sd, (1 - s)d)] + \tilde{E}_{m,K}(R_b, R_m; e + d) + \tilde{E}_b(R_b; e + d) = \left( \int_{\phi_K}^{+\infty} \phi dF(\phi) \right) (e + d)^\zeta - \phi_K (1 - F(\phi_K))(e + d)^\zeta + \chi \left( \int_{0}^{\phi_K} \phi dF(\phi) \right) (e + d)^\zeta - (1 - \chi)F(\phi_K)(1 - \delta)(e + d)$$

$$= \mathbb{E}(\pi(\phi)) - (1 - \chi) \int_{0}^{\phi_K} \pi(\phi)dF(\phi).$$

Since $\tilde{E}_{m,K}(R_b, R_m; e + d) = (1 + r_m)(1 - s)d$ and $\tilde{E}_b(R_b; e + d) = (1 + r_b)sd$, this in turn implies that:

$$\mathbb{E}[\tilde{\pi}(\phi; e, sd, (1 - s)d)] = \mathbb{E}(\pi(\phi)) - (1 - \chi) \int_{0}^{\phi_K} \pi(\phi)dF(\phi) - ((1 + r_m)(1 - s) + (1 + r_b)s)d.$$ 

When instead $((1 + r_m)(1 - s) + (1 + r_b)s)d < (1 - \delta)(e + d)$, the entrepreneur never defaults, and moreover $R_b = (1 + r_b)sd$ and $R_m = (1 + r_m)(1 - s)d$. Thus:

$$\mathbb{E}[\tilde{\pi}(\phi; e, sd, (1 - s)d)] = \mathbb{E}(\pi(\phi)) - (1 + r_m)(1 - s)d - (1 + r_b)sd.$$
This proves the lemma’s claim in the case \((b = sd, m = (1 - s)d) \in \mathcal{S}_K(e)\), with:

\[
\overline{\phi}(e, d, s) = \frac{R_K(d, s, e) - (1 - \delta)(e + d)}{(e + d)\zeta},
\]

where note that \(\overline{\phi}(e, b, m) = 0\), if and only if, \((1 - \delta)(e + b + m) \geq (1 + r_b)b + (1 + r_m)m\).

The proof is similar when \((b = sd, m = (1 - s)d) \in \mathcal{S}_R(e)\), with in that case:

\[
\overline{\phi}(e, d, s) = \frac{R_{R,1}(d, s, e) - (1 - \chi)(1 - \delta)(e + d)}{(1 - \chi)(e + d)\zeta}.
\]
## Appendix to section 5

### C.1 Calibrations

<table>
<thead>
<tr>
<th>Parameters:</th>
<th>Baseline</th>
<th>High cost spread</th>
<th>High cost levels (1)</th>
<th>High cost levels (2)</th>
<th>Low average productivity</th>
<th>High productivity dispersion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_m )</td>
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<td>0.05</td>
<td>0.06</td>
<td>0.06</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
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<td>0.07</td>
<td>0.07</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td>( \chi )</td>
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<td>0.60</td>
<td>0.60</td>
<td>0.60</td>
<td>0.60</td>
<td>0.60</td>
</tr>
<tr>
<td>( \xi )</td>
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<td>1.60</td>
<td>1.60</td>
<td>1.60</td>
<td>1.60</td>
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</tr>
<tr>
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<td>1.84</td>
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<td>1.82</td>
</tr>
<tr>
<td>( \zeta )</td>
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<td>0.92</td>
<td>0.92</td>
<td>0.92</td>
<td>0.92</td>
<td>0.92</td>
</tr>
</tbody>
</table>

| Moments of \( F(\cdot) \): |
|-----------------------------|----------|-------------------|----------------------|----------------------|--------------------------|----------------------------|
| \( \mathbb{E}(\phi) \)     | 1.65     | 1.65              | 1.67                 | 1.65                 | 1.63                     | 1.65                       |
| \( \sigma(\phi) \)         | 1.06     | 1.06              | 1.07                 | 1.06                 | 1.05                     | 1.22                       |
| \( \frac{\sigma(\phi)}{\mathbb{E}(\phi)} \) | 0.64     | 0.64              | 0.64                 | 0.64                 | 0.64                     | 0.74                       |

| Shape of solution:         |
|-----------------------------|----------|-------------------|----------------------|----------------------|--------------------------|----------------------------|
| \( s^*(0) \)               | 0.78     | 0.78              | 0.78                 | 0.78                 | 0.78                     | 0.79                       |
| \( \xi \)                  | 29.19    | 25.20             | 29.24                | 26.03                | 25.81                    | 29.82                      |
| \( s^*(\xi) \)             | 0.97     | 0.96              | 0.97                 | 0.97                 | 0.97                     | 0.98                       |
| \( \tau \)                 | 71.90    | 55.87             | 72.08                | 64.18                | 63.56                    | 74.88                      |
| \( s^*(\tau) \)            | 0.90     | 0.90              | 0.90                 | 0.90                 | 0.90                     | 0.92                       |
| \( k \)                    | 88.94    | 79.31             | 89.04                | 79.28                | 78.63                    | 88.90                      |
| \( e^* \)                  | 100.00   | 100.00            | 100.00               | 89.04                | 88.40                    | 100.00                     |

Table 1: Calibrations of the model discussed in section 5. The parameters \( \xi \) and \( \lambda \) are the location and scale parameters of the Weibull distribution. \( e^* \) refers to the level of equity such that the firm will choose not to borrow from either sources.
C.2 Comparative statics

Figure 13: Effects of an increase in the lending costs: first case Graphs 13(a) and 13(b) report the characteristics of the debt structure in the "Baseline" calibration (black line), and in the "High cost levels (1)" calibration (grey line); see table 1 for details.
Figure 14: **Effects of an increase in the lending costs: second case** Graphs 14(a) and 14(b) report the characteristics of the debt structure in the "Baseline" calibration (black line), and in the "High cost levels (2)" calibration; see table 1 for details.
Figure 15: **Effects of a fall in average productivity** Graphs 15(a) and 15(b) report the characteristics of the debt structure in the "Baseline" calibration (black line), and in the "Low average productivity" calibration; see table 1 for details.
Figure 16: **Effects of an increase in the dispersion of productivity** Graphs 16(a) and 16(b) report the characteristics of the debt structure in the "Baseline" calibration (black line), and in the "High productivity dispersion" calibration; see table 1 for details.
Appendix to section 2

D.1 Variable definitions in the QFR

The QFR survey is constructed on a quarterly basis from two separate samples. The first sample (the "short-form" sample) contains 5000 manufacturing firms with assets less than or equal to 25m$. The second sample (the "long-form" sample) contains manufacturing firms with at least 25m$. Firms with between 25m$ and 250m$ are sampled from the universe of manufacturing firms, while all existing manufacturing firms with more than 250m$ in assets are included. For both samples, the QFR contains a variety of information on firms’ real and financial variables. The samples differ in the detail with which firms report their balance sheets, as detailed below.

<table>
<thead>
<tr>
<th>Asset size class</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assets (m$)</td>
<td>&lt; 5</td>
<td>[5,10)</td>
<td>[10,25]</td>
<td>[25,50]</td>
<td>[50,100]</td>
<td>[100,250]</td>
<td>[250,1000]</td>
<td>≥ 1000</td>
</tr>
</tbody>
</table>

Table 2: Definition of asset size brackets in the QFR.

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>STBANK</td>
<td>Short-term bank debt</td>
<td>STBANK</td>
<td>Short-term bank debt</td>
</tr>
<tr>
<td>INSTBANK</td>
<td>Long-term bank debt with maturity ≤ 1 year</td>
<td>INSTBANK</td>
<td>Long-term bank debt with maturity ≤ 1 year</td>
</tr>
<tr>
<td>STDEBTOTH</td>
<td>Other short-term debt (incl. commercial paper)</td>
<td>STDEBTOTH</td>
<td>Other short-term debt (excl. commercial paper)</td>
</tr>
<tr>
<td>INSTOTH</td>
<td>Other long-term debt with maturity ≤ 1 year (incl. bonds)</td>
<td>INSTOTH</td>
<td>Other long-term debt with maturity ≤ 1 year (excl. bonds)</td>
</tr>
<tr>
<td>LTBNKDEBT</td>
<td>Long-term bank debt with maturity &gt; 1 year</td>
<td>LTBNKDEBT</td>
<td>Long-term bank debt with maturity &gt; 1 year</td>
</tr>
<tr>
<td>LOTHDEBT</td>
<td>Other debt with maturity &gt; 1 year (incl. bonds)</td>
<td>LOTHDEBT</td>
<td>Other debt with maturity &gt; 1 year (excl. bonds)</td>
</tr>
<tr>
<td>LTBNDDEBT</td>
<td>Bonds maturing in ≤ 1 year</td>
<td>LTBNDDEBT</td>
<td>Bonds maturing in &gt; 1 year</td>
</tr>
</tbody>
</table>

Table 3: Financial liabilities reported in the QFR, short and long form samples.

<table>
<thead>
<tr>
<th>Name</th>
<th>Definition for short form sample</th>
<th>Name</th>
<th>Definition for long form sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>CB</td>
<td>STBANK + INSTBANK</td>
<td>CB</td>
<td>LTBNKDEBT</td>
</tr>
<tr>
<td>CNB</td>
<td>STDEBTOTH + INSTOTH</td>
<td>CNB</td>
<td>CBB + LTBNKDEBT + LOTHDEBT</td>
</tr>
<tr>
<td>CNBX</td>
<td>n.a.</td>
<td>CNBX</td>
<td>CBB + LTBNKDEBT + LOTHDEBT</td>
</tr>
<tr>
<td>CFIN</td>
<td>CB + CNB</td>
<td>CFIN</td>
<td>CB + LOTHDEBT</td>
</tr>
</tbody>
</table>

Table 4: Definition of aggregates of financial liabilities.
D.2 Robustness

D.2.1 Excluding commercial paper from non-bank liabilities

Figure 17: Robustness check 1: no commercial paper. Graphs 17(a), 17(b) and 17(c) report the same series as graphs 1(a), 2 and 3, but excluding commercial paper from non-bank debt in the case of long-form sample firms.
D.2.2 An alternative definition of small and large firms

To examine whether facts 2 and 3 are robust to other groupings of "small" and "large" firms, I use the definitions of Gertler and Gilchrist (1994). They use information on total sales of different asset size brackets in order to determine a cutoff between small and large firms (in terms of asset size). The (gross) growth rate of sales of small firms for a quarter is computed as a weighted average of the growth rate of total sales in the two categories that straddle the thirtieth percentiles of cumulative sales. A series for total sales in each size category is then constructed by taking the cumulative sum of the log of the quarterly gross growth rates obtained in this fashion.

I describe this method more formally. Fix \( t \). Let \( \{ x^{(1)}, \ldots, x^{(n)} \} \) denote the asset size brackets for quarter \( t \), and let \( s_{i,t} \) and \( x_{i,t} \) denote sales and total assets of firm \( i \) at time \( t \). First, let \( S_t = \sum_i s_{i,t} \) and define a cutoff in terms of assets, \( x_t \), by:

\[
x_t = \max \left\{ x \in \{ x^{(1)}, \ldots, x^{(n)} \} \left/ \frac{\sum_{i:x_{i,t} \leq x} s_{i,t}}{S_t} \leq 0.3 \right. \right\}.
\]

Furthermore, let \( x_t^+ \) be the cutoff immediately above \( x_t \) in the list \( \{ x^{(1)}, \ldots, x^{(n)} \} \). The cumulative sales of all firms with at most \( x_t^+ \) are less than 30 % of the total of sales of manufacturing firms. Second, compute weights \( w_t \) such that:

\[
w_t \frac{\sum_{x_{i,t} \leq x_t} s_{i,t}}{S_t} + (1 - w_t) \frac{\sum_{x_{i,t} \leq x_t^+} s_{i,t}}{S_t} = 0.3.
\]

Third, for any series \( y_{i,t} \), compute "weighted" growth rates according to:

\[
G_{y,S,t}^{j+1} = w_t \frac{\sum_{i:x_{i,t} \leq x_j} y_{i,t}}{\sum_{i:x_{i,t} \leq x_j} y_{i,t-1}} + (1 - w_t) \frac{\sum_{i:x_{i,t} \geq x_j^+} y_{i,t}}{\sum_{i:x_{i,t} \geq x_j^+} y_{i,t-1}}.
\]

Roughly speaking, the goal of this approximation is to compute the growth rate of the variable \( y \) for a synthetic size class which represents, on average, 30% of total sales. These weighted growth rates form the basis of a synthetic series for small firms for variable \( y \), \( y_{S,t} \), through:

\[
y_{S,t} = \sum_{j=0}^t \log(G_{y,S,j-1,j}^t).
\]

For large firms, Gertler and Gilchrist (1994) use\(^{20}\):

\[
G_{L,t}^{j+1} = w_t \frac{\sum_{i:x_{i,t} \geq x_j} y_{i,t}}{\sum_{i:x_{i,t} \geq x_j} y_{i,t-1}} + (1 - w_t) \frac{\sum_{i:x_{i,t} \geq x_j^+} y_{i,t}}{\sum_{i:x_{i,t} \geq x_j^+} y_{i,t-1}}.
\]

\(^{20}\)In the later part of the sample, the cutoff \( x_t \) turns out to be 168, the largest asset class. For these years, I set the weight \( w_t \) in \( G_{L,t}^{j+1} \) to be equal to 1.
where \( w_t \) is the same value as above, and likewise define a synthetic series for large firms, for the variable \( y \), \( y_{L,t} \), as:

\[
y_{L,t} = \sum_{j=0}^{t} \log(G_{L,j-1,j}).
\]

The results are reported in figures 19(a) and 19(b), when \( y \) corresponds to the series \( CB, CNB, TB \) and \( TNB \) defined above. Consistently with facts 2 and 3, these figures suggest that, while the "synthetic" small firm experiences a contraction in her liabilities driven entirely by a reduction in bank loan, the large firm experiences a substitution of market credit for bank credit. Notably, the upward trend in non-bank liabilities is less noticeable in definition of small and large firms; in turn, over the first year after the recession, the substitution is less than one for one.

Figure 18: **Robustness check 2: broader definition of large firms.** Graphs 19(a) and 19(b) report the same series as graphs 2 and 3, but for the definition of "large" and "small" developed by Gertler and Gilchrist (1994); see text for details.
D.3 Facts 2 and 3 excluding the recent recession

The following graph reports changes in the composition of liabilities of small and large firms in recessions, excluding the 2007-2009 recession. The series are thus the average of the change in liabilities for the 1990-1991 and 2000 recession.

(a) Fact 2: the debt structure of small firms during recessions

(b) Fact 3: the debt structure of large firms during recessions

Figure 19: Robustness check 3: excluding the 2007 recession. Graphs 19(a) and 19(b) report the same series as graphs 2 and 3, excluding the 2007-2009 recession.
D.4 Aggregate financial liabilities in levels

Figure 20: Financial liabilities in levels, baseline small/large classification. All series are smoothed with a 2 by 4 MA smoother to remove seasonal variation. Shaded areas indicate NBER recessions.
Figure 21: Financial liabilities in levels, using the Gertler and Gilchrist (1994) definition of small and large firms. All series are smoothed with a 2 by 4 MA smoother to remove seasonal variation. Shaded areas indicate NBER recessions. Units on the y-axis are arbitrary, since the series are computed from synthetic growth rates; see text for details.